Exercise 2.2

Are the random variables X and Y in the joint ensemble of figure 2.2 independent?

Solution:

No, $P(x, y) \neq P(x)P(y)$ since each row or column are not proportional to each other.

Exercise 2.4

An urn contains K balls, of which B are black and $W = K - B$ are white. Fred draws a ball at random from the urn and replaces it, N times.

(a) What is the probability distribution of the number of times a black ball is drawn, n_B?

(b) What is the expectation of n_B? What is the variance of n_B? What is the standard deviation of n_B? Give numerical answers for the cases $N = 5$ and $N = 400$, when $B = 2$ and $K = 10$.

Solution:

(a) $P(n_B) = \binom{N}{n_B}(\frac{B}{K})^{n_B}(\frac{K-B}{K})^{N-n_B}$

(b) $\frac{B}{K} = \frac{1}{5}$, $\frac{K-B}{K} = \frac{4}{5}$ so the distribution is $\mathrm{B}(n, \frac{1}{5})$.

Hence, $\mathrm{E}[n_b] = \frac{1}{5}n$, $\mathrm{Var}[n_B] = \frac{1}{5}n(1 - \frac{1}{5})$, $\mathrm{Std}[n_B] = \sqrt{\mathrm{Var}[n_B]}$

For n = 5: $\mathrm{E}[n_b] = 1$, $\mathrm{Var}[n_B] = \frac{4}{5}$, $\mathrm{Std}[n_B] = \sqrt{\frac{4}{5}}$

For n = 400: $\mathrm{E}[n_b] = 80$, $\mathrm{Var}[n_B] = 64$, $\mathrm{Std}[n_B] = 8$

Exercise 2.5

An urn contains K balls, of which B are black and $W = K - B$ are white. We define the fraction $f_B \equiv \frac{B}{K}$. Fred draws N times from the urn, exactly as in exercise 2.4, obtaining n_B blacks, and computes the quantity

$$z = \frac{(n_B - f_B N)^2}{N f_B (1 - f_B)}. \tag{2.19}$$

What is the expectation of z? In the case $N = 5$ and $f_B = \frac{1}{5}$, what is the probability distribution of z? What is the probability that $z < 1$? [Hint: compare z with the quantities computed in the previous exercise.]

Solution:

$$\mathrm{E}[z] = \frac{1}{N f_b (1 - f_B)} (\mathrm{E}[n_B^2] - 2\,\mathrm{E}[f_B n_B N] + \mathrm{E}[f_B^2 N^2]) \quad = \frac{1}{N f (1 - f)} (N f (1 - f) + N^2 f^2 - 2 N^2 f$$

With $N = 5$ and $f_B = \frac{1}{5}$,

$$z = \frac{(n_B - 1)^2}{\frac{4}{5}}$$

so

n_B	0	1	2	3	4	5
z	$\frac{5}{4}$	0	$\frac{5}{4}$	5	$\frac{45}{4}$	20

Hence,

$$P(z < 1) = P(n_B = 1) = \binom{5}{1} \cdot \left(\frac{1}{5}\right) \left(\frac{4}{5}\right)^4 = \frac{256}{625} = 0.4096$$

Exercise 2.8

Assuming a uniform prior on f_H, $P(f_H) = 1$, solve the problem posed in example 2.7 (p.30). Sketch the posterior distribution of f_H and compute the probability that the $N+1$th outcome will be a head, for

(a) $N = 3$ and $n_H = 0$;

(b) $N = 3$ and $n_H = 2$;

(c) $N = 10$ and $n_H = 3$;

(d) $N = 300$ and $n_H = 29$.

Solution:

$$P(f_H|n_H, N) = \frac{P(n_H|f_H, N)P(f_H)}{P(n_H|N)} = \frac{P(n_H|f_H, N)}{P(n_H|N)} = \frac{\binom{N}{n_H} f_H^{n_H}(1-f_H)^{N-n_H}}{P(n_H|N)}$$

Now,

$$\begin{aligned}
&\int P(f_H|n_H, N)df_H = 1 \\
\iff &\frac{\binom{N}{n_H}}{P(n_H|N)} \int_1^0 f_H^{n_H}(1-f_H)^{N-n_H}df_H = 1 \\
\iff &P(n_H|N) = \binom{N}{n_H}\frac{\Gamma(n_H+1)\Gamma(N-n_H+1)}{\Gamma(N+2)} \\
&= \binom{N}{n_H}\frac{n_H!(N-n_H)!}{(N+1)!} \\
&= \frac{n_H!(N-n_H)!}{(N+1)n_H!(N-n_H)!} = \frac{1}{N+1}
\end{aligned}$$

so

$$P(f_H|n_H, N) = \frac{(N+1)!}{n_H!(N-n_H)!} f_H^{n_H}(1-f_H)^{N-n_H}$$

The graph looks like this: www.desmos.com/calculator/bjjlgnqg87
Now,

$$\begin{aligned}
\mathrm{E}[f_H] &= \int_0^1 f_H P(f_H|n_H, N)df_H \\
&= \frac{(N+1)!}{n_H!(N-n_H)!}\int_0^1 f_H^{n_H+1}(1-f)^{N-n_H}df_H \\
&= \frac{(N+1)!}{n_H!(N-n_H)!}\frac{(n_H+1)!(N-n_H)!}{(N+2)!} \\
&= \frac{n_H+1}{N+2}
\end{aligned}$$

Thus,

(a) $\frac{1}{5}$

(b) $\frac{3}{5}$

(c) $\frac{4}{12} = \frac{1}{3}$

(d) $\frac{30}{302} = \frac{15}{151}$

Exercise

Solution:

Exercise

Solution: