Symbolic Logic: Theory and Applications

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Chapter 1

Basic Concepts

1.1 What is Logic?

I don't know that there is any accepted definition of logic. Here are four that I've heard:

1. The study of reasoning
2. The analysis and evaluation of arguments
3. The study of formal languages
4. The study of patterns of truth

Although I don't think that any of the four are successful definitions, it is certainly the case that they each represent some important part of the study of logic. We will see all four of these topics, in various forms, in later chapters. This chapter introduces some important concepts that are fundamental to the study of logic. The first is truth preservation.

The study of logic provides a set of tools for reasoning. As you continue in the study of logic, you will find that there are different systems of logic, each system being a different tool that can be used. There are two obvious, but important, things to note — first, the particular tool that should be used is relative to the context or task at hand. Hammers and saws are both tools used in carpentry. Which one should you use? Is your goal to drive a nail or cut a board? We'll see that the system of logic that should be used depends on the context in which one is reasoning. A good rule of thumb is that one should use the simplest tool that is adequate for the task. For example, a doctor shouldn't use an electrocardiogram or MRI when a stethoscope will do. Systems of logic also differ in degrees of complexity, and we should use the simplest system that will be effective in the circumstances.

Second, a tool that leaves you in a worse situation than you were in before you used it is certainly *not* good. The purpose of a toaster is to make a nicely toasted piece of bread. A toaster that produces a smoldering, inedible, piece of charcoal is not a good toaster. You were better off just eating the non-toasted bread. A central purpose of a logic system is to preserve truth. When the tools of logic are applied to truths, the results should still be true. A system of logic that turns truth into falsehoods is a failure as a tool of reasoning. Another way to state this is that the rules of our logic systems should be truth-preserving.

1.2 Arguments

One fundamental concept in logic is that of the argument. For a good example of what we are not talking about, consider a bit from a famous sketch by *Monty Python's Flying Circus* (Cleese and Chapman 1980):

> Man: (Knock)
> Mr. Vibrating: Come in.
> Man: Ah, Is this the right room for an argument?
> Mr. Vibrating: I told you once.
> Man: No you haven't.
> Mr. Vibrating: Yes I have.
> Man: When?
> Mr. Vibrating: Just now.
> Man: No you didn't.
> Mr. Vibrating: Yes I did.
> Man: You didn't!
> Mr. Vibrating: I did!
> Man: You didn't!
> Mr. Vibrating: I'm telling you I did!
> Man: You did not!!
> Mr. Vibrating: Oh, I'm sorry, just one moment. Is this a five minute argument or the full half hour?

People often use "argument" to refer to a dispute or quarrel between people. For our purposes, an argument is defined as

Argument A set of statements, one of which is taken to be supported by the remaining sentences.

Conclusion The statement in the argument that is being supported.

Premises The statements in the argument that provide the support for the conclusion.

There are three important things to remember here:

1. Arguments contain statements.
2. They have a conclusion.
3. They have at least one premise

This might get more complicated later, but for now, think of a statement as a sentence that has a truth-value. Also, for now, we will assume that there are only two truth values: true and false, which we will abbreviate with **T** and **F**. Think of a statement as an attempt to describe the world; when it gets it right, the statement is true, otherwise, it is false. Since George Washington was in fact the first President of the United States, the statement 'George Washington was the first President of the United States' has the truth value of **T**. Likewise, the statement 'John Adams was the first President of the United States' has the truth value of **F**.

Some sentences don't have truth values, that is to say that they are neither true or false. Examples are questions ('What did you do last summer?') and commands ('Please take a seat.'). Such sentences can be neither premises nor conclusions of arguments. It is important to note that, even though a statement has a truth value, we may not necessarily know what that truth value is. The rules of logic will still apply, though.

An inference is the process of reasoning from the premises to the conclusion. That is, inference is the psychological process by which one draws a conclusion from some given premises. The argument just is the premises and conclusion.

A distinction needs to be made between deductive arguments and inductive arguments. It's very hard to make this distinction precise, but it's not hard to have an informal understanding of the difference. Think of the difference in terms of what the argument is intended to establish. If the argument is only intended to establish the probable truth of the conclusion, then it is inductive. If the argument is intended to guarantee the truth of the conclusion, then it is deductive.

Every argument has exactly one conclusion. Very complex arguments may have sub-conclusions, which are themselves inferred from premises. These sub-conclusions then serve as premises for the main conclusion of the argument. Let's keep things simple for now. Consider this argument:

> Calculus II will be no harder than Calculus I. Susan did well in Calculus I. So, Susan should do well in Calculus II.

Conclusion	Premise
Therefore	Since
So	Because
Thus	For
Hence	Is implied by
Consequently	For the reason that
Implies that	
It follows that	

Here the conclusion is that Susan should do well in Calculus II. The other two sentences are premises. These premises are the reasons offered for believing that Susan should do well in the course..

Now, to make the argument easier to evaluate, we will put it into what is called "standard form." To put an argument in standard form, write each premise on a separate, numbered line. Draw a line underneath the last premise, the write the conclusion underneath the line.

1. Calculus II will be no harder than Calculus I.
2. Susan did well in Calculus I.

∴ Susan will do well in Calculus II.

Now that we have the argument in standard form, we can talk about premise 1, premise 2, and all clearly be referring to the same statement.

Unfortunately, when people present arguments, they rarely put them in standard form. So, we have to decide which statement is intended to be the conclusion, and which are the premises. Don't make the mistake of assuming that the conclusion comes at the end. The conclusion is often at the beginning of the passage, but could even be in the middle. A better way to identify premises and conclusions is to look for indicator words. Indicator words are words that signal that statement following the indicator is a premise or conclusion. The example above used a common indicator word for a conclusion, 'so.' The other common conclusion indicator, as you can probably guess, is 'therefore.' This table lists the indicator words you might encounter.

Each argument will likely use only one indicator word or phrase. When the conlusion is at the end, it will generally be preceded by a conclusion indicator. Everything else, then, is a premise. When the conclusion comes at the beginning, the next sentence will usually be introduced by a premise indicator. All of the following sentences will also be premises.

For example, here's our previous argument rewritten to use a premise indicator:

> Susan should do well in Calculus II, because Calculus II will be no harder than Calculus I, and Susan did well in Calculus I.

Sometimes, an argument will contain no indicator words at all. In that case, the best thing to do is to determine which of the premises would logically follow from the others. If there is one, then it is the conclusion. Here is an example:

> Spot is a mammal. All dogs are mammals, and Spot is a dog.

The first sentence logically follows from the others, so it is the conclusion. When using this method, we are forced to assume that the person giving the argument is rational and logical, which might not be true.

One thing that complicates our task of identifying arguments is that there are many passages that, although they look like arguments, are not arguments. The most common types are:

1. Explanations
2. Mere asssertions
3. Conditional statements
4. Loosely connected statements

Explanations can be tricky, because they often use one of our indicator words. Consider this passage:

> Abraham Lincoln died because he was shot.

If this were an argument, then the conclusion would be that Abraham Lincoln died, since the other statement is introduced by a premise indicator. If this is an argument, though, it's a strange one. Do you really think that someone would be trying to prove that Abraham Lincoln died? Surely everyone knows that he is dead. On the other hand, there might be people who don't know how he died. This passage does not attempt to prove that something is true, but instead attempts to explain why it is true. To determine if a passage is an explanation or an argument, first find the statement that looks like the conclusion. Next, ask yourself if everyone likely already believes that statement to be true. If the answer to that question is yes, then the passage is an explanation.

Mere assertions are obviously not arguments. If a professor tells you simply that you will not get an A in her course this semester, she has not given you an argument. This is because she hasn't given you any reasons to believe that the statement is true. If there are no premises, then there is no argument.

Conditional statements are sentences that have the form "If…, then…" A conditional statement asserts that *if* something is true, then something else would be true also. For example, imagine you are told, "If you have the winning lottery ticket, then you will win ten million dollars." What is being claimed to be true, that you have the winning lottery ticket, or that you will win ten million dollars? Neither. The only thing claimed is the entire conditional. Conditionals can be premises, and they can be conclusions. They can be parts of arguments, but that cannot, on their own, be arguments themselves.

Finally, consider this passage:

> I woke up this morning, then took a shower and got dressed. After breakfast, I worked on chapter 1 of the logic text. I then took a break and drank some more coffee….

