CS&SS 321 - Data Science and Statistics for Social Sciences

Module IV - Hypothesis test and multivariate regression

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Module IV

- This module introduces and reviews the topic of causation in science.
  - *Statistical Inference*.
  - *Hypothesis test*.
  - *Multivariate regression*.

Statistical inference: estimation

- In statistical inference, we are concerned with making **predictions** (inferences) about a **DGP** or *population* based on information obtained from a *sample*.
- This involves the following key concepts:
  - **Estimand**: The **quantity of interest** from the data-generating process that we aim to estimate or infer.
  - **Estimator**: A statistical **method** or **formula** used to estimate the estimand based on sample data.
  - **Estimate**: it is the calculated value that serves as the **best guess** or approximation of the estimand based on the available information from the sample.

Estimand, estimator, and esitmate

- Statistical inference involves using **estimators** to obtain **estimates** of **estimands** from sample data to make predictions about the population.

- Analogy: have you ever heard about the *ecce homo*?

Estimand, estimator, and esitmate

Estimand, estimator, and esitmate

Estimand, estimator, and esitmate

Estimand, estimator, and esitmate

- **Estimates** are *best guesses*, but they never return you the "*true*".

Populations and samples

Population Parameter:

- A population **parameter** is a numerical value that describes a characteristic of a **population**.
- It is a **fixed and unknown** value that we aim to estimate or infer using statistical methods.

Sample Statistic:

- A sample **statistic** is a numerical value that describes a characteristic of a **sample**.
- It is calculated from the data of a sample and is used to estimate or make **inferences** about population parameters.

Sample statistics

- A **sample mean** that represents a social process:

$$\bar{X} = \frac{1}{n}(X_1 + X_2 + ... + X_n) = \frac{1}{n}\sum_{i=1}^{n} X_i \tag{1}$$

- The **sample variance** that we estimate:

$$S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 \tag{2}$$

Sample statistics

	Grade_i	Grade_i - Grade_Mean	(Grade_i - Grade_Mean)^2
Student 1	2.4	-0.76	0.5776
Student 2	2	-1.16	1.3456
Student 3	3.8	0.64	0.4096
Student 4	3.6	0.44	0.1936
Student 5	3.4	0.24	0.0576
Student 6	2.9	-0.26	0.0676
Student 7	3.3	0.14	0.0196
Student 8	3.8	0.64	0.4096
Student 9	3.4	0.24	0.0576
Student 10	3	-0.16	0.0256
n	**Mean**		**Variance**
10	**3.16**		**0.3164**

Populations and samples

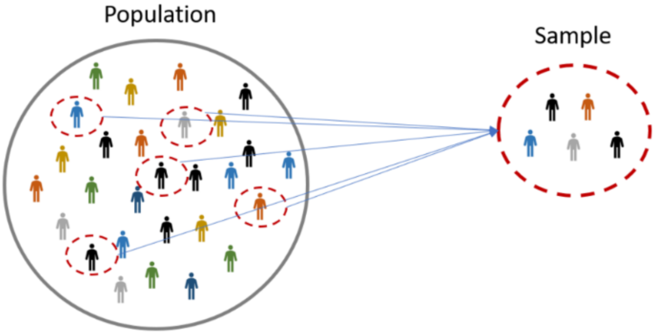
- Typically, we seek to learn features from **populations**, but studying the entire population is unfeasible.
- Thus, we rely on **samples** to make **inferences** under different **assumptions**.

Parameter/Statistic	Population	Sample
Mean	μ	$\bar{X}$
Variance	σ^2	$\hat{\sigma}^2$ or s^2
Standard deviation	σ	$\hat{\sigma}$ or s
Slope/coefficient	β	$\hat{\beta}$ or b

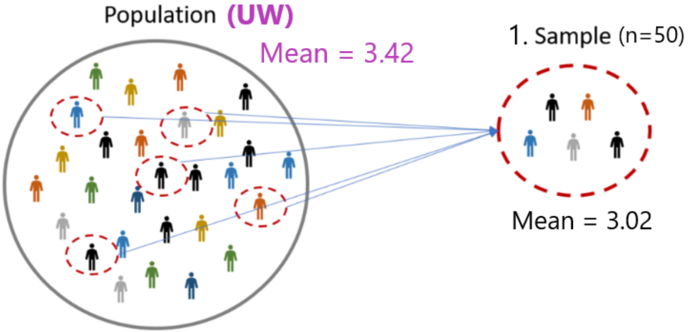
Table 1: Comparison of Population Parameters and Sample Statistics

