Extended Essay in Mathematics

Title: A comparison of different methods to find the shortest distance on the Earth ellipsoid

Research Question: To what extent can great circle and great elliptic arc be used to find the shortest distance between 2 points on an ellipsoid compared with the geodesic method?

Word count: 3861

Contents

1. Introduction

When I was young, I was confused by the curved flight lines between the departures and destinations on the 2D Map. To me, flight lines should be as straight as possible since the straight line is the shortest distance between 2 points. Later, I learned that since Earth has curved surfaces, those curved flight lines are closer to the shortest distance between departures and destinations on the surface.

The flight lines inspired me to solve this problem:

> Given the coordinates of the departure and destination points on the Earth and find the shortest distance between the 2 points on the Earth surface.

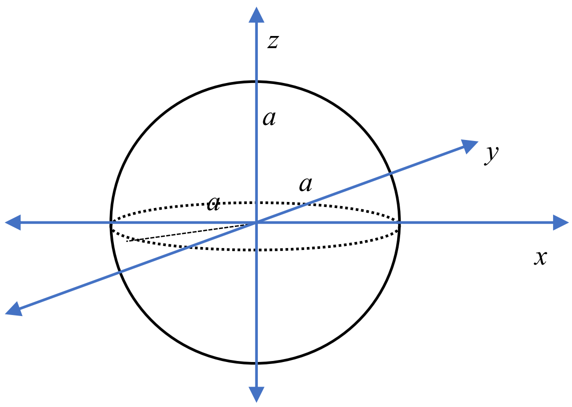
The desired shortest distance is called geodesic distance. It can be solved in many ways, most commonly approximated by great circle method. Other methods include great ellipse method and geodesic method. However, the geodesic distance calculation using geodesic method is quite complicated. Using great circle or great elliptic arc between 2 points, the arc distance and path between 2 points can also be calculated. The difference between the distance calculated using the great elliptic arc or great circle and geodesic method is quite small (Tseng et al. 287). This essay aims to present the great elliptic arc and great circle method to find the shortest distance between 2 points and compare its calculated distance and path with that obtained using geodesic method with regard to the accuracies. To compare, part of the geodesic method is also presented with formulas that will be used in comparison calculation.

2. Preliminaries

2.1 Modelling the Earth

The Earth is a planet with rough surfaces and shapes. It is usually modelled into sphere and ellipsoid for easier calculation.

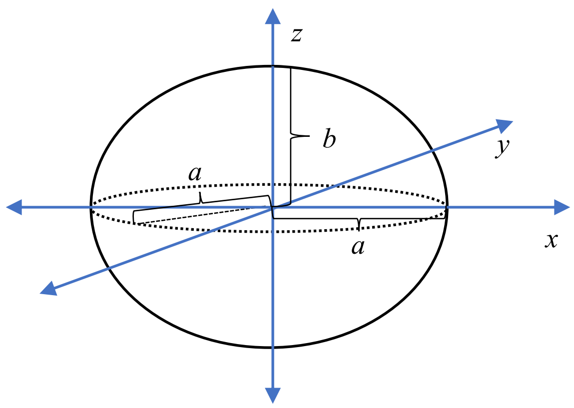
Spherical model is used in the great circle method. The radius[1] a is taken to be the average radius of the Earth.



Sphere:
$a = 6371393m$

Figure 1: Sphere

Ellipsoid model is used in the great ellipse method. The data for polar radius b and equatorial radius a use the data for Bessel ellipsoid[2].



Ellipsoid:
$a = 6377397.155m$
$b = 6356078.963km$

Figure 2: Ellipsoid

[1] "地球半径(radius of the Earth)." Baidubaike, baike.baidu.com/item/%E5%9C%B0%E7%90%83%E5%8D%8A%E5%BE%84.

[2] "Bessel Ellipsoid." *Wikipedia*, Wikimedia Foundation, 27 Aug. 2019, en.wikipedia.org/wiki/Bessel_ellipsoid. Accessed 12 Jan. 2021.

2.2 Normal sections & meridian section & prime vertical section[3] [4]

For a point A on the ellipsoid, there is an infinite number of planes passing through this point. All the planes except the tangential plane intersect with the ellipsoid and the intersecting-ellipse has a radius of curvature. The radius of curvature can be thought as the radius of the circle best approximates the elliptic curve at the point. The planes that contain the normal to the tangential plane at A is called normal sections.

Meridian section: a normal section that contains ellipsoid's axis of revolution. The radius of curvature is shown as M. This section is perpendicular to the XOY plane.

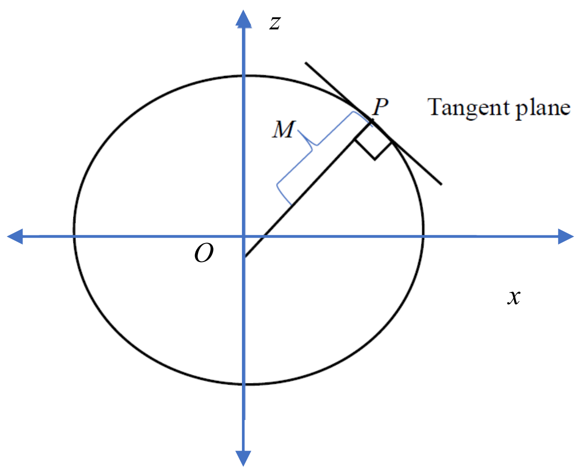


Figure 3: Meridian Section

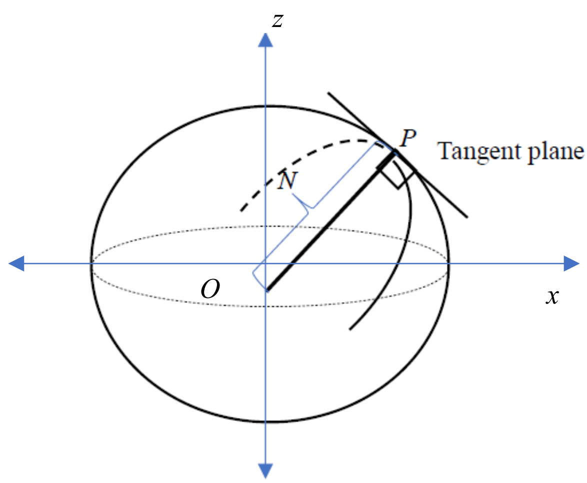


Figure 4: Prime Vertical Section

Prime vertical section: a normal section that is perpendicular to the meridian plane, radius of curvature is shown as N.

[3] Deakin, Rod, and M. Hunter. "GEODESICS ON AN ELLIPSOID - BESSEL'S METHOD." *ResearchGate*, 2009, www.researchgate.net/profile/Rod_Deakin/publication/267986122_GEODESICS_ON_AN_ELLIPSOID_-BESSEL%27S_METHOD. Accessed 20th Aug, 2020.

[4] Krakiwsky, E. J., and D. B. Thomson. GEODETIC POSITION COMPUTATIONS, Dec. 1995, www2.unb.ca/gge/Pubs/LN39.pdf. Accessed 20th Aug. 2020.

2.3 Conversion Between Parametric and Geodesic latitudes[5]

As shown in figure 5, geodesic latitude for P is the angle from the prime meridian to the current position, β. Parametric latitude for P is angle φ. P and P' has Cartesian coordinates.

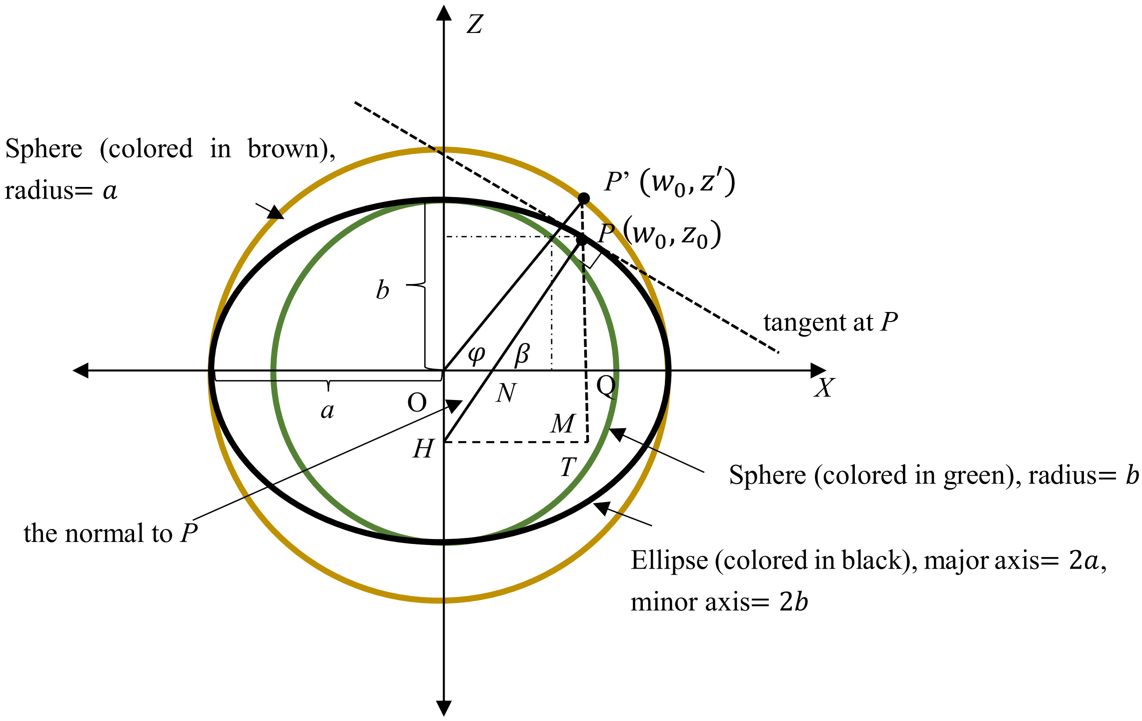


Figure 5: The ellipse when Earth ellipsoid is cut from *XOZ* plane.

$$\because \frac{{w_x}^2}{a^2} + \frac{{z_y}^2}{b^2} = 1$$
$$\therefore \frac{{w_0}^2}{a^2} + \frac{{z_0}^2}{b^2} = 1$$
$$\therefore {w_0}^2 + \frac{{z_0}^2}{b^2} \times a^2 = a^2$$

$$\because w_x^2 + {z'}_y^2 = a^2$$
$$\therefore {w_0}^2 + z'^2 = a^2$$
$$\therefore \frac{{z_0}^2}{b^2} \times a^2 = z'^2$$
$$\because a > 0, b > 0, z_0 \cdot z' \geq 0$$
$$\therefore \frac{a}{b} \times z_0 = z'$$

[5] Deakin, Rod, and M. Hunter. "GEODESICS ON AN ELLIPSOID - BESSEL'S METHOD." *ResearchGate*, 2009, www.researchgate.net/profile/Rod_Deakin/publication/267986122_GEODESICS_ON_AN_ELLIPSOID_-BESSEL%27S_METHOD. Accessed 20th Aug, 2020.

$$\because \frac{{w_x}^2}{a^2}+\frac{{z_y}^2}{b^2}=1$$

$$\therefore \frac{d}{dw}\left(\frac{{w_x}^2}{a^2}+\frac{{z_y}^2}{b^2}\right)=\frac{d}{dw}(1)$$

$$\frac{dz}{dw}=-\frac{w_x}{z_y}\times\frac{b^2}{a^2}$$

$$k=\frac{-1}{\frac{dz}{dw}}=\frac{z_y}{w_x}\times\frac{a^2}{b^2}$$

$$PH: y=\frac{z_y}{w_x}\times\frac{a^2}{b^2}\times x+b_0$$

substitute $w_x=w_0, z_y=z_0$ into $y=\frac{z_y}{w_x}\times\frac{a^2}{b^2}\times x+b_0$:

$$z_0=\frac{z_0}{w_0}\times\frac{a^2}{b^2}\times w_0+b_0$$

$$OH: b_0=\left(1-\frac{a^2}{b^2}\right)\times z_0$$

$$PM: l=-\left(1-\frac{a^2}{b^2}\right)\times z_0+z_0=z_0\times\frac{a^2}{b^2}$$

let $PH=v$

$\because \Delta NQP\sim\Delta HTP$

$\therefore NQ=m$

$$\frac{z_0}{l}=\frac{m}{w_0}=\frac{b^2}{a^2}$$

$$l=v\times\sin\beta$$

$$z_0=b\sin\varphi=l\times\frac{b^2}{a^2}=v\times\sin\beta\times\frac{b^2}{a^2}=v\times\sin\beta\times(1-e^2)$$

$$\because e^2=\frac{a^2-b^2}{a^2}, (1-e^2)^{\frac{1}{2}}=(1-f)$$

$$\therefore \tan\varphi=\frac{\sin\varphi}{\cos\varphi}=\frac{\frac{z}{b}}{\frac{w}{a}}=\frac{\sin\beta}{\cos\beta}\times\frac{a}{b}\times(1-e^2)=\tan\beta\times(1-e^2)^{\frac{1}{2}}$$

2.4 Radius of curvature[6]

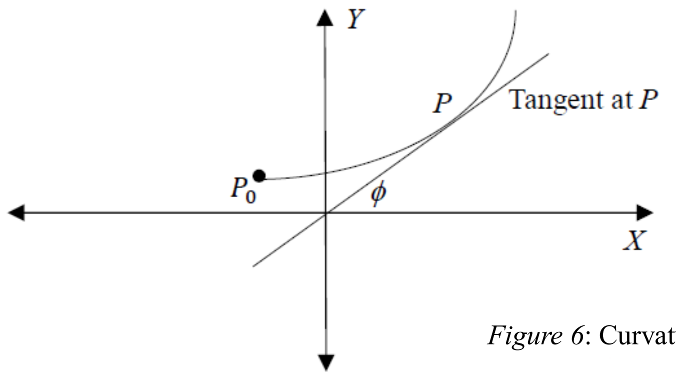


Figure 6: Curvature

Definition of curvature at P is $\dfrac{d\phi}{ds}$.

where s is the distance from curve's starting point P_0 to P

$$\phi = \tan^{-1}(\frac{dy}{dx})$$

$$\because ds = \sqrt{1+(\frac{dy}{dx})^2}\,dx$$

$$\therefore \frac{d\phi}{ds} = \frac{d\phi}{dx} \times \frac{dx}{ds} = \frac{\frac{d\phi}{dx}}{\frac{ds}{dx}} = \frac{\frac{1}{1+(\frac{dy}{dx})} \times (\frac{d^2y}{dx^2})}{\sqrt{1+(\frac{dy}{dx})^2}} = \frac{\frac{d^2y}{dx^2}}{[1+(\frac{dy}{dx})^2]^{\frac{3}{2}}}$$

Definition of radius of curvature $\mathrm{r} = \dfrac{[1+(\frac{dy}{dx})^2]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$

[6] Analytic Geometry and Calculus, by Henry Bayard Phillips, Wiley, 1957, pp. 194–197.

