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On WKB Expansions for Alfvén Waves in the Solar Wind

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We reexamine the WKB expansion for "toroidal" Alfvén waves in the solar wind, as described by equations (9) of Heinemann and Olbert (1980). Our principal conclusions are as follows: (1) The WKB expansion used by Belcher (1971) and Hollweg (1973) is nonuniformly convergent. (2) Using the method of multiple scales (Nayfeh, 1981), we obtain an expansion which is uniform. (3) The uniform expansion takes into account the small modification to the Alfvén wave phase speed due to spatial gradients of the background. (4) Both the uniform and nonuniform expansions reveal that each "normal mode" has both Elsässer variables $\delta z^+ \neq 0$ and $\delta z^- \neq 0$. Thus if δz^- corresponds to the outgoing mode in a homogeneous background, an observation of $\delta z^+ \neq 0$ does not necessarily imply the presence of the inward propagating mode, as is commonly assumed. (5) Even at the Alfvén critical point (where $V = v_A$) we find that $\delta z^+ \neq 0$. Thus incompressible MHD turbulence, which requires both $\delta z^+ \neq 0$ and $\delta z^- \neq 0$, can proceed at the Alfvén critical point (cf. Roberts, 1989). (6) With very few exceptions, the predictions of these calculations do not agree with recent observations (Marsch and Tu, 1990) of the power spectra of δz^+ and δz^- in the solar wind. Thus the evolution of Alfvén waves in the solar wind is governed by dynamics not included in the Heinemann and Olbert equations.

INTRODUCTION

The propagation of Alfvén waves in the solar wind has been of continuing interest since their ubiquitous presence was demonstrated by *Belcher et al.* [1969] and *Belcher and Davis* [1971]. They have usually been described by some variation of the WKB approximation, which is an appropriate procedure for waves having high frequency and short wavelength [*Weinberg*, 1962; *Parker*, 1965; *Belcher*, 1971; *Dewar*, 1970; *Hollweg*, 1973, 1974, 1978; *Völk and Alpers*, 1973; *Barnes and Hollweg*, 1974; *Jacques*, 1977; *Isenberg and Hollweg*, 1982]. (Close to the Sun, however, the WKB procedure is inappropriate, and a more general treatment, based on the full wave equation, is needed [*Heinemann and Olbert*, 1980].)

Although one usually deals with only the leading terms, the WKB method is really an expansion, as has been explicitly represented by *Belcher* [1971] and *Hollweg* [1973]. The purpose of this paper is to reexamine the expansion given by these authors and to discuss some of its implications, particularly concerning recent observations of the Elsässer variables (see below) in the solar wind.

Our principal point will be that the expansion used by Belcher and Hollweg is "nonuniformly convergent," or "pedestrian" [*Nayfeh*, 1981], and we will show how a multiple space scale expansion can be used to obtain a "uniform expansion" [*Nayfeh*, 1981] which takes into account corrections to the wave phase speed resulting from inhomogeneity of the background plasma and magnetic field. We will also reiterate the known fact [*Heinemann and Olbert*, 1980; *Hollweg*, 1973; *Velli et al.*, 1989, 1990; *Weinberg*, 1962] that the two simple Alfvén modes which exist in a uniform plasma and magnetic field become coupled (i.e., they are no longer the normal modes) when the plasma and field are not homogeneous, and we will show how this result can be obtained from the higher-order terms in our uniform WKB expansion. This latter point has important implications for the development of turbulence in the solar wind, since it has been shown that incompressible MHD turbulence cannot develop if the wave field consists of only one of the simple homogeneous Alfvén modes [*Dobrowolny et al.*, 1980*a*, *b*; *Matthaeus and Goldstein*, 1982; *Tu et al.*, 1984; *Velli et al.*, 1989, 1990].

