Policy Gradient Methods

Reinforcement Learning Seminar

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Policy-Based Reinforcement Learning

- In the last lecture we approximated the Value or Action-value function (Q-factor) using parameters θ,

$$V_\theta(s) \approx V^\pi(s)$$
$$Q_\theta(s,a) \approx Q^\pi(s,a)$$

- A policy was generated directly from the value function (e.g. using ϵ-greedy)
- In this lecture we will directly parameterize the policy

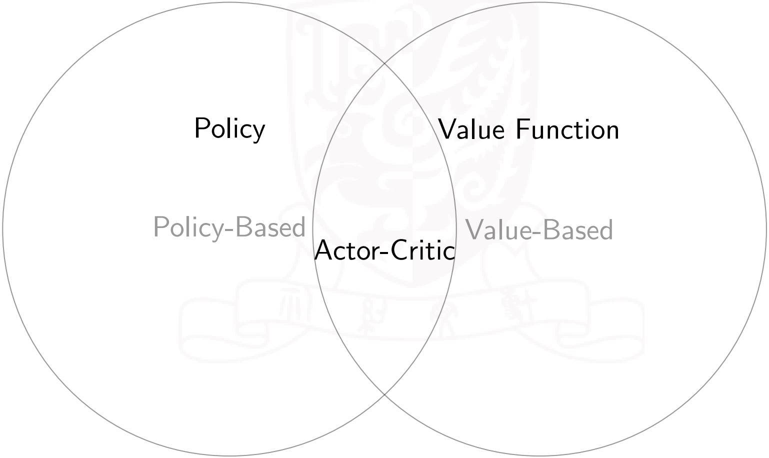
$$\pi_\theta(s,a) = \mathbb{P}[a \mid s, \theta]$$

- We will focus again on model-free reinforcement learning

Value-Based and Policy-Based RL

- Value Based
  - Learnt Value Function
  - Implicit policy (ϵ-greedy)
- Policy Based
  - No Value Function
  - Learnt Policy
- Actor-Critic
  - Learnt Value Function
  - Learnt Policy

Value-Based and Policy-Based RL



Advantages of Policy-Based RL

Advantages:

- Better convergence properties
- Effective in high-dimensional or continuous action spaces
- Can learn stochastic policies

Advantages of Policy-Based RL

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Disadvantages:

- Typically converge to a local rather than global optimum
- Evaluating a policy is typically inefficient and high variance

Example: Rock-Paper-Scissors

- Two-player game of rock-paper-scissors
  - Scissors beats paper
  - Rock beats scissors
  - Paper beats rock
- Consider policies for *iterated* rock-paper-scissors
  - A deterministic policy is easily exploited
  - A uniform random policy is optimal (i.e. Nash equilibrium)

Policy Objective Functions

- Goal: given policy $\pi_\theta(s, a)$ with parameters θ, find best θ
- But how do we measure the quality of a policy π_θ?
- In episodic environments we can use the start value

$$J_1(\theta) = V^{\pi_\theta}(s_1) = \mathbb{E}_{\pi_\theta}[v_1]$$

- In continuing environments we can use the average value

$$J_{avV}(\theta) = \sum_s d^{\pi_\theta}(s) V^{\pi_\theta}(s)$$

- Or the average reward per time-step

$$J_{avR}(\theta) = \sum_s d^{\pi_\theta}(s) \sum_a \pi_\theta(s, a) \mathcal{R}_s^a$$

- where $d^{\pi_\theta}(s)$ is stationary distribution of Markov chain for π_θ

Policy Optimization

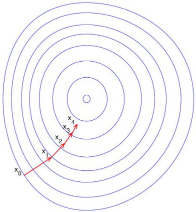
- Policy based reinforcement learning is an optimization problem
- Find θ that maximises $J(\theta)$
- Some approaches do not use gradient (gradient-free)
  - Hill climbing
  - Simplex / amoeba / Nelder Mead
  - Genetic algorithms
- Greater efficiency often possible using gradient
  - Gradient descent
  - Conjugate gradient
  - Quasi-newton
- We focus on gradient descent, many extensions possible
- And on methods that exploit sequential structure

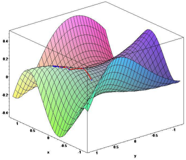
Policy Gradient

- Let $J(\theta)$ be any policy objective function
- Policy gradient algorithms search for a local maximum in $J(\theta)$ by ascending the gradient of the policy, w. r.t. parameters θ
- Where $\nabla_\theta J(\theta)$ is the policy gradient

$$\nabla_\theta J(\theta) = \begin{pmatrix} \frac{\partial J(\theta)}{\partial \theta_1} \\ \vdots \\ \frac{\partial J(\theta)}{\partial \theta_n} \end{pmatrix}$$