2.4.1 r for meridian section[7]

The intersection of ellipsoid with *XOZ* plane is shown in figure 7. It is expressed using:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

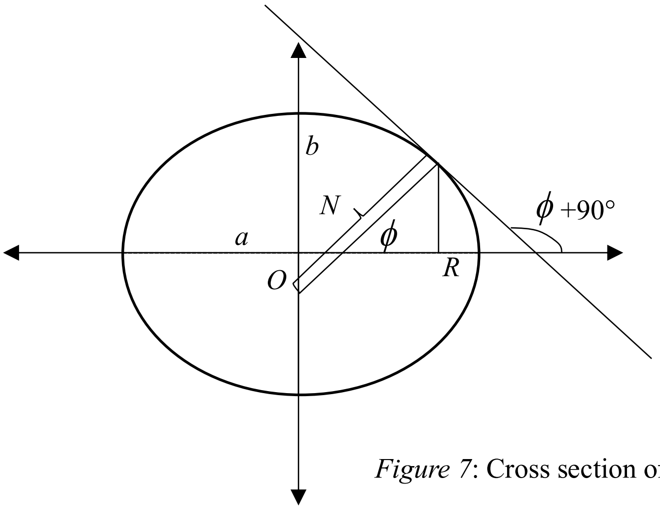


Figure 7: Cross section of the ellipsoid

$$\because \tan\phi = \tan(\phi+90^{\circ})$$

$$\because \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\therefore \frac{dy}{dx}=-\frac{x}{y}\cdot\frac{b^2}{a^2} \qquad \tan\phi=\frac{a^2}{b^2}\cdot\frac{y}{x}$$

$$\frac{d^2y}{dx^2}=-\frac{b^2}{a^2}\cdot\left(\frac{y-x\cdot\frac{dy}{dx}}{y^2}\right)=-\frac{b^2}{a^2y^2}\left(y+\frac{x^2}{y}\cdot\frac{b^2}{a^2}\right)$$

$$\because b=a(1-e^2)^{\frac{1}{2}} \quad \therefore \tan\phi=\frac{a^2}{a^2(1-e^2)}\cdot\frac{y}{x} \qquad y=\tan\phi\cdot x\cdot(1-e^2)$$

Substitute y and b into $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$:

[7] Krakiwsky, E. J., and D. B. Thomson. GEODETIC POSITION COMPUTATIONS, Dec. 1995, www2.unb.ca/gge/Pubs/LN39.pdf. Accessed 20th Aug. 2020.

$$x = \frac{a}{(1+\tan^2\phi)^{\frac{1}{2}}} \qquad y = \frac{a(1-e^2)\sin\phi}{(1+e^2\sin^2 e^2)^{\frac{1}{2}}}$$

Substitute $x, y,$ $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ into $\mathrm{r} = \frac{[1+(\frac{dy}{dx})^2]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$:

$$\rho = M = \frac{a(1-e^2)}{(1-e^2\sin^2\phi)^{3/2}} = \frac{a(1-e^2)}{W^3} \quad \text{where } W^2 = 1-e^2\sin^2\phi$$

2.4.2 r of prime vertical section[8]

$$\text{Let } OR = x$$

$$\therefore \cos\phi = \frac{x}{N} \qquad N = \frac{x}{\cos\phi}$$

Substitute $x = \frac{a}{(1+\tan^2\phi)^{\frac{1}{2}}}$ into $N = \frac{x}{\cos\phi}$:

$$\nu = N = \frac{a}{(1-e^2\sin^2\phi)^{1/2}} = \frac{a}{W}$$

$$\because w_0 = \nu \times \cos\phi = a \times \cos\varphi \text{ from figure 5}$$

$$\therefore \cos\varphi = \frac{\cos\phi}{\left(1-e^2\sin^2\phi\right)^{\frac{1}{2}}}$$

8 Krakiwsky, E. J., and D. B. Thomson. GEODETIC POSITION COMPUTATIONS, Dec. 1995, www2.unb.ca/gge/Pubs/LN39.pdf. Accessed 20th Aug. 2020.

2.5 Ellipsoid equation[9]

$$\begin{cases} x = v\cos\phi\cos\lambda \\ y = v\cos\phi\sin\lambda \\ z = v(1-e^2)\sin\phi \end{cases} \qquad v = \frac{a}{(1-e^2\sin^2\phi)^{\frac{1}{2}}}$$

$$\begin{cases} x = a\cos\varphi\cos\lambda \\ y = a\cos\varphi\sin\lambda \\ z = b\sin\varphi \end{cases}$$

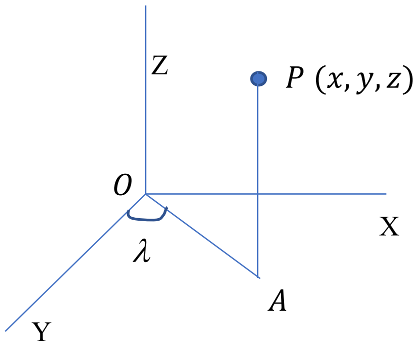


Figure 8: Axis

At any point P on the ellipsoid:

φ is the parametric latitude, ϕ is the geodesic latitude and λ is the longitude: the angle measured from YOZ plane to POA plane.

[9] Deakin, Rod, and M. Hunter. "GEODESICS ON AN ELLIPSOID - BESSEL'S METHOD." *ResearchGate*, 2009, www.researchgate.net/profile/Rod_Deakin/publication/267986122_GEODESICS_ON_AN_ELLIPSOID_-BESSEL%27S_METHOD. Accessed 20th Aug, 2020.

3. Methods to calculate the path and distance

3.1 Great elliptic arc[10]

Given an ellipsoid and 2 points $\vec{A}$, $\vec{B}$ on the ellipsoid. The great ellipse is the intersection between the ellipsoid and the plane formed by the $\vec{A}$, $\vec{B}$.

Suppose the normal vector of the plane the great ellipse is $\vec{n}=\begin{pmatrix} l \\ m \\ 1 \end{pmatrix}$

The plane can be described as: $l\cdot x+m\cdot y+1\cdot z=0$

Let $\vec{P}$ represent the points on the great ellipse between $\vec{A}$ and $\vec{B}$.

$$\because \begin{cases} x=v\cos\varphi\cos\lambda \\ y=v\cos\varphi\sin\lambda \\ z=v(1-e^2)\sin\varphi \end{cases}$$

$$\because \vec{A},\ \vec{B} \text{ and } \vec{P} \text{ are on the great ellipse } \therefore \begin{cases} \vec{A}\cdot\vec{n}=0 \\ \vec{B}\cdot\vec{n}=0 \\ \vec{P}\cdot\vec{n}=0 \end{cases}$$

$$\because \begin{cases} \vec{A}\cdot\vec{n}=0 \\ \vec{B}\cdot\vec{n}=0 \end{cases} \qquad \therefore \begin{cases} \begin{pmatrix} v_A\cos\varphi_A\cos\lambda_A \\ v_A\cos\varphi_A\sin\lambda_A \\ v_A\sin\varphi_A(1-e^2) \end{pmatrix}\cdot\begin{pmatrix} l \\ m \\ 1 \end{pmatrix}=0 \\ \begin{pmatrix} v_B\cos\varphi_B\cos\lambda_B \\ v_B\cos\varphi_B\sin\lambda_B \\ v_B\sin\varphi_B(1-e^2) \end{pmatrix}\cdot\begin{pmatrix} l \\ m \\ 1 \end{pmatrix}=0 \end{cases}$$

$$\therefore \begin{cases} v_A\cos\varphi_A\cos\lambda_A\bullet l+v_A\cos\varphi_A\sin\lambda_A\bullet m+v_A\sin\varphi_A(1-e^2)=0 \\ v_B\cos\varphi_B\cos\lambda_B\bullet l+v_B\cos\varphi_B\sin\lambda_B\bullet m+v_B\sin\varphi_B(1-e^2)=0 \end{cases}$$

[10] Tseng, Wei-Kuo, and Hsuan-Shih Lee. "NAVIGATION ON A GREAT ELLIPSE." Journal of Marine Science and Technology, vol. 18, no. 3, 2010, pp. 369–375.

$$\therefore \begin{cases} \cos\varphi_A \cos\lambda_A \bullet l + \cos\varphi_A \sin\lambda_A \bullet m + \sin\varphi_A (1-e^2) = 0 \\ \cos\varphi_B \cos\lambda_B \bullet l + \cos\varphi_B \sin\lambda_B \bullet m + \sin\varphi_B (1-e^2) = 0 \end{cases}$$

$$\therefore \begin{cases} l = \dfrac{(1-e^2)\cdot\left(\cos\varphi_A \sin\varphi_B \sin\lambda_B - \cos\varphi_B \sin\varphi_A \sin\lambda_A\right)}{\cos\varphi_B \sin\lambda_B \cos\varphi_A \cos\lambda_A - \cos\varphi_A \sin\lambda_A \cos\varphi_B \cos\lambda_B} \\ m = \dfrac{(1-e^2)\cdot\left(\sin\varphi_A \cos\varphi_B \cos\lambda_B - \sin\varphi_B \cos\varphi_A \cos\lambda_A\right)}{\cos\varphi_B \sin\lambda_B \cos\varphi_A \cos\lambda_A - \cos\varphi_A \sin\lambda_A \cos\varphi_B \cos\lambda_B} \end{cases}$$

$$\because \ \vec{P}\cdot\vec{n} = 0$$

$\therefore$ great ellipse equation: $\cos\varphi_p \cos\lambda_p \bullet l + \cos\varphi_p \sin\lambda_p \bullet m + \sin\varphi_p (1-e^2) = 0$

$$\therefore \tan\varphi_p = -\frac{\cos\lambda_p \bullet l + \sin\lambda_p \bullet m}{1-e^2} \quad (1)$$

differentiate (1): $\dfrac{d\varphi}{d\lambda} = \dfrac{(l\cdot\sin\lambda - m\cdot\cos\lambda)}{\left(1-e^2\right)\left(1+\tan^2\varphi\right)}$

3.1.1 Arc length

For space parametric curve:

$$\because \ \Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$$

$$\therefore ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$

$$s = \int \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))}^2 dt$$

$$\because \begin{cases} x = v\cos\varphi\cos\lambda \\ y = v\cos\varphi\sin\lambda \\ z = v\sin\varphi(1-e^2) \end{cases} \qquad \begin{cases} x = a\cos\beta\cos\lambda \\ y = a\cos\beta\sin\lambda \\ z = b\sin\beta \end{cases} \quad (2)$$

differentiate (2) with respect to λ $\begin{cases} \dfrac{dx}{d\lambda} = -a\sin\beta\cos\lambda\dfrac{d\beta}{d\lambda} - a\cos\beta\sin\lambda \\ \dfrac{dy}{d\lambda} = -a\sin\beta\sin\lambda\dfrac{d\beta}{d\lambda} + a\cos\beta\cos\lambda \\ \dfrac{dz}{d\lambda} = b\cos\beta\dfrac{d\beta}{d\lambda} \end{cases}$

$$\therefore s=\int_{\lambda_1}^{\lambda_2}\sqrt{(a^2\sin^2\beta+b^2\cos^2\beta)(\frac{d\beta}{d\lambda})^2+a^2\cos^2\beta d\lambda}$$

$$\because \tan\beta=\tan\varphi\times(1-e^2)^{\frac{1}{2}}$$

$$\therefore \sec^2\beta\frac{d\beta}{d\varphi}=\left(1-e^2\right)^{\frac{1}{2}}\sec^2\varphi$$

$$\therefore \frac{d\beta}{d\varphi}=\left(1-e^2\right)^{\frac{1}{2}}\frac{\sec^2\varphi}{\sec^2\beta}=\left(1-e^2\right)^{\frac{1}{2}}\frac{1+\tan^2\varphi}{1+\left(1-e^2\right)\tan^2\varphi}$$

$$\therefore\quad s=\int_{\lambda_1}^{\lambda_2}\sqrt{\left(a^2\sin^2\beta+b^2\cos^2\beta\right)\left(\frac{d\beta}{d\lambda}\right)^2+a^2\cos^2\beta d\lambda}$$

$$\because \cos^2\beta=\frac{\cos^2\varphi}{1-e^2\sin^2\varphi}$$

$$\therefore \cos^2\beta=\frac{\cos^2\varphi}{1-e^2\sin^2\varphi}=\frac{\frac{\cos^2\varphi}{\cos^2\varphi}}{\frac{1-e^2\sin^2\varphi}{\cos^2\varphi}}=\frac{1}{\sec^2\varphi-e^2\tan^2\varphi}$$

$$=\frac{1}{1+\tan^2\varphi-e^2\tan^2\varphi}=\frac{1}{1+(1-e^2)\tan^2\varphi}$$

let $\gamma=1-e^2$:

$$\therefore s=a\int_{\lambda_1}^{\lambda_2}\sqrt{\frac{1}{1+\gamma\tan^2\varphi}\left(1+\gamma^2\left(\frac{\left(1+\tan^2\varphi\right)^3}{\left(1+\gamma\tan^2\varphi\right)^2}\left(\frac{d\varphi}{d\lambda}\right)^2\right)\right)}d\lambda$$

$$s=a\int_{\lambda_1}^{\lambda_2}\sqrt{\frac{1}{1+\gamma\tan^2\varphi}\left(1+\frac{1+\tan^2\varphi}{\left(1+\gamma\tan^2\varphi\right)^2}(l\sin\lambda-m\cos\lambda)^2\right)}d\lambda$$

3.2 Great circle[11]

Great circle distance between 2 points is calculated from spherical trigonometry. The formulas are listed below:

Given points: $P(\varphi_1, \lambda_1), \mathrm{Q}(\varphi_2, \lambda_2)$

$\Delta\lambda = \lambda_2 - \lambda_1$

$\sigma = \arccos\left(\sin\varphi_1 \sin\varphi_2 + \cos\varphi_1 \cos\varphi_2 \cos(\Delta\lambda)\right)$

$S = \sigma \times r_{Earth}$

$sin\alpha_1 = \dfrac{\cos\varphi_2 \bullet \sin\Delta\lambda}{\sin S}$

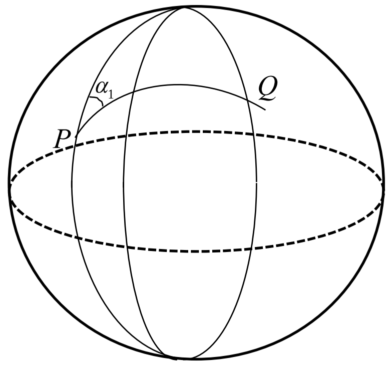


Figure 9: Great circle distance

For sphere, $e^2 = 0$

Great circle equation is:

$$\cos\varphi_p \cos\lambda_p \bullet l + \cos\varphi_p \sin\lambda_p \bullet m + \sin\varphi_p = 0$$

$$\begin{cases} l_c = \dfrac{\cos\varphi_A \sin\varphi_B \sin\lambda_B - \cos\varphi_B \sin\varphi_A \sin\lambda_A}{\cos\varphi_B \sin\lambda_B \cos\varphi_A \cos\lambda_A - \cos\varphi_A \sin\lambda_A \cos\varphi_B \cos\lambda_B} \\ m_c = \dfrac{\sin\varphi_A \cos\varphi_B \cos\lambda_B - \sin\varphi_B \cos\varphi_A \cos\lambda_A}{\cos\varphi_B \sin\lambda_B \cos\varphi_A \cos\lambda_A - \cos\varphi_A \sin\lambda_A \cos\varphi_B \cos\lambda_B} \end{cases}$$

$$\tan\varphi_p = -(\cos\lambda_p \bullet l + \sin\lambda_p \bullet m)$$

Intersection of great circle arc at the equator:

$$\because \varphi_p = 0$$

$$\therefore \sin\lambda_p \bullet m = -\cos\lambda_p \bullet l \qquad \tan\lambda_p = \frac{\sin\lambda_p}{\cos\lambda_p} = -\frac{l}{m}$$

Great circle coordinates calculated using this method is the parametric coordinate of the great ellipse. This is because the conversion of geodesic coordinate on the ellipse into parametric coordinate will cancel out the term $1-e^2$. For later comparison on the path of the great circle and great ellipse method with the geodesic method, the coordinates used are the parametric coordinates.