This might be a good description of my day, but it's not an argument. There's nothing in the passage that plays the role of a premise or a conclusion. The passage doesn't attempt to prove anything. Remember that arguments need a conclusion, there must be something that is the statement to be proved. Lacking that, it simply isn't an argument, no matter how much it looks like one.

1.3 Deductive Validity

Deductive arguments are intended to be fully truth-preserving. A deductively valid argument is one that is in fact completely truth-preserving. That is, a deductively valid argument will never have all true premises and a false conclusion. It is important to understand how strong this claim is. It is not merely the case a valid argument happens to have true premises and a true conclusion. The relationship between the premises and the conclusion is so strong that it is not possible for the premises to be true and the conclusion false.

Deductive validity An argument is deductively valid if and only if it is not possible for the premises to be true and the conclusion to be false.

Deductive invalidity An argument is deductively invalid if and only if it is not deductively valid.

Here is an example of a deductively valid argument:

1. All dogs are mammals.
2. Lucy is a dog.

∴ Lucy is a mammal.

Since the first premise is true, being a dog is enough to guarantee that the animal is a mammal. So, the only way that the conclusion could be false is if Lucy is not a dog. It is impossible that the premises be true and the conclusion false.

Compare the previous argument to this one:

1. All dogs are mammals.
2. Lucy is a mammal.

∴ Lucy is a dog.

Since Lucy is the name of my dog, both the premises and the conclusion are in fact true. Note, though, that the premises are not enough to guarantee the truth of the conclusion. Lucy could have been the name of a cat. If so, the premises would have been true and the conclusion false. This argument is therefore invalid.

If we know that an argument is invalid, then we know that there is a special logical relationship between the premises and the conclusion such that, if the premises are true, then the conclusion must also be true. It is important to understand that validity alone does not mean that the premises and conclusion are true. If I know only that an argument is valid, I know that the *if* the premises are true, *then* the conclusion must also be true. Valid arguments cannot have a combination of true premises and false conclusion, but they can have any other combination. It can be reasonable to doubt that a conclusion is true, even if the argument is valid. What is not reasonable is to grant the argument is valid and has true premises and still doubt that the conclusion is true. This means that validity alone is not enough to guarantee that a conclusion is true. What guarantees that a conclusion is true is deductive validity along with true premises. This is called deductive soundness.

Deductive soundness An argument is deductively sound if and only if it is deductively valid and has all true premises.

1.4 Inductive Arguments

Later chapters will cover inductive arguments. For now it is enough to say that inductive arguments are not intended to be deductive valid. That is, inductive arguments are those whose premises do not guarantee the truth of the conclusion. For inductive arguments, it is always possible that the premises be true and the conclusion false. Here is an example of an inductive argument:

1. A random sample of 100 students at the university unanimously reported preferring traditional classes to online instruction.

∴ The majority of all students at the university prefer traditional classes to online instruction.

The truth of the premise does not guarantee the truth of the conclusion. It is certainly possible that the sample managed to include the only students that don't prefer online courses. Still, though, it seems that, if the premise is in fact true, then the conclusion should be highly likely to be true. So, this is a good inductive argument, one that we call inductively strong.

Inductive strength An argument is inductively strong to the extent that the conclusion is probably true given the truth of the premises.

Another difference between inductive and deductive arguments is that inductive strength is a matter of degree. The argument above is inductively strong, but doubling the sample size would make it even stronger.

1.5 Logical Consistency and Logical Truth

Consistency is a property of sets of statements:

Logical consistency A set is logically consistent if and only if it is possible for all of the members of the set to be true at the same time.

Logical inconsistency A set is logically inconsistent if and only if it is not logically consistent.

It is not necessary for all of the statements to be true in order for the set to be consistent. Here is an example:

Oklahoma is south of Texas. There are 125 members of the U.S. Senate.

Neither of the statements in this set are true. The set is consistent, though, because there is nothing about either sentence that prevents the other from possibly being true. Here is an example of an inconsistent set:

No student will make an A in Logic this semester. At least one student will make an A in Logic this semester.

The truth of one of those statements is incompatible with the truth of the other. So, the set containing both is inconsistent. Logical consistency is a very important concept, because, once defined, other logical concepts can be defined in terms of consistency. We'll do this in a later chapter.

For the most part, logic alone is not enough to determine if a statement is true. It is not logic that makes it true that Topeka is the capital of Kansas. What makes that true is something about the political structure and history of Kansas. There are two important exceptions to this, though. There are some statements that are true simply because of their logical structure. An important example is something like this: Either Susan will pass logic this semester or Susan will not pass logic this semester. No matter how well Susan does in the class, it must be true that she either passes or not. Sentences like these are called logical truths, or tautologies. Logical truths *must* be true. On the other hand, there are sentences that *cannot* be true. For example, Susan will both pass logic this semester and not pass logic this semester. Sentences like these are called logical falsehoods, or contradictions. Most statements are neither logical truths nor logical falsehoods. Such statements are logically indeterminate.

Logical truth A statement is logically true if and only if it is not possible for the statement to be false.

Logical falsity A statement is logically false if and only if it is not possible for the statement to be true.

Logical indeterminacy A statement is logically indeterminate if and only if it is neither logicall true nor logically false.

Finally, there are sentences that are related in such a way so that, if one is true, the second must also be true, and vice versa. These sentences are called logically equivalent. Here is a simple example:

Susan will pass.
Susan will not fail.

Since failing is just not passing, then any situation in which Susan passes is a situation in which she does not fail. So, the two statements are true in exactly the same situations, and false in exactly the same situations. They always have the same truth values.

Logical equivalence Two sentences are logically equivalent if and only if it is impossible for one to be true and the other to be false.

Chapter 2

Categorical Logic

Now we turn to some structured logic systems. The first, categorical logic, is one of the oldest. It dates back at least to Aristotle (384–322 BCE). Categorical logic is a fairly simple logic of categories or classes. A class is a group of things that we designate with a common noun: students, teachers, dogs, politicians, etc. Each sentence will use two different classes. One is the subject class, and the other is the predicate class. In this logic, we can say something about all members of a class, called a universal sentence, or we can say something about some members of a class, called a particular sentence. We can also make a positive claim, called an affirmation, or we can make a negative claim, called a negation.

With these two distinctions, universal/particular and affirnation/negation, we can make four kinds of sentences. S and P stand for the subject class and the predicate class, respectively.

A : All S are P (universal affirmation)

E : No S are P (universal negation)

I : Some S are P (particular affirmation)

O : Some S are not P (particular negation)[1]

Here are some examples of categorical statements, some true and some false.

1. All dogs are mammals.
2. All mammals are dogs.

[1]The letters A, E, I, and O, are thought to come from the first two vowels of the Latin words *affirmo* and *nego*, meaning "I affirm" and "I deny."

3. No reptiles are dogs.
4. No politicians are honest people.
5. Some politicians are honest people.
6. Some cats are amphibians.
7. Some dogs are not beagles.
8. Some beagles are not dogs.

Look at the sentences carefully. You should be able to tell that the odd-numbered ones are true and the even-numbered ones are false.

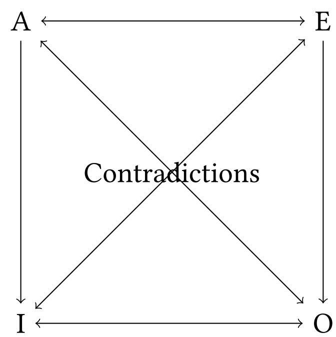
2.1 The Square of Opposition

We can visualize interesting logical relationships between these four types of sentences with something called "The Square of Opposition."

