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QUALIFY FOR THE FUTURE WORLD
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Tick this box if there is no writing in this booklet

Scholarship 2020
Calculus

9.30 a.m. Monday 16 November 2020
Time allowed: Three hours
Total score: 40

ANSWER BOOKLET

There are five questions in this examination. Answer ALL FIVE questions.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

Write ALL your answers in this booklet.

Make sure that you have Formulae and Tables Booklet S–CALCF.

Show ALL working. Start your answer to each question on a new page. Carefully number each question.

Answers developed using a CAS calculator require **ALL commands to be shown**. Correct answers only will not be sufficient.

Check that this booklet has pages 2–27 in the correct order and that none of these pages is blank.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Question	Score
ONE	
TWO	
THREE	
FOUR	
FIVE	
TOTAL	/40

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1d

$$\frac{3x^2+2x-4}{5x^2+8x-1} = \frac{\frac{3}{5}(5x^2+8x-1)}{5x^2+8x-1}$$

We get $3x^2+2x-4 = \frac{3}{5}(5x^2+8x-1) - 2.8x - 3.4$

So $$\frac{3x^2+2x-4}{5x^2+8x-1} = \frac{3}{5} - \frac{2.8x+3.4}{5x^2+8x-1}.$$

Consider $\lim_{x\to\infty} \frac{2.8x+3.4}{5x^2+8x-1}$. I claim this is 0.

This is because, for all $\varepsilon > 0$, $\exists M$ s.t. $\left|\frac{2.8x+3.4}{5x^2+8x-1}\right| < \varepsilon \ \forall x > M$.

Indeed, choose $M = \left(\frac{1}{\varepsilon}\right) \times 10^{100} + 10^{100}$

Then, for $x > M$,

$8x > 1$ as $x > \frac{1}{8}$,

and $2.8x + 3.4 < 7x$ as $x > 1$.

Thus, $$\left|\frac{2.8x+3.4}{5x^2+8x-1}\right| < \left|\frac{7x}{5x^2}\right| = \left|\frac{7}{5}\cdot\frac{1}{x}\right| < \left|\frac{7}{5}\cdot\varepsilon\cdot\frac{1}{10^{100}}\right| < \varepsilon.$$

So, $\lim_{x\to\infty} \frac{2.8x+3.4}{5x^2+8x-1} = 0$, and so

$$\lim_{x\to\infty} \frac{3x^2+2x-4}{5x^2+8x-1} = \lim_{x\to\infty}\left(\frac{3}{5} + \frac{2.8x+3.4}{5x^2+8x-1}\right) = \frac{3}{5}.$$

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1b

if $v = \sqrt{a^4+x^4}$, $\frac{dv}{dx} = \frac{1}{2} \times \frac{1}{\sqrt{a^4+x^4}} \times 4x^3 = \frac{2x^3}{\sqrt{a^4+x^4}}$

$$\therefore \int_0^a \frac{x^3}{\sqrt{a^4+x^4}}\,dx = \frac{1}{2}\int_0^a \frac{2x^3}{\sqrt{a^4+x^4}}\,dx$$

$$= \frac{1}{2}\left[\sqrt{a^4+x^4}\right]_0^a$$

$$= \frac{1}{2}\left(\sqrt{2a^4} - \sqrt{a^4}\right)$$

$$= \frac{1}{2}(\sqrt{2}-1)a^2.$$

1c i

$P(x) = ax^4 + bx^3 + cx^2 + dx + e$

$= a(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$ by FTOA.

$= a(x^2 - (\alpha+\beta)x + \alpha\beta)(x^2 - (\gamma+\delta)x + \gamma\delta)$

$= a(x^4 - (\alpha+\beta+\gamma+\delta)x^3 + (\alpha\beta+\gamma\delta+(\alpha+\beta)(\gamma+\delta))x^2 - (\alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta))x + \alpha\beta\gamma\delta)$

$= a(x^4 - (\alpha+\beta+\gamma+\delta)x^3 + (\alpha\beta+\gamma\delta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta)x^2 - (\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)x + \alpha\beta\gamma\delta)$.

Comparing x^3 coefficients, $b = a[-(\alpha+\beta+\gamma+\delta)]$

$\therefore (\alpha+\beta+\gamma+\delta) = -\frac{b}{a}$.

" x^2 " $c = a(\alpha\beta+\gamma\delta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta)$

$\therefore \beta\gamma+\gamma\alpha+\alpha\beta+\alpha\delta+\beta\delta+\gamma\delta = \frac{c}{a}$

" x " $d = a[-(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)]$

$\therefore \beta\gamma\delta+\gamma\alpha\delta+\alpha\beta\delta+\alpha\beta\gamma = -\frac{d}{a}$.

" unit " $e = a(\alpha\beta\gamma\delta)$

$\therefore \alpha\beta\gamma\delta = \frac{e}{a}$.

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1(ii) Let roots be $\alpha, \beta, \gamma, \delta$.

Let $\alpha+\beta = \gamma+\delta$. From $\alpha+\beta+\gamma+\delta = -\frac{b}{a} = -\frac{(-8)}{1} = 8$,

$\alpha+\beta = \frac{1}{2}(2\alpha+\beta) = \frac{1}{2}(\alpha+\beta+\gamma+\delta) = \frac{1}{2}\cdot 8 = 4$,

and also $\gamma+\delta = 8-\alpha+\beta = 4$.

Note that $\alpha\beta+\gamma\delta+(\alpha+\beta)(\delta+\gamma) = \alpha\beta+\gamma\delta+\alpha\delta+\alpha\gamma+\beta\delta+\beta\gamma$

$= \frac{c}{a} = 14$, since $\alpha+\beta = \delta+\gamma = 4$.

$\alpha\beta+\gamma\delta+4^2 = 14$

$\therefore \alpha\beta+\gamma\delta = 14-4^2 = 3$.

Finally, $\alpha\beta\gamma\delta = \frac{e}{a} = 2$.

Let $y = \alpha\beta$, $z = \gamma\delta$. Then $y+z = 3$, ① $yz = 2$ ②

~~ie.~~ $\therefore z = 3-y$, substituting into ②,

$y(3-y) = 2$, $y^2-3y+2=0$

$\Rightarrow (y-1)(y-2) = 0$.

$\therefore$ either $y=1 \Rightarrow z=2$

or $y=2 \Rightarrow z=1$.

Let $y=1$, $z=2$ (we ~~have~~ are free to do this since $4 = \alpha\beta = \gamma\delta$. So by symmetry both choices give the same).

Then $\alpha+\beta = 4$, $\alpha\beta = 1$.

$\therefore \beta = 4-\alpha$, $\alpha(4-\alpha) = 1$, $\alpha^2-4\alpha+1=0$.

Has roots $\frac{4\pm\sqrt{12}}{2} = 2\pm\sqrt{3}$

So let $\alpha = 2+\sqrt{3}$, $\beta = 2-\sqrt{3}$.

also $\gamma+\delta = 4$, $\gamma\delta = z = 2$

$\therefore \delta = 4-\gamma$, $\gamma(4-\gamma) = 2$, $\gamma^2-4\gamma+2=0$

Has roots $\frac{4\pm\sqrt{8}}{2} = 2\pm\sqrt{2}$.

Let $\gamma = 2+\sqrt{2}$, $\delta = 2-\sqrt{2}$

$\therefore$ roots are $2\pm\sqrt{3}, 2\pm\sqrt{2}$.

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Now $p(x) = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$

$= (x-(2+\sqrt{3}))(x-(2-\sqrt{3}))(x-(2+\sqrt{2}))(x-(2-\sqrt{2}))$

$= (x^2-4x+1)(x^2-4x+2)$

$= x^4-8x^3+14x-12x+2$

$= x^4-8x^3+14x+px+2$,

so $p = -12$.

2a $\left(x^4+\frac{1}{x^4}\right)^2 = x^8+2\cdot x^4\cdot\frac{1}{x^4}+\frac{1}{x^8} = x^8+\frac{1}{x^8}+2$

But $\left(x^4+\frac{1}{x^4}\right)^2 = 7^2 = 49$

$\therefore x^8+\frac{1}{x^8} = 49-2 = 47$.

bi $f(x) = \frac{\cos x}{2+\sin x}$. $f'(x) = \frac{(2+\sin x)(-\sin x)-\cos x(\cos x)}{(2+\sin x)^2}$

$= \frac{-2\sin x-\sin^2 x-\cos^2 x}{(2+\sin x)^2} = \frac{-2\sin x-1}{(2+\sin x)^2}$

Turning points are when $f'(x)=0$

i.e. $-\frac{(2\sin x+1)}{(2+\sin x)^2} = 0$, or when $2\sin x = -1$.

(note $(2+\sin x)^2 \geq (1)^2 \geq 1$).

$\therefore \sin x = -\frac{1}{2}$, which occurs when $x = \frac{7\pi}{6}$ or $x = \frac{11\pi}{6}$.