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Recently, solar wind fluctuations have been studied in terms of the Elsässer variables $\mathbf{z}^\pm \equiv \mathbf{V} \pm \mathbf{V}_A$ where $\mathbf{V}$ is the plasma flow velocity and $\mathbf{v}_A$ is the Alfvén speed [*Marsch and Tu*, 1990; *Roberts et al.*, 1987*a*, *b*; *Tu et al.*, 1989, 1990; *Tu and Marsch*, 1990]. In a homogeneous configuration the two Alfvén modes are represented by $\delta\mathbf{z}^+ = 0$ or $\delta\mathbf{z}^- = 0$, where the prefix δ denotes the fluctuation; these modes propagate in opposite directions in the frame in which $\mathbf{V} = 0$. However, we shall see that in the nonhomogeneous solar wind, each mode has both $\delta\mathbf{z}^+ \neq 0$ and $\delta\mathbf{z}^- \neq 0$. It is thus in general incorrect to make the (common) assumption that an observation of $\delta\mathbf{z}^-$ (say) represents an outward propagating Alfvén wave while $\delta\mathbf{z}^+$ necessarily represents an inward propagating Alfvén wave which is locally generated by reflection, instability, or turbulence. (The simultaneous presence of $\delta\mathbf{z}^+$ and $\delta\mathbf{z}^-$ may indeed indicate inward and outward propagating Alfvén waves, it may also indicate the presence of the fast, slow, or surface MHD modes, or it may indicate effects of "geometrical mode coupling" [e.g., *Hollweg and Lilliequist*, 1978; *Scheurwater*, 1990; *Wentzel*, 1989] which will not be treated here.) We will emphasize that the simultaneous presence of $\delta\mathbf{z}^+$ and $\delta\mathbf{z}^-$ is implied by the uniform WKB expansion to be presented below, and we will compare this prediction with recent observations of the Elsässer variables reported by *Marsch and Tu* [1990].

In this paper we confine our attention to the simple configuration studied by *Heinemann and Olbert* [1980]. They considered a nonrotating sun with an axisymmetric background magnetic field $\mathbf{B}_0$ and background flow velocity $\mathbf{V}_0$ confined to meridional planes. The vectors $\mathbf{V}_0$ and $\mathbf{B}_0$ are locally aligned. The small-amplitude fluctuations are also axisymmetric, with $\delta\mathbf{V}$ and $\delta\mathbf{B}$ locally perpendicular to the meridional planes. This configuration describes pure "toroidal" Alfvén waves. A more general treatment should include effects of refraction and geometrical mode coupling, which are beyond the scope of this study [*Scheurwater*, 1990].

Heinemann and Olbert give the following basic linearized equations for fluctuations varying as exp $(-i\omega t)$:

$$\left(-i\omega + V_s \frac{\partial}{\partial s}\right) g = f V_s \psi \qquad (1a)$$

$$\left(-i\omega + V_d \frac{\partial}{\partial s}\right) f = g V_d \psi \qquad (1b)$$

where $V_s = V + v_A$, $V_d = V - v_A$, s is distance along the background magnetic field, and we have dropped the subscript zero. The quantities f and g are defined by

$$f = (1 - \eta^{1/2})\delta z^+/\eta^{1/4} \qquad (2a)$$

$$g = (1 + \eta^{1/2})\delta z^-/\eta^{1/4} \qquad (2b)$$

and $\eta \equiv \rho/\rho_A$, where ρ is mass density and ρ_A is ρ at the Alfvénic critical point where $V = v_A$ (we will without loss of generality take V and v_A to be positive). The quantity ψ is

$$\psi \equiv -\frac{d}{ds} \ln (R\eta^{1/4}) \qquad (3)$$

where R is distance from the symmetry axis to the field line in question; for a purely radial field, $\psi = (1/2v_A)\, dv_A/ds$. In a homogeneous medium, $\psi = 0$, and then f and g represent the inward and outward propagating Alfvén modes, respectively.

In the next section we shall consider a contrived problem, which retains the same mathematical structure as equations (1) but which turns out to conveniently illustrate the basic points we want to make in this paper. Following that we will present a proper multiple space scale analysis of (1) leading to a uniform WKB expansion. Our results are summarized in the final section, where we also compare the predictions of this model with observations. We will find that the WKB predictions do not agree with both quantitative and qualitative aspects of recent data.

An Instructive Exercise

Consider the contrived problem where V_d, V_s, and ψ are all taken to be constants. This allows us to obtain an exact solution with which two approximate procedures can be compared. We then seek normal mode solutions to (1) varying as exp (iks). We find the dispersion relation

$$kV_sV_d/\omega = V \pm v_A[1 - (V_sV_d\psi/\omega v_A)^2]^{1/2} \qquad (4)$$

Note that the waves take on a growing or decaying character when $|V_sV_d\psi| > |\omega v_A|$. This qualitatively corresponds to the "power law growth" far from the Sun at low frequencies obtained by *Heinemann and Olbert* [1980] (see their equation (51) et seq.). We will not consider this point further.