Populations and samples

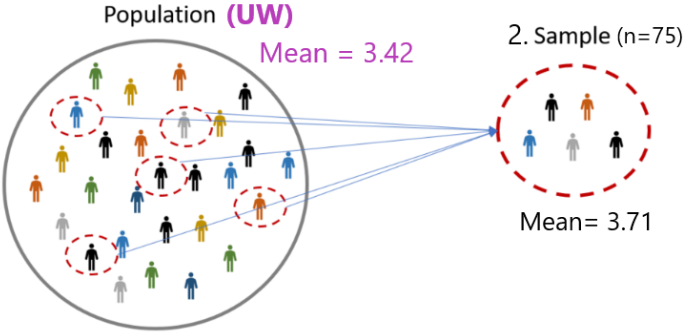
- *Example*: We want to learn the mean GPA of the University of Washington (population) through random sampling students.



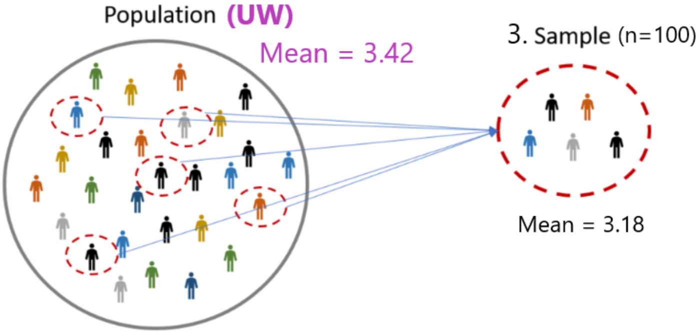
Populations and samples



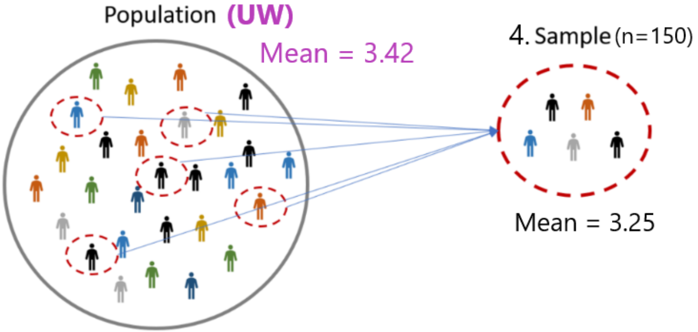
Populations and samples



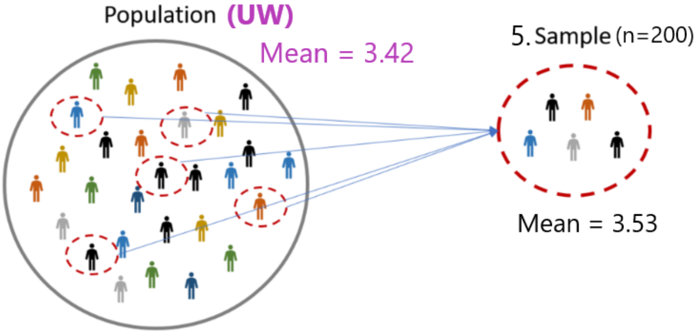
Populations and samples



Populations and samples



Populations and samples



Estimation: Bias

- However, how can we tell if these are good estimates?
  - Ideally, we would compute the estimation error or **bias**.

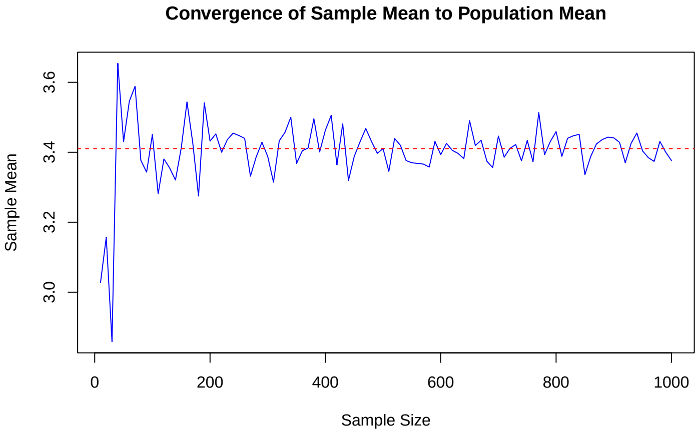
$$\text{bias} = \textit{estimate} - \textit{truth} = \bar{X} - \mu \tag{3}$$

n	**bias**	$\bar{X} - \mu$
50	-0.40	$3.02 - 3.42$
75	0.29	$3.71 - 3.42$
100	-0.24	$3.18 - 3.42$
150	-0.17	$3.25 - 3.42$
200	0.11	$3.53 - 3.42$

Table 2: What is the extent of bias in our estimates?