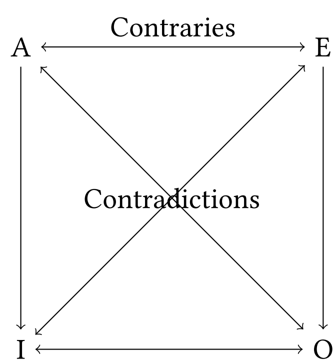
The first step is to place the sentence types in the corners of an imaginary square. A is at the upper left; E, the upper right; I, the lower left, and O, the lower right. Next, draw arrows on the diagonals, pointing to the sentences in the corners. Then, draw an arrow between the two at the top, and another one between the two at the bottom. Finally, draw an arrow on each side, going from top to bottom. When finished, you should have something like this:

$$\begin{tikzcd} \mathrm{A} \arrow[rr, leftrightarrow] \arrow[dd] \arrow[ddrr, leftrightarrow] & & \mathrm{E} \arrow[dd] \arrow[ddll, leftrightarrow] \\ & & \\ \mathrm{I} \arrow[rr, leftrightarrow] & & \mathrm{O} \end{tikzcd}$$

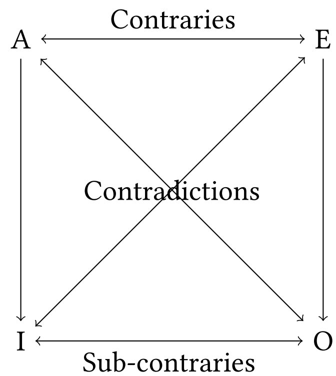
The next step is to note the relationship between the diagonals. The diagonals are contradictories, meaning they always have opposite truth values. They can't both be true, and they also can't both be false. If the A sentence is true, the O sentence must be false—if it is true that all dogs are mammals, it cannot be true that some dogs are not mammals. If the O sentence is true, then the A sentence must be false. It is the same for the E and the I.



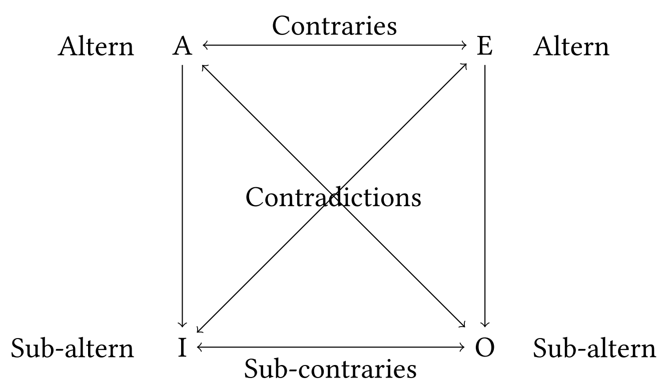
Next, note the relationship between the A sentences and the E sentences, called contraries. Like the contradictories, they cannot both be true. Unlike the contradictories, they can both be false. If it's true that all critical thinking students are good students, then it must be false that no critical thinking students are good students. If it's false that all critical thinking students are good students, then it can be false that critical thinking students are good students. In fact, they are both false, because some critical thinking students are good and others are not.



At the bottom, we have sub-contraries. They can both be false, but cannot both be true.



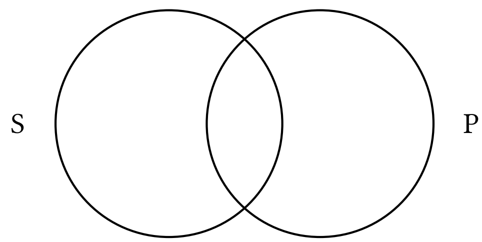
Finally, we have the relationship between the top level sentences and the bottom level sentences on the same side. This is called alternation. The universal is called the superaltern and the particular is called the subaltern. If the superaltern is true, then the subaltern must also be true. If the superaltern is false, then the subaltern can be either true or false. If the subaltern is false, then the superaltern must be false. If the subaltern is true, then the superaltern can be either true or false. It is easy to remember this way: truth goes down, falsity goes up.



2.2 Diagramming Sentences

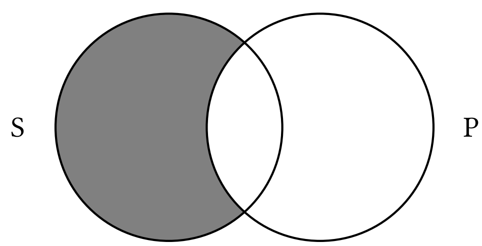
We diagram sentences and arguments in categorical logic using Venn diagrams. You've probably used these in a math class at some time. Before we can use these to evaluate arguments in categorical logic, we first have to learn how to diagram individual sentences.

The first step is to draw two interlocking circles. Label the left circle with an "S" and the right circle with "P"—standing for the subject term and predicate term, respectively.



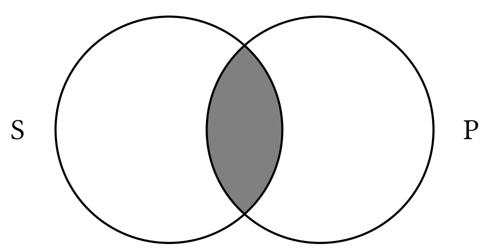
A-Sentences

Remember that the A-sentence has the form All S are P. That means that everything that is in the S circle must also be in the P circle. To diagram this, we shade the region of the S circle that is not contained in the P circle. If a region is shaded, that means that nothing is in that region.



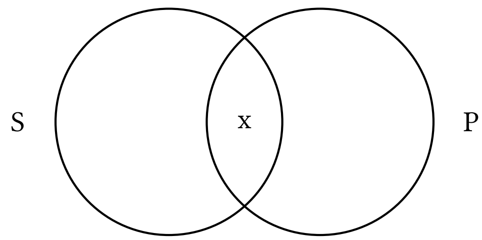
E-Sentences

To shade the universal negation, we shade the region that is shared by both S and P:



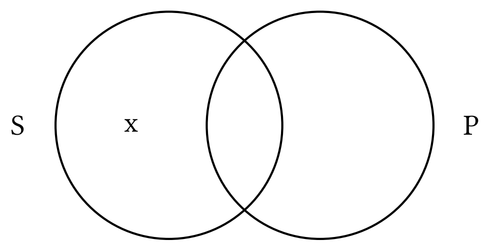
I-Sentences

To diagram a particular affirmation, we place an x in the region shared by S and P:



O-Sentences

Finally, to diagram an O-sentence, we place an x in S, but not in P:



Evaluating Categorical Syllogisms

A syllogism is an argument that has two premises and a conclusion. A categorical syllogism is a syllogism that contains only categorical sentences. Here is an example:

1. All Dogs are mammals.
2. <u>All mammals are animals.</u>

∴ All dogs are animals

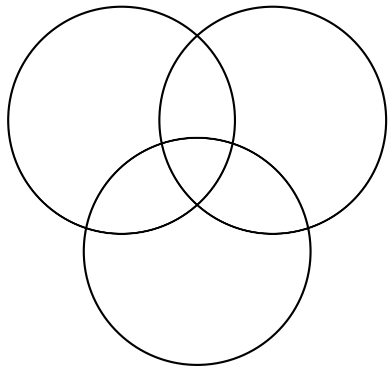
Both premises and the conclusion are A-sentences. Notice that we have three terms in the argument: dogs, mammals, and animals. Every categorical syllogism, in proper form, has three terms. Each term occurs in two sentences. Two of those terms will be found in the conclusion, and one term is only in the premises. The predicate term of the conclusion is called the major term. The subject of the conclusion is called the minor term. The term that is not in the conclusion is called the middle term.

There are two ways to determine if a categorical syllogism is valid. One way uses Venn diagrams, and the other involves applying some simple rules.

Diagram Method

Since we have three terms in the argument, we'll need three intersecting circles. We'll start by drawing two circles for the conclusion, just as we did before. Then, in the middle and below, we'll draw another circle for the middle term. For labels, use letters that correspond to the classes in the argument. Here, we'll use D for dogs, M for mammals, and A for animals.

D

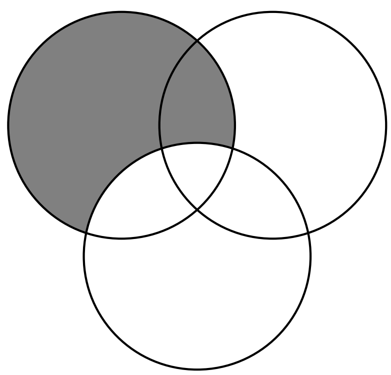


A

M

Next, we finish diagramming the premises by shading or placing an x. Since our first premise is "All dogs are mammals", we need to shade everything in the D circle that is not in the M circle.

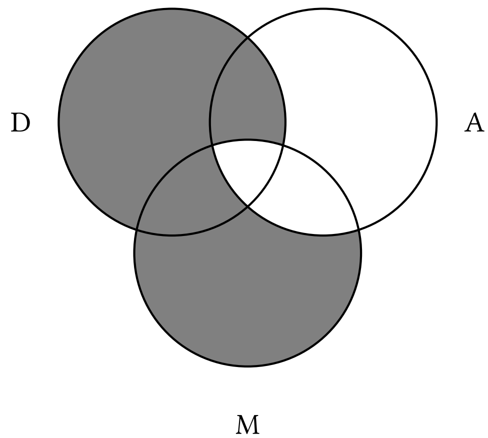
D



A

M

Next, we diagram the second premise by shading everything that is in the M circle but not in the A circle.