At high frequencies we expand the square root in (4). For the purposes of this exercise it is sufficient to take the lower sign in (4), which gives

$$kV_s/\omega \approx 1 + v_A\alpha^2/2V_d + v_A\alpha^4/8V_d \qquad (5)$$

where

$$\alpha \equiv V_sV_d\psi/\omega v_A$$

Equation (5) is the dispersion relation for the outward propagating mode, but its phase speed is modified by the presence of ψ, which represents the inhomogeneity.

From (1a) we obtain

$$f/g \approx i[\alpha/2 + (\alpha/2)^3] \qquad (6)$$

Note that the mode is dominated by g but contains a small contribution from f. Remarkably, for a purely radial background magnetic field the lead term in (6) agrees precisely with equation (23) of *Hollweg* [1973], which was obtained from the pedestrian WKB expansion.

For the purpose of comparing with the pedestrian expansion, we write $g = G$ exp (iks) and expand the exponential taking $|s\psi\alpha/2| \ll 1$. We obtain, to $O(\psi^3)$,

$$g \approx G \exp (is\omega/V_s)(1 + is\psi\alpha/2) \qquad (7)$$

$$f \approx G \exp (is\omega/V_s)[i\alpha/2 + i(\alpha/2)^3 - s\psi(\alpha/2)^2] \qquad (8)$$

The failure of this expansion at large $|s|$ means that it is nonuniformly convergent, or pedestrian.

Our goal now is to show that the procedure used by *Belcher* [1971] and *Hollweg* [1973] yields the pedestrian equations (7) and (8). In essence they took

$$f = (f_0 + \varepsilon f_1 + \cdots) \exp (iS) \qquad (9a)$$

$$g = (g_0 + \varepsilon g_1 + \cdots) \exp (iS) \qquad (9b)$$

where $S = S(s)$ and ε is a small expansion parameter defined by taking ψ to be of $O(\varepsilon)$; in addition, terms like $(d/ds) \ln g_1$ are also taken to be $O(\varepsilon)$.

In lowest order, equations (1) require either

$$\omega/k = V_s \qquad f_0 = 0 \qquad (10a)$$

or

$$\omega/k = V_d \qquad g_0 = 0 \qquad (10b)$$

where $k \equiv dS/ds$. We will make the first choice, which corresponds to taking the lower sign in (4); making the other choice corresponds to the upper sign in (4). Note that we have only obtained the first term on the right-hand side of (5).

In next order we obtain, after using (10a),

$$g_0 = \text{const} \qquad (11)$$

$$\varepsilon f_1 = i\alpha g_0/2 \qquad (12)$$

In next order we obtain two more equations. One of them is

$$dg_1/ds = \psi f_1 \qquad (13)$$

Using (12), we obtain

$$\varepsilon g_1 = is\psi\alpha g_0/2 \qquad (14)$$

The appearance of s represents a "secularity" [*Nayfeh*, 1981] indicating that this procedure yields the pedestrian expansion. The other equation is

$$i\varepsilon f_2(k - \omega/V_d) = \psi g_1 \qquad (15)$$

or

$$\varepsilon^2 f_2 = -s\psi(\alpha/2)^2 g_0 \qquad (16)$$

and we again obtain a secular term.

Equation (16) is $O(\psi^3)$, but it turns out that another $O(\psi^3)$ term comes from the next order in our expansion. Proceeding in the same fashion, we find

$$\varepsilon^3 f_3 = i(\alpha/2)^3 g_0 + O(\psi^5) \tag{17}$$

Combining (9), (10*a*), (11), (12), (14), (16), and (17) is easily shown to reproduce (7) and (8), which were obtained from the exact solution to our contrived problem. Thus the expansion represented by (9) "works," but it has inadequacies. The appearance of secular terms in (14) and (16) means that it is nonuniformly convergent and fails at large $|s|$. Also, the ordering of terms is clumsy; for example, the ψ^3 terms come from two different orders in the expansion. This is a signature of the fact that (7) and (8) really involve two expansions of the exact solution: first we expanded the square root in (4), and then we expanded part of exp (iks). In addition, this method does not explicitly reveal the modification of k due to the presence of ψ. Finally, equations (10) require that the analysis of the two normal modes be done separately.

A multiple-scale analysis [*Nayfeh*, 1981] resolves these difficulties. It is convenient to rewrite (1) as

$$\left(-i + \frac{V_s}{\omega}\frac{\partial}{\partial s}\right) g = f V_s \psi/\omega \tag{18a}$$

$$\left(-i + \frac{V_d}{\omega}\frac{\partial}{\partial s}\right) f = g V_d \psi/\omega \tag{18b}$$

We write

$$f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots \tag{19a}$$

$$g = g_0 + \varepsilon g_1 + \varepsilon^2 g_2 + \cdots \tag{19b}$$

and

$$\frac{\partial}{\partial s} = \frac{\partial}{\partial s_0} + \varepsilon\frac{\partial}{\partial s_1} + \varepsilon^2\frac{\partial}{\partial s_2} + \cdots \tag{20}$$

where

$$s_0 = s \qquad s_1 = \varepsilon s \qquad s_2 = \varepsilon^2 s$$

etc. The small expansion parameter is ε, and we shall order the expansion by again taking ψ to be $O(\varepsilon)$. To make contact with what has gone before, for the time being we will take $f_0 = 0$; note, however, that this is not necessary, in contrast to (10).

In lowest order we have

$$-ig_0 + \frac{V_s}{\omega}\frac{\partial g_0}{\partial s_0} = 0 \tag{21}$$

which gives

$$g_0 = \gamma_0 \exp(i\omega s_0/V_s) \tag{22}$$

where $\gamma_0 = \gamma_0(s_1, s_2, \cdots)$.

In next order we obtain the following two equations:

$$-ig_1 + \frac{V_s}{\omega}\left[\frac{\partial g_1}{\partial s_0} + \frac{\partial g_0}{\partial s_1}\right] = 0 \tag{23}$$

$$-if_1 + \frac{V_d}{\omega}\frac{\partial f_1}{\partial s_0} = g_0 V_d \psi_1/\omega \tag{24}$$

where $\psi_1 \equiv \psi/\varepsilon$. The solution to (23) will contain a secular term unless $\partial g_0/\partial s_1 = 0$, implying $\gamma_0 = \gamma_0(s_2, s_3, \cdots)$. Since we are interested only in the particular solution to (23), we then have

$$g_1 = 0 \tag{25}$$

The solution to (24) is

$$\varepsilon f_1 = (i\alpha/2)\gamma_0 \exp(i\omega s_0/V_s) \tag{26}$$

In next order we note that $g_1 = 0$ and $\partial f_1/\partial s_1 = 0$. We then obtain a homogeneous equation for f_2 and take

$$f_2 = 0 \tag{27}$$

We also obtain

$$-ig_2 + \frac{V_s}{\omega}\left[\frac{\partial g_2}{\partial s_0} + \frac{\partial g_0}{\partial s_2}\right] = f_1 V_s \psi_1/\omega \tag{28}$$

To eliminate a secular term, we must have

$$\frac{\partial g_0}{\partial s_2} = f_1 \psi_1 \tag{29}$$

and (28) then gives

$$g_2 = 0 \tag{30}$$

The solution to (29) is

$$\gamma_0 = \gamma_{00} \exp(is_2 V_d V_s \psi_1^2/2\omega v_A) \tag{31}$$

where $\gamma_{00} = \gamma_{00}(s_3, s_4, \cdots)$. We now see the advantage of this procedure. The first correction to k, given by (5), is recovered, and it properly appears in the exponential; there are no secular terms, and the expansion is uniform.

We have carried out the expansion to two more orders, and we have verified that it agrees precisely with (5) and (6). We have also verified that the expansion proceeds naturally if we carry along f_0 with g_0; in that case we obtain both normal modes given by (4) (with the square root expanded).

In summary, we have considered a contrived problem having the same mathematical structure as (1) but which allows us to obtain an exact solution which can be compared with two approximate procedures. We found that the expansion used by *Belcher* [1971] and *Hollweg* [1973] contains secular terms and is thus nonuniformly convergent. But we also found that a multiple-scale analysis conveniently yields a uniform expansion, and other features of the exact solution, such as the modification of k due to ψ, and the presence of both f and g (i.e., δz^+ and δz^-) in each normal mode. Thus in the next section we will apply the multiple-scale analysis to (1) allowing V, v_A, and ψ to be spatially variable.

