Convergence and Spectral Radius

COMP 767 - Reinforcement Learning

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Policy Evaluation: Iterative Method (Matrix Form)

Bellman equation

$$v_{k+1}(s) = \sum_{a} \pi(a|s) \sum_{s',r} p(s', r|s, a)[r + \gamma v_k(s')]$$

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Optimal Solution

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$$v^* - v_{k+1} = R_\pi + \gamma P_\pi v^* - R_\pi - \gamma P_\pi v_k$$
$$v^* - v_{k+1} = \gamma P_\pi (v^* - v_k)$$
$$v^* - v_{k+1} = (\gamma P_\pi)^{k+1} (v^* - v_0)$$
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Convergence

$$\lim_{k \to \infty} v_k = v^* \text{ if } \lim_{k \to \infty} (\gamma P_\pi)^k = 0$$

Spectral Radius

For a linear operator T, the spectral radius is the maximal magnitude of its eigenvalues.

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In our case,

$$\rho(\gamma P_\pi) = \gamma < 1$$

because P_π is stochastic!

Proof of Convergence

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$$e_0 = c_1 x_1 + \ldots + c_N x_N$$

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This assumption is not necessary, but it helps visualize the proof. Without this assumption, the proof would be essentially the same, at a more abstract level.

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Again...

$$e_k = c_1 \lambda_1^k x_1 + \ldots + c_N \lambda_N^k x_N$$

Proof of Convergence

To Infinity!

$$\begin{aligned}\lim_{k\to\infty} ||e_k|| &= \lim_{k\to\infty} ||c_1\lambda_1^k x_1 + \ldots + c_N\lambda_N^k x_N|| \\ &\leqslant |c_1|(\lim_{k\to\infty} |\lambda_1|^k)||x_1|| + \ldots + |c_N|(\lim_{k\to\infty} |\lambda_N|^k)||x_N||\end{aligned}$$

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Use the spectral radius

$$\begin{aligned}\rho(\gamma P_\pi) &< 1 & & \Leftrightarrow \\ |\lambda_i| &< 1 & \forall i & \Leftrightarrow \\ \lim_{k\to\infty} |\lambda_i|^k &= 0 & \forall i &\end{aligned}$$

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Conclusion

$$\lim_{k\to\infty} ||e_k|| \leqslant |c_1| \times 0 \times ||x_1|| + \ldots + |c_N| \times 0 \times ||x_N|| = 0$$

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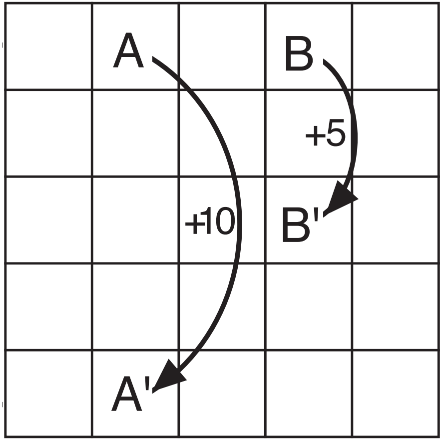
- Convergence is linear at worst (and in the general case)
- Small spectral radius (γ) = Fast convergence

Experience: Gridworld (Example 3.8 of Sutton & Barto)



- 25 states
- 4 actions
- Rewards: -1/0/+5/+10

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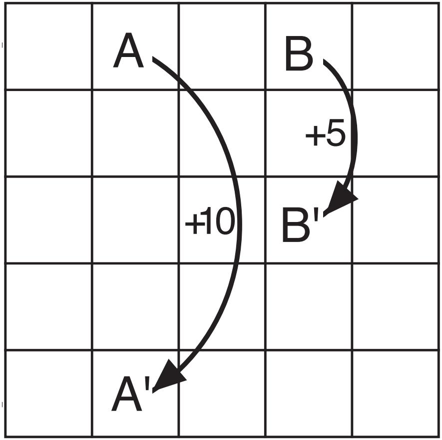

