Exercise 1.2

Show that the error probability is reduced by the use of R_3 by computing the error probability of this code for a binary symmetric channel with noise level f.

Solution:

Without R_3, we have error probability f.
With R_3, we have error probability

$$\binom{3}{2}(1-f)f^2 + f^3 = 3f^2 - 3f^3 + f^3 = 3f^2 - 2f^3$$

Now,

$$\begin{aligned} f - (3f^2 - 2f^3) &= f - 3f^2 + 2f^3 \\ &= f(1-2f)(1-f) > 0 \quad (\because f < \frac{1}{2}) \end{aligned}$$

so $f > 3f^2 - 2f^3$ hence error probability reduced.

Exercise 1.3

(a) Show that the probability of error of R_N, the repetition code with N repetitions, is

$$p_{\mathrm{b}} = \sum_{n=(N+1)/2}^{N} \binom{N}{n} f^n (1-f)^{N-n}, \tag{1.24}$$

for odd N.

(b) Assuming $f = 0.1$, which of the terms in this sum is the biggest? How much bigger is it than the second-biggest term?

(c) Use Stirling's approximation (p.2) to approximate the $\binom{N}{n}$ in the largest term, and find, approximately, the probability of error of the repetition code with N repetitions.

(d) Assuming $f = 0.1$, find how many repetitions are required to get the probability of error down to 10^{-15}. [Answer: about 60.]

Solution:

(a) If N is odd, error when $\frac{N+1}{2}$ or more bits flipped. Hence,

$$p_b = \sum_{n=(N+1)/2}^{N} \binom{N}{n} f^n (1-f)^{N-n}$$

(b) Let each term a_n.

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{N!n!(N-n)!f^{n+1}(1-f)^{N-n-1}}{(n+1)!(N-n-1)!N!f^n(1-f)^{N-n}} \\ &= \frac{(N-n)f}{(n+1)(1-f)} = \frac{N-n}{9(n+1)}\end{aligned}$$

Now,

$$\begin{aligned}&\frac{N-n}{9(n+1)} > 1 \\ \Longleftrightarrow\ & 10n + 9 < n \\ \Longleftrightarrow\ & n < \frac{N-9}{10} \quad (< \frac{N+1}{2})\end{aligned}$$

so ratio < 1 for all $n \geq \frac{N+1}{2}$, meaning the series is decreasing, hence largest term is when $n = \frac{N+1}{2}$.
Ratio is $\frac{N-\frac{N+1}{2}}{9(\frac{N+1}{2}+1)} = \frac{N-1}{9N+11} \simeq \frac{1}{9}$.
The largest term is approximately 9 times larger than second-largest term.

(c)

$$\binom{N}{n} \approx 2^{NH_2(\frac{n}{N})}$$

Now,

$$\begin{aligned}H_2(\frac{\frac{N+1}{2}}{N}) &= \frac{N+1}{2N}\log_2(\frac{2N}{N+1}) + \frac{N-1}{2N}\log_2(\frac{2N}{N-1}) \\ &\simeq \frac{1}{2}\log_2(2) + \frac{1}{2}\log_2(2) = 1\end{aligned}$$

So,

$$p_b \approx p_B = \binom{N}{\frac{N+1}{2}} f^{\frac{N+1}{2}}(1-f)^{N-\frac{N+1}{2}} \quad \simeq 2^N f^{\frac{N}{2}}(1-f)^{\frac{N}{2}} = \{4f(1-f)\}^{\frac{N}{2}}$$