Estimation: Consistency

- What may happen if we repeat this "experiment" and we increase the sample in each iteration?



Estimation: Bias and Consistency

- **Unbiasedness**: an estimator $\bar{X}$ of a parameter μ is unbiased if and only if:

$$E(\bar{X}) = \mu \tag{4}$$

- **Consistency**: an estimator is consistent if for a sequence $\{X_n\}$ to converge to a limit μ as $n \to \infty$, we have:

$$\lim_{n \to \infty} X_n = \mu \tag{5}$$

However, an unbiased estimator with high variability is impractical because it will return **high prediction error** (MSE) as:

$$MSE = Var + bias^2 \tag{6}$$

Estimation

- Furthermore, they do not provide information about the **uncertainty** or precision of the estimate.
- **Confidence intervals** (CIs) address this issue by providing a range of plausible values for the **estimate**.
  - CIs are based on the principles of probability and sampling variability.
  - Different samples from the **same** population will yield different confidence intervals.

To construct **confidence intervals**, we need to estimate the standard deviation to determine the standard error.

Uncertainty: standard errors.

- The **sample standard deviation** is simply the square root of the variance (see second slide).

$$\hat{\sigma} = \sqrt{\hat{\sigma}^2} \tag{7}$$

- To characterize the variability of an estimator, we compute the **standard error**:

$$SE(\bar{X}) = \frac{\hat{\sigma}}{\sqrt{n}} \tag{8}$$

Uncertainty: critical values.

To calculate the margin of error, we need to choose a **critical value**. Critical values influence the interpretation and outcome of the analysis because:

- constructing **confidence intervals**, and
- determining the **significance level** in hypothesis tests.

Significance Level	**Critical Value**	**Confidence Interval**
0.1	1.645	$1 - 0.1 = 0.9 \quad (90\%)$
0.05	1.96	$1 - 0.05 = 0.95 \quad (95\%)$
0.01	2.576	$1 - 0.01 = 0.99 \quad (99\%)$

Table 3: Common Critical Values and Confidence Intervals

Uncertainty: margin of error.

Once we have the standard error and select a critical value, the **margin error**, ME, and the **confidence intervals** are estimated as follows:

$$ME = \text{critical value} \times SE(\bar{X}) \tag{9}$$

$$\begin{aligned} \text{Confidence Interval} &= \left(\bar{X} - ME, \bar{X} + ME\right) \\ &= (CI_{lower}, CI_{upper}) \end{aligned} \tag{10}$$

Uncertainty: example

```
dat <- read_csv("data/students.csv")
names(dat)
```

```
## [1] "GPA"    "gaming" "study"  "quiz"
```

```
# Randomly sample 40 observations
sampled_data <- sample(dat$GPA, size = 40, replace = F)

(GPA_mean <- mean(sampled_data) ) # sample mean
```

```
## [1] 3.132462
```

```
(GPA_sd <- sd(sampled_data)) # sample standard deviation
```

```
## [1] 1.348265
```

```
(GPA_se <- GPA_sd / sqrt( length(sampled_data) ) ) # sample standard errors
```

```
## [1] 0.2131794
```

Uncertainty: example

Question: Is the sample mean biased estimator? Is the population mean within the confidence interval of our estimator?

```
statistics <- tibble(
  mean = GPA_mean,
  CI_lwr = GPA_mean - (1.96 * GPA_se),
  CI_upr = GPA_mean + (1.96 * GPA_se)
)

mean(dat$GPA) # population mean of GPA
```

```
## [1] 3.203627
```

```
statistics
```

```
## # A tibble: 1 x 3
##    mean CI_lwr CI_upr
##   <dbl>  <dbl>  <dbl>
## 1  3.13   2.71   3.55
```

Uncertainty: interpreting confidence intervals

- In most settings, we rely on a single sample for making **inferences**. To determine if our estimates fall within the range of the true population parameter, we use **confidence intervals.**
- A confidence interval is a computed range of values from the sample data that is **likely** to contain the true population parameter with a specified level of confidence.
- The **confidence level**, denoted as $(1 - \alpha)$ or simply $1 -$ significance level, indicates the probability that the confidence interval **will encompass** the true population parameter over hypothetical replications.
  - *For example*: a 95% confidence interval implies that if we were to repeat the sampling process many times and construct confidence intervals for each sample, approximately 95% of those intervals **would** contain the true parameter.

