

# **CS&SS 321 - Data Science and Statistics for Social Sciences**

## **Module III - Introduction to causal inference and linear models**

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# Module III

- ▶ This module introduces and reviews the topic of causation in science.
  - ▶ *randomization.*
  - ▶ *applied causal inference.*
- ▶ It also introduces the **linear regression model** and the method of **least squares** (LS).

# The statistics war of the late XXth century



# The statistics war of the XXIth century

- Causal inferences requires a model outside of the statistical model.



## *Causes in, causes out*

- ▶ Why do experiments work? When do they work?
- ▶ What if treatment is imperfect assigned?
- ▶ Should you *control* for anything? Everything?

Answers depend upon **causal assumptions** ( $\rightarrow$ ).

- ▶ An **assumption** is a premise or supposition that is accepted *without direct evidence*, often forming the basis for reasoning or an argument.

## *Causes in, causes out*

- ▶ Causal assumptions requires **causal knowledge** of social systems.
- ▶ For example, where  $X$  represents **rain** and  $Y$  represents **puddles**.
  - ▶ What **causal assumption** ( $\rightarrow$ ) you find more reasonable?
    - (i)  $X \leftarrow Y$
    - (ii)  $X \rightarrow Y$



# Causal design

- ▶ **Step 1:** sketch a (scientific) causal model:  $X \rightarrow Y$ .
  - ▶ *Causes in:* assumptions reflect **background knowledge** (*theory and literature review*).
- ▶ **Step 2:** use the model to design **data collection** and **statistical procedures**.
- ▶ **Step 3:** use statistical analyses to **hypothesis test** and report results.
  - ▶ *Causes out:* test assumptions' implications about the **causal mechanism**.

# Causal design: intervention

- ▶ In causal inference, an **intervention** is a deliberate and controlled manipulation of one or more variables in a system to assess their **causal impact** on the outcome of interest.
  - ▶ *Example:* Pouring a bucket of water on the floor creates a puddle; does rain follow?
- ▶ We formalize this via the **potential outcomes** framework.



# Causation in science

Treatment indicator:  $T_i \in \{0, 1\}$ , where  $i$  refers respondents.

- ▶ (1) example:
  - ▶  $T_i = 0$  indicates no membership in a union.
  - ▶  $T_i = 1$  indicates membership in a union.
- ▶ (2) example:
  - ▶  $T_i = 0$  indicates no daughters.
  - ▶  $T_i = 1$  indicates having daughters.

Outcome:  $Y_i$

- ▶ (1) example: redistribution attitudes (*gincdif*).
- ▶ (2) example: pro-feminist attitudes (*progressive.vote*).

# Causation in science

- ▶ Consider the treatments' ( $T$ ) **causal mechanisms** ( $\rightarrow$ ) that drives the **outcome** ( $Y$ ).
  - ▶ **Why** does labor **union membership** increase support for redistribution?
  - ▶ **Why** does having a **daughter** increase pro-feminist attitudes?

Potential outcomes  $Y_i(0)$ ,  $Y_i(1)$ , where:

- ▶ **(1) example:**
  - ▶  $Y_i(0)$  represents redistribution attitudes *without* membership.
  - ▶  $Y_i(1)$  represents redistribution attitudes *with* membership.
- ▶ **(2) example:**
  - ▶  $Y_i(0)$  represents pro-feminist attitudes *without* daughters.
  - ▶  $Y_i(1)$  represents pro-feminist attitudes *with* daughters.

## Causation in science

The **fundamental problem of causality** posits that we cannot observe two outcomes at the same time:

$$\text{individual treatment effect} = Y_{\text{Ramses}}(1) - Y_{\text{Ramses}}(0) \quad (1)$$

Instead, we **estimate** group-level effects by taking the differences in means between **treatment**,  $\bar{Y}(1)$ , and **control**,  $\bar{Y}(0)$ , groups.

$$\text{average treatment effect} = \bar{Y}(1) - \bar{Y}(0) \quad (2)$$

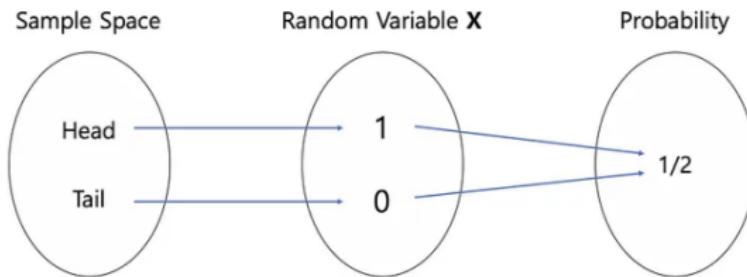
However, we can identify **ATE** if, and only if, the treatment  $D$  has been **randomly assigned** to each respondent i. Formally,

$$T_i \perp\!\!\!\perp (Y_i(0), Y_i(1)) \quad (3)$$

# Causation in science

- ▶ Think about random assignment as flipping a coin.
  - ▶ In **expectation** (as  $n \rightarrow \infty$ ), a fair coin has a probability of 0.5 to show tails (0) or heads (1).
  - ▶ By definition, a random event has a probability of 0.5.

## Toss 1 Coin Example



- ▶ What if, in expectation, a coin has a probability of 0.7 ?

# Causation in science

- ▶ Is labor union membership a random occurrence?



# Causation in science

- Is having a girl (instead of a boy) a random occurrence?



Boy

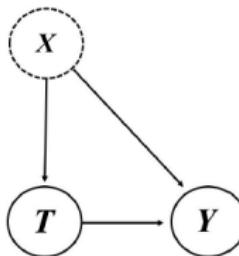
Girl

# Causation in science

- ▶ **Selection bias:** Self-selection and unbalanced factors introduce bias in our statistical estimations.
  - ▶ *Self-selection:* Left-wing individuals are more likely to become labor union activists.
  - ▶ *Unbalanced factors:* Labor union members may systematically differ from non-union members in terms of other variables such as occupation and income.

# Causation in science

- ▶ In observational studies, unconditional treatment effects are unlikely due to the influence of **confounding** factors, both **observed** and **unobserved**.



- ▶ However, sometimes we can assume **conditional random effects**.

$$T_i \perp\!\!\!\perp (Y_i(0), Y_i(1)) | X_i. \quad (4)$$

# Causation in science

- ▶ Let's work a short coding example.
- ▶ Open the file `unions_sweden.Rmd`, we will do only the **first** section.
- ▶ We will finish the remaining section next week.

## From previous model: Data Generating Process

- ▶ Two very useful pieces of information from a DGP are its **mean** and **standard deviation**.

$$\bar{X} = \frac{1}{n} \sum_{i=1}^N X_i \quad ; \quad S = \sqrt{\frac{1}{N} \sum_{i=1}^n (X_i - \bar{X})^2}$$

where

- ▶  $\bar{X}$  represents the **sample mean**.
- ▶  $N$  is the number of **observations** in the sample.
- ▶  $X_i$  represents **values** from a variable in the sample.
- ▶  $S$  represents the **sample standard deviation**.

# Standard deviation and variance

- The **standard deviation** and **variance** are both measures of the spread of a distribution.
  - To estimate the variance ( $S^2$ ), we simply take the **square** of the standard deviation ( $S$ ).

$$S^2 = \left( \sqrt{\frac{1}{N} \sum_{i=1}^n (X_i - \bar{X})^2} \right)^2$$

$$S^2 = \frac{1}{N} \sum_{i=1}^n (X_i - \bar{X})^2$$

- $S^2$  is the **sample** variance.
- Q: Why choose the standard deviation over the variance to report **summary statistics**?

# Mean and variance

$$\bar{X} = \frac{1}{N} \sum_{i=1}^n X_i \quad ; \quad S^2 = \frac{1}{N} \sum_{i=1}^n (X_i - \bar{X})^2$$

- ▶ The **sample mean** ( $\bar{X}$ ) describes the location (*the center*) of the data (*distribution*).
- ▶ The **sample variance** ( $S^2$ ) measures the variability in the data (*distribution*).
  - ▶ The variance describes the **average deviation** in a distribution.

