Partially Ordered Monads for Monadic Topologies, Rough Sets and Kleene Algebras

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Abstract

In this paper we will show that partially ordered monads contain sufficient structure for modelling monadic topologies, rough sets and Kleene algebras. Convergence represented by extension structures over partially ordered monads includes notions of regularity and compactness. A compactification theory can be developed. Rough sets [23] are modelled in a generalized setting with set functors. Further, we show how partially ordered monads can be used in order to obtain monad based examples of Kleene algebras building upon a wide range of set functors far beyond just strings [19] and relations [27].

Keywords: Partially ordered monads, Monadic topologies, Rough sets, Kleene algebras

1 Introduction

Partially ordered monads can be seen as derived from studies around convergence structures, originally involving filters [21]. Cauchy structures, again based on filters were initiated in [18]. Steps departing from filters and using more general

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set functors for convergence were taken in [10] including a general structure theory. The observation that functors extended to monads [22] provide more power to convergence is due to [11]. Concrete examples, in particular involving the fuzzy filter monad, were developed in [7], followed by monad techniques for compactification constructions in [8] based on a compactification construction [24] for filter based limit spaces. The introduction of partially ordered monads and its use within extension structures is due to [13,14], with a follow-up on considerations for compacatifications in [15].

Topology and convergence were driving forces in the development of partially ordered monads and the demonstration of their power. However, these monads are useful also in other areas. We will show that they contain sufficient structure for modelling rough sets [23] in a generalized setting with set functors. Even for the ordinary relations, the adaptations through partially ordered monads increases the understanding of rough sets in a basic many-valued logic [16] setting. Further we show how these partially ordered monads contribute to providing a generalised notion of powerset Kleene algebras. This generalisation builds upon a more general powerset functor setting far beyond just strings [19] and relational algebra [27]. Kleene algebras are widely used e.g. in formal languages [26] and analysis of algorithms [1].

The paper is organized as follows. In Section 2 we introduce partially ordered monads. Section 3 is devoted to topology and we show how extension structures allow compactification constructions. Section 4 presents rough sets in a categorical framework entirely new to the rough set community. This opens up several avenues for applications involving more than just relations on sets, i.e. the power set partially ordered monad. In Section 5, we see how Kleene algebra multiplications are Kleisli compositions and Kleene asterates can be defined using the partial order within the monad. Finally, Section 6 concludes the paper.

2   Partially ordered monads

An almost complete semilattice is a partially ordered set $(X, \leq)$ such that the suprema $\sup \mathcal{M}$ of each non-empty subsets $\mathcal{M}$ of X exists. A *basic triple* ([13]) is a triple $(\varphi, \leq, \eta)$, where $(\varphi, \leq) : \mathsf{Set} \to \mathsf{acSLAT}$, $X \mapsto (\varphi X, \leq)$ is a covariant functor, with $\varphi : \mathsf{Set} \to \mathsf{Set}$ as the underlying set functor, and $\eta : \mathrm{id} \to \varphi$ is a natural transformation. A *monad* (φ, η, μ) over a category C consists of a covariant functor $\varphi : \mathsf{C} \to \mathsf{C}$, together with natural transformations $\eta : id \to \varphi$ and $\mu : \varphi \circ \varphi \to \varphi$ fulfilling the conditions $\mu \circ \varphi\mu = \mu \circ \mu\varphi$ and $\mu \circ \varphi\eta = \mu \circ \eta\varphi = id_\varphi$.

Partially ordered monads are monads [22], where the underlying endofunctor is equipped with an order structure. Some additional structure is imposed. Partially ordered monads are useful for various generalized topologies and convergence structures [13,15], and have also been used for generalisation of Kleene algebras [27,19,9].

Partially ordered monads over the category Set of sets are defined by means of functors from Set to the category acSLAT of almost complete semilattices. A

partially ordered monad is a quadruple $(\varphi, \leq, \eta, \mu)$, where $(\varphi, \leq, \eta)$ is a basic triple, (φ, η, μ) is a monad (over Set), and further, for all mappings $f, g : Y \to \varphi X$, $f \leq g$ implies $\mu_X \circ \varphi f \leq \mu_X \circ \varphi g$, where $\leq$ is defined argumentwise with respect to the partial ordering of φX. We also require that for each set X, $\mu_X : (\varphi \varphi X), \leq) \to (\varphi X, \leq)$ preserves non-empty suprema.

The classical example of a partially ordered monad is the power set partially ordered monad $(P, \leq, \eta, \mu)$, where PX is the ordinary power set of X and $\leq$ is set inclusion $\subseteq$ making $PX, \leq$ a partially ordered set. The unit $\eta : X \to PX$ is given by $\eta(x) = \{x\}$ and the multiplication $\mu : PPX \to PX$ by $\mu(\mathcal{B}) = \cup \mathcal{B}$.