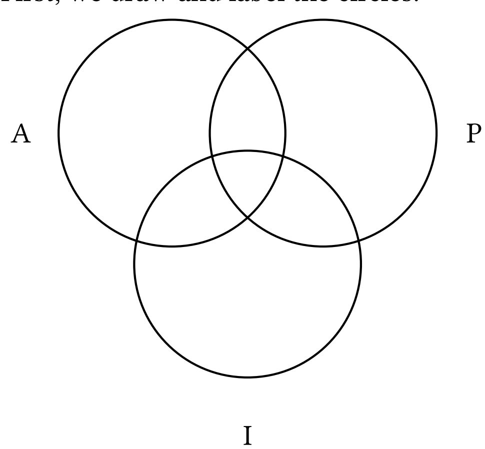


If there is any circle that has only one region left unshaded, you can place an 'X' in that region. This is because categorical logic assumes that there are no empty categories, meaning that every category has at least one thing in it. This is really only important for arguments that have an I or an O-sentence for a conclusion. In this case, we won't worry about it. Now that the premises are diagrammed, check to see if the conclusion has also been diagrammed, which in this case means that everything in the D circle that is not also in the A circle is shaded out. If so, then the argument is valid. This shows that making the premises true was enough to make the conclusion true also.

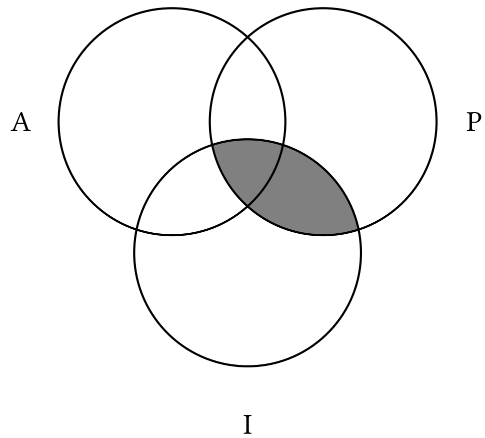
Let's try to diagram this argument:

1. No introverts are politicians
2. <u>All artists are introverts</u>
3. No artists are politicians

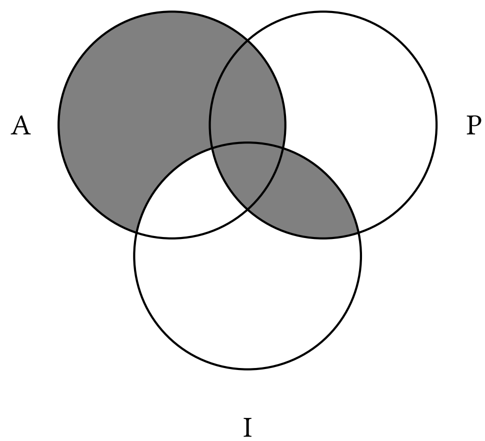
First, we draw and label the circles:



Then we diagram the premises, always doing the universals before any particulars. In this case, we have two universal premises, so we will just begin with the first premise:



Now, we'll diagram the second premise:

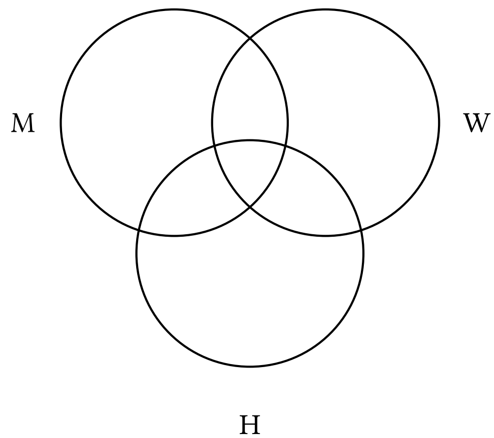


Diagramming the conclusion would require the intersection of A and P to be shaded. Notice, though, that the region between A and P has already been shaded by just diagramming the premises. That means that making the premises true was enough to guarantee that the conclusion would also be true, and the argument is valid.

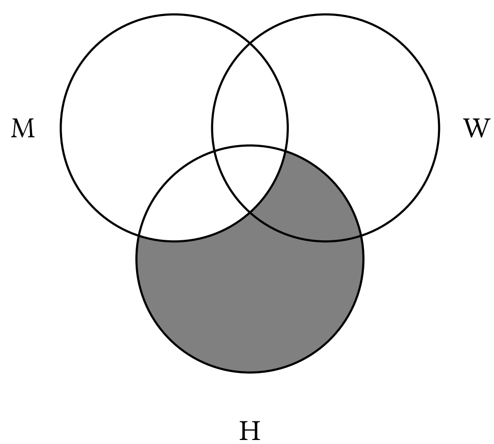
Let's try one more argument.

1. Some horses are things that weigh over 2,000 pounds.
2. All horses are mammals.
3. Some mammals are things that weigh over 2,000 pounds.

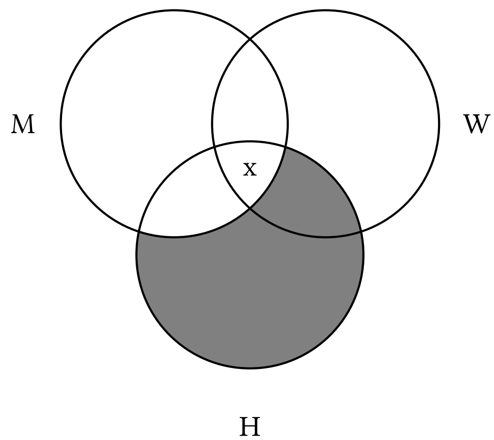
Again, we begin by drawing and labeling the circles.



The we diagram any universal premises, which, in this case, is the second premise.



Then, we diagram any particular premises.



Finally, we check to see if diagramming the premises was enough to make the conclusion also diagrammed. In this case, it was, so the argument is valid.

Hints for Diagramming Categorical Syllogisms

1. Diagram universals before particulars (shade before making an x.)
2. If it is not clear where the x goes, then put it on the line.

2.3 Rules for Categorical Syllogisms

There is another way to determine validity for categorical syllogisms. Every valid syllogism must meet three conditions:

1. There must be the same number of negations in the conclusion as in the premises.
2. The middle term must be distributed at least once.
3. Any term distributed in the conclusion must be distributed in the premises.

Before these rules can be applied, we'll have to explain what distribution is. Every categorical statement says something about a category or class. A statement distributes a term just in case what it says about that class is true of every subset of the class. For example, it is true that all dogs are mammals. It's also true that all members of any subset of the set of dogs are mammals—all dogs in Oklahoma are mammals, and all dogs in Greece are mammals, and so on. All dogs are not necessarily members of every subset of the class of mammals, however. The class of cats is a subset of the class of mammals, and no dog is a cat. So, the subject of an A-sentence is distributed, but the predicate is not. To remember when something is distributed, keep this in mind:

1. Universals distribute subjects, and
2. Negations distribute predicates.

So, A-sentences distribute the subject, E-sentences distribute both terms, I-sentences don't distribute anything, and O sentences distribute the predicate.

The rules are easy to apply. First, put the argument in standard form:

1. All A are B.
2. All B are C.

∴ All A are C.

Then, circle all of the distributed terms.

1. All A are B.
2. All B are C.

∴ All A are C.

Now, just check to see if there are any violations of the rules:

1. Are there the same number of negations in the conclusion as in the premises? Yes, since there are no negations at all.
2. Is the middle term distributed at least once? Yes, the middle term is B and it is distributed in the second premise.
3. Is any term that distributed in the conclusion also distributed in the premises? Yes, A is distributed in the conclusion, but it is also distributed in the first premise.

So, since the argument breaks none of the rules, it is valid.

2.4 Relations of Equivalence

Properly formed categorical syllogisms have only three terms. Unfortunately, some arguments that you will encounter won't always be in proper form. One common way this happens is for a person to use a term like "Americans" in one premise, but use "non-Americans" in another. This can result in a syllogism with four or more terms, making it impossible to evaluate using either of our two methods. What we then need to do is to convert the sentence using one of the terms into a logically equivalent sentence that uses the other term.