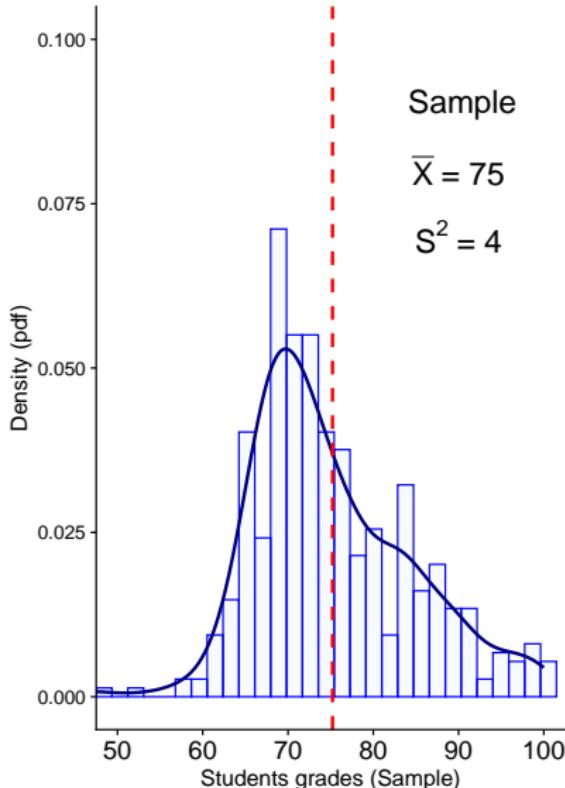
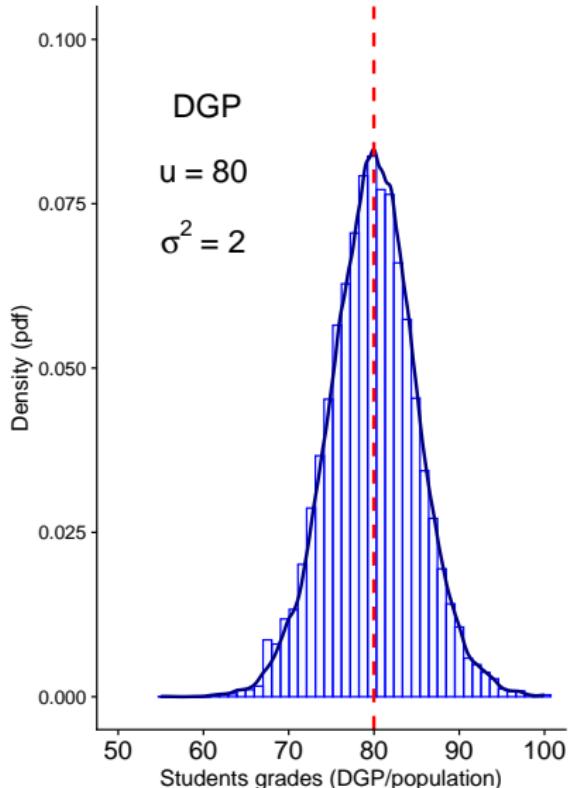
# DGP vs. sample

We distinguish between the **Data Generating Process** (DGP) and the data **sample**.

- ▶ DGP or *population* is a **theoretical** concept describing how observed/sampled data is generated.
  - ▶ It follows a **distribution**, typically depicted as the *TRUE* (!?).
  - ▶ Its parameters, mean ( $\mu$ ) and variance ( $\sigma^2$ ), are **fixed**.
- ▶ The sample is an **empirical** construct, representing realizations/occurrences of a data process.
  - ▶ Sample data maps into **distributions** of *random variables*.
  - ▶ Its parameters, mean ( $\bar{X}$ ) and variance ( $S^2$ ), are **random**.

*Note:* we use the sample to infer (**approach**) the underlying *TRUE* of a DGP.

# DGP vs. sample



# Unconditional distributions

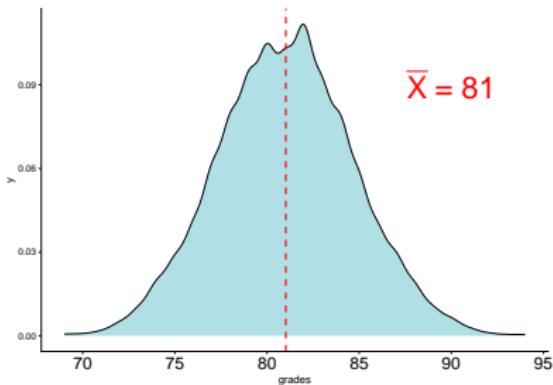
- ▶ The **expectation**  $E[.]$  of a random variable  $X$ , denoted as  $E[X]$ , is a useful measure of central tendency of the DGP.
  - ▶ The expectation is also called the **expected value** or **mean**.
  - ▶ In the case of the normal distribution, the expectation is the first **central moment** and is denoted as  $\mu$ .
- ▶ In general, a natural estimator of the expectation is the **sample mean**.

$$\mu = E[X] = \bar{X} = \frac{1}{n} \sum_{i=1}^N X_i$$

# Unconditional distributions

- ▶ We have a sample of UW students' grades.
- ▶ What may be a good candidate to estimate the mean of this population?

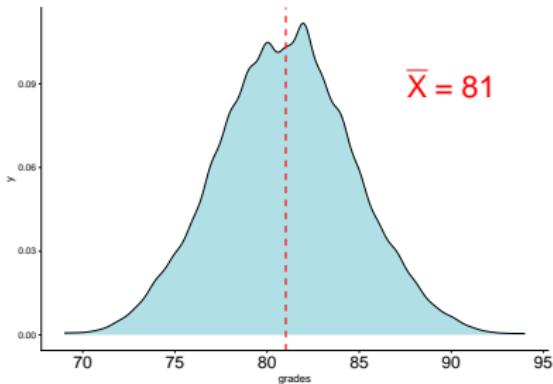
$$E[\text{grades}] = ?$$



# Unconditional distributions

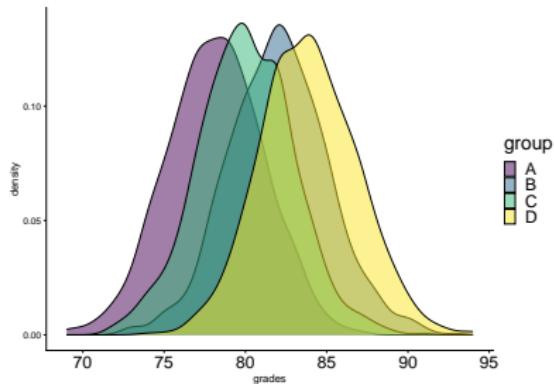
- We have a sample of UW students' grades.
- What may be a good candidate to estimate the mean of this population?

$$E[\text{grades}] = 81$$



# Conditional distributions

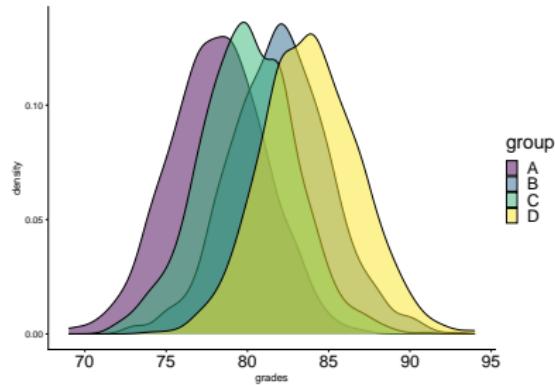
- We can compare the grade distribution for these different **sub-populations**.
  - Group A
  - Group B
  - Group C
  - Group D



# Conditional distributions

- We can **condition** grades on a fixed value ( $x$ ) of the group random variable.
- We call this the **conditional mean** (or **conditional expectation**).

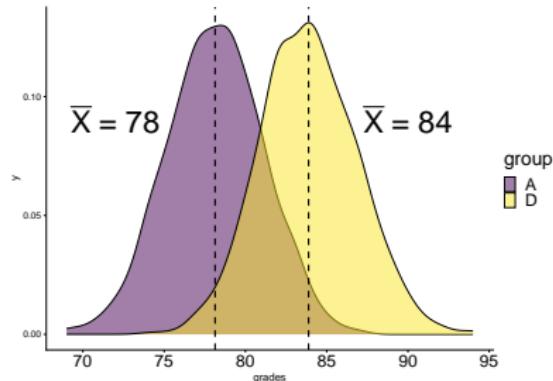
$$E[\text{grades} \mid \text{group} = x]$$



# Conditional distributions

- ▶ For example, take the conditional mean of groups A and D.

$$E[\text{grades} \mid \text{group} = A] = 78$$



$$E[\text{grades} \mid \text{group} = D] = 84$$

# Conditional distributions

- ▶ When **conditioning** a distribution (*grades*), we **adjust** it to a second variable (*group*).
- ▶ This offers more insight into the **variance** of the outcome (*grades*).

$$E[\text{grades} \mid \text{group} = D] - E[\text{grades} \mid \text{group} = A] = 84 - 78 = 6$$

- ▶ However, it is crucial to note that we **cannot** attribute *causality* or interpretation to these differences.
- ▶ Conditioning helps in **describing variation** but does not constitute a **model** or explanation by itself.