When $x = \frac{7\pi}{6}$, $\cos x = -\frac{\sqrt{3}}{2}$, so

$f(x) = \frac{-\frac{\sqrt{3}}{2}}{2-\frac{1}{2}} = \frac{-\frac{\sqrt{3}}{2}}{\frac{3}{2}} = -\frac{\sqrt{3}}{3}$

When $x = \frac{11\pi}{6}$, $\cos x = \frac{\sqrt{3}}{2}$, so $f(x) = \frac{\frac{\sqrt{3}}{2}}{2-\frac{1}{2}} = \frac{\frac{\sqrt{3}}{2}}{\frac{3}{2}} = \frac{\sqrt{3}}{3}$.

Turning points $\left(\frac{7\pi}{6}, -\frac{\sqrt{3}}{3}\right)$ and $\left(\frac{11\pi}{6}, \frac{\sqrt{3}}{3}\right)$.

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ii $f'(x) = -\dfrac{(2\sin x+1)}{(2+\sin x)^2}$.

$f''(x) = \dfrac{(2+\sin x)^2(-2\cos x) + (2\sin x+1)\cdot[2(2+\sin x)\cos x]}{(2+\sin x)^4}$

$= \dfrac{(4+4\sin x+\sin^2 x)(-2\cos x) + (2\sin x+1)(4\cos x+2\sin x\cos x)}{(2+\sin x)^4}$

$= \dfrac{(-8\cos x - 8\sin x\cos x - 2\sin^2 x\cos x) + (8\sin x\cos x + 4\cos x + 4\sin^2 x\cos x + 2\sin x\cos x)}{(2+\sin x)^4}$

$= \dfrac{2\sin^2 x\cos x + 2\sin x\cos x - 4\cos x}{(2+\sin x)^4}$

$= \dfrac{2\cos x(\sin^2 x + \sin x - 2)}{(2+\sin x)^4}$.

$= \dfrac{2\cos x(\sin x - 1)(\sin x + 2)}{(2+\sin x)^4}$

Points of inflection are when $f''(x) = 0$, i.e.

either $\cos x = 0$, $\sin x = 1$, or $\sin x = -2$ (impossible)

$\cos x = 0 \Rightarrow x = \frac{\pi}{2}$ or $x = \frac{3\pi}{2}$,

$\sin x = 1 \Rightarrow x = \frac{\pi}{2}$, so $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$ are inflection points

Between $[0, \frac{\pi}{2})$, the function is concave down (concave) as $\cos x > 0$, $\sin x < 1$, and $\sin x > -2$ so $\dfrac{2\cos x(\sin x-1)(\sin x+2)}{(2+\sin x)^4} < 0$

Between $(\frac{\pi}{2}, \frac{3\pi}{2})$, the function is concave up (convex) as $\cos x < 0$, $\sin x < 1$, $\sin x > -2$, so

$\dfrac{2\cos x(\sin x - 1)(\sin x+2)}{(2+\sin x)^4} > 0$

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Between $(\frac{3\pi}{2}, 2\pi]$, the function is concave down (concave) as $\cos x > 0$, $\sin x < 1$ and $\sin x > -2$

$\therefore \dfrac{2\cos x(\sin x - 1)(\sin x+2)}{(2+\sin x)^4} > 0$.

+ Note that $(2+\sin x) \geq 1$, so $(2+\sin x)^4 > 0$

2c We prove that $\angle ABC = \angle ADC$ first.

Because $AB = AD = a$, A lies on perp. bisector of BD.

Similarly, $CB = CD = b \Rightarrow C$ lies on perp. bisector

$\therefore$ AC is the perpendicular bisector of BD

$\therefore$ Reflecting B over AC gives D, so $\angle ABC = \angle ADC$ by reflection. But $\angle ABC = 180° - \angle ADC$ by cyclic quadrilateral, so

$180° - \angle ADC = \angle ADC \Rightarrow \angle ADC = 90°$, and $\angle ABC = 90°$ as well

So AC is a diameter.

and $\angle CAD = \frac{1}{2}\angle BAD = \frac{\theta}{2}$ by symmetry.

Note $\frac{b}{a} = \tan\frac{\theta}{2} = \dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$.

$1 - 2\sin^2\frac{\theta}{2} = \cos\theta \therefore \sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$.

$2\cos^2\frac{\theta}{2} - 1 = \cos\theta \therefore \cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}$.

$\therefore \left(\dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right)^2 = \dfrac{1-\cos\theta}{1+\cos\theta}$

Note $\left(\dfrac{\sin\theta}{1+\cos\theta}\right)^2 = \dfrac{\sin^2\theta}{(1+\cos\theta)^2} = \cancel{\dfrac{\ }{4+4\cos\theta+\cos^2\theta}}$

$= \dfrac{(1-\cos^2\theta)}{(1+\cos\theta)^2} = \dfrac{(1-\cos\theta)(1+\cos\theta)}{(1+\cos\theta)^2} = \dfrac{(1-\cos\theta)}{(1+\cos\theta)}$

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so $\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \pm\frac{\sin\theta}{1+\cos\theta}$. Since $\theta < 180°$ by triangle.