(d)

$$\begin{aligned} 10^{-15} &= \{4f(1-f)\}^{\frac{N}{2}} \\ N &= 2\frac{-15}{\log_{10} 4 \cdot 0.1 \cdot 0.9} = 67.91 \approx 68 \end{aligned}$$

or more accrately,

$$\begin{aligned} \ln \binom{N}{n} &\simeq (N-n)\log_2 \frac{N}{N-n} + n \log_2 \frac{N}{n} - \frac{1}{2}\log_2(2\pi\frac{(N-n)n}{N}) \\ &= NH_2(\frac{n}{N}) - \frac{1}{2}\log(2\pi\frac{(N-n)n}{N}) \\ &\xrightarrow{n \simeq \frac{N}{2}} N - \frac{1}{2}\log_2(\frac{\pi N}{2}) \end{aligned}$$

$$\binom{N}{n} \simeq 2^{N - \frac{1}{2}\log_2(\frac{\pi N}{2})} = \frac{2^N}{\sqrt{\frac{\pi N}{2}}}$$

Then,

$$\begin{aligned} p_b \approx p_B &\simeq \binom{N}{\frac{N+1}{2}} f^{\frac{N+1}{2}}(1-f)^{\frac{N-1}{2}} \\ &\simeq \frac{2^N}{\sqrt{\frac{\pi N}{2}}} f\{f(1-f)\}^{\frac{N-1}{2}} = \frac{2f}{\sqrt{\frac{\pi N}{2}}} f\{4f(1-f)\}^{\frac{N-1}{2}} \end{aligned}$$

so

$$\begin{aligned} 10^{-15} &= \frac{0.2}{\sqrt{\frac{\pi N}{2}}}\{4 \cdot 0.1 \cdot 0.9\}^{\frac{N-1}{2}} \\ \frac{N-1}{2} &= \frac{-15 - \log_1 0\frac{0.2}{\sqrt{\frac{\pi N}{2}}}}{\log_1 00.36} \end{aligned}$$

Iterate from $\hat{N}_1 = 67.6$

$$\begin{aligned} \frac{\hat{N}_2 - 1}{2} &= 29.95 \\ \hat{N}_2 &= 60.90 \\ \hat{N}_3 &= 61.00 \\ \hat{N}_3 &= 61.00 \end{aligned}$$

which results in 61.

Exercise 1.4

Prove that this is so ($\mathbf{Ht} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$) by evaluating the 3×4 matrix $\mathbf{HG}^{\mathrm{T}}$

Solution:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Exercise 1.5

This exercise and the next three refer to the (7, 4) Hamming code. Decode the received strings:

(a) r = 1101011

(b) r = 0110110

(c) r = 0100111

(d) r = 1111111

Solution:

The following are symdrome, unflipped bit and answer.

(a) $z = 011, r_4, 1100011$

(b) $z = 111, r_3, 0100110$

(c) $z = 001, r_7, 0100110$

(d) $z = 000, \text{None}, 1111111$

Exercise 1.6

(a) Calculate the probability of block error p_B of the (7, 4) Hamming code as a function of the noise level f and show that to leading order it goes as $21f^2$.

(b) Show that to leading order the probability of bit error p_b goes as $9f^2$

Solution:

(a)

$$
\begin{aligned}
p_B &= 1 - \sum_{k=0}^{1} \binom{7}{k} f^k (1-f)^{7-k} \\
&= 1 - (1-f)^7 - 7f(1-f)^6 \\
&= 1 - (1+6f)(1-f)^6 \\
&= 1 - (1+6f)(1-6f+15f^2+O(f^3)) \\
&= 1 - (1-6f+15f^2+6f-36f^2+O(f^3)) \\
&= 21f^2 + O(f^3)
\end{aligned}
$$

(b) Let X = the number of bits flipped in a block after decoding.

$$
\begin{aligned}
E[p_b] &= \frac{1}{7} E[X] \\
&= \frac{1}{7} (3 \binom{7}{2} f^2 (1-f)^5 + \sum_{3}^{7} a_k \binom{7}{k} f^k (10f)^{7-k}) \\
&= \frac{3}{7} \times \frac{7 \cdot 6}{2 \cdot 1} f^2 (1-f)^5 + O(f^3) \\
&= 9f^2 (1-f)^5 \\
&= 9f^2 + O(f^3)
\end{aligned}
$$

Exercise 1.7

Solution:

Exercise

Solution: