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ASTRONOMY AND ASTROPHYSICS

Dependence of MHD turbulence spectra on the velocity field-magnetic field correlation

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Summary. Steady-state MHD turbulence with non-zero velocity v-magnetic field b correlation is studied using both a two-point turbulence closure and phenomenological arguments. We obtain that equilibrium spectra have power-laws which may strongly depart from the 3/2 law derived by Kraichnan (1965) in the uncorrelated case. This departure stems from the broken symmetry between the two fields $z^{\pm} = v \pm b$ when the correlation is non zero. For strong correlations, the spectral index for the total energy spectrum may lie anywhere between 3/2 and 3, depending on the correlation level whose maximum is determined by the molecular dissipation coefficients.

Key words: magnetic field – magnetohydrodynamics – turbulence

1. Introduction

Strong correlations between all three components of the fluctuating velocity v and the magnetic field b are observed for long periods of time in the solar wind (Belcher and Davis, 1971). These correlations may either be intrinsic properties of the source, or they may arise from non-linear interactions occurring during propagation from the Sun. The latter was conjectured by Dobrowolny et al. (1980) on a phenomenological basis. A detailed study of the growth of the correlation between v and b has been given in Grappin et al. (1982) (hereafter referred to as Paper I) using a two-point closure, the Eddy Damped Quasi Normal Markovian (EDQNM) approximation. Direct numerical simulations both in two and in three dimensions of incompressible MHD with periodic boundary conditions have also been performed. They confirm that non-linear interactions alone tend to make correlation coefficient grow in time, where

$$\varrho = 2\langle v \cdot b\rangle / (\langle v^2\rangle + \langle b^2\rangle), \tag{1.1}$$

both for deterministic and for random initial conditions (Frisch, Meneguzzi, Pouquet, private communication; see also Images de la Physique, 1981).

The equations for the velocity and magnetic field in the incompressible case are written below (b has been dimensionned to a velocity):

$$\begin{aligned} &\partial v/\partial t + v \cdot \nabla v = -\nabla p + b \cdot \nabla b + \nu \nabla^2 v + F^v, \\ &\partial b/\partial t + v \cdot \nabla b = b \cdot \nabla v + \lambda \nabla^2 b + F^M, \\ &\nabla \cdot v = 0, \ \nabla \cdot b = 0. \end{aligned} \tag{1.2}$$

p is the total pressure, ν is the kinematic viscosity and λ is the magnetic diffusivity. We shall take here the magnetic Prandtl number, $Pr = \nu/\lambda$ equal to unity. The increase of the correlation coefficient $\varrho = 2E^C/E^T$ is related to the evolution of the total energy $E^T = 1/2\langle v^2 + b^2\rangle$ and the correlation $E^C = 1/2\langle v \cdot b\rangle$. Both quantities are exact invariants of the MHD equations in absence of dissipation and forcing (note that E^C is not definite positive, whereas E^T is).

We are interested in this paper by the effect of the growth of the correlation coefficient on the energy spectra. No significant departure from the classical $-3/2$ law of Kraichnan (1965) was found in Paper I for correlation coefficients up to 90%. However, it is to be expected that, for still higher correlation coefficients ($|\varrho| \to 1$) as could be the case in the Solar Wind, the flux of energy to small scales vanish and thus the spectra may undergo some modifications (recall that when $|\varrho|$ is equal to 1, non linear terms vanish).

We investigate this problem here, both numerically and phenomenologically. We adopt the standard way to study spectral slopes in fully developed turbulence, initiated by Kolmogorov (1941), extended to MHD by Kraichnan (1965) and also by Dobrowolny et al. (1981). Assume stationarity for all quantities, and the existence of a large range of wavenumbers (the so called "inertial" range) where non linear interactions are dominant. In this inertial range, the flux of energy (usually from large to small scales) is independent of wavenumber, being far from both the source (large scale) and the dissipative sink (small scales). This flux represent either the energy injected at large scale by a truly external source ("forcing") or, in the unforced problem which interests us here, the energy released by the decay of large scale eddies.

We follow this method also in our numerical study: kinetic, magnetic energies and correlation are injected at large wavenumbers.

We stress what does not concern us in this paper:

a) we are not concerned with dynamo effects, i.e. time growth of magnetic energy with kinetic energy forcing (this dynamo effect has been already studied; see Léorat et al., 1981; Pouquet et al., 1976).

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b) We are not concerned with modelization of any source of correlation (the growth of correlation due to non-linear interactions has been studied in Paper I).

Forcing is only a mean to study quasi-equilibria in a highly correlated, decaying MHD turbulence as in the Solar Wind. It speeds up numerical computations, gives clear-cut inertial ranges, allowing easier comparison between numerical and phenomenological results. We will come back in the conclusion to the time dependent, decaying turbulence, and discuss there the validity of the quasi-stationary approximation.

The paper proceeds as follows. Section 2 gives results obtained within the framework of the EDQNM closure, for various correlation injection rates. In Sect. 3, which can be read independently, we present a modification of the MHD phenomenology derived by Kraichnan (1965) which takes into account the effect of correlation. The last section is a discussion.

2. Closure results

The method

We use the Eddy Damped Quasi Normal Markovian (EDQNM) approximation to study homogeneous isotropic non helical MHD turbulence with velocity-magnetic field correlation. The EDQNM closure leads to a set of coupled integrodifferential equation for the components of the Fourier spectra of the second order moments of the fluctuating velocity field $v(r, t)$ and magnetic field $b(r, t)$. It is sometimes convenient to use the Elsässer variables $z^{\pm} = v \pm b$, and the various spectra that can thus be constructed are listed in Table 1. The closure equations for MHD turbulence with correlation were defined in Paper I and are given for convenience in Appendix A.

The relative correlation in the flow may be measured by the coefficient

$$\varrho = 2\langle v \cdot b\rangle / \langle v^2 + b^2\rangle = 2E^C/(E^V + E^M) = (E^+ - E^-)/(E^+ + E^-). \tag{2.1}$$

As soon as equipartition of kinetic and magnetic energy is reached, this definition is equivalent to the usual one, namely:

$$\gamma = \langle v \cdot b\rangle / [\langle v^2\rangle \langle b^2\rangle]^{1/2}. \tag{2.2}$$

Note that $|\varrho| \leqq |\gamma| \leqq 1$ and that vanishing non linear interactions obtain when $|\varrho| = 1$.