$\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2} > 0$, so $\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} > 0$.

Also, $1+\cos\theta > 0$ as $\cos\theta \neq -1$, ~~true~~ ($\theta < 180°$)

and $\sin\theta > 0$, as $0<\theta<180°$,

so $\pm\frac{\sin\theta}{1+\cos\theta} > 0$ implies the positive root,

$\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \frac{\sin\theta}{1+\cos\theta}$. Then $\frac{a}{b} = \frac{1+\cos\theta}{\sin\theta}$.

ii If $b = \frac{a}{\sqrt{3}}$, $\sqrt{3} = \frac{a}{\frac{a}{\sqrt{3}}} = \frac{1+\cos\theta}{\sin\theta}$, or

$\Rightarrow \sqrt{3}\sin\theta = 1+\cos\theta$.

$\Rightarrow \sqrt{3}\left(\sqrt{1-\cos^2\theta}\right) = 1+\cos\theta$

$\Rightarrow 3(1-\cos^2\theta) = 1+2\cos\theta+\cos^2\theta$.

$\Rightarrow 4\cos^2\theta+2\cos\theta-2=0$

$\Rightarrow 2\cos^2\theta+\cos\theta-1=0$.

$\cos\theta = \frac{-1\pm\sqrt{9}}{4} = \frac{-1\pm 3}{4}$,

i.e. $\cos\theta = \frac{1}{2}$ or $\cos\theta = -1$.

$\cos\theta = -1 \Rightarrow \theta = (2n+1)\pi$, which is impossible as $0<\theta<180°$.

So $\cos\theta = \frac{1}{2}$. $\Rightarrow \theta = \frac{\pi}{3}$.

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3a If $g(x) = e^x + \sin x$,

$g'(x) = e^x + \cos x$,

$g''(x) = e^x - \sin x$.

If $h(x) = x^2+ax+b$,

$h'(x) = 2x+a$,

$h''(x) = 2$.

$f''(0)$ exist $\Rightarrow f'(0)$ exists.

Thus, $f(0) = b$,

$b = \cancel{f(0)} = \lim_{x\to 0^+} f(x) = e^0+\sin 0 = 1$, so $b=1$.

$f''(0)$ exists $\Rightarrow f'(0) = \lim_{x\to 0^+} f'(0)$.

$f'(0) = 2x+a = a$,

$\lim_{x\to 0^+} f'(x) = e^0+\cos 0 = 2$.

so $a=2$.

Thus $a=2$, $b=1$

b If at point $(x, f(x))$, ~~the~~ normal to curve has gradient $-\frac{1}{\frac{dy}{dx}} = -\frac{dx}{dy}$, and passes through $(x, f(x))$, or (x,y) and the origin.

$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ is known

Note that the line through (a,b) and $(0,0)$ has gradient $\frac{b}{a}$, so we must have $\frac{y}{x} = -\frac{dx}{dy} = \frac{y}{x}$

$\therefore$ $y\,dy = -x\,dx$

$\therefore \frac{y^2}{2} = -\frac{x^2}{2}+C$.

$y^2 = -x^2+C$.

$y = \sqrt{C-x^2}$, or $y = -\sqrt{C-x^2}$

(only one since it is smooth).

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Since as $x=\sqrt{2}$, $y=-\sqrt{2}$,

$-\sqrt{2}=\pm\sqrt{C-2}$, so $C=4$ and

$y=-\sqrt{4-x^2}$.

c $\frac{dl}{dt}=2$, $\frac{dw}{dt}=3$. diagonal is $\sqrt{l^2+w^2}$.

differentiate wrt time

$\frac{d}{dt}(\sqrt{l^2+w^2})=\frac{1}{2}\frac{1}{\sqrt{l^2+w^2}}\cdot(2l\cdot\frac{dl}{dt}+2w\cdot\frac{dw}{dt})$

$=(l\cdot\frac{dl}{dt}+w\cdot\frac{dw}{dt})\div\sqrt{l^2+w^2}$

When $l=12, w=9$, $\sqrt{l^2+w^2}=\sqrt{144+81}=\sqrt{225}=15$,

so $\frac{d}{dt}(\sqrt{l^2+w^2})=(12\cdot2+9\cdot3)\div15$

$=(24+27)\div15$

$=\frac{51}{15}$.

d $x=y^2$, distance from (y^2,y) to $(1,0)$ is $\sqrt{(y^2-1)^2+y^2}$

Let $f(y)=\sqrt{(y^2-1)^2+y^2}=\sqrt{y^4-y^2+1}$

Then, $f'(y)=\frac{1}{2}\cdot\frac{1}{\sqrt{y^4-y^2+1}}\cdot(4y^3-2y)$

$=\frac{2y^3-y}{\sqrt{y^4-y^2+1}}$

Note that the min. satisfies $f'(y)=0$

$\therefore\frac{2y^3-y}{\sqrt{y^4-y^2+1}}=0$, or $2y^3-y=0$.

