In this section we provide the derivation for the integrated likelihood for the Bayes ZIP (zero inflated poisson) tree model. Now let us define some notation: j will index either all observations or only the observed zero count observations if the upper limit is n_0 then j will index observed zero counts only, if the upper index limit is n then all observations are indexed. Also j' will index the non zero observations. the total number of nonzero observations is denoted n_+, so that $n_+ + n_0 = n$. Finally let $\bar{y}_{i+}$ denote the sample mean of the nonzero observed counts.

$$\begin{aligned}
\Pr(Y|X,\mathcal{T}) &= \prod_{i=1}^{b}\int_0^1\int_0^\infty \prod_{j=1}^{n_i}\left[\mathbb{1}[y_{ij}=0](\phi+(1-\phi)\exp(-\lambda)) + \mathbb{1}[y_{ij}>0]\frac{\exp(-\lambda)\lambda^{y_{ij}}}{y_{ij}!}\right]\pi(\phi_i,\lambda_i)d\lambda_i d\phi_i \\
&= \prod_{i=1}^{b}\int_0^1\int_0^\infty \left(\underbrace{\prod_{j=1}^{n_0}(\phi+(1-\phi)\exp(-\lambda))\pi(\phi_i,\lambda_i)d\lambda_i d\phi_i}_{=(1)} + \underbrace{\prod_{j'=1}^{n_+}\frac{\exp(-\lambda)\lambda^{y_{ij'}}}{y_{ij'}!}\pi(\phi_i,\lambda_i)d\lambda_i d\phi_i}_{=(2)}\right)
\end{aligned}$$

We will first tackle (1), then tackle (2).

$$\begin{aligned}
(1) &= \int_0^1\int_0^\infty \left(\prod_{j=1}^{n_0}(\phi+(1-\phi)\exp(-\lambda))\pi(\phi_i,\lambda_i)d\lambda_i d\phi_i\right. \\
&= \int_0^1\int_0^\infty ((\phi+(1-\phi)\exp(-\lambda))^{n_0}\pi(\phi_i,\lambda_i)d\lambda_i d\phi_i \\
&= \int_0^1\int_0^\infty \sum_{j=1}^{n_0}\binom{n_0}{j}\phi^j(1-\phi)^{n_0-j}\exp(-(n_0-j)\lambda))\pi(\phi_i)\pi(\lambda_i)d\lambda_i d\phi_i
\end{aligned}$$

Now we take $\pi(\phi_i)$ to be a beta(α,β) prior and $\pi(\lambda_i)$ to be a gamma$(\alpha_\lambda,\beta_\lambda)$ prior. This simplifies matters greatly.

$$\begin{aligned}
&\int_0^1\int_0^\infty \sum_{j=1}^{n_0}\binom{n_0}{j}\phi^j(1-\phi)^{n_0-j}\exp(-(n_0-j)\lambda))\frac{\Gamma(\alpha+\beta)\phi^{\alpha-1}(1-\phi)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}\frac{\lambda^{\alpha_\lambda-1}\exp(-\lambda/\beta_\lambda)}{\Gamma(\alpha_\lambda)\beta_\lambda^{\alpha_\lambda}}d\lambda_i d\phi_i \\
&= \sum_{j=1}^{n_0}\binom{n_0}{j}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha_\lambda)\beta_\lambda^{\alpha_\lambda}}\underbrace{\int_0^1\phi^{j+\alpha-1}(1-\phi)^{\beta+n_0-j-1}d\phi_i}_{\text{a beta kernel}}\underbrace{\int_0^\infty \lambda^{\alpha_\lambda-1}\exp(-(n_0-j+\beta_\lambda^{-1})\lambda)d\lambda_i}_{\text{a gamma kernel}} \\
&= \underbrace{\sum_{j=1}^{n_0}\binom{n_0}{j}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha_\lambda)\beta_\lambda^{\alpha_\lambda}}\frac{\Gamma(\alpha+j)\Gamma(\beta+n_0-j)\Gamma(\alpha_\lambda)}{\Gamma(\alpha+\beta+n_0)}(n_0-j+\beta_\lambda^{-1})^{\alpha_\lambda}}_{=(1)}
\end{aligned}$$

Now with the first piece simplified we move on to piece (2).

$$
\begin{aligned}
(2) &= \int_0^1 \int_0^\infty \prod_{j'=1}^{n_+} \frac{\exp{(-\lambda)}\lambda^{y_{ij'}}}{y_{ij'}!} \pi(\phi_i, \lambda_i) d\lambda_i d\phi_i \\
&= \int_0^\infty \prod_{j'=1}^{n_+} \frac{\exp{(-\lambda)}\lambda^{y_{ij'}}}{y_{ij'}!} \pi(\lambda_i) d\lambda_i \\
&= \int_0^\infty \frac{\exp{(-n_+\lambda)}\lambda^{n_+\bar{y}_{i+}}}{\prod_{j'=1}^{n_+} y_{ij'}!} \pi(\lambda_i) d\lambda_i \\
&= \int_0^\infty \frac{\exp{(-n_+\lambda)}\lambda^{n_+\bar{y}_{i+}}}{\prod_{j'=1}^{n_+} y_{ij'}!} \frac{\lambda^{\alpha_\lambda - 1}\exp{(-\lambda/\beta_\lambda)}}{\Gamma(\alpha_\lambda)} d\lambda_i \\
&= \frac{\int_0^\infty \exp{(-(n_+ + \beta_\lambda^{-1})\lambda)}\lambda^{n_+\bar{y}_{i+} + \alpha_\lambda - 1} d\lambda_i}{\Gamma(\alpha_\lambda)\prod_{j'=1}^{n_+} y_{ij'}!} \\
&= \underbrace{\frac{\Gamma(n_+\bar{y}_{i+} + \alpha_\lambda)(n_+ + \beta_\lambda^{-1})^{n_+\bar{y}_{i+} + \alpha_\lambda}}{\Gamma(\alpha_\lambda)\prod_{j'=1}^{n_+} y_{ij'}!}}_{=(2)}
\end{aligned}
$$

And the result is shown.