MULTIPLE-SCALE ANALYSIS

Rewrite equations (1) as

$$\left(-i + \frac{\partial}{\partial \Gamma}\right) g = f\frac{dP}{d\Gamma} \tag{32a}$$

$$\left(-i + \frac{\partial}{\partial \Phi}\right) f = g \frac{dP}{d\Phi} \tag{32b}$$

where $P \equiv -\ln (R\eta^{1/4})$ and Γ and Φ are defined by

$$d\Gamma = \omega \, ds/V_s \tag{33a}$$

$$d\Phi = \omega \, ds/V_d \tag{33b}$$

We again use expansion (19), and in analogy with (20) we write

$$\frac{\partial}{\partial \Gamma} = \frac{\partial}{\partial \Gamma_0} + \varepsilon \frac{\partial}{\partial \Gamma_1} + \cdots \tag{34a}$$

$$\Gamma_0 = \Gamma \tag{34b}$$

$$\Gamma_1 = \varepsilon \Gamma \tag{34c}$$

etc., with similar expressions for $\partial/\partial\Phi$. The quantity P varies on the slow space scale, so we take

$$dP/d\Gamma = \varepsilon \, dP/d\Gamma_1 \tag{35}$$

and similarly for $dP/d\Phi$. The analysis then proceeds much as in the preceding section, and we will omit many of the details.

In lowest order we obtain

$$g_0 = \gamma_0(\Gamma_1, \cdots) \exp (i\Gamma_0) \tag{36a}$$

$$f_0 = \phi_0(\Phi_1, \cdots) \exp (i\Phi_0) \tag{36b}$$

In the next order we first remove secular terms. This requires

$$\gamma_0 = \gamma_0(\Gamma_2, \cdots) \tag{37a}$$

$$\phi_0 = \phi_0(\Phi_2, \cdots) \tag{37b}$$

Thus γ_0 and ϕ_0 are constants on the Γ_1 and Φ_1 scales. This is equivalent to the usual lowest-order WKB result for the evolution of the amplitudes of the inward and outward propagating waves (e.g., equations (20) and (21) of *Hollweg* [1973]). After eliminating the secular terms we find

$$g_1 = \gamma_1 \exp (i\Phi_0) \tag{38a}$$

$$f_1 = \phi_1 \exp (i\Gamma_0) \tag{38b}$$

where

$$\gamma_1 = -\frac{iV_d}{2v_A} \frac{dP}{d\Gamma_1} \phi_0 \tag{39a}$$

$$\phi_1 = \frac{iV_s}{2v_A} \frac{dP}{d\Phi_1} \gamma_0 \tag{39b}$$

and it has been assumed that V_d and V_s do not vary on the Γ_0 or Φ_0 scales. Note that f_1 has the same outward propagating phase factor as g_0, and it is incorrect to think of f_1 as a reflected or inward propagating component of the wave field. For purely radial V_0 and B_0, equation (39*b*) agrees precisely with equation (23) of *Hollweg* [1973], and for the more general case considered here it agrees with equation (4*b*) of *Velli et al.* [1989]. Similar remarks apply to g_1.

In next order we obtain two equations involving (g_0, g_1, g_2) and (f_0, f_1, f_2). To eliminate secularities, we require

$$\frac{\partial g_0}{\partial \Gamma_2} = f_1 \frac{dP}{d\Gamma_1} \tag{40a}$$

$$\frac{\partial f_0}{\partial \Phi_2} = g_1 \frac{dP}{d\Phi_1} \tag{40b}$$

Using equations (36)–(39), we obtain

$$\gamma_0 = \gamma_{00}(\Gamma_3, \cdots) \exp \left[\int \frac{iV_s V_d}{2\omega v_A} \left(\frac{dP}{ds}\right)^2 ds \right] \tag{41a}$$

$$\phi_0 = \phi_{00}(\Phi_3, \cdots) \exp \left[-\int \frac{iV_s V_d}{2\omega v_A} \left(\frac{dP}{ds}\right)^2 ds \right] \tag{41b}$$

The appearance of the exponentials is basically equivalent to the corrections to k found in the contrived problem of the previous section.

With the secularities eliminated we obtain

$$-ig_2 + \frac{\partial g_2}{\partial \Gamma_0} = -\frac{\partial g_1}{\partial \Gamma_1} \tag{42}$$

and a similar equation for f_2. In contrast to the previous section, g_2 and f_2 are no longer zero. If we write

$$\frac{\partial}{\partial \Gamma_0} = \frac{V_s}{V_d} \frac{\partial}{\partial \Phi_0}$$

and use (38*a*) and (39*a*), we obtain

$$g_2 = \gamma_2 \exp [i\Phi_0] \tag{43}$$

with

$$\gamma_2 = \frac{V_d}{2v_A} \frac{\partial}{\partial \Gamma_1} \left[\frac{V_d}{2v_A} \frac{dP}{d\Gamma_1} \phi_0 \right] \tag{44}$$

Similarly,

$$f_2 = \phi_2 \exp (i\Gamma_0) \tag{45}$$

with

$$\phi_2 = \frac{V_s}{2v_A} \frac{\partial}{\partial \Phi_1} \left[\frac{V_s}{2v_A} \frac{dP}{d\Phi_1} \gamma_0 \right] \tag{46}$$

This completes the expansion to $O(\varepsilon^2)$. At this level, $\phi_{00}(\Phi_3, \cdots)$ and $\gamma_{00}(\Gamma_3, \cdots)$ are to be regarded as the constants of integration. If we convert back to the spatial variable s and assemble the terms in the expansion, we obtain finally

$$g = \gamma_{00} e^{i\Gamma} e^{(+)} - i(\alpha/2)\phi_{00} e^{i\Phi} e^{(-)} + \phi_{00}(V_d V_s/4\omega v_A) \cdot (\partial\alpha/\partial s) e^{i\Phi} e^{(-)} \tag{47}$$

$$f = \phi_{00} e^{i\Phi} e^{(-)} + i(\alpha/2)\gamma_{00} e^{i\Gamma} e^{(+)} + \gamma_{00}(V_d V_s/4\omega v_A) \cdot (\partial\alpha/\partial s) e^{i\Gamma} e^{(+)} \tag{48}$$

where $e^{(+)}$ and $e^{(-)}$ represent the exponentials appearing in (41*a*) and (41*b*), respectively. (We have taken $e^{(+)}$ and $e^{(-)}$ out of the derivatives appearing in (44) and (46), since their differentiation leads to higher-order terms.) We have verified by direct substitution that (47) and (48) satisfy (1) to the required order.

The validity of the expansion requires that the terms in (47) and (48) be successively smaller in magnitude. One condition is $|\alpha/2| \ll 1$ or

$$\left|\frac{V^2 - v_A^2}{2\omega v_A}\frac{dP}{ds}\right| \ll 1 \tag{49}$$

(see also equation (31) of *Heinemann and Olbert* [1980]). If $V \ll v_A$, (49) requires a short wavelength, i.e., $(\lambda/4\pi) \ll L$, where λ is the wavelength and $L \equiv |ds/dP|$ is the length scale for variations of the background, If $V \gg v_A$, (49) can be interpreted by viewing the waves in the flowing plasma frame; one then has the requirement that the time rate of change of the background in that frame, i.e., $|VdP/ds|$, should be small compared to $2\omega'$, where ω' is frequency in the plasma frame. These intuitively reasonable interpretations become muddied when $V \approx v_A$; indeed, (49) is perfectly satisfied when $V = v_A$. In terms of time scales a more general interpretation of (49) is as follows: Consider an outward propagating wave moving nearly at speed V_s. The rate of change of the background seen by the wave is $dP/dt \equiv |V_s dP/ds|$. Condition (49) can then be written as

$$\frac{dP}{dt} \gg |k\Delta V_{ph}| \tag{50}$$

where ΔV_{ph} is the modification to the local phase speed implied by (5). Now $|k\Delta V_{ph}|^{-1}$ is essentially the time required for the phase to differ by 1 rad from what it would have been if the phase speed were exactly V_s. Thus (50) implies that the time required for a 1-rad phase shift must be long compared to the time required for the wave to see a change in the background. (This argument applies also to an inward propagating wave.) Finally, (49) can be viewed as simply a requirement that the two modes be nearly "pure" in the sense that the f part of the outward propagating mode be small compared to g, and similarly for the inward mode.