$\therefore y(2y^2-1)=0$, so $y=0$, or $y^2=\frac{1}{2}$, giving $y=\pm\frac{1}{\sqrt{2}}$.

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$y=0$ gives a distance of 1, $y=\frac{1}{\sqrt{2}}$ gives a distance of

$\sqrt{(\frac{1}{2})^2-\frac{1}{2}+1}=\sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{2}<1$.

Checking for minimum, note that at $y=1$, distance is 1, at $y=0$ distance is 1, so since it is a stationary point between ~~two~~ equal values with a lower value, it must be a minimum, and thus global minimum.

4a $\frac{d}{dx}(f(x)\cdot g(x))=\lim_{h\to0}\frac{f(x+h)\cdot g(x+h)-f(x)\cdot g(x)}{h}$

$=\lim_{h\to0}\frac{f(x+h)\cdot g(x+h)-f(x)\cdot g(x+h)}{h}$

$+\lim_{h\to0}\frac{f(x)\cdot g(x+h)-f(x)\cdot g(x)}{h}$

$=\lim_{h\to0}\left(\frac{f(x+h)-f(x)}{h}\right)\cdot g(x+h)+\lim_{h\to0}\left(\frac{g(x+h)-g(x)}{h}\right)\cdot f(x)$

$=\frac{df(x)}{dx}\cdot g(x)+\frac{d(g(x))}{x}+f(x)$.

bi $\int e^{-x}\cos x\,dx$. Let $f(x)=-e^{-x}$, $f'(x)=e^{-x}$, $g(x)=\cos x$, $g'(x)=-\sin x$

$\int e^{-x}\cos x\,dx=-e^{-x}\cdot\cos x-\int -e^{-x}(-\sin x)\,dx$

$=-e^{-x}\cdot\cos x-\int e^{-x}\sin x\,dx$.

Note that if $h(x)=\sin x$, $h'(x)=\cos x$, so

$\int e^{-x}\sin x\,dx=-e^{-x}\cdot\sin x-\int -e^{-x}\cdot\cos x\,dx$

$=-e^{-x}\cdot\sin x+\int e^{-x}\cdot\cos x\,dx$.

So $\int e^{-x}\cos x\,dx=-e^{-x}\cdot\cos x-\int e^{-x}\sin x\,dx=$

$-e^{-x}\cdot\cos x+e^{-x}\cdot\sin x-\int e^{-x}\cdot\cos x\,dx$.

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Thus $2\int e^{-x}\cos x\,dx = e^{-x}\sin x - e^{-x}\cos x$.

$\Rightarrow \int e^{-x}\cos x\,dx = \frac{e^{-x}\sin x - e^{-x}\cos x}{2}$

ii ~~This is~~

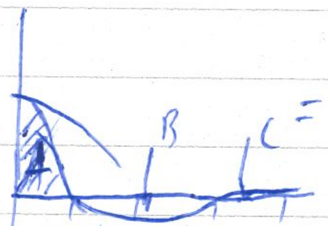
$\cancel{\int_0^{2\pi} e^{-x}\cos x\,dx = \left[\frac{e^{-x}\sin x - e^{-x}\cos x}{2}\right]_0^{2\pi}}$

Note that between $0 \le x \le \frac{\pi}{2}$ and $\frac{3\pi}{2} \le x \le 2\pi$, $\cos x \ge 0$, and between $\frac{\pi}{2} \le x \le \frac{3\pi}{2}$, $\cos x \le 0$.

Thus the area is a sum of areas:

A = 1st (positive) area: $\int_0^{\frac{\pi}{2}} e^{-x}\cos x\,dx$

$= \left[\frac{e^{-x}\sin x - e^{-x}\cos x}{2}\right]_0^{\frac{\pi}{2}}$



graph of $e^{-x}\cos x$.