There are three operations that can be applied to categorical sentences: conversion, obversion, and contraposition. It is important to know both how to apply them and in what cases does an operation result in an equivalent sentence. We're particularly interested in the conditions that those different operations are *truth-preserving*. An operation is truth preserving when, applied to a true sentence, it always results in a true sentence.

Conversion

Conversion is the simplest of the three. The converse of a sentence simply exchanges the subject and predicate terms of the original sentence. Conversion applied to A-sentences is *not* truth-preserving. "All dogs are mammals" is true, but "All mammals are dogs" is not. Conversion is truth-preserving for E-sentences and I-sentences. If it is true that no dogs are reptiles, it must be true that no reptiles are dots. Likewise, if it is true that some dogs are brown things, it must be true that some brown things are dogs.

Another way to think about this is to consider what the diagrams would like before the change and after the change. Before the change, the diagram looks like figure below, with the intersection of the S and P circles shaded.

After the change, the diagram looks like figure , with the intersection of the S and P circles shaded. Essentially, there's been no change. Imagine what it would like to view the first diagram from behind, or upside-down. In either case, what you would see is the same as the first diagram.

Obversion

Take another look at the square of opposition in figure 4.1. Note that the A and the E are straight across from each other, as are the I and the O. The first step in forming the obverse is to first change the sentence into the type that is straight across the square of opposition. That is, if you started with an A-sentence, then make it into and E. The O becomes and I, and so on.

Once you've changed the sentence type, the next step is to change predicate into its complement. The complement of a class C is the class of everything that is not in C. The easiest way to form a complement is to prefix the class with 'non'. For example, the complement of the class of students is the class of non-students.

So, the obverse of all dogs are mammals is no dogs are non-mammals. The obverse of no OBU students are martians is all OBU students are non-martians. Obversion is truth-preserving in all cases.

Contraposition

The last of our three relations is contraposition. To form the contrapositive of a sentence, first form the converse, then exchange both terms for their complements.

The contrapositive of all dogs are mammals is all non-mammals are non-dogs. Contraposition is truth-preserving for A-sentences and O-sentences only.

Original	Converse	Obverse	Contrapositive
All S are P	All P are S	**No S are non-P**	**All non-P are non-S**
No S are P	**No P are S**	**All S are non-P**	No non-P are non-S
Some S are P	**Some P are S**	**Some S are not non-P**	Some non-P are non-S
Some S are not P	Some P are not S	**Some S are non-P**	**Some non-P are not non-S**

Here's a table to help keep this straight (operations that are truth-preserving are in bold type):

Example

Look at the following argument:

1. All Catholics are non-Protestants.
2. <u>All Lutherans are Protestants.</u>
3. No Catholics are Lutherans.

Note that this argument has four terms:

1. Catholics
2. Non-Protestants
3. Lutherans
4. Protestants

To evaluate the argument, we will first have to either change "non-Protestants" to "Protestants" in the first premise, or "Protestants" to "non-Protestants" in the second premise and conclusion. To minimize errors, we should probably try the option requiring the fewest changes. The only two truth-preserving operations on A-sentences are obversion and contraposition. The contrapositive of "All Catholics are non-Protestants" is "All non-non-Protestants are non-Catholics." The double-non will cancel out, which will fix our original problem, but it will leave us with a new term, "non-Catholic." So, let's try the obverse. The obverse of "All Catholics are non-Protestants" is "No Catholics are Protestants." So, using that for our first premise, the argument becomes:

1. No Catholics are Protestants.
2. <u>All Lutherans are Protestants.</u>
3. No Catholics are Lutherans.

Now, we can check for validity — I'll leave that for you.

Chapter 3

Sentential Logic: Syntax

Sentential logic (*SL*) is a system of logic that treats statements, or propositions, as fundamental units.[1] *SL* is not just a system of logic, however, it is also a language. There are two very general types of languages, natural and formal. Natural languages are the languages that we ordinarily use to communicate with each other. Examples of natural languages include English, French, German, Mandarin, etc. Computer programming languages like Python and Java, on the other hand, are formal languages.

There are certain features that every language, natural or formal, must have. Those features govern how expressions are formed in the language, and determine what those expressions mean. Let's see how those features work in the English language. It begins with a character set, which includes the twenty-six characters of the Latin alphabet and various punctuation marks. The characters are put together to form words; the meaning of a word is determined by its definition. These words are combined in various ways to form sentences. Not just any string of English words can be a meaningful sentence in English. There are certain rules that must be followed—articles must precede nouns, for instance. The characters and the formation rules are the syntax of the language, the rules that determine meaning are the semantics. *SL* is a language, and, like any other language, it has a syntax and semantics. This chapter will focus on the syntax; the semantics of *SL* will be the subject of the next chapter.

Natural languages have many advantages over formal languages, the greatest of which is their expressive power. This expressive power, however, can also be

[1]Informally, we use 'proposition' and 'statement' interchangeably. Strictly speaking, the proposition is the content, or meaning, that the statement expresses. When different sentences in different languages mean the same thing, it is because they express the same proposition. Sentential logic is also called 'propositional logic.'

a disadvantage. It is possible to express more than one thought with exactly the same sentence using a natural language. For example, consider the following English sentence:

> "I don't beat my dog."

Now, imagine that same sentence uttered several times, stressing a different word each time. "I don't beat my *dog*" means something very different from "I don't beat *my* dog." This is an example of ambiguity, a word or an expression having more than one meaning. Another feature of natural languages is vagueness; an expression is vague when it doesn't have a precise meaning. Think of vague expressions as those having fuzzy boundaries that make it impossible to draw a precise line between the times that the expression is true of something and the times when it is not. Examples of vague terms are rich, bald, and young. Formal languages eliminate, as much as possible, ambiguity and vagueness. Every term in a formal language should have exactly one meaning and that meaning should be precise.

Every language has ways of using sentences in the language to create other sentences in the language. This is done by applying sentence formation operators or connectives. One way to do this is by joining two sentences with terms like 'and', 'or', 'but', 'however', or 'unless'. So, we can make the sentence

> All of Plato's know works are dialogues, but none of Aristotle's known works are dialogues.

by putting 'but' between 'All of Plato's know works are dialogues' and 'None of Aristotle's known works are dialogues.'

Although we call these sentence-forming operators "connectives," it is important to understand that some of them don't actually connect different sentences. Some of them apply to just one sentence. Examples of these are 'it is not the case that', and 'it is possible that'.

All of the connectives in *SL* will be truth-functional connectives. A connective is truth-functional if the truth value of the resulting sentence is determined completely by the truth values of the connected sentences and the definition of the connective. So, if we know the truth values of the connected sentences and the operator used to connect them, we will always be able to determine the truth value of the resulting sentence.

Formal languages are used for many things, but an important use in logic is analyzing the logical relationships between sentences and sets of sentences in natural languages, in our case, English. Doing this with *SL* will require translating sentences from English into *SL* and vice versa. Later, I'll state the syntax of *SL* in a very precise way. For now, let me introduce the language by discussing how English is translated into *SL*.

3.1 *SL* Translations

Upper-case letters of the Latin alphabet are used to symbolize simple, or atomic, English sentences. Remember that a simple sentence is one that contains no other sentence as a component. So, 'Socrates is a philosopher' is a simple sentence, but 'Socrates is a philosopher, but Cicero is an orator' is not, since it contains two sentences as components.

Any letter can be used to symbolize a sentence, but we should pick one that will help us to remember the English sentence that is translated. For example, the English sentence

> Socrates is a philosopher

could be symbolized with

> S

Of course, we only have twenty-six upper-case letters, but we need a potentially infinite supply. So, we'll allow letters with positive integer subscripts to be used as sentences in *SL*. So, these are all simple sentences in *SL*:

> A, B, C, P, Q, Z, D_1, F_{412}

One could use a sentence letter to symbolize a complex, or molecular, sentence of English, but it's not a good idea. By doing so, we would be hiding some of the logical structure of the sentence, run the risk of translating a valid argument into an invalid argument in *SL*. So, be sure to only use letters for the simple sentences.

Conjunctions

Our first kind of complex sentence is the conjunction. Conjunctions are sentences formed by combining two sentences with a conjunction connective. The sentences that are joined are called 'conjuncts'. Conjunctions are true whenever both of the conjuncts are true. Although the most common conjunction operator in English is 'and', any operator that implies that both of the connected sentences are true is a conjunction operator. The list of English conjunction operators includes and

but

also

however

yet

still

moreover

although

nevertheless

both

The two most common ways of symbolizing a conjunction operator are '&' (ampersand), '·' (dot), and '∧' (wedge). We'll use '&'.