# Best predictor

- ▶ In statistics, we model data to **predict quantities** of interest.
  - ▶ *What is the causal effect of a cancer treatment?*
  - ▶ *What will be the stock market price next month?*
- ▶ Prediction is the closest **best guess** (*estimate*) among all data realizations in a distribution.
  - ▶ *What is the best estimate in predicting the midterm grades of all students in CS&SS321?*

## Best predictor

- The **best predictor**, denoted as  $\theta$ , minimizes **prediction error** ( $e$ ), which is the distance of each data point from our best guess:  $e = Y_i - \theta$ .
- **Mean Squared Error** (MSE) quantifies the magnitude of prediction error.

$$\text{MSE} : E[(Y_i - \theta)^2]$$

*Note:* The notation  $\theta$  is arbitrary and denotes the optimal or best predictor.

## Prediction error: first guess

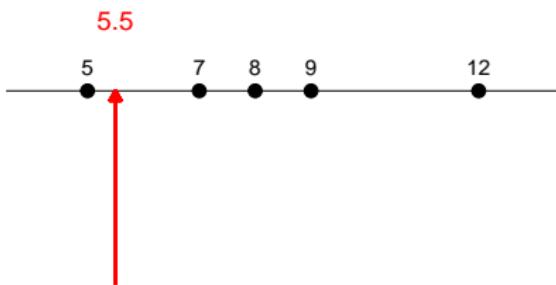
What is your **best guess** ( $\theta$ ) that **minimizes** the prediction error ( $MSE$ )?



$N_i$	$Y_i$	$\theta$	$Y_i - \theta$	error
1	5			
2	7			
3	8			
4	9			
5	12			

$$MSE = E[(Y_i - \theta)^2]$$

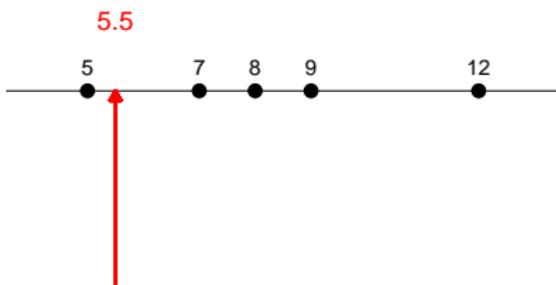
# Prediction error: first guess



$N_i$	$Y_i$	$\theta$	$Y_i - \theta$	error
1	5	5.5	-0.5	-0.5
2	7	5.5	1.5	1.5
3	8	5.5	2.5	2.5
4	9	5.5	3.5	3.5
5	12	5.5	6.5	6.5

$$MSE = E[(Y_i - 5.5)^2]$$

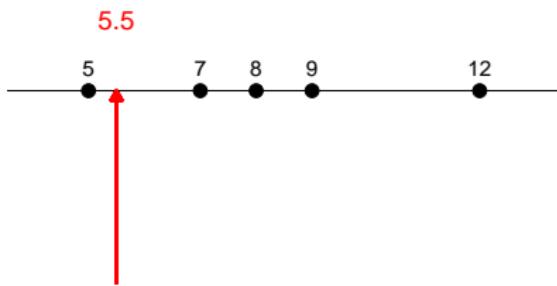
## Prediction error: first guess



$N_i$	$Y_i$	$\theta$	$Y_i - \theta$	error
1	5	5.5	5-5.5	5.5
2	7	5.5	7-5.5	5.5
3	8	5.5	8-5.5	5.5
4	9	5.5	9-5.5	5.5
5	12	5.5	12-5.5	5.5

$$MSE = E[(Y_i - 5.5)^2]$$

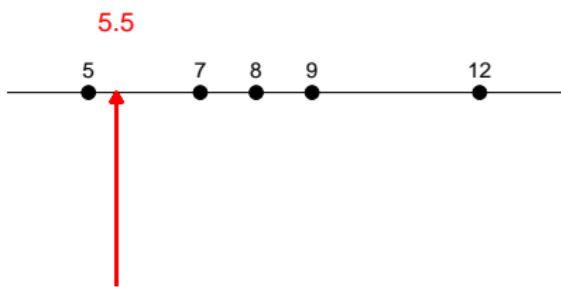
## Prediction error: first guess



$N_i$	$Y_i$	$\theta$	$Y_i - \theta$	error
1	5	5.5	5-5.5	-0.5
2	7	5.5	7-5.5	1.5
3	8	5.5	8-5.5	2.5
4	9	5.5	9-5.5	3.5
5	12	5.5	12-5.5	6.5

$$MSE_1 = \frac{1}{5}(-0.5 + 1.5 + 2.5 + 3.5 + 6.5)^2$$

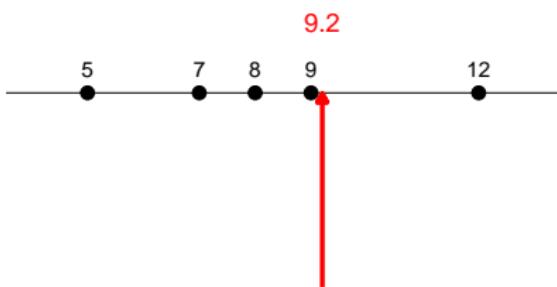
## Prediction error: first guess



$N_i$	$Y_i$	$\theta$	$Y_i - \theta$	error
1	5	5.5	5-5.5	-0.5
2	7	5.5	7-5.5	1.5
3	8	5.5	8-5.5	2.5
4	9	5.5	9-5.5	3.5
5	12	5.5	12-5.5	6.5

$$\begin{aligned}MSE_1 &= \frac{1}{5}(-0.5 + 1.5 + 2.5 + 3.5 + 6.5)^2 \\&= \frac{(13.5)^2}{5} = \frac{182.25}{5} = 36.45\end{aligned}$$

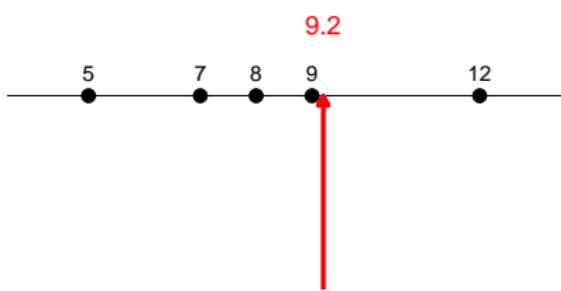
## Prediction error: second guess



$N_i$	$Y_i$	$\theta$	$Y_i - \theta$	error
1	5	9.2	5-9.2	
2	7	9.2	7-9.2	
3	8	9.2	8-9.2	
4	9	9.2	9-9.2	
5	12	9.2	12-9.2	

$$MSE_2 = E[(Y_i - 9.2)^2]$$

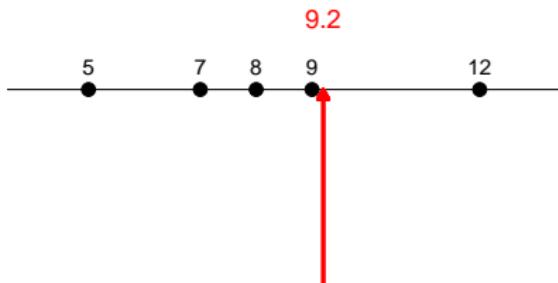
## Prediction error: second guess



$N_i$	$Y_i$	$\theta$	$Y_i - \theta$	result
1	5	9.2	5-9.2	-4.2
2	7	9.2	7-9.2	-2.2
3	8	9.2	8-9.2	-1.2
4	9	9.2	9-9.2	-0.2
5	12	9.2	12-9.2	2.8

$$MSE_2 = \frac{1}{5}(-4.2 + -2.2 + -1.2 + -0.2 + 2.8)^2$$

## Prediction error: second guess



$N_i$	$Y_i$	$\theta$	$Y_i - \theta$	result
1	5	9.2	5-9.2	-4.2
2	7	9.2	7-9.2	-2.2
3	8	9.2	8-9.2	-1.2
4	9	9.2	9-9.2	-0.2
5	12	9.2	12-9.2	2.8

$$\begin{aligned}MSE_2 &= \frac{1}{5}(-4.2 - 2.2 - 1.2 - 0.2 + 2.8)^2 \\&= \frac{(-5)^2}{5} = \frac{25}{5} = 5\end{aligned}$$

## Best predictor and prediction error

- ▶ Two best guesses are provided:  $\theta_1 = 5.5$  and  $\theta_2 = 9.2$ .
- ▶ From these best guesses, two measures of prediction error are retrieved:  $MSE_1 = 36.45$  and  $MSE_2 = 5$ .
- ▶ The best predictor minimizes prediction error given the data.
  - ▶ Which was the **best predictor**,  $\theta_1$  or  $\theta_2$ ?
  - ▶ It's evident that  $MSE_1 > MSE_2$ .
  - ▶ Therefore, 9.2 better predicts this DGP than 5.5.

