We compared the density function of the optimal proposal and of the gamma proposal.

The improper optimal proposal is:

$$\begin{aligned}
p_{t|t-1}(n_t|n_{t-1}, y_t) &\propto p(y_t|n_t)f(n_t|n_{t-1}) \\
&\propto e^{-\phi n_t} n_t^{y_t} \frac{1}{n_t} e^{-\frac{1}{2\sigma^2}(\ln(n_t)-\ln(rn_{t-1}e^{-n_{t-1}}))^2} \\
&\propto e^{-\phi n_t} n_t^{y_t-1} e^{-\frac{1}{2\sigma^2}(\ln(\frac{n_t}{rn_{t-1}})+n_{t-1})^2}
\end{aligned}$$

We found the normalizing constant by finding numerically the value of $\int_0^\infty e^{-\phi n_t} n_t^{y_t-1} e^{-\frac{1}{2\sigma^2}(\ln(\frac{n_t}{rn_{t-1}})+n_{t-1})^2} \mathrm{d}n_t$

Figure 1 shows the result of the comparison when we use the approximation which minimizes $D_{KL}(P||Q)(\alpha, \theta) = \int_0^\infty p(z|\mu, \sigma^2) \log(\frac{p(z|\mu,\sigma^2)}{q(z|\alpha,\theta)})\mathrm{d}z$ where p is the probability density function of a $\log \mathcal{N}(\mu, \sigma^2)$ and q of a Gamma with shape α and scale θ (Approximation 1). We took $\ln(r) = 3.0$, $\sigma^2 = 0.3$ and $\phi = 10$ and set $n_{t-1} = 5$ and $y_t = 7$ when calculating $p_{t|t-1}(n_t|n_{t-1}, y_t)$. The optimal density is blue whereas the density of the gamma approximation is green.
It can be seen that the right tail of the gamma approximation is thinner than the one the optimal proposal whereas the approximation rises sharper near 0. Therefore the relative error is quickly converging to -1 when we are in the right tail of the densities as there is quickly a difference of several orders of magnitude between the two and there is spike near 0. The absolute error is bounded in absolute value by 0.2 and the maximum is attained around the mode.

Figure 2 shows the result of the comparison when we use the approximation which minimizes $D_{KL}(P||Q)(\alpha, \theta) = \int_0^\infty q(z|\alpha, \theta) \log(\frac{q(z|\alpha,\theta)}{p(z|\mu,\sigma^2)})\mathrm{d}z$ where p and q are the same as above (Approximation 2). We took $\ln(r) = 3.0$, $\sigma^2 = 0.3$ and $\phi = 10$ and set $n_{t-1} = 5$ and $y_t = 7$ when calculating $p_{t|t-1}(n_t|n_{t-1}, y_t)$. The optimal density is blue whereas the density of the gamma approximation is green.
Now, it can be seen that right tail of the gamma approximation is thicker than the one of the optimal proposal. The mode of the approximation is also significantly smaller than the optimal one. Here the relative error increases sharply as the approximation becomes orders of magnitudes greater than the optimal proposal in the tail. Once again the relative error shows a spike near 0 for the same reason as above.

Figure 3 compares the optimal proposal density and the gamma approximation Approximation 1 density for different values of $\ln(r)$ and σ^2. When $\ln(r)$ becomes smaller (σ^2 constant) the overall match between the two densities worsens (the approximation goes more quickly towards 0), but the relative error has a smaller spike near 0 because there the approximation is closer to the optimal proposal. When σ becomes smaller ($\ln(r)$ constant) the overall match between the two densities seems better, but the relative error near zero becomes even spikier. When $\ln(r)$ or σ increases the overall match gets better, the shape of the relative error stays the same although increasing σ reduces the amplitude of this spike.

Figure 4 compares the optimal proposal density and the gamma approximation Approximation 2 density for different values of $\ln(r)$ and σ^2. Both when $\ln(r)$ and σ decrease (one at a time) the overall match worsens. In the first case the approximation becomes more right left skewed, and in the second case more left skewed. When $\ln(r)$ or σ increases the conclusions are the same as above.