The many-valued extension of P to L_{id} is as follows. Let L be a complete and completely distributive lattice, with 0 and 1, respectively, as the smallest and largest elements of L. The functor L_{id} is obtained by $L_{id}X = L^X$, i.e. the set of mappings $A : X \to L$. The partial order $\leq$ on $L_{id}X$ is given pointwise. Morphisms $f : X \to Y$ in Set are mapped according to

$$L_{id}f(A)(y) = \bigvee_{f(x)=y} A(x).$$

Here we adopt the convention that $\vee_\emptyset = 0$. Finally $\eta_X : X \to L_{id}X$ is given by

$$\eta_X(x)(x') = \begin{cases} 1 & \text{if } x' \leq x \\ 0 & \text{otherwise} \end{cases}$$

and $\mu_X : L_{id}X \circ L_{id}X \to L_{id}X$ by

$$\mu_X(\mathcal{M})(x) = \bigvee_{A \in L_{id}X} A(x) \wedge \mathcal{M}(A).$$

Proposition 2.1 *[5]* $\mathbf{L}_{id} = (L_{id}, \leq, \eta, \mu)$ *is a partially ordered monad.*

Given a monad, there are two standard ways of defining a category associated with it, namely the *Eilenberg-Moore* construction and the *Kleisli* construction. The latter is used in this paper.

For historical remarks let us mention the important role that Kleisli and Eilenberg-Moore categories play in the study of the relation between adjunctions and monads [2]. Every adjunction defines a monad, and conversely every monad can be seen as generated by an adjunction, called a *resolution* for the monad. Eilenberg-Moore and Kleisli categories give rise to resolutions, e.g., they are respectively the terminal and initial objects in the category of all resolutions.

A Kleisli category C_Φ for a monad Φ over a category C is defined as follows: Objects in C_Φ are the same as in C, and the morphisms are defined as $hom_{C_\Phi}(X, Y) = hom_C(X, \Phi Y)$, that is morphisms $f : X \to Y$ in C_Φ are simply morphisms $f : X \to \Phi Y$ in C, with $\eta_X^\Phi : X \to \Phi X$ being the identity morphism. Composition of morphisms is defined as $(X \xrightarrow{f} Y) \diamond (Y \xrightarrow{g} Z) = X \xrightarrow{\mu_Z^\Phi \circ \Phi g \circ f} \Phi Z$.

The Kleisli category is equivalent to the full subcategory of free Φ-algebras of the monad, and its definition makes it clear that the arrows are substitutions. Indeed, the categorical unification algorithm in [25] is based on the Kleisli category of the term monad.

A monad (Φ, η, μ) written as $(\Phi, \eta, \diamond)$, where $\diamond$ is the composition of morphisms in the corresponding Kleisli category, is said to be a monad in *clone form*. In fact, there is a one-to-one correspondence between monads, respectively, in monoid and clone forms [22].

The Kleisli category of the powerset monad (for $L = 2$) is the category of sets and relations, `SetRel`. This is a consequence of the fact that given a relation $R \subseteq X \times Y$, we can define a morphism $\varphi(R) : X \to PY$, $\varphi(R)(x) = \{y \in Y \mid (x, y) \in R\}$ associated to R; and, given the morphism $f : X \to PY$, we can define the relation $R_f \subseteq X \times Y$, $R_f = \{(x, y) \in X \times Y \mid y \in f(x)\}$.

3 Monadic topologies

Extension structures [12] generalize the notion of Cauchy structures and are functorially defined by basic triples. For more elaborate notions, like regularity, basic triples need to be enriched to become partially ordered monads. In this enrichment we are able to provide compactification constructions of related limit structures. The idea is to use completion constructions and to show that completion of some special Cauchy structures coincide with compactifications of their corresponding limit structures.

In the following we will show how these techniques are used and combined, and indeed our aim is to demonstrate the power of the underlying partially ordered monads. Proofs can be found in [15].

3.1 Extension structures

An *φ-extension structure* [12] on a set X is a triple $(S, t, \sim)$ such that $S \subseteq \varphi X$ and $t \subseteq \varphi X \times X$, and $\sim$ is an equivalence relation on S. We write $\mathcal{M} \xrightarrow{t} x$ instead of $(\mathcal{M}, x) \in t$, and the following conditions must be fulfilled.

(ext 1) $\mathcal{M} \xrightarrow{t} x$ implies $\mathcal{M} \in S$

(ext 2) $\mathcal{M} \xrightarrow{t} x$, $\mathcal{M} \sim \mathcal{N}$ imply $\mathcal{N} \xrightarrow{t} x$

(ext 3) $\mathcal{M} \xrightarrow{t} x$, $\mathcal{N} \xrightarrow{t} x$ imply $\mathcal{M} \sim \mathcal{N}$

A *morphism* (in the category of extension structures) $f : (X, (S_1, t_1, \sim_1)) \to (Y, (S_2, t_2, \sim_2))$ between φ-extension structures is a mapping $f : X \to Y$, where

(map 1) $\mathcal{M} \in S_1$ implies $\varphi f(\mathcal{M}) \in S_2$

(map 2) $\mathcal{M} \xrightarrow{t_1} x$ implies $\varphi f(\mathcal{M}) \xrightarrow{t_2} f(x)$

(map 3) $\mathcal{M} \sim_1 \mathcal{N}$ implies $\varphi f(\mathcal{M}) \sim_2 \varphi f(\mathcal{N})$

A φ-extension structure $(X, (S, t, \sim))$ is said to be *separated*, whenever $\mathcal{M} \xrightarrow{t} x$ and $\mathcal{M} \xrightarrow{t} y$ only if $x = y$, and *complete* if for each $\mathcal{M} \in S$ we have $\mathcal{M} \xrightarrow{t} x$ for some $x \in X$. The category of all separated and complete φ-extension structures is an epireflective subcategory of the category of all separated φ-extension structures [14].