Another condition for validity of the expansion comes from comparing the second and third terms on the right-hand sides of (47) and (48). We require

$$\left|\frac{V^2 - v_A^2}{2\omega v_A}\frac{d\ln\alpha}{ds}\right| \ll 1 \tag{51}$$

which is the same as (49) with P replaced by $\ln \alpha$. Since $\alpha = 0$ when $V = v_A$, a better form is

$$\left|\frac{d\alpha}{ds}\right| \ll 2\left|\frac{dP}{ds}\right| \tag{52}$$

Thus in contrast to the analyses of *Belcher* [1971] and *Hollweg* [1973], we have obtained a uniformly convergent "WKB" solution to the equations of Heinemann and Olbert, which simultaneously handles the two wave modes and which includes in a natural way the modification to k resulting from inhomogeneity. Each mode contains both f and g components, and a measurement of $f \neq 0$ in the solar wind does not necessarily require the presence of the inward propagating mode.

The fact that f and g are in general simultaneously present is related to a recent paper by *Zhou and Matthaeus* [1989]. They presented equations for the evolution of the power spectra of δz^+ and δz^- in the solar wind. Their equation (6) contains a term which is closely related to $\langle\delta z^+\delta z^-\rangle$, or $\langle fg\rangle$, where the angle brackets denote an ensemble average (which can be adequately performed by time averaging in the spacecraft frame). They argue that this term introduces [*Zhou and Matthaeus*, 1989, pp. 756, 755] "completely new kinds of effects" which "do not appear in the usual WKB orderings." We should like to point out here that at least for the special geometry and wave polarization considered here, their equation (6) is in fact already included in our equations (1) and that there are no new effects which cannot be uncovered by solving (1). First replace $-i\omega$ by $\partial/\partial t$ in (1), thus fully recovering equations (9) of Heinemann and Olbert. Then multiply (1*a*) and (1*b*) by g and f, respectively, and average over time (or over the fast time scale if we wish to allow for a situation in which the wave amplitudes are slowly varying in time). We then have

$$\frac{\partial\langle g^2\rangle}{\partial t} + V_s\frac{\partial\langle g^2\rangle}{\partial s} = 2\langle fg\rangle V_s\psi \tag{53}$$

and a similar equation for $\langle f^2\rangle$. For the special case at hand we have verified that these equations are equivalent to equation (6) of Zhou and Matthaeus if they are rewritten in terms of power spectra. Zhou and Matthaeus discuss the effect of $\langle fg\rangle$. However, we can use our solutions (47) and (48) to evaluate $\langle fg\rangle$. If the lowest-order solutions, f_0 and g_0, are uncorrelated, then $\langle fg\rangle$ will be of $O(\varepsilon^2)$, in virtue of the 90° phase shift in the middle terms on the right-hand sides of (47) and (48). Thus the presence of $\langle fg\rangle$ will introduce only a small correction, of $O(\varepsilon^3)$, to the "usual" WKB result in which the right-hand side of (53) is zero. (W. H. Matthaeus (private communication, 1989) has suggested that the $\langle fg\rangle$ term will be completely different from its WKB form if the local wave vector is perpendicular to $\mathbf{B}_0$. Within the framework of (1), this can only be described by taking $\omega = 0$, which is an uninteresting limit. Thus we cannot address Matthaeus' point here.)

Finally, we consider conditions at the solar wind Alfvén critical point where $V = v_A$ and $\eta = 1$. Heinemann and Olbert showed on energetic grounds that $f = 0$ there. However, from the definition of f it is not necessarily true that $\delta z^+ = 0$ when $\eta = 1$ (equation (2*a*)). For example, for the outgoing mode in a radial field we have

$$\frac{\delta z^+}{\delta z^-} = \frac{iV_s^2}{4\omega v_A^2}\frac{dv_A}{ds} \tag{54}$$

which is well behaved at the critical point. Thus incompressible MHD turbulence, which requires $\delta z^+ \neq 0$ and $\delta z^- \neq 0$, can in principle evolve at the Alfvén critical point.

SUMMARY AND DISCUSSION

Our conclusions can be summarized as follows:

1. The WKB expansion used by *Belcher* [1971] and *Hollweg* [1973] for Alfvén waves in the solar wind is nonuniformly convergent.

2. For the special case studied by *Heinemann and Olbert* [1980] we have used the method of multiple scales [*Nayfeh*, 1981] to obtain an expansion which is uniform.

3. The uniform expansion takes into account the small

modification to the Alfvén wave phase speed due to spatial gradients of the background.

4. Both the uniform and nonuniform expansions reveal that each "normal mode" has $f \neq 0$ and $g \neq 0$, or, in terms of Elsässer variables, $\delta z^+ \neq 0$ and $\delta z^- \neq 0$. Thus if δz^- corresponds to the outgoing mode in a homogeneous background, an observation of $\delta z^+ \neq 0$ does not necessarily imply the presence of the inward propagating mode, as is commonly assumed.