Imai (2018, p. 328) - critical values

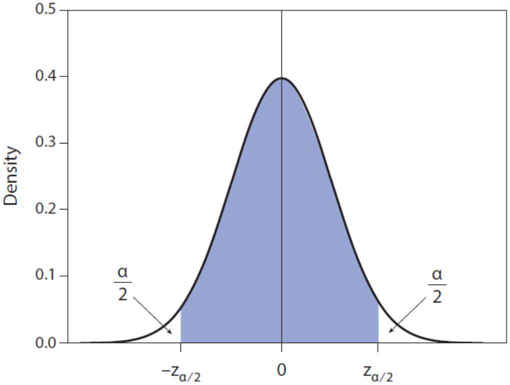
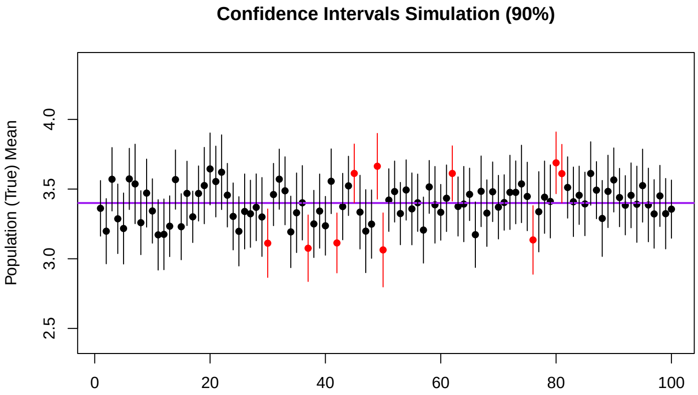


Figure 7.1. Critical Values Based on the Standard Normal Distribution. The lower and upper critical values, $-z_{\alpha/2}$ and $z_{\alpha/2}$, are shown on the horizontal axis. The area under the density curve between these critical values (highlighted in blue) equals $1 - \alpha$. These critical values are symmetric.

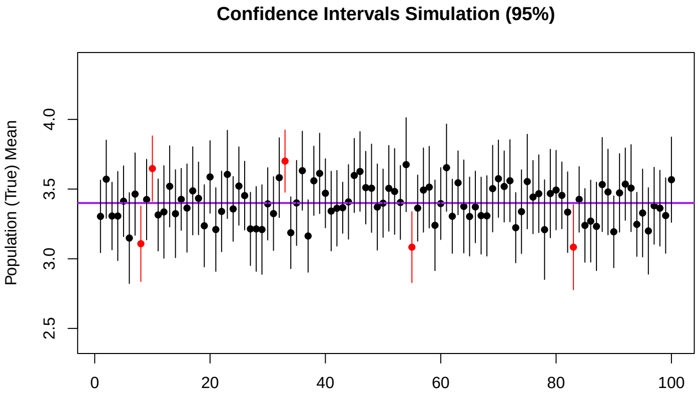
Uncertainty: interpreting confidence intervals

- Resampling and estimating the GPA of the same population (UW) over 100 iterations with a significance **level of 0.1** (90% confidence intervals).



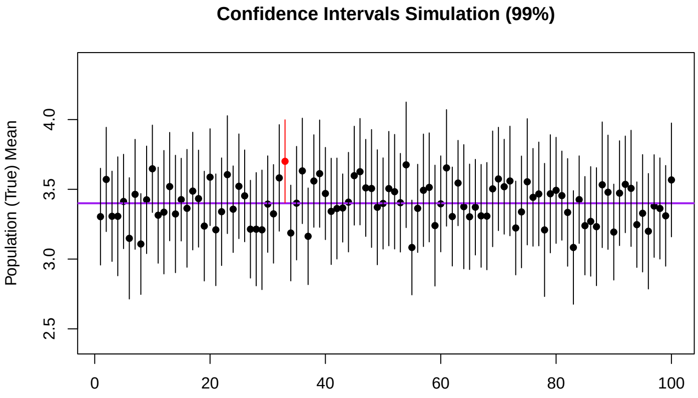
Uncertainty: interpreting confidence intervals

- Resampling and estimating the GPA of the same population (UW) over 100 iterations with a significance **level of 0.05** (95% confidence intervals).



