KYLE STAHL - MN (/)

STOCK PORTFOLIO OPTIMIZATION

CONVEX OPTIMIZATION IN PYTHON

Kyle Stahl

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INTRODUCTION

Thinking about managing your own stock portfolio? This is going to illustrate how to implement the Mean-Variance portfolio theory (aka the markowitz model) in python to minimize the variance of your portfolio given a set target average return. The higher of a return you want, the higher of a risk (variance) you will need to take on. This optimization problem will find the optimal weights for each assest in the portfolio.

This analysis will make two different optimized portfolios: one with the ability to short sell stocks, and the other without the ability to short sell. Then we will compare how both of these portfolios performed compared to the NASDAQ index over 2017. The NASDAQ increased ~27% through 2017 (https://seekingalpha.com/article/4135478-2017-nasdaq-return (https://seekingalpha.com/article/4135478-2017-nasdaq-return)), which is pretty good. We set our portfolios to attempt to get a 7.0% percent return, which is the average annual return of the stock market (https://www.thesimpledollar.com/where-does-7-come-from-when-it-comes-to-long-term-stock-returns/ (https://www.thesimpledollar.com/where-does-7-come-from-when-it-comes-to-long-term-stock-returns/)).

IMPORT PACKAGES

In [1]:
```
import numpy as np
import pandas as pd
from scipy.optimize import minimize

import quandl
quandl.ApiConfig.api_key = "FZCjpYTSpS7TQrKFZA
t-"
```

I used the quandl API to get the stock market data. All you need to do to use it is get a key from https://www.quandl.com/tools/python (https://www.quandl.com/tools/python). You can also get stock market prices fairly easily from pandas-datareader, yahoo finance, and google finance.

DOWNLOAD STOCK DATA

Here we define a list of a bunch of stock tickers to pick from to put in our portfolio.

In [2]:
```
stocks = ['AAPL','F','TSLA','MSFT','GIS',
          'AMZN','DIS','BBY','BA','CBS',
          'DAL','EFX','UNH','GE','GM',
          'HON','IBM','JNJ','NKE',
          'SBUX','TGT','USB','VZ','WFC']

stock_prices = pd.DataFrame()
```

Then we loop through that list, the grab the stock prices from 2012, and append those prices to a data frame.

In [3]:
```
for stock in stocks:
    new_stock = quandl.get('WIKI/'+stock,
    start_date="2012-01-01")['Adj. Close']
    stock_prices[stock] = new_stock
```

In [4]:
```
stock_prices.head(5)
```

Out[4]:

	AAPL	F	TSLA	MSFT	GIS
Date					
2012-01-03	52.848787	8.772734	28.08	22.792249	33.:
2012-01-04	53.132802	8.906729	27.71	23.332995	33.:
2012-01-05	53.722681	9.135309	27.12	23.571435	33.:
2012-01-06	54.284287	9.229893	26.91	23.933352	33..
2012-01-09	54.198183	9.300832	27.25	23.622529	33.:

5 rows × 24 columns

Then we are going to split the stock prices into two sets. The *train* set will be used to find the optimal weights, and the *test* set will be used to test and see how the portfolio performed based off of the weights decided by the markowitz model.

For the markowitz optimization model, we are trying to minimize the variance of ***returns***... not of stock prices. We will use the `pct_change()` method to find the returns for each day.

In [5]:
```
train = stock_prices[stock_prices.index < '2017-01-01']
test = stock_prices[stock_prices.index >= '2017-01-01']
```

Note: If the below text does not show up as bold centered equations, refresh the page and it should render properly. I think this happens due to some bug with square space code embeding.

For day i... $$Return_i = \frac{Price_i}{Price_{i-1}} \qquad \qquad Pct. Change = \frac{Price_i - Price_{i-1}}{Price_{i-1}} \qquad \Rightarrow \qquad Return_i = Pct. Change + 1$$

In [6]:
```
returns = train.pct_change()
returns = returns.iloc[1:] # Day 1 does not have a return so we remove it from the dataframe
returns = returns + 1
returns.head(5)
```

Out[6]:

	AAPL	F	TSLA	MSFT	GI
Date					
2012-01-04	1.005374	1.015274	0.986823	1.023725	1.(
2012-01-05	1.011102	1.025664	0.978708	1.010219	1.(
2012-01-06	1.010454	1.010354	0.992257	1.015354	0.!
2012-01-09	0.998414	1.007686	1.012635	0.987013	0.!
2012-01-10	1.003580	1.000000	1.013578	1.003605	1.(

5 rows × 24 columns

INITIALIZE PARAMETERS

This is the optimization problem that we want to solve. We want to minimize the projected variance (Σ) of the portfolio for a given projected return (μ). The portfolio weights must sum up to 1. And for this problem we will say all of the weights must be greater than 0 (this is the function equivalent to not allowing short selling, we will relax this later). As stated above we are aiming for a 7% return for the year, we need to find the daily return needed to hit that mark and set that as μ.

- w = ***vector of weights of each stock***
- r = ***vector of mean returns for each stock***
- Σ = ***Covariance matrix of stock returns***
- e = ***vector of 1s***

\begin{align} minimize \qquad w^T\Sigma w \qquad \qquad \qquad \\ subject\ to\qquad \qquad \qquad \qquad \qquad \\ r^T w = \mu \qquad \qquad \quad \\ e^T w = 1 \qquad \qquad \quad \\ w \ge 0 \quad\forall \ i \qquad \qquad \end{align}

In [7]:
```
# Mean return for each stock (in array)
r = np.array(np.mean(returns, axis=0))

# Covariance matrix between stocks (in array)
S = np.array(returns.cov())

# Vector of 1's equal in length to r
e = np.ones(len(r))

# Set the projected mean return for the portfo
lio
mu = 1+(0.07/252) # 7% rate annually per day
```

DEFINE OPTIMIZATION PROBLEM

Here we define the objective function that we want to minimize (variance of the portfolio). Here is a quick rundown on how the variance is calculated. https://www.investopedia.com/ask/answers/071515/how-can-i-measure-portfolio-variance.asp (https://www.investopedia.com/ask/answers/071515/how-can-i-measure-portfolio-variance.asp). The we initialize the weights of the portfolio randomly, so that we have a python array for the values.

$$Variance = \sum_{j=1}^{n}\sum_{i=1}^{n} \sigma_{i,j} w_i w_j \quad \forall \ i,j \in \ set \ of \ stocks$$

In [8]:
```
def objective(w):
    return np.matmul(np.matmul(w,S),w)

# Set initial weight values
w = np.random.random(len(r))
```

Now we define the constraints for the optimization problem. It could happen that all the weights are set to be 0, then the variance of returns would also be 0. Our constraints are inplace to prevent this. The SciPy convex optimization solver accepts a tuple of dictionary constraints, which can be kinda complicated to think about so take a second to look what is going on.

We specify the type of constraint, `ineq` for $\ge$ and `eq` for =. The inequality is automatically $\ge$ because it is a minimization problem. There is a little more to it than that, but that is beyond the scope of this analysis. Here is the documentation of how the SciPy minimize function:
https://docs.scipy.org/doc/scipy/reference/generated/scipy.optimize.min
(https://docs.scipy.org/doc/scipy/reference/generated/scipy.optimize.mi
And some information on optimization problems in general:
https://www.math.ucla.edu/~tom/LP.pdf
(https://www.math.ucla.edu/~tom/LP.pdf)

For the non-negativity constraints, we can specify the `bounds` arugment of the `minimize` function. This argument will take a tuple that is the same length of the number of deicision variables. Each element of that tuple, will be another tuple of length 2: with the lower and upper bounds for each decision variable. So we have a tuple of dictionaries for the constraints, and a tuple of tuples for the variable bounds.

$$Returns:\qquad \sum_{i=1}^{n}w_ir_i \ge \mu$$

$$\quad \quad Weights\ sum\ to\ 1:\quad \quad \sum_{i=1}^{n} w_i = 1.0\qquad \qquad$$

$$\qquad \qquad \qquad \qquad Bounds:\quad 0\ \le \ w_i\ \le \ \infty \qquad \forall \ i \in \ set \ of \ stocks$$

In [9]:
```
# Define Constraints
const = ({'type' : 'ineq' , 'fun' : lambda w: np.dot(w,r) - mu}, # returns - mu >= 0
         {'type' : 'eq' , 'fun' : lambda w: np.dot(w,e) - 1})    # sum(w) - 1 = 0

# Create Bounds
# Creates a tuple of tuples to pass to minimize
# to ensure all weights are betwen [0, inf]
non_neg = []
for i in range(len(r)):
    non_neg.append((0,None))
non_neg = tuple(non_neg)
```

Next step is to solve the Linear Program. The attributes of the optimization program are saved in the varaible named `solution`. And `solution.x` has the weights for each stock. We print out the weights and make sure they sum to 1. As you can see, not all of the stocks need to be invested in. This could be because stock returns can sometimes be exteremly co-linear, therefore buying some stocks that are very similar can be redundant. Then we print out the names of the stocks that the algorithm chose to buy.