We are interested in situations in which both the kinetic and magnetic Reynolds numbers, R^V and R^M, are equal and take large values. More precisely, we suppose that R^M is far above the critical value for dynamo action to take place (Léorat et al., 1981) i.e. that large scale magnetic energy is present in the system. We therefore choose identical kinetic and magnetic injection spectra (see Table 1 for definitions)

$$F_k^V = F_k^M \sim k^4 \exp - 2k^2; \tag{2.3}$$

with identical injection rates:

$$\varepsilon^V = \varepsilon^M = \int dk\, F_k^V\, dk = 1. \tag{2.4}$$

In the numerical calculations to be described below, we then vary the injection rate of correlation ε^C (see Table 1). Note that the kinetic and magnetic energy injection rates and the correlation injection rate obey a Schwartz inequality: $(\varepsilon^C)^2 \leqq \varepsilon^V \varepsilon^M$. We take a correlation injection spectrum different from the one used for the energy injection [Eq. (2.3)], namely:

$$F_k^C \sim k^6 \exp - k^2, \tag{2.5}$$

to ensure that the non linear terms do not vanish identically for zero initial conditions (cf. Paper I).

Results will be given in terms of the ratio $\varepsilon^+/\varepsilon^-$ for the $\pm$ energy injection rates defined in Table 1. We shall always take a positive injection of correlation, which corresponds to $\varepsilon^+/\varepsilon^- > 1$.

The numerical integration of the equations of Appendix A is performed using an exponential discretization in wavenumber space, $k_i = k_0\, \varepsilon^{i/F}$ and we take $F = 4$ points per octave (see Paper I for more details). When the correlation coefficient is close to one, typical time scales are slowed down and the integration time of the closure equations becomes prohibitive. Stationary states are better studied using an iterative technique which is described for example

Table 1. Definitions of the different spectra and useful relations

Fields	$z^{\pm} = v \pm b$	$v = (z^+ + z^-)/2$ $b = (z^+ - z^-)/2$
Energy densities and spectra	$E^{\pm} = \int dk\, E_k^{\pm} = \frac{1}{2}\langle (z^{\pm})^2\rangle$ $E^R = \frac{1}{2}\int d^3 r \langle z^+ \cdot z^-\rangle$ $E^R = \int dk\, E_k^R$	$E^V = \int dk\, E_k^V = \frac{1}{2}\langle (v)^2\rangle$ $E^M = \int dk\, E_k^M = \frac{1}{2}\langle (b)^2\rangle$ $E^C = \int dk\, E_k^C = \frac{1}{2}\langle v \cdot b\rangle$
Identities	$E^{\pm} = E^V + E^M \pm 2E^C$ $E^R = E^V - E^M$ $E^T = E^V + E^M = (E^+ + E^-)/2$	$E^V = (E^+ + E^- + 2E^R)/4$ $E^M = (E^+ + E^- - 2E^R)/4$ $E^C = (E^+ - E^-)/4$
Invariants of the 3-dimensional non dissipative MHD flow	E^+ and E^- ($\pm$ energies)	$E^V + E^M$ (total energy) and E^C (correlation)
Forcing rates and spectra	$\varepsilon^{\pm} = \varepsilon^V + \varepsilon^M \pm 2\varepsilon^C$ $\varepsilon^R = \varepsilon^V - \varepsilon^M$	$\varepsilon^V = \int dk\ F_k^V = \frac{1}{2}\langle (f^V)^2\rangle$ $\varepsilon^M = \int dk\, F_k^M = \frac{1}{2}\langle (f^M)^2\rangle$ $\varepsilon^C = \int dk\ F_k^C = \frac{1}{2}\langle f^V \cdot f^M\rangle$

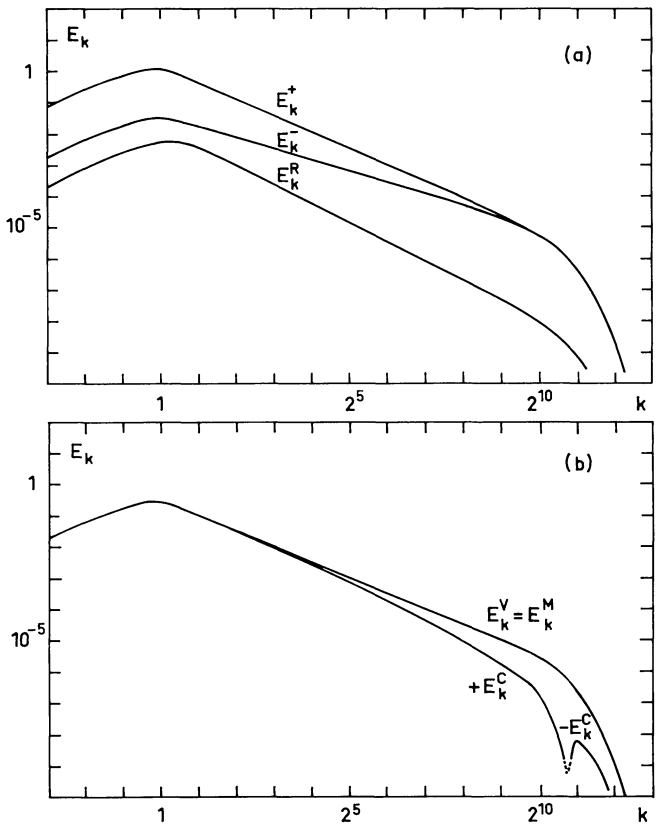

Fig. 1a and b. Equilibrium spectra with a relative correlation injection rate $r=\varepsilon^+/\varepsilon^-=10\%$ ($\nu=\lambda=10^{-6}$). Kinetic (or magnetic) energy = 1.2; kinetic (or magnetic) Reynolds number = 1.5 10^6. Correlation coefficient $\varrho=88\%$. **a** (+, −, R) spectra; spectral indices: $m^+\simeq 1.8$, $m^-\simeq 1.2$, $m^R\simeq 2$. Notice the equipartition of ± energies in the dissipative range. **b** (V, M, C) spectra; spectral indices: $m^V\simeq m^M\simeq 1.8$. Notice the change of sign of the correlation spectrum at the dissipation scale