# **Best predictor and conditional means**

- ▶ Let's work a short coding example.
- ▶ Open the file BestGuess.Rmd, and complete all the exercises.

# Causality review

- ▶ Effective research designs can aid in identifying **causal effects** from **associations**, but they also come with their own set of **assumptions**.
- ▶ Experimental designs:
  - ▶ Randomization (e.g., RCT).
- ▶ Observational studies:
  - ▶ Confounding adjustment (via causal modeling).
  - ▶ “Natural” experiments (as if random).
- ▶ Even if **assumptions** are met, and often can **never** be completely confirmed, there is a trade-off in **conclusions validity**.

# Coding exercise

- ▶ Open the file CausRev.Rmd and complete as many sections as possible.
  - ▶ The four sections are **not cumulative**; you can proceed to the next one if you feel stuck or encounter unfamiliar functions.
  - ▶ Refer to my **Module 2 slides** for explanations and detailed examples of any new functions.

## Causality review: randomization

- ▶ In randomized experiments, we can identify average treatment effects (**ATE**) only if the **intervention** and treatment  $T$  are randomly assigned to each respondent  $i$ .
  - ▶ This relies on the **exchangeability** or exogeneity assumption:

$$T_i \perp\!\!\!\perp (Y_i(0), Y_i(1)) \quad (5)$$

- ▶ This assumption implies that all other variables/factors, both observables (like income) and non-observables (like ideology), are **balanced**.
- ▶ However, in practice, randomization is never perfectly implemented, and some imbalance may occur.

## Causality review: randomization

- If, and only if, **randomization** has been *perfectly* implemented **and** there is **covariate balance**, we can **estimate** the causal effect of the treatment by computing the following:

$$\begin{aligned}\text{DiD} &= E[Y | T = 1] - E[Y | T = 0] \\ &= \bar{Y}_{1T} - \bar{Y}_{0T}\end{aligned}$$

- Under **ideal** randomization, no statistical modeling is necessary.
- A simple **differences-in-means** (*conditional means*) estimator provides the causal effect of interest.

## Causality review: observational studies

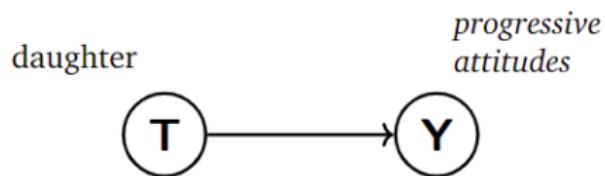
- ▶ In observational research designs, we cannot randomize an intervention, but we can identify causal effects by **conditioning** on confounders and making some (*heroic*) assumptions.
  - ▶ **Unconfoundedness** or selection on observable assumption.

$$T_i \perp\!\!\!\perp (Y_i(0), Y_i(1) | X_i) \quad (6)$$

- ▶ Unconfoundedness implies that causal effects can be identified if we **adjust** for a set of variables that bias the causal effect.
- ▶ **Causal modeling** (Module 4) can help identify unconfoundedness, but it is practically impossible to meet in most applications.

## Causality review: PS2, Q5

- ▶ Think about the **causal assumptions/mechanism.**
- ▶ Can someone be **biased** to have girls (instead of boys)?
- ▶ Having a girl is an **event** (*coin flip*), however, what is a **pre-condition** to having a daughter?



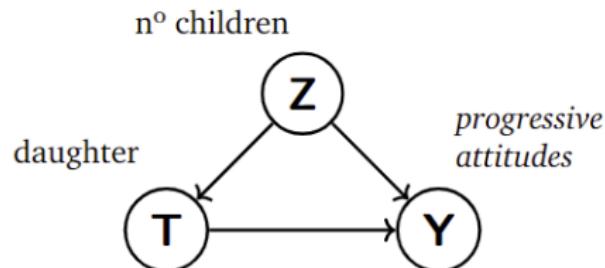
## Causality review: PS2, Q5

- ▶ Conditional on children, having a daughter *may* be a random occurrence.

$$girl_i \perp\!\!\!\perp (PA_i(0), PA_i(1) | child_i)$$

*PA: Progressive Attitudes.*

- ▶ However, we need to provide **evidence** that supports this assumption.



# Research design: natural experiments

- ▶ Over the past two decades, there has been an explosion in **applied causal inference**.
  - ▶ It relies on finding observational research designs with features that make it easier to assume *as-if randomness*.
    - ▶ Instrumental regression.
    - ▶ Discontinuous regression.
    - ▶ Differences-in-Differences, etc.
  - ▶ These are known as **natural experiments** because *nature* randomly assigns the **intervention**.
    - ▶ **Strong (heroic!) assumptions** must be met to infer causality.
    - ▶ For example, in time-series/panel studies, causal estimation requires the assumption of **parallel trends**.

## Research design: Differences-in-Differences

- ▶ A study conducted by [Card and Krueger \(1994\)](#) analyzed the impact of minimum wage laws (**T**) on unemployment (**Y**) in two neighboring American states.
  - ▶ *Natural experiment*: New Jersey increased its minimum wage (MW) while Pennsylvania did not.
- ▶ The underlying **assumption** is that New Jersey and Pennsylvania have **similar** economic systems (**as-if random**).
- ▶ If the assumption holds, and the **only** difference between the states is the intervention (minimum wage law), we can estimate the causal effect with a **Diff-in-Diff** estimator.

## Research design: Differences-in-Differences

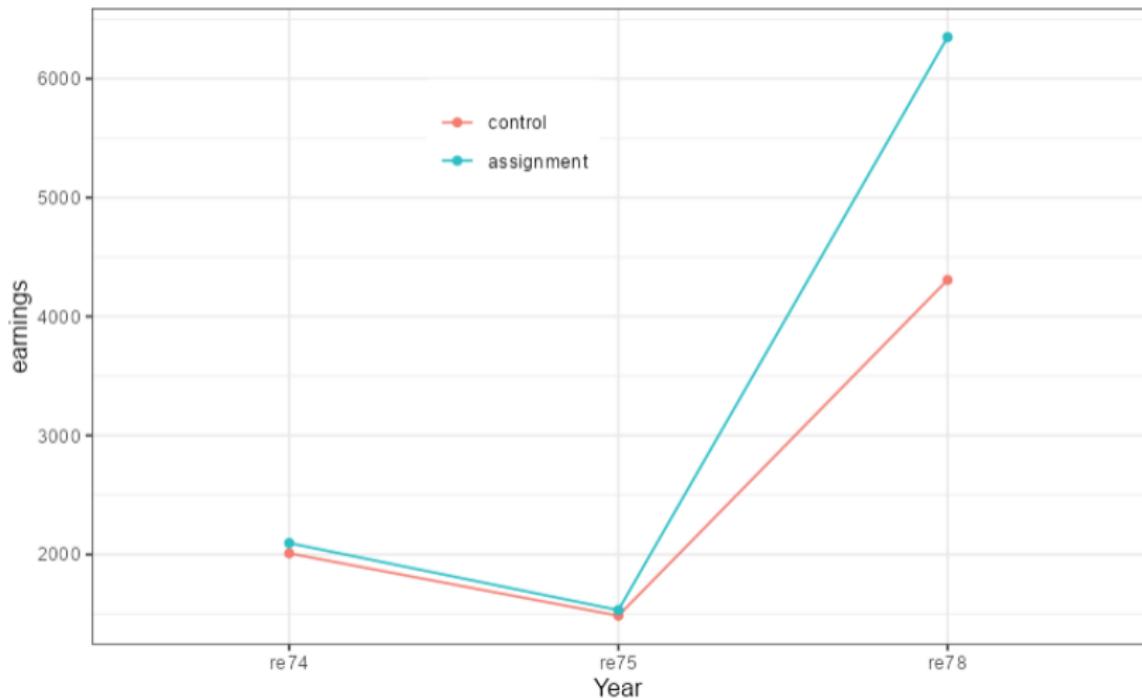
$$\text{DiD} = [\bar{Y}(1)_{\text{after}} - \bar{Y}(0)_{\text{after}}] - [\bar{Y}(1)_{\text{before}} - \bar{Y}(0)_{\text{before}}] \quad (7)$$