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[11] "The Great Circle Distance" *The Great Circle Distance | Trigonometry: Compound Angles | Underground Mathematics*, undergroundmathematics.org/trigonometry-compound-angles/the-great-circle-distance. Accessed 23rd Jan. 2021.

3.3 Geodesic method[12]

3.3.1 Clairaut equation $v\cos\phi\sin\alpha = C$

The French mathematician Alexis-clade Clairaut derived this equation which states that $v\cos\phi\sin\alpha{=}C$ at points on the geodesic.

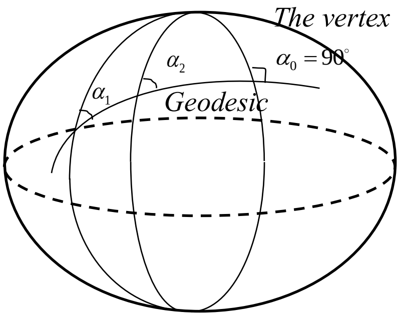

Figure 10: Geodesic

ϕ is the geodesic latitude at a point

α is the azimuth

For the point on the geodesic where $\sin\alpha{=}1$, the point is called geodesic vertex and $v\cos\phi$ will be the smallest value. From section 2.2, another relationship between parametric latitude and geodesic latitude can be developed:

$$v\cos\phi\sin\alpha = a\cos\psi\sin\alpha = a\cos\psi_0$$

$$\therefore \cos\psi_0 = \cos\psi\sin\alpha$$

where ψ_0 is parametric latitude for geodesic vertex

$$\cos\alpha = \sqrt{1\text{-}\sin^2\alpha} = \frac{\sqrt{\cos^2\psi - \cos^2\psi_0}}{\cos\psi}$$

[12] Deakin, Rod, and M. Hunter. "GEODESICS ON AN ELLIPSOID - BESSEL'S METHOD." *ResearchGate*, 2009, www.researchgate.net/profile/Rod_Deakin/publication/267986122_GEODESICS_ON_AN_ELLIPSOID_-BESSEL%27S_METHOD. Accessed 20th Aug, 2020.

3.3.2 Auxiliary sphere

Consider the equation $\cos\psi_0 = \cos\psi \sin\alpha$

To find the shortest path and its distance between 2 points (P, Q) on the ellipsoid, we need to find intermediate points which satisfies above equation and calculate the distance of this path.

Consider the case of sphere: the shortest distance is found using sphere trigonometry. The great circle arc is also the geodesic.

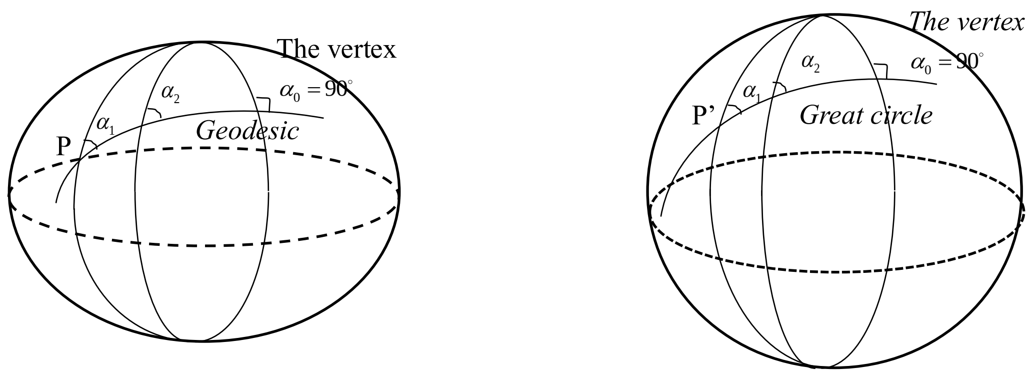

Figure 11: Geodesic and great circle

Using auxiliary sphere, we can relate the points on the ellipsoid and sphere to solve Clairaut's equation.

Statement: related points on the ellipsoid and sphere has same parametric latitude.
Reasoning: consider P on the geodesic on the ellipsoid, when it's transformed to P'on the auxiliary sphere. It can be seen as "elevated". Mapped points form a great circle on the sphere. Therefore, the geodesic vertex on the ellipsoid is also the geodesic vertex on the sphere.

Geodesic on the ellipsoid satisfy:

$$\cos\psi_1 \sin\alpha_1 = \cos\psi_2 \sin\alpha_2 = \cos\psi_0$$

On the auxiliary sphere:

$$\cos\psi_1 \sin A_1 = \cos\psi_2 \sin A_2 = \cos\psi_3$$

$$\therefore \ \sin A_1 = \sin\alpha_1$$

3.3.3 Relationship between geodesic distance (s) and great circle distance (σ)

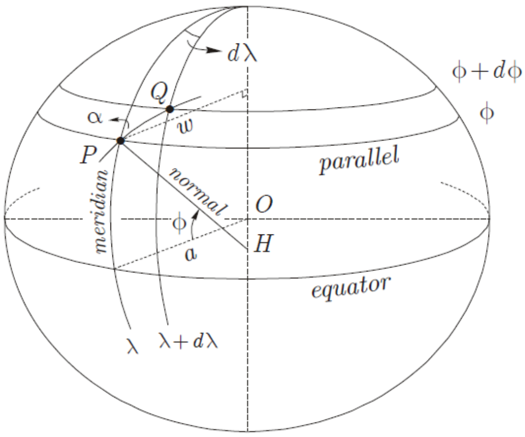

Figure 12: Ellipsoid surface[13]

$d\phi$ is the change in geodetic latitude from P to O

ϕ is the geodetic latitude at P

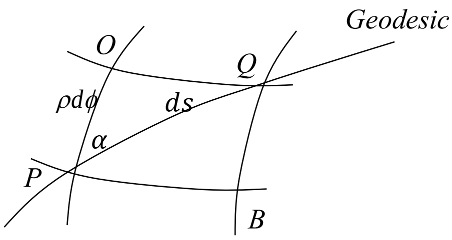

Figure 13: Differential triangle on ellipsoid surface

On the ellipsoid, the differential triangle can be approximated as a plane triangle when $PQ = ds$.

$$PO = ds\cos\alpha$$

$$ds\cos\alpha = \rho d\phi \quad (1)$$

$$ds\sin\alpha = \nu\cos\phi d\lambda \quad (2)$$

auxiliary sphere surface

$d\varphi$ is the change in parametric latitude from P to O

φ is the parametric latitude at P

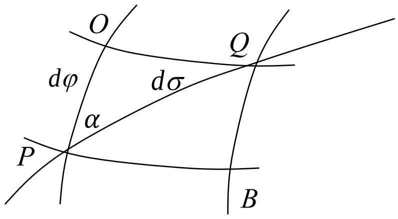

Figure 14: Differential triangle on sphere surface

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[13] Deakin, Rod, and M. Hunter. “GEODESICS ON AN ELLIPSOID - BESSEL'S METHOD.” *ResearchGate*, 2009, www.researchgate.net/profile/Rod_Deakin/publication/267986122_GEODESICS_ON_AN_ELLIPSOID_-BESSEL%27S_METHOD. Accessed 20th Aug, 2020.

Similarly, for sphere:

$$d\sigma \cos\alpha = d\psi \quad (3)$$

$$d\sigma \sin\alpha = \cos\psi\, d\omega \quad (4)$$

Divide (1) by (3) and (2) by (4):

$$\frac{ds}{d\sigma} = \rho\frac{d\phi}{d\psi} = a\frac{d\lambda}{d\omega}$$
$$\frac{d\lambda}{d\omega} = \frac{1}{a}\frac{ds}{d\sigma}$$

According to reference for section 3.3[14], using further ellipsoid relations, we have:

$$\frac{ds}{d\sigma} = a(1-e^2\cos^2\psi)^{\frac{1}{2}}$$

3.3.4 Geodesic distance

Consider ΔNPH on the auxiliary sphere $NHP = 90°$. P_E is the intersection of great circle arc through P and Q with the equator. P_2 is an arbitrary point along geodesic PQ.

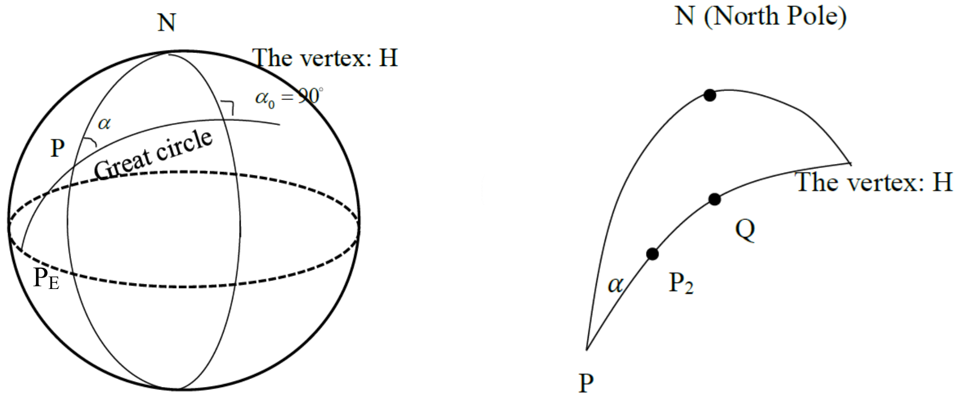


Figure 15: Ellipsoid triangle

[14] Deakin, Rod, and M. Hunter. "GEODESICS ON AN ELLIPSOID - BESSEL'S METHOD." *ResearchGate*, 2009, www.researchgate.net/profile/Rod_Deakin/publication/267986122_GEODESICS_ON_AN_ELLIPSOID_-BESSEL%27S_METHOD. Accessed 20th Aug, 2020.

Let angular distance of different arcs be:

$$P_2P = \sigma$$
$$P_EP_2 = \sigma_1$$

According to reference for section 3.3[15], using Napier's rules:

$$\sin\psi_2 = \sin(\sigma_1 + \sigma)\sin\psi_0$$

$$\because \frac{ds}{d\sigma} = a(1 - e^2\cos^2\psi)^{\frac{1}{2}}$$

$$\therefore ds = a[1 - e^2(1 - \sin^2(\sigma_1 + \sigma)\sin^2\psi_0)]^{\frac{1}{2}} d\sigma$$

$$\text{Let} \qquad x = \sigma_1 + \sigma \qquad \therefore \frac{dx}{d\sigma} = 1$$

$$ds = a[1 - e^2(1 - \sin^2 x\sin^2\psi_0)]^{\frac{1}{2}} dx = a[1 - e^2 + e^2\sin^2 x\sin^2\psi_0)]^{\frac{1}{2}} dx$$

$$ds = a\left(\frac{1}{1+e^{'2}} + \frac{e^{12}}{1+e^{'2}}\sin^2 x\sin^2\psi_0\right)^2 dx$$

$$ds = \frac{a}{\left(1+e^{'2}\right)^{\frac{1}{2}}}\left(1 + e^{'2}\sin^2 x\sin^2\psi_0\right)^{\frac{1}{2}} dx$$

$$ds = b\left(1 + e^{'2}\sin^2 x\sin^2\psi_0\right)^{\frac{1}{2}} dx$$

$$\because \cos\psi_1\sin\alpha_1 = \cos\psi_2\sin\alpha_2$$
$$\therefore \cos\psi_0 = \sin\alpha_E$$

$$\text{Let} \;\; u^2 = e^{'2}\sin^2\psi_0 = e^{'2}\cos^2\alpha_E$$

$$\therefore \;\; ds = b\left(1 + u^2\sin^2 x\right)^{\frac{1}{2}} dx$$

The integral is the arc length of the geodesic:

$$s = b\int_{x=\sigma_1}^{x=\sigma_1+\sigma}\left(1 + u^2\sin^2 x\right)^{\frac{1}{2}} dx$$

[15] Deakin, Rod, and M. Hunter. "GEODESICS ON AN ELLIPSOID - BESSEL'S METHOD." *ResearchGate*, 2009, www.researchgate.net/profile/Rod_Deakin/publication/267986122_GEODESICS_ON_AN_ELLIPSOID_-BESSEL%27S_METHOD. Accessed 20th Aug, 2020.

3.3.5 Longitude difference

$d\omega$ is the change in longitude from P to P'

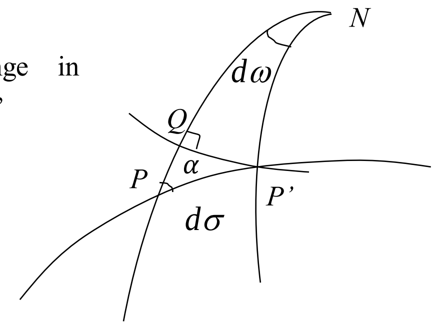


Figure 16: Spherical triangle

Consider $\Delta NQP'$, $\Delta QP'P$ $\quad \angle P'QP = 90^\circ$

$$PP' = d\sigma$$

$$\mathrm{QP} = \sin\alpha_i \bullet d\sigma = \cos\psi_P \bullet d\omega$$

$$\therefore d\omega = \frac{\sin\alpha_i}{\cos\psi_i} d\sigma$$

$$\because \cos\psi_0 = \cos\psi \sin\alpha \qquad \sin\alpha_i = \frac{\cos\psi_0}{\cos\psi_i}$$

$$\therefore d\omega = \frac{\cos\psi_0}{\cos^2\psi} d\sigma$$

$$\because \frac{d\lambda}{d\omega} = \frac{1}{a}\frac{ds}{d\sigma} \qquad \frac{ds}{d\sigma} = a(1-e^2\cos^2\psi)^{\frac{1}{2}}$$

$$\therefore d\lambda = \cos\psi_0 \frac{\left(1-e^2\cos^2\psi\right)^{\frac{1}{2}}}{\cos^2\psi} d\sigma$$

$$d\lambda - d\omega = \cos\psi_0 \left[\frac{\left(1-e^2\cos^2\psi\right)^{\frac{1}{2}}}{\cos^2\psi} - \frac{1}{\cos^2\psi}\right] d\sigma$$

$$\because \sin\psi = \sin\left(\sigma_1 + \sigma\right)\sin\psi_0$$
$$\because \sin^2\psi = \sin^2 x \sin^2\psi_0 = 1 - \cos^2\psi$$

$$\therefore \Delta\lambda - \Delta\omega = \cos\psi_0 \int_{x=\sigma_1}^{x=\sigma_1+\sigma} \left[\frac{\left(1-e^2\cos^2\psi\right)^{\frac{1}{2}}}{\cos^2\psi} - \frac{1}{\cos^2\psi} \right] dx$$

To use this method, we assume the departure point on the ellipsoid and sphere has the same longitude since Earth is an oblate ellipsoid.

Thus, longitude and parametric latitude coordinates of geodesic path can be calculated. Using great circle equation in section 3.2, the coordinates of related great circle arc can be calculated. Parametric latitude on the auxiliary sphere and the ellipsoid is the same. $\Delta\omega$ is obtained, and thus the longitude coordinate for geodesic path λ using (1).

$$\lambda = \omega_{start} + \Delta\lambda = \omega_{start} + \Delta\omega + \cos\psi_0 \int_{x=\sigma_1}^{x=\sigma_1+\sigma} \left[\frac{\left(1-e^2\cos^2\psi\right)^{\frac{1}{2}}}{\cos^2\psi} - \frac{1}{\cos^2\psi} \right] dx \quad (1)$$

4. Comparison methods

The method to find the geodesic is called Bessel's method. It would be desired to start solve the inverse geodesic problem directly, e.g., starting with 2 points on the oblate ellipsoid and work the way to the path and geodesic distance. However, the integration of geodesic distance needs the value of σ. To find σ, we need the longitude difference on the axillary sphere since the longitude on the axillary sphere is different from that on the ellipsoid. The longitude difference uses integration involves σ. The only way is iteration. This method is complicated and my purpose is the compare the methods. So, another approach is discussed below.

4.1 Description

The purpose of this essay is to find how well we can use the great elliptic arc or great circle to approximate the geodesic since the computing of geodesic distance and path requires the value σ. Following steps will be used for comparison.

1. Start with 2 points on the sphere.
2. Calculate the great circle distance and σ using spherical trigonometry.
3. The great ellipse distance is calculated using great ellipse method.
4. The great ellipse coordinates on the ellipsoid are calculated using the great ellipse equation. The coordinates are calculated by inputting 3000 equally spaced longitude value between departure and destination coordinates into the equation to obtain latitude value. For later comparison, the geodesic latitudes are transformed into parametric latitudes.
5. The great elliptic path formula can also be applied into the great circle path formula, taking $e^2=0$. Similar to 4, great circle latitudes are obtained by inputting longitude values into the equation. For the sphere, the parametric latitude and geodesic latitude is the same since $e^2=0$.
6. For the geodesic distance, it is calculated using formula.
7. For the geodesic path, the longitude is calculated from $\Delta\lambda$ by setting the departure coordinate on ellipsoid same as that of sphere.
8. Difference of distance between great ellipse method and great circle method from geodesic method is taken the absolute value.

9. Calculate the difference in longitudes coordinates of geodesic and great elliptic arc for same parametric latitude.

4.1.1 Calculation formulas

1. Great circle

Given $P(\beta_1,\lambda_1),\mathrm{Q}(\beta_2,\lambda_2)$ where $\beta_1 \le \beta_2, \lambda_1 \le \lambda_2$.

$$\sigma = \arccos\left(\sin\beta_1 \sin\beta_2 + \cos\beta_1 \cos\beta_2 \cos(\Delta\lambda)\right)$$

Note: from spherical trigonometry, $\Delta\lambda$ need to be a positive value.

Great circle distance $d = \sigma \times r_{Earth}$ (r is 6371393m)

Azimuth of departing angle: $\alpha = \arcsin\dfrac{\cos\beta_2 \bullet \sin\Delta L}{\sin S}$

Path

$$\beta_p = \arctan(-(\cos\lambda_p \bullet l + \sin\lambda_p \bullet m))$$

Where $\begin{cases} l_c = \dfrac{\cos\beta_A \sin\beta_B \sin\lambda_B - \cos\beta_B \sin\beta_A \sin\lambda_A}{\cos\beta_B \sin\lambda_B \cos\beta_A \cos\lambda_A - \cos\beta_A \sin\lambda_A \cos\beta_B \cos\lambda_B} \\ m_c = \dfrac{\sin\beta_A \cos\beta_B \cos\lambda_B - \sin\beta_B \cos\beta_A \cos\lambda_A}{\cos\beta_B \sin\lambda_B \cos\beta_A \cos\lambda_A - \cos\beta_A \sin\lambda_A \cos\beta_B \cos\lambda_B} \end{cases}$

Input β_p ranging from β_1 to β_2 to obtain β_p total 3000 points taken

2. Great ellipse

Convert parametric latitude into geodesic latitude: $\varphi = \arctan\dfrac{\tan\beta}{(1-e^2)^{\frac{1}{2}}}$

Path

$$\varphi_p = \arctan\left(-\frac{\cos\lambda_p \bullet l + \sin\lambda_p \bullet m}{1-e^2}\right)$$

$$\begin{cases} l = \dfrac{(1-e^2)\cdot\left(\cos\varphi_A \sin\varphi_B \sin\lambda_B - \cos\varphi_B \sin\varphi_A \sin\lambda_A\right)}{\cos\varphi_B \sin\lambda_B \cos\varphi_A \cos\lambda_A - \cos\varphi_A \sin\lambda_A \cos\varphi_B \cos\lambda_B} \\ m = \dfrac{(1-e^2)\cdot\left(\sin\varphi_A \cos\varphi_B \cos\lambda_B - \sin\varphi_B \cos\varphi_A \cos\lambda_A\right)}{\cos\varphi_B \sin\lambda_B \cos\varphi_A \cos\lambda_A - \cos\varphi_A \sin\lambda_A \cos\varphi_B \cos\lambda_B} \end{cases}$$

Input λ_p ranging from λ_1 to λ_2 to obtain φ_p total 3000 points taken

Convert φ into $\beta = \arctan((1-e^2)^{\frac{1}{2}} \bullet \tan\varphi)$ for later coordinate comparison

Great ellipse distance using MATLAB numerical integration

$$s_{ellipse} = a\int_{\lambda_1}^{\lambda_2} \sqrt{\frac{1}{1+\gamma\tan^2\varphi}\left(1+\gamma^2\left(\frac{\left(1+\tan^2\varphi\right)^3}{\left(1+\gamma\tan^2\varphi\right)^2}\left(\frac{d\varphi}{d\lambda}\right)^2\right)\right)}\, d\lambda$$

here $\gamma = 1-e^2$ $\qquad \dfrac{d\varphi}{d\lambda} = \dfrac{(l\cdot\sin\lambda - m\cdot\cos\lambda)}{\left(1-e^2\right)\left(1+\tan^2\varphi\right)}$

3. <u>Geodesic</u>

$\psi_0 = \arccos(\cos\psi \sin\alpha)$

$u^2 = e'^2 \sin^2\psi_0$

$\lambda_{equator} = \arctan(-\dfrac{l}{m})$

Great circle distance from equator intersection to departure

$\beta_E = 0$

$$\Delta\sigma = \arccos\left(\sin\beta_1 \sin\beta_E + \cos\beta_1 \cos\beta_E \cos(\Delta\lambda)\right)$$
$$\sigma_1 = \arccos\left(\cos\beta_E \cos(\Delta\lambda)\right)$$

$\Delta\lambda = \left|\lambda_1 - \lambda_E\right|$

Geodesic distance $s_{geodesic} = b\int_{x=\sigma_1}^{x=\sigma_1+\sigma} \left(1+u^2\sin^2 x\right)^{\frac{1}{2}} dx$

Path

$$\lambda_{geodesic} = \omega_{start} + \Delta\lambda = \omega_{start} + \Delta\omega + \Delta\lambda = \omega_{current} + \cos\psi_0 \int_{x=\sigma_1}^{x=\sigma_1+\sigma} \left[\frac{\left(1-e^2\cos^2\psi\right)^{\frac{1}{2}}}{\cos^2\psi} - \frac{1}{\cos^2\psi} \right] dx$$

Where $\cos^2\psi = 1 - \sin^2 x \sin^2\psi_0$ $\quad \omega_{start} = \lambda_1$

$\omega_{current}$ is the corresponding longitude coordinates of the great circle path

Parametric latitude is the same $\beta_{geodesic} = \beta_P$

Difference calculation

Great ellipse $\Delta_a = \left|S_{ellipse} - S_{geodesic}\right|$

Great circle $\Delta_a = \left|d - S_{geodesic}\right|$

Difference in path

Parametric latitude is the same for great circle, great ellipse and geodesic method, so the path difference is calculated using difference in longitude coordinates for same parametric latitude.

Difference $\delta = \sqrt{\dfrac{\sum_{i=1}^{i=3000} (\lambda_{iellipse} - \lambda_{igeodesic})^2}{3000}}$

Note:

1. All the figures are converted into radian in calculation.
2. Although inverse trigonometric functions have certain range, but coordinate systems are chosen so values do not need to consider $\theta + k\pi$.
3. The absolute difference in the path is calculated instead of percentage difference. This is because in real cases the actual difference between the methods is of more concern than percentage difference for sailing and flight lines.
4. The great circle arc distance is also compared since the intermediary steps of geodesic calculation requires great circle arc distance.

4.2 Different cases

The Earth is an oblate ellipsoid. So, the calculated results of error should be nearly the same for same longitude difference of 2 chosen points, except minor computation errors of numerical methods in integration and trigonometry.

However, the latitude parts are not symmetrical for same latitude differences. So, several departure and destination scenarios are listed below for latitude difference comparison.

Latitude differences:
The hemisphere here refers to Northern and Southern hemispheres.

1. The departure and destination points locate in the same hemisphere.
2. The departure and destination points locate in different hemisphere, with departure from Southern hemisphere, destination on Northern hemisphere.

Table 1: Different latitudes for departure and destination

Departure	0.1°	10°	20°	30°	40°	…	60°	70°	80°	89.9°
Destination	0.1°	6°	12°	18°	24°	…	72°	78°	84°	89.9°

Longitude differences:
The last difference is 179.5° because the integration function of MATLAB reports error for longitude difference of 180°.

Table 2: Longitude difference

Longitude difference between the 2 points						
0°	30°	60°	90°	120°	150°	179.5°

Methods for comparison in section 4.4 and 4.5:
Within each longitude difference value, the departure has 11 different latitudes and destination has 16 different latitudes. The comparison cases are:

$$11 \times 16 = 176$$

For each longtitude differnce, the range, sum and standard deivation of δ for 176 cases are calculated. They are then ploted on the same graph for each in section 4.4.1 and 4.5.1.

For each longtitude differnce, the range, average and standard deivation of Δ_a for 176 cases are calculated. They are then ploted on the same graph for great ellipse and great circle method respectively.

<u>Flaws in the computation:</u>

1. Some of the value of the trigonometry is zero. $\sin 0 = 0$ and $\cos\frac{\pi}{2} = 0$.
   These values make it impossible to calculate the great elliptic path using the vector method. As l or m will be zero or the denominator of l or m will be 0. So, the ending points is taken as 89.9° and starting points is 0.1°.

2. In reality, departure along the equator is the great circle so the paths of geodesic and great ellipse are the same.

3. The path for geodesic and great ellipse from 0° latitude to 90° latitude is also the same.

4.3 Sample Calculation

This sample calculation shows the path and distances calculated using 3 methods between 2 chosen points. The graph is generated using the path data of different methods. The data is listed in the appendix in section 7.1.1.

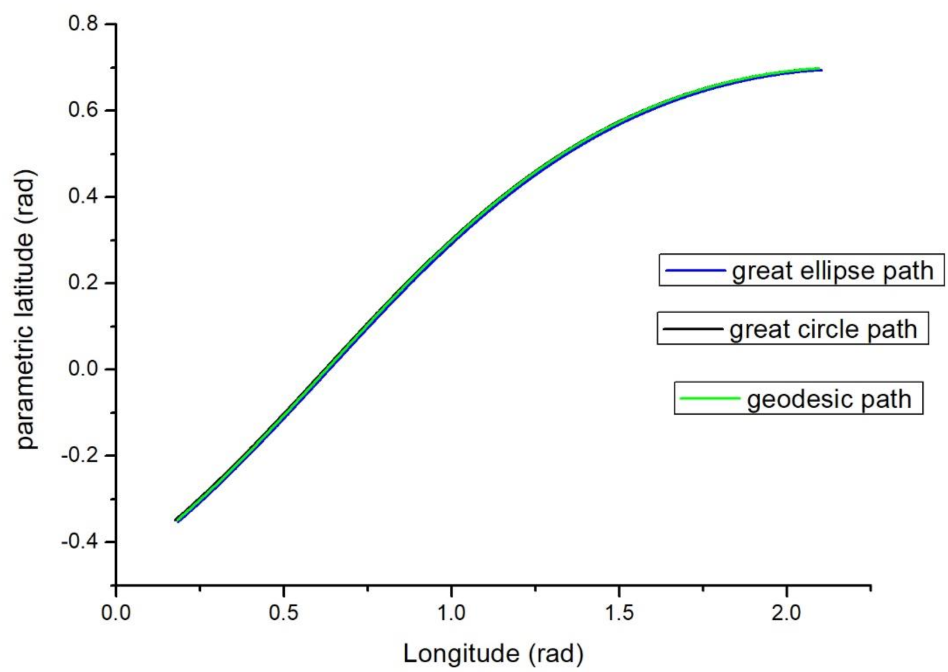
Table 3: Coordinates for departure and destination

	Departure	Destination
Parametric latitude	-20°	40°
Longitude	10°	120°

Table 4: Computed data

	Distance for each method	Δ_a for each method
Great circle	13097133.2694556m	0.001410341m
Great ellipse	13098064.1005729m	0.001481513m
Geodesic method	13078687.858281m	/

The path difference (δ) for great circle and great ellipse method is 0.003270351355463.



Graph 1: Path for different methods

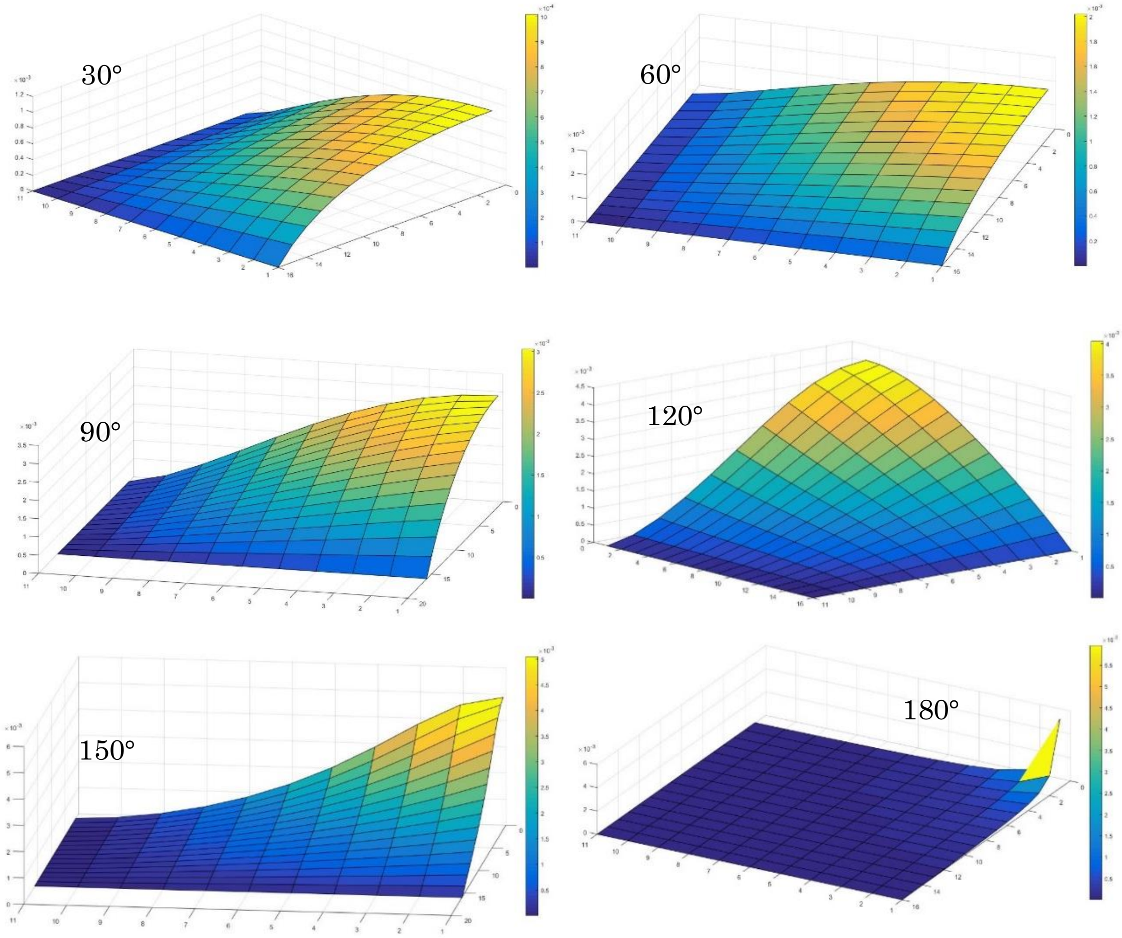
As shown in table 4 and graph 1, the results for 3 methods are very close.

