Randomized Elliptical Potential Lemma with an Application to Linear Thompson Sampling

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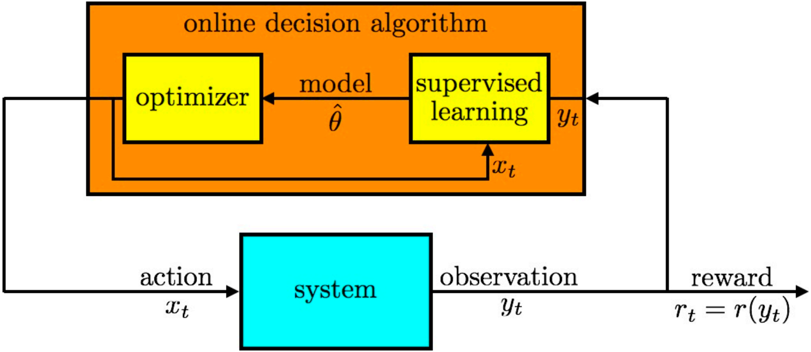
Mainly based on:
Hamidi, Nima, and Mohsen Bayati. "The Randomized Elliptical Potential Lemma with an Application to Linear Thompson Sampling." arXiv preprint arXiv:2102.07987 (2021).
Kalkanlı, Cem, and Ayfer Özgür. "An Improved Regret Bound for Thompson Sampling in the Gaussian Linear Bandit Setting." 2020 IEEE International Symposium on Information Theory (ISIT). IEEE, 2020.

Overview

- Generalize the elliptical potential bound to any arbitrary prior and noise distributions.

- The second contribution of this note is to apply the aforementioned randomized elliptical potential lemma in combination with the proof techniques in (Dong and Van Roy, 2018; Kalkanli and Özgür, 2020) to prove an $\mathcal{O}(d\sqrt{T \log T})$ bound for the Bayesian regret of the well-known linear Thompson sampling (LinTS).

Online Decision Making



Problem Formulation

- Action set $\mathcal{A}_t$ (compact (infinitely many elements) and possibly changing at each round)
- Set of all possible outcomes $\mathcal{Y}$ (assume $\subseteq \mathbb{R}^N$)
- For $t = 1, 2, \ldots,$ a decision maker sequentially chooses actions $A_1, A_2, \ldots$ and observes outcomes $Y_{1,A_1}, Y_{2,A_2}, \ldots$
- The decision maker assumes that there is a "true" outcome distribution p^*, and imposes a prior on p^*.
- Conditioned on p^*, the outcomes $Y_1, Y_2, \ldots$ are iid from p^*.
- Reward function $R : \mathcal{Y} \to \mathbb{R}$, known and fixed.

Problem Formulation

- $A_t \in \mathcal{A}_t$ is chosen based on the history of observations $H_t = \left(A_1, Y_{1,A_1}, \ldots, A_{t-1}, Y_{t-1,A_{t-1}}\right)$ up to time t.
- A randomized policy (an algorithm) $\pi^{\text{alg}} = (\pi_t)_{t\in\mathbb{N}}$ is a sequence of deterministic functions, where $\pi_t\left(H_t\right)$ specifies a probability distribution over the action set $\mathcal{A}_t$.
- Let $\mathscr{D}(\mathcal{A})$ denote the set of probability distributions over $\mathcal{A}$.
- $A_t \sim \pi_t\left(H_t\right) \in \mathscr{D}(\mathcal{A}_t)$
- With some abuse of notation, write $\pi_t = \pi_t(H_t)$, where $\pi_t(a) = \mathbb{P}\left(A_t = a \mid H_t\right)$.

Problem Formulation

- Optimal action
$$A_t^* \in \arg\max_{a \in \mathcal{A}_t} \mathbb{E}\left[R\left(Y_{t,a}\right) \mid p^*\right]$$
(Note: A_t^* is a random variable.)
- The objective is to maximize the expected cumulative reward
$$\mathbb{E}\left[\sum_{t=1}^{T} R\left(Y_{t,A_t}\right)\right]$$
- Equivalently, we minimize the expected (Baysian) regret
$$\mathbb{E}\left[\operatorname{Regret}\left(T, \pi^{\mathrm{alg}}\right)\right] = \mathbb{E}\left[\sum_{t=1}^{T} R\left(Y_{t,A_t^*}\right) - R\left(Y_{t,A_t}\right)\right]$$

Part 1: Thompson Sampling

$$\mathbb{P}(A_t^* = \cdot \mid \text{history}, \mathcal{A}_t)$$

Thompson sampling review

- Filtration $(\mathcal{F}_t)_{t\in\mathbb{N}}$, where $\mathcal{F}_t$ is the sigma-algebra generated by the history

$$H_t = \left(A_1, Y_{1,A_1}, \ldots, A_{t-1}, Y_{t-1,A_{t-1}}\right)$$

- For $t = 1, 2, \ldots, T$, Thompson sampling samples

$$A_t \overset{i.i.d}{\sim} \mathbb{P}\left(A_t^* = \cdot \mid \mathcal{F}_t, \mathcal{A}_t\right)$$

- "Probability matching": $\mathbb{P}\left(A^* = \cdot \mid \mathcal{F}_t\right) = \mathbb{P}\left(A^* = \cdot \mid \sigma(H_t)\right) = \mathbb{P}\left(A^* = \cdot \mid H_t\right)$

Thompson sampling review - Simple Implementation

- Suppose that the true outcome distribution p^* is from some parametric family of distributions $\{p_\theta\}_{\theta \in \theta}$
- Then p^* corresponds to some $\theta^* \in \theta$.

- Thompson sampling starts with a prior on θ^*.
- During each time period, Thompson sampling samples $\hat{\theta}_t \sim \mathbb{P}\left(\theta^* = \cdot \mid \mathcal{F}_t\right)$
- Then, it selects $A_t \in \arg\max_{a \in \mathcal{A}_t} \mathbb{E}\left[R\left(Y_{t,a}\right) \mid \theta^* = \hat{\theta}_t\right]$

Thompson Sampling review - Linear-Gaussian Bandit

- Action set $\mathcal{A} \subset \mathbb{R}^d$
- Model parameter $\theta^* \in \mathbb{R}^d$ distributed as $N(\mu_0, \boldsymbol{\Sigma}_0)$
- If $a \in \mathcal{A}$ is selected, observe outcome $Y_{t,a} = a^\top \theta^* + w_t$, where $w_t \sim N\left(0, \sigma_w^2\right)$.
- Reward $R(Y_{t,a}) = Y_{t,a}$
- By conjugacy, the posterior distribution of θ^* is also normal.

Algorithm 1 Linear–Gaussian Thompson Sampling

1: **Sample Model:**
   $\hat{\theta}_t \sim N(\mu_{t-1}, \Sigma_{t-1})$
2: **Select Action:**
   $A_t \in \arg\max_{a \in \mathcal{A}} \langle a, \hat{\theta}_t \rangle$
3: **Update Statistics:**
   $\mu_t \leftarrow \mathbb{E}[\theta^* | \mathcal{F}_t]$
   $\Sigma_t \leftarrow \mathbb{E}[(\theta^* - \mu_t)(\theta^* - \mu_t)^\top | \mathcal{F}_t]$
4: **Increment** t **and Goto Step 1**

Thompson Sampling review - Generic linear bandit

- General prior and noise distribution

Algorithm Linear Thompson Sampling

for $t = 1, 2, \dots,$ **do**
    Observe action set $\mathcal{A}_t$
    Sample $\hat{\theta}_t \sim \mathbb{P}(\theta^* \mid \mathcal{F}_t)$
    $A_t \leftarrow \arg\max_{a \in \mathcal{A}_t} \langle a, \hat{\theta} \rangle$
    Observation Y_{t, A_t}
end for

literature review

TODO: literature review to be shortened and crystallized!

- (Rusmevichientong, and Tsitsiklis, 2010) showed that in this setting, the Bayesian regret of any strategy is lower bounded by $\Omega(\sqrt{T})$, where T stands for the number of time steps, or equivalently, the number of actions taken.
- (Russo, Van Roy, 2014) proved that the Bayesian regret of Thompson sampling in the Gaussian linear bandit setting was no more than $O(\sqrt{T}\log(T))$, which put Thompson sampling within $O(\log(T))$ of optimality.
- On the other hand, (Russo, Van Roy, 2016) considered Thompson sampling for the general multi armed bandit, and they derived a tighter bound of order $O(\sqrt{T})$ in the Gaussian linear bandit setting by assuming that the set of actions the algorithm can choose from is finite, while the earlier work (Russo, Van Roy, 2016) allowed for a compact action set, i.e. uncountably many actions.

Literature review

- (Russo, Van Roy, 2014) analyze Thompson sampling in the case of the Gaussian linear bandit. They reformulate the Bayesian regret of Thompson sampling by using upper confidence bounds, and show that it is bounded by $O(\sqrt{T}\log(T))$ in this setting.
- On the other hand, (Russo, Van Roy, 2016) considers Thompson sampling for general multi armed bandits, and uses the concept called the information ratio to derive the performance bounds. Their results when specialized to the Gaussian linear bandit yield a regret bound of order $O(\sqrt{\log(|U|)dT})$. As a result, this bound strengthenes the one given by [6] in the case when the action set, U, is finite.
- To extend this result to the action sets with infinitely many elements, (Dong and Van Roy, 2018) propose the use of rate distortion besides information ratio. They prove a regret bound of order $O(d\sqrt{T\log(T)})$ for Thompson sampling in the linear bandit setting through the discretization of θ, but as a result they require θ to have a bounded support-eventually excluding the Gaussian linear bandit setup.

literature review

- In the setting of the linear bandit, (Dong and Van Roy, 2018) showed that Thompson sampling achieves a bound of order $O(\sqrt{T \log(T)})$ even if the action set contains infinitely many actions. However, they restricted the main system variable to have bounded support, which excludes the Gaussian linear bandit setting.
- (Kalkanli and Özgür, 2020) show that the Bayesian regret of Thompson sampling is bounded by $O(\sqrt{T \log(T)})$ in the Gaussian linear bandit setting with a compact (possibly infinite) action set.

Literature review

- **This work** is to apply the aforementioned randomized elliptical potential lemma in combination with the proof techniques in (Russo and Van Roy, 2016, Dong and Van Roy, 2018; Kalkanli and Özgür, 2020) to prove an $\mathcal{O}(d\sqrt{T \log T})$ bound for the Bayesian regret of the well-known linear Thompson sampling (LinTS). The proof idea was originated from Proposition 5 in (Russo and Van Roy, 2016) and was later generalized by (Kalkanli and Özgür, 2020), which avoids constructing confidence sets around $\theta^\star - \widehat{\theta}_t$ that introduce an additional $\sqrt{\log T}$ term.
- This result is proved under mild distributional assumptions and allows the action sets to change at each round. This result extends the regret bound of Dong and Van Roy (2018) as they require action sets to be fixed (which excludes for example the k -armed contextual bandit problem). Our result also generalizes the bound of Kalkanl and Özgür (2020) by relaxing the Gaussian assumption.

Classical Elliptical Potential Lemma

To state the elliptical potential lemma, let $A_1, A_2, \cdots$ be a sequence of vectors in $\mathbb{R}^d$ that satisfy $\|A_t\|_2 \le 1$ for all $t \ge 1$. For a fixed constant λ with $\lambda \ge 1$, define the sequence of covariance matrices $\{\boldsymbol{\Sigma}_t\}_{t\ge 0}$ as follows:

$$\boldsymbol{\Sigma}_1^{-1} := \lambda \boldsymbol{I}_d \quad , \quad \boldsymbol{\Sigma}_t^{-1} := \lambda \boldsymbol{I}_d + \sum_{\tau=1}^{t-1} A_\tau A_\tau^\top$$

The elliptical potential lemma then asserts that

$$\sum_{t=1}^{T} A_t^\top \boldsymbol{\Sigma}_t A_t \le 2 \log \frac{\det \boldsymbol{\Sigma}_1}{\det \boldsymbol{\Sigma}_{T+1}} \le 2d \log\left(1 + \frac{T}{\lambda d}\right)$$

Randomized Elliptical Potential - Notations and Assumptions

- Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}$ be an increasing sequence of σ-algebras that are meant to encode the information available up to time t.
- Conditional expectation $\mathbb{E}_t\left[\cdot\right] \equiv \mathbb{E}\left[\cdot \mid \mathcal{F}_t\right]$ and conditional variance $\mathbb{V}_t\left[\cdot\right] \equiv \mathbb{V}\left[\cdot \mid \mathcal{F}_t\right]$
- Let $\theta^\star : \Omega \to \mathbb{R}^d$ be the true parameters vector and assume that $\|\theta^\star\|_2 \leq 1$ a.s.
- A_t is $\mathcal{F}_{t+1}$-measurable and $\|A_t\|_2 \leq 1$ a.s.
- Y_t is $\mathcal{F}_{t+1}$-measurable and for all $t \geq 1$:

$$\mathbb{E}_t\left[Y_{t,A_t} \mid A_t, \theta^\star\right] = \langle \theta^\star, A_t \rangle \quad \text{and} \quad \mathbb{V}_t\left[Y_{t,A_t} \mid A_t, \theta^\star\right] \leq \sigma^2 \quad a.s.$$

Posterior covairance

- Posterior covariance matrix of $\theta^\star$ at time t,

$$\mathbf{\Gamma}_t := \mathrm{Var}\left(\theta^\star \mid \mathcal{F}_t\right)$$

$\spadesuit$: $\mathbf{\Gamma}_t$ is a stochastic PSD matrix in $\mathbb{R}^{d\times d}$ that is $\mathcal{F}_t$-adapted.