- ▶ Where
  - ▶ DiD is the differences-in-differences estimator,
  - ▶  $\bar{Y}(1)_{\text{after}}$  is the average unemployment for New Jersey **after** increasing the MW,
  - ▶  $\bar{Y}(0)_{\text{after}}$  is the average unemployment for Pennsylvania **after not** increasing the MW,
  - ▶  $\bar{Y}(1)_{\text{before}}$  is the average unemployment for New Jersey **before not** increasing the MW,
  - ▶  $\bar{Y}(0)_{\text{before}}$  is the average unemployment for Pennsylvania **before not** increasing the MW.

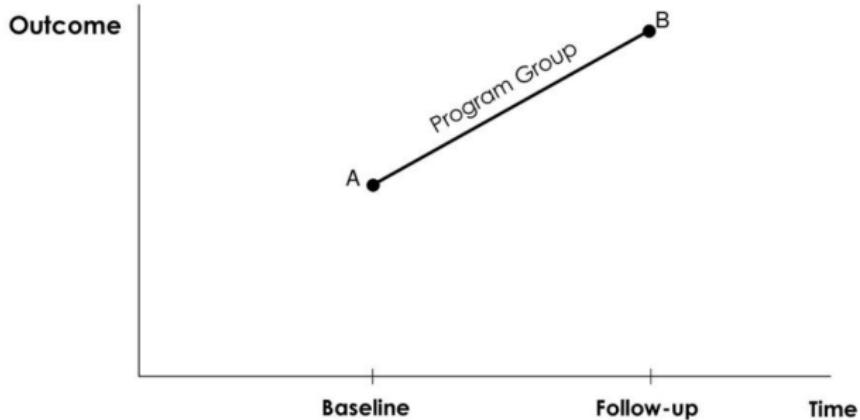
## Research design: Differences-in-Differences



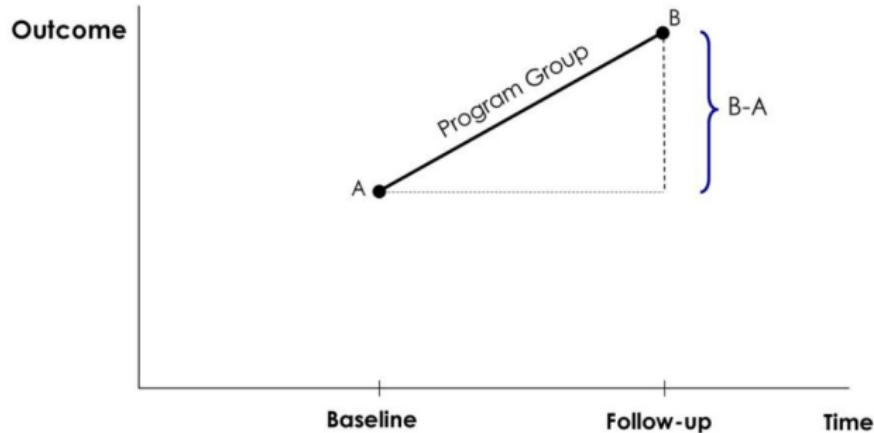
# Research design: Parallel trends



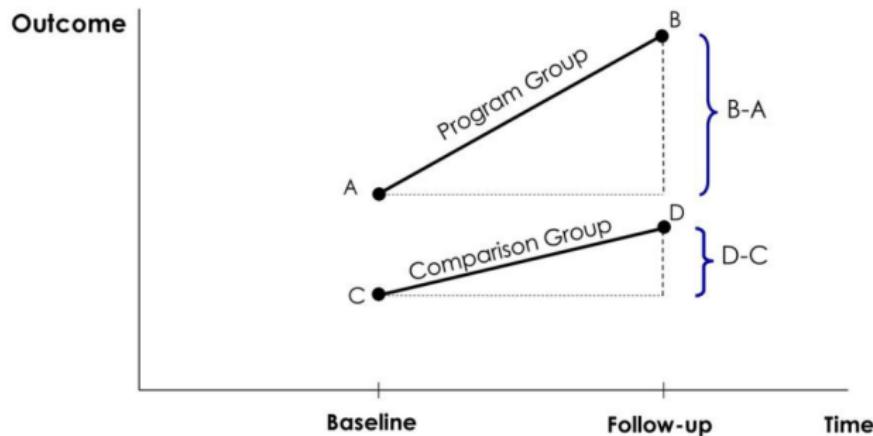
# Research design: Differences-in-Differences



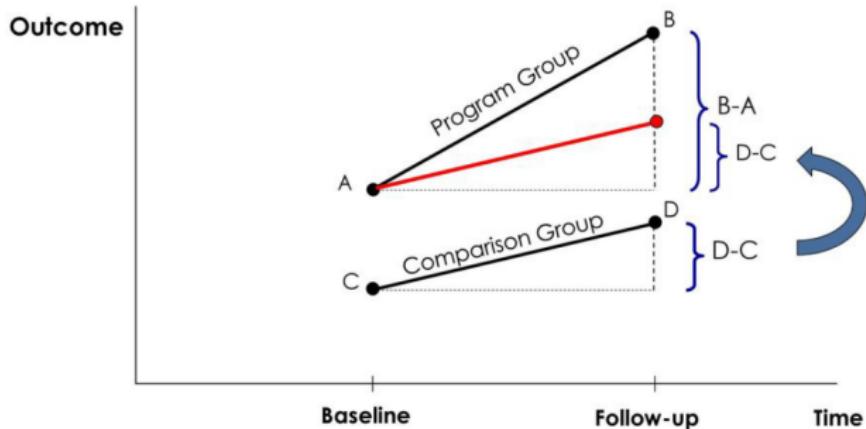
# Research design: Differences-in-Differences



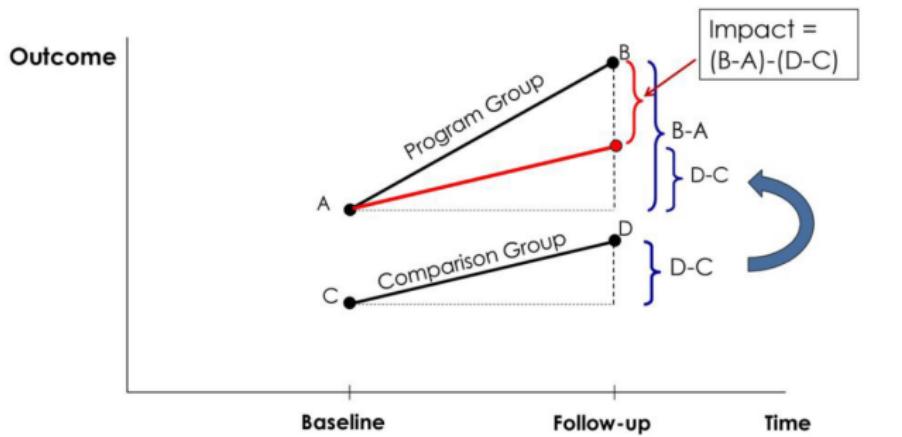
# Research design: Differences-in-Differences



# Research design: Differences-in-Differences



# Research design: Differences-in-Differences



# Causality review: key points

- ▶ To identify a causation in experimental settings, *perfect randomization* provides **covariate balance** between treatment and control groups (*exchangeability*), considering both **observed** and **unobserved** variables.
  - ▶ However, in practice, even in experimental designs, practitioners often **adjust** by conditioning on **unbalanced confounders** (*unconfoundedness*).
- ▶ In some **observational studies**, it is possible to estimate causal effects in **research designs** that mimic *natural experiments*.

