

Exercise 1.2

Show that the error probability is reduced by the use of R_3 by computing the error probability of this code for a binary symmetric channel with noise level f .

Solution:

Without R_3 , we have error probability f .

With R_3 , we have error probability

$$\binom{3}{2}(1-f)f^2 + f^3 = 3f^2 - 3f^3 + f^3 = 3f^2 - 2f^3$$

Now,

$$\begin{aligned} f - (3f^2 - 2f^3) &= f - 3f^2 + 2f^3 \\ &= f(1 - 2f)(1 - f) > 0 \quad (\because f < \frac{1}{2}) \end{aligned}$$

so $f > 3f^2 - 2f^3$ hence error probability reduced.

Exercise 1.3

- (a) Show that the probability of error of R_N , the repetition code with N repetitions, is

$$p_b = \sum_{n=(N+1)/2}^N \binom{N}{n} f^n (1-f)^{N-n}, \quad (1.24)$$

for odd N .

- (b) Assuming $f = 0.1$, which of the terms in this sum is the biggest? How much bigger is it than the second-biggest term?
- (c) Use Stirling's approximation (p.2) to approximate the $\binom{N}{n}$ in the largest term, and find, approximately, the probability of error of the repetition code with N repetitions.
- (d) Assuming $f = 0.1$, find how many repetitions are required to get the probability of error down to 10^{-15} . [Answer: about 60.]

Solution:

- (a) If N is odd, error when $\frac{N+1}{2}$ or more bits flipped. Hence,

$$p_b = \sum_{n=(N+1)/2}^N \binom{N}{n} f^n (1-f)^{N-n}$$

(b) Let each term a_n .

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{N!n!(N-n)!f^{n+1}(1-f)^{N-n-1}}{(n+1)!(N-n-1)!N!f^n(1-f)^{N-n}} \\ &= \frac{(N-n)f}{(n+1)(1-f)} = \frac{N-n}{9(n+1)} \end{aligned}$$

Now,

$$\begin{aligned} \frac{N-n}{9(n+1)} &> 1 \\ \iff 10n+9 &< n \\ \iff n &< \frac{N-9}{10} \quad (< \frac{N+1}{2}) \end{aligned}$$

so ratio < 1 for all $n \geq \frac{N+1}{2}$, meaning the series is decreasing, hence largest term is when $n = \frac{N+1}{2}$.

Ratio is $\frac{N-\frac{N+1}{2}}{9(\frac{N+1}{2}+1)} = \frac{\frac{N-1}{2}}{9\frac{N+1}{2}+9} \simeq \frac{1}{9}$.

The largest term is approximately 9 times larger than second-largest term.

(c)

$$\binom{N}{n} \approx 2^{NH_2(\frac{n}{N})}$$

Now,

$$\begin{aligned} H_2\left(\frac{\frac{N+1}{2}}{N}\right) &= \frac{N+1}{2N} \log_2\left(\frac{2N}{N+1}\right) + \frac{N-1}{2N} \log_2\left(\frac{2N}{N-1}\right) \\ &\simeq \frac{1}{2} \log_2(2) + \frac{1}{2} \log_2(2) = 1 \end{aligned}$$

So,

$$p_b \approx p_B = \binom{N}{\frac{N+1}{2}} f^{\frac{N+1}{2}} (1-f)^{N-\frac{N+1}{2}} \simeq 2^N f^{\frac{N}{2}} (1-f)^{\frac{N}{2}} = \{4f(1-f)\}^{\frac{N}{2}}$$

(d)

$$10^{-15} = \{4f(1-f)\}^{\frac{N}{2}}$$

$$N = 2 \frac{-15}{\log_{10} 4 \cdot 0.1 \cdot 0.9} = 67.91 \approx 68$$

or more accurately,

$$\begin{aligned} \ln \binom{N}{n} &\simeq (N-n) \log_2 \frac{N}{N-n} + n \log_2 \frac{N}{n} - \frac{1}{2} \log_2 (2\pi \frac{(N-n)n}{N}) \\ &= NH_2(\frac{n}{N}) - \frac{1}{2} \log(2\pi \frac{(N-n)n}{N}) \\ &\xrightarrow{n \simeq \frac{N}{2}} N - \frac{1}{2} \log_2 (\frac{\pi N}{2}) \\ \binom{N}{n} &\simeq 2^{N - \frac{1}{2} \log_2 (\frac{\pi N}{2})} = \frac{2^N}{\sqrt{\frac{\pi N}{2}}} \end{aligned}$$