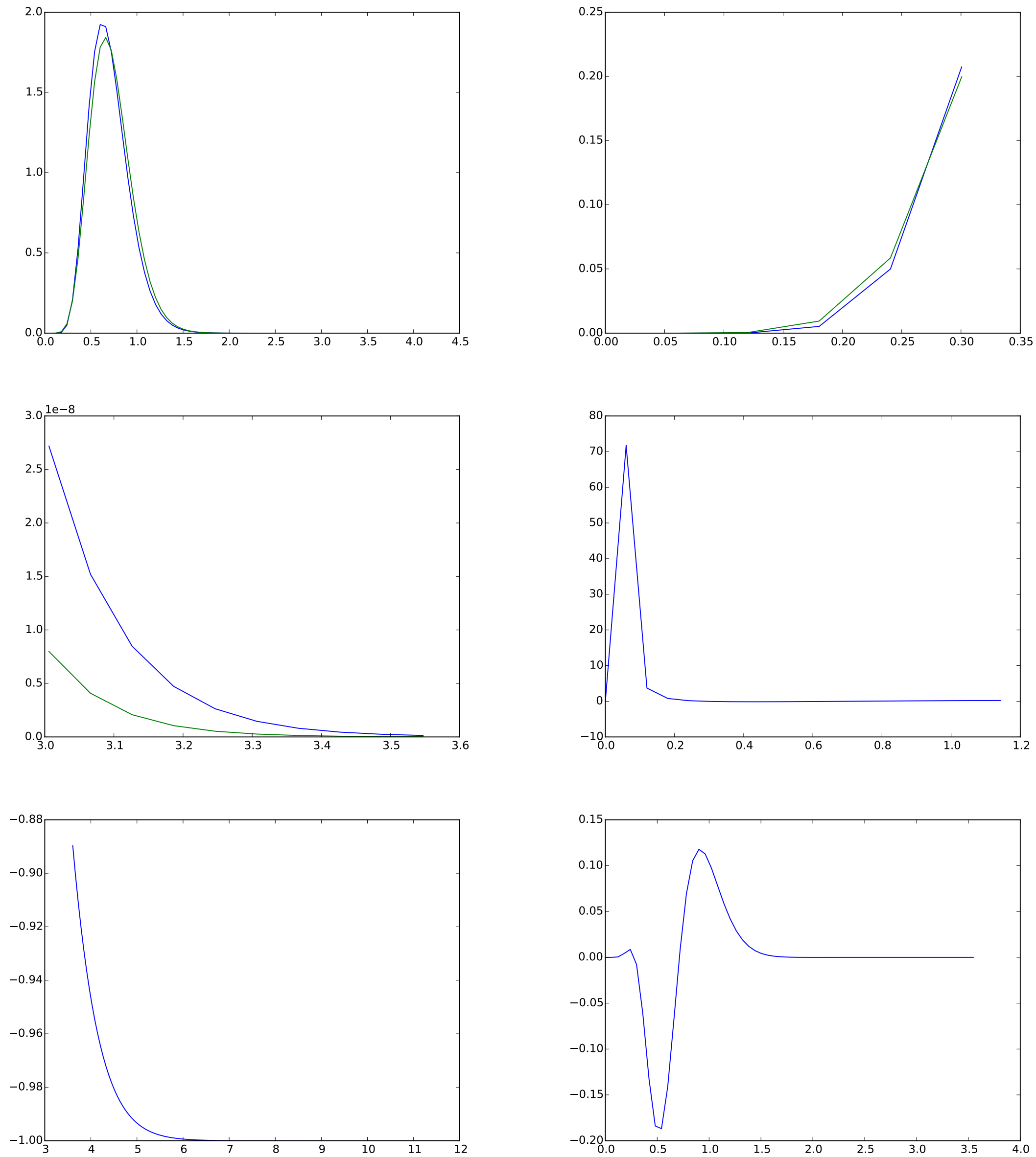

Figure 1: **(first row left)** Comparison between **(blue)** the optimal proposal density and **(green)** the gamma approximation density, **(first row right)** comparison between the left tail of **(blue)** the optimal proposal density and of **(green)** the gamma approximation density, **(second row left)** comparison between the right tail of **(blue)** the optimal proposal density and of **(green)** the gamma approximation density, **(second row right)** relative error between values near 0 of the gamma approximation density and of the optimal proposal density, **(third row left)** relative error between the right tails of the gamma approximation density and of the optimal proposal density, **(third row right)** absolute error between the gamma approximation density and the optimal proposal density.

