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Time correlations and random sweeping in isotropic turbulence

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The Lagrangian and Eulerian frequency spectrum in isotropic turbulence is considered without a mean flow, concentrating on its second moment, the mean-square acceleration. The pressure and viscous contributions are reviewed and the scaling properties of the advective term $\langle[(\mathbf{v}\cdot\nabla)\mathbf{v}]^2\rangle$ are examined. Random sweeping from this term is shown to be dominant at large Reynolds numbers if the fluctuation spectrum of the kinetic energy $v^2(x)$ scales as $k^{-5/3}$. If on the other hand it satisfies the same Kolmogorov scaling as the pressure going as $k^{-7/3}$, then the recent renormalization group prediction of no sweeping is recovered. This question is subject to direct experimental resolution. The experiments of Van Atta and Wyngaard [J. Fluid Mech. **72**, 673 (1975)] strongly indicate that the spectrum of v^2 goes as $k^{-5/3}$ at high Reynolds numbers, thereby supporting the sweeping hypothesis.

I. INTRODUCTION

At sufficiently high Reynolds numbers, the small-scale fluctuations in incompressible fluid turbulence are universal and locally isotropic. These fluctuations are described, to a good first approximation, by the 1941 Kolmogorov theory,[1] in which the average rate of energy dissipation per unit mass ϵ and the kinematic viscosity ν are assumed to be the only relevant dynamical parameters. The most familiar consequence of this theory is the spectral distribution of energy in wavenumber

$$E(k) = \epsilon^{2/3} k^{-5/3} f(k\eta), \tag{1}$$

where

$$\eta = (\nu^3/\epsilon)^{1/4} \tag{2}$$

is the Kolmogorov microscale, and the scaling function $f(x)$ is constant at small x and decays approximately exponentially at large x. We will not be concerned in this paper with possible corrections to the 1941 Kolmogorov theory because of intermittency.

Similar arguments[1] can be applied to the frequency spectrum following the motion of a fluid particle, which we call the Lagrangian frequency spectrum. This is given by

$$E_L(\omega) = \epsilon\omega^{-2} g(\omega/\omega_0), \tag{3}$$

where

$$\omega_0 = (\epsilon/\nu)^{1/2} \tag{4}$$

is the only characteristic frequency that can be constructed from ϵ and ν. Equations (1)–(4) are consequences of dimensional analysis plus the 1941 Kolmogorov assumption. To calculate the Eulerian frequency spectrum $E(\omega)$ evaluated at a fixed spatial point, the Taylor frozen turbulence assumption is usually invoked, which states that the turbulent structure is advected past the measuring probe at the mean speed U. Thus $E(\omega)$ is given by (1) with the identification

$$\omega = kU. \tag{5}$$

The Eulerian frequency spectrum in the absence of a mean flow is more controversial. The most frequently stated view is that the dominant effect at high frequencies is the sweeping of small eddies past the observation point by the large energy containing eddies. This "random sweeping" hypothesis has a long history.[2] We apply it here in the form presented by Tennekes.[3] The frequency spectrum is thus given by

$$E(\omega) = \epsilon^{2/3} u^{2/3} \omega^{-5/3} f(\omega/\omega_c), \tag{6}$$

with the cutoff frequency

$$\omega_c = u/\eta. \tag{7}$$

In Eqs. (6) and (7), u is a typical velocity for an energy containing eddy and u^2 is proportional to the turbulent kinetic energy per unit mass. With this hypothesis of random sweeping, the power laws for the spectrum are still universal, but the 1941 Kolmogorov assumption breaks down since the velocity u, which is a property of the nonuniversal large scales, enters the result.