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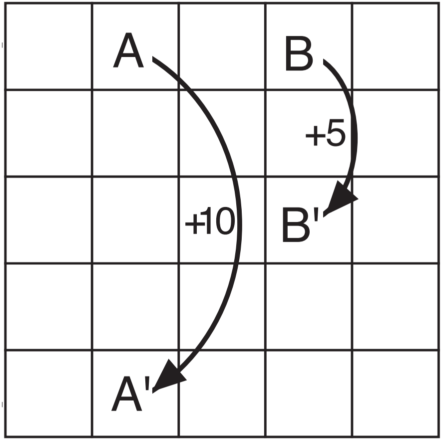

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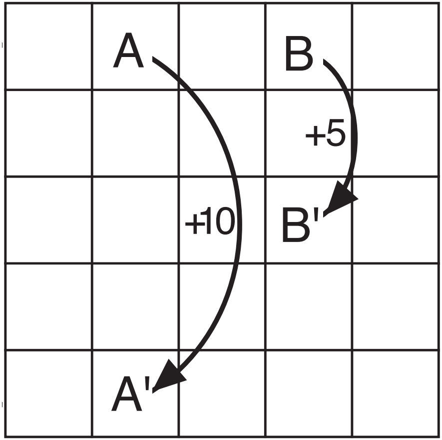

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- γ: variable

Successive Iterations of the Algorithm

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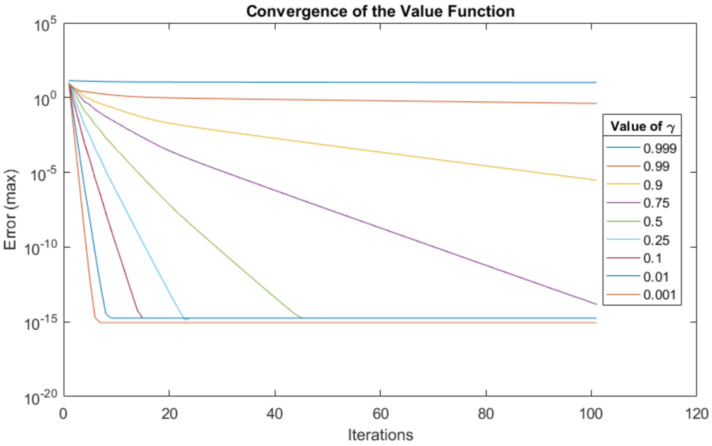

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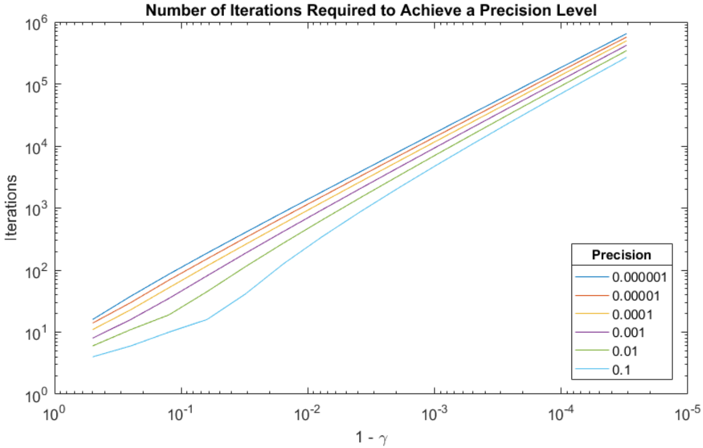

Proof of Convergence (General Case)

Relies on the following result

$$\rho(M) < 1 \qquad \Leftrightarrow \qquad \lim_{k\to\infty} M^k = 0$$

Then,

$$\lim_{k\to\infty} e_k = \lim_{k\to\infty} (\gamma P_\pi)^k e_0 = 0$$

Proof of Convergence (General Case)

Express M in its Jordan normal form

$$M = VJV^{-1}$$

$$M^k = VJ^kV^{-1}$$

$$\lim_{k\to\infty} J^k = 0$$

$$\lim_{k\to\infty} M^k = 0$$