Turbulent Cascade of Incompressible Unidirectional Alfvén Waves in the Interplanetary Medium

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The large-scale inhomogeneity of the solar wind is taken into account to estimate the turbulent flux due to nonlinear interactions among purely outward-traveling waves. The nonlinear interactions are mediated by secondary, incoming waves generated by the linear coupling of the dominant species to the large-scale gradients. A quasistationary self-similar turbulent cascade is possible, with a spectrum scaling as k^{-1}, close to what is found in the low-frequency range of solar-wind fluctuations near the sun.

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Two prominent and seemingly contradictory features of solar-wind fluctuations are, first, that they seem for a large part of the time to be made up of incompressible, almost pure Alfvén waves propagating outwards from the sun along the average spiral magnetic field; and, second, that they present a well developed power spectrum over many frequency decades. The spectrum varies with heliocentric distance in the inner heliosphere: Far from the sun, it is well represented by a quasi-Kolmogorov power law, while it is much less steep near the sun.[1]

A natural hypothesis to explain the apparent contradiction is that the outgoing part of the turbulent spectrum formed beyond the Alfvénic point is simply advected without modifications by the solar wind.[2] The difficulty with this idea is that purely outgoing waves do not interact nonlinearly (in the limit of incompressible fluctuations) so that the extremely inhomogeneous region near the sun should leave its imprint on the spectrum in the form of, say, preferred excited scales or absorption lines.[3] Such features are not seen in the observed spectrum. A second problem with purely outgoing Alfvén waves is that their spectrum should evolve with heliocentric distance only because of large-scale WKB effects, which have been shown to be insufficient to account for the observed changes.[4]

In order for nonlinear interactions to occur starting from a spectrum of purely outgoing waves, some source of incoming waves is necessary. The source may be the large-scale shear instabilities at the interaction regions of high- and low-speed streams.[5] Although this hypothesis of *in situ* generation may explain, for example, the observed variation of of the velocity–magnetic-field correlation with distance,[6] it cannot in itself predict a spectral slope different from that of the classical MHD phenomenology.[7]

Indeed, the above explanations do not take into account the inhomogeneity deriving from the radial expansion of the solar wind. Tu, Pu, and Wei[8] have modeled the effect of large-scale systematic gradients on turbulence by mixing together the usual phenomenological expressions for the turbulent flux[3] (valid for homogeneous turbulence) and large-scale linear WKB terms. This allows them to predict the steepening of the spectrum between 0.3 and 1 AU (astronomical unit) but it overlooks the problem of why the spectrum is initially so flat near the sun. In fact, the self-similar solution associated with the turbulent flux[3] is a $k^{-3/2}$ spectrum.[7] The question which we shall investigate here is the following: How is the turbulent flux itself modified by the large-scale inhomogeneity?

To proceed we perform a two-time-scale analysis of the solar-wind dynamic variables; i.e., we split the fields into a large-scale, quasistationary, compressible part, and a small-scale, incompressible, fluctuating component. The equations for the velocity and magnetic field fluctuations have been written by Whang.[9] We write them here in terms of Elsasser variables $\mathbf{z}^{\pm}=\mathbf{v}\pm\mathbf{b}/(4\pi\rho)^{1/2}$, where ρ is the density of the medium (with U and V_a, respectively, for the average bulk and Alfvén velocity, P for pressure, and $\langle\,\rangle$ for large-scale average):

$$d^{\pm}\mathbf{z}^{\pm}/dt=-\mathbf{z}^{\mp}\cdot\nabla(\mathbf{U}\pm\mathbf{V}_a)+\tfrac{1}{2}(\mathbf{z}^{-}-\mathbf{z}^{+})\nabla\cdot(\mathbf{V}_a\pm\mathbf{U}/2)+(1/\rho)\nabla P-\mathbf{z}^{\pm}\cdot\nabla\mathbf{z}^{\mp}+\langle\mathbf{z}^{\pm}\cdot\nabla\mathbf{z}^{\mp}\rangle\,. \quad (1)$$