A contrary point of view has recently been proposed by Yakhot, Orszag, and She[4] using the renormalization group (RNG) approach to fluid turbulence. Their form of the RNG theory has no random sweeping effect, so that the Eulerian frequency spectrum is essentially the same as the Lagrangian spectrum of Eq. (3). Recently, Chen and Kraichnan[5] have given strong theoretical arguments in favor of the random sweeping hypothesis. In the present paper we address this controversy by studying the second frequency moment of $E(\omega)$, which is, of course, the mean-square acceleration. In the Lagrangian case, this can be estimated from the Navier–Stokes equations plus the usual Kolmogorov scaling arguments. This is discussed in some detail in Ref. 1. The dimensional analysis and dynamical arguments agree. They depend in an essential way on an assumption about the spectrum of pressure fluctuations. In the Eulerian case, the mean-square acceleration is dominated in the case of random sweeping by the advective term

$$A^2 = \langle[(\mathbf{v}\cdot\nabla)\mathbf{v}]^2\rangle. \tag{8}$$

Tennekes estimates A^2 by a simple argument of statisti-

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cal independence, and finds that it overwhelms the Lagrangian mean-square acceleration at high Reynolds numbers. We take a different point of view, converting Eq. (8) into the mean-square divergence of the Reynolds stress tensor. If the spectrum of Reynolds stress fluctuations has the same universal form as the pressure fluctuations, which seems reasonable at first sight, then Tennekes' estimate would be wrong and the Lagrangian and Eulerian mean-square accelerations would scale in the same way with Reynolds number. This would agree with the RNG prediction and is consistent with the additional RNG result that the spectrum of pressure fluctuations and kinetic energy fluctuations both scale as $k^{-7/3}$ in the inertial subrange. This RNG result is the same that one obtains by a naive use of dimensional analysis in the 1941 Kolmogorov theory. Treating the velocity as a Gaussian process, however, leads to the equally plausible conclusion that the spectrum of kinetic energy fluctuations should scale as $k^{-5/3}$ in the inertial subrange. Fortunately these two predictions are easily distinguished by an experimental determination of the universal exponent for the spectrum of kinetic energy fluctuations. We develop these arguments below.

II. MEAN-SQUARE ACCELERATION

Simple scaling arguments for mean-square derivatives are often useful. For example,

$$\left\langle\left(\frac{\partial u}{\partial x}\right)^2\right\rangle = \int k^2 E(k)dk = \epsilon^{2/3}\eta^{-4/3} = \frac{\epsilon}{\nu} \quad (9)$$

is a standard result of turbulence theory.[1] We neglect constants of order unity throughout. Applying the same argument to the Lagrangian mean-square acceleration, we obtain

$$\langle A_{\rm L}^2\rangle = \left\langle\left(\frac{Du}{Dt}\right)^2\right\rangle = \int \omega^2 E_{\rm L}(\omega)d\omega = \epsilon\omega_0 = \epsilon^{3/2}\nu^{-1/2}. \quad (10)$$

In Eq. (10) the symbol D/Dt represents differentiation with respect to time while moving with a fluid particle. From the Navier–Stokes equations, there are two contributions to Eq. (10):

$$\langle A_{\rm L}^2\rangle = \langle A_p^2\rangle + \langle A_v^2\rangle. \quad (11)$$

The viscous contribution is

$$\langle A_v^2\rangle = \nu^2\langle(\nabla^2\mathbf{v})^2\rangle = \nu^2\int k^4 E(k)dk. \quad (12)$$

Using the 1941 Kolmogorov scaling, this is given by

$$\langle A_v^2\rangle = \epsilon^{2/3}\nu^2\eta^{-10/3} = \epsilon^{3/2}\nu^{-1/2}, \quad (13)$$

in agreement with Eq. (10). The pressure contribution is given by

$$\langle A_p^2\rangle = \rho^{-2}\langle(\nabla p)^2\rangle = \rho^{-2}\int k^2 E_p(k)dk, \quad (14)$$

where $E_p(k)$ is the spectrum of pressure fluctuations. In the 1941 Kolmogorov theory,[1] this is given by

$$E_p(k) = \rho^2\epsilon^{4/3}k^{-7/3}h(k\eta), \quad (15)$$

where the scaling function $h(x)$ is similar to $f(x)$. If Eqs. (14) and (15) are combined, we obtain

$$\langle A_p^2\rangle = \epsilon^{4/3}\eta^{-2/3} = \epsilon^{3/2}\nu^{-1/2}, \quad (16)$$

which scales with Reynolds number in the same way as Eqs. (10) and (13). In Ref. 1, a more detailed calculation is given, which indicates that the numerical coefficient we have omitted is substantially larger in Eq. (16) than in Eq. (13). For isotropic turbulence[1] the cross term $\langle A_p A_v\rangle$ vanishes.