In [10]:
```
# Run optimization with SLSQP solver
solution = minimize(fun=objective, x0=w, metho
d='SLSQP',constraints=const,bounds=non_neg)
w = solution.x.round(6)
print w
print w.sum()
print list(returns.columns[w > 0.0])
```

```
[ 0.        0.048917  0.        0.178053  0.
0.282629  0.        0.
  0.19074   0.        0.        0.        0.
0.        0.        0.
  0.        0.        0.        0.        0.06
2392  0.        0.        0.237269]
1.0
['F', 'MSFT', 'AMZN', 'BA', 'TGT', 'WFC']
```

```
C:₩Users₩kyles₩.conda₩envs₩Python2₩lib₩site-pa
ckages₩scipy₩optimize₩slsqp.py:341: RuntimeWar
ning: invalid value encountered in greater
  bnderr = bnds[:, 0] > bnds[:, 1]
```

Now we will invest \$100,000 in this portfolio for the prices at January 1st, 2017, and track how it performed over the year.

In [11]:
```
# Invest $100,000 on Jan. 1st 2017
num_shares = w * 100000 / test.iloc[0,]
np.dot(num_shares, test.iloc[0,])
no_short = test.dot(num_shares)
```

If we relax the bounds of the weights and allow them to be negative (and have the ability to short sell stocks), this optimization problem has a closed form solution that can be found using Lagrangian Multipliers and KKT conditions. It is out of scope to deep dive into the mathematics of non-linear optimization problems, but if you want to learn more wikipedia is an good place to start: (https://en.wikipedia.org/wiki/Lagrange_multiplier (https://en.wikipedia.org/wiki/Lagrange_multiplier)) (https://en.wikipedia.org/wiki/Karush%E2%80%93Kuhn%E2%80%93Tu (https://en.wikipedia.org/wiki/Karush%E2%80%93Kuhn%E2%80%93Tu (https://www.math.ust.hk/~maykwok/courses/ma362/Topic2.pdf (https://www.math.ust.hk/~maykwok/courses/ma362/Topic2.pdf)). We are going to add the following constraint as an optimality condition using the lagrange multipliers: this will allow us to find the closed form solution for `w`.

$$Optimality\ Condition:\qquad \sum_{j=1}^{n} \sigma_{ij} w_j\ -\ \lambda_1\ -\ \lambda_2 \overline{R}_i \quad \forall \ i \ \in 1...n$$

VECTORIZED CLOSED FORM SOLUTION

$$w=\lambda_1 \Sigma^{-1} \overline{r}\ +\ \lambda_2 \Sigma^{-1} e$$

$$where:\qquad \lambda_1 = \frac{c\mu - b}{ac - b^2} \qquad \lambda_2 = \frac{-b\mu + a}{ac - b^2} \qquad$$

$$and: \qquad a=\overline{r}^{\ \ T}\Sigma^{-1}\overline{r} \qquad b=\overline{r}^{\ \ T}\Sigma^{-1}e \qquad c=e^{\ T}\Sigma^{-1}e$$

In [12]:
```
# Mean return for each stock (in array)
r = np.array(np.mean(returns, axis=0))

# Covariance matrix between stocks (in array)
S = np.array(returns.cov())

# Inverse of the covariance matrix
Si = np.linalg.inv(S)

# Vector of 1's equal in length to r
e = np.ones(len(r))

# a, b, c coefficients
a = np.matmul(np.matmul(r,Si),r)
b = np.matmul(np.matmul(r,Si),e)
c = np.matmul(np.matmul(e,Si),e) # same as Si.
sum()

# Lambda1 and Lambda2 coefficients
l1 = (c*mu - b) / (a*c - b*b)
l2 = (-b*mu + a) / (a*c - b*b)

# Calculate weights
w = l1*(np.matmul(Si,r)) + l2*(np.matmul(Si,e
))
print w
print w.sum()
print list(returns.columns[w != 0.0])
```

```
[ 0.04458599  0.06738652 -0.02410659 -0.042511
22  0.25483255 -0.02965159
  0.00375714 -0.0071139  -0.01019564 -0.012834
82 -0.05897875 -0.0154145
  0.00708143  0.00289147 -0.01635133 -0.067649
75  0.18989133  0.29394998
  0.04803156  0.01442399  0.12106522  0.096348
78  0.1944815  -0.05391938]
0.99999999986
['AAPL', 'F', 'TSLA', 'MSFT', 'GIS', 'AMZN',
'DIS', 'BBY', 'BA', 'CBS', 'DAL', 'EFX', 'UN
H', 'GE', 'GM', 'HON', 'IBM', 'JNJ', 'NKE', 'S
BUX', 'TGT', 'USB', 'VZ', 'WFC']
```