Now, we can translate our sentence

Socrates is a philosopher, but Cicero is an orator

as

S & C

and

Plato is from Athens and Aristotle is from Macedonia

can be translated as

A & M

Most of the time, these translations are simple. Unfortunately, the complexity of natural languages can sometimes result in some trick translation issues. There are times when the conjunction operator doesn't appear to be joining two sentences. Here is an example:

Plato and Aristotle are philosophers.

The best thing to do in these cases is to paraphrase the English sentence into a sentence that joins two statements with the conjunction operator. If the resulting paraphrase is true under exactly the same conditions as the original, then they are equivalent. If so, then simply translate the paraphrase into *SL*.

The paraphrase of the preceding sentence would be

Plato is a philosopher and Aristotle is a philosopher.

and the *SL* translation is

P & A

On the other hand, consider this sentence:

Abbot and Costello made a good team.

Paraphrasing this results in

Abbot made a good team and Costello made a good team.

This can't be right though. Neither one, individually, made a team at all. So, this sentence, even though it contains the word 'and', is a simple sentence.

The list of English words that gets translated as a conjunction shows that the *SL* translations often fail to capture the full sense of the original English sentences. When I say, "Although John studied hard for his logic exam, he failed", I don't just mean to say that John studied hard *and* he failed, which is how we would paraphrase it before symbolizing the sentence in *SL*. Instead, I mean to say that *in spite* of his studying, he failed. Some nuance is lost in the translation. That nuance is not important however, for our purposes. The conjunction captures everything that is important for logical analysis.

Disjunctions

Another type of complex sentence is the disjunction, commonly expressed in English with 'or', as in this sentence:

Either the Democrat will win the election or the Republican will win.

Disjuncts are true whenever at least one of the connected sentences, called disjuncts, are true. They are only false when both disjuncts are false. The symbol '$\vee$' (wedge) is used for disjunctions in *SL*. So, the sentence above is translated

D ∨ R

As was the case with the conjunction, disjunctions in English are not always two complete sentences joined with an ‘or’. The example above is more likely to be stated, “Either the Democrat or the Republican will win.” Sometimes, disjunctions in English don’t use ‘or’ at all. Here’s an example:

> Alice and Bob are running a race tomorrow. At least one of them will finish.

That second sentence can be paraphrased

> Alice will finish the race, or Bob will finish the race.

It can then be symbolized

A ∨ B

A very important thing to remember when translating disjunctions into *SL*, is that the English word ‘unless’ express a disjunction. The sentence,

> Charlie will fail logic unless he drops the course before the deadline

is equivalent to

> Either Charlie will fail logic or he drops the course before the deadline

and symbolized as

F ∨ D

One thing that is tricky about disjunctions is distinguishing inclusive disjunctions from exclusive disjunctions. An inclusive disjunction is true when *at least one* of the disjuncts is true. An exclusive disjunction is true when one, *and only one*, of the disjuncts is true. The *SL* symbol ‘∨’ is always an inclusive disjunction. In English, however, the word ‘or’ is often used to express exclusive disjunctions. For example, you might see this in a menu:

> Your meal comes with either baked potato or french fries.

The probably don’t mean “Your meal comes with either baked potato, french fries, or both.” So, this sentence should *not* be translated as

P ∨ F

Later, we will see how to translate exclusive disjunctions in *SL*.

Negations

The English phrase 'it is not the case that' generates a kind of complex sentence called a negation. Negations are true if the negated sentences are false. For example,

> It is not the case that Aristotle was from Athens.

This sentence is true just in case it is false that Aristotle was from Athens, and it would be true if it were the case that Aristotle was not from Athens.

The symbol for negations is '¬'. A sentence is negated by placing the negation operator in front of the sentence. This is the one logical connective that doesn't connect two sentences. So, the negation is known as a "unary connective", and the others are called "binary connectives." The phrase 'it is not the case at' is convenient to use when paraphrasing negations because it precedes negated sentences in the same way that the negation symbol in *SL* does. Most of the time, though, 'not' will be somewhere in the English sentence, like this:

Aristotle was not from Athens.

This is naturally equivalent to 'It is not the case that Aristotle was from Athens' and is translated in *SL* as

> ¬A

Other ways that negations are expressed in English besides 'it is not the case that' and 'not' are prefixes like 'non', 'in', and 'un'. These, however, require some care. The sentence

> John is unmarried

is equivalent to

> It is not the case that John is married

and is symbolized as '¬J'. On the other hand,

> Some students are unmarried

is *not* equivalent to

> It is not the case that some students are married.

Instead, it is equivalent to

> It is not the case that *all* students are married.

Combining Logical Connectives

Before I introduce the last two connectives in *SL*, let's look at how the connectives are combined to form more complex sentences. Imagine that I want to symbolize this sentence:

> Either I will stay home tonight, or I will both go out for dinner and go to the movie theater.

Let's start with this:

$$H \vee D \,\&\, M$$

There's a problem, though. Remember that a goal of formal languages is to eliminate ambiguity, and this sentence is definitely ambiguous. Is it a disjunction that has a conjunction as one of its disjuncts, or is a conjunction with a disjunction as one of its conjuncts? It it's a disjunction, then my staying home would be enough to make it true. If it's a conjunction, then it's being true requires that I go to the movies. In *SL*, the ambiguity is cleared up with parentheses. If we decide that the English sentence is a conjunction, then we'll symbolize it like this:

$$(H \vee D) \,\&\, M$$

And if it is a disjunction, we'll symbolize it like this:

$$H \vee (D \,\&\, M)$$

In *SL*, and all of the other systems that we will encounter, every sentence has a main connective. The main connective of a sentence determines what kind of sentence it is, and will always be outside of the parentheses—in this case, should we make the main connective the '$\vee$' or the '&'?

There are two things to look for when trying to determine the main logical operator in an English sentence:

1. Puncation, such as commas and semicolons.
2. Words like 'either... or', 'both... and', and 'if... then'.

The first strategy is to group things together that are on the same side of a punctuation mark. Since 'I will go out for dinner' and 'I will go to the movie theater' are both on the right side of the comma, we should group them together with parentheses, like this:

> H ∨ (D & M)

The second strategy is that anything that is grouped together with parentheses should read naturally as a complete English sentence. If we translated the sentence as

> (H ∨ D) & M

then ‘Either I will stay home tonight, or I will both go out for dinner’ should make sense as a stand-alone sentence, but it doesn’t because of the ‘both’ that is missing an ‘and’. ‘I will both go out for dinner and go to the movie theater’ does make sense, though. So this strategy also requires us to translate the sentence as

> H ∨ (D & M)

These two strategies should be enough to help us correctly translate any well-formed, well-punctuated English sentence. Unfortunately, natural languages are often simply ambiguous, and, in those cases, you must simply do your best. Here are some more examples.

> It’s not the case that both Plato and Aristotle are from Athens.

First, we’ll paraphrase the sentence to make the connected sentences clear:

> It’s not the case that both Plato is from Athens and Aristotle is from Athens.

Since we need to keep the ‘both... and’ together, we should symbolize this as

> ¬(P & A)

Since the only connective that is outside the parentheses is the ‘¬’, it is the main operator and the sentence is a negation.

> Plato is from Athens, but Aristotle is not.

The ‘but’ is a conjunction, so paraphrasing this results in

> Plato is from Athens, and it is not the case that Aristotle is from Athens

and is symbolized like this:

P & ¬A

Parentheses are not needed here, since there's no ambiguity. The '¬' modifies what it immediate precedes, which in this case is the simple sentence 'A'. The qtt& joins the sentence is immediately before it with the one that is immediately after it, which in this case are 'P' and '¬A'. Intuitively, the main connective is the '&', and the sentence is a conjunction.

Here is a rule for determining the main connective of a sentence in symbolic logic: The main connective is the one outside the parentheses, unless there is more than one connective outside the parentheses, then the main connective is not a negation. Just keep in mind that the only time that there can be more than one connective outside the parentheses is when negations are involved, and in that case, the main connective will never be a negation. The main connective can be a negation only when it is the only one outside the parentheses.

Here's one a bit more complex:

> Both either Plato was the greatest metaphysician or Aristotle was the greatest logician, and either Plato was the greatest epistemologist or Aristotle was the greatest moral philosopher.