III. RANDOM SWEEPING

In the Eulerian case, there is a third term in the mean-square acceleration, namely,

$$\langle A_s^2\rangle = \langle[(\mathbf{v}\cdot\nabla)\mathbf{v}]^2\rangle. \quad (17)$$

A reasonable estimate of this term is given by Tennekes.[3] He assumes that the small-scale velocity gradients are swept by the large-scale velocities without dynamical distortion so that Eq. (17) can be written as

$$\langle A_s^2\rangle = \langle\mathbf{v}^2\rangle\langle(\nabla\mathbf{v})^2\rangle = u^2\epsilon\nu^{-1}. \quad (18)$$

Since Eq. (18) is proportional to ν^{-1} it grows much more rapidly with Reynolds number than Eq. (13) or Eq. (16). Random sweeping, if present, will dominate the mean-square Eulerian acceleration. The same result is obtained directly from Eqs. (6) and (7), which give

$$\langle A_s^2\rangle = u^{2/3}\epsilon^{2/3}\omega_c^{4/3} = u^2\epsilon\nu^{-1}. \quad (19)$$

IV. REYNOLDS STRESS AND KINETIC ENERGY SPECTRA

An alternative point of view to Eq. (17) can be given using the familiar Reynolds stress equation

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = v_\beta\left(\frac{\partial v_\alpha}{\partial x_\beta}\right) = \frac{\partial(v_\alpha v_\beta)}{\partial x_\beta}, \quad (20)$$

which depends only on the incompressibility condition $\nabla\cdot\mathbf{v} = 0$. The sweeping term in the mean-square acceleration is then given by

$$\langle A_s^2\rangle = \int k^2 E_R(k)dk, \quad (21)$$

where the Reynolds stress tensor R is $v_\alpha v_\beta$. First we consider the diagonal part of this tensor and assume that its scaling properties are the same as the off-diagonal part. We analyze this assumption later. The fluctuation spectrum of the diagonal part is $E_{\rm K}(k)$, where $K = v^2$ is the local kinetic energy per unit mass. If we apply straightforward Kolmogorov scaling to this quantity, we have

$$E_{\rm K}(k) = \epsilon^{4/3}k^{-7/3}q(k\eta), \quad (22)$$

where $q(x)$ is yet another scaling function of the same general form. If Eq. (22) is substituted into Eq. (21) we obtain

$$\langle A_s^2\rangle = \epsilon^{4/3}\eta^{-2/3} = \epsilon^{3/2}\nu^{-1/2}, \quad (23)$$

which is of the same order as the pressure contribution and is much smaller at high Reynolds number than Tennekes' estimate. This is in agreement with the RNG result, which predicts no random sweeping effects. In fact, Eq. (22) is an explicit prediction of the RNG theory in Ref. 4, as is the statement that the decorrelation effects included by Tennekes are absent in the RNG theory. The RNG theory gives an internally consistent set of predictions about the absence of random sweeping, and we should look for a way to test these predictions for real turbulent flows.

How would the kinetic energy spectrum have to behave to recover the Tennekes sweeping result? It is easily seen that the choice

$$E_K(k) = u^2\epsilon^{2/3}k^{-5/3}f(k\eta), \tag{24}$$

when substituted into Eq. (21), will give the random sweeping result of Eq. (18). Thus the controversy over random sweeping is closely related to the proper scaling of the kinetic energy spectrum. We cannot resolve this issue definitively here, but we can give several useful comments. One could test the Reynolds number dependence of the various contributions to the mean-square acceleration by direct numerical simulation, but the Reynolds numbers that can be studied may not be high enough to settle this question definitively.

Fortunately a direct experimental resolution is feasible. Consider an atmospheric boundary layer at very high Reynolds number where at least two decades of a $-5/3$ law can be found for the velocity spectrum $E(k)$. If the velocity signal in this situation is squared before spectrally analyzing it, the spectrum $E_K(k)$ is directly obtained. This experiment has in fact been performed by Van Atta and Wyngaard,[6] and the results strongly support an inertial range exponent of $-5/3$ for the kinetic energy fluctuations.