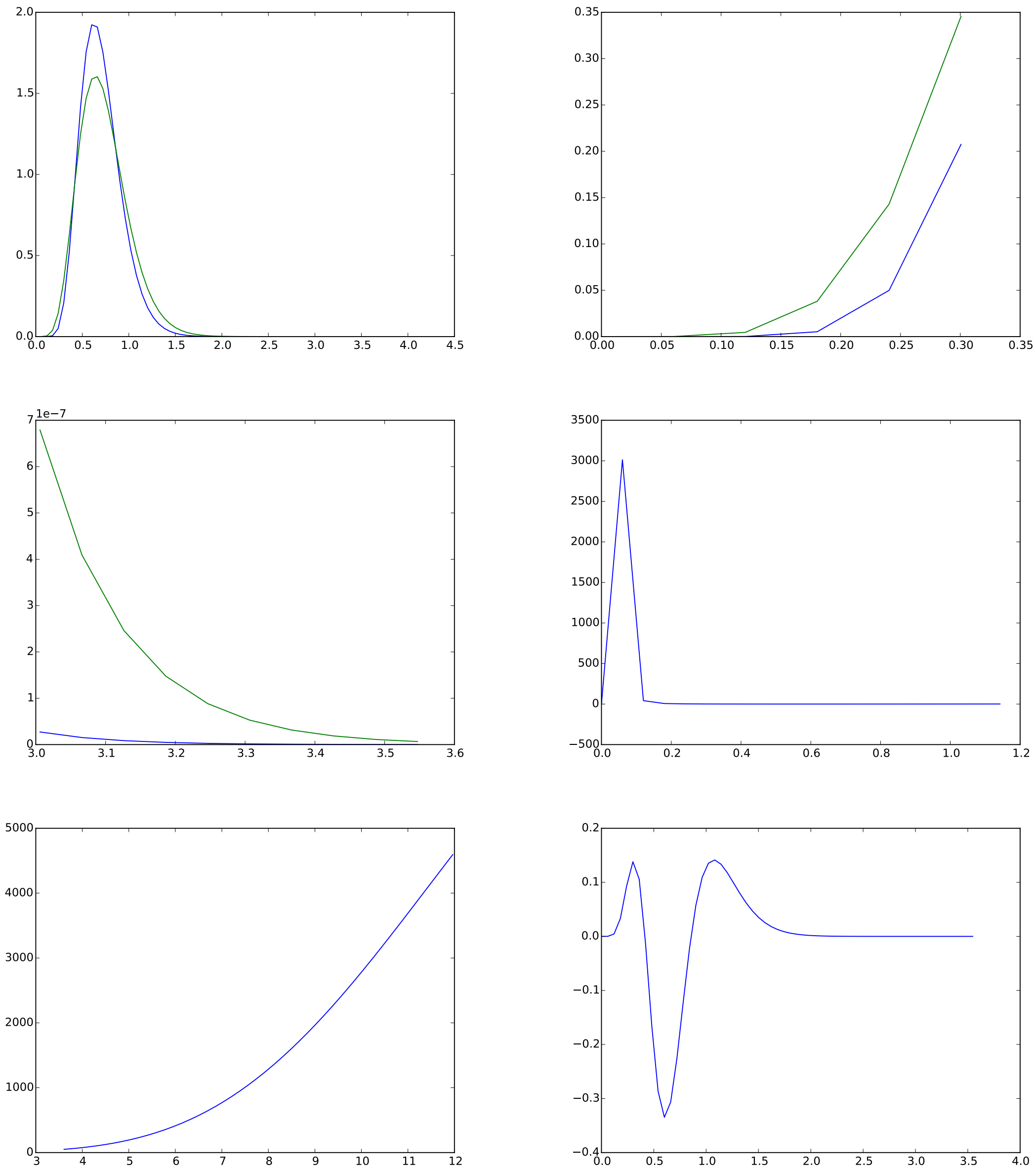

Figure 2: **(first row left)** Comparison between **(blue)** the optimal proposal density and **(green)** the gamma approximation density, **(first row right)** comparison between the left tail of **(blue)** the optimal proposal density and of **(green)** the gamma approximation density, **(second row left)** comparison between the right tail of **(blue)** the optimal proposal density and of **(green)** the gamma approximation density, **(second row right)** relative error between values near 0 of the gamma approximation density and of the optimal proposal density, **(third row left)** relative error between the right tails of the gamma approximation density and of the optimal proposal density, **(third row right)** absolute error between the gamma approximation density and the optimal proposal density.