- and α is a step-size parameter





Computing Gradients By Finite Differences

- To evaluate policy gradient of $\pi_\theta(s, a)$
- For each dimension $k \in [1, n]$
  - Estimate k-th partial derivative of objective function w.r.t. θ
  - By perturbing θ by small amount ϵ in k-th dimension

$$\frac{\partial J(\theta)}{\partial \theta_k} \approx \frac{J(\theta + \epsilon u_k) - J(\theta)}{\epsilon}$$

where u_k is unit vector with 1 in kth component, 0 elsewhere

- Uses n evaluations to compute policy gradient in n dimensions
- Simple, noisy, inefficient - but sometimes effective
- Works for arbitrary policies, even if policy is not differentiable

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Score Function

- We now compute the policy gradient analytically
- Assume policy π_θ is differentiable whenever it is non-zero
- and we know the gradient $\nabla_\theta \pi_\theta(s, a)$
- Likelihood ratios exploit the following identity

$$\begin{aligned}
\nabla_\theta \pi_\theta(s, a) &= \pi_\theta(s, a) \frac{\nabla_\theta \pi_\theta(s, a)}{\pi_\theta(s, a)} \\
&= \pi_\theta(s, a) \nabla_\theta \log \pi_\theta(s, a)
\end{aligned}$$

- The score function is $\nabla_\theta \log \pi_\theta(s, a)$

Softmax Policy

- We will use a softmax policy as a running example
- Weight actions using linear combination of features $\phi(s,a)^T\theta$
- Probability of action is proportional to exponential weight

$$\pi_\theta(s,a) \propto e^{\phi(s,a)^T\theta}$$

- The score function is

$$\nabla_\theta \log \pi_\theta(s,a) = \phi(s,a) - \mathbb{E}_{\pi_\theta}[\phi(s,\cdot)]$$

Gaussian Policy

- In continuous action spaces, a Gaussian policy is natural
- Mean is a linear combination of state features $\mu(s) = \phi(s)^T \theta$
- Variance may be fixed σ^2, or can also parameterized
- Policy is Gaussian, $a \sim \mathcal{N}(\mu(s), \sigma^2)$
- The score function is

$$\nabla_\theta \log \pi_\theta(s, a) = \frac{(a - \mu(s))\phi(s)}{\sigma^2}$$

One-Step MDPs

- Consider a simple class of one-step MDPs
- Starting in state $s \sim d(s)$
- Terminating after one time-step with reward $r = \mathcal{R}_{s,a}$
- Use likelihood ratios to compute the policy gradient

$$
\begin{aligned}
J(\theta) &= \mathbb{E}_{\pi_\theta}[r] \\
&= \sum_{s \in \mathcal{S}} d(s) \sum_{a \in \mathcal{A}} \pi_\theta(s,a) \mathcal{R}_{s,a} \\
\nabla_\theta J(\theta) &= \sum_{s \in \mathcal{S}} d(s) \sum_{a \in \mathcal{A}} \pi_\theta(s,a) \nabla_\theta \log \pi_\theta(s,a) \mathcal{R}_{s,a} \\
&= \mathbb{E}_{\pi_\theta}[\nabla_\theta \log \pi_\theta(s,a) r]
\end{aligned}
$$

Policy Gradient Theorem

- The policy gradient theorem generalises the likelihood ratio approach to multi-step MDPs
- Replaces instantaneous reward r with long-term value $Q^{\pi}(s, a)$

Policy Gradient Theorem

For any differentiable policy $\pi_\theta(s, a)$, for any of the policy objective functions $J = J_1, J_{avR}$, or $\frac{1}{1-\gamma} J_{aw}$, the policy gradient is

$$\begin{aligned}\nabla_\theta J(\theta) &= \sum_{s \in \mathcal{S}} d^{\pi_\theta}(s) \sum_{a} \nabla_\theta \pi_\theta(s, a) Q^{\pi_\theta}(s, a) \\ &= \mathbb{E}_{\pi_\theta} \nabla_\theta \log \pi_\theta(s, a) Q^{\pi_\theta}(s, a)\end{aligned}$$

Monte-Carlo Policy Gradient (REINFORCE)