# Causality review: key points

- ▶ Understand the trade-offs between **internal** and **external** validity when interpreting research design and statistical results (see Professor Ainsley's Week 3 slides).
- ▶ Adjusting for **confounding** in observational studies through linear regression does not guarantee identification of a causal effect.
  - ▶ Identification of a causal effect requires **balanced unobservable** characteristics or assumptions as **as-if random**, like in [Card and Krueger \(1994\)](#).

# Statistics: recap

So far, we have seen:

- The **population** mean and variance:

$$\mu = E[X] \quad ; \quad \sigma^2 = V[X] = E[(X - \mu)^2]$$

- The **sample** mean and variance:

$$\bar{X} = \frac{1}{N} \sum_{i=1}^n X_i \quad ; \quad S^2 = \frac{1}{N} \sum_{i=1}^n (X_i - \bar{X})^2$$

*Note:*

- the **expectation**  $E[.]$  is an operator that calculates the **average value** of a function of a random variable.
- *disclaimer:* it is actually more than an average, but for now it is "fine".

# Statistics: recap

- ▶ the **population** mean ( $\mu$ ) and variance ( $\sigma^2$ ) are fixed quantities (*TRUEs*) of a **data generating process**.
- ▶ the **sample** mean ( $\bar{X}$ ) and variance ( $S^2$ ) are random variables, and **estimators** of the population parameters ( $\mu$  and  $\sigma^2$ ).

In addition, we have seen:

- ▶ The conditional expectation (or mean):  $E[Y|X]$ .
- ▶ The mean squared error, a measurement of **prediction error**:
  - ▶ MSE :  $E[(Y_i - \theta)^2]$

# Statistics: covariance

Note:

$$V[X] = E[(X - \mu)^2] = E[(X - \mu)(X - \mu)]$$

We can ask how much **two variables** vary together with the covariance:

$$\text{Cov}[Y, X] = E[(Y - \mu_Y)(X - \mu_X)]$$

- ▶ **Covariance** measures the degree to which two random variables change (*vary*) together.
- ▶ It quantifies the extent of **linear association** between two variables.

# Statistics: correlation

- ▶ A drawback of **covariance** is its sensitivity to the original numeric **scale** of each variable ( $Y$  and  $X$ ).
- ▶ To normalize its scale, we can compute the ratio of each variable's **standard deviation**, resulting in Pearson's correlation:

$$\rho = \frac{\text{Cov}[Y, X]}{S(Y)S(X)}$$

- ▶ It offers a standardized measure of the **strength** and **direction** of the linear relationship between two variables.

## Linear model: intercept only

A special case of the regression model is when there are no regressors

$$Y = \mu + e$$

In the **intercept only model**, we find out that the best predictor is  $\mu$ !

Hence, the best predictor of an unconditional distribution is its **mean**. We can show this by computing the MSE:

$$\text{MSE} : E[(Y - \theta)^2] = E[(Y - \mu)^2]$$

# Bivariate regression

$$Y_i = \alpha + \beta_1 X_i + e_i \quad (8)$$

Notation:

- ▶  $Y$  is the **outcome** or dependent variable.
- ▶  $X$  (or  $T$ ) is the **predictor** or independent variable.
- ▶  $\alpha$  (or sometimes  $\beta_0$ ) is the **intercept**.
- ▶  $\beta$  are **coefficients** or slopes that we estimate.
- ▶  $e$  is the **error** term or disturbance.
- ▶ Subscript  $i$  refers to each observation (row).

*Research question:* what is the relationship between fertility and education?

- ▶  $Y$ : Fertility rates.
- ▶  $X$ : Education Beyond Primary School.

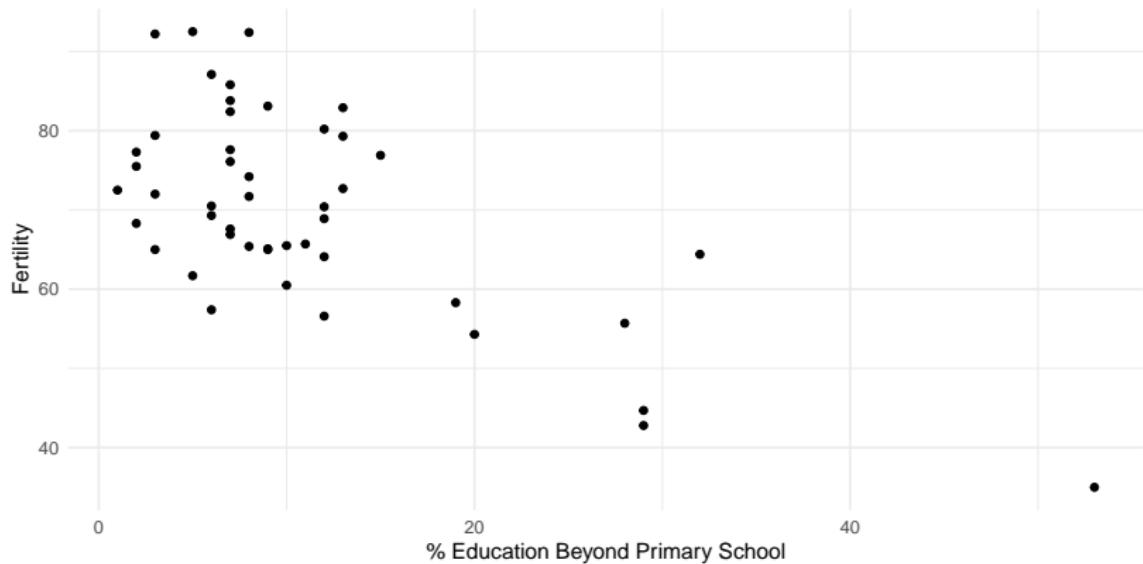
# Bivariate regression

- **Note:** in this case, each *i*th refers to a municipality from Switzerland.

	Fertility	Education	Agriculture	Examination	Catholic
## Courtelary	80.2	12	17.0	15	9.96
## Delemont	83.1	9	45.1	6	84.84
## Franches-Mnt	92.5	5	39.7	5	93.40
## Moutier	85.8	7	36.5	12	33.77
## Neuveville	76.9	15	43.5	17	5.16
## Porrentruy	76.1	7	35.3	9	90.57

# Bivariate regression

Is there a negative or positive relationship between education and fertility? How strong is this relationship? What would “no relationship” look like visually?



# Bivariate regression

We can quantify this direction and strength by **correlation**:

```
cor(swiss$Education, swiss$Fertility)
```

```
## [1] -0.6637889
```

```
swiss %>%
  select(Education,Fertility) %>%
  cor() %>% round(digits=2)
```

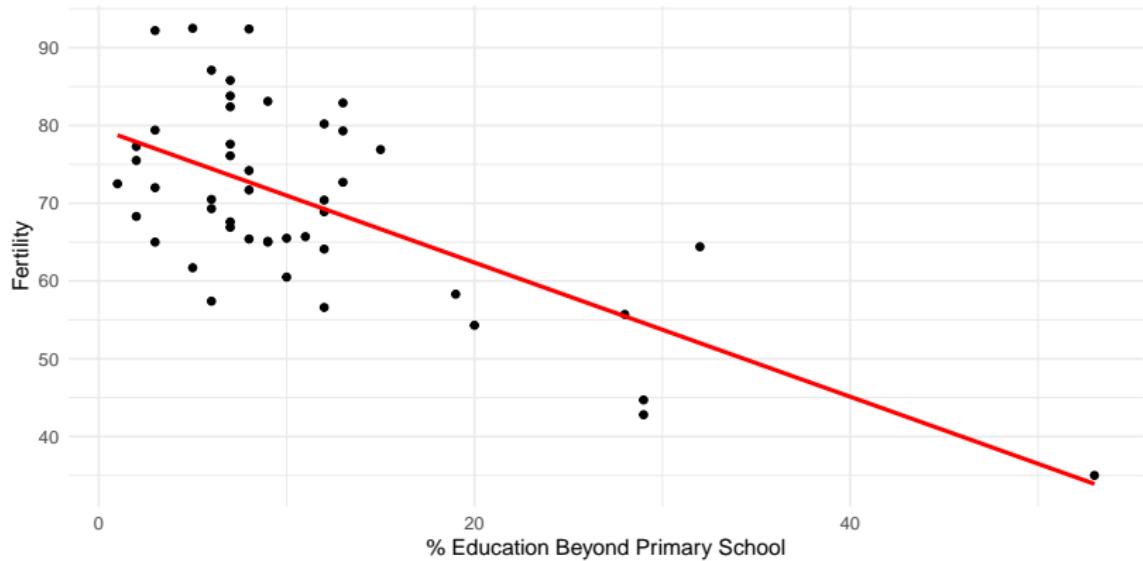
```
##          Education Fertility
## Education      1.00     -0.66
## Fertility     -0.66      1.00
```

# Bivariate regression

- ▶ Assumes **linear** relationship: it measures the **strength** and **direction** of a linear association between variables.
  - ▶ Not optimal for **non-linear** relationships.
- ▶ Values range from -1 to 1.
- ▶ Interpreting the magnitude of the coefficient: in general, **larger** absolute values of the correlation coefficient indicate **stronger** relationships.