5. Although $f = 0$ at the Alfvén critical point (where $V = v_A$ and $\eta = 1$), it is not true that $\delta z^+ = 0$ there. Thus incompressible MHD turbulence, which requires both $\delta z^+ \neq 0$ and $\delta z^- \neq 0$, can proceed at the Alfvén critical point.

6. Since in general both $\delta z^+ \neq 0$ and $\delta z^- \neq 0$, incompressible turbulence can ubiquitously evolve in the solar wind. This fact has been exploited recently by *Velli et al.* [1989, 1990], although we shall see in the next paragraph that a basic prediction of their papers, and this one, is not supported by the data.

Thus the principal goal of this paper has been to make some formal points about the simultaneous presence of δz^+ and δz^- and the uniformity of the WKB expansion. One prediction of this description, which can be tested by observation, is as follows: Suppose both δz^+ and δz^- are solely due to the outward propagating mode. Then $\phi_{00} = 0$, and the first two terms on the right-hand sides of (47) and (48) suggest that the power spectrum for f (or δz^+) should fall off faster with frequency, by the factor ω^{-2}, than the power spectrum for g (or δz^-). *Marsch and Tu* [1990] have compared power spectra for δz^+ and δz^- (δz^+ in their notation corresponds to δz^- in our notation, and vice versa), using Helios data inside 1 AU. The power spectra and their ratios are given in their Figures 2 and 3, respectively. In high-speed streams, at frequencies (in the spacecraft frame) below about 10^{-4} Hz, the ratio of the power spectra does indeed fall off with increasing frequency, but the falloff is closer to ω^{-1} than the predicted ω^{-2}; a possible exception is the data set obtained at 0.87 AU which shows an approximately ω^{-2} falloff between 3×10^{-5} and 10^{-4} Hz. In slow wind, there is also a falloff of the power spectrum ratio below about 3×10^{-4} Hz, but except at 0.31 AU the falloff does not go as ω^{-2}; however, in slow wind the power spectra for the two Elsässer variables have comparable magnitudes, and comparison with our WKB expansion is not appropriate. At higher frequencies the ratio of the power spectra either is nearly constant in frequency (slow wind) or increases with frequency (fast wind). This behavior is even qualitatively at variance with the predictions of the WKB expansion. Clearly, the dynamics of g and f (in particular) are not adequately described by equations (1) in the WKB approximation. Since the predictions of our analysis have the greatest disagreement with the data at high frequencies, where the WKB expansion should be best, we suspect that the problem is not with WKB, but with the dynamical content of equations (1). Geometrical mode coupling, which is not contained in (1), should also become less relevant at higher frequencies, so it is probable that geometrical mode coupling is not the source of the discrepancy. We thus concur with *Marsch and Tu* [1990] and other authors that the data can only be understood in terms of nonlinear (turbulent) processes, or perhaps in terms of local generation of waves via Kelvin-Helmholtz instabilities [*Bavassano and Bruno,* 1989; *Roberts et al.,* 1987*b*] or other instabilities generated by velocity shear [*Hollweg et al.,* 1990].

Finally, we should point out that there is a fundamental quantitative disagreement with the data. For radial flow at constant speed in a radial magnetic field, equation (54) gives $|\delta z^+|/|\delta z^-| \approx 5 \times 10^{-6} f_{\rm Hz}^{-1}$ if we take $V_s = 750$ km s^{-1} and $v_A r = 4.5 \times 10^9$ km^2 s^{-1} (corresponding to $v_A = 100$ km s^{-1} at $r = 0.3$ AU), where $f_{\rm Hz} = \omega/2\pi$. Even with $f_{\rm Hz} = 3 \times 10^{-5}$ Hz we obtain $|\delta z^+|/|\delta z^-| = 0.17$. The normalized cross helicity, i.e., $(|\delta z^-|^2 - |\delta z^+|^2)/(|\delta z^-|^2 + |\delta z^+|^2)$, would then be 0.94 at this frequency; the cross helicity approaches 1.0 at higher frequencies. The observed cross helicities in the solar wind are considerably smaller than these computed values, and we conclude that the effects discussed in this paper are unable to explain the observed behavior of $|\delta z^+|/|\delta z^-|$ in the solar wind.

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