In Eq. (1), the total time derivative operator $d^{\pm}/dt$ is defined by

$$d^{\pm}/dt=\partial/\partial t+(\mathbf{U}\mp\mathbf{V}_a)\cdot\nabla\,.$$

In the case of homogeneous MHD turbulence, the $z^{\pm}$ fields can be thought of as wave packets propagating in opposite directions in a frame of reference traveling with the average wind velocity along the average (uniform) magnetic field, i.e., as a superposition of Alfvén modes $z^{\pm}=\exp[i\mathbf{k}\cdot\mathbf{r}-i\mathbf{k}\cdot(\mathbf{U}\mp\mathbf{V}_a t)]$: A $k^{-3/2}$ spectrum results from the repeated interactions of such wave packets, assuming $|z^{\pm}|<V_a$.[3,7] To see how the standard Alfvén modes are modified by the inhomogeneity, let us

measure the magnitude of large-scale gradients by a small parameter ϵ:

$$|\nabla(\mathbf{U}\pm\mathbf{V}_a)| \approx |\nabla\cdot(\mathbf{V}_a/2\pm\mathbf{U}/4)| \approx \epsilon/T^0, \qquad (2)$$

where T^0 is a characteristic time (the Alfvén time at a characteristic scale L^0). The usual WKB solution, $\mathbf{z}^{\pm} = \mathbf{P}^{\pm}(\epsilon t,\epsilon\mathbf{r})\exp(i\mathbf{k}\cdot\mathbf{r}+i\phi^{\pm})$, allows for a slow change in amplitude of the Alfvén wave packets with time, as we follow the plasma convected by the wind.[10] This solution, however, is not complete:[11,12] The $O(\epsilon)$ variation in z^+ generates a secondary component in the field z^-, which travels in the same direction as the z^+ primary field (instead of the opposite direction as does the primary z^- component). Writing both fields $z^{\pm}$ as a superposition of primary ($P^{\pm}$) and secondary ($\epsilon S^{\pm}$) components,

$$\mathbf{z}^{\pm} = \mathbf{P}^{\pm}(\epsilon t,\epsilon\mathbf{r})\exp(i\mathbf{k}\cdot\mathbf{r}+i\phi^{\pm}) + \epsilon\mathbf{S}^{\pm}(\epsilon t,\epsilon\mathbf{r})\exp(i\mathbf{k}\cdot\mathbf{r}+i\phi^{\mp}), \qquad (3)$$

where $\omega^{\pm} = -\mathbf{k}\cdot(\mathbf{U}\mp\mathbf{V}_a) = \partial\phi^{\pm}/\partial t$, we obtain to first order in ϵ

$$\epsilon d^{\pm}\mathbf{P}^{\pm}/dt = -(\mathbf{P}^{\pm}/2)\nabla\cdot(\mathbf{V}_a\pm\mathbf{U}/2), \qquad (4a)$$

$$\epsilon\mathbf{S}^{\pm} = \frac{i}{2\mathbf{k}\cdot\mathbf{V}_a}\{-\mathbf{P}^{\mp}\cdot\nabla(\mathbf{U}\pm\mathbf{V}_a) + \mathbf{P}^{\mp}\nabla\cdot(\mathbf{V}_a/2\pm\mathbf{U}/4)\}. \qquad (4b)$$

The first equation gives the usual WKB amplitude variation. Equation (4b) has been written in a different form by Hollweg,[11] to study the wave contribution to the transport of angular momentum by the solar wind. In the present context, it is important to note that the secondary component depends algebraically on the primary component, which at this order evolves independently. As a consequence, assuming that the secondary fields are given by the linear relation (4b), the nonlinear evolution of the primary components will be determined by nonlinear coupling terms which may easily be written solely in terms of the primary components themselves.

Consider the particular case of the super-Alfvénic wind near the sun and suppose that it contains only outgoing waves. If, for example, the average magnetic field is pointing towards the sun, the plus and minus fields reduce, respectively, to their primary and secondary components. The resulting nonlinear coupling term is non- zero, being of the form $z^+z^- = \epsilon S^- P^+ + O(\epsilon^2)$. Let us analyze the consequences on the nonlinear energy transfer. We define the local (at a given heliocentric distance) energy spectrum $E^{\pm}(k)$ as the energy per unit volume and unit wave number k in plus and minus eddies, and consider a discrete sequence of scales $l_n = l_0 2^{-n}$, $n = 0,1,\ldots$, and of wave numbers $k_n = 1/l_n$. The kinetic energy per unit mass in scales $\approx l_n$ is defined as

$$E_n^{\pm} = \int_{k_n}^{k_{n+1}} E^{\pm}(k)\,dk. \qquad (5)$$

We assume that nonlinear interactions are between nearest-neighbor scales only, and that we are at a scale far from the dissipation scales. In the following, we also place ourselves in the frame of reference moving with the outward-traveling waves. Then the energy variation rate may be written from Eq. (1) as

$$dE_n^+/dt = -\eta E_n^+ + F_{n-1}^+ - F_n^+, \qquad (6)$$

where $\eta = \nabla\cdot(3\mathbf{U}/2-\mathbf{V}_a) = O(\epsilon)$ and F_{n-1}^+ is the nonlinear energy flux which comes into scale n from scale $n-1$. When $\eta = 0$, this relation ensures total conservation of energy over the whole inertial range. Equation (6) is an alternate (discrete) form of the one used by Tu, Pu, and Wei[8] (who have written their equation in terms of the frequency power spectrum).