A φ-extension structure $(X, (S, t, \sim))$ is called *η-stable* whenever $\eta_X(x) \xrightarrow{t} x$ for all $x \in X$. For each η-stable φ-extension structure $(X, (S, t, \sim))$ we have $t = \{(\mathcal{M}, x) \mid \mathcal{M} \sim \eta_X(x)\}$, and t is then completely determined by S and $\sim$.

3.2 Φ-Cauchy structures and their completions

Let $\Phi = (\varphi, \leq, \eta)$ be a basic triple. A *Φ-Cauchy structure* on a set X is a subset S of φX such that

(Cau 1) $\eta_X(x) \in S$ for all $x \in X$

(Cau 2) $\mathcal{M} \in S$ and $\mathcal{N} \leq \mathcal{M}$ imply $\mathcal{N} \in S$

(Cau 3) for $\mathcal{M}, \mathcal{N} \in S$, if the infimum $\mathcal{M} \wedge \mathcal{N}$ exists, then $\mathcal{M} \vee \mathcal{N} \in S$

If φ is the filter functor, then the Φ-Cauchy structures are the Cauchy structures in the usual sense.

A mapping $f : (X, S) \to (Y, S')$ between Φ-Cauchy structures is called *Φ-Cauchy continuous*, whenever $s \in S$ implies $\varphi f(s) \in S'$.

Each Φ-Cauchy structure (X, S) is associated with a η-stable φ-extension structure $(S, t, \sim)$, where S and t are given by

(1) $\quad \mathcal{M} \sim \mathcal{N} \Longleftrightarrow \mathcal{M} \vee \mathcal{N} \in S$

(2) $\quad \mathcal{M} \xrightarrow{t} x \Longleftrightarrow \mathcal{M} \vee \eta_X(x) \in S$

A mapping between Φ-Cauchy structures is Φ-Cauchy continuous if and only if it is a morphism between the associated η-stable φ-extension structures.

For each Φ-Cauchy structure (X, S), the set t defined by (2) is a *Φ-limit structure*, that is, the following conditions are fulfilled:

(lim 1) $\eta_X(x) \xrightarrow{t} x$ for all $x \in X$

(lim 2) $\mathcal{M} \xrightarrow{t} x$ and $\mathcal{N} \leq \mathcal{M}$ imply $\mathcal{N} \xrightarrow{t} x$

(lim 3) if $\mathcal{M} \xrightarrow{t} x$ and $\mathcal{N} \xrightarrow{t} x$, then $\mathcal{M} \vee \mathcal{N} \xrightarrow{t} x$

It is easily seen that Φ-Cauchy continuity of mappings $f : (X, S) \to (Y, S')$ amounts to continuity for corresponding mappings $f : (X, t) \to (Y, t')$ between the related Φ-limit structures as given by $\mathcal{M} \xrightarrow{t} x$ implying $\varphi f(\mathcal{M}) \xrightarrow{t'} f(x)$. Separatedness and completeness can be similarly introduced for Φ-Cauchy and Φ-limit structures, and, as easily seen, specialized to classical situations they correspond in the classical sense. A notion of regularity can also be introduced. Let (X, t) be a Φ-limit structure and let $t_1 : \mathrm{gr}[t] \to \varphi X$ and $t_2 : \mathrm{gr}[t] \to X$ be the projections $(\mathcal{M}, x) \mapsto \mathcal{M}$ and $(\mathcal{M}, x) \mapsto x$, respectively. Then (X, t) is said to be *regular* if for every $\mathcal{L} \in \varphi \mathrm{gr}[t]$ and every $x \in X$ we have $\varphi t_2(\mathcal{L}) \xrightarrow{t} x$ whenever $(\mu_X \circ \varphi t_1)(\mathcal{L}) \xrightarrow{t} x$.

Completions of Φ-Cauchy structures can be derived from completions of extension structures [6,14]. However, some conditions have to be imposed:

(cpl 1) for $\mathcal{M}, \mathcal{N} \in \varphi X$ and $x, y \in X$, if the infimum

$$(\mathcal{M} \vee \eta_X(x)) \wedge (\mathcal{N} \vee \eta_X(y))$$

exists, then at least one of the infima $\mathcal{M} \wedge \mathcal{N}$, $\mathcal{M} \wedge \eta_X(y)$ and $\mathcal{N} \wedge \eta_X(x)$ exists, or else $x = y$

(cpl 2) for $f : X \to Y$, $\mathcal{M} \in \varphi X$ and $y \in Y$, if the infimum $\eta_Y(y) \wedge \varphi f(\mathcal{M})$ exists, then $y = f(x)$ for some $x \in X$

We say $\mathcal{M} \in \mathcal{S}$ is *stratified* if $\mathcal{M} \leq \bigvee_{x \in X} \eta_X(x)$, and Φ fulfills the *stratification property* if, for each $\mathcal{M} \in \mathcal{S}$, the infimum $\mathcal{M}^{+} = \mathcal{M} \wedge \bigvee_{x \in X} \eta_X(x)$ exists, and for each mapping $f : X \to Y$ and $\mathcal{M} \in \varphi X$ we have $\varphi f(\mathcal{M}^{+}) = \varphi f(\mathcal{M})^{+}$.

