We compared the density function of the optimal proposal and of the gamma proposal.

The improper optimal proposal is:

$$\begin{aligned} p_{t|t-1}(n_t|n_{t-1}, y_t) &\propto p(y_t|n_t) f(n_t|n_{t-1}) \\ &\propto e^{-\phi n_t} n_t^{y_t} \frac{1}{n_t} e^{-\frac{1}{2\sigma^2}(\ln(n_t) - \ln(r n_{t-1} e^{-n_{t-1}})^2} \\ &\propto e^{-\phi n_t} n_t^{y_t - 1} e^{-\frac{1}{2\sigma^2}(\ln(\frac{n_t}{r n_{t-1}}) + n_{t-1})^2} \end{aligned}$$

We found the normalizing constant by finding numerically the value of $\int_0^\infty e^{-\phi n_t} n_t^{y_t - 1} e^{-\frac{1}{2\sigma^2}(\ln(\frac{n_t}{r n_{t-1}}) + n_{t-1})^2} \mathrm{d}n_t$

Figure 1 shows the result of the comparison when we use the approximation which minimizes $D_{KL}(P||Q)(\alpha, \theta) = \int_0^\infty p(z|\mu, \sigma^2) \log(\frac{p(z|\mu,\sigma^2)}{q(z|\alpha,\theta)}) \mathrm{d}z$ where p is the probability density function of a $\log \mathcal{N}(\mu, \sigma^2)$ and q of a Gamma with shape α and scale θ (Approximation 1). We took $\ln(r) = 3.0$, $\sigma^2 = 0.3$ and $\phi = 10$ and set $n_{t-1} = 5$ and $y_t = 7$ when calculating $p_{t|t-1}(n_t|n_{t-1}, y_t)$. The optimal density is blue whereas the density of the gamma approximation is green.
It can be seen that the right tail of the gamma approximation is thinner than te one the optimal proposal whereas the approximation rises sharper near 0. Therefore the relative error is quickly converging to -1 when we are in the right tail of the densities as there is quickly a difference of several orders of magnitude between the two and it shows spike near 0. The absolute error is bounded in absolute value by 0.2 and the maximum is attained around the mode.

Figure **??** shows the result of the comparison when we use the approximation which minimizes $D_{KL}(P||Q)(\alpha, \theta) = \int_0^\infty q(z|\alpha, \theta) \log(\frac{q(z|\alpha,\theta)}{p(z|\mu,\sigma^2)}) \mathrm{d}z$ where p and q are the same as above (Approximation 2). We took $\ln(r) = 3.0$, $\sigma^2 = 0.3$ and $\phi = 10$ and set $n_{t-1} = 5$ and $y_t = 7$ when calculating $p_{t|t-1}(n_t|n_{t-1}, y_t)$. The optimal density is blue whereas the density of the gamma approximation is green.
It can now be seen that right tail of the gamma approximation is thicker than the one of the optimal proposal. The mode of the approximation is also significantly smaller than the optimal one. Here the relative error increases sharply as the approximation becomes orders of magnitudes greater than the optimal proposal in the tail. Once again the relative error shows a spike near 0 for the same reason as above.

Figure 3 compares the optimal proposal density and the gamma approximation Approximation 1 density for different values of $\ln(r)$ and σ^2.

Figure 4 compares the optimal proposal density and the gamma approximation Approximation 2 density for different values of $\ln(r)$ and σ^2.