- Update parameters by stochastic gradient ascent
- Using return v_t as an unbiased sample of $Q^{\pi_\theta}(s_t, a_t)$

$$\Delta\theta_t = \alpha \nabla_\theta \log \pi_\theta(s_t, a_t) v_t$$

Algorithm 1 REINFORCE

1: Init θ arbitrarily
2: **for** each episode $\{s_1, a_1, r_2, \cdots, s_{T-1}, a_{T-1}, r_T\} \sim \pi_\theta$ **do**
3:     **for** $t = 1$ to $T - 1$ **do**
4:         $\theta \leftarrow \theta + \alpha \nabla_\theta \log \pi_\theta(s_t, a_t) v_t$
5:     **end for**
6: **end for**
7: **return** θ

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Reducing Variance Using a Critic

- Monte-Carlo policy gradient still has high variance
- We use a critic to estimate the action-value function,

$$Q_w(s, a) \approx Q^{\pi_\theta}(s, a)$$

- Actor-critic algorithms maintain *two* sets of parameters
  Critic Updates action-value function parameters w
  Actor Updates policy parameters θ, in direction suggested by critic
- Actor-critic algorithms follow an *approximate* policy gradient

$$\nabla_\theta J(\theta) \approx \mathbb{E}_{\pi_\theta}[\nabla_\theta \log \pi_\theta(s, a) Q_w(s, a)]$$
$$\Delta\theta = \alpha \nabla_\theta \log \pi_\theta(s, a) Q_w(s, a)$$

Estimating the Action-Value Function

- The critic is solving a familiar problem: policy evaluation
- How good is policy π_θ for current parameters θ?
- This problem was explored in previous two lectures, e.g.
  - Monte-Carlo policy evaluation
  - Temporal-Difference learning
  - $TD(\lambda)$
- Could also use e.g. least-squares policy evaluation

Action-Value Actor-Critic

Using linear value fn approx. $Q_w(s,a) = \phi(s,a)^T w$

Critic Updates w by linear $TD(0)$

Actor Updates θ by policy gradient

Algorithm 2 QAC

Init s, θ, Sample $a \sim \pi_\theta$

for each step **do**

  Sample reward $r = \mathcal{R}_s^a$, $s' \sim \mathcal{P}_{s,\cdot}^a$ and $a' \sim \pi_\theta(s', a')$

  $\theta = \theta + \alpha \nabla_\theta \log \pi(s,a) Q_w(s,a)$

  $w \leftarrow w + \beta \left(r + \gamma Q_w(s', a') - Q_w(s,a)\right) \phi(s,a)$

  $a \leftarrow a', s \leftarrow s'$

end for

Bias in Actor-Critic Algorithms

- Approximating the policy gradient introduces bias
- A biased policy gradient may not find the right solution
- Luckily, if we choose value function approximation carefully
- Then we can avoid introducing any bias
- i.e. We can still follow the exact policy gradient

Compatible Function Approximation

Compatible Function Approximation Theorem

If the following two conditions are satisfied:

- Value function approximator is compatible to the policy

$$\nabla_w Q_w(s,a) = \nabla_\theta \log \pi_\theta(s,a)$$

- Value function parameters w minimize the mean-squared error

$$\epsilon = \mathbb{E}_{\pi_\theta}\left[\left(Q^{\pi_\theta}(s,a) - Q_w(s,a)\right)^2\right]$$

Then the policy gradient is exact,

$$\nabla_\theta J(\theta) = \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s,a) \, Q_w(s,a)\right]$$

Proof of Compatible Function Approximation Theorem

If w is chosen to minimise mean-squared error, gradient of ϵ w r.t. w must be zero,

$$
\begin{aligned}
\nabla_w \epsilon &= 0 \\
\mathbb{E}_{\pi_\theta}\left[(Q^\theta(s,a) - Q_w(s,a))\nabla_w Q_w(s,a)\right] &= 0 \\
\mathbb{E}_{\pi_\theta}\left[(Q^\theta(s,a) - Q_w(s,a))\nabla_\theta \log \pi_\theta(s,a)\right] &= 0 \\
\mathbb{E}_{\pi_\theta}\left[Q^\theta(s,a)\nabla_\theta \log \pi_\theta(s,a)\right] &= \mathbb{E}_{\pi_\theta}\left[Q_w(s,a)\nabla_\theta \log \pi_\theta(s,a)\right]
\end{aligned}
$$

So $Q_w(s,a)$ can be substituted directly into the policy gradient,

$$
\nabla_\theta J(\theta) = \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s,a) Q_w(s,a)\right]
$$