The energy flux F_n^+ is approximated by the energy E_n^+ divided by the nonlinear time τ_n^+ which is itself built on the z^- field (see for instance Ref. 13 for the hydrodynamic case, and Ref. 3 for the homogeneous MHD case):

$$F_n^+ \sim E_n^+/\tau_n^+ \sim (z_n^+)^2/\tau_n^+, \qquad (7)$$

$$\tau_n^+ = 1/k_n z_n^-, \qquad (8)$$

where $z_n^{\pm}$ is a typical rms fluctuation across an eddy of size l_n, so that $E_n^{\pm} \sim (z_n^{\pm})^2$. Now, z^- reduces to its secondary component and, using Eqs. (2) and (4b), may be related to z^+ as

$$z_n^- \sim (\epsilon/T^0)z_n^+/k_n V_a. \qquad (9)$$

Replacing in Eqs. (8) and (7), the energy flux finally becomes

$$F_n^+ \sim \epsilon(z_n^+)^3/V_a T^0 \sim (\epsilon/V_a T^0)E_n^{+3/2}. \qquad (10)$$

The self-similar solution is one in which the energy is independent of the scale n: $F_n^+ = F(t)$. As is seen from Eq. (10), this leads to $E_n^+ = E(t)$, i.e., to an energy spectrum $E^+(k) \sim k^{-1}$ [cf. Eq. (6)]. To obtain the temporal variation, we rewrite Eq. (6) as

$$d(E_n^+/\omega_n)/dt = (1/\omega_n)^*(F_{n-1}^+ - F_n^+), \qquad (11)$$

where $\omega_n = \mathbf{k}_n\cdot\mathbf{V}_a$. We thus see that the self-similar solution conserves the wave action E_n^+/ω_n at each scale and has a spectrum of the form

$$E^+(k) \sim V_a(t)k^{-1}. \qquad (12)$$

We have found that a self-similar spectrum with k^{-1} power law is a possible outcome of nonlinear interactions in a population of waves propagating in a single direction in a medium with a large-scale gradient. The energy varies with time (since the wave action is conserved), but the spectrum varies in a self-similar way, maintaining its k^{-1} shape.

In Eq. (7) we have assumed that the time necessary to transfer energy towards smaller scales is the same as the nonlinear turnover time τ_n^+. A different choice can (and should) be made in the case of homogeneous MHD turbulence. Indeed, in this case, z^+z^- nonlinear interactions occur only via collisions of wave packets propagating in opposite directions along a uniform (large) average magnetic field, and since the collision (or Alfvén) time is small, repeated collisions are necessary for the energy to cascade to smaller scales. The resulting effective transfer time is thus longer than the nonlinear turnover time,[3,7] and the resulting self-similar solution is a $k^{-3/2}$ spectrum. Now, in the situation of purely outgoing waves we are considering here, the z^+ and z^- wave packets interact coherently, since they propagate in the same direction with the same velocity, and so the energy transfer time is given by the nonlinear turnover time.

How does our model compare with observational constraints? An approximate k^{-1} spectrum is found at frequencies (in the satellite's frame of reference) between 10^{-4} and 10^{-3} Hz at 0.3 AU.[1,14] On the other hand, there are three characteristic times in our model: the adiabatic time $\tau_{ad} = T^0/\epsilon$, the Alfvén time $\tau_A = 1/k_n V_a$, and the nonlinear time $\tau_n^+ = 1/kz_n^-$, associated, respectively, with large-scale gradients, the linear propagation along the average magnetic field, and the establishment of a turbulent spectrum. We have assumed that turbulence was created by the nonlinear interaction of a mixture of "primary" and "secondary" Alfvén waves: This makes sense only if the Alfvén time is smaller than the nonlinear turnover time, $\tau_A/\tau_n^{\pm} < 1$, or

$$z_n^{\pm}/V_a < 1 . \tag{13a}$$

This condition is satisfied most of the time in the solar wind (see Figs. 1 and 2, which show sample spectra of z_n^+/V_a at frequencies between 10^{-5} and 10^{-2} Hz at various heliocentric distances, as well as a plot of this ratio versus time in a medium frequency band, using Helios data analyzed in Ref. 14). Moreover, in order that the WKB solutions (3) be valid, the adiabatic time should be larger than the Alfvén time, $\tau_A/\tau_{ad} < 1$, which implies that