4.4 Comparison of difference on same hemisphere.

For the following colored 3D graphs that shows the trend, each graph shows the trend for a certain longitude difference. Different departure and destination latitudes are represent by *x-axis* and *y-axis*.

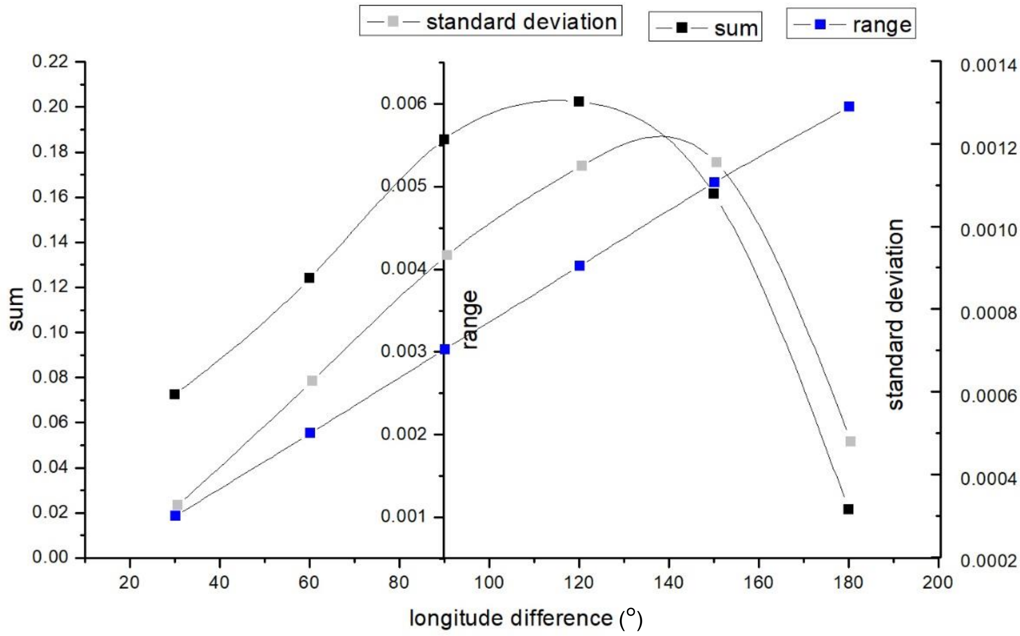
Raw data for calculation is put in appendix in section 7.1.

4.4.1 Path difference



Graph 2: Trends for path difference at different longitude differences

Graph 2 shows data for δ. δ is plotted on *z-axis*. It suggests that when longitude differences are the same, the higher the departing and destination latitudes are, the smaller the difference between the great elliptic path and geodesic path.



Graph 3: Sum, standard deviation and range for path difference at different longitude differences

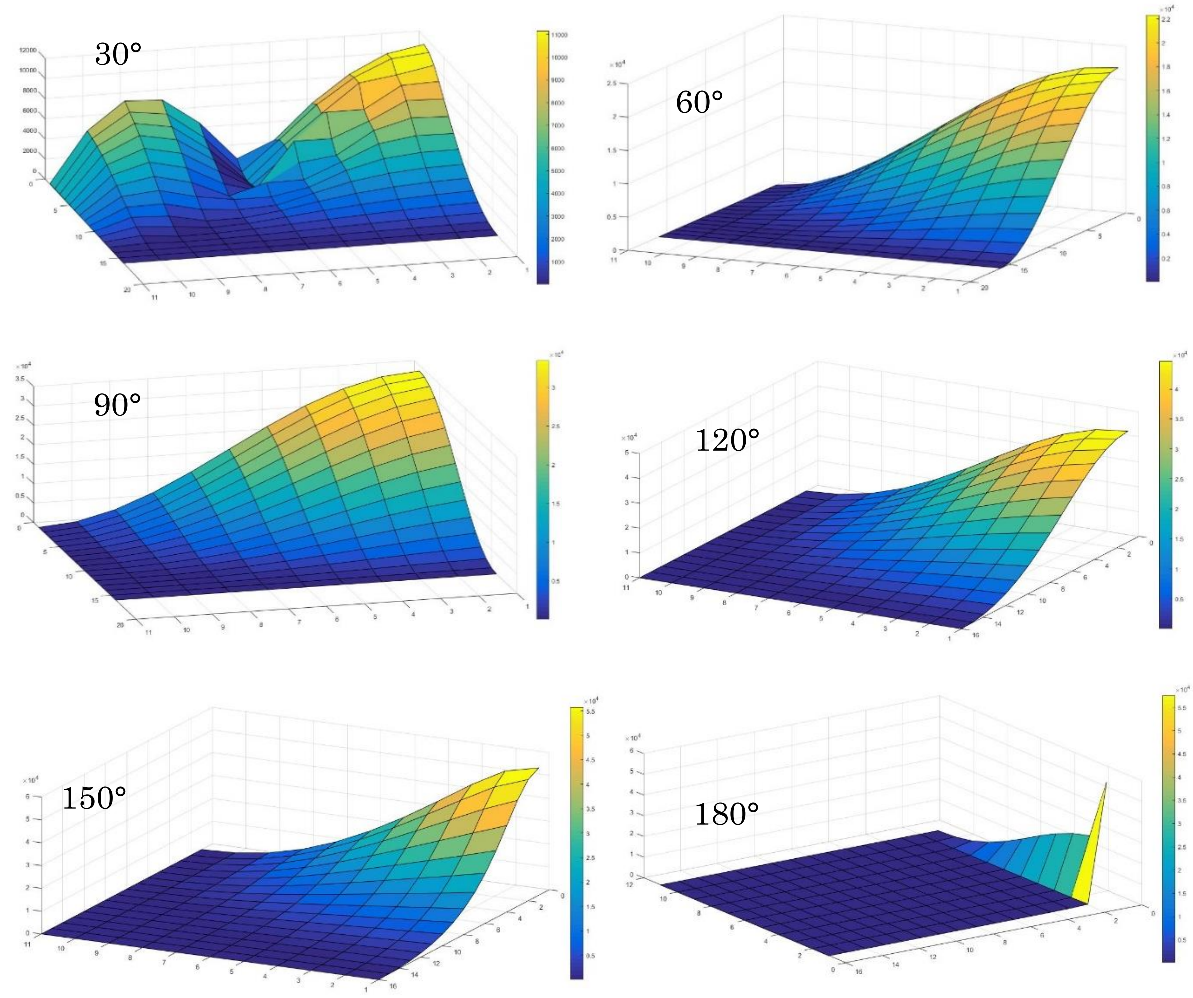
For the longitudes, largest deviation of great elliptic path from the geodesic occurs at 120°. Graph 3 suggests that smaller path difference would be obtained at smaller or larger longitude differences. It also suggests that range of δ increases follow an increase in the longitude difference. It is worth noting that although the range for longitude difference 180° is large, the sum at 180° is small, indicating that results from 3 methods at 180° longitude differences is very close at higher latitudes.

Overall, from the values of the sum and standard deviation of the data, the path calculated from the great ellipse or great circle method is quite close to that of geodesic method.

4.4.2 Absolute difference between distance in great ellipse and great circle method with geodesic method

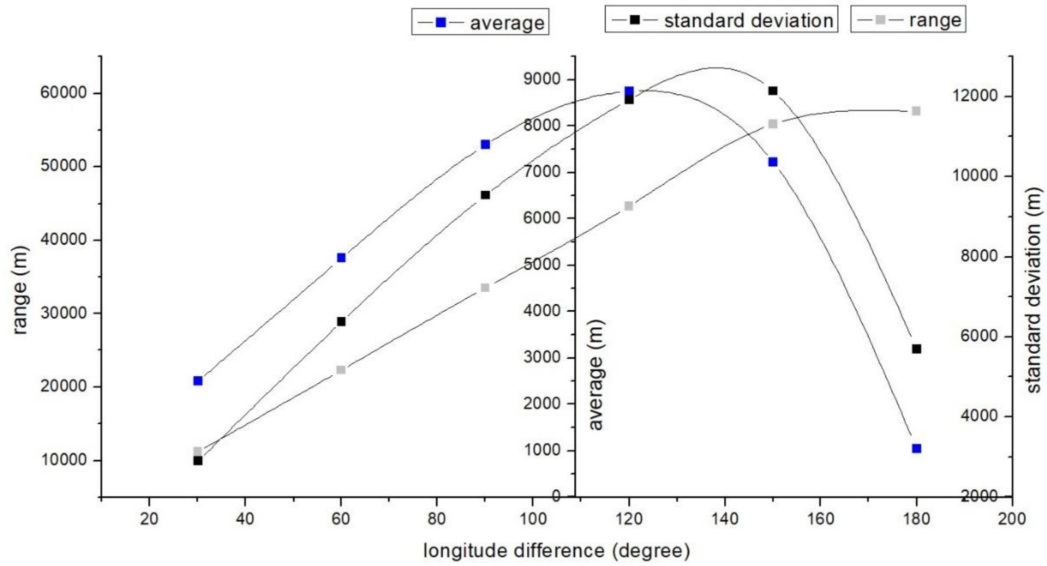
<u>Great ellipse</u>

Graph 4 shows data for Δ_a. Δ_a is plotted on *z-axis*.



Graph 4: Trends for great ellipse method distance difference at different longitude differences

When the longitude difference is the same, there is a larger difference at smaller latitudes. This might due to longer absolute length for the path at lower latitudes. But for longitude difference of 180°, the data seems to be a very good approximation when latitudes are both higher than about 15°. While for longitude difference of 30°, the trend is different. Its more accurate when destination latitudes are over 70° or departure latitudes around 60° to 80°.



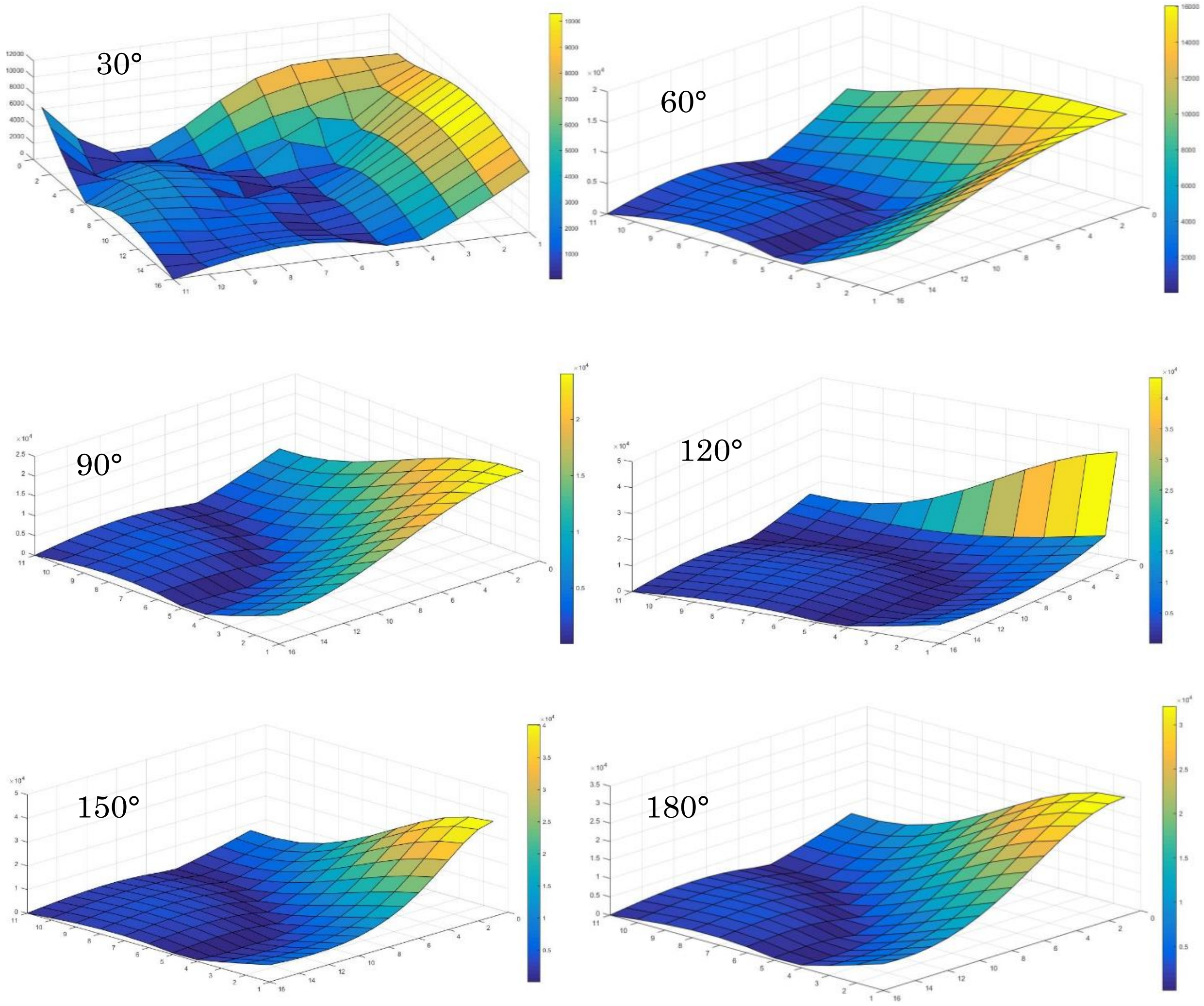
Graph 5: Average, standard deviation and range for great ellipse method at different longitude differences

The average value of absolute difference and standard deviation of these data suggests that the great elliptic distance is less accurate around 120°. The difference is smaller at smaller longitude difference is not surprising since the total geodesic distance would be smaller. The range between the differences increases as longitude differences increases. It is worth noting that although the range for longitude difference 180° is large, the average at 180° is small, indicating that results from 3 methods at 180° longitude differences is very close at higher latitudes.

Also, from the values of the average and standard deviation, the difference from great ellipse method distance and geodesic method is on average around 1000m to 9000m, standard deviation around 3000m to 13000m. These figures are not large when put on the scale of the Earth, suggesting great ellipse method is a good approximation of the geodesic method.

Great circle

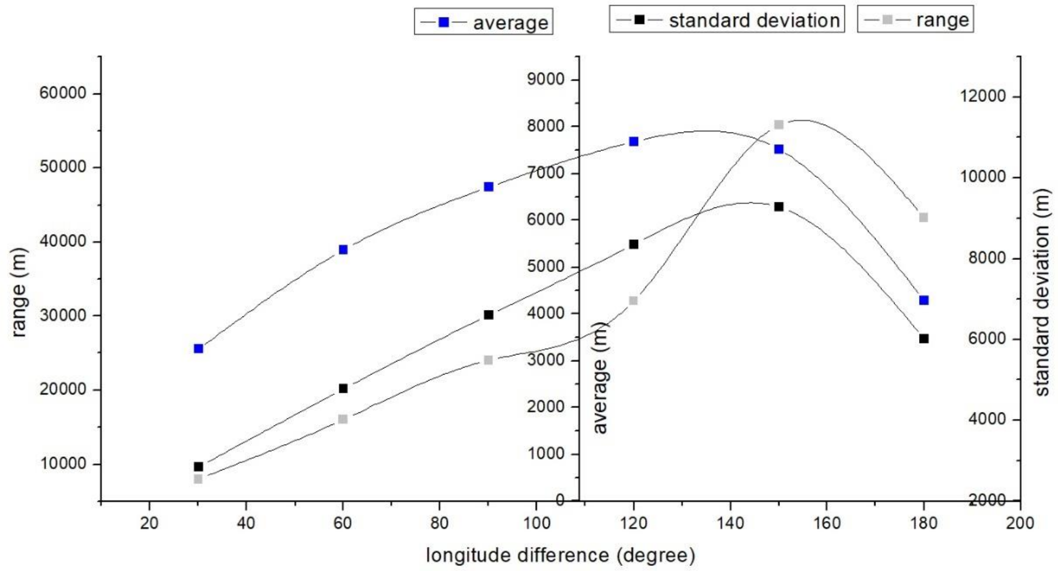
Graph 6 shows data for Δ_a. Δ_a is plotted on *z-axis*.