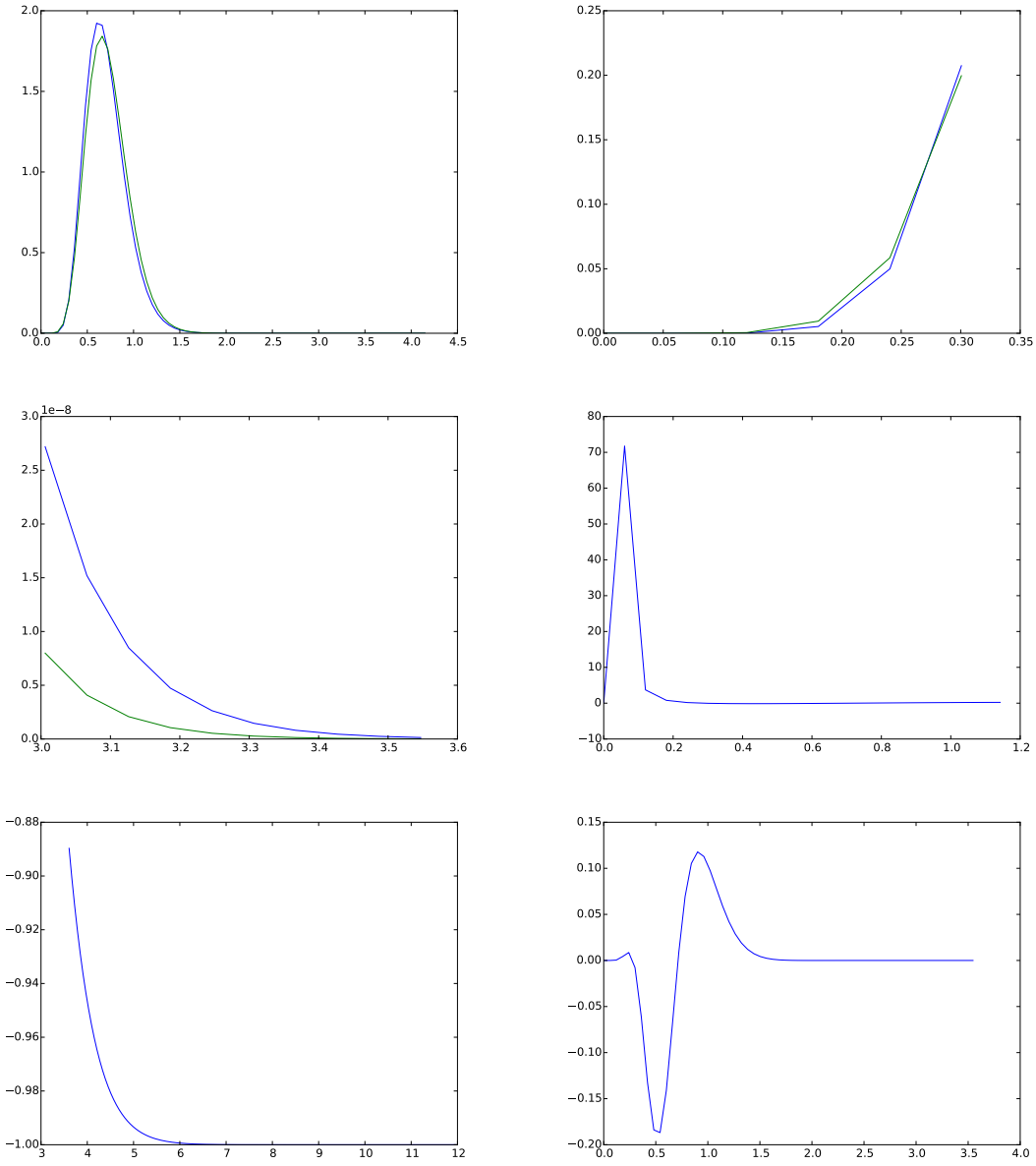

Figure 1: **(first row left)** Comparison between **(blue)** the optimal proposal density and **(green)** the gamma approximation density, **(first row right)** comparison between the left tail of **(blue)** the optimal proposal density and of **(green)** the gamma approximation density, **(second row left)** comparison between the right tail of **(blue)** the optimal proposal density and of **(green)** the gamma approximation density, **(second row right)** relative error between values near 0 of the gamma approximation density and of the optimal proposal density, **(third row left)** relative error between the right tails of the gamma approximation density and of the optimal proposal density, **(third row right)** absolute error between the gamma approximation density and the optimal proposal density.

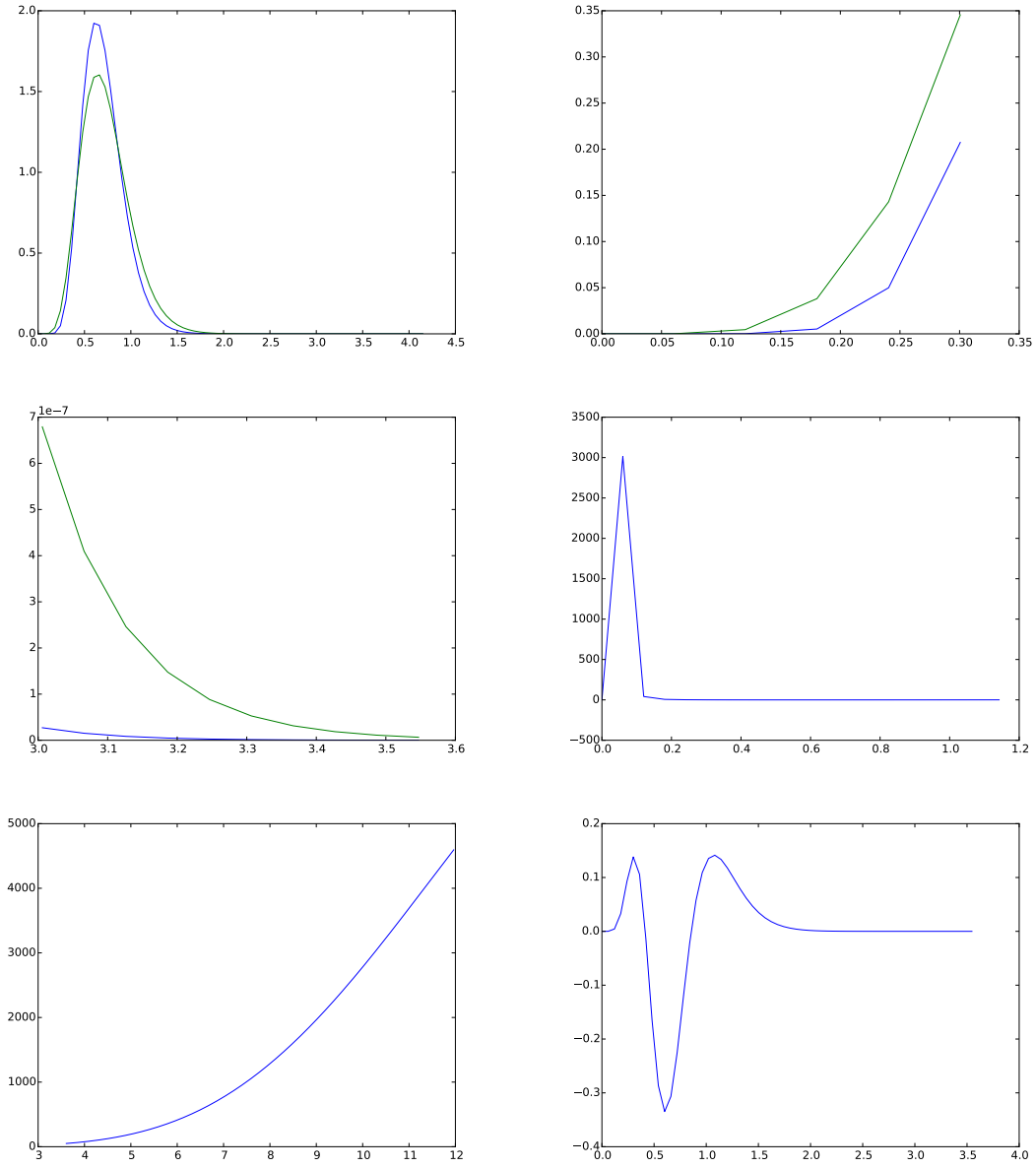

Figure 2: **(first row left)** Comparison between **(blue)** the optimal proposal density and **(green)** the gamma approximation density, **(first row right)** comparison between the left tail of **(blue)** the optimal proposal density and of **(green)** the gamma approximation density, **(second row left)** comparison between the right tail of **(blue)** the optimal proposal density and of **(green)** the gamma approximation density, **(second row right)** relative error between values near 0 of the gamma approximation density and of the optimal proposal density, **(third row left)** relative error between the right tails of the gamma approximation density and of the optimal proposal density, **(third row right)** absolute error between the gamma approximation density and the optimal proposal density.