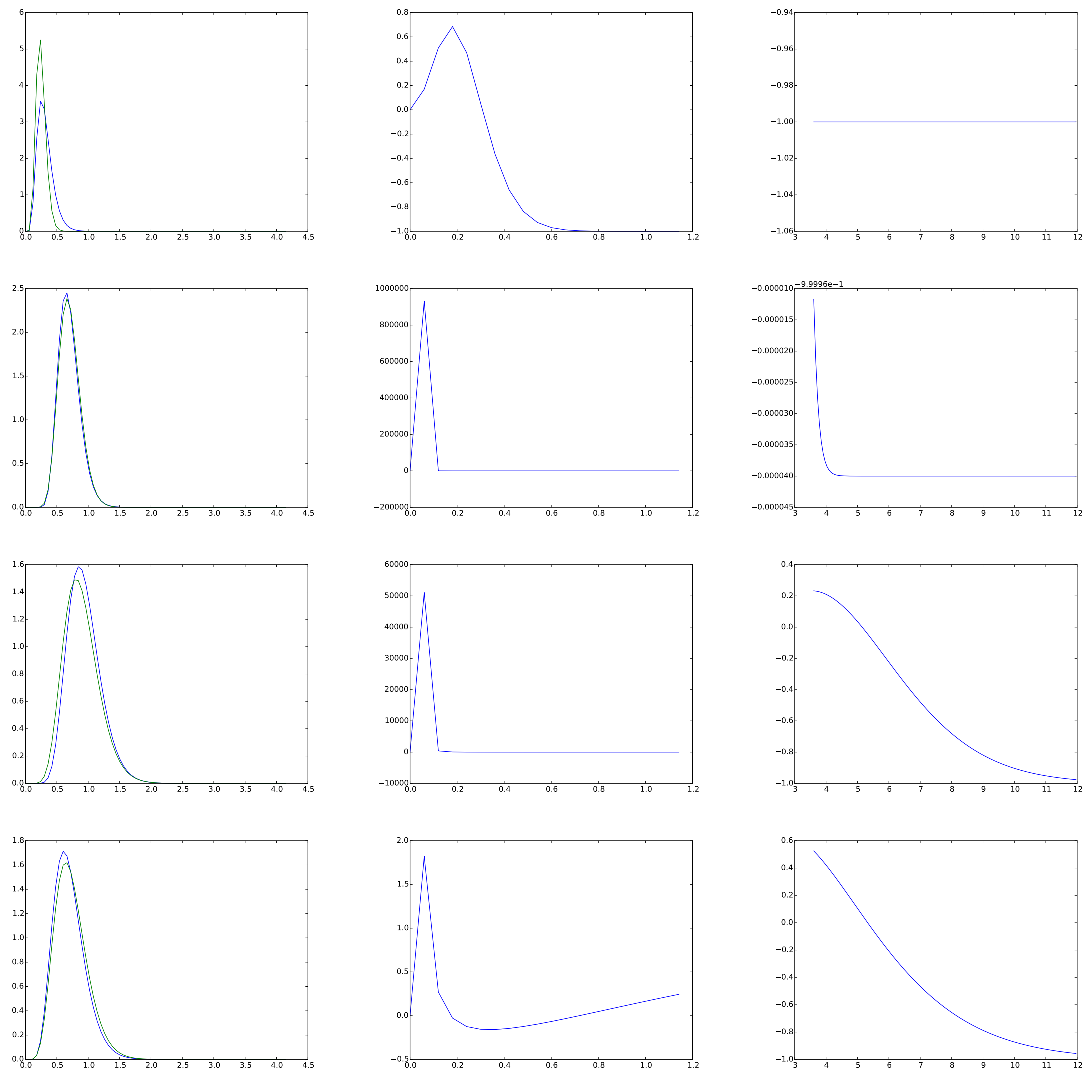

Figure 3: **(left)** Comparison between **(blue)** the optimal proposal density and **(green)** the gamma approximation density, **(middle)** relative error between values near 0 of the gamma approximation density and the optimal proposal density, **(right)** relative error between the right tails of the gamma approximation density and the optimal proposal density for **(first row)** $\ln(r) = 1$ and $\sigma^2 = 0.3$, **(second row)** $\ln(r) = 3$ and $\sigma^2 = 0.1$, **(third row)** $\ln(r) = 4$ and $\sigma^2 = 0.3$, **(fourth row)** $\ln(r) = 3$ and $\sigma^2 = 1$.