- **Remark**: it is not true in general that $\mathbf{\Gamma}_{t+1} \preccurlyeq \mathbf{\Gamma}_t$.

Posterior covairance does not reduce in general

- To see this, let $d = 1$ and $\theta^\star \in \{0, 1/4, 3/4\}$
- Prior on $\theta^\star$: $\mathbb{P}(\theta^\star = 1/4) = 3p$ and $\mathbb{P}(\theta^\star = 3/4) = p$ for some small $p > 0$.
- Define $A_t := 1$ for all $t \geq 1$ and assume $Y_{t,A_t} \sim \text{Bernoulli}(\theta^\star \cdot A_t)$.
- $\boldsymbol{\Gamma}_1 = \text{Var}(\theta^\star)$ can be arbitrarily small by choosing a sufficiently small $p > 0$.
- whenever $Y_{1,1} = 1$, $\theta^\star \mid \mathcal{F}_2 \sim \text{unif}\{1/4, 3/4\}$ which gives us $\boldsymbol{\Gamma}_2 = 1/16 > \boldsymbol{\Gamma}_1$.
- This can be shown by noting that $\mathbb{P}(\theta^\star = 0 \mid Y_1 = 1) = 0$ and

$$\frac{\mathbb{P}\left(\theta^\star = \frac{1}{4} \mid Y_1 = 1\right)}{\mathbb{P}\left(\theta^\star = \frac{3}{4} \mid Y_1 = 1\right)} = \frac{\mathbb{P}\left(\theta^\star = \frac{1}{4}\right) \cdot \mathbb{P}\left(Y_1 = 1 \mid \theta^\star = \frac{1}{4}\right)}{\mathbb{P}\left(\theta^\star = \frac{3}{4}\right) \cdot \mathbb{P}\left(Y_1 = 1 \mid \theta^\star = \frac{3}{4}\right)} = \frac{3p \cdot \frac{1}{4}}{p \cdot \frac{3}{4}} = 1$$

Posterior covairance reduces in expectation

- Apply the law of total variance to get

$$\begin{aligned}\mathbb{E}\left[\mathbf{\Gamma}_{t+1} \mid \mathcal{F}_t\right] &\preccurlyeq \mathbb{E}\left[\mathbf{\Gamma}_{t+1} \mid \mathcal{F}_t\right] + \operatorname{Var}\left(\mathbb{E}\left[\theta^\star \mid \mathcal{F}_{t+1}\right] \mid \mathcal{F}_t\right) \quad (1)\\ &= \operatorname{Var}\left(\theta^\star \mid \mathcal{F}_t\right) \quad (2)\\ &= \mathbf{\Gamma}_t \quad (3)\end{aligned}$$

- Law of total variance:

$$\mathbb{V}\left[X\right] = \mathbb{E}\left[\mathbb{V}\left[X \mid Y\right]\right] + \mathbb{V}\left[\mathbb{E}\left[X \mid Y\right]\right]$$

- This inequality is not useful as it does not tell us how much the expected variance decreases at each round.

Quick verification of Law of Total Variance

- $\mathbb{V}[X \mid Y = y] = \mathbb{E}\left[X^2 \mid Y = y\right] - (\mathbb{E}[X \mid Y = y])^2$ for all y
- $\mathbb{V}[X \mid Y] = \mathbb{E}\left[X^2 \mid Y\right] - (\mathbb{E}[X \mid Y])^2$
- $\mathbb{E}[\mathbb{V}[X \mid Y]] = \mathbb{E}\left[\mathbb{E}\left[X^2 \mid Y\right]\right] - \mathbb{E}\left[(\mathbb{E}[X \mid Y])^2\right]$
- $\mathbb{V}[\mathbb{E}[X \mid Y]] = \mathbb{E}\left[(\mathbb{E}[X \mid Y])^2\right] - (\mathbb{E}[\mathbb{E}[X \mid Y]])^2$
- $\mathbb{E}[\mathbb{V}[X \mid Y]] + \mathbb{V}[\mathbb{E}[X \mid Y]] = \mathbb{E}\left[X^2\right] - (\mathbb{E}[X])^2$
- $\mathbb{V}[X] = \mathbb{E}\left[X^2\right] - (\mathbb{E}[X])^2$

Variance reduction on Posterior Covairance

Lemma 1 (Variance reduction).

Whenever the above-mentioned assumptions hold, for all $t \geq 1$, we have

$$\mathbb{E}\left[\boldsymbol{\Gamma}_{t+1} \mid \mathcal{F}_t, A_t\right] \preccurlyeq \boldsymbol{\Gamma}_t - \frac{\boldsymbol{\Gamma}_t^\top A_t A_t^\top \boldsymbol{\Gamma}_t}{\sigma^2 + A_t^\top \boldsymbol{\Gamma}_t A_t} \quad a.s.$$

- **Remark**: Lemma 1 demonstrates that the posterior covariance decays in expectation. As we discussed before, this does not necessarily hold for $\boldsymbol{\Gamma}_t$ almost surely.

Proof of variance reduction

- First, we prove the claim for $\theta^\star$ with $\mathbb{E}_t[\theta^\star] = 0$.

- It suffices to prove that for any $V \in \mathbb{R}^d$,

$$\begin{aligned} V^\top \mathbb{E}_t\left[\boldsymbol{\Gamma}_{t+1} \mid A_t\right] V &= \mathbb{E}_t\left[\mathrm{Var}_{t+1}\left(\langle \theta^\star, V\rangle\right) \mid A_t\right] \\ &\le V^\top \boldsymbol{\Gamma}_t V - \frac{\left(A_t^\top \boldsymbol{\Gamma}_t V\right)^2}{\sigma^2 + A_t^\top \boldsymbol{\Gamma}_t A_t} \end{aligned}$$