Proposition 3.1 ([14]) *If Φ fulfills the conditions* (cpl 1) *and* (cpl 2), *then the category of all separated and complete Φ-Cauchy structures is an epireflective subcategory of the category of all separated Φ-Cauchy structures.*

3.3   Monadic compactifications

We begin by observing how compactness is seen as completeness of some dedicated Cauchy structures. Let $\Phi = (\varphi, \leq, \eta)$ be a basic triple. A Φ-limit structure (X, t) is called *compact* provided that all ultra objects $\mathcal{M} \in \varphi X$ converge with respect to t, and *weakly separated* if for all $x, y \in X$, whenever there are objects in φX converging to x and to y with respect to t, then

$$\{\mathcal{M} \mid \mathcal{M} \xrightarrow{t} x\} = \{\mathcal{M} \mid \mathcal{M} \xrightarrow{t} y\}$$

Proposition 3.2 *Let (X, t) be a separated Φ-limit structure and let*

$$S_t = \{\mathcal{M} \mid \mathcal{M} \xrightarrow{t} x \text{ for some } x\} \ \cup \ \{\mathcal{M} \in \varphi X \mid \mathcal{M} \text{ ultra}\}$$

Then (X, S_t) is a Φ-Cauchy structure and (X, t) is the associated Φ-limit structure of (X, S_t). Further, (X, t) is compact if and only if (X, S_t) is complete.

For the compactification construction, let $\Phi = (\varphi, \leq, \eta, \mu)$ be a partially ordered monad, and assume that Φ fulfills the following conditions:

(cpt 1) For each φ-object $\mathcal{M}$ on a set X there exists an ultra φ-object $\mathcal{U}$ on X such that $\mathcal{U} \leq \mathcal{M}$.

(cpt 2) For each mapping $f : X \to Y$ and each ultra φ-object $\mathcal{U}$ on X, $\varphi f(\mathcal{U})$ is an ultra φ-object on Y.

(cpt 3) For each set X and each ultra φ-object $\mathcal{U}$ on $\varphi^u X$, $(\mu_X \circ \varphi e_X)(\mathcal{U})$ is an ultra φ-object on X.

In the following, let (X, t) be a separated Φ-limit structure with (X, S_t) be the corresponding Φ-Cauchy structure. Further, let $\sim$ be the related equivalence relation on S_t defined by (1) and let $X^{\wedge} = \{\mathcal{M}^{\sim} \mid \mathcal{M} \in S\}$ be the set of all equivalence classes with respect to $\sim$.

We can now define a mapping $\kappa : X^{\wedge} \to \varphi X$ by

$$\kappa(\mathcal{M}^{\sim}) \ = \ \begin{cases} \eta_X(x) & \text{if } \mathcal{M} \xrightarrow{t} x \\[2mm] \mathcal{M} & \text{if } \mathcal{M} \text{ is a non-converging ultra object} \end{cases}$$

Proposition 3.3 *For an ultra object* $\mathcal{K} \in \varphi X^{\wedge}$, *we have that* $(\mu_X \circ \varphi \kappa)(\mathcal{K}) \in \varphi X$ *is also an ultra object. Further,* $(X^{\wedge}, t^{\wedge})$ *where*

$$t^{\wedge} = \{(\mathcal{K}, \mathcal{M}^{\sim}) \in \varphi X^{\wedge} \times X^{\wedge} \mid (\mu_X \circ \varphi \kappa)(\mathcal{K}) \in \mathcal{M}^{\sim}\}$$

is a separated and compact Φ*-limit structure.*

In the filter monad case, this corresponds precisely to the diagonalization technique given in [21].

Proposition 3.4 ([15]) *If* $\iota : X \to X^{\wedge}$ *be the mapping* $x \mapsto \eta_X(x)^{\sim}$, *then* $\iota : (X, t) \to (X^{\wedge}, T^{\wedge})$ *is an epimorphism of the category of separated* Φ*-limit structures and for each continuous mapping* $f : (X, t) \to \mathcal{Y}$ *of* (X, t) *into a separated, compact and regular* Φ*-limit structure* $\mathcal{Y}$, *there exists one and only one continuous mapping* $f^{\wedge} : (X^{\wedge}, T^{\wedge}) \to \mathcal{Y}$ *such that* $f = f^{\wedge} \circ \iota$.

Note that the classical *Čech-Stone-compactification* leads to the result that the category of compact Hausdorff (separated) topologies is an epireflective subcategory of the category of all Hausdorff topologies. However, as shown in [3], the more general category of separated compact filter limit structures fails to be an epireflective subcategory of the category of all separated filter limit structures. The restrictive result in the proposition above reflects this fact.

4 Rough sets

Traditionally, rough sets build upon relations based on ordinary sets, i.e. relations on X as subsets of $X \times X$. Rough sets provide a good environment to deal with vagueness and uncertainty situations. We show, in this section, how monads can be used to generalize and interpret rough situations. In particular, the partially ordered ordinary power set monad turns out to contain sufficient structure in order to provide rough set operations.

However, we can go far beyond ordinary relations as we show how more general power sets, i.e. partially ordered monads built upon a wide range of set functors, can be used to provide what we call *rough monads*.