$$k \geq k_{\min} = \epsilon/V_a T^0 , \tag{13b}$$

which gives for the corresponding frequency in the satellite's frame $k_{\min}U/2\pi \approx (144 \text{ h})^{-1}$ (with $\epsilon/T^0 \sim \nabla \cdot \mathrm{U}/4 \geq 150 \text{ km s}^{-1} \text{AU}^{-1}$ and $V_a \approx 50$ km/s). The observed low-frequency cutoff for the k^{-1} spectrum is at a much higher frequency, about 10^{-4} Hz, possibly because at lower frequencies the temporal fluctuations reflect the rotation of coherent spatial structures such as coronal holes near the surface of the sun, rather than random turbulent processes.

We know from Helios and Voyager observations that the actual spectrum is not self-similar: Spectral shapes vary both with distance and with wind structure.

(a) The systematic steepening of the spectrum during its transport from the sun[1] has been modeled by Tu, Pu, and Wei[8] by integrating Eq. (6), using for the energy flux the expression appropriate for the homogeneous case. A natural physical explanation of this evolution is given by the relative decay with heliocentric distance of secondary-primary interactions $\epsilon S^- P^+$, which are responsible for the k^{-1} spectrum, compared to the primary-primary terms P^+P^-, associated with the formation of homogeneous turbulence, which should grow due to local production of both Alfvén species in shear structures. Notice that a small P^- term will dominate the secondary component S^- at sufficiently small scale, due to the very steep spectrum of the S^- component [Eqs. (9) and (12) lead to $E^-(k) \sim k^{-3}$]; hence it is not surprising that the spectrum has a standard shape for high frequencies at all distances from the sun. On the other hand, at the lowest frequencies, a k^{-1} spectrum should be recovered and the cutoff separating the two regions in the power spectrum should decrease with increasing heliocentric distance, as is, in fact, observed.[15]

(b) The variation with wind structure is observed in both the incoming and outgoing species, respectively correlated with temperature and the density fluctuations,[14] in the medium frequency range between 10 and $\frac{1}{2}$ h. In the high-velocity regions, the outgoing-com-

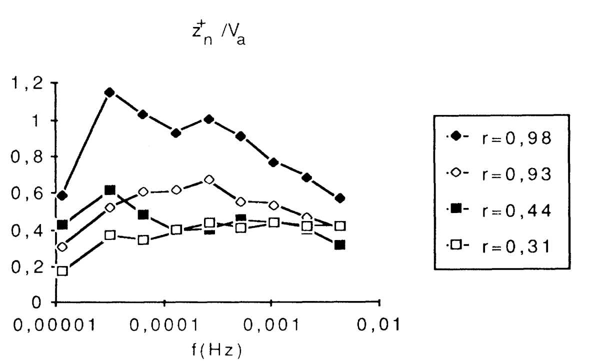

FIG. 1. Sample spectra of z_n^+/V_a, $n=1$ to 9, corresponding to the fundamental frequency $f_1 = 1 \text{ day}^{-1} = 1.16 \times 10^{-5}$ Hz and eight octaves between f_1 and 5.9×10^{-3} Hz, using Helios data analyzed in Ref. 14.

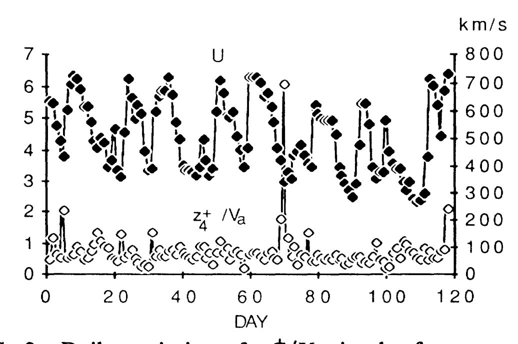

FIG. 2. Daily variation of z_n^+/V_a, in the frequency band $n=4$: [8 day^{-1}, 16 day^{-1}].

ponent spectrum falls approximately as k^{-1}, while the ingoing-component spectrum falls down much more rapidly than the dominant one: This could possibly correspond to the "slave" secondary component as described above. On the contrary, in the compressive, low-temperature and -velocity regions of the wind, one observes similar spectra for both fields: This could be due to an enhancement of the primary components by local production of both E^- and E^+ energies, leading to more conventional MHD turbulence at all scales with slopes near k^{-m}, $m \sim 1.5$ to 1.7.

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