Proof of variance reduction

- For any fixed $V \in \mathbb{R}^d$. Denoting by $\mathcal{F}_t^A$ the set of $\mathcal{F}_t$-adaptable random variables,

$$\begin{aligned}
\mathbb{E}_t\left[\mathrm{Var}_{t+1}\left(\langle\theta^\star, V\rangle\right) \mid A_t\right] &= \mathbb{E}_t\left[\inf_{Z\in\mathcal{F}_{t+1}^A} \mathbb{E}_{t+1}\left[\left(\langle\theta^\star, V\rangle - Z\right)^2\right] \mid A_t\right] \\
&\leq \mathbb{E}_t\left[\inf_{a\in\mathbb{R}} \mathbb{E}_{t+1}\left[\left(\langle\theta^\star, V\rangle - aY_t\right)^2\right] \mid A_t\right] \\
&\leq \inf_{a\in\mathbb{R}} \mathbb{E}_t\left[\mathbb{E}_{t+1}\left[\left(\langle\theta^\star, V\rangle - aY_t\right)^2\right] \mid A_t\right]
\end{aligned}$$

- $\mathbb{E}_t\left[\mathbb{E}_{t+1}\left[\left(\langle\theta^\star, V\rangle - aY_t\right)^2\right] \mid A_t\right] = \mathbb{E}_t\left[\left(\langle\theta^\star, V\rangle - aY_t\right)^2 \mid A_t\right]$

$$\inf_{a\in\mathbb{R}}\left(\mathbb{E}_t\left[\left(\langle\theta^\star, V\rangle - aY_t\right)^2 \mid A_t\right]\right) = \mathbb{E}_t\left[\langle\theta^\star, V\rangle^2 \mid A_t\right] - \frac{\mathbb{E}_t\left[\langle\theta^\star, V\rangle Y_t \mid A_t\right]^2}{\mathbb{E}_t\left[Y_t^2 \mid A_t\right]}$$

Proof of variance reduction

- For the first term, we have

$$\mathbb{E}_t\left[\langle\theta^\star, V\rangle^2 \mid A_t\right] = \mathbb{E}_t\left[V^\top \theta^{\star\top}\theta^\star V\right] = V^\top \mathbf{\Gamma}_t V$$

- For the second expectation, the numerator can also be computed in the following way

$$\begin{aligned}
\mathbb{E}_t\left[\langle\theta^\star, V\rangle Y_t \mid A_t\right] &= \mathbb{E}_t\left[\mathbb{E}_t\left[\langle\theta^\star, V\rangle Y_t \mid A_t, \theta^\star\right] \mid A_t\right] \\
&= \mathbb{E}_t\left[\langle\theta^\star, V\rangle \cdot \mathbb{E}_t\left[Y_t \mid A_t, \theta^\star\right] \mid A_t\right] \\
&= \mathbb{E}_t\left[\langle\theta^\star, V\rangle \langle\theta^\star, A_t\rangle \mid A_t\right] \\
&= \mathbb{E}_t\left[A_t^\top \theta^\star \theta^{\star\top} V \mid A_t\right] \\
&= A_t^\top \mathbb{E}_t\left[\theta^\star \theta^{\star\top} \mid A_t\right] V \\
&= A_t^\top \mathbb{E}_t\left[\theta^\star \theta^{\star\top}\right] V = A_t^\top \mathbf{\Gamma}_t V.
\end{aligned}$$

Proof of variance reduction

- Finally, for the denominator of the second expectation we have

$$\begin{aligned}\mathbb{E}_t\left[Y_t^2 \mid A_t\right] &= \mathbb{E}_t\left[\mathrm{Var}_t\left(Y_t \mid A_t, \theta^\star\right) + \mathbb{E}_t\left[Y_t \mid A_t, \theta^\star\right]^2 \mid A_t\right] \\ &\leq \sigma^2 + \mathbb{E}_t\left[\langle \theta^\star, A_t\rangle^2 \mid A_t\right] \\ &= \sigma^2 + A_t^\top \mathbf{\Gamma}_t A_t\end{aligned}$$

Proof of variance reduction

- Finally, whenever $\mathbb{E}_t[\theta^\star] \neq 0$,
- Define $\mu^\star := \theta^\star - \mathbb{E}_t[\theta^\star \mid A_t]$ and $Z_t := Y_t - \mathbb{E}_t[Y_t \mid A_t] = Y_t - \langle \mathbb{E}_t[\theta^*], A_t \rangle$.
- Note that $\mathrm{Var}_t(\mu^\star) = \mathrm{Var}_t(\theta^\star) = \mathbf{\Gamma}_t$
- $\mathbb{E}_t[Z_t \mid A_t, \mu^\star] = \mathbb{E}_t[Z_t \mid A_t, \theta^\star] = \langle \theta^*, A_t \rangle - \langle \mathbb{E}_t[\theta^*], A_t \rangle = \langle \mu^\star, A_t \rangle$
- $\mathrm{Var}_t(Z_t \mid A_t, \mu^\star) = \mathrm{Var}_t(Z_t \mid A_t, \theta^\star) = \mathrm{Var}_t(Y_t \mid A_t, \theta^\star) \leq \sigma^2$.
- We can apply the result (for the case $\mathbb{E}_t[\theta^\star] = 0$) to $\mu^\star$ and Z_t and get

$$\mathbb{E}_t[\mathbf{\Gamma}_{t+1} \mid A_t] = \mathbb{E}_t[\mathrm{Var}_{t+1}(\mu^\star) \mid A_t] \preccurlyeq \mathbf{\Gamma}_t - \frac{\mathbf{\Gamma}_t^\top A_t A_t^\top \mathbf{\Gamma}_t}{\sigma^2 + A_t^\top \mathbf{\Gamma}_t A_t}$$

Key Lemma

Lemma 2.
For $x > 0$ and positive semi-definite matrix $\boldsymbol{\Sigma}$, define $f(\boldsymbol{\Sigma}, x) = \log\det(\boldsymbol{I} + x\boldsymbol{\Sigma})$. Then, $f(\cdot, \cdot)$ satisfies the following properties:

(1) *For any fixed $x > 0, f(\cdot, x)$ is a concave function on the positive semi-definite cone.*

(2) *If $\boldsymbol{\Sigma}$ is an invertible and positive semidefinite matrix then $f(\boldsymbol{\Sigma}, x)$ satisfies the variational representation*

$$f(\boldsymbol{\Sigma}, x) = \log\det\left(\boldsymbol{\Sigma}^{\frac{1}{2}}\left(\boldsymbol{\Sigma}^{-1} + x\boldsymbol{I}\right)\boldsymbol{\Sigma}^{\frac{1}{2}}\right) = \sup_{\Lambda \preccurlyeq x\boldsymbol{I}} \log\det\left(\boldsymbol{\Sigma}^{\frac{1}{2}}\left(\boldsymbol{\Sigma}^{-1} + \Lambda\right)\boldsymbol{\Sigma}^{\frac{1}{2}}\right)$$

Key Lemma

Lemma 3.

For $x > 0$ and positive semi-definite matrix $\boldsymbol{\Sigma}$, define $f(\boldsymbol{\Sigma}, x) = \log\det(\boldsymbol{I} + x\boldsymbol{\Sigma})$. Then, $f(\cdot,\cdot)$ satisfies the following properties:

(3) *For any vector $V \in \mathbb{R}^d$, we have*

$$\log\left(1 + V^\top \boldsymbol{\Sigma} V\right) + f\left(\boldsymbol{\Sigma}', x\right) \leq f\left(\boldsymbol{\Sigma}, x + V^\top V\right)$$

where

$$\boldsymbol{\Sigma}' := \boldsymbol{\Sigma} - \frac{\boldsymbol{\Sigma} V V^\top \boldsymbol{\Sigma}}{1 + V^\top \boldsymbol{\Sigma} V} = \boldsymbol{\Sigma}^{\frac{1}{2}} \left(\boldsymbol{I} - \frac{\boldsymbol{\Sigma}^{\frac{1}{2}} V V^\top \boldsymbol{\Sigma}^{\frac{1}{2}}}{1 + V^\top \boldsymbol{\Sigma} V}\right) \boldsymbol{\Sigma}^{\frac{1}{2}}$$

and equivalently $\boldsymbol{\Sigma}'^{-1} = \boldsymbol{\Sigma}^{-1} + VV^\top$, using Sherman-Morrison formula.

Proof of Key Lemma

(1) The concavity of $f(\cdot, x)$ follows from the fact that $\log\det(\cdot)$ is concave over the positive definite cone (see, e.g., (Boyd et al., 2004, page 74)) and $f(\cdot, x)$ is obtained by composing log det $(\cdot)$ with a linear function of $\boldsymbol{\Sigma}$.

(2) The variational representation can be obtained by noting that $\log\det(\cdot)$ is increasing with respect to the positive semi-definite order '$\preccurlyeq$'

Proof of Key Lemma

(3) First, assume that $\boldsymbol{\Sigma}$ is invertible.

- In this case, we have $\boldsymbol{\Sigma}'^{-1} = \boldsymbol{\Sigma}^{-1} + VV^\top$, using Sherman-Morrison formula.

$$
\begin{aligned}
f\left(\boldsymbol{\Sigma}, x + V^\top V\right) &= \sup_{\Lambda \preccurlyeq (x+V^\top V)\boldsymbol{I}} \log\det\left(\boldsymbol{\Sigma}^{\frac{1}{2}}\left(\boldsymbol{\Sigma}^{-1} + \Lambda\right)\boldsymbol{\Sigma}^{\frac{1}{2}}\right) \\
&\overset{(a)}{\geq} \sup_{\Lambda' \preccurlyeq x\boldsymbol{I}} \log\det\left(\boldsymbol{\Sigma}^{\frac{1}{2}}\left(\boldsymbol{\Sigma}^{-1} + VV^\top + \Lambda'\right)\boldsymbol{\Sigma}^{\frac{1}{2}}\right) \\
&= \sup_{\Lambda' \preccurlyeq x\boldsymbol{I}} \log\det\left(\boldsymbol{\Sigma}^{\frac{1}{2}}\left(\boldsymbol{\Sigma}'^{-1} + \Lambda'\right)\boldsymbol{\Sigma}^{\frac{1}{2}}\right)
\end{aligned}
$$

(a) by $\Lambda' + VV^\top \succeq V^\top V\boldsymbol{I}$

Proof of Key Lemma

$$
\begin{aligned}
&\sup_{\Lambda' \preccurlyeq x\boldsymbol{I}} \log\det\left(\boldsymbol{\Sigma}^{\frac{1}{2}}\left(\boldsymbol{\Sigma}'^{-1}+\Lambda'\right)\boldsymbol{\Sigma}^{\frac{1}{2}}\right) \\
&= \log\det(\boldsymbol{\Sigma}) + \log\det\left(\boldsymbol{\Sigma}'^{-1}+x\boldsymbol{I}\right) \\
&= \log\det\left(\boldsymbol{\Sigma}'\right) - \log\det\left(\boldsymbol{I} - \frac{\boldsymbol{\Sigma}^{\frac{1}{2}}VV^\top\boldsymbol{\Sigma}^{\frac{1}{2}}}{1+V^\top\boldsymbol{\Sigma}V}\right) + \log\det\left(\boldsymbol{\Sigma}'^{-1}+x\boldsymbol{I}\right) \\
&\overset{(b)}{=} \log\det\left(\boldsymbol{\Sigma}'\right) - \log\left(1-\frac{V^\top\boldsymbol{\Sigma}V}{1+V^\top\boldsymbol{\Sigma}V}\right) + \log\det\left(\boldsymbol{\Sigma}'^{-1}+x\boldsymbol{I}\right) \quad \left(\det\left(\boldsymbol{I}+ZZ^\top\right)=1+Z^\top Z\right) \\
&= \log\left(1+V^\top\boldsymbol{\Sigma}V\right) + \underbrace{\log\det\left(\boldsymbol{\Sigma}'^{\frac{1}{2}}\left(\boldsymbol{\Sigma}'^{-1}+x\boldsymbol{I}\right)\boldsymbol{\Sigma}'^{\frac{1}{2}}\right)}_{f(\boldsymbol{\Sigma}',x)}
\end{aligned}
$$

Proof of Key Lemma

- Remain to prove the argument for the case of non-invertible matrix $\boldsymbol{\Sigma}$.
- In this case, for $\epsilon > 0$, we define $\boldsymbol{\Sigma}_\epsilon = \boldsymbol{\Sigma} + \epsilon \boldsymbol{I}$ and

$$\boldsymbol{\Sigma}'_\epsilon := \boldsymbol{\Sigma}_\epsilon - \frac{\boldsymbol{\Sigma}_\epsilon V V^\top \boldsymbol{\Sigma}_\epsilon}{1 + V^\top \boldsymbol{\Sigma}_\epsilon V}.$$

- Clearly, $\boldsymbol{\Sigma}_\epsilon$ is invertible. Therefore, we can apply the previous results to $\boldsymbol{\Sigma}_\epsilon$ to obtain

$$\log\left(1 + V^\top \boldsymbol{\Sigma}_\epsilon V\right) + f\left(\boldsymbol{\Sigma}'_\epsilon, x\right) \leq f\left(\boldsymbol{\Sigma}_\epsilon, x + V^\top V\right)$$

The claim then follows the continuity of the above expressions with respect to ϵ on $[0, \infty]$

Randomized Elliptical Potential

Theorem 4 (Randomized Elliptical Potential).