Reducing Variance Using a Baseline

- We subtract a baseline function $B(s)$ from the policy gradient
- This can reduce variance, without changing expectation

$$
\begin{aligned}
\mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s,a) B(s)\right] &= \sum_{s\in\mathcal{S}} d^{\pi_\theta}(s) \sum_a \nabla_\theta \pi_\theta(s,a) B(s) \\
&= \sum_{s\in\mathcal{S}} d^{\pi_\theta} B(s) \nabla_\theta \sum_{a\in\mathcal{A}} \pi_\theta(s,a) = 0
\end{aligned}
$$

- A good baseline is the state value function $B(s) = V^{\pi_\theta}(s)$
- So we can rewrite the policy gradient using the advantage function $A^{\pi_\theta}(s,a)$

$$
\begin{aligned}
A^{\pi_\theta}(s,a) &= Q^{\pi_\theta}(s,a) - V^{\pi_\theta}(s) \\
\nabla_\theta J(\theta) &= \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s,a) A^{\pi_\theta}(s,a)\right]
\end{aligned}
$$

Estimating the Advantage Function (1)

- The advantage function can significantly reduce variance of policy gradient
- So the critic should really estimate the advantage function
- For example, by estimating *both* $V^{\pi_\theta}(s)$ *and* $Q^{\pi_\theta}(s,a)$
- Using two function approximators and two parameter vectors,

$$
\begin{aligned}
V_v(s) &\approx V^{\pi_\theta}(s) \\
Q_w(s,a) &\approx Q^{\pi_\theta}(s,a) \\
A(s,a) &= Q_w(s,a) - V_v(s)
\end{aligned}
$$

- And updating *both* value functions by e.g. TD learning

Estimating the Advantage Function (2)

- For the true value function $V^{\pi_\theta}(s)$, the *TD* error δ^{π_θ}

$$\delta^{\pi_\theta} = r + \gamma V^{\pi_\theta}(s') - V^{\pi_\theta}(s)$$

- is an unbiased estimate of the advantage function

$$\begin{aligned}\mathbb{E}_{\pi_\theta}[\delta^{\pi_\theta}|s,a] &= \mathbb{E}_{\pi_\theta}[r + \gamma V^{\pi_\theta}(s')|s,a] - V^{\pi_\theta}(s) \\ &= Q^{\pi_\theta}(s,a) - V^{\pi_\theta}(s) = A^{\pi_\theta}(s,a)\end{aligned}$$

- So we can use the TD error to compute the policy gradient

$$\nabla_\theta J(\theta) = \mathbb{E}_{\pi_\theta}[\nabla_\theta \log \pi_\theta(s,a)\delta^{\pi_\theta}]$$

- In practice we can use an approximate TD error

$$\delta_v = r + \gamma V_v(s') - V_v(s)$$

- This approach only requires one set of critic parameters v

Critics at Different Time-Scales

- Critic can estimate value function $V_\theta(s)$ from many targets at different time-scales
  - For MC, the target is the return v_t

$$\Delta\theta = \alpha({\color{red}v_t} - V_\theta(s))\phi(s)$$

  - For TD(0) , the target is the TD target $r + \gamma V(s')$

$$\Delta\theta = \alpha({\color{red}r + \gamma V(s')} - V_\theta(s))\phi(s)$$

  - For forward-view $TD(\lambda)$, the target is the λ-return v_t^λ

$$\Delta\theta = \alpha({\color{red}v_t^\lambda} - V_\theta(s))\phi(s)$$

  - For backward-view $TD(\lambda)$, we use eligibility traces

$$\begin{aligned}
\delta_t &= r_{t+1} + \gamma V(s_{t+1}) - V(s_t) \\
e_t &= \gamma\lambda e_{t-1} + \phi(s_t) \\
\Delta\theta &= \alpha\delta_t e_t
\end{aligned}$$

Actors at Different Time-Scales

- The policy gradient can also be estimated at many time-scales

$$\nabla_\theta J(\theta) = \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s,a) A^{\pi_\theta}(s,a)\right]$$

- Monte-Carlo policy gradient uses error from complete return

$$\Delta\theta = \alpha(v_t - V_v(s_t))\nabla_\theta \log \pi_\theta(s_t, a_t)$$

- Actor-critic policy gradient uses the one-step TD error

$$\Delta\theta = \alpha(r + \gamma V_v(s_{t+1}) - V_v(s_t))\nabla_\theta \log \pi_\theta(s_t, a_t)$$