A starting point of this section is the equivalent view on relations as mappings from X to the (ordinary) power set PX.

Let R be a relation on X, i.e. $R \subseteq X \times X$. We represent the relation as a mapping $\rho_X : X \to PX$, where $\rho_X(x) = \{y \in X | xRy\}$. The corresponding inverse relation R^{-1} is then represented as $\rho_X^{-1}(x) = \{y \in X | xR^{-1}y\}$.

Based on indistinguishable relations, *rough sets* are introduced by defining the upper and lower approximation of sets. These approximations represent uncertain or imprecise knowledge. To be more formal, given a subset A of X, the lower approximation of A correspond to the objects that surely (with respect to an indistinguishable relation) are in A.

The lower and upper approximations of A are respectively obtained by

$$A^{\downarrow} = \{x \in X | \rho_X(x) \subseteq A\}$$

and

$$A^\uparrow = \{x \in X | \rho_X(x) \cap A \neq \emptyset\}.$$

The boundary of A is the (set) difference between the upper and lower approximations.

Compositional modifiers can also be found within the theory of rough sets yielding the modified sets defined by:

$$A^\Downarrow = \{x \in X | \rho_X^{-1}(x) \subseteq A\}$$

and

$$A^\Uparrow = \{x \in X | \rho_X^{-1}(x) \cap A \neq \emptyset\}.$$

In the case that the relation is reflexive, $A^\Downarrow$ and $A^\Uparrow$ are called *R-weakened* and *R-substantiated* sets of a subset A of X.

4.1   The powerset situation

In what follows we will assume that the underlying almost complete semilattice has finite infima, i.e. is a join complete lattice.

Considering P as the functor in its corresponding partially ordered monad we then immediately have

Proposition 4.1 *The upper and lower approximations of a subset A of X are respectively given by*

$$A^\uparrow = \bigvee_{\rho_X(x) \wedge A > 0} \eta_X(x) = \mu_X \circ P\rho_X^{-1}(A) \ \ and \ \ A^\downarrow = \bigvee_{\rho_X(x) \leq A} \eta_X(x)$$

For the corresponding *R*-weakened and *R*-substantiated sets of a subset A of X we have the following proposition

Proposition 4.2 *The R-weakened and R-substantiated sets of a subset A of X are given by*

$$A^\Uparrow = \mu_X \circ P\rho_X(A)$$

and

$$A^\Downarrow = \bigvee_{\rho_X^{-1}(x) \leq A} \eta_X(x),$$

respectively.

Duality conditions can be shown for the upper and lower approximations of powersets. We will denote the complement of a set A by $\neg(A)$.

Lemma 4.3 *Lower and upper approximations of powersets are dual, i.e.*
$\neg(A^\downarrow) = \neg(A)^\uparrow$ *and* $\neg(A^\uparrow) = \neg(A)^\downarrow$.

Proof.
   $x \in \neg(A^\downarrow)$ iff $x \notin A^\downarrow$ iff $\rho_X(x) \not\subseteq A$ iff $\rho_X(x) \cap \neg(A) \neq \emptyset$ iff $x \in \neg(A)^\uparrow$.

$x \in \neg(A^\uparrow)$ iff $x \notin A^\uparrow$ iff $\rho_X(x) \cap A = \emptyset$ iff $\rho_X(x) \subseteq \neg(A)$ iff $x \in \neg(A)^\downarrow$.

$\square$

Lemma 4.4 *If $A \subseteq B$ then $A^\uparrow \subseteq B^\uparrow$, $A^\downarrow \subseteq B^\downarrow$, $A^\Uparrow \subseteq B^\Uparrow$, $A^\Downarrow \subseteq B^\Downarrow$.*

Proof. The proof is straightforward as e.g. $A^\uparrow = \mu_X \circ P\rho_X^{-1}(A) \subseteq \mu_X \circ P\rho_X^{-1}(B) = B^\uparrow$ by using Proposition 4.1 and properties of the underlying lattice.

$\square$

4.2 Inverse relations

Inverse relations in the ordinary case means to mirror pairs around the diagonal. Generally, in the case of $\varphi = L_{id}$ we would accordingly define $\rho_X^{-1}(x)(x') = \rho_X(x')(x)$. The following propositions relate inverses to the multiplication of the corresponding monads. Proofs can be found in [4].

Proposition 4.5 *In the case of P,*

$$\bigvee_{\rho_X(x) \wedge A > 0} \eta_X(x) = \mu_X \circ P\rho_X^{-1}(A)$$

if and only if

$$\rho_X^{-1}(x) = \bigcup_{\eta_X(x) \le \rho_X(y)} \eta_X(y).$$

In the case of L_{id}, we obtain the following result:

Proposition 4.6

$$\mu_X \circ L\rho_X^{-1}(A)(x) = \bigvee_{x' \in X} (\rho_X(x) \wedge A)(x').$$

The generalization from the ordinary power set monad to involve a wide range of set functors and their corresponding partially ordered monads requires an appropriate management of relational inverses and complement. Obviously, for more complicated set functors, the corresponding relational views no longer rest upon 'mirroring over the diagonal'. The general representation of inverses is still an open question and for the the purpose of this paper we specify inverses *in casu*.

4.3 Similarities for rough powersets

Rough sets are, traditionally, based on indistinguishable relations in the form of equivalence relations. Some extension contemplate relations where the transitivity property is not a must. A *tolerance relation*, is a reflexive and symmetric relation. Further, some authors even consider indistinguishable relations that are far from being an equivalence relation. This, in fact can be necessary in many situations where our knowledge is categorized as vague.

In the many valued situation we often find in the literature concepts such as *fuzzy equality relation*, *fuzzy equivalence relation* and *similarity relation*. We adopt the latter terminology.

Definition 4.7 A similarity on X is a mapping $E \colon X \times X \to L$ satisfying the following axioms,

$$E(x, x) = 1 \qquad\qquad\qquad\qquad\qquad\qquad \text{(reflexivity)}$$

$$E(x, y) = E(y, x) \qquad\qquad\qquad\qquad\qquad \text{(symmetry)}$$

$$E(x, y) \wedge E(y, z) \leq E(x, z) \qquad\qquad\qquad \text{(transitivity)}$$

for all $x, y, z \in X$.

Definition 4.8 In the powerset situation, $\rho_X : X \to PX$ is *reflexive* if $\eta_X \subseteq \rho_X$, *symmetric* if $\rho = \rho^{-1}$ and *transitive* if $y \in \rho(x)$ implies $\rho(y) \subseteq \rho(x)$.

Lemma 4.9 *There is a one-to-one correspondence between* $a : X \times X \to L$ *and* $b : X \to LX$.

Proof. Let us consider $v(a) : X \to LX$ defined by $v(a)(x)(x') = a(x, x')$, and $w(b) : X \times X \to L$ defined by $w(b)(x, x') = b(x)(x')$.

We easily see that v and w are well defined:

$$v(w(b))(x)(x') = w(b)(x, x') = b(x)(x')$$

$$w(v(a))(x, x') = v(a)(x)(x') = a(x, x')$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Proposition 4.10 *Definitions 4.7 and 4.8 are equivalent.*

Proof. As shown in previous lemma, given $\rho_X : X \to LX$ and $E : X \times X \to L$. To see that both definitions are equivalent we consider $\rho_X = v(E)$ and $w(\rho_X) = E$.

For reflexivity we have:

$$E(x, x') = v(E)(x)(x') \geq \eta_X(x)(x')$$

Now for symmetry, we have that $E^{-1}(x, x') = E(x', x)$. To see that $E = E^{-1}$ iff $\rho_X = \rho_X^{-1}$ is equivalent, we have to show that $w(\rho_X)^{-1} = w(\rho_X^{-1})$.

Indeed, we can easily see:

$$
\begin{aligned}
w(\rho_X)^{-1}(x, x') &= E^{-1}(x, x')\\
&= E(x', x)\\
&= w(\rho_X)(x', x)\\
&= \rho_X(x')(x)\\
&= \rho_X^{-1}(x)(x')\\
&= w(\rho_X^{-1})(x, x')
\end{aligned}
$$

Finally, for transitivity, $y \in \rho_X(x)$ is equivalent to $v(E)(x)(y) = E(x, y) = 1$. Therefore, if $y \in \rho_X(x)$, we have that $E(x, y) \wedge E(y, z) = E(y, z)$ and the similarity

condition for transitivity is equivalent to $v(E)(y)(z) \leq v(E)(x)(z)$ for any $z \in X$ i.e. $\rho_X(y) \leq \rho_X(x)$. □

In the following we will see some characterizations of the properties of similarity relations wrt the upper and lower approximations.

Proposition 4.11 *For the powerset monad (crisp situation) we obtain the following results:*

(i) *If ρ_X is reflexive $A^\downarrow \subseteq A$ and $A \subseteq A^\uparrow$.*

(ii) *If ρ_X is symmetric $A^{\downarrow\uparrow} \subseteq A$ and $A \subseteq A^{\uparrow\downarrow}$.*

(iii) *If ρ_X is transitive $A^{\uparrow\uparrow} \subseteq A^\uparrow$ and $A^\downarrow \subseteq A^{\downarrow\downarrow}$.*

Proof. For (i) by Proposition 4.1 and reflexivity we immediately obtain:

$$A^\downarrow = \bigcup_{\rho_X(x) \subseteq A} \eta_X(x) \subseteq \bigcup_{\rho_X(x) \subseteq A} \rho_X(x) \subseteq A$$

Note that if the relation is reflexive then also the inverse relation is reflexive. Thus, applying properties of the underlying lattice and one of the monad's identities, we have:

$$A = \mu_X \circ P\eta_X(A) \subseteq \mu_X \circ P\rho_X^{-1}(A) = A^\uparrow$$

To show (ii), symmetry together with naturality of η_X and the monad's conditions yield:

$$\begin{aligned} A^{\downarrow\uparrow} &= \mu_X \circ P\rho_X^{-1}(A^\downarrow)) \\ &= \bigcup_{\rho_X(x) \subseteq A} \mu_X \circ P\rho_X^{-1}(\eta_X(x)) \\ &= \bigcup_{\rho_X(x) \subseteq A} \rho_X(x) \\ &\subseteq A. \end{aligned}$$

On the other hand, to show $A \subseteq A^{\uparrow\downarrow}$ we use Lemma 4.3

$$\neg(A^{\uparrow\downarrow}) = \neg(A^\uparrow)^\uparrow = \neg(A)^{\downarrow\uparrow}$$

Applying now the property for symmetry proved above, $\neg(A)^{\downarrow\uparrow} \subseteq \neg(A)$. What remains is to apply the complementary to the inequality obtained and we get: $A = \neg(\neg(A)) \subseteq \neg(\neg(A)^{\downarrow\uparrow}) = \neg(\neg(A^{\uparrow\downarrow})) = A^{\uparrow\downarrow}$.