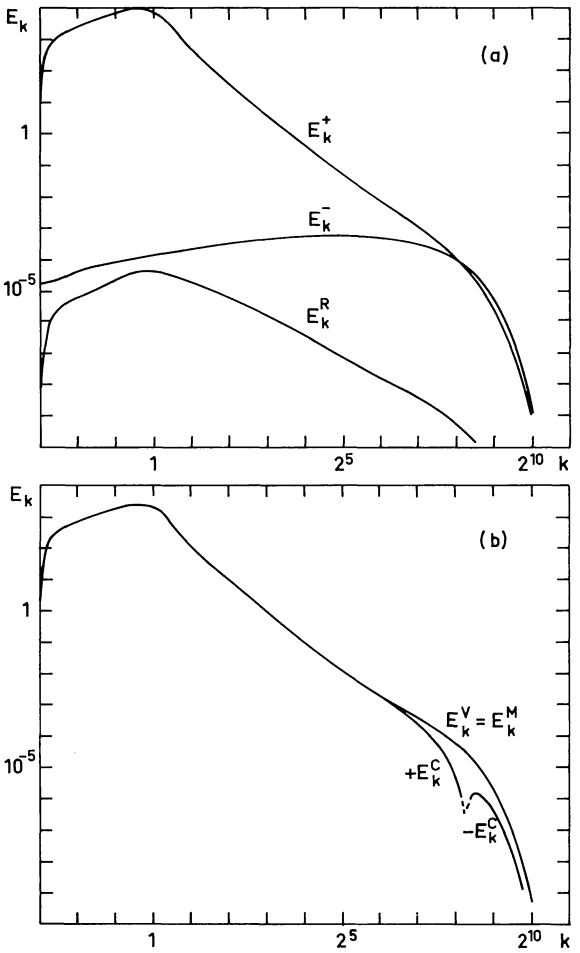

Fig. 2 a and b. Equilibrium spectra with a relative correlation injection rate $r=80\%$ ($\nu=\lambda=10^{-6}$). Kinetic (or magnetic) energy = 4.3 10^3 kinetic (or magnetic) Reynolds number = 0.9 10^8. Correlation coefficient: $\varrho=0.99998$. **a** (+, −, R) spectra; spectral indices: $m^+\simeq 3$, $m^R\simeq 2$. Notice that the − energy spectrum is flat only in a part of the inertial range. **b** (V, M, C) spectra; spectral indices: $m^V\simeq m^M\simeq 3$

Table 2. Numerical data obtained from the integration of the EDQNM spectral equations of Appendix A, for two values of the injection correlation coefficient, $r=\varepsilon^+/\varepsilon^-$ ($R^M=R^V=10^6$)

Injection correlation coefficients [cf. Eq. (2.6)]		$r=10\%$ (Fig. 1)	$r=80\%$ (Fig. 2)
Injection rates (cf. Table 1)	ε^+	2.2	3.6
	ε^-	1.8	0.4
	$\varepsilon^+/\varepsilon^-$	1.22	9
Total energies (cf. Table 1)	E^+	2.4 10^2	8.6 10^5
	E^-	0.16 10^2	8.7
	E^+/E^-	15	10^5
Correlation coefficient $\varrho=(E^+-E^-)/(E^++E^-)$		0.88	0.99998
Spectral indexes in the inertial range [cf. Eq. (2.7)]	m^+	1.8	2.99
	m^-	1.2	≈0
	m^R	2	2

in Léorat et al. (1981). We verify that the spectra thus obtained are also solutions to the time dependent problem, by integrating equations of Appendix A for a few large scale characteristic times.

Numerical results

Figures 1 and 2 show the stationary spectra for two characteristic sets of data corresponding to an injection correlation coefficient

$$r=2\varepsilon^C/(\varepsilon^V+\varepsilon^M)=(\varepsilon^+-\varepsilon^-)/(\varepsilon^++\varepsilon^-) \qquad (2.6)$$

equal to 10% and 80%, respectively. Spectral indices in the inertial range are defined as

$$E_k^X\sim k^{-m^X}, \qquad (2.7)$$

where $X=(\pm, R)$ or (V, M, C). Typical numbers for the runs are given in Table 2.

A striking feature of the numerical results, as shown in Figs. 1 and 2, is the dependence of the slopes of the energy spectra on the ratio $\varepsilon^+/\varepsilon^-$. Instead of the relation $m^+ = m^- = 3/2$ which obtains for Alfvénic turbulence in the uncorrelated case, we have now

$$m^+ + m^- \simeq 3 \tag{2.8}$$

with $m^\pm$ varying with the correlation injection and reading $m^+ \simeq 3$, $m^- \simeq 0$ in the extreme case $\varepsilon^+/\varepsilon^- \gg 1$ (the other possible choice of injection rates, namely $\varepsilon^-/\varepsilon^+ \gg 1$ would lead to $m^- \simeq 3$, $m^+ \simeq 0$). In contrast, the slope of the residual energy spectrum E_k^R remains remarkably constant ($m^R \simeq 2$) for various correlation injection rates r, including $r = 0$. A k^{-2} range for the residual energy spectrum was in fact obtained in Pouquet et al. (1976) as a first order correction to the solution of equal kinetic and magnetic spectra with slopes $m^V = m^M = 3/2$. The same result still holds here since, as is shown further below, the k^{-2} range is a consequence of the competition between the action of large scale magnetic eddies on small scales (Alfvén effect) and the interaction of + and − eddies of comparable sizes; the resulting residual energy spectrum is independent of the ± slopes as soon as they satisfy relation (2.8).

Spectral indices in MHD *turbulence with correlation: the reduced equations*

Let us now explain why relation (2.8) holds, for a large domain of values and not strictly for $m^+ = m^- = 3/2$. To simplify the analysis, we shall first neglect the residual energy, E^R. This is justified by the very low level of E^R, due to the Alfven effect, as compared to the + and − spectra (see Figs. 1 and 2). The relevant equations ("reduced equations") were obtained in Paper I and read (see Appendix A for definition of Δ_k and $m_{k_{pq}}$):

$$\partial E_k^\pm/\partial t = T_k^\pm - 2\nu k^2 E_k^\pm + F_k^\pm, \tag{2.9}$$

$$T_k^\pm = \int_{\Delta k} dp\,dq\,(k+p+q)^{-1}(m_{kpq}/p)(k^2 E_p^\pm E_q^\mp - p^2 E_q^\mp E_k^\pm).$$

Note that the transfer integrals in (2.9) converge in the interval $0 < m^\pm < 3$; the lower bound corresponds to a divergence of the eddy viscosity (with the triad relaxation time dominated by the Alfvén sweeping time) and the upper bound to a divergence of the enstrophy.