Policy Gradient with Eligibility Traces

- Just like forward-view $TD(\lambda)$, we can mix over time-scales

$$\Delta\theta = \alpha(v_t^{\lambda} - V_v(s_t))\nabla_\theta \log \pi_\theta(s_t, a_t)$$

- where $v_t^{\lambda} - V_v(s_t)$ is a biased estimate of advantage fn
- Like backward-view $TD(\lambda)$, we can also use eligibility traces
- By equivalence with $TD(\lambda)$, substituting $\phi(s) = \nabla_\theta \log \pi_\theta(s, a)$

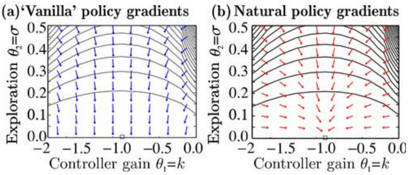
$$\begin{aligned} \delta &= r_{t+1} + \gamma V_v(s_{t+1}) - V_v(s_t) \\ e_{t+1} &= \gamma\lambda e_t + \nabla_\theta \log \pi_\theta(s, a) \\ \Delta\theta &= \alpha\delta e_t \end{aligned}$$

- This update can be applied online, to incomplete sequences

Alternative Policy Gradient Directions

- Gradient ascent algorithms can follow any ascent direction
- A good ascent direction can significantly speed convergence
- Also, a policy can often be reparametrized without changing action probabilities
- For example, increasing score of all actions in a softmax policy
- The vanilla gradient is sensitive to these reparametrizations

Natural Policy Gradient



- The natural policy gradient is parametrization independent
- It finds ascent direction that is closest to vanilla gradient, when changing policy by a small, fixed amount

$$\nabla_\theta^{nat}\pi_\theta(s,a) = G_\theta^{-1}\nabla_\theta\pi_\theta(s,a)$$

- where G_θ is the Fisher information matrix

$$G_\theta = \mathbb{E}_{\pi_\theta}\left[\nabla_\theta \log \pi_\theta(s,a)\nabla_\theta \log \pi_\theta(s,a)^T\right]$$

Natural Policy Gradient

- Using compatible function approximation,

$$\nabla_w A_w(s, a) = \nabla_\theta \log \pi_\theta(s, a)$$

- So the natural policy gradient simplifies,

$$\begin{aligned}
\nabla_\theta J(\theta) &= \mathbb{E}_{\pi_\theta}[\nabla_\theta \log \pi_\theta(s, a) A^{\pi_\theta}(s, a)] \\
&= \mathbb{E}_{\pi_\theta}[\nabla_\theta \log \pi_\theta(s, a) \nabla_\theta \log \pi_\theta(s, a)^T w] \\
&= G_\theta w \\
\nabla_\theta^{nat} J(\theta) &= w
\end{aligned}$$

- i.e. update actor parameters in direction of critic parameters

Summary of Policy Gradient Algorithms

- The policy gradient has many equivalent forms

$$
\begin{aligned}
\nabla_\theta J(\theta) &= \mathbb{E}_{\pi_\theta}[\nabla_\theta \log \pi_\theta(s,a) v_t] && \text{REINFORCE} \\
&= \mathbb{E}_{\pi_\theta}[\nabla_\theta \log \pi_\theta(s,a) Q_w(s,a)] && \text{Actor-Critic} \\
&= \mathbb{E}_{\pi_\theta}[\nabla_\theta \log \pi_\theta(s,a) A_w(s,a)] && \text{Advantage actor-Critic} \\
&= \mathbb{E}_{\pi_\theta}[\nabla_\theta \log \pi_\theta(s,a) \delta] && \text{TD Actor-Critic} \\
&= \mathbb{E}_{\pi_\theta}[\nabla_\theta \log \pi_\theta(s,a) \delta e] && \text{TD}(\lambda)\text{ Actor-Critic} \\
G_\theta^{-1} \nabla_\theta J(\theta) &= w && \text{Natural Actor-Critic}
\end{aligned}
$$

- Each leads a stochastic gradient ascent algorithm
- Critic uses policy evaluation (e.g. MC or TD learning) to estimate $Q^\pi(s,a)$, $A^\pi(s,a)$ or $V^\pi(s)$

State of the art not yet covered

- Deep Deterministic Policy Gradient (DDPG)
- Asynchronous Advantage Actor-Critic Algorithm (A3C), Importance Weighted Actor-Learner Architectures (IMPALA)
- Trust Region Policy Optimization (TRPO), Proximal Policy Optimization (PPO)
- Soft Actor-Critic

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