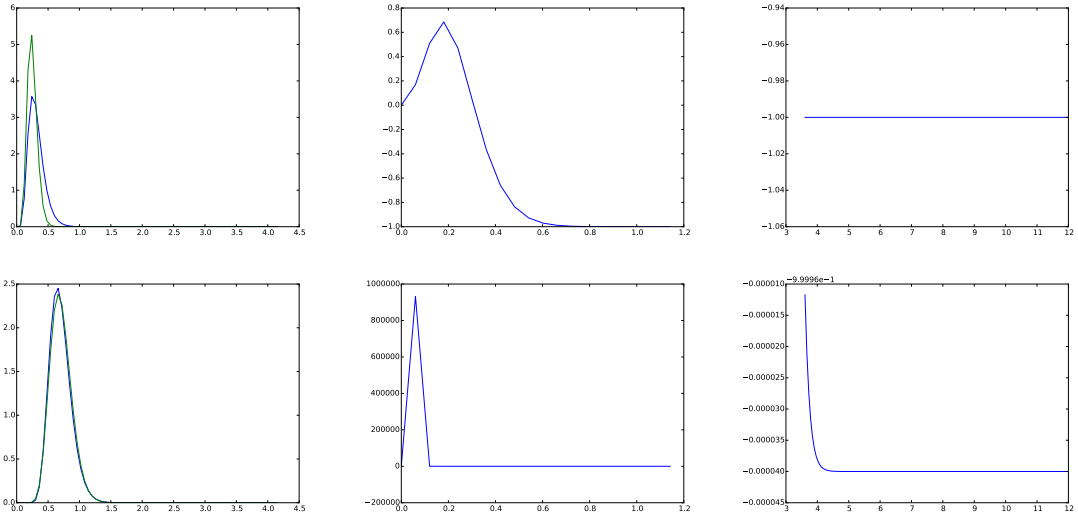

Figure 3: **(first row left)** Comparison between **(blue)** the optimal proposal density and **(green)** the gamma approximation density with $\ln(r) = 0.1$ and $\sigma^2 = 0.3$, **(first row middle)** relative error between values near 0 of the gamma approximation density and the optimal proposal density with $\ln(r) = 0.1$ and $\sigma^2 = 0.3$, **(first row right)** relative error between the right tails of the gamma approximation density and the optimal proposal density with $\ln(r) = 0.1$ and $\sigma^2 = 0.3$, **(second row left)** comparison between **(blue)** the optimal proposal density and **(green)** the gamma approximation density with $\ln(r) = 3$ and $\sigma^2 = 0.1$, **(second row middle)** relative error between values near 0 of the gamma approximation density and the optimal proposal density with $\ln(r) = 3$ and $\sigma^2 = 0.1$, **(second row right)** relative error between the right tails of the gamma approximation density and the optimal proposal density with $\ln(r) = 3$ and $\sigma^2 = 0.1$.

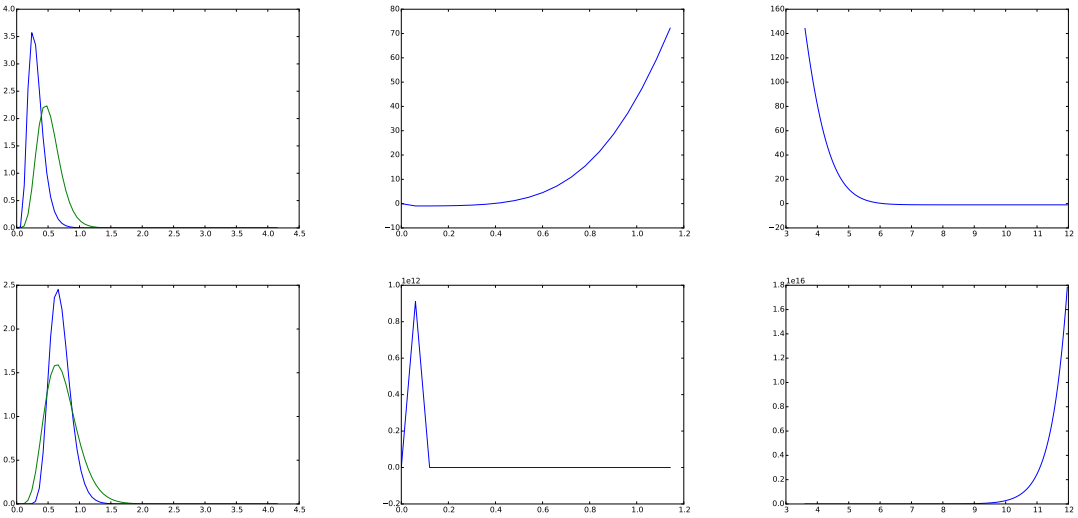

Figure 4: **(first row left)** Comparison between **(blue)** the optimal proposal density and **(green)** the gamma approximation density with $\ln(r) = 0.1$ and $\sigma^2 = 0.3$, **(first row middle)** relative error between values near 0 of the gamma approximation density and the optimal proposal density with $\ln(r) = 0.1$ and $\sigma^2 = 0.3$, **(first row right)** relative error between the right tails of the gamma approximation density and the optimal proposal density with $\ln(r) = 0.1$ and $\sigma^2 = 0.3$, **(second row left)** comparison between **(blue)** the optimal proposal density and **(green)** the gamma approximation density with $\ln(r) = 3$ and $\sigma^2 = 0.1$, **(second row middle)** relative error between values near 0 of the gamma approximation density and the optimal proposal density with $\ln(r) = 3$ and $\sigma^2 = 0.1$, **(second row right)** relative error between the right tails of the gamma approximation density and the optimal proposal density with $\ln(r) = 3$ and $\sigma^2 = 0.1$.