The equilibrium condition at a scale K^{-1} in the inertial range (far from both dissipative and injection scales) may be written:

$$\Pi_K^{\pm\,(0)} = -\int_0^K dk\, T_k^\pm = \varepsilon^\pm, \tag{2.10}$$

where $\Pi_K^{\pm\,(0)}$ is the flux through wave number K in the reduced equations. If we now look for power-law spectra $E_k^\pm = C^\pm k^{-m^\pm}$ (where $C^\pm$ are properly dimensioned), and we use Eqs. (2.9) and (2.10), we obtain:

$$\varepsilon^\pm = k^{3-m^+-m^-} C^+ C^- G(m^\pm, m^\mp), \tag{2.11}$$

where $G(m,n)$ is independent of k and is *not* symmetric under exchange of m, n. From the two equations, (2.11), we immediately deduce

$$m^+ + m^- = 3 \tag{2.12a}$$

and

$$\varepsilon^+/\varepsilon^- = G(m^+, m^-)/G(m^-, m^+), \tag{2.12b}$$

where G is a complicated integral expression.

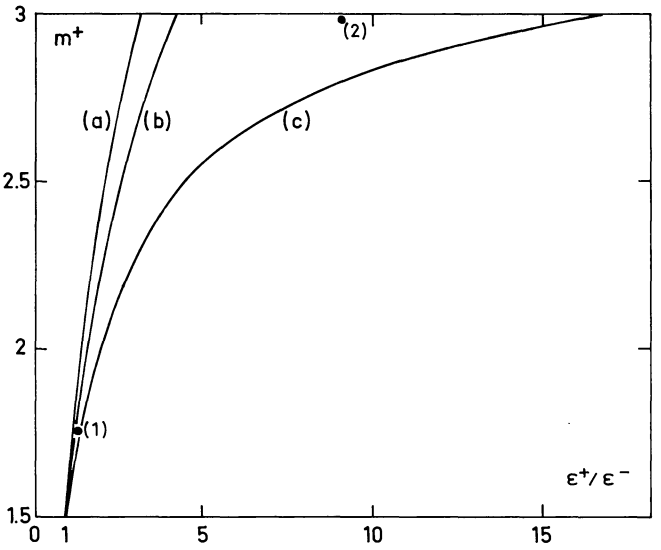

Fig. 3. Relation between the slopes $m^\pm$ of the ± energy spectra in the inertial range and the relative correlation injection rate $r = \varepsilon^+/\varepsilon^-$; fluxes ε^+ and ε^- are calculated for different slopes m^+ and $m^- = 3 - m^+$ over 16 octaves. (*a*) $F = 4$ wavenumbers per octave. No non local term included. (*b*) $F = 8$ wavenumbers per octave. No non local term included. (*c*) $F = 4$ wavenumbers per octave. Non-local terms explicitly included. (1) and (2) correspond to the complete calculations of Figs. 1 and 2 (iterative calculations with the full equations using 8 to 10 octaves)

We give in Fig. 3 a numerical evaluation of relation (2.12b): the reduced fluxes (2.10) produced by given power law spectra $E_k^\pm \sim k^{-m^\pm}$ extending on 16 octaves have been calculated, varying m^+ between 3/2 and 3 with $m^+ + m^- = 3$. When $m^- \to 0$ (or $m^+ \to 0$), the contribution to the ± fluxes of widely separated scales becomes important: in fact, at infinite Reynolds numbers, the transfer integrals $T_k^\pm$ diverge when $m^\pm \to 3$. In the numerical calculations, these non-local contributions have to be introduced explicitly (see Pouquet et al., 1976) since the limited number of points F per octave of wavenumbers ($F = 4$) excludes most of these interactions. Curves (*a*) and (*b*) were obtained without introducing the non-local terms in the calculation, with $F = 4$ (*a*) and $F = 8$ (*b*) points per octave respectively. Curve (*c*) includes explicitly non-local interactions, with $F = 4$. Figure 3 also reproduces the two values of the flux associated with the equilibrium spectra of Figs. 1 and 2. The discrepancy between curve (*c*) and points (1) and (2) arises from the fact that iterative solutions displayed in Figs. 1 and 2 had an inertial range restricted to only 8 to 10 octaves. The relation between inertial ranges and fluxes will be discussed further in the next section.

Spectral indices: the full equations

Relations (2.12) were obtained starting from the reduced equations. Let us now examine the complete equations given in Appendix A. When the residual energy is identically zero, only one constraint for the two slopes ($m^+ + m^- = 3$) obtains by imposing that the + and − fluxes are independent of wavenumber. If we now apply the same argument to the non-reduced fluxes Π_k^+, Π_k^-

and Π_k^R we obtain the three conditions: $m^+ + m^- = 3$, $m^\pm + m^R = 3$, $2m^R = 3$, which lead to all indices equal to 3/2. However, as already noticed, this power law solution is not reached in the numerical calculations performed with the complete equations. Concerning the residual energy, a simple argument can be given which yields $E_k^R \sim k^{-2}$. Let us write the equations for the residual energy in the following way:

$$\partial E_k^R/\partial t = T_k^{\text{local}} + T_k^{\text{nonlocal}},$$

where dissipative and forcing terms have been omitted. T_k^{local} represents the interaction between modes of comparable wavenumbers ($p \approx q \approx k$). T_k^{nonlocal} may be written, to zeroth-order in an expansion in powers of $a = \inf(k,p,q)/\sup(k,p,q)$ (cf. Pouquet et al., 1976),

$$T_k^{\text{nonlocal}} \simeq -E_k^R/\tau_k^A,$$

where $\tau_k^A = k^{-1}\left(\int_0^{ak} E_{k'}^M\, dk'\right)^{-1/2} \simeq (k B_0)^{-1}$ is the Alfvén time. If one were to discard the local terms, one would obtain complete equipartition between kinetic and magnetic energy ($E_k^R = 0$) after an Alfvén time. A dimensional evaluation of the local terms yields $T_k^{\text{local}} \approx \tau_k^A k^3 E_k^+ E_k^-$. We thus find for the equilibrium residual spectrum a low, but non vanishing value

$$E_k^R \approx (\tau_k^A)^2 k^3 E_k^+ E_k^-.$$

Using again $E^\pm = C^\pm k^{-m^\pm}$ with $m^+ + m^- = 3$, we obtain

$$E_k^R \approx C^+ C^- k^{-2} B_0^{-2} \tag{2.13}$$

which is the announced k^{-2} spectrum. We found numerically that both the slope and the level at which the relative energy spectrum saturates agree with relation (2.13), with an excess of magnetic energy in the inertial range ($E_k^R < 0$).