$= \frac{e^{-\frac{\pi}{2}}}{2} - \left(\frac{-e^{-0}}{2}\right) = \frac{e^{-\frac{\pi}{2}}}{2} + \frac{1}{2} = \frac{1+e^{-\frac{\pi}{2}}}{2}$.

B = 2nd (negative) area:

$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{-x}\cos x\,dx$

$= \left[\frac{e^{-x}\sin x - e^{-x}\cos x}{2}\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$

$= \frac{e^{-\frac{3\pi}{2}}(-1)}{2} - \frac{e^{-\frac{\pi}{2}}\cdot(1)}{2} = -\frac{\left(e^{-\frac{\pi}{2}} + e^{-\frac{3\pi}{2}}\right)}{2}$.

This has area $\frac{e^{-\frac{\pi}{2}} + e^{-\frac{3\pi}{2}}}{2}$.

C = 3rd (positive) area:

$\int_{\frac{3\pi}{2}}^{2\pi} e^{-x}\cos x\,dx$

$\left[\frac{e^{-x}\sin x - e^{-x}\cos x}{2}\right]_{\frac{3\pi}{2}}^{2\pi}$

$= \frac{-e^{-2\pi}}{2} - \frac{e^{-\frac{3\pi}{2}}(-1)}{2} = \frac{e^{-\frac{3\pi}{2}} - e^{-2\pi}}{2}$.

Total area $= \frac{1}{2} + e^{-\frac{\pi}{2}} + e^{-\frac{3\pi}{2}} - e^{-2\pi}$ #

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$xy + e^y = 2x+1$ (1)

c differentiating wrt x,

$y + \frac{dy}{dx}\cdot x + \frac{dy}{dx}\cdot e^y = 2.$ (2)

$xy + e^y = 2x+1$ (1), $x=0$ gives $e^y = 1$, or $y=0$.

From (2), $y + \frac{dy}{dx}\cdot x + \frac{dy}{dx}\cdot e^y = 2$. $x=0$ gives

$y + \frac{dy}{dx}\cdot e^y = 2$. $y=0$ implies

$\frac{dy}{dx} = 2$ at $x=0$

Differentiating (2) wrt x,

$\frac{dy}{dx} + \left(\frac{d^2y}{dx^2}\cdot x + \frac{dy}{dx}\right) + \frac{d^2y}{dx^2}\cdot e^y + \left(\frac{dy}{dx}\right)^2\cdot e^y = 0.$

$x=0$, $y=0$, $\frac{dy}{dx}=2$ implies

$2 + (0+2) + \frac{d^2y}{dx^2}\cdot 1 + 4\cdot 1 = 0.$

$\Rightarrow \frac{d^2y}{dx^2} = -8$ at $x=0$

5ai Note $e^{i\theta} = \cos\theta + i\sin\theta$, $e^{i(-\theta)} = \cos(-\theta) + i\sin(-\theta)$

$= \cos(\theta) - i\sin\theta$

So $e^{i\theta} + e^{-i\theta} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$ as $\cos(x) = \cos(-x)$, $\sin(-x) = -\sin(x)$

$= 2\cos\theta$

$\therefore \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$.

ii From 5ai, $e^{i\theta} - e^{-i\theta} = \cos\theta + i\sin\theta - \cos\theta + i\sin\theta = 2i\sin\theta$,

so $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$.

Then, $\sin^3\theta = \frac{(e^{i\theta} - e^{-i\theta})^3}{(2i)^3} = \frac{(e^{i\theta})^3 - 3(e^{i\theta})^2(e^{-i\theta}) + 3(e^{i\theta})(e^{-i\theta})^2 - (e^{-i\theta})^3}{-8i}$

$= \frac{e^{i3\theta} - 3e^{i\theta} + 3e^{-i\theta} - e^{-i3\theta}}{-8i}$

$= \frac{\left(\frac{e^{i3\theta} - e^{-i3\theta}}{2i}\right)}{-4} + \frac{3}{4}\left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right) = -\frac{\sin 3\theta}{4} + \frac{3\sin\theta}{4}$

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b $\int \cos x \cdot e^x\,dx = \frac{1}{2}\int (e^{ix} + e^{-ix}) e^x\,dx = \frac{1}{2}\int e^{ix}\cdot e^x\,dx + \frac{1}{2}\int e^{-ix}\cdot e^x\,dx$

$= \frac{1}{2}\cdot\frac{1}{i+1}\cdot e^{(i+1)x}$

Consider $\int e^{-ix}\cdot e^x\,dx$.
$= \int e^{(1-i)x}\,dx$.