Equation (24) and its experimental support are at first sight surprising, since it suggests that pressure and kinetic energy scale in *different* ways. Clearly, random sweeping from the pressure term makes little physical sense. To understand why pressure and kinetic energy might scale differently, we take the divergence of the Navier–Stokes equations and use incompressibility to write

$$\nabla\cdot[(\mathbf{v}\cdot\nabla)\mathbf{v}] = \left(\frac{\partial v_\alpha}{\partial x_\beta}\right)\left(\frac{\partial v_\beta}{\partial x_\alpha}\right) = -\rho^{-1}\nabla^2 p. \tag{25}$$

It is of course familiar that the pressure is a solution of a Poisson equation with the divergence of $(\mathbf{v}\cdot\nabla)\mathbf{v}$ as the source, but the essential point here is that the source term can be written entirely in terms of velocity derivatives, and therefore refers entirely to the small-scale velocity fluctuations. Thus the case for universality is quite strong. This is exploited on p. 407 of Ref. 1 to explicitly calculate the spectrum of pressure fluctuations making a quasinormal approximation in which fourth-order cumulants are discarded.

An elementary argument suggests that the Kolmogorov scaling of Eq. (23) should be replaced by Eq. (24) for the spectrum of kinetic energy fluctuation. Consider the correlation function $\langle v^2(x)\, v^2(x+r)\rangle$ and make the simplifying assumption that $v(x)$ is a one-dimensional Gaussian random process. This assumption is wrong, but may not lead to qualitatively wrong conclusions. We then have

$$\langle v^2(x)\, v^2(x+r)\rangle = u^4 + 2[\langle v(x)\, v(x+r)\rangle]^2. \tag{26}$$

In the Kolmogorov theory, the velocity structure function

$$\langle [v(x+r) - v(x)]^2\rangle = (\epsilon r)^{2/3} \tag{27}$$

in the inertial range $\eta \ll r \ll L$. The integral length scale L is defined by $\epsilon = u^3/L$. Using this result, Eq. (26) becomes

$$\langle v^2(x)v^2(x+r)\rangle = u^4[3 - 2(r/L)^{2/3} + \tfrac{1}{2}(r/L)^{4/3}]. \tag{28}$$

The second term on the right-hand side of Eq. (28) is dominant in the inertial range and will also be dominant when taking a Fourier transform, leading to Eq. (24). We note that the experiments of Van Atta and Wyngaard give surprisingly detailed agreement with the idea that $v(x)$ is a one-dimensional Gaussian process. It has, furthermore, been emphasized by Kraichnan[5] that the 1941 Kolmogorov theory is quasi-Gaussian in an essential way, so the scaling of higher-order quantities should be estimated from a Gaussian approximation rather than from dimensional analysis.

Finally we return to the issue of the off-diagonal elements of the Reynolds stress. From the standard vector identity

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = \tfrac{1}{2}\nabla(v^2) - \mathbf{v}\times\boldsymbol{\omega} \tag{29}$$

the diagonal elements are identified with the gradient of the kinetic energy, and the off-diagonal elements with the cross-product of velocity and vorticity. Our argument has been based solely on the scaling properties of the diagonal part. Since the off-diagonal parts contain the product of two distinct stochastic variables, their scaling properties could well be different. They too might be subject to the decorrelation assumption, in which case we could write

$$\langle(\mathbf{v}\times\boldsymbol{\omega})\rangle^2 = \langle v\rangle^2\langle\omega\rangle^2. \tag{30}$$

This would make a contribution to the mean-square acceleration of the same order as the kinetic energy term, but much larger than the pressure term. Thus the scaling properties of the off-diagonal parts are not likely to affect the essential physical conclusions of this paper.

V. CONCLUSIONS

We have shown that the presence or absence of random sweeping effects in the Eulerian frequency spectrum depends on the scaling assumption that one makes about the fluctuation spectrum of the kinetic energy $v^2(x)$. The RNG prediction that this scales as $k^{-7/3}$ is consistent with the RNG prediction that sweeping effects are absent. The Tennekes prediction of random sweeping is consistent with the spectrum of the kinetic energy being proportional to $k^{-5/3}$. A simple Gaussian model supports this prediction. Unlike most problems in turbulence, this question has a clean experimental answer. Experiment strongly supports random sweeping, and emphasizes that the inertial range of the 1941 Kolmogorov theory corresponds to a quasi-Gaussian process in which higher-order correlations are not to be calculated from simple dimensional arguments.

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