Uncertainty: interpreting confidence intervals

- Resampling and estimating the GPA of the same population (UW) over 100 iterations with a significance **level of 0.01** (99% confidence intervals).



Takeaways

- Understand **bias** and **consistency**.
- Represent **uncertainty** in estimates.
- The impact of the **critical value** (α) on constructing confidence intervals.
- Wider confidence intervals increase the likelihood of the *"true value"* being within the intervals over **hypothetical replications**.
  - **Question**: Why might someone want to calculate narrower confidence intervals?

Time to code a little bit!

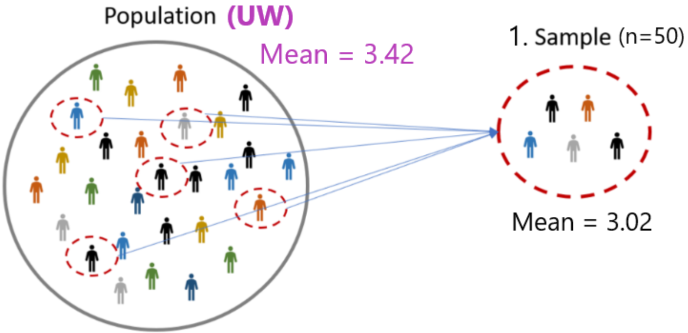
- Open the file `W8D1_confint.rmd`

Hypothesis Testing: motivation

- In any statistical analysis, including hypothesis testing, we work with a **sample** from a larger **population**.
- Due to inherent **variability** in the data, the sample may **not perfectly reflect** the entire population.
- This variability can cause the **sample mean** to differ from the true **population mean**, even if there is no real difference in the population.
- Through a **t-test**, we assess whether the observed difference between the sample mean and the hypothesized value exceeds what is expected due to chance (aka random sampling variability alone).

Hypothesis Testing: motivation

- Recall from the last lab:



Hypothesis Testing

- Hypothesis testing is used to make inferences about population **parameters** based on **sample** data.
- It involves formulating **null** and **alternative hypotheses** and evaluating the evidence against the null hypothesis.
  - **Null Hypothesis** (H_0): a statement of no effect or no difference between groups or variables (*proof by contradiction*).
  - **Alternative Hypothesis** (H_a): contradicts the null hypothesis and suggests the presence of an effect or a difference between groups or variables.
- **Goal**: to determine whether the evidence from the sample supports the null hypothesis or provides evidence for the alternative hypothesis.

Hypothesis Testing

- **T-test**: calculates the **t-value**, which quantifies the difference between the sample statistic and the hypothesized value relative to the variability within the data.
  - It takes into account the **sample size** and the **standard error** of the statistic to assess the likelihood of observing such a difference by chance.
- **Significance Level** (α): The predetermined threshold for rejecting the null hypothesis.
- **P-value**: it measures the **strength of evidence** against the null hypothesis, we compare it with the significance level to determine if we **reject or fail to reject** the null (H_0).
  - p-value is **large**: suggest insufficient evidence to reject the null hypothesis.
  - p-value is **low**: stronger evidence against the null, favoring the alternative(H_a).

Hypothesis Testing: error types

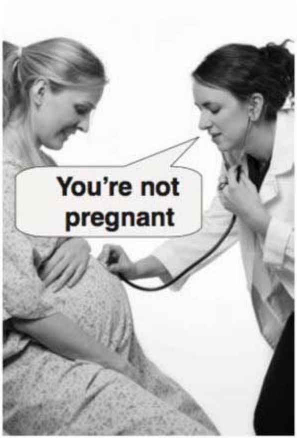
- There is a clear trade-off between **Type I** and **Type II** errors in that minimizing type I error usually increases the risk of type II error.