I claim this is $\frac{1}{2}\cdot\frac{1}{1-i}e^{(1-i)x}$

so

$\int \cos x \cdot e^x\,dx = \frac{1}{2}\left(\frac{1}{i+1}e^{(i+1)x} + \frac{1}{1-i}e^{(1-i)x}\right)$

We prove that the conjugate of $\frac{1}{i+1}e^{(i+1)x}$ is $\frac{1}{1-i}e^{(1-i)x}$.
Indeed, the conjugate of $\frac{1}{i+1}$ is $\frac{1}{\overline{i+1}} = \frac{1}{\overline{i+1}} = \frac{1}{1-i}$,
so we are RTP the conjugate of $e^{(i+1)x}$ is $e^{(1-i)x}$
$\Rightarrow$ ~~...~~ e^{ix}.
the conjugate of $e^{ix}\cdot e^x = e^{-ix}\cdot e^x$. But
notice the conjugate of e^{ix} is e^{-ix}, as
$|e^{ix}| = |e^{-ix}| = 1$, and ~~the~~ $e^{ix}\cdot e^{-ix} = e^0 = 1$, and
e^x is real, so has conjugate equal to itself.
Thus, $\overline{e^{(i+1)x}} = e^{(1-i)x}$. so

$\overline{\frac{1}{(i+1)}e^{(i+1)x}} = \overline{\frac{1}{(i+1)}}\cdot\overline{e^{(i+1)x}} = \frac{1}{1-i}\cdot e^{(1-i)x}$.

Since $\cos(x) = Re(e^{ix})$,

$\int \cos x\, e^x\,dx = Re\left(\int e^{ix}\cdot e^x\,dx\right)$
$= Re\left(\frac{1}{i+1}e^{(i+1)x}\right)$
$= \dfrac{\frac{1}{i+1}e^{(i+1)x} + \overline{\frac{1}{i+1}e^{(i+1)x}}}{2}$

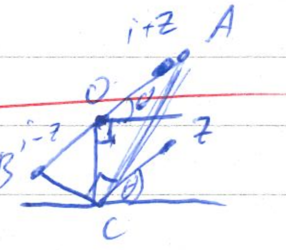
$= \frac{1}{2}\left(\frac{1}{i+1}e^{(i+1)x} + \frac{1}{1-i}e^{(1-i)x}\right)$, as desired.

QUESTION NUMBER

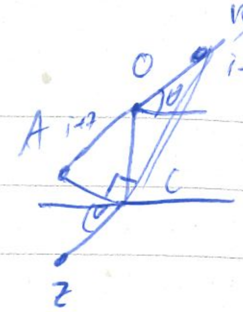
c $z = a+bi$, $a^2+b^2 = 1$.

$w = \frac{i+a+bi}{i-a-bi} = \frac{(a+(b+1)i)(-a+(b-1)i)}{(-a-(b-1)i)(-a+(b-1)i)}$

$= \frac{\quad}{a^2+(b-1)^2}$



Consider the complex plane.
Let $O = i$, $A = i+z$, $B = i-z$, and
$C = 0$. Notice that $|OA| = |OB| = |OC| = 1$,
as $|OA| = |i+z-i| = |z| = 1$,
$|OB| = |i-z-i| = |-z| = 1$, and
$|OC| = |i| = 1$.
So O is the circumcentre of ABC. Moreover, O is
the midpoint of AB, $\left(\frac{(i-z)+(i+z)}{2}\right) = \left(\frac{2i}{2}\right) = i = O$.
So, AB is a diameter in circle (ABC), so
$\angle ACB = 90°$.
Thus, $\arg\left(\frac{i+z}{i-z}\right) = \frac{\pi}{2}$, if $Re(z) \geq 0$, and
$\arg\left(\frac{i+z}{i-z}\right) = -\frac{\pi}{2}$ if $Re(z) \leq 0$. Either way,
w is imaginary. Then, the ~~...~~ modulus
of $\frac{AC}{AB} = \tan\angle ABC$ by ABC is a right triangle
If $Re(z) \geq 0$, Note $\angle AOC = 90° + \theta$, where $\theta = \arg(z)$, so
$\angle ABC = \frac{1}{2}\angle AOC = 45° + \frac{\theta}{2}$*, where ~~...~~ $-90° \leq \theta \leq 90°$
i.e. $0° \leq \angle ABC \leq 90°$,
*angles at middle subtend half the arc.
so $\tan(\angle ABC)$ can take any value between
$[0, \infty]$, so the locus of w is all imaginary
no.s with non-negative imaginary part.



If $Re(z) \leq 0$, then $\arg\left(\frac{i+z}{i-z}\right) = \frac{?\pi}{2}$.

i.e. w is ~~a~~ an ~~nega~~ imaginary no. with non-positive imaginary part.

Let θ = angle of z $(90° \leq \theta \leq -90°)$

Then $\angle BOC = 90° + \theta$, $\therefore$ $\angle BAC = 45° + \frac{\theta}{2}$ and $\angle ABC = 45° - \frac{\theta}{2}$

Since $-90° \leq \theta \leq 90°$, $0° \leq \angle ABC \leq 90°$,

so again $\tan(\angle ABC)$ has a range $[0, \infty]$,

meaning w can be any imaginary no. with non-positive imaginary part.

Both parts put together, the locus of w is all imaginary numbers, with zero real part.

QUESTION NUMBER

d $x^2 - yz = 1$ ①, $y^2 - zx = 2$ ②, $z^2 - xy = 3$. ③

~~$x^2y^2 - (x^3+y^3)z$~~

~~$x^2 + y^2 - (x+y)z = 3$.~~

~~$xy = z^2 - 3$.~~

~~$(x+y)^2 - 2xy - (x+y)z = 3$.~~

~~$(x+y)^2 - 2z^2 + 6 - (x+y)z = 3$.~~

~~$(x+y)^2 - (x+y)z - 2z^2$~~

Adding ① and ③, $x^2 + z^2 - (x+z)y = 4$. ④

② gives $xz = y^2 - 2$.

④ gives $(x+z)^2 - 2xz - (x+z)y = 4$

$\Rightarrow (x+z)^2 - 2(y^2-2) - (x+z)y = 4$

$\Rightarrow (x+z)^2 - (x+z)y - 2y^2 + 4 = 4$

$\Rightarrow (x+z)^2 - (x+z)y - 2y^2 = 0$.

$\Rightarrow (x+z-2y)(x+z+y) = 0$.

So either $y = \frac{x+z}{2}$, or $y = -x-z$.

The first case $y = \frac{x+z}{2}$, substituting into ②,

$\frac{x^2+2xz+z^2}{4} - xz = 2$, $\left(\frac{x-z}{2}\right)^2 = 2$, ~~so either~~

$x - z = \pm 2\sqrt{2}$.

$x = z \pm 2\sqrt{2}$.

Then $y = \frac{x+z}{2} = \frac{2z \pm 2\sqrt{2}}{2} = z \pm \sqrt{2}$.

So, either $x = z + 2\sqrt{2}$, $y = z + \sqrt{2}$, or $x = z - 2\sqrt{2}$, $y = z - \sqrt{2}$.

First case gives ①, $(z+2\sqrt{2})^2 - z(z+\sqrt{2}) = 1$

$\Rightarrow z^2 + 4\sqrt{2} \cdot z + 8 - z^2 - z\sqrt{2} = 1$,

$3\sqrt{2}z = -7$.

$z = \frac{-7}{3\sqrt{2}}$

~~$(z+2\sqrt{2})^2$~~

② gives

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$(z+\sqrt{2})^2 - z(z+2\sqrt{2}) = 2$,

$z^2+2\sqrt{2}z+2-z^2-2\sqrt{2}z=2$,

③ gives $z^2-(z+\sqrt{2})(z+2\sqrt{2})=3$,

$z^2-z^2-3\sqrt{2}z-4=3$,

$z=\frac{-7}{3\sqrt{2}}$, which works

So $z=\frac{-7}{3\sqrt{2}}$, $y=\frac{-7}{3\sqrt{2}}+\sqrt{2}$, $x=\frac{-7}{3\sqrt{2}}+2\sqrt{2}$.

If ~~$x=z-2\sqrt{2}$~~ $x=z-2\sqrt{2}$.

① gives $(z-2\sqrt{2})^2-(z-\sqrt{2})z=1$

$z^2-4\sqrt{2}z+8-z^2+\sqrt{2}z=1$

$z=\frac{7}{3\sqrt{2}}$,

which gives $y=\frac{7}{3\sqrt{2}}-\sqrt{2}$ and $x=\frac{7}{3\sqrt{2}}-2\sqrt{2}$.

These are the values above, multiplied by -1.

These all work as the above values work, and multiplying x,y,z by -1 in the given doesn't change anything.

If $y=-x-z$,

② gives $x^2+xz+z^2=2$,

③ gives $z^2-x(-x-z)=3$,

or $z^2+x^2+xz=3$,

this is a contradiction, as $x^2+xz+z^2=2$ not 3.

So the only values which work are

$x=\pm\frac{7}{3\sqrt{2}}\mp2\sqrt{2}$, $y=\pm\frac{7}{3\sqrt{2}}\mp\sqrt{2}$,

$z=\pm\frac{7}{3\sqrt{2}}$ me

Yay

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