I don't think there's any need for paraphrasing here. The comma tells us that the 'and' is the main connective, and it joins two disjunctions. So, the translation is

(M ∨ L) & (E ∨ P)

Now, here's something that can be a little trick:

> Neither Aristotle nor Epictetus were from Athens.

This can be paraphased as Both it is not the case that Aristotle is from Athens and it is not the case that Epictetus is from Athens. So, the translation would be

¬A & ¬E

An equally acceptable paraphase is this:

> It is not the case that either Aristotle is from Athens or Epictetus is from Athens.

This would naturally be translated as

¬(A ∨ E)

It should be easy to see that these are equivalent. The first says that both of the sentences represented by 'A' and 'E' are false. The second says that it is not the case that either one is true. We'll prove that they are equivalent in the next chapter.

Material Conditionals

A conditional is an 'if... then' sentence. There are many ways to express conditionals in English, including

if

if… then

only if

whenever

when

only when

implies

provided that

means

entails

is a sufficient condition for

is a necessary condition for

given that

on the condition that

in case

A conditional claims that something is true, if something else is also. Examples of conditionals are

"If Sarah makes an A on the final, then she will get an A for the course."

"Your car will last many years, provided you perform the required maintenance."

"You can light that match only if it is not wet."

The symbol that we will use in *SL* for conditionals is called a horseshoe, '⊃'. Another commonly used symbol is the right arrow, '→'. We can translate the above examples like this:

$F \supset C$

$M \supset L$

$L \supset \neg W$

One big difference between conditionals and other sentences, like conjunctions and disjunctions, is that order matters. Notice that there is no logical difference between the following two sentences:

Albany is the capital of New York and Austin is the capital of Texas.

Austin is the capital of Texas and Albany is the capital of New York.

They essentially assert exactly the same thing, that both of those conjuncts are true. So, changing order of the conjuncts or disjuncts does not change the meaning of the sentence, and if meaning doesn't change, then truth value doesn't change either. That's not true of conditionals. Note the difference between these two sentences:

If you drew a diamond, then you drew a red card.

If you drew a red card, then you drew a diamond.

The first sentence *must* be true. if you drew a diamond, then that guarantees that it's a red card. The second sentence, though, could be false. Your drawing a red card doesn't guarantee that you drew a diamond, you could have drawn a heart instead. So, we need to be able to specify which sentence goes before the arrow and which sentence goes after. The sentence before the arrow is called the antecedent, and the sentence after the arrow is called the consequent.

Look at those three examples again:

If Sarah makes an A on the final, then she will get an A for the course.

Your car will last many years, provided you perform the required maintenance.

You can light that match only if it is not wet.

The antecedent for the first sentence is 'Sarah makes an A on the final'. The consequent is 'She will get an A for the course'. Note that the 'if' and the 'then' are not parts of the antecedent and consequent.

In the second sentence, the antecedent is 'You perform the required maintenance'. The consequent is 'Your car will last many years'. This tells us that the antecedent won't always come first in the English sentence.

The third sentence is tricky. The antecedent is 'You can light that match'. Why? The explanation involves understanding the difference between necessary and sufficient conditions.

A sufficient condition is something that is enough to guarantee the truth of something else. For example, getting a 95 on an exam is sufficient for making an A on the exam, assuming the usual grading scale and that exam is worth 100 points. A necessary condition is something that must be true in order for something else to be true. Making a 95 on an exam is not necessary for making an A—a 94 would have still been an A. Taking the exam is necessary for making an A, though. You can't make an A if you don't take the exam, or, in other words, you can make an A only if you enroll in the course.

Here are some important rules to keep in mind:

> 'If' introduces an antecedent, but 'only if' introduces a consequent.
>
> If A is a sufficient condition for B, then A → B.
>
> If A is a necessary condition for B, then B → A.

All of the connectives in *SL* are truth-functional. That is, the truth value of the complex sentence is a function solely of the simple sentences and the logical connectives contained in the sentence. The truth-functional conditional is called the material conditional, and it has some puzzling consequences. Consider this sentence again:

> If Sarah makes an A on the final, then she will get an A for the course.

Intuitively, this sentence is false when Sarah makes an A on the final, but does *not* get an A for the course, and true when Sarah makes an A on the final and also makes an A for the course. What if Sarah does not get an A on the final? In that case, the material conditional is true no matter whether she gets an A for the course or not. So far, this doesn't seem to be a problem. Let's change the consequent of the conditional, however.

> If Sarah makes an A on the final, then she will get an F for the course.

If this is a material conditional, it will be false only when the antecedent is true and the consequent is false, true whenever either the antecedent is false or the consequent is true. In any case in which Sarah does not make an A on the final, the sentence is true. Now, imagine that the final is worth 100 of the 1,000 points possible in the course. Also, imagine that Sarah has made a perfect score on everything in the course before the final. Sarah then takes the final and makes a score of 85. That lowers her perfect average of 100 to an almost perfect 98.5, which is surely still deserving of an A for the course. Since she didn't make an A on the final, however, the material conditional 'If Sarah makes an A on the final, then she will get an F for the course' is still true. This is called the paradox of the material conditional, something that we will revisit in later chapters.

Most of the time, when we assert a conditional in English, we don't intend it to be a material conditional. Unfortunately, incorporating other kinds of conditionals would complicate our logic. The good news, though, is that when it comes to analyzing logical relations between sentences, we can usually just pretend that every conditional is a material conditional. For now, that just what we'll do.

Material Biconditionals

Our final connective is expressed by the English phrases 'if and only if', 'when and only when', and 'just in case'. These are used to express conditions that are both necessary and sufficient, such as making at least a 90 and getting an A. So this sentence,

> Students make an A if and only if they average at least 90% on all work

can be paraphrased as

> If students make an A, then they average at least 90% on all work; and students average at least 90% on all work only if they make an A

and symbolized as a conjunction of two conditionals like this:

$$(A \supset W) \,\&\, (W \supset A)$$

Although we could certainly translate biconditionals this way, they occur often enough that it is useful to have a symbol dedicated for that purpose. It's common to use either '≡' (triple bar) or '↔' (double arrow). We'll use the triple bar, and translate the previous example like this:

$$A \equiv W$$

Material biconditionals are true whenever the two sentences joined by the biconditional operator have the same truth value, and false whenever they have different truth values. For biconditionals, as for conjunctions and disjunctions, the order of the connected sentences doesn't matter.

Complex Symbolizations

Now that we've introduced the five truth-functional connectives, let's practice some more complex translations into *SL*. We'll use these simple sentences:

> F: The Federal Reserve will raise interest rates.
> C: Congress will increase spending.
> I: The inflation rate goes up.
> U: There is increased unemployment.
> P: Prices of consumer goods increase.

Here's the first one:

> If the inflation rate goes up, then either the Federal Reserve will raise interest rates or Congress will increase spending.

We start by putting in the sentence letters and the connective symbols.

$$I \supset F \vee C$$

Then, we put in the parentheses, making sure to keep the sentences on the same side of the comma together.

$$I \supset (F \vee C)$$

The second example is

> Both the Federal Reserve will raise interest rates and Congress will increase spending if and only if either prices of consumer goods increase or there is increased unemployment.

Filling in the sentence letters and connectives gives us this:

$$F \mathbin{\&} C \equiv P \vee U$$

We don't have any punctuation to help us here, but we do have a 'both... and' and an 'either... or' that will guide our punctuation, resulting in this:

$$(F \mathbin{\&} C) \equiv (P \vee U)$$

One more:

> Congress will not increase spending, unless neither prices of consumer goods increase nor there is increased unemployment; but if the inflation rate goes up, then the Federal Reserve will raise interest rates.

The semicolon tells us that the 'but' is the main connective, and the second conjunct is pretty simple:

> I ⊃ F)

The first conjunct is a disjunction. The first disjunct is

> ¬C

The second disjunct, is a 'neither, nor' sentence, which is best translated as a negation of a disjunction. Putting the two disjuncts together, we have

> ¬C ∨ ¬(P ∨ U)

So, we just need to put those two conjuncts together with an '&', being sure to put brackets around the first to avoid any ambiguity:

> [¬C ∨ ¬(P ∨ U)] & (I ⊃ F)

3.2 Rules of Syntax

Use and Mention

We use words to talk about other things. For instance, in the sentence, 'The 46th President of the United States is from Delaware', the phrase 'The 46th President of the United States' refers to Joe Biden. In the sentence 'Red is the color at the long wavelength end of the visible spectrum', the word 'red' refers, unsurprisingly to the color red. When we use the word 'red', we refer to the color red. When we use the name 'Joe Biden', we refer to President Biden.