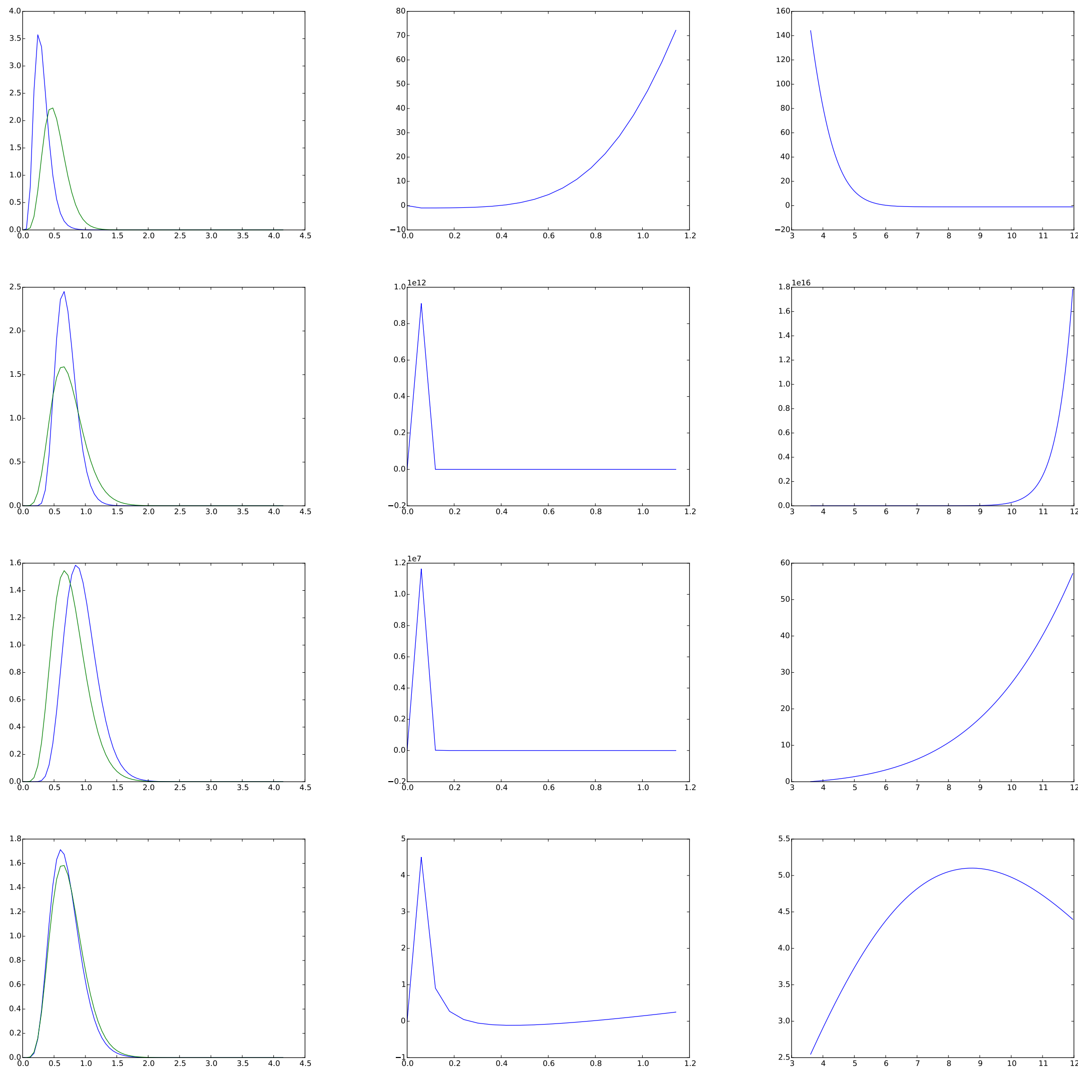

Figure 4: **(left)** Comparison between **(blue)** the optimal proposal density and **(green)** the gamma approximation density, **(middle)** relative error between values near 0 of the gamma approximation density and the optimal proposal density, **(right)** relative error between the right tails of the gamma approximation density and the optimal proposal density for **(first row)** $\ln(r) = 1$ and $\sigma^2 = 0.3$, **(second row)** $\ln(r) = 3$ and $\sigma^2 = 0.1$, **(third row)** $\ln(r) = 4$ and $\sigma^2 = 0.3$, **(fourth row)** $\ln(r) = 3$ and $\sigma^2 = 1$.