We may now formally write the non-reduced fluxes as a sum of three terms in ascending powers of the residual energy:

$$\Pi_k^\pm = \sum_{n=0}^{2} \Pi_k^{\pm(n)} O(E_k^R)^n),$$

where $\Pi_k^{\pm(0)}$ are the reduced fluxes (2.10). We now have

$$m^+ + m^- = 3 + O(E^R), \tag{2.14a}$$

$$\varepsilon^+/\varepsilon^- = [G(m^+, m^-)/G(m^-, m^+)](1 + O(E^R)). \tag{2.14b}$$

This relation shows that, since the residual energy is negligible (although not quite zero) in the inertial range because of the Alfvén effect, the flux of correlation cannot adjust to large injection of correlation without adopting different slopes for + and − spectra. The slopes given by Eq. (2.14) should be close to those obtained in the reduced case [Eq. (2.12)], as is indeed seen in Figs. 1 and 2 and Table 2.

A few comments

According to Eq. (2.12), we may have power law stationary solutions for Alfvénic turbulence that differ from the 3/2 law first obtained by Kraichnan (1965) in the uncorrelated case. Relation (2.12) is compatible with Kraichnan's result since, when $m^+ = m^- = 3/2$, the total (kinetic plus magnetic) energy spectra, $E_k^T = (E_k^+ + E_k^-)/2$, also obeys a 3/2 law. When $m^+ \neq m^-$, the total energy will no longer follow a power law anymore; however, when the correlation between the velocity and magnetic field is high, i.e. when the asymmetry between the ± variables is strong, the total energy will again follow approximately the power law of the dominant species. This can be readily observed in Figs. 1 and 2 where $E^+ \gg E^-$, corresponding to a positive correlation; we see in Figs. 1 and 2 that the total energy and the + spectra have identical spectral indices for a wide range of wavenumbers.

The main content of Eq. (2.12) is that the 3/2 law with $m^+ = m^-$ obtains only when the correlation flux; $\varepsilon^c = (\varepsilon^+ - \varepsilon^-)/4$, is zero[1]. This can in fact be readily seen on the reduced equations. Take for the ± energy spectra power laws $E_k^\pm \sim k^{-m^\pm}$ with equal spectral indices. Direct examination of the transfer terms in Eq. (2.9) then yields $T_k^+ = T_k^-$ and thus $\varepsilon^+ = \varepsilon^-$. This in turn implies that when the injection rates of the ± spectra are different ($\varepsilon^+ \neq$ spectra are different ($\varepsilon^+ \neq \varepsilon^-$) the spectral indices cannot be equal. This departure from the 3/2 law arises from the asymmetry between the two variables $z^\pm = v \pm b$ when the correlation between the velocity and magnetic field is non zero [since $E^c = (E^+ - E^-)/4$].

Finally, we note that relations (2.12) indicate that the respective values of the spectral indices m^+ and m^- depend on the ratio of the injection rates. The phenomenological study presented in the next section will give a simple evaluation of such a relation.

We now look at the equilibrium correlation coefficient ϱ or equivalently at the ratio $E^+/E^- = (1+\varrho)/(1-\varrho)$. Let us first remark that the correlation coefficient reached in a stationary state is substantially greater than the injection coefficient r, or equivalently that for $\varepsilon^+/\varepsilon^- > 1$ we have:

$$E^+/E^- \gg \varepsilon^+/\varepsilon^-. \tag{2.15}$$

Since $\varepsilon^\pm$ are equal to the dissipation rates of ± energies,

$$D^\pm = 2\nu \int k^2 E_k^\pm\, dk \quad \text{we can rewrite} \tag{2.15}$$

as

$$E^+ D^- - E^- D^+ > 0. \tag{2.16}$$

Transposing now this inequality to the decay problem for which $dE^\pm/dt = -D^\pm$; we get:

$$d(E^+/E^-)/dt > 0 \tag{2.17}$$

or, equivalently, $d\varrho/dt > 0$. Thus, the inequality (2.15) is consistent with the growth of the correlation coefficient in the decay problem (cf. Paper I).

Another feature of the numerical results of Figs. 1 and 2 is that the ± spectra become equal in the dissipation range which begins at wavenumber k_D. It follows that the ratio E^+/E^- (and hence the correlation coefficient) will depend on k_D, that is on viscosity. On the other hand, since the non-linear transfer terms converge in the range $0 < m^\pm < 3$, the spectral indices of the ± spectra are independent of viscosity for given fluxes. We thus may conjecture that the correlation coefficient tends to unity when the dissipation wavenumber tends to infinity (vanishing viscosity). Note that this implies that the coefficients $C^\pm$ appearing in the expression of the ± energy spectra are viscosity-dependent if we are to have a universal relation between fluxes $\varepsilon^\pm$ and slopes $m^\pm$. We show in the next section that such ideas may be incorporated into a phenomenological description of MHD turbulence when the role of correlation is taken into account.

1 When there is no correlation injection, the correlation spectrum decays to zero. Notice that this fact is not contradictory with the growth of correlation coefficient in the unforced case studied in Paper I

3. Phenomenology of correlated stationary MHD turbulence

A phenomenology of Alfvénic turbulence was first proposed by Kraichnan (1965) for uncorrelated MHD. We derive in this section a modified phenomenology which takes into account the presence of velocity-magnetic field correlation. We shall show that its predictions are in general agreement with the EDQNM results of Sect. 2, namely that (i) for a given Reynolds number, the spectral index of the energy spectrum and the correlation coefficient grow together, (ii) to a given relative correlation injection r corresponds a much higher steady state correlation coefficient ϱ.