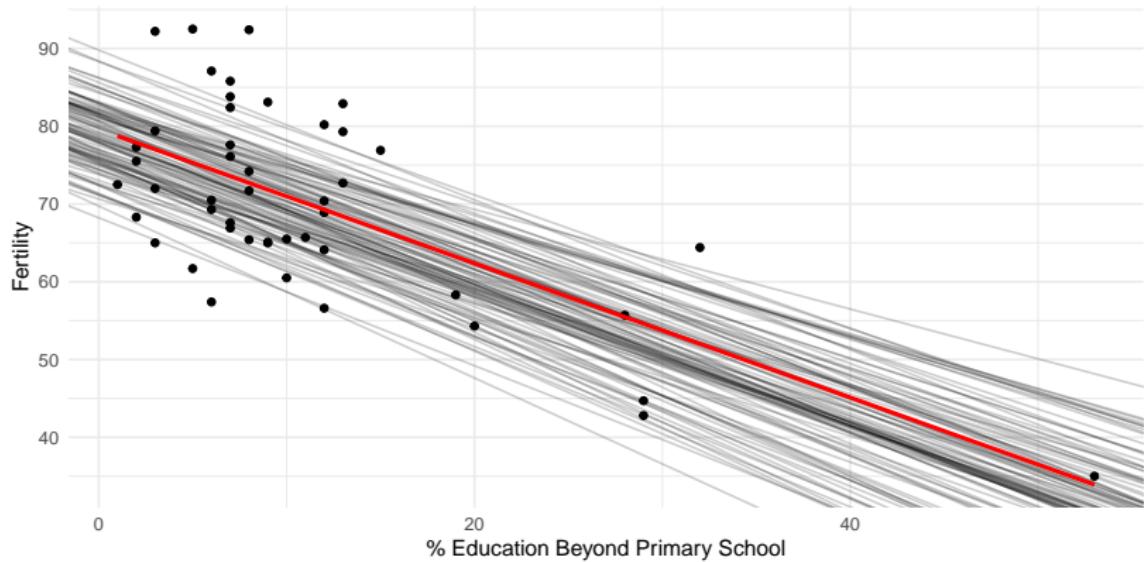
# Bivariate regression

- We can ask ggplot to plot a regression line fit on top of our scatter.
  - `geom_smooth(method="lm")`



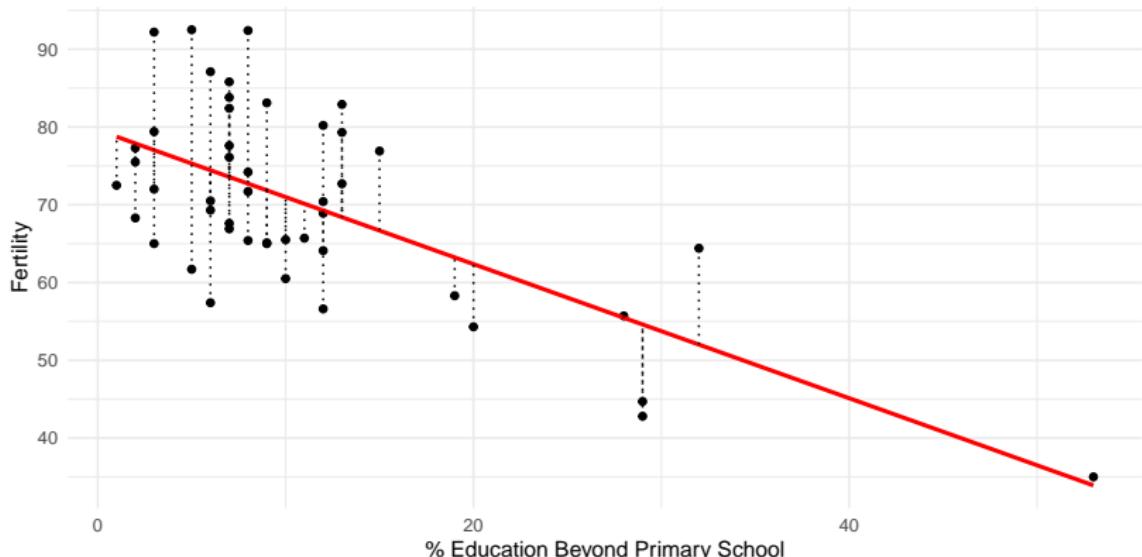
# Bivariate regression

- ▶ How does R draw this regression line?
  - ▶ In fact, you can draw many lines that “pass through” those points:



## Bivariate regression

- If I ask you to draw only one line that “*best describe the relationship.*” How do we pick the “*best fitting*” line?
- The clue is in the **OLS** (ordinary least squares): **minimizing** the **sum of squared distance (error)** of all points to a line.



# Bivariate regression

- ▶ How do we run regression to produce the best fitting line?

```
res <- lm(Fertility ~ Education, data = swiss)
coef(res)
```

```
## (Intercept)   Education
## 79.6100585 -0.8623503
```

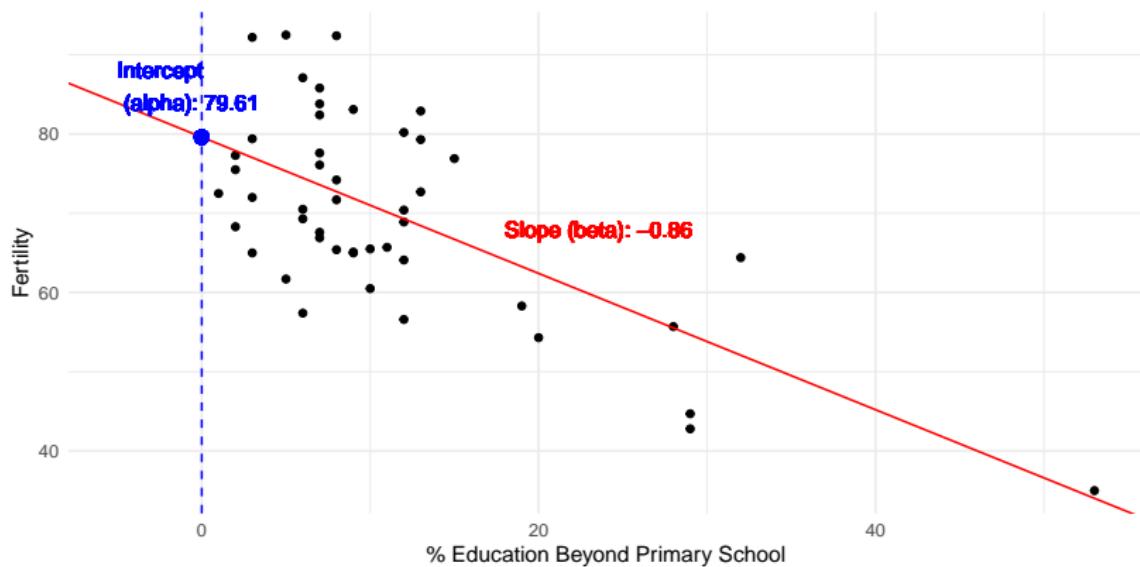
$$\hat{Y}_i = \hat{\alpha} + \hat{\beta}_1 X_i \quad (9)$$

$$Fertility_i = (79.61) + (-0.86)Education_i \quad (10)$$

- ▶ Prediction: if education = 0, fertility on average  $\hat{Y}$  is 79.6.
- ▶ Prediction: If education goes up by 1 unit, all else equal, change on fertility ( $\hat{Y}$ ) is -0.86 units.