In [13]:
```
# Invest $100,000 on Jan. 1st 2017
num_shares = w * 100000 / test.iloc[0,]
np.dot(num_shares, test.iloc[0,])
short_sell = test.dot(num_shares)
```

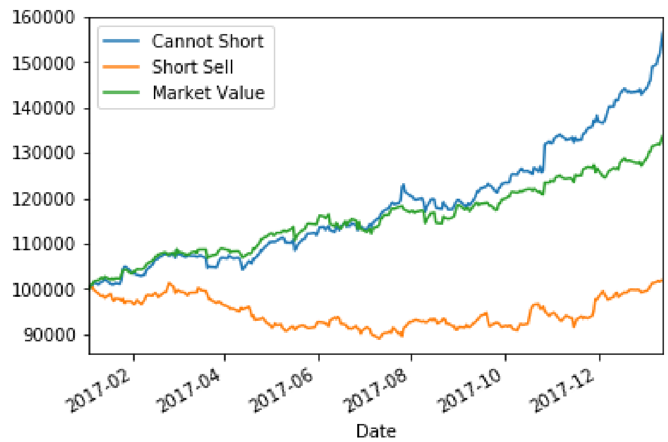
COMPARE AGAINST MARKET INDEX

In [14]:
```
# Get Market Index
Market = quandl.get("NASDAQOMX/COMP", start_da
te="2017-01-01")["Index Value"]

# Invest $100,000 on Jan. 1st 2017
market_shares = 100000 / Market.iloc[0,]
Market_Overall = Market * market_shares

# Combine portfolios into one dataframe
portfolio = pd.DataFrame({'Short Sell' : short
_sell, 'Cannot Short':no_short})
portfolio = portfolio.join(Market_Overall)
portfolio.rename(columns={'Index Value':'Marke
t Value'},inplace=True)
```

EVALUATE RESULTS

In [15]:
```
#Plot Results
%matplotlib inline
portfolio.plot(kind='line')
```

Out[15]: <matplotlib.axes._subplots.AxesSubplot at 0xa915b00>



In [16]:
```
results = pd.DataFrame()
portfolio_returns = portfolio.pct_change().iloc[1:] + 1
results['Overall Return'] = portfolio.apply(lambda x: x[len(portfolio.index)-1]/100000.0)
results['Mean Daily Return'] = portfolio_returns.mean()
results['Variance Daily Return'] = portfolio_returns.var()
results.sort_values(by='Overall Return', ascending=False,inplace=True)
results.head()
```

Out[16]:

	Overall Return	Mean Daily Return	Variance Daily Return
Cannot Short	1.564761	1.001769	0.000050
Market Value	1.337438	1.001150	0.000037
Short Sell	1.019725	1.000097	0.000043

Since the beginning of 2017, the stock market index has increased by 33.7%, which is towards the top of historical annual returns of the NASDAQ index (http://www.1stock1.com/1stock1_142.htm (http://www.1stock1.com/1stock1_142.htm)). We would normally expect the portfolio that has the ability to short sell to perform the best since it has the most flexibility. But 2017 was such an up year for the market, shorting the popular stocks I included in the portfolio was not a good idea, the variance of the short sell portfolio was even lower than the market index. The no-short portfolio out performed the market, but it is odd that the variance of that portfolio is so high considering we are trying to minimize variance. I would've expected the returns, and the variance to be slightly lower than the market.

FUTURE ANALYSIS AND NEXT STEPS

Just like every mathematical model, the markowitz model has it's flaws/assumptions. The biggest assumption of this model is that the past is truely indicitive of the future. Specifically asset returns and variance. This assumption shows up a lot, and I think is flawed especially when talking about stock prices. Another view of stock returns is that the only thing that matters is the stock's current price, along with other metrics and valuations of the company's financial statements (a.k.a the previous price movements are mute).

I think this assumption is a better representation of reality. For a future analysis, I think it would be good to try and apply the same optimization problem, but calculate ***r*** and Σ differently; and in a way that does not rely on historical data. I would try to use predictive modeling and machine learning based off of company financial statements, and external factors to predict the mean return and covariances of the stocks. This might require calculating returns on a quarterly level instead of every single day. This is fine, it will just reduce the number of observations of returns for each stock.

I also want to try to calculate the weights dynamically. Every single day (or quarter in the case above) there is new information that you could use to recalucate the weights for your portfolio. Every day/week/quarter you could recalculate your optimal combination of assests, and then buy and sell accordingly.

In [ ]:

```

```