When the correlation is not zero, it is convenient to work with the $z^{\pm}$ variables, corresponding to the two $\pm$ energy invariants ($E^{\pm}$) in the non-dissipative case (see Table 1). Let us first recall the main ingredients of the phenomenology of Alfvénic turbulence; by Alfvénic turbulence, we mean a turbulence in which the Alfvén time $\tau_A = l/B_0$ at scale l (B_0 = magnetic field of large scale eddies $= (2E^M)^{1/2}$, with E^M the total magnetic energy) is much smaller than the non-linear interaction time of $\pm$ eddies $\tau^{\pm}_{NL} = l/[2E^{\mp}(l)]^{1/2}$ ($E(l)$ = energy contained in scale l). Because of the Alfvén waves, the time of one coherent interaction between $\pm$ eddies cannot be larger than τ_A and the typical time to transfer energy $\tau^{\pm}_*$ at scale l is much longer than the non-linear time $\tau^{\pm}_{NL}$:

$$\tau^{\pm}_* = \tau^{\pm}_{NL}(\tau^{\pm}_{NL}/\tau_A) = lB_0/E^{\pm}(l). \tag{3.1}$$

Espressing the $\pm$ energy fluxes $\varepsilon^+ = E^+(l)/\tau^+_*$ and $\varepsilon^- = E^-(l)/\tau^-_*$, one finds that:

$$\varepsilon^+ = \varepsilon^- = E^+(l) \cdot E^-(l)/(lB_0) \tag{3.2}$$

(Dobrowolny et al., 1980).

Relation (3.2) led Dobrowolny et al. (1980) to conjecture that the velocity magnetic field correlation (whose rate of transfer is $\varepsilon^C = (\varepsilon^+ - \varepsilon^-)/4$) is not dissipated, whereas the total energy is. Relation (3.2) gives Kraichnan (1965) 3/2 range since equal fluxes imply identical + and − spectra.

However, we know from Sect. 2 that equilibria with non-zero correlation flux ($\varepsilon^+ = \varepsilon^-$) do exist, in the framework of the EDQNM approximation. To be able to describe these states, we need to modify the phenomenology; in particular, we must break the $\pm$ symmetry in relation (3.2) due to the local (in wavenumber space) evaluation of the transfer times $\tau^{\pm}_*$. In fact, a given + mode at wavenumber k interacts with a set of − modes whose characteristic wavenumbers lie in a band ($k_{\text{inf}} \leqq k \leqq k_{\text{sup}}$); we may suppose that $k_{\text{inf}}/k \approx k/k_{\text{sup}}$ independent of k if self-similarity holds. We now replace the local expression $\varepsilon^+ \approx k E^+_k/\tau^+_*$ (with τ_* given by (3.2)) by

$$\varepsilon^+ \approx k E^+_k \tau_A \langle (\tau^+_{NL})^{-2} \rangle, \tag{3.3}$$

where brackets mean averaging of the non-linear time in the band (k_{inf}, k_{sup}). Evaluating (3.3) yields:

$$\varepsilon^+ \approx F^+ k^3 E^+_k E^-_k / B_0, \tag{3.4}$$

where F^+ is independent of wavenumber but depends on the spectral indices $m^{\pm}$ (notice that when $m^+ \to 3$ and $m^- \to 0$, non-local interactions should become important and k_{sup}/k or k/k_{inf} should increase). Relation (3.4) shows that the only constraint on the slopes $m^{\pm}$ ($E^{\pm}_k = C^{\pm} k^{-m^{\pm}}$) is

$$m^+ + m^- = 3, \tag{3.5}$$

in agreement with the closure result. To evaluate the coefficients $F^{\pm}$, let us write that in the stationary state, injection and dissipation rates of the $\pm$ energies are equal, namely

$$\varepsilon^{\pm} = 2\nu \int_0^{\infty} dp p^2 E^{\pm}_p \approx 2\nu \int_{k_0}^{k_{\pm}} dp p^2 E^{\pm}_p,$$

where $l_0 = k_0^{-1}$ is the scale at which energy is injected. Here $k_{\pm}$ denotes the wavenumbers at which dissipation of + or − energy dominates its non-linear transfer. It is evaluated by writing that at k_+ (say) the viscous time $\tau^+_{\text{vis}} = 1/(\nu k_+^2)$ (for a unit Prandtl number) is equal to the transfer time τ^+_* [see Eq. (3.1)]; this yields

$$E^-_{k^+} = E^+_{k^-} = \nu B_0. \tag{3.6}$$

This expression is further simplified by noting that k_+ and k_- are comparable in magnitude. Indeed, let us suppose that they are not e.g. $k_- \ll k_+$. At an intermediate scale k; $k_- < k = 1/l < k_+$ the − energy $E^-(l)$ becomes negligible since it has an exponential decay in the dissipation zone; hence the transfer time for the + energy $\tau^+_* = B_0/E^-(l)$ rapidly becomes larger than the dissipation time $\tau^+_{\text{diss}} = 1/(\nu k_+^2)$. Thus we are led to conclude that

$$k_+ \approx k_- \equiv k_D. \tag{3.7}$$

Equation (3.6) then says that at the dissipative scale $1/k_D$, the $\pm$ energy spectra become equal to νB_0, a feature that was observed on the numerical calculations of the EDQNM closure (see Figs. 1 and 2). Finally we find from Eq. (3.5) through (3.7) that in the stationary state, the ratio of the $\pm$ energy fluxes varies as

$$\frac{\varepsilon^+}{\varepsilon^-} = \frac{(3-m^-)(1-(k_0/k_D)^{3-m^+})}{(3-m^+)(1-(k_0/k_D)^{3-m^-})}. \tag{3.8}$$

When $k_D/k_0 \to \infty$ (infinite Reynolds number), the expression (3.8) reduces to

$$\varepsilon^+/\varepsilon^- = m^+/m^-. \tag{3.9}$$

When k_D/k_0 is large but finite, Eq. (3.9) is a good approximation, except when $m^{\pm}$ approach 0 and/or 3. For example, in the case $m^- = 3 - m^+ \to 0$, we obtain:

$$\varepsilon^+/\varepsilon^- \to 3\log(k_D/k_0). \tag{3.10}$$

We have thus shown that the spectral indices grow with the corresponding injection rates and hence that the total energy slope grows with the correlation injection.

We can now estimate the level at which the ratio E^+/E^- of the total energies will saturate, using (3.6) and (3.7) and the expression for the energies $E^{\pm}$ integrated from k_0 to k_D. We find

$$E^+/E^- \approx (k_D/k_0)^{\hat{m}^+ - \hat{m}^-}, \tag{3.11}$$

where $\hat{m} = \sup(1, m)$ (a logarithmic expression arises when $m^{\pm} = 1$). This formula is verified in the EDQNM calculations (within a factor 2) for values of E^+/E^- differing by fours orders of magnitude. Relation (3.11) predicts that the correlation coefficient $(E^+ - E^-)/(E^+ + E^-)$ saturates at values ϱ_s depending on viscosity ν ($|\varrho_s| < 1$ for $\nu \neq 0$). However, in the limit of zero viscosity, the dissipation wavenumber k_D should go to infinity and the correlation coefficient then reaches its maximum value ± 1.