Finally for (iii), we have

$$A^\uparrow = \mu_X \circ P\rho_X^{-1}(A) = \bigvee_{\rho_X(x) \wedge A > 0} \eta_X(x) = \bigcup_{\rho_X(x) \cap A \neq \emptyset} \{x\}$$

$$A^{\uparrow\uparrow} = \mu_X \circ P\rho_X^{-1}(A^\uparrow) = \bigcup_{\rho_X(x) \cap A^\uparrow \neq \emptyset} \{z\}$$

Take $t \in A^{\uparrow\uparrow}$. Therefore $\rho_X(t) \cap A^\uparrow \neq \emptyset$.

We want to show that $t \in A^\uparrow$, e.g. $\rho_X(t) \cap A \neq \emptyset$. Since $\rho_X(t) \cap A^\uparrow \neq \emptyset$ this implies there is an r such that $r \in \rho_X(t)$ and $r \in A^\uparrow$(and therefore $\rho_X(r) \cap A \neq \emptyset$).

Now by transitivity we obtain that $\rho_X(r) \subseteq \rho_X(t)$ and thus, $\rho_X(r) \cap A \subseteq \rho_X(t) \cap A$ and $\rho_X(t) \cap A \neq \emptyset$.

For the lower approximation,

$A^\downarrow = \bigcup_{\rho_X(x) \subseteq A} \{x\}$ and $A^{\downarrow\downarrow} = \bigcup_{\rho_X(z) \subseteq A^\downarrow} \{z\}$

Consider $t \in A^\downarrow$. Then $\rho_X(t) \subseteq A$. Assume that $t \notin A^{\downarrow\downarrow}$, i.e. $\rho_X(t) \subsetneqq A^\downarrow$ so there is an $r \in \rho_X(t)$ such that $r \notin A^\downarrow$.

By transitivity $\rho_X(r) \subseteq \rho_X(t)$. Therefore, $\rho_X(r) \subseteq A$ which is a contradiction to the fact that $r \notin A^\downarrow$. Therefore $t \in A^{\downarrow\downarrow}$ and $A^\downarrow \subseteq A^{\downarrow\downarrow}$.   □

Corollary 4.12 *If ρ_X is an equivalence relation, $A^{\downarrow\uparrow} = A^\downarrow$ and $A^{\uparrow\downarrow} = A^\uparrow$.*

Proof. We use here Lemma 4.4 and Proposition 4.11.

Applying reflexivity we have $A^\uparrow \subseteq A^{\uparrow\uparrow}$. By transitivity we immediately obtain $A^{\uparrow\uparrow} = A^\uparrow$ and thus, $A^{\uparrow\uparrow\downarrow} = A^{\uparrow\downarrow}$. Now we can apply symmetry to the set $A^\uparrow$ obtaining that $A^\uparrow \subseteq A^{\uparrow\downarrow}$.

On the other hand, reflexivity applied to the set $A^\uparrow$ yields the other inequality, $A^{\uparrow\downarrow} \subseteq A^\uparrow$.

The other equality, $A^{\uparrow\downarrow} = A^\uparrow$ can be proved similarly.

                                                                                     □

4.4   Generalizations

Rough sets can now be generalized from a set-theoretic framework. This generalization does not, necessarily, make use of an equivalence relation. In fact, theoretical and practical needs might require generalizations of the approximation operators using non-equivalence relations.

Let $\Phi = (\varphi, \leq, \eta, \mu)$ be a partially ordered monad. We say that $\rho_X : X \to \varphi X$ is a Φ-*relation* on X, and by $\rho_X^{-1} : X \to \varphi X$ we denote its *inverse*. The inverse must be specified for the given set functor φ.

For any $f \colon X \to \varphi X$, the following condition is required:

$$\varphi f(\bigvee_i a_i) = \bigvee_i \varphi f(a_i)$$

This condition is valid both for P as well as for L_{id}.

Remark 4.13 Let ρ_X and ρ_Y be relations on X and Y, respectively. Then the mapping $f : X \to Y$ is a congruence, i.e. $x' \in \rho_X(x)$ implies $f(x') \in \rho_Y(f(x))$, if and only if $Pf \circ \rho_X \leq \rho_Y \circ f$. Thus, congruence is related to a kind of weak naturality.