Sometimes, though, we want to talk about the words themselves. What should we do then? Just as we use names of persons to talk about the persons, we need names for words and expressions to be able to talk about those words and expressions. We form the names of expressions by placing those expressions in single quotes. Without the quotes, we are using the expresion. When the expression is enclosed in quotes, it is being mentioned, not used. For example,

> 'Red' has three letters.

In this case, the word ‘red’ is being mentioned, not used. In the sentence,

> Red is often used to signify passion.

the word ‘red’ is being used. The use-mention fallacy is committed whenever a word is used when it should have been mentioned, or vice versa. An example is the sentence,

> Red has three letters.

This sentence asserts that the color red has three letters, but colors aren’t the kind of things that can have letters. Instead it should be

> ‘Red’ has three letters.

The first is nonsense, the second is a perfectly meaningful, and true, English sentence. When we use a word, we are referring to the thing that the word stands for, when we mention a word, we are referring to the word itself.

Object Language and Metalanguage

The ability to form names of expressions by enclosing them in single quotes enables us to talk about the expressions of on language while using another language. Here is an example:

> ‘Ich’ is the German first-person singular pronoun.

This is a sentence in English, so all of its components must be in English. By enclosing the German word in single quotes, we have formed the English name for that German expression. So, we are using English to talk about an expression in German. The language that is being discussed is called the *object language*. The language that is being used to talk about the object language is called the *metalanguage*. In this case, the object language is German, and the metalanguage is English. As we formulate the rules for *SL*, the metalanguage will be English, but the object language will be *SL*.

Metalinguistic Variables

To refer to specific expressions of *SL*, we can simply enclose them in single quotes, as we have been doing. The sentence

> ‘A & B’ is a conjunction.

claims that a particular expression of *SL* is a conjunction. It is an English sentence using the English name for an expression of *SL*. To formulate the rules of *SL*, however, we will need to be able to discuss general types of expressions of *SL*, not just particular expressions of *SL*. We do this using *metalinguistic variables*, or *metavariables*, for short.

A metalinguistic variable is an expression in the metalanguage that stands for *any* expression in the object language. We will use lowercase Greek letters, such as 'α', 'β', and 'γ', for metavariables. The preceding sentence encloses the metavariables in single quotes because it is talking about the metavariables. When the metavariables are being used, they won't need to be enclosed in quotes since they are expressions in the metalanguage.

The Syntax of *SL*

We now have everything we need to carefully define the rules of syntax of *SL*. These rules will tell us what counts as a proper expression of *SL*. Not every string of English words will be a proper English sentence. For instance 'ball the red is blue and' is composed of perfectly good English words, but any English speaker will know that the sentence is nonsense because it violates the conventions for making proper English expressions. We need to formulate rules for making proper expressions of *SL* that are rigorous and precise, such that, for any string of symbols of *SL*, will be able to determine if that string forms a proper expression of *SL*. We'll call a proper expression a 'well-formed formula', or 'wff' for short.

Here are our rules of syntax for *SL*:

1. Every uppercase letter of the Latin alphabet, with or without a numerical subscript, is a well-formed formula.
2. If α is a well-formed formula, then $\neg\alpha$ is a well-formed formula.
3. If α and β are well-formed formulas, then so is $(\alpha \mathbin{\&} \beta)$.
4. If α and β are well-formed formulas, then so is $(\alpha \vee \beta)$.
5. If α and β are well-formed formulas, then so is $(\alpha \supset \beta)$.
6. If α and β are well-formed formulas, then so is $(\alpha \equiv \beta)$.
7. Nothing is well-formed formula unless it can be formed by repeated application of 1–6.

These are *recursive* rules. That is, they specify what counts as the simplest type of well-formed formula in *SL*, then show how increasingly more complex formulas are generated by applying the same rules again and again.

If you have paid close attention to the previous sections, you might have recognized a problem with the rules. Consider the expression formed with the negation symbol and a metalinguistic variable:

$\neg\alpha$

It consists of a metalinguistic variable and a symbol of *SL*. Thus, it's a mixture of the object language and the metalanguage. That means it is a well-formed formula of neither language. We're committing the use-mention fallacy—the '¬' is being used, when instead it should be merely mentioned. So, we need to enclose it in single quotes, but where should the quotes go? It looks like we have two options, but neither are good. The first is to enclose the whole formula, including the metavariable, in quotes:

'$\neg\alpha$'.

If we do that, then, we are no longer using qttα as a metavariable. Instead, we would be treating it as a symbol in *SL*, but it's not a symbol in *SL*. So, we haven't avoided the use-mention fallacy; we've merely committed another instance of the use-mention fallacy.

Another option is to enclose only the negation symbol in single quotes, like this:

'¬'α

This is better, but it's not clear what it means. A precise way to state rule 2 without committing the use-mention fallacy is this:

> 2*: If α is a well-formed formula, then the formula expressed by concatenating '¬' and α is a well-formed formula.

That's not bad, but fixing rules 3–7 in the same way would be somewhat awkward. The solution is to use something called "quasi-quotation".[2] Quasi-quotation encloses the string formed by the symbols of the object-language and the metavariables in corner-quotes:

$\ulcorner\neg\alpha\urcorner$

Quasi-quotation is just shorthand for a statement about the concatenation of symbols and formulas. So $\ulcorner\neg\alpha\urcorner$ is shorthand for

The formula expressed by concatenating '¬' and α.

[2] This is also sometimes called "Quine-quotation", because it was first introduced by W.V.O. Quine in his book *Mathematical Logic*.

To make things simpler, we will adopt this as a convention: whenever we use expressions that consist of both metavariables and formulas of an object language, we will understand the entire expression as being enclosed in corner-quotes. So, if we were to write this,

$(\alpha \mathbin{\&} (\neg\beta \lor \gamma))$

we would understand it as expressing this:

$\ulcorner(\alpha \mathbin{\&} (\neg\beta \lor \gamma))\urcorner$

and when we quote an expression consisting of a mixture of object language symbols and metavariables, we will understand that quotation to be quasi-quotation. So, when we write '$\alpha \mathbin{\&} \beta$', we will understand it as $\ulcorner\alpha \mathbin{\&} \beta\urcorner$.

There are two more conventions regarding the use of parentheses in *SL*. First, notice that the only punctuation defined in the rules of syntax are parentheses; there are no brackets or braces. This is because we need to be able to form indefinitely large expressions of *SL*, which requires nested expressions. If we demanded a different punctuation character at each level, we would quickly run out of punctuation symbols. This can, though, make things difficult to read. Since we will never, in this book, encounter nesting of more than three levels, we will use braces and brackets for levels two and three. So, instead of writing

$\neg((\alpha \supset (\beta \mathbin{\&} \gamma)) \equiv ((\alpha \lor \delta) \lor (\gamma \supset \epsilon)))$

we will write

$\neg\{[\alpha \supset (\beta \mathbin{\&} \gamma)] \equiv [(\alpha \lor \delta) \lor (\gamma \supset \epsilon)]\}$

which should be easier to read.

The second convention is that we will drop the outermost set of parentheses that encloses the entire complex formula, since those parentheses aren't needed to avoid ambiguities. So, instead of

$(\alpha \mathbin{\&} \beta)$

we'll simply write

$\alpha \mathbin{\&} \beta$

and instead of

[α & (¬β ∨ γ)]

we can write

α & (¬β ∨ γ)

Now, we'll use the rules to determine if an expression of *SL* is a well-formed formula. We start with the sentence letters used in the expression, then show, by using repeated instances of the rules of syntax, that the expression can be generated. Here's a simple example:

Show that '¬(A & B) ⊃ C' is a well-formed formula, or sentence, in *SL*.

First, by rule 1, both 'A' and 'B' are wff's. By rule 3, 'A & B' is a wff. By rule 2, '¬(A & B)' is a wff. Finally, by rule 5, '¬(A & B) ⊃ C' is a wff.

Finally, here are some expression that are *not* well-formed formulas of *SL*. See if you can determine what is wrong with each one.

A & B ∨ C

(P ⊃ Q)R

¬ ≡ Z

¬[(E & F) ⊃ D