Relation (3.8) indicates an asymmetry in the $\pm$ fluxes and suggests the choice $F^{\pm} = 1/m^{\pm}$ in Eq. (3.4). The fluxes then become

$$\varepsilon^{\pm} = E^{\pm}(l)/(m^{\pm}\tau^{\pm}_*) = k^3 E^+_k E^-_k/(m^{\pm} B). \tag{3.12}$$

This relation between $\pm$ fluxes and slopes can be recovered by evaluating the flux at scale $1/k$, using a turbulent viscosity, viz:

$$\varepsilon^+_k \approx \nu^+_{\text{turb}} k^3 E^+_k, \quad \text{where} \quad \nu^+_{\text{turb}} = \int_k^{\infty} dp\, \tau_A E^-_p = \int_k^{\infty} dp\, E^-_p/(pB_0)$$

takes into account the contribution of small scales.

We finally determine the dependence of total energies $E^{\pm}$ on viscosity. For this, we need to express the $C^{\pm}$ factor appearing in front of the energy spectra $E_k^{\pm} = C^{\pm} k^{-m^{\pm}}$. We find after some algebra that for $m^+ > 3/2$, and in the limit of vanishing viscosity:

$C^+ \sim \nu^{-2(2m^+-3)/(2m^++3)} \to \infty$;

$C^- \sim (C^+)^{-1/2} \to 0$.

This result strongly departs from the usual rule that the spectral coefficients are independent of Reynolds number (in the limit of high Reynolds). We thus find that the total + energy $E^+ \sim C^+ k_0^{1-m^+}$ goes to infinity when $\nu \to 0$. When $m^- > 1$, we readily have $E^- \sim C^- k_0^{1-m^-} \to 0$ when $\nu \to 0$ (when $m^- < 1$, the analysis is a bit longer but yields the same result).

4. Discussion

Summary

We give in this paper an extension of the phenomenology of Alfvénic turbulence (Kraichnan, 1965) when velocity magnetic-field correlations are taken into account. In terms of the Elsässer variables $z^{\pm} = v \pm b$, the main feature of the proposed phenomenology is that a non-zero injection rate of correlation ε^C modifies the spectral indices $m^{\pm}$ of the $\pm$ steady state energy spectra defined in Table 1 ($m^+ + m^- = 3$ with $0 < m^{\pm} < 3$). The equilibrium values of $m^{\pm}$ depend on the ratio of the $\pm$ injection rates (or, equivalently, on ε^C). The $\pm$ energy spectra become equal at the onset of the dissipation range. The ratio of the $\pm$ total energies, and the correlation coefficient ϱ defined in (2.1), grow with increasing Reynolds number. Thus, for a given (however small) correlation injection rate, one may always obtain a steady-state value of the correlation coefficient arbitrarily close to unity granted the viscosity be small enough (this paper considers only the case of unit magnetic Prandtl number). For such strong correlations, the slope of the total (kinetic plus magnetic) energy lies in the range between 3/2 and 3. In contrast to hydrodynamic and uncorrelated MHD turbulence, we note that the equilibrium total energy does depend on the molecular dissipation parameters, although the spectral index does not in the limit of vanishing viscosity.

Considerations on the decay problem

The above results may be partially extended to the decay problem. In Paper I we studied the decay problem for either low correlation or rather short times, in which cases no departure from the $-3/2$ slope was found. We thus made a run starting with 80% correlation, which led to a significant increase both of the correlation and of the slope of the + spectrum: from $m^+ \simeq 1.5$ at time 13 to $m^+ \simeq 1.7$ at time 38 (correlation 98%) (see Fig. 4). On the other hand, the same run shows that the − spectrum has a peculiar behaviour: a plateau develops in the large scales followed by a zone in which relation (3.5) holds approximately, down to the dissipation range. However, we find that this zone shrinks with time, at the expense of the plateau. The flux of − energy increases with wavenumber in the zone corresponding to the plateau whereas the flux of + energy is fairly independent of wavenumber.

This shows that the cascade process is far from stationary, as far as the − energy is concerned. Let us comment this last point. Note that a typical time for transferring + energy varies as $1/E^-(l)$ (and vice versa). For a correlation of 98% (as is the case in Fig. 4 at $t = 38$), the relative ratio of $\pm$ energies (and thus of $\pm$ characteristic times) is of the order of 1%: one is approximately in a steady state for the + energy but a quasi stationnary phenomenology cannot apply to the − energy since its reservoir is small and the characteristic time to empty it is short. For example relation (3.11) cannot hold for long times since on the one hand the dissipation wavenumber decreases with time (as the reservoir of energy diminishes) and yet the correlation coefficient probably continues to grow, however slowly. For high correlations, the non-linear terms are almost vanishing and, as conjectured in Paper I, the turbulence is frozen in, even though the viscosity and magnetic diffusivity take finite values. Notice however that a phenomenology of such non stationary turbulence is still missing.

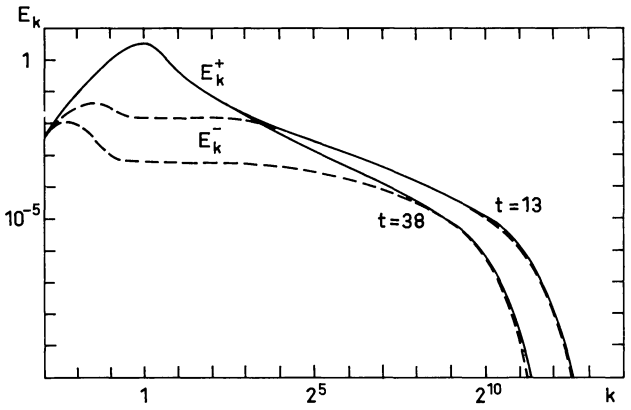

Fig. 4. Decaying + (solid line, and − (broken line) energy spectra starting from high correlation state (no forcing). The correlation coefficient is $\varrho = 80\%$ at $t = 0$ and reaches $\varrho = 98\%$ at $t = 38$. Note the steepening of the + energy spectrum slope and the corresponding V–M pectra slopes: $m^{\pm} \simeq 1.5$ at $t = 13$ and $m^+ \simeq 1.7$ at $t = 38$

Conclusion

We find in this paper a good agreement between the phenomenology of correlated MHD turbulence and the numerical results obtained with the EDQNM approximation. Closures are known to capture inertial scale properties of the flow such as the direction of cascades. It can be remarked that nonlinear iteractions lead to two direct cascades (energy and correlation) at different rates. Closures preserve enough of the structure of the primitive MHD equations to be able to capture this phenomenon (the symmetry by exchange of the magnetic field b into $-b$ may be the root of the growth of correlation coefficient; see Frisch, 1981). It would be worth considering the $2D$ problem since in this case there is a known discrepancy, concerning the behaviour of small scales, between closures (Pouquet, 1978) and direct numerical simulation at high resolution (Frisch et al., 1982, in preparation).