Under the above assumptions, it holds that

$$\mathbb{E}\left[\sum_{t=1}^{T} A_t^\top \mathbf{\Gamma}_t A_t\right] \leq 2 \log\det\left(\boldsymbol{I} + T\mathbf{\Gamma}_1\right)$$

Proof of Randomized Elliptical Potential

- First, notice that since $\|\theta^\star\|_2 \le 1$ and $\|A_t\| \le 1$ almost surely, we have $A_t^\top \mathbf{\Gamma}_t A_t \le 1$ for all $t \in [T]$ almost surely. Next, the fact that $x \le 2\log(1+x)$ for all $x \in [0,1]$ implies that

$$A_t^\top \mathbf{\Gamma}_t A_t \le 2\log\left(1 + A_t^\top \mathbf{\Gamma}_t A_t\right) \tag{4}$$

- We now prove this lemma inductively. For $T = 1$, it suffices to note that

$$\boldsymbol{I} + \mathbf{\Gamma}_1^{\frac{1}{2}} A_1 A_1^\top \mathbf{\Gamma}_1^{\frac{1}{2}} \preccurlyeq \boldsymbol{I} + \mathbf{\Gamma}_1^{\frac{1}{2}} \boldsymbol{I} \mathbf{\Gamma}_1^{\frac{1}{2}} \preccurlyeq \boldsymbol{I} + \mathbf{\Gamma}_1$$

- For $T > 1$, we can use the induction hypothesis for $T - 1$ and get that

$$\mathbb{E}\left[\sum_{t=2}^{T} A_t^\top \mathbf{\Gamma}_t A_t \mid A_1, Y_1\right] \le 2\log\det\left(\boldsymbol{I} + (T-1)\mathbf{\Gamma}_2\right) \quad a.s.$$

Proof of Randomized Elliptical Potential

- Using the concavity of $\log\det(\cdot)$, it follows from Jensen's inequality that
-

$$
\begin{aligned}
\mathbb{E}\left[\sum_{t=2}^{T} A_t^\top \mathbf{\Gamma}_t A_t \mid A_1\right] &= \mathbb{E}\left[\mathbb{E}\left[\sum_{t=2}^{T} A_t^\top \mathbf{\Gamma}_t A_t \mid A_1, Y_t\right] \mid A_1\right] \\
&\le \mathbb{E}\left[\log\det\left(\boldsymbol{I} + (T-1)\mathbf{\Gamma}_2\right) \mid A_1\right] \\
&\le 2\log\det\left(\boldsymbol{I} + (T-1)\mathbb{E}\left[\mathbf{\Gamma}_2 \mid A_1\right]\right) \\
&\le 2\log\det\left(\boldsymbol{I} + (T-1)\left(\mathbf{\Gamma}_1 - \frac{\mathbf{\Gamma}_1^\top A_1 A_1^\top \mathbf{\Gamma}_1}{1 + A_1^\top \mathbf{\Gamma}_1 A_1}\right)\right) \\
&\le 2f\left(\mathbf{\Gamma}_1', T-1\right)
\end{aligned}
$$

where $\mathbf{\Gamma}_1' := \mathbf{\Gamma}_1 - \frac{\mathbf{\Gamma}_1^\top A_1 A_1^\top \mathbf{\Gamma}_1}{1+A_1^\top \mathbf{\Gamma}_1 A_1}$.

Proof of Randomized Elliptical Potential

Finally, we apply Lemma 3 and equation 4 to get that

$$\begin{aligned}
\mathbb{E}\left[\sum_{t=1}^{T} A_t^\top \mathbf{\Gamma}_t A_t\right] &\leq 2\mathbb{E}\left[\log\left(1 + A_1^\top \mathbf{\Gamma}_1 A_1\right) + f\left(\mathbf{\Gamma}_1', T-1\right)\right] \\
&\leq 2\mathbb{E}\left[f\left(\mathbf{\Gamma}_1, T\right)\right] \quad \text{(by the fact } \|A_t\| \leq 1) \\
&= 2\log\det\left(1 + T\mathbf{\Gamma}_1\right)
\end{aligned}$$

Proposition 2 in (Kalkanli and Özgür, ISIT 20')

Lemma 5 (Proposition 2 in (Kalkanli and Özgür, ISIT 20')).
Let X_1 and X_2 be arbitrary i.i.d., $\mathbb{R}^m$ valued random variables and f_1, f_2 measurable maps such that $f_1, f_2 : \mathbb{R}^m \to \mathbb{R}^d$ with $\mathbb{E}\left[\|f_1(X_1)\|_2^2\right], \mathbb{E}\left[\|f_2(X_1)\|_2^2\right] < \infty$, then

$$\left|\mathbb{E}\left[f_1(X_1)^T f_2(X_1)\right]\right| \leq \sqrt{d\mathbb{E}\left[\left(f_1(X_1)^T f_2(X_2)\right)^2\right]}$$

- **Remark**: Proposition 2 relates the inner product of two random vectors, possibly correlated ones, to a function of independent random vectors with the same marginal distributions as the previous ones.

Special case of the proposition

- When $d = 1$, Proposition recovers the Cauchy-Schwarz inequality for random variables.

$$X_1 = (Y_1, Y_2),$$
$$f_1(y_1, y_2) = y_1, \quad f_2(y_1, y_2) = y_2 \text{ for any } (y_1, y_2) \in \mathbb{R}^2.$$

- Also let X_2 be an i.i.d. copy of X_1. Then with this setup, the proposition implies

$$\begin{aligned} |\mathbb{E}[Y_1 Y_2]| &= |\mathbb{E}[f_1(X_1) f_2(X_1)]| && (5) \\ &\leq \sqrt{\mathbb{E}\left[f_1(X_1)^2 f_2(X_2)^2\right]} && (6) \\ &= \sqrt{\mathbb{E}\left[f_1(X_1)^2\right] \mathbb{E}\left[f_2(X_2)^2\right]} && (7) \\ &= \sqrt{\mathbb{E}[Y_1^2]\,\mathbb{E}[Y_2^2]} && (8) \end{aligned}$$

Proof of the proposition

- Let $\{e_i\}_{i=1}^d$ be a set of eigenvectors of $\mathbb{E}\left[f_1(X_1) f_1(X_1)^T\right]$ and an orthonormal basis for $\mathbb{R}^d$, i.e. $I_{d\times d} = \sum_{i=1}^d e_i e_i^T$

$$\mathbb{E}\left[\left(f_1(X_1)^T f_2(X_2)\right)^2\right] = \mathbb{E}\left[\left(\sum_{n=1}^d \left(f_1(X_1)^T e_n\right)\left(f_2(X_2)^T e_n\right)\right)^2\right]$$

$$= \sum_{n_1,n_2} \left(\mathbb{E}\left[\left(f_1(X_1)^T e_{n_1}\right)\left(f_2(X_2)^T e_{n_1}\right)\left(f_1(X_1)^T e_{n_2}\right)\left(f_2(X_2)^T e_{n_2}\right)\right]\right) \tag{9}$$

$$= \sum_{n_1,n_2} \left(\mathbb{E}\left[\left(f_1(X_1)^T e_{n_1}\right)\left(f_1(X_1)^T e_{n_2}\right)\right]\mathbb{E}\left[\left(f_2(X_2)^T e_{n_1}\right)\left(f_2(X_2)^T e_{n_2}\right)\right]\right) \tag{10}$$