Decision	H_0 is True	H_0 is False
Retain H_0	Correct	**Type II Error**
Reject H_0	**Type I Error**	Correct

Hypothesis Testing: error types

Type I error
(false positive)

Type II error
(false negative)

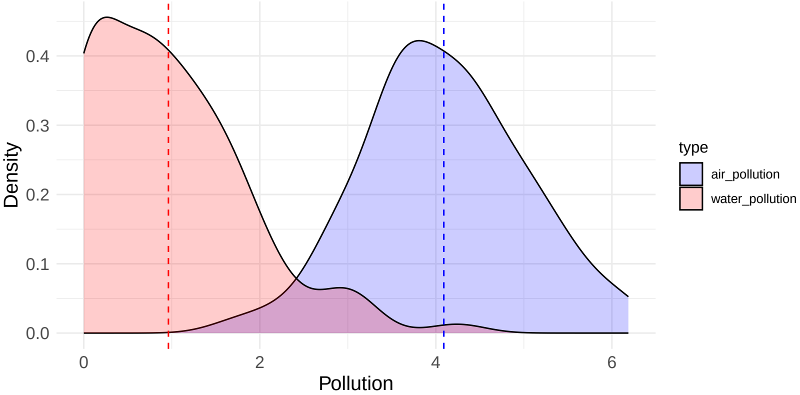


Hypothesis Testing Process

1. State the **null** and **alternative** hypotheses.
2. Choose a test statistic and the **significance level** (α).
3. Estimate the test statistic, in our case the **t-value**.
4. Compute the **p-value**, and compare it with with the significance level.
   - For example, is *p*-value $< \alpha$?
5. Reject the null hypothesis if the *p*-value is less than or equal to α.

Hypothesis Testing Process

- We will focus on a scenario where we want to assess the **association** of air and water **pollution** on **climate change**.
  - *Disclaimer*: this data was simulated.



Hypothesis Testing Process

- We define a theoretical model:

$$cc = \alpha + \beta_1 \text{air} + \beta_2 \text{water} + \epsilon \tag{11}$$

1. State the null and alternative hypotheses:
   - **Null Hypothesis** (H_0): air (β_1) or water (β_2) pollution are **not** associated with climate change. In other words, $\beta_1 = 0$ or $\beta_2 = 0$.
   - **Alternative Hypothesis** (H_a): air or water are associated with climate change. In other words, $\beta_1 \neq 0$ or $\beta_2 \neq 0$
2. Set the **significance level**, the default in social sciences is 0.05.

Hypothesis Testing Process

- The `lm()` function estimates the t-statistic and p-values (steps 3 and 4) using the fitted model and sample data argument.

```
model <- lm(climate_change ~ air_pollution + water_pollution)
round(coef(model), digits=2)
```

```
##     (Intercept)   air_pollution water_pollution
##            0.65            1.87            0.18
```

- Estimated model, are the coefficients statistically significant?

$$cc = 0.65 + 1.87\text{air} + 0.18\text{water} \tag{12}$$

Model summary

- Use the function `summary()` for the t-test and the p-value.
- Can we reject H_0?
  - Remember that the **significant level** that we choose was 0.05 *(critical value = 1.96)*.

```r
summary(model)
```

```
##
## Call:
## lm(formula = climate_change ~ air_pollution + water_pollution)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -1.8735 -0.6615 -0.1320  0.6208  2.0701
##
## Coefficients:
##                 Estimate Std. Error t value Pr(>|t|)
## (Intercept)       0.6491     0.4555   1.425   0.1574
## air_pollution     1.8663     0.1048  17.802   <2e-16 ***
## water_pollution   0.1840     0.1093   1.683   0.0956 .
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.9514 on 97 degrees of freedom
## Multiple R-squared:  0.7662, Adjusted R-squared:  0.7614
## F-statistic: 158.9 on 2 and 97 DF,  p-value: < 2.2e-16
```

Hypothesis Testing and Confidence Intervals

- Can we reject H_0?

```
(p_value <- summary(model)$coefficients[, "Pr(>|t|)"])
```

```
##     (Intercept)   air_pollution water_pollution
##    1.573807e-01    2.525692e-32    9.558931e-02
```

```
(t_value <- summary(model)$coefficients[, "t value"])
```

```
##     (Intercept)   air_pollution water_pollution
##        1.424952       17.802076        1.683010
```

```
p_value < 0.05 # is p-value < significant level?
```

```
##     (Intercept)   air_pollution water_pollution
##           FALSE            TRUE           FALSE
```

```
t_value > 1.96 # is t-value > critical value?
```

```
##     (Intercept)   air_pollution water_pollution
##           FALSE            TRUE           FALSE
```

Hypothesis Testing and Confidence Intervals

- Can we reject H_0?
  - H_0 air pollution: sufficient evidence to reject the null hypothesis.
  - H_0 water pollution: insufficient evidence to reject the null hypothesis.
- **Conclusion**: air pollution has a positive significant **association** with climate change. However, water pollution is **not statistically significant**.
  - When an estimated coefficient is not statistically significant, we mean that it is not **significantly different from 0**. In this case, $\beta_2 = 0 \neq 0.18$, because we fail to reject the null H_0 for water pollution.
- However...

