Homework 3
Due: End of Day Friday Nov 14, 2025

1. Two events E and F are independent if $Pr(E \cap F) = Pr(E)Pr(F)$. Now consider tossing a 6-sided fair die. Are the events $E = \{1, 2, 3\}$ and $F\{4, 5, 6\}$ independent?

   The conditional probability for an event E given another event F is the probability that E happens given that F happens, and is denoted by $P(E|F)$. Calculate the conditional probability for events $E = \{1, 2, 3\}$ and $F\{4, 5, 6\}$. Recall that $P(E|F) = \frac{P(E \cap F)}{P(F)}$.

2. Perform the basic calculations for the following random variables.

   (a) Let X be an r.v. for tossing a fair coin, where $X(Head) = 1$ and $X(Tail) = -1$. Calculate $E[X]$ and $Var(X)$.

   (b) Let Y be an r.v., for tossing a 11 sided fair die, where $Y(j) = j$ for $j = 1, 2, \ldots, 11$. Calculate $E[Y]$ and $Var(Y)$.

   (c) Let Z be an r.v. defined as $Z = X + Y$. Assuming that X and Y are independent, calculate the probability mass function P_Z for Z.

3. Use Markov's Inequality or Chebyshev's Inequality to estimate the tail probabilities below for the r.v.'s as defined in Problem 2. Does the resulting probability make sense? Why or why not?

   (a) $Pr(X \geq 1)$

   (b) $Pr(Y \geq 9)$

   (c) $Pr(|Y - 6| \geq 2$

4. Poker is a game where players compare hands, and usually whoever has the hand of lower probability wins. There are many poker games. In the most common one, a hand is a set of 5 playing cards out of a deck of 52 cards. (To simplify discussion, we remove the jokers and assume that we are playing single-deck of cards here.)

   A **flush** is a hand that contains five cards all of the same suit (e.g., spades, hearts, clubs, and diamonds), but not necessarily of a sequential rank (e.g., A, K, Q, J, 10, 9, 8, 7, 6, 5, 4, 3 and 2).

   A **straight** is a hand that contains five cards of sequential rank, not necessarily of the same suit.

   Calculate the probability of flush hand and the probability of a straight hand and conclude which one is the better hand based on their probabilities. (You should assume that any 5-card hand is equally likely.)

5. In class, we have seen that the worst case running time of insertion sort is $O(n^2)$. Suppose that we randomly perturb the input sequence just like in randomized quicksort, what is the expected running time of randomized insertion sort now?

6. For a given a graph $G = (V, E)$, a vertex cover of G is a subset of vertices $C \subseteq V$ such that each edge of G has at least one endpoint in C. The goal of the vertex cover problem is to find the optimal vertex cover C^* with the minimum number of vertices.

   Consider the following randomized algorithm for vertex cover.

Step 1: Start with $C = \phi$.

Step 2: Pick an edge e uniformly at random from the edges that are not covered by C (i.e., if e has endpoints u and v, then $\{u, v\} \cap C = \phi$. and add a random endpoint of e to C.

Step 3: If C is a vertex cover, terminate and output C; else go to Step 2.

Answer the following questions:

(a) Consider the very first iteration of the algorithm. What is the probability that a vertex from the smallest vertex cover C^* is added to C? (Hint: for each edge $e \in E$, at least one endpoint of e must be in C^*.)

(b) Consider the second iteration of the algorithm. What is the probability that a vertex from the smallest vertex cover C^* is added to C? (Hint: you should discuss the two scenarios of whether a vertex from C^* is added to C in the first iteration or not.)

(c) Let k be the number of vertices in the smallest vertex cover C^*. Show that the expectation of the number of vertices of C is $2k$.

7. It is said that the great mathematician Kolmogorov once conjectured that it takes $\Omega(n^2)$ time to multiply two $n-$digit numbers. (Note that, here we are talking about arbitrary large numbers not the "standard type" numbers from the RAM model.) Answer the following questions:

- Explain the mathematical meaning of $\Omega(n^2)$ time.
- Can you show that multiplying two $n-$digit numbers is actually similar to multiplying 2 polynomials and can be viewed as a convolution?
- Show that with Fast Fourier Transform, we can multiply two $n-$digit numbers in $O(n \log n)$ time.