Proof of propostion

- Since $\{e_i\}_{i=1}^{d}$ are orthogonal eigenvectors of $\mathbb{E}\left[f_1\left(X_1\right) f_1\left(X_1\right)^T\right]$, we have

$$\mathbb{E}\left[\left(f_1\left(X_1\right)^T e_{n_1}\right)\left(f_1\left(X_1\right)^T e_{n_2}\right)\right] = e_{n_1}^T \mathbb{E}\left[f_1\left(X_1\right) f_1\left(X_1\right)^T\right] e_{n_2} = 0$$

  for any $n_1 \neq n_2$.
- Then we can simplify equation 10 to:

$$\mathbb{E}\left[\left(f_1\left(X_1\right)^T f_2\left(X_2\right)\right)^2\right] = \sum_{n=1}^{d}\left(\mathbb{E}\left[\left(f_1\left(X_1\right)^T e_n\right)^2\right] \mathbb{E}\left[\left(f_2\left(X_1\right)^T e_n\right)^2\right]\right)$$

Proof of the proposition

- Consequently, we have

$$
\begin{aligned}
\left(\frac{1}{d}\mathbb{E}\left[f_1\left(X_1\right)^T f_2\left(X_1\right)\right]\right)^2 &= \left(\frac{1}{d}\sum_{n=1}^{d}\mathbb{E}\left[\left(f_1\left(X_1\right)^T e_n\right)\left(f_2\left(X_1\right)^T e_n\right)\right]\right)^2 \\
&\leq \frac{1}{d}\sum_{n=1}^{d}\left(\mathbb{E}\left[\left(f_1\left(X_1\right)^T e_n\right)\left(f_2\left(X_1\right)^T e_n\right)\right]^2\right) \quad \text{Jensen} \\
&\leq \frac{1}{d}\sum_{n=1}^{d}\left(\mathbb{E}\left[\left(f_1\left(X_1\right)^T e_n\right)^2\right]\mathbb{E}\left[\left(f_2\left(X_1\right)^T e_n\right)^2\right]\right) \quad \text{CS} \\
&= \frac{1}{d}\mathbb{E}\left[\left(f_1\left(X_1\right)^T f_2\left(X_2\right)\right)^2\right]
\end{aligned}
$$

Alternative statement of the propostion

Lemma 6.

Let X, Z be two random vectors in $\mathbb{R}^d$. Then, we have

$$\mathbb{E}\left[X^\top Z\right]^2 \leq d\,\mathrm{Tr}\left(\mathbb{E}\left[XX^\top\right]\mathbb{E}\left[ZZ^\top\right]\right)$$

- Proof by the proposition 2: let $X_1 = (X, Z)$ and X_2 is a i.i.d. copy of X_1. Let $f_1(x,z) = x, f_2(x,z) = z$.

$$\begin{aligned}\mathbb{E}\left[f_1\left(X_1\right)^T f_2\left(X_1\right)\right]^2 &\leq d\mathbb{E}\left[\left(f_1\left(X_1\right)^T f_2\left(X_2\right)\right)^2\right] \\ &= d\mathbb{E}\left[\mathrm{Tr}\left(f_1(X_1)f_1(X_1)^\top f_2(X_2)f_2(X_2)^\top\right)\right] \\ &= d\,\mathrm{Tr}\left(\mathbb{E}\left[f_1(X_1)f_1(X_1)^\top\right]\mathbb{E}\left[f_2(X_2)f_2(X_2)^\top\right]\right)\end{aligned}$$

Alternative proof of the lemma 6

- First, observe that for any unitary matrix U, if one defines $X' := UX$ and $Z' := UZ$, then

$$\mathbb{E}\left[X^\top Z\right] = \mathbb{E}\left[X'^\top Z'\right]$$

and

$$\begin{aligned}
\operatorname{Tr}\left(\mathbb{E}\left[X'X'^\top\right]\mathbb{E}\left[Z'Z'^\top\right]\right) &= \operatorname{Tr}\left(\mathbb{E}\left[UXX^\top U^\top\right]\mathbb{E}\left[UZZ^\top U^\top\right]\right) \\
&= \operatorname{Tr}\left(U\mathbb{E}\left[XX^\top\right]U^\top U\mathbb{E}\left[ZZ^\top\right]U^\top\right) \\
&= \operatorname{Tr}\left(U^\top U\mathbb{E}\left[XX^\top\right]U^\top U\mathbb{E}\left[ZZ^\top\right]\right) \\
&= \operatorname{Tr}\left(\mathbb{E}\left[XX^\top\right]\mathbb{E}\left[ZZ^\top\right]\right)
\end{aligned}$$

- **Implication**: it suffices to prove the statement for (X', Z') instead of (X, Z).

Alternative proof of the lemma 6

- Choose U so that $\mathbb{E}\left[Z'Z'^\top\right] = U\mathbb{E}\left[ZZ^\top\right]U^\top$ is diagonal. This can be done through the singular value decomposition of $\mathbb{E}\left[ZZ^\top\right]$. Then, notice that

$$\begin{aligned}
d\,\mathrm{Tr}\left(\mathbb{E}\left[X'X'^\top\right]\mathbb{E}\left[Z'Z'^\top\right]\right) &= d\sum_{i=1}^{d}\mathbb{E}\left[X'X'^\top\right]_{ii}\mathbb{E}\left[Z'Z'^\top\right]_{ii} \\
&= d\sum_{i=1}^{d}\mathbb{E}\left[X_i'^2\right]\mathbb{E}\left[Z_i'^2\right] \geq d\sum_{i=1}^{d}\mathbb{E}\left[X_i'Z_i'\right]^2 \quad \text{(CS)} \\
&= \left(\sum_{i=1}^{d}1\right)\left(\sum_{i=1}^{d}\mathbb{E}\left[X_i'Z_i'\right]^2\right) \geq \mathbb{E}\left[\sum_{i=1}^{d}X_i'Z_i'\right]^2 \quad \text{(CS)} \\
&= \mathbb{E}\left[X'^\top Z'\right]^2
\end{aligned}$$

Regret bound

Theorem 7.