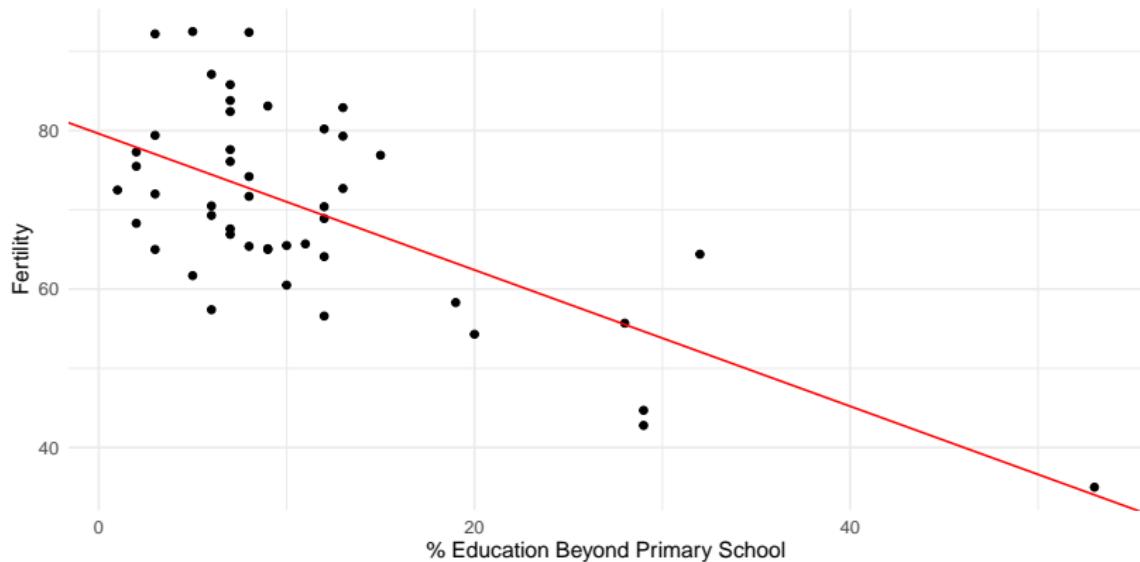
# Bivariate regression

- Visualizing  $\hat{\alpha}$  and  $\hat{\beta}$ :



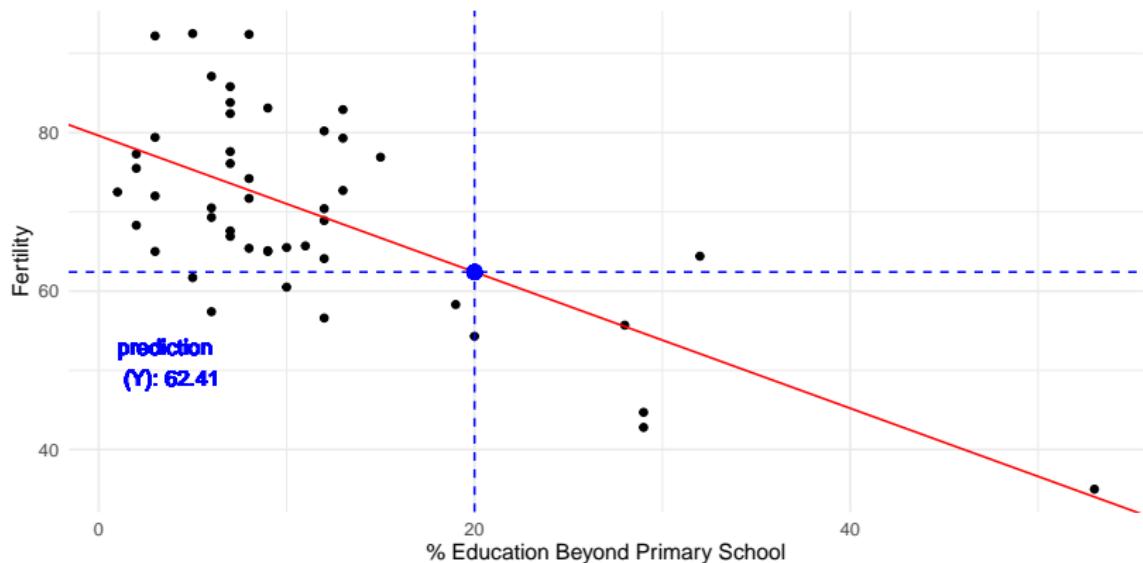
# Bivariate regression: prediction

- ▶ what is the average fertility rate when the rate of education beyond primary school is 20?



# Bivariate regression: prediction

- ▶ what is the average fertility rate when the rate of education beyond primary school is 20?



# Bivariate regression

- You can use the function `summary()` to display most of the necessary results.

```
summary(res)

## 
## Call:
## lm(formula = Fertility ~ Education, data = swiss)
## 
## Residuals:
##      Min       1Q   Median       3Q      Max
## -17.036  -6.711  -1.011   9.526  19.689
## 
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)    
## (Intercept) 79.6101    2.1041  37.836 < 2e-16 ***
## Education   -0.8624    0.1448  -5.954 3.66e-07 ***
## ---        
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
## 
## Residual standard error: 9.446 on 45 degrees of freedom
## Multiple R-squared:  0.4406  Adjusted R-squared:  0.4282
```

## Bivariate regression: intercept.

- ▶ The intercept, denoted by  $\alpha$ , represents the **predicted value** of the outcome variable  $\hat{Y}$  when all covariates on the left-hand side of the equation are set to 0.
- ▶ The intercept is estimated as function of the **estimated slopes** and **sample means**:

Bivariate model:

$$\hat{\alpha} = \bar{Y} - \hat{\beta}_1 * \bar{X}_1 \quad (11)$$

- ▶ The intercept is **not equivalent** to the sample mean value of the outcome,  $\bar{Y}$ , when all covariates are 0.
  - ▶ **Exception:** If the covariates are **centered**, which means they are transformed to have a mean of 0. E.g.,  $X_i - \bar{X}$ .

# Bivariate regression: intercept.

Bivariate model:  $\hat{\alpha} = \bar{Y} - (\hat{\beta}_1 * \bar{X}_1)$

```
(Y_mean <- mean(swiss$Fertility)) # sample mean of Y
```

```
## [1] 70.14255
```

```
(X_mean <- mean(swiss$Education)) # sample mean of X
```

```
## [1] 10.97872
```

```
Y_mean - (beta_1 * X_mean) # estimating the intercept
```

```
## Education  
## 79.61006
```

```
intercept
```

```
## (Intercept)  
## 79.61006
```

## Bivariate regression: slope.

- ▶ In a **bivariate regression**, the estimated slope coefficient represents the change in the dependent variable (Y) associated with a unit increase in the independent variable (X).
  - ▶ Empirical model:  $\hat{Fertility}_i = 79.61 - 0.86 * Education_i$ .
  - ▶ Interpretation:  $\beta_1$  has a slope of  $-0.86$ , and represents the **average** change in *Fertility* for every unit of increase in *Education*.
- ▶ In the **multivariate regression**, the estimated slope gives us the expected change in Y for each unit increase in X, holding all other variables constant (at their means).

## Bivariate regression: slope.

- ▶ In a bivariate regression model, the estimated coefficient of X,  $\hat{\beta}$ , with OLS is equal to the
  1. **correlation between X and Y**,  $r(X, Y)$ , multiplied by
  2. the **ratio** of the standard deviation of Y and the standard deviation of X,  $sd(Y)/sd(X)$ .

$$\hat{\beta} = r(X, Y) * (sd(Y)/sd(X)) \quad (12)$$

- ▶ **Standard deviation** is a measure of the amount of **variation or dispersion** of a set of data values around their **mean**.
- ▶ This **only** applies to the **bivariate model**.
- ▶ In the multivariate model, while there is interdependence among variables variation, it is distinct.

## Bivariate regression: slope.

```
(correl <- cor(swiss$Fertility,swiss$Education)) # correlation between Y and X  
  
## [1] -0.6637889  
  
(sd_Fert <- sd(swiss$Fertility)) # standard deviation of Y  
  
## [1] 12.4917  
  
(sd_Edu <- sd(swiss$Education)) # standard deviation of X  
  
## [1] 9.615407  
  
correl * (sd_Fert / sd_Edu)  
  
## [1] -0.8623503  
  
beta_1  
  
## Education  
## -0.8623503
```