**Exercise 1.2**

Show that the error probability is reduced by the use of $R_3$ by computing the error probability of this code for a binary symmetric channel with noise level $f$.

**Solution:**

Without $R_3$, we have error probability $f$.
With $R_3$, we have error probability

$$\binom{3}{2}(1-f)f^2 + f^3 = 3f^2 - 3f^3 + f^3 = 3f^2 - 2f^3$$

Now,

$$f - (3f^2 - 2f^3) = f - 3f^2 + 2f^3$$
$$= f(1-2f)(1-f) > 0 \quad (\because f < \frac{1}{2})$$

so $f > 3f^2 - 2f^3$ hence error probability reduced.

**Exercise 1.3**

(a) Show that the probability of error of $R_N$, the repetition code with $N$ repetitions, is

$$p_{\mathrm{b}} = \sum_{n=(N+1)/2}^{N} \binom{N}{n} f^n (1-f)^{N-n}, \qquad (1.24)$$

for odd $N$.

(b) Assuming $f = 0.1$, which of the terms in this sum is the biggest? How much bigger is it than the second-biggest term?

(c) Use Stirling's approximation (p.2) to approximate the $\binom{N}{n}$ in the largest term, and find, approximately, the probability of error of the repetition code with $N$ repetitions.

(d) Assuming $f = 0.1$, find how many repetitions are required to get the probability of error down to $10^{-15}$. [Answer: about 60.]

**Solution:**

(a) If $N$ is odd, error when $\frac{N+1}{2}$ or more bits flipped. Hence,

$$p_b = \sum_{n=(N+1)/2}^{N} \binom{N}{n} f^n (1-f)^{N-n}$$

(b) Let each term $a_n$.

$$\frac{a_{n+1}}{a_n} = \frac{N! n! (N-n)! f^{n+1}(1-f)^{N-n-1}}{(n+1)!(N-n-1)! N! f^n (1-f)^{N-n}}$$
$$= \frac{(N-n)f}{(n+1)(1-f)} = \frac{N-n}{9(n+1)}$$

Now,

$$\frac{N-n}{9(n+1)} > 1$$
$$\iff 10n + 9 < n$$
$$\iff n < \frac{N-9}{10} \quad (< \frac{N+1}{2})$$

so ratio $< 1$ for all $n \geq \frac{N+1}{2}$, meaning the series is decreasing, hence largest term is when $n = \frac{N+1}{2}$.

Ratio is $\frac{N - \frac{N+1}{2}}{9(\frac{N+1}{2}+1)} = \frac{N-1}{9N+11} \simeq \frac{1}{9}$.

The largest term is approximately 9 times larger than second-largest term.

(c)

$$\binom{N}{n} \approx 2^{NH_2(\frac{n}{N})}$$

Now,

$$H_2(\frac{\frac{N+1}{2}}{N}) = \frac{N+1}{2N} \log_2(\frac{2N}{N+1}) + \frac{N-1}{2N} \log_2(\frac{2N}{N-1})$$
$$\simeq \frac{1}{2} \log_2(2) + \frac{1}{2} \log_2(2) = 1$$

So,

$$p_b \approx p_B = \binom{N}{\frac{N+1}{2}} f^{\frac{N+1}{2}}(1-f)^{N-\frac{N+1}{2}} \simeq 2^N f^{\frac{N}{2}}(1-f)^{\frac{N}{2}} = \{4f(1-f)\}^{\frac{N}{2}}$$

(d)

$$10^{-15} = \{4f(1-f)\}^{\frac{N}{2}}$$

$$N = 2\frac{-15}{\log_{10} 4 \cdot 0.1 \cdot 0.9} = 67.91 \approx 68$$

or more accrately,

$$\ln \binom{N}{n} \simeq (N-n)\log_2 \frac{N}{N-n} + n\log_2 \frac{N}{n} - \frac{1}{2}\log_2(2\pi\frac{(N-n)n}{N})$$

$$= NH_2(\frac{n}{N}) - \frac{1}{2}\log(2\pi\frac{(N-n)n}{N})$$

$$\xrightarrow{n\simeq\frac{N}{2}} N - \frac{1}{2}\log_2(\frac{\pi N}{2})$$

$$\binom{N}{n} \simeq 2^{N-\frac{1}{2}\log_2(\frac{\pi N}{2})} = \frac{2^N}{\sqrt{\frac{\pi N}{2}}}$$

Then,

$$p_b \approx p_B \simeq \binom{N}{\frac{N+1}{2}} f^{\frac{N+1}{2}}(1-f)^{\frac{N-1}{2}}$$

$$\simeq \frac{2^N}{\sqrt{\frac{\pi N}{2}}} f\{f(1-f)\}^{\frac{N-1}{2}} = \frac{2f}{\sqrt{\frac{\pi N}{2}}} f\{4f(1-f)\}^{\frac{N-1}{2}}$$

so

$$10^{-15} = \frac{0.2}{\sqrt{\frac{\pi N}{2}}}\{4 \cdot 0.1 \cdot 0.9\}^{\frac{N-1}{2}}$$

$$\frac{N-1}{2} = \frac{-15 - \log_1 0\frac{0.2}{\sqrt{\frac{\pi N}{2}}}}{\log_1 00.36}$$

Iterate from $\hat{N}_1 = 67.6$

$$\frac{\hat{N}_2 - 1}{2} = 29.95$$

$$\hat{N}_2 = 60.90$$

$$\hat{N}_3 = 61.00$$

$$\hat{N}_3 = 61.00$$

which results in 61.

**Exercise**

**Solution:**