Introduction to Predictive State Representation

Hossein Aboutalebi

March 10, 2017

Introduction

- **HMM** and **POMDP** are used for state representation
- **PSR:** Predictive state representations are a recent models for discrete-time dynamical systems.
- **Distinct feature:** They represent the state of the system as a set of predictions of observable outcomes.
- **PSR** represents the state as a set of predictions of observable outcomes of tests on the system.

Uncontrolled System

- The Uncontrolled system can be viewed as a generator of observations.
- At time step i, it produces an observation o_i from set O.
- The system is the probability distribution over all sequence of observations.
- The prediction of a length-k $p(t)$ for $t = o^1 o^2 ... o^k$ means the first k observation is t or:

$$p(t) = prob(o_1 = o^1, ..., o_k = o^k)$$

Controlled System

- Controlled System takes inputs from some set A (our actions) and generates observations from set O.
- The Controlled system involves a sequence of action-observation pairs
- The system is the probability distribution over all sequence of action observations.
- The prediction of a length-k $t = a^1 o^1 a^2 o^2 ... a^2 o^k$ means the first k observations are $o^1 o^2 .. o^k$ given actions $a^1 a^2 ... a^k$ or:

$$p(t) = prob(o_1 = o^1, ..., o_k = o^k | a_1 = a^1, ..., a_k = a^k)$$

- We call t our **test data**

System Dynamic Matrix

- Consider a matrix, D, whose columns correspond to tests and whose rows correspond to histories
- This matrix is System Dynamic Matrix

$$D_{ij} = p(t_j|h_i) = \frac{p(h_i t_j)}{p(h_i)}$$

$$\begin{array}{c|cccc}
 & t_1 & \cdots & t_j & \cdots \\
\hline
h_1 = \phi & p(t_1|h_1) & & p(t_j|h_1) & \\
h_2 & & & & \\
\vdots & & & & \\
h_i & p(t_1|h_i) & & p(t_j|h_i) & \\
\vdots & & & &
\end{array}$$

Properties Of System Dynamic Matrix

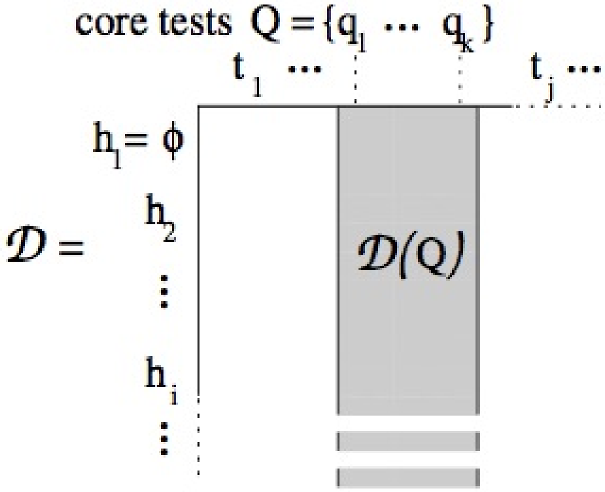
- $0 \leq d_{ij} \leq 1$
- $\forall a, \forall t, \forall h : p(t|h) = \sum_{o \in O} p(aot|h)$
- if $p(u|\epsilon) = 0$ then $p(.|u) = 0$ we call u unreachable
- $\forall h : p(\epsilon|h) = 1$ if h is reachable
- **Note that System Dynamic Matrix is not a model of the system but rather it is the system itself.**
- Thus, the **model of the system** is sth that can generate system dynamic matrix

System Dynamic Matrix

- Define **linear dimension** of a dynamical system as the **rank** of its corresponding system-dynamics matrix.
- For any D with rank k, there must be k linearly independent columns and rows; **not necessarily unique**.