Let $\theta^\star$ be such that $\|\theta^\star\|_2 \leq 1$ almost surely and $\mathcal{F}_t$ be the σ -algebra generated by $\left(\mathcal{A}_1, \widetilde{A}_1, Y_1, \mathcal{A}_2, \cdots, \mathcal{A}_t, \widetilde{A}_t\right)$. Furthermore, assume that

$$\mathbb{E}\left[Y_t \mid \mathcal{F}_t, \theta^\star\right] = \left\langle \theta^\star, \widetilde{A}_t \right\rangle \quad \textit{and} \quad \operatorname{Var}\left(Y_t \mid \mathcal{F}_t, \theta^\star\right) \leq \sigma^2$$

almost surely. Then, the following regret bound holds for LinTS (Algorithm 1) when it has access to the true prior and noise distributions:

$$\textit{BayesRegret}\ \left(T, \pi^{\textit{LinTS}}\right) \leq \sqrt{2dT \log \det\left(1 + T\mathbf{\Gamma}_1\right)}$$

Regret Bound - Remark

- Remark 3.1. The assumption that $\|\theta^\star\|_2 \leq 1$ almost surely implies that $\mathbf{\Gamma}_1 \preccurlyeq \boldsymbol{I}$. Hence, we have the trivial bound $\log\det(1 + T\mathbf{\Gamma}_1) \leq d\log(1+T)$ which in turn leads to

$$\text{BayesRegret}\ \left(T, \pi^{\text{LinTS}}\right) \leq d\sqrt{2T\log(1+T)}$$

- Remark 3.2. As shown in Hamidi and Bayati (2020), the assumption that LinTS uses the true posterior distribution for $\theta^\star$ is crucial, as the Bayesian regret of LinTS can grow linearly for $\exp(Cd)$ rounds for some constant $C > 0$
- Remark 3.3. An interesting aspect of this result is that it does not require the noise to be bounded or sub-Gaussian. Having a bounded second moment suffices for Eq. (3.1) to hold.

Proof of the regret bound

- Observe that $(\theta^\star, A_t^\star)$ and $\left(\widetilde{\theta}_t, \widetilde{A}_t\right)$ are exchangeable conditional on $(\mathcal{F}_t, \mathcal{A}_t)$.
- Then, defining $\mu_t := \mathbb{E}\left[\theta^* \mid \mathcal{F}_t\right] = \mathbb{E}\left[\theta^\star \mid \mathcal{F}_t, \mathcal{A}_t\right]$, we have

$$
\begin{aligned}
\text{BayesRegret}\ \left(T, \pi^{\mathrm{LinTS}}\right) &= \sum_{t=1}^{T} \mathbb{E}\left[\langle \theta^\star, A_t^\star \rangle - \left\langle \theta^\star, \widetilde{A}_t \right\rangle\right] \\
&= \sum_{t=1}^{T} \mathbb{E}\left[\langle \theta^\star, A_t^\star \rangle\right] - \mathbb{E}\left[\mathbb{E}\left[\left\langle \theta^\star, \widetilde{A}_t \right\rangle \mid \mathcal{F}_t, \mathcal{A}_t\right]\right] \\
&= \sum_{t=1}^{T} \mathbb{E}\left[\langle \theta^\star, A_t^\star \rangle\right] - \mathbb{E}\left[\left\langle \mathbb{E}\left[\theta^\star \mid \mathcal{F}_t\right], \mathbb{E}\left[\widetilde{A}_t \mid \mathcal{F}_t, \mathcal{A}_t\right] \right\rangle\right] \\
&= \sum_{t=1}^{T} \mathbb{E}\left[\langle \theta^\star, A_t^\star \rangle\right] - \mathbb{E}\left[\langle \mu_t, \mathbb{E}\left[A_t^\star \mid \mathcal{F}_t, \mathcal{A}_t\right] \rangle\right] \\
&= \sum_{t=1}^{T} \mathbb{E}\left[\mathbb{E}\left[\langle \theta^\star, A_t^\star \rangle - \langle \mu_t, A_t^\star \rangle \mid \mathcal{F}_t\right]\right]
\end{aligned}
$$

Proof of the regret bound

- Define $\boldsymbol{\Gamma}_t := \mathbb{E}\left[(\theta^\star - \mu_t)(\theta^\star - \mu_t)^\top \mid \mathcal{F}_t\right]$. Then, it follows from Lemma 3.1 and the independence of $\widetilde{A}_t$ and $\theta^\star$ conditional on $\mathcal{F}_t$ that

$$\text{BayesRegret}\left(T, \pi^{\text{LinTS}}\right) \leq \sqrt{d} \sum_{t=1}^{T} \mathbb{E}\left[\operatorname{Tr}\left(\boldsymbol{\Gamma}_t \cdot \mathbb{E}\left[A_t^\star A_t^{\star\top} \mid \mathcal{F}_t\right]\right)^{\frac{1}{2}}\right]$$

$$\begin{aligned}\operatorname{Tr}\left(\boldsymbol{\Gamma}_t \cdot \mathbb{E}\left[A_t^\star A_t^{\star\top} \mid \mathcal{F}_t\right]\right)^{\frac{1}{2}} &= \operatorname{Tr}\left(\boldsymbol{\Gamma}_t \cdot \mathbb{E}\left[\widetilde{A}_t \widetilde{A}_t^\top \mid \mathcal{F}_t\right]\right)^{\frac{1}{2}} = \operatorname{Tr}\left(\mathbb{E}\left[\boldsymbol{\Gamma}_t \cdot \widetilde{A}_t \widetilde{A}_t^\top \mid \mathcal{F}_t\right]\right)^{\frac{1}{2}} \\ &= \mathbb{E}\left[\operatorname{Tr}\left(\boldsymbol{\Gamma}_t \cdot \widetilde{A}_t \widetilde{A}_t^\top\right) \mid \mathcal{F}_t\right]^{\frac{1}{2}} = \mathbb{E}\left[\widetilde{A}_t^\top \boldsymbol{\Gamma}_t \widetilde{A}_t \mid \mathcal{F}_t\right]^{\frac{1}{2}}\end{aligned}$$

$$\sqrt{d} \sum_{t=1}^{T} \mathbb{E}\left[\mathbb{E}\left[\widetilde{A}_t^\top \boldsymbol{\Gamma}_t \widetilde{A}_t \mid \mathcal{F}_t\right]^{\frac{1}{2}}\right] \leq \sqrt{d} \sum_{t=1}^{T} \mathbb{E}\left[\widetilde{A}_t^\top \boldsymbol{\Gamma}_t \widetilde{A}_t\right]^{\frac{1}{2}} \leq \sqrt{dT}\, \underbrace{\mathbb{E}\left[\sum_{t=1}^{T} \widetilde{A}_t^\top \boldsymbol{\Gamma}_t \widetilde{A}_t\right]^{\frac{1}{2}}}_{\text{Elliptical Potential}}$$