Graph 6: Trend graph for great circle method at different longitude differences

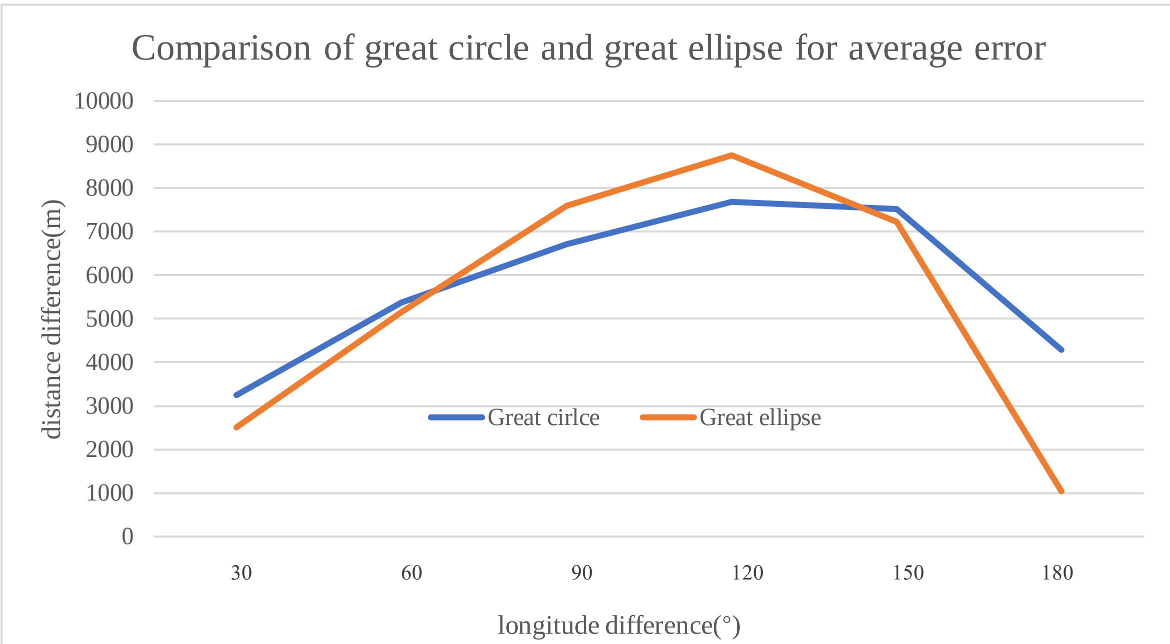
When longitude difference is the same, the great circle method to calculate the distance is still more close to the result of the geodesic method at higher latitudes. However, the smaller values occur mostly around 35° to 55° latitude of departure and destination latitudes.

From Graph 7 (overleaf), the average value of absolute difference and standard deviation of these data suggests that the great elliptic distance is less accurate around 120° to 150° longitude differnces. The difference is smaller at smaller longitude difference. The range between the differences increases as longitude differences increases.



Graph 7: Average, standard deviation and range for great ellipse method at different longitude difference

Also, similarly, from the values of the average and standard deviation of different longitude difference. It shows that great circle method is a good approximation of the geodesic method.



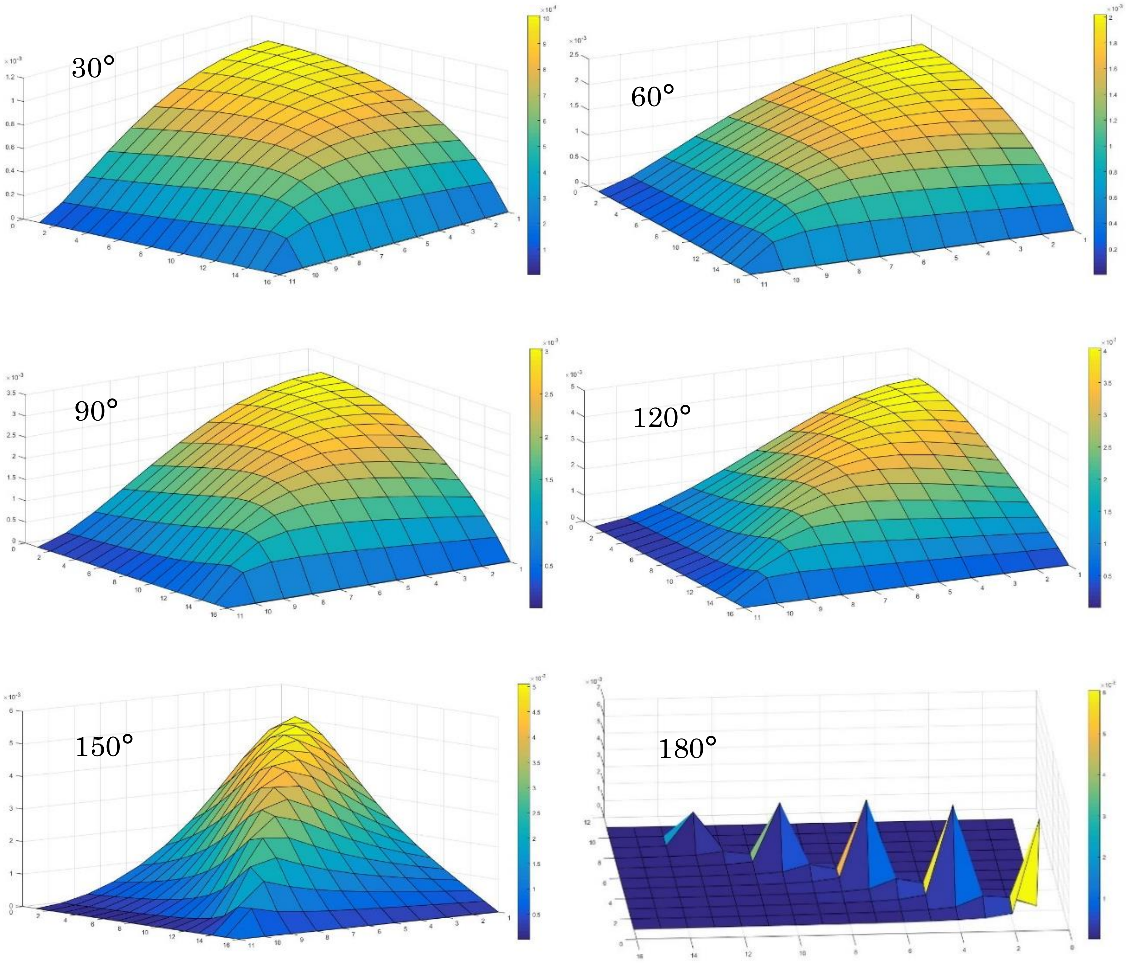
Graph 8: Comparison of great circle and great ellipse for average error

Compare the great ellipse and great circle method, great ellipse distance is more accurate at higher or lower longitude difference than great circle method.

4.5 Comparison of difference on different hemispheres

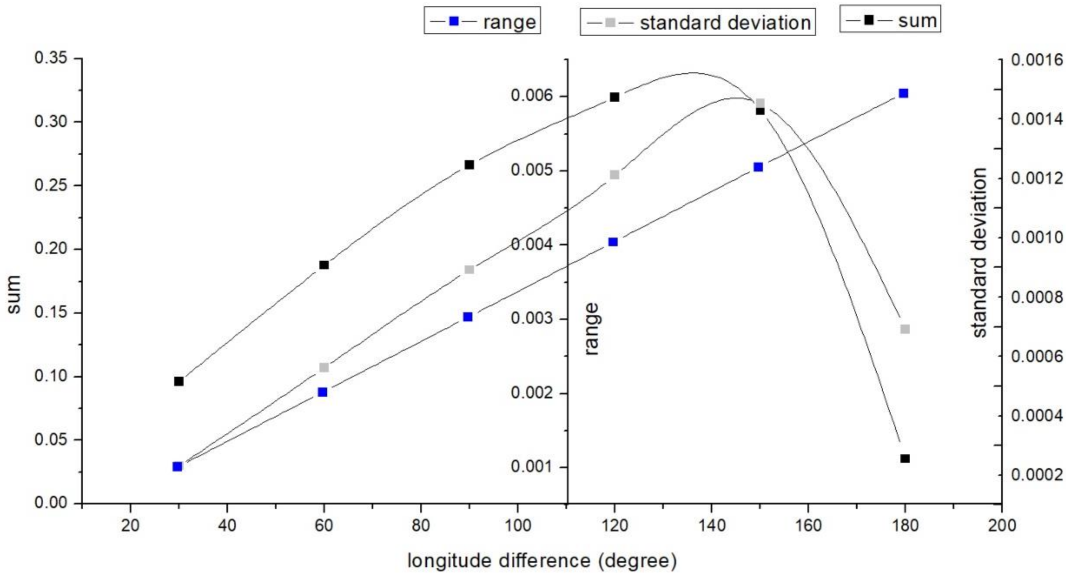
4.5.1 Path difference

Graph 9 shows data for δ. δ is plotted on *z-axis*.



Graph 9: Trend graph for path difference at different longitude differences

When longitude difference is the same, the higher the departing and destination latitudes are, the smaller the difference between the great elliptic path and geodesic path. This is similar to the trend when 2 points are located on the same hemisphere.



Graph 10: Sum, standard deviation and range for path difference at different longitude differences

The path calculated using great circle or great ellipse method is more accurate at lower longitude differences. Similar to the trend when 2 points are located on the same hemisphere. The 150° longitude difference is the most diverse one as it has the largest standard deviation. The sum is bigger for 120° and 150° longitude difference.

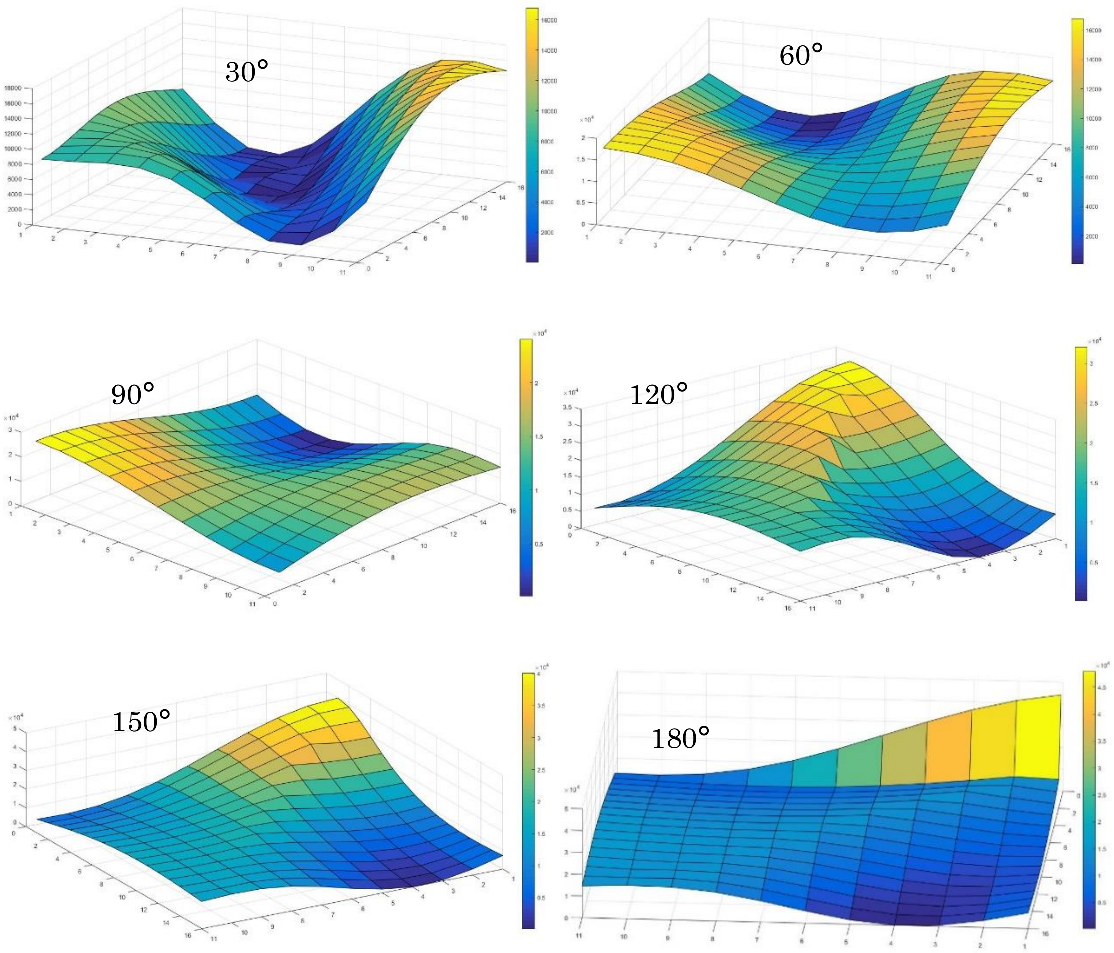
The range of path difference increases follow an increase in the longitude difference. Similarly, although the range for 180° longitude difference is large, the sum at 180° is small, indicating that data at 180° is very accurate at higher latitudes.

Also, from the values of the sum of average difference and standard deviation. It shows that great circle or great ellipse method is a good approximation of the geodesic path.

4.5.2 Absolute difference between distance in great ellipse and great circle method with geodesic method

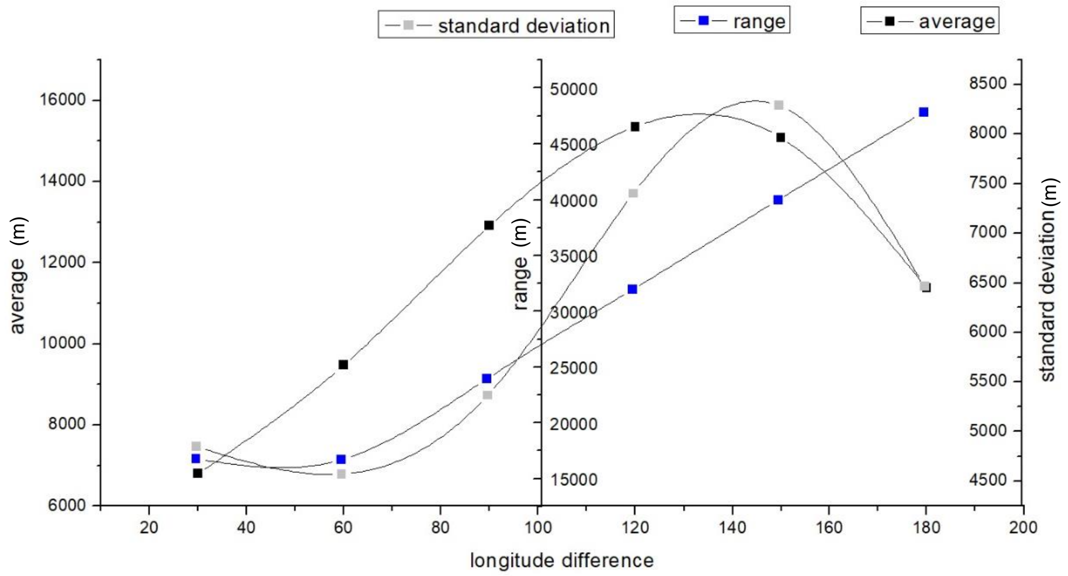
<u>Great circle</u>

Graph 11 shows data for Δ_a. Δ_a is plotted on *z-axis*.