Finally, we would like to stress that the $z^{\pm}$ fields (as opposed to velocity v and magnetic field b) seem to be an essential ingredient in the description of correlated MHD turbulence. In particular, the symmetry between v and b variables due to Alfven waves exist for all correlations; but the global symmetry between the $\pm$ variables is unstable to small perturbations and evolves towards a state in which the large scales of the velocity and the magnetic fields are correlated. We therefore suggest that both observational data

(obtained in the solar wind) and numerical data of two and three dimensional simulations of MHD turbulence should be analyzed in terms of the Elsässer variables and not, as is customary, in terms of the kinetic and magnetic variables. For example, the various values of spectral indices for the energy spectra in the solar wind that are found in the literature are in general consistent with relation (3.5) which predicts a power law k^{-m} with m between 3/2 and 3 for the total (kinetic plus magnetic) energy spectrum at high correlations. One should look in particular for parallel variations of the correlation coefficient ϱ and of the spectral indices of the $(v \pm b)$ spectra.

Appendix A

The EDQNM spectral equations for homogeneous non-helical MHD turbulence with velocity magnetic field correlations are given below, taken from Table 2 of Grappin et al. (1982)

$$\left(\frac{\partial}{\partial t}+(\nu+\lambda)k^2\right)E_k^+ +(\nu-\lambda)k^2 E_k^R$$

$$= \int_{\Delta k} dp\,dq\,\theta_{kpq}(T_{+-}^+ + T_{RR}^+ + T_{+R}^+) + F_k^+,$$

$$\left(\frac{\partial}{\partial t}+(\nu+\lambda)k^2\right)E_k^- +(\nu-\lambda)k^2 E_k^R$$

$$= \int_{\Delta k} dp\,dq\,\theta_{kpq}(T_{+-}^- + T_{RR}^- + T_{-R}^-) + F_k^-,$$

$$\left(\frac{\partial}{\partial t}+(\nu+\lambda)k^2\right)E_k^R + (\nu-\lambda)k^2\left(\frac{E_k^+ + E_k^-}{2}\right)$$

$$= \int_{\Delta k} dp\,dq\,\theta_{kpq}(T_{RR}^R + T_{+R}^R + T_{-R}^R + T_{+-}^R),$$

$$T_{+-}^+ = \frac{m_{kpq}}{p}(k^2 E_p^+ E_q^- - p^2 E_q^- E_k^+),$$

$$T_{RR}^+ = \frac{S_{kpq}}{k} k^2 E_p^R E_q^R - \frac{t_{kpq}}{p} p^2 E_q^R E_k^R,$$

$$T_{+R}^+ = -\frac{s_{kpq}}{k} p^2 E_q^+ E_k^R + \frac{r_{kpq}}{q} p^2 E_q^R E_k^+.$$

One finds the $T_{..}^-$ terms by exchanging + and − superscripts in the above expression of the $T_{..}^+$ terms.

$$T_{RR}^R = \frac{m_{kpq}}{p} k^2 E_p^R E_q^R + \frac{r_{kpq}}{q} \cdot p^2 E_q^R E_k^R,$$

$$T_{+R}^R = -m_{kpq} p \frac{E_q^+ E_k^R}{2} - t_{kpq} p \frac{E_q^R E_k^+}{2},$$

$$T_{-R}^R = -m_{kpq} p \frac{E_q^- E_k^R}{2} - t_{kpq} p \frac{E_q^R E_k^-}{2},$$

$$T_{+-}^R = \frac{s_{kpq}}{k}\left\{k^2 E_p^+ E_q^- - p^2\left(\frac{E_q^- E_k^+ + E_q^+ E_k^-}{2}\right)\right\}.$$

Δ_k is a subset of the p, q plane such that k, p and q can form a triangle, the cosines of the interior angles opposite to k, p, q being x, y and z.

$$m_{kpq} = (k/2q)(1-y^2)(1+z^2),\ s_{kpq} = -yz(1-x^2)/2,$$
$$r_{kpq} = -xy(1-z^2)/2,\ t_{kpq} = -zx(1-y^2)/2;$$

with:

$$m_{kpq}/p = m_{pkq}/k,\ s_{kpq} = r_{pqk},\ t_{kpq} = s_{pqk}.$$

The triad relaxation $\theta_{kpq}(t)$ and the relaxation rate μ_k are given by:

$$\theta_{kpq}(t) = t/(1+(\mu_k+\mu_p+\mu_q)t),$$

where μ_k is the sum of the characteristic frequencies of the three main effects contained in the primitive equations (1), namely nonlinear scrambling of the flow (τ_{NL}^{-1}), the Alfvén effect (τ_A^{-1}), and the viscous and Joule dissipation (τ_D^{-1}):

$$\mu_k = \tau_{NL}^{-1} + \tau_A^{-1} + \tau_D^{-1} = C_{NL}\int_0^k q^2(E^M(q)+E^V(q))\,dq$$
$$+ (1/\sqrt{3})k\left\{\int_0^k 2E^M(q)\,dq\right\}^{1/2} + (\lambda+\nu)k^2.$$

The numerical constant C_{NL} is chosen so as to agree with experimental data on the Kolmogorov constant in the non-magnetic case ($C_{NL}=0.26$). The constant $1/\sqrt{3}$ is given by an explicit calculation of the relaxation rate of triple correlation in the case of a gaussian large scale field (cf. Pouquet et al., 1976)

All calculations are performed with an exponential discretization in wavenumber $k_i = k_0 2^{i/F}$, with $F=4$. Non local terms, as described in Paper I, were included explicitly.

Note that $F_k^{\pm} = F_k^V + F_k^M \pm 2F_k^C$, where the identical kinetic and magnetic energy injection spectra are defined in Eq. (2.3) and the correlation injection spectrum F_k^C is defined in Eq. (2.5).

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