Hypothesis Testing and Confidence Intervals

- Can we really reject H_0 if we instead use a significant level of **0.10**?

```
p_value < 0.1 # is p-value < significant level?
```

```
##     (Intercept)   air_pollution water_pollution
##           FALSE            TRUE            TRUE
```

```
t_value > 1.645 # is t-value > critical value?
```

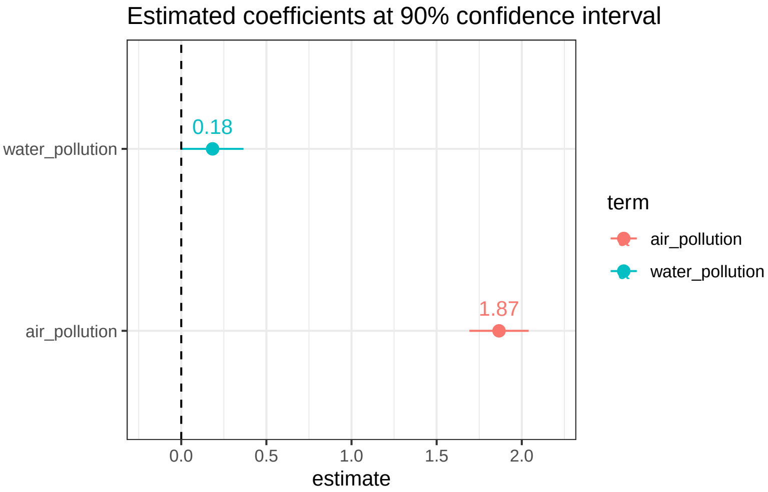
```
##     (Intercept)   air_pollution water_pollution
##           FALSE            TRUE            TRUE
```

- Type I and II error trade-off.

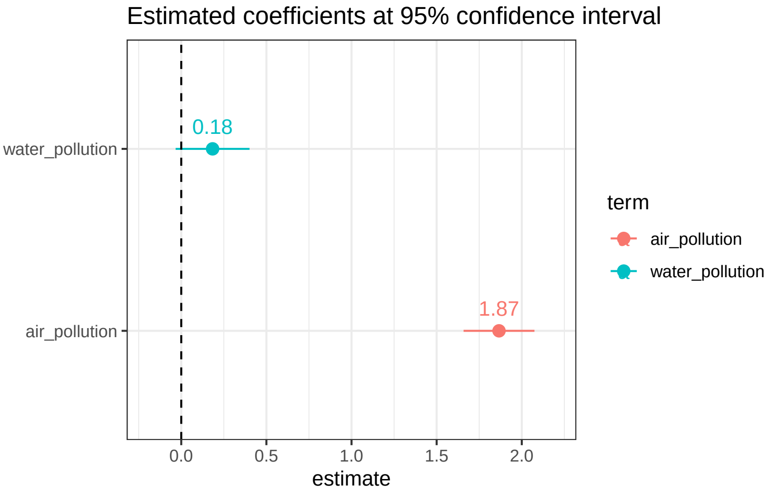
Hypothesis Testing and Confidence Intervals

- **Confidence intervals** and **hypothesis testing** are closely related.
- If the confidence interval **contains the null** value, $\beta_2 = 0$, the null hypothesis cannot be rejected.
- If the confidence interval does not contain the null value, the corresponding hypothesis test would lead to rejecting the null hypothesis in favor of the alternative hypothesis, $\beta_2 \neq 0$.
- The p-value in hypothesis testing **quantifies** the strength of evidence against the null hypothesis, similar to how confidence intervals provide a range of **plausible** parameter values.
  - **Important**: the p-value is **NOT** the probability that the null is true.

Hypothesis Testing and Confidence Intervals



Hypothesis Testing and Confidence Intervals



Hypothesis Testing and Confidence Intervals