Graph 11: Trend graph for great circle method at different longitude differences

When the longitude difference is the same, except for longitude difference of 30°, as the departure and destination latitudes become larger, the error decreases. All graphs except for longitude difference of 30° show that start at 20° to 40° latitude and end in 70° to 90° latitude is smaller in error.



Graph 12: Average, standard deviation and range for great circle method at different longitude

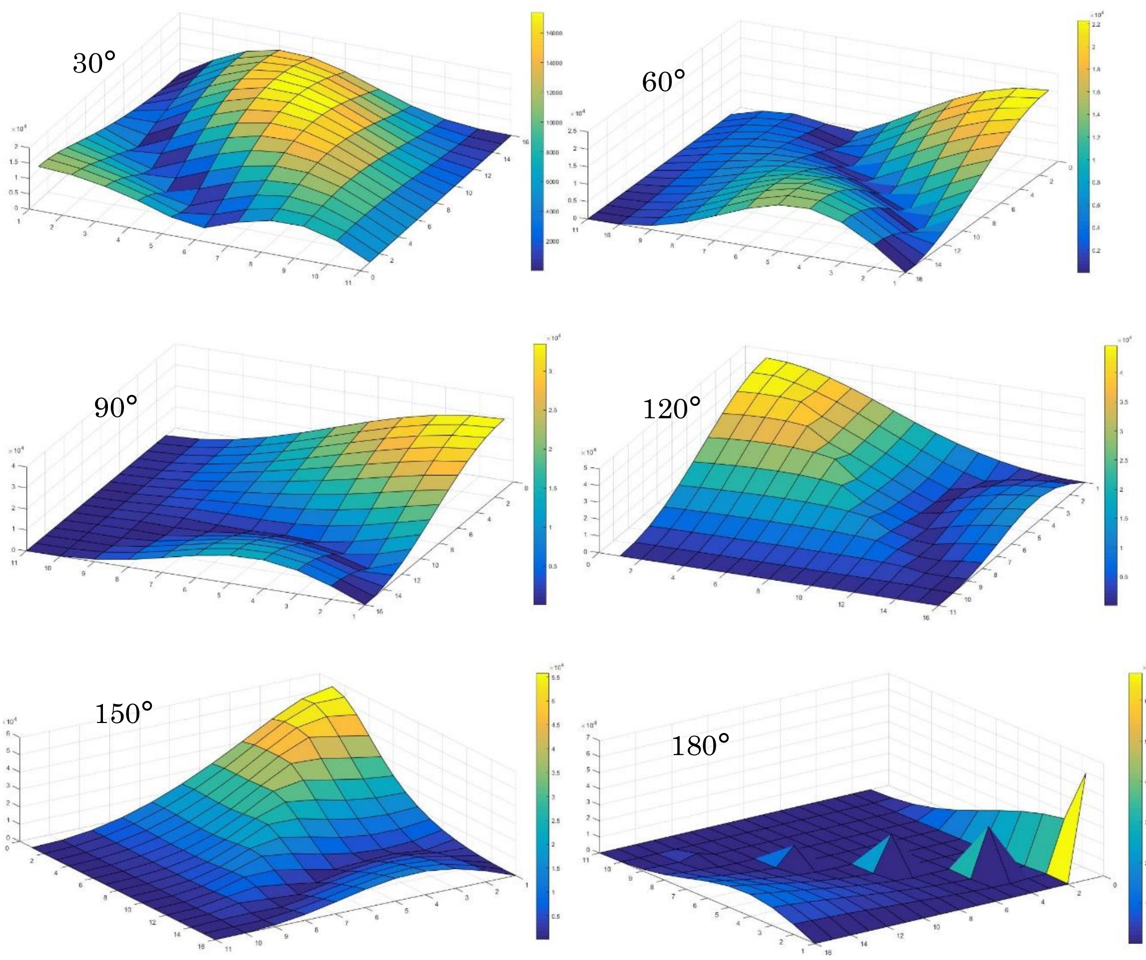
The average error is bigger in longitude difference 120° and 150°, smaller in 30°, 60°. Standard deviation is significant larger at larger longitude differences.

Range for error increases as longitude difference increases. Combine this with the trend graph, longitude difference of 30° and 60° produce more accurate distance than larger longitude differences. For 180° longitude difference, except ending on 0° latitude, other points are also considerably accurate.

The great circle method to calculate the distance is quite accurate as the average difference is 7000m to 17000m with standard deviation of 4000m to 8500m, range from 16000m to 50000m compared to the scale of the Earth. Since when departure and destination are on different hemisphere, the distance between them is longer, it is not surprising that great circle method results deviate more in this case.

Great ellipse

Graph 13 shows data for Δ_a. Δ_a is plotted on *z-axis*.

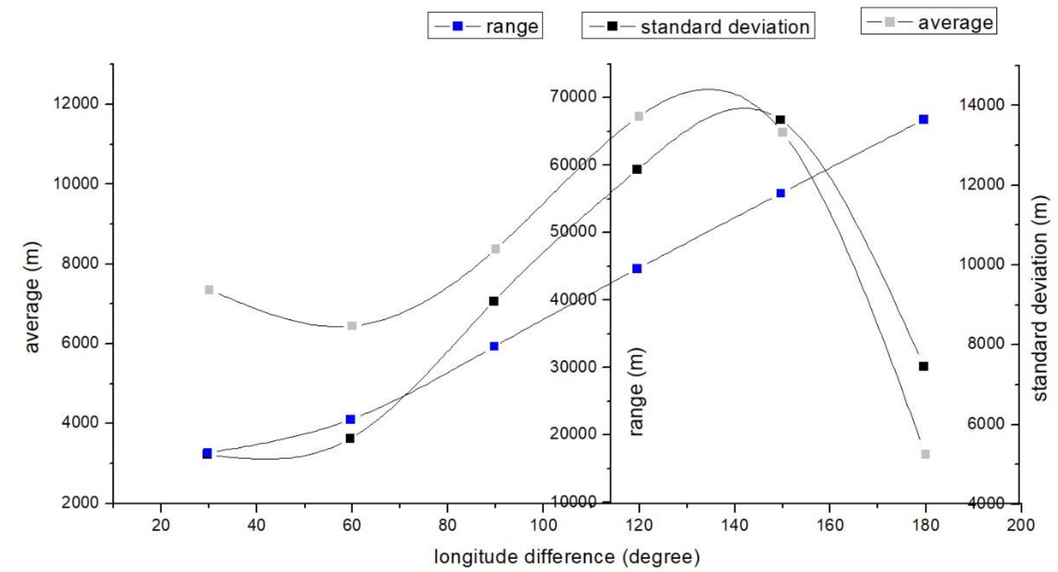


Graph 13: Trend graph for great ellipse method at different longitude differences

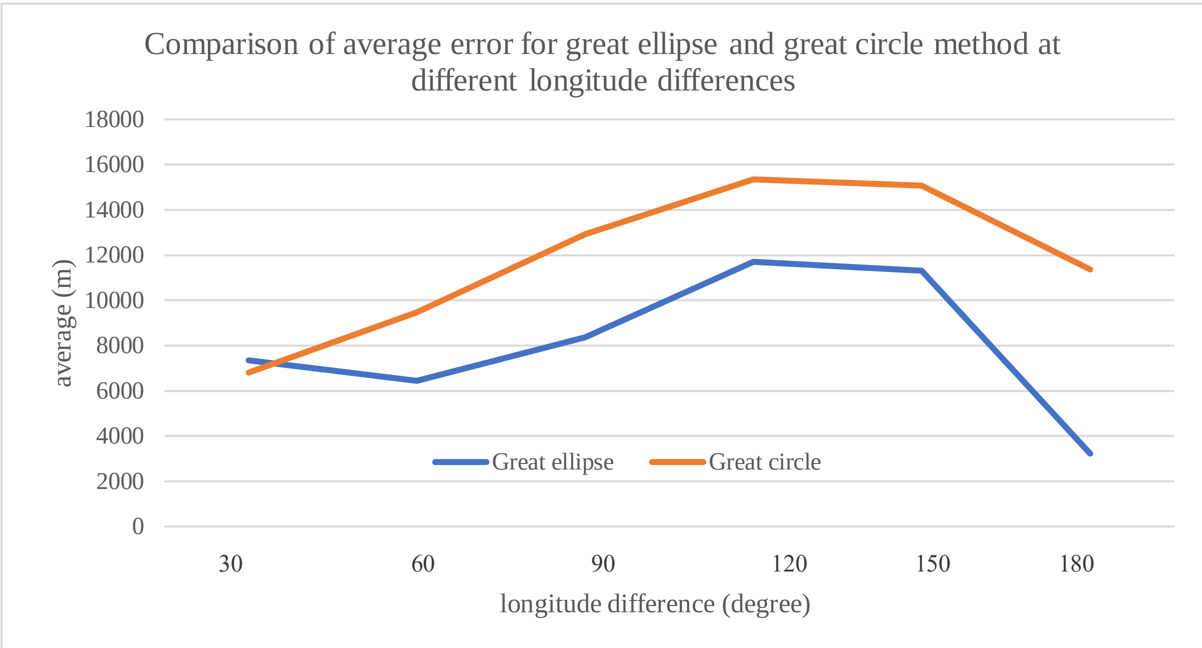
Except for longitude difference of 30°, for same longitude differences, the higher the departure and destination latitudes are, the smaller the difference. This trend is similar for other cases and also for great circle method. It is because at higher latitudes, the distance between the departure and destination points are closer.

From Graph 15 (overleaf), the average error is bigger in longitude difference 120° and 150°, smaller in 180°. Standard deviation is larger at longitude difference of 120° and 150°.

Range for error increases as longitude difference increases. Similarly, combine this with the trend graph, 30° and 60° longitude difference produce more accurate distance than larger longitude differences. For 180° longitude difference, except ending on 0° latitude, other points are also considerably accurate.



Graph 14: Average, standard deviation and range for great ellipse method at different longitude



Graph 15: Comparison of average error for great ellipse and great circle method at different longitude differences

The average error of great ellipse method is smaller than that calculated using great circle method, especially at larger longitude differences. This is because that compared with approximate the Earth as a sphere, using an ellipsoid is closer to the reality. Since the Earth is large, nuances will be magnified.

5. Conclusion

In this essay, I linked geometry with linear algebra and calculus to compare the great ellipse, great circle and geodesic methods. I presented preliminaries of geodesy. I also checked and elaborated related works researched from different sources and formulated an efficient method to compare the difference in different methods. The traditional way of calculating the geodesic uses iteration. Since my research focuses on the comparison of different methods, I started the geodesy calculation with known points on the great circle. This result is then compared with great circle and great ellipse sailing. 3D graphs are presented to show the difference in methods of different departure and destination latitudes.

From the comparisons of path and distances, both the great circle and great ellipse method can be considered as good approximations for the complex geodesic method to calculated the shortest distance.

The general conclusions for how the difference vary with different departures and destinations are: the higher the departure and destination latitudes are, the closer great elliptic sailing is to geodesic sailing. The path calculated using great circle and great ellipse method is the same when using parametric latitudes. The further the departure and destination are apart, the larger the absolute differences are. The great ellipse method's results are closer to geodesic method. However, this error should be examined case by case for different longitude differences and latitudes.

The limitations in the research are the numerical integrations used in MATLAB will introduce some errors to the path and distance calculate in all methods. Most of the integrals used do not have a closed form.

Further investigations on the difference between geodesic sailing and great circle, great ellipse sailing can be made using the method I presented. Using parameters of longitude difference and latitude difference with smaller gaps, more accurate diagrams can be plotted. The difference for specific parameters can also be calculated.

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7. Appendices

7.1 Data used in comparison section

7.1.1 Sample calculation

All figures are in radian.

Great ellipse method			Great circle method		Geodesic method	
Geodesic latitude	Longitude	Parametric latitude	Parametric latitude	Longitude	Parametric latitude	Longitude
-0.350143377	0.174532925	-0.34906585	-0.34906585	0.174532925	-0.34906585	0.174532925
-0.3497124	0.175172879	-0.348635977	-0.348635977	0.175172879	-0.348635977	0.175170991
-0.349281156	0.175812833	-0.348205837	-0.348205837	0.175812833	-0.348205837	0.175809056
-0.348849644	0.176452787	-0.347775432	-0.347775432	0.176452787	-0.347775432	0.17644712
-0.348417865	0.177092741	-0.347344761	-0.347344761	0.177092741	-0.347344761	0.177085184
-0.347985819	0.177732695	-0.346913825	-0.346913825	0.177732695	-0.346913825	0.177723247
-0.347553507	0.17837265	-0.346482623	-0.346482623	0.17837265	-0.346482623	0.178361309
-0.347120928	0.179012604	-0.346051156	-0.346051156	0.179012604	-0.346051156	0.178999371
-0.346688082	0.179652558	-0.345619424	-0.345619424	0.179652558	-0.345619424	0.179637433
-0.34625497	0.180292512	-0.345187427	-0.345187427	0.180292512	-0.345187427	0.180275493
-0.345821592	0.180932466	-0.344755166	-0.344755166	0.180932466	-0.344755166	0.180913554
-0.345387947	0.18157242	-0.34432264	-0.34432264	0.18157242	-0.34432264	0.181551613
-0.344954037	0.182212374	-0.34388985	-0.34388985	0.182212374	-0.34388985	0.182189672
-0.344519862	0.182852328	-0.343456795	-0.343456795	0.182852328	-0.343456795	0.182827731
-0.34408542	0.183492282	-0.343023477	-0.343023477	0.183492282	-0.343023477	0.183465788
-0.343650713	0.184132236	-0.342589894	-0.342589894	0.184132236	-0.342589894	0.184103846
-0.343215741	0.18477219	-0.342156048	-0.342156048	0.18477219	-0.342156048	0.184741902
-0.342780504	0.185412144	-0.341721938	-0.341721938	0.185412144	-0.341721938	0.185379958
-0.342345002	0.186052098	-0.341287564	-0.341287564	0.186052098	-0.341287564	0.186018014
-0.341909235	0.186692052	-0.340852928	-0.340852928	0.186692052	-0.340852928	0.186656069

7.1.2 Comparison section

Path difference data for departure and destination on same hemisphere

Longitude difference	Sum	Standard deviation	Average
30°	0.072471387	0.000325	0.000411769
60°	0.124134619	0.000626	0.00070531
90°	0.185674523	0.00093	0.001054969
120°	0.202397688	0.001147	0.001149987
150°	0.1613955	0.001154	0.00091702
180°	0.021439074	0.00048	0.000121813

Longitude difference (°)	Max	Min	Range
30	0.0010106	3.93E-07	0.0010106
60	0.002015	2.12E-07	0.002015
90	0.003032	1.41E-08	0.003032
120	0.004042	9.97E-08	0.004042
150	0.005053	8.48E-08	0.005053
180	0.005964	6.29E-08	0.005964

Great ellipse data for departure and destination on same hemisphere

Longitude difference	Average(m)	Standard deviation(m)
30°	2500.94	2894.82
60°	5151.81	6380.7
90°	7597.73	9534.56
120°	8753.3	11925.72
150°	7220.24	12137.74
180°	1039.24	5686.4