Predictive State Representation

- Consider a set of k linearly independent columns of D and let the corresponding tests be $Q = \{q_1, q_2, ..., q_k\}$
- Call Q the core tests
- Let $D(Q)$ be the submatrix of D containing the columns in Q
- The state representation of the PSR model is just the set of predictions for the core tests.
- For any history h the state of PSR is given by the vector $p(Q|h) = [p(q_1|h) \;\; p(q_2|h) \;\; ... \;\; p(q_k|h)]$.
- The initial state is $p(Q|\epsilon)$, the entries of the first row in D(Q).


core tests Q = {q1 ... qk}
t1 ...
tj ...
h1 = φ
h2
hi
D =
D(Q)

Predictive State Representation

- let D(t) be the column corresponding to test t in Dynamic Matrix D
- By the definition of rank of a matrix, all the columns of D are a linear combination of the columns in D(Q)
- For every test t, there is a weight vector of length k, m_t, such that D(t) is given by $D(t) = D(Q)m_t$
- Or for any history h $p(t|h) = p(Q|h)^T m_t$
- This Type of PSR is called a linear PSR

State Update

- Assume we are in a state $p(Q|h) = [p(q_1|h) \;\; \dots \;\; p(q_k|h)]$. Then the action-observation pair (a,o) happens.
- The new history is *hao* instead of *h*.
- We have for any $q_i \in Q$:

$$p(q_i|hao) = \frac{p(aoq_i|h)}{p(ao|h)} = \frac{p(Q|h)^T m_{aoq_i}}{p(Q|h)^T m_{ao}}$$

- Consider a matrix M_{ao} where the jth column is m_{aoq_j}:

$$p(Q|hao) = \frac{p(Q|h)^T M_{ao}}{p(Q|h)^T m_{ao}}$$

- model parameters are m_{aoq} and m_{ao}. Parameters can be negative which is different from other models like POMDP.

Expressiveness of PSR model

Theorem 1: *A POMDP with k nominal states cannot model a dynamical system with dimension greater than k.*

Theorem 2: *Linear PSRs with k core tests are equivalent to dynamical systems with linear dimension k.*

Corollary 1: *Linear PSRs with k core tests has at least the representation power of a POMDP with k nominal states.*

Expressiveness of PSR model

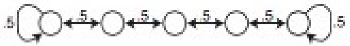
- There exist linear PSRs no more complex than the minimal POMDP for an environment,
- In some cases the minimal linear PSR seems to be much smaller, even exponentially

Example

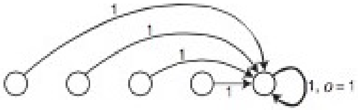
Comparison of POMDP & PSR:

- Consider the float/reset problem consisting of a linear string of 5 states with a distinguished reset state on the far right.
- One action, **f** (float), causes the system to move uniformly at random to the right or left by one state.
- Action **r** (reset), causes a jump to the reset state irrespective of the current state.
- The observation is always 0 unless the **r** action is taken when the system is already in the reset state, in which case the observation is 1.

Example



a) float action



b) reset action

- on **f** action, the correct prediction is always 0
- with **r** action, the correct prediction depends on how many fs there have been since the last r.
- for zero fs, it is 1; for one or two fs, it is 0.5; for three or four fs, it is 0.375; for five or six fs, it is 0.3125, and so on decreasing after every other number of fs, asymptotically until 0.2.

Example

- A POMDP model it by maintaining a belief-state representation over a set of 5 states.
- A PSR can use just two tests: r1 and f0r1.
- The correct predictions for these two tests are always two successive probabilities in the sequence (1, 0.5, 0.5, 0.375,...),
- So, it is always a sufficient statistic to predict the next pair in the sequence.

References

- Littman, Michael L., Richard S. Sutton, and Satinder Singh. "Predictive representations of state." Advances in neural information processing systems 2 (2002): 1555-1562.
- Singh, Satinder, Michael R. James, and Matthew R. Rudary. "Predictive state representations: A new theory for modeling dynamical systems." Proceedings of the 20th conference on Uncertainty in artificial intelligence. AUAI Press, 2004.