Let $\rho_X : X \to \varphi X$ be a Φ-relation and let $a \in \varphi X$. The Φ-ρ-upper and Φ-ρ-lower approximations, and further the Φ-ρ-weakened and Φ-ρ-substantiated sets, now define rough monads using the following monadic instrumentation:

$\Uparrow_X (a) = \mu_X \circ \varphi \rho_X(a)$

$\downarrow_X (a) = \bigvee_{\rho_X(x) \leq a} \eta_X(x)$

$$\uparrow_X (a) = \mu_X \circ \varphi \rho_X^{-1}(a)$$
$$\Downarrow_X (a) = \bigvee_{\rho_X^{-1}(x) \leq a} \eta_X(x)$$

Proposition 4.14 *[4] If $a \leq b$, then $\Uparrow_X a \leq \Uparrow_X b$, $\downarrow_X a \leq \downarrow_X b$, $\uparrow_X a \leq \uparrow_X b$, $\Downarrow_X a \leq \Downarrow_X b$.*

Definition 4.15 $\rho_X : X \rightarrow \varphi X$ *is* reflexive *if $\eta_X \leq \rho_X$, and* symmetric *if $\rho = \rho^{-1}$.*

Proposition 4.16 *[4] The following properties hold:*

(i) *If ρ is reflexive, then $a \leq \Uparrow_X (a)$.*

(ii) *ρ is reflexive iff $\downarrow_X (a) \leq a$.*

(iii) *ρ_X^{-1} is reflexive iff $a \leq \uparrow_X (a)$.*

(iv) *If ρ is symmetric, then $\uparrow_X (\downarrow_X (a)) \leq a$.*

Traditionally for the study of rough approximations, the use of equivalence relation was commonly assumed. Nevertheless, one can find situations where tolerance relations are more suitable for the particular study being considered.

When going to the many-valued situation, tolerance relation can, in fact, be very convenient, leaving the transitivity property as an "in-casu" situation due to its strong requirements. For instance, in order to place a student from another school in the right level of knowledge we might need to define a relation that will be able to compare different courses. We could accept that a student from another school has achieved the requirement of a given course if (s)he has passed another course whose contents are at least of an 80% of the one considered, e.g. two courses will be in the same class if they have at least 80% of their contents in common. In this situation we could easily find a situation where two courses are in the same class, i.e. course A and B have an 80% of common contents, course C has also an 80% of common content with course B but less than 80% of the contents in common with course A. Hence we do not have transitivity.

5 Kleene algebras

Let $\Phi = (\varphi, \leq, \eta, \mu)$ be a partially ordered monad where $\emptyset \in \varphi X$. Denote by 0_X, or 0 for short, the morphism $0 : X \rightarrow \varphi X$ satisfying $0(x) = \emptyset$ for all $x \in X$, and let $1 = \eta_X$. Further, for $f_1, f_2 \in Hom(X, \varphi X)$, define

$$f_1 + f_2 = f_1 \vee f_2,$$

and

$$f_1 \cdot f_2 = f_1 \diamond f_2$$

where $f_1 \diamond f_2 = \mu_X \circ \varphi f_2 \circ f_1$ is the composition of morphisms in the corresponding Kleisli category of Φ.

A partial order $\leq$ on $Hom(X, \varphi X)$ is, for $f_1, f_2 \in Hom(X, \varphi X)$, defined by $f_1 \leq f_2$ whenever $f_1(x) \leq f_2(x)$ for all $x \in X$. Obviously, $f_1 \leq f_2$ if and only if $f_1 + f_2 = f_2$.

The partially ordered monad $\Phi = (\varphi, \leq, \eta, \mu)$ is said to be a *Kleene monad*, if the following conditions are fulfilled:

$$\varphi \mathbf{0}_X = \mathbf{0}_{\varphi X}$$
$$\varphi f(\emptyset) = \emptyset$$
$$\mu_X(\emptyset) = \emptyset$$
$$\varphi(\vee_i f_i) = \vee_i \varphi f_i$$
$$\varphi f \circ (\vee_i g_i) = \vee_i(\varphi f \circ g_i)$$
$$\mu_X \circ (\vee_i g_i) = \vee_i(\mu_X \circ g_i)$$

Proposition 5.1 *Let* $\Phi = (\varphi, \leq, \eta, \mu)$ *be a Kleene monad.*
Then $(Hom(X, \varphi X), +, \cdot, \mathbf{0}, \mathbf{1})$ *is an idempotent semiring.*

The Kleene asterate, for mappings $f : X \to \varphi X$, is defined by

$$f^* = \bigvee_{k=0}^{\infty} f^k$$

where $f^0 = 1$ and $f^{k+1} = \mu_X \circ \varphi f^k \circ f$. Suprema of mappings is defined pointwise. We then have

Proposition 5.2 *[9] Let* $\Phi = (\varphi, \leq, \eta, \mu)$ *be a Kleene monad.*
Then $(Hom(X, \varphi X), +, \cdot, *, \mathbf{0}, \mathbf{1})$ *is a Kleene algebra.*

6 Conclusions

Partially ordered monads enrich monads with an order structure powerful enough to generalize convergence, rough sets and Kleene algebras. Monadic topologies are the result of decades of investigations on functor and monad based convergence. On the other hand, rough and Kleene monads are novelties where their acceptance and potential for applications is yet to be seen. Clearly, both in the case of rough sets as well as for Kleene algebras, when we generalize from ordinary relations and strings to enable more elaborate set functors extendable to partially ordered monads (and there are quite a few of them!), we open up entirely new possibilities for semantic considerations of programming languages and decision support models involving formal logic and uncertainties. Composing partially ordered monads will be as important as composing monads. The term monad, and its generalizations through compositions, will continue to play an important role.

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