Longitude difference (°)	Max	Min	Range
30	11162.125	3.93E-07	11162.125
60	22324.23	2.12E-07	22324.23
90	33486.27	1.41E-08	33486.27
120	44648.06	9.97E-08	44648.06
150	55808.12	8.48E-08	55808.12
180	57562.55	6.29E-08	57562.55

Great circle data for departure and destination on same hemisphere

Longitude difference (°)	Max	Min	Range
30	8018.363	5.424443	8012.938
60	16036.71	10.47921	16026.23
90	24055.02	14.81984	24040.2
120	32073.18	18.1505	32055.03
150	55808.12	20.24424	55787.87
180	43340.72	20.95818	43319.77

longitude difference (°)	Average(m)	Standard deviation(m)
30	3246.7	2841.55
60	5367.4	4772.74
90	6711.96	6600.27
120	7683.26	8350.89
150	7517.69	9274.48
180	4287.96	6011.91

Path difference data for departure and destination on different hemispheres

Longitude difference (°)	Sum	Standard deviation	Average
30	0.09632202	0.000331	0.170454545
60	0.187604732	0.000662	0.340909091
90	0.266695079	0.000992	0.511363636
120	0.319672597	0.001311	0.681818182
150	0.309460436	0.001552	0.852272727
180	0.035318111	0.000791	1.022727273

Longitude difference (°)	Max	Min	Range
30	0.001011	1.85E-07	0.00101
60	0.002021	3.01E-07	0.002021
90	0.003032	3.12E-07	0.003031
120	0.004042	2.28E-07	0.004042
150	0.005053	1.03E-07	0.005053
180	0.006047	1.46E-09	0.006047

Great circle method data for departure and destination on different hemispheres

Longitude difference (°)	Average	Standard deviation
30	6804.71	4,848
60	9478.9286	4,572
90	12912.999	5364.868
120	15348.398	7399.96
150	15077.572	8287.087
180	11374.58	6463.007

Longitude difference (°)	Max	Min	Range
30	16782.38245	11.75021637	16770.63
60	16785.39942	94.72175312	16690.68
90	24055.11249	114.4009738	23940.71
120	32073.61776	121.3215821	31952.3
150	40091.99378	126.3764331	39965.62
180	47976.72454	128.223704	47848.5

Great ellipse method data for departure and destination on different hemispheres

Longitude difference (°)	Average	Standard deviation
30	7356.483	5250.325
60	6444.336	5658.205
90	8377.7056	9095.314
120	11703.525	12385.7
150	11297.84	13636.33
180	3221.58	7457.811

Longitude difference (°)	Max	Min	Range
30	17380.74	0.006024	17380.74
60	22324.03	0.024935	22324.01
90	33116.62	0.050726	33116.57
120	44648.57	0.076568	44648.5
150	55810.73	0.026506	55810.7
180	66786.84	1.69E-06	66786.84

7.2 MATLAB code

7.2.1 Computing the geodesic and great ellipse route

```
format long

fileofell='E:\eedata\H11elip.xlsx';
fileofcircle='E:\eedata\H11cir.xlsx';
fileofgeo='E:\eedata\H11geo.xlsx';

%great cicle para
a=6377397.155;
b=6356078.963;
covt=1/180*pi;
plax=-20*covt;
longx=10*covt;
play=40*covt;
longy=120*covt;
delta_lam=acos(sin(plax).*sin(play)+cos(plax).*cos(play).*cos(longy-
longx));
disp(delta_lam);
azi_circle=asin(cos(play)*sin(longy-longx)./sin(delta_lam));
disp(azi_circle);
radius=6371393;
distance_arc=radius.*delta_lam;
disp('circle arc length');
disp(distance_arc);
es=(a.^2-b.^2)./(a.^2);
eso=(b.^2)./(a.^2);
disp(eso);

%covrt parametric to geodesic coordinate
geola=zeros(3003,3);
pala=zeros(3003,1);
geolo=zeros(3003,1);

%analysis of function domain latitude is -/+(pi/2)
%longtitude is -/+(pi)
%conversion below with not be the wrong value
glax=atan(tan(plax)./sqrt(eso));
disp(glax);
disp(plax);
glay=atan(tan(play)./sqrt(eso));
```

```
disp(glay);
disp(play);
cons=(cos(glax).*cos(longx).*cos(glay).*sin(longy)-
cos(glax).*sin(longx).*cos(glay).*cos(longy));
lp=(cos(glax).*sin(longx).*eso.*sin(glay)-
cos(glay).*sin(longy).*eso.*sin(glax))./cons;
mp=(-
cos(glax).*cos(longx).*eso.*sin(glay)+cos(glay).*cos(longy).*eso.*sin
(glax))./cons;
fun=@(x)sqrt(1./(1+eso.*((cos(x).*lp+sin(x).*mp)./eso).^2).*(1+(1+((c
os(x).*lp+sin(x).*mp)./eso).^2)./(((1+eso.*((cos(x).*lp+sin(x).*mp)./
eso).^2)).^2).*(lp.*sin(x)-mp.*cos(x)).^2));
arc_ellipse=a.*integral(fun,longx,longy);
disp('arc length');
disp(arc_ellipse);
inc_step=(longy-longx)./3000;
disp('geodesic');

for i = 1:3001
    xx=longx+(i-1).*inc_step;
    pos_p_arc=atan(-(cos(xx).*lp+sin(xx).*mp)./eso);
    geola(i,1)=pos_p_arc;
    geola(i,2)=xx;
    geola(i,3)=atan(tan(pos_p_arc).*(eso.^(0.5)));
end

%1-geola,2-geolo,3-parala into excel table in rad
xlswrite(fileofell,geola);

%above is the coordinate for great ellipse, below is for great
circle,
%and e^2=0
cgeola=zeros(3003,3);
cpala=zeros(3000,1);
cgeolo=zeros(3000,1);
cglax=plax;
cglay=play;
clp=(cos(cglax).*sin(longx).*sin(cglay)-
cos(cglay).*sin(longy).*sin(cglax))./(cos(cglax).*cos(longx).*cos(cgl
ay).*sin(longy)-cos(cglax).*sin(longx).*cos(cglay)*cos(longy));
cmp=(-
cos(cglax).*cos(longx).*sin(cglay)+cos(cglay).*cos(longy).*sin(cglax)
)./(cos(cglax).*cos(longx).*cos(cglay).*sin(longy)-
cos(cglax).*sin(longx).*cos(cglay)*cos(longy));
```

```
inc_step=(longy-longx)./3000;
disp('circle geodesic');
for i = 1:3001
    xx=longx+(i-1).*inc_step;
    pos_p_arc=atan(-(cos(xx).*clp+sin(xx).*cmp));
    cgeola(i,1)=pos_p_arc;
    cgeola(i,2)=xx;
end
xlswrite(fileofcircle,cgeola);

%geodesic
paravertex=acos(cos(plax).*(cos(play).*sin(longy-
longx)./sin(delta_lam)));
us=(a.^2./b.^2-1).*(sin(paravertex)).^2;

%find o1 using intersection
lavertex=0;
logvertex=atan(-clp./cmp);
cir_delta_lam=acos(cos(plax).*cos(abs(longx-logvertex)));
func=@(x)sqrt(1+us.*power(sin(x),2));

%geolengthofnumericalintegral
length_geo=b.*integral(func,cir_delta_lam,cir_delta_lam+delta_lam);
disp(length_geo);

%percentage difference of circle and ellipse to geodesic
disp((distance_arc-length_geo)./length_geo);
disp((arc_ellipse-length_geo)./length_geo);

%geopath
%egoe 1 for latitude 2 for longtitude
egoe=zeros(3003,2);
sinazi_depart=cos(play)*sin(longy-longx)./sin(delta_lam);
cosla_highvertex=sinazi_depart*cos(plax);
sinla_highvertexsqr=1-cosla_highvertex.^2;
egoe(1,1)=cgeola(1,1);
egoe(1,2)=cgeola(1,2);

for i= 2:3001
delta_lam1=acos(sin(plax).*sin(cgeola(i,1))+cos(plax).*cos(cgeola(i,1
)).*cos(cgeola(i,2)-longx));
    clongd=cgeola(i,2)-longx;
    funcc=@(x)(sqrt(1-es*((1-(sin(x).^2).*sinla_highvertexsqr).^2))-
1)./((1-(sin(x).^2).*sinla_highvertexsqr).^2);
```

```
elongd=clongd+cosla_highvertex.*integral(funcc,cir_delta_lam,cir_delt
a_lam+delta_lam1);
    egoe(i,1)=cgeola(i,1);
    egoe(i,2)=elongd+longx;
end;
xlswrite(fileofgeo,egoe);

diff=zeros(3003,1);
s=0;
for i=1:3001
    diff(i)=(egoe(i,2)-geola(i,2)).^2;
    s=s+diff(i);
end
s=s/3001;
as=sqrt(s);

disp(as);
```

7.2.2 Comparison of difference between great circle, great ellipse and geodesic sailing

```
format long
fileofgeo='E:\eedata\startg12geo.xlsx';
a=6377397.155;
b=6356078.963;
covt=1/180*pi;
radius=6371393;
es=(a.^2-b.^2)./(a.^2);
eso=(b.^2)./(a.^2);

delta_lam=zeros(100,1);
azi_circle=zeros(100,1);
distance_arc=zeros(100,1);
length_geo=zeros(100,1);
arc_ellipse=zeros(100,1);
as=zeros(100,1);
s=zeros(100,1);
ecd=zeros(100,1);
ged=zeros(100,1);
```

```
%same starting point
dlax=0.5;
dlox=0;
dloy=90;
plax=dlax*covt;
longx=dlox*covt;
longy=dloy*covt;
eposes=0;
icre=6;
endp=zeros(100,1);
aaa=zeros(10,1);
aaa(1)=plax;
aaa(2)=longx;
aaa(3)=longy;

%different ending point
for k=1:16
    endp(k)=eposes+(k-1).*icre;
end
endp(16)=89.5;
endp(1)=0.5;

for k=1:16

    %great cicle para
play=endp(k).*covt;
delta_lam(k)=acos(sin(plax).*sin(play)+cos(plax).*cos(play).*cos(long
y-longx));
azi_circle(k)=asin(cos(play)*sin(longy-longx)./sin(delta_lam(k)));
distance_arc(k)=radius.*delta_lam(k);

%covrt parametric to geodesic coordinate
geola=zeros(3003,3);
pala=zeros(3003,1);
geolo=zeros(3003,1);

%analysis of function domain latitude is -/+(pi/2)
%longtitude is -/+(pi)
%conversion below with not be the wrong value
glax=atan(tan(plax)./sqrt(eso));
glay=atan(tan(play)./sqrt(eso));
cons=(cos(glax).*cos(longx).*cos(glay).*sin(longy)-
cos(glax).*sin(longx).*cos(glay).*cos(longy));
```

```
lp=(cos(glax).*sin(longx).*eso.*sin(glay)-
cos(glay).*sin(longy).*eso.*sin(glax))./cons;
mp=(-
cos(glax).*cos(longx).*eso.*sin(glay)+cos(glay).*cos(longy).*eso.*sin
(glax))./cons;
fun=@(x)sqrt(1./(1+eso.*((cos(x).*lp+sin(x).*mp)./eso).^2).*(1+(1+((c
os(x).*lp+sin(x).*mp)./eso).^2)./(((1+eso.*((cos(x).*lp+sin(x).*mp)./
eso).^2)).^2).*(lp.*sin(x)-mp.*cos(x)).^2));

arc_ellipse(k)=a.*integral(fun,longx,longy);
inc_step=(longy-longx)./3000;

for i = 1:3001
    xx=longx+(i-1).*inc_step;
    pos_p_arc=atan(-(cos(xx).*lp+sin(xx).*mp)./eso);
    geola(i,1)=pos_p_arc;
    geola(i,2)=xx;
    geola(i,3)=atan(tan(pos_p_arc).*(eso.^(0.5)));
end

cgeola=zeros(3003,3);
cpala=zeros(3000,1);
cgeolo=zeros(3000,1);
cglax=plax;
cglay=play;
clp=(cos(cglax).*sin(longx).*sin(cglay)-
cos(cglay).*sin(longy).*sin(cglax))./(cos(cglax).*cos(longx).*cos(cgl
ay).*sin(longy)-cos(cglax).*sin(longx).*cos(cglay)*cos(longy));
cmp=(-
cos(cglax).*cos(longx).*sin(cglay)+cos(cglay).*cos(longy).*sin(cglax)
)./(cos(cglax).*cos(longx).*cos(cglay).*sin(longy)-
cos(cglax).*sin(longx).*cos(cglay)*cos(longy));

inc_step=(longy-longx)./3000;
for i = 1:3001
    xx=longx+(i-1).*inc_step;
    pos_p_arc=atan(-(cos(xx).*clp+sin(xx).*cmp));
    cgeola(i,1)=pos_p_arc;
    cgeola(i,2)=xx;
end


%geodesic
paravertex=acos(cos(plax).*(cos(play).*sin(longy-
```

```
longx)./sin(delta_lam(k))));
us=(a.^2./b.^2-1).*(sin(paravertex)).^2;

%find o1 using intersection
lavertex=0;
logvertex=atan(-clp./cmp);
cir_delta_lam=acos(cos(plax).*cos(abs(longx-logvertex)));
func=@(x)sqrt(1+us.*power(sin(x),2));

%geolengthofnumericalintegral
length_geo(k)=b.*integral(func,cir_delta_lam,cir_delta_lam+delta_lam(
k));


%percentage difference of circle and ellipse to geodesic
ecd(k)=abs((distance_arc(k)-length_geo(k))./length_geo(k));
ged(k)=abs((arc_ellipse(k)-length_geo(k))./length_geo(k));

%geopath
%egoe 1 for latitude 2 for longtitude
egoe=zeros(3003,2);
sinazi_depart=cos(play)*sin(longy-longx)./sin(delta_lam(k));
cosla_highvertex=sinazi_depart*cos(plax);
sinla_highvertexsqr=1-cosla_highvertex.^2;
egoe(1,1)=cgeola(1,1);
egoe(1,2)=cgeola(1,2);
for i= 2:3001
delta_lam1=acos(sin(plax).*sin(cgeola(i,1))+cos(plax).*cos(cgeola(i,1
)).*cos(cgeola(i,2)-longx));
    clongd=cgeola(i,2)-longx;
    funcc=@(x)(sqrt(1-es*((1-(sin(x).^2).*sinla_highvertexsqr).^2))-
1)./((1-(sin(x).^2).*sinla_highvertexsqr).^2);

elongd=clongd+cosla_highvertex.*integral(funcc,cir_delta_lam,cir_delt
a_lam+delta_lam1);
    egoe(i,1)=cgeola(i,1);
    egoe(i,2)=elongd+longx;
end;

diff=zeros(3003,1);
s(k)=0;
for i=1:3001
    diff(i)=(egoe(i,2)-geola(i,2)).^2;
    s(k)=s(k)+diff(i);
```

```
end
s(k)=s(k)/3001;
as(k)=sqrt(s(k));


end
xlswrite(fileofgeo,as,1);
xlswrite(fileofgeo,ecd,2);
xlswrite(fileofgeo,ged,3);
xlswrite(fileofgeo,aaa,4);
```

7.2.3 Absolute difference

```
ecd(k,kk)=abs(distance_arc(k,kk)-length_geo(k,kk));
ged(k,kk)=abs(arc_ellipse(k,kk)-length_geo(k,kk));
```

7.2.4 Latitude across the hemisphere

```
ddlax=zeros(30,1);
ddlax(1)=0;

for kk=2:11
    ddlax(kk)=ddlax(kk-1)+9;
end

for kk=2:11
    ddlax(kk)=-ddlax(kk);
end

ddlax(1)=-0.1;
ddlax(11)=-89.9;
disp(ddlax);
```