Stage-Structured Healthy Herds: 2 Analytical Solution Problems

robert.richards

April 2023

1 Equations

$$\frac{dR}{dt} = rR(1 - \frac{R}{K}) - f_j R(S_j + I_j) - f_a R(S_a + I_a) \tag{1}$$

$$\frac{dS_j}{dt} = ef_a R(S_a + I_a) - d_j * S_j - u_j \sigma_j I_j S_j - u_j \sigma_a I_a S_j - p_j S_j - g_s f_j R S_j \tag{2}$$

$$\frac{dS_a}{dt} = g_i f_j R I_j + g_s f_j R S_j - d_a * S_a - u_a \sigma_j I_j S_a - u_a \sigma_a I_a S_a - p_a S_a \tag{3}$$

$$\frac{dI_j}{dt} = u_j \sigma_j I_j S_j + u_j \sigma_a I_a S_j - (d_j + v_j) * I_j - p_j I_j - g_i f_j R I_j \tag{4}$$

$$\frac{dI_a}{dt} = u_a \sigma_a I_j S_a + u_a \sigma_a I_a S_a - (d_a + v_a) * I_j - p_a I_a \tag{5}$$

2 Simplified Equations

If we assume that juveniles and adults are similar in terms of mortality and spore yield and that infection doesn't affect maturation we get:

$$\frac{dR}{dt} = rR(1 - \frac{R}{K}) - f_j R(S_j + I_j) - f_a R(S_a + I_a) \tag{6}$$

$$\frac{dS_j}{dt} = ef_a R(S_a + I_a) - d * S_j - u_j \sigma(I_j S_j + I_a S_j) - p_j S_j - g f_j R S_j \tag{7}$$

$$\frac{dS_a}{dt} = gf_jR(I_j + S_j) - d * S_a - u_a\sigma(I_jS_a + I_aS_a) - p_aS_a \quad (8)$$

$$\frac{dI_j}{dt} = u_j\sigma(I_jS_j + I_aS_j) - (d + v) * I_j - p_jI_j - gf_jRI_j \quad (9)$$

$$\frac{dI_a}{dt} = u_a\sigma(I_jS_a + I_aS_a) - (d + v) * I_j - p_aI_a \quad (10)$$

3 Parameters

$f_{j,a}$	Feeding rate (juvenile and adult)
$u_{j,a}$	Per spore infection rate (juvenile and adult)
$d_{j,a}$	Mortality rate (juvenile and adult)
$v_{j,a}$	Disease induced additional mortality rate (juvenile and adult)
e	Conversion efficiency of resources to offspring
$g_{s,i}$	Maturation rate per resource consumed (susceptible and infected)
$\sigma_{j,a}$	spore yield at death (juvenile and adult)
m	Resource background mortality rate
r	Resource intrinsic growth rate
K	Resource carrying capacity
$p_{j,a}$	Predation rate (on juveniles/adults)

4 Problems

4.1 Analytical Solution for Age Skew

We are interested in generating populations that skew from highly adult dominated to highly juvenile dominated, but with the same overall population size. We can do this computationally by varying maturation rate (g) and fecundity (e) with larger values of maturation rate and lower levels of fecundity producing adult dominated populations and vice-versa. We assume there should be an analytical solution for the curve in maturation-fecundity space that varies age skew but keeps population size constant.

4.2 Analytical Solution for R_0 using Next Generation Matrix Approach

We are interested in producing an analytical solution for R_0 in our system of equations. This ought to be straightforward with a next generation matrix approach. But my attempts (both by hand and in Mathematica) have been hampered by a number of the transitions in this matrix due to the combination of infection and age structure. I suspect there is some simplification trick but I have not been able to figure it out. I will attach my best Mathematica attempt.