Then,

$$\begin{aligned} p_b \approx p_B &\simeq \binom{N}{\frac{N+1}{2}} f^{\frac{N+1}{2}} (1-f)^{\frac{N-1}{2}} \\ &\simeq \frac{2^N}{\sqrt{\frac{\pi N}{2}}} f \{f(1-f)\}^{\frac{N-1}{2}} = \frac{2f}{\sqrt{\frac{\pi N}{2}}} f \{4f(1-f)\}^{\frac{N-1}{2}} \end{aligned}$$

so

$$\begin{aligned} 10^{-15} &= \frac{0.2}{\sqrt{\frac{\pi N}{2}}} \{4 \cdot 0.1 \cdot 0.9\}^{\frac{N-1}{2}} \\ \frac{N-1}{2} &= \frac{-15 - \log_1 0 \frac{0.2}{\sqrt{\frac{\pi N}{2}}}}{\log_1 0.36} \end{aligned}$$

Iterate from $\hat{N}_1 = 67.6$

$$\begin{aligned} \frac{\hat{N}_2 - 1}{2} &= 29.95 \\ \hat{N}_2 &= 60.90 \\ \hat{N}_3 &= 61.00 \\ \hat{N}_3 &= 61.00 \end{aligned}$$

which results in 61.

Exercise 1.4

Prove that this is so ($\mathbf{Ht} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$) by evaluating the 3×4 matrix \mathbf{HG}^T

Solution:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Exercise 1.5

This exercise and the next three refer to the $(7, 4)$ Hamming code. Decode the received strings:

- (a) $r = 1101011$
- (b) $r = 0110110$
- (c) $r = 0100111$
- (d) $r = 1111111$

Solution:

The following are syndrome, unflipped bit and answer.

- (a) $z = 011, r_4, 1100011$
- (b) $z = 111, r_3, 0100110$
- (c) $z = 001, r_7, 0100110$
- (d) $z = 000, \text{None}, 1111111$

Exercise 1.6

- (a) Calculate the probability of block error p_B of the $(7, 4)$ Hamming code as a function of the noise level f and show that to leading order it goes as $21f^2$.
- (b) Show that to leading order the probability of bit error p_b goes as $9f^2$

Solution:

(a)

$$\begin{aligned}
 p_B &= 1 - \sum_{k=0}^1 \binom{7}{k} f^k (1-f)^{7-k} \\
 &= 1 - (1-f)^7 - 7f(1-f)^6 \\
 &= 1 - (1+6f)(1-f)^6 \\
 &= 1 - (1+6f)(1-6f + 15f^2 + O(f^3)) \\
 &= 1 - (1-6f + 15f^2 + 6f - 36f^2 + O(f^3)) \\
 &= 21f^2 + O(f^3)
 \end{aligned}$$

(b) Let X = the number of bits flipped in a block after decoding.

$$\begin{aligned}
 E[p_b] &= \frac{1}{7} E[X] \\
 &= \frac{1}{7} (3 \binom{7}{2} f^2 (1-f)^5 + \sum_3^7 a_k \binom{7}{k} f^k (10f)^{7-k}) \\
 &= \frac{3}{7} \times \frac{7 \cdot 6}{2 \cdot 1} f^2 (1-f)^5 + O(f^3) \\
 &= 9f^2 (1-f)^5 \\
 &= 9f^2 + O(f^3)
 \end{aligned}$$

Exercise 1.7

Find some noise vectors that give the all-zero syndrome (that is, noise vectors that leave all the parity checks unviolated). How many such noise vectors are there?

Solution:

There are $2^4 = 16$ codewords, so 16 such noise vectors. This is because, by definition, codewords are the only vectors that have the property of $\mathbf{Hn} = 0$. This can be seen from the Venn diagram being uniquely filled if the first 4 bits are determined.

Exercise 1.8

I asserted above that a block decoding error will result whenever two or more bits are flipped in a single block. Show that this is indeed so. [In principle, there might be error patterns that, after decoding, led only to the corruption of the parity bits, with no source bits incorrectly decoded.]

Solution:

Assume there is a block error but no decoding error. That is, the source bit is unchanged but the parity bit is changed. The parity bit is deterministic from the source bit, so if the source bit is the same, the parity bit must also be the same, meaning there would be no block error. This contradicts with the assumption that the code has a block error, so if there is a block error, there must also be a decoding error.

Exercise**Solution:**