NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

Assessment Report

Scholarship, 2006

Mathematics with Calculus

Mathematics with Calculus, Scholarship, 2006

Commentary

The 2006 examination was more accessible than those in the previous two years. However, while it discriminated well at the top end, and the top scholars were easily discernible, there were still a sizeable number of students who were unable to make substantial progress on any of the questions. It was clear there were many candidates who sat this examination that did not have the mathematical background required to succeed at this level. The general level of algebraic ability displayed was pleasing and this year there was a noticeable improvement in candidates' ability to simplify working as they progressed through a problem, including manipulation of trig expressions. There was a decline this year in candidates' ability both to write and to use correct mathematical statements. This was exemplified by solutions to questions 1b, 2a and 3b.

Presentation of working was often poor. Candidates should be advised to number the question they are working on, especially if parts of questions are attempted in several places, and not to cross out work unless it has been replaced, since crossed out work is not marked.

The best performing candidates most commonly demonstrated the following skills and / or knowledge:

- excellent levels of algebraic manipulation and conceptual thinking
- application of mathematics to unfamiliar contexts
- good understanding of geometric principles
- good understanding of trigonometry, and ability to apply it
- understanding of the need to simplify expressions before proceeding to work with them
- ability to produce novel methods leading to correct solutions
- high level of skills, such as differentiation and integration
- ability to use diagrams and graphs to structure their thinking
- ability to use knowledge of the mathematics at NCEA Levels 1, 2 and 3
- understanding and application of the rules of logarithms
- an understanding of methods of proof and the ability to support mathematical statements with written reasons.

Candidates who did NOT achieve scholarship lacked some or all of the skills and knowledge above and in addition they:

- lacked basic technical and manipulation skills, such as trigonometric functions
- preferred to work with degrees or decimals rather than radians or exact rational or algebraic values
- failed to use given results to continue with the rest of the question
- did not understand that calculus requires angles to be measured in radians and not degrees
- were not able to write mathematical notation correctly
- appeared not to use the given formula sheet
- used inappropriate simplification of problems
- did not understand the concept of proof.

Comments on individual questions:

Q1(a) This turned out not to be as easy as anticipated and may have been better placed as Q1(c). Those who understood rates of change were able to solve the problem successfully, but the standard of differentiation following use of the cosine rule was disappointing at this level. The most common error was for candidates to use the arc length as the distance and give the minute hand moving at a rate of $\frac{4\pi}{15}$ cm/min (some called this radians per min) and the hour hand at $\frac{\pi}{60}$ cm/min then subtracting to get $\frac{\pi}{4}$ and using this as the angle. Progress was often hampered by inconsistent or wrong use of units, such as rad/min, rad/hr, or °/min, with a number of candidates working in degrees, which is not acceptable in a calculus question.

Q1(b) This question was done well by many who attempted it. The method for the first part was clearly unfamiliar to many but they worked at it, often employing novel, and sometimes correct methods of finding A, B, and C, rather than by differentiating and letting $x = 0$. Results left as algebraic expressions were not sufficient to gain credit. Most achieved some credit, although there were candidates who were unable to differentiate $\ln(1+x)$ correctly, even with basic derivatives given on the formula sheet. The second part of this question was attempted by most candidates, but many failed to manipulate logs and inequalities correctly to end up with the inequality $4n^2 - 23n + 7 > 0$ to solve. Some candidates wrote $\log\left(1+\frac{1}{2n}\right) = \log 1 + \log\frac{1}{2n}$, or similar things, while others incorrectly cancelled logs : $\frac{n+3}{n-1} < \frac{\ln(1+\frac{1}{n})}{\ln(1+\frac{1}{2n})}$ by removing the ln, or subtracted to get $\frac{n+3}{n-1} < \ln(1+\frac{1}{n} - (1+\frac{1}{2n}))$. A sizeable minority of candidates did not see the use of the first part of the question and so used trial and error at an early stage. This gained some credit, but not all.

Q1(c) This question was accessible to most candidates and was one of the best-attempted questions on the paper. Different methods were attempted but the standard $V = \int_0^{\pi} \pi x^2 dy$ was the most successful. Those who attempted a shortcut using symmetry or tried dividing the volume into parts usually did not succeed. Many candidates got the correct expression for x^2 but some could not cope with integrating $\cos^2(2y)$ and wrote versions of $\frac{\cos^3(2y)}{3}$, or similar, or could not work out the correct limits and used *b* – 2*a* and *b* + 2*a*. Candidates should have the ability to integrate functions such as $\cos^2(x)$ and $\sin^2(x)$ at scholarship level.

Q2(a) There were many methods used to prove the result $\cos\frac{\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2}$, often using the double angle formula, but others, especially the geometric proofs, were ingenious, if a little long at times. The second part was generally well done, but some candidates did not have the negative sign in $z^2 = \sqrt{2}cis\left(-\frac{\pi}{4}\right)$, and others did not realise there would be a second root. Use of De Moivre's theorem was the most common method, but some managed to reason successfully that the roots would be *z* and –*z*. Those who chose to use an algebraic route by setting the given complex number to $(a+ib)^2$ often got lost in a maze of algebra. Difficulties often arose when solving for *z* and having to find an expression for $\sin\frac{\pi}{8}$ to get the two solutions required. Candidates need to be aware that decimals are inappropriate when answers are specifically asked for in surd form.

Q2(b) This was a difficult question since it required candidates to think about an unfamiliar area of mathematics. However, it was pleasing to see the range of methods used to attempt the question, and the success some candidates achieved, most often by translating to the origin and reflecting the graph. However, some reasoned that a point (*a, b*) on the graph would end up at (2–*a*, –2–*b*) and were able to complete the solution quickly. A few weaker candidates interpreted the use of the term rotation to mean a volume and tried integrating the function.

Q3(a) The first part of this question getting $k = 0$ was successfully done by most candidates, although a few did not realise they needed to use the product or quotient rule to find $\frac{d^2y}{dx^2}$ from $\frac{dy}{dx} = \frac{\cos(\ln x)}{x}$, or could not correctly apply it. The second part was done well by those who used the differential equation from the first part to help them. Unfortunately a number of candidates assumed that $y = \sin(\ln x)$ instead, which was not appropriate since the question was to find y subject to the given condition. Some candidates did not understand that $\frac{d\left(x^2 \frac{dy}{dx}\right)}{dx}$ is a product and so failed to get $\frac{d\left(x^2 \frac{dy}{dx}\right)}{dx} = 2x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2}$.

Q3(b) It was good to see that the level of algebraic, log and trigonometric manipulation of many candidates was up to the task of this question, and it was successfully done by many, using a variety of different approaches. Many worked with sin and cos instead of tan, thus making the working much harder. Those who circumvented the intended route of the question by equating the result to 1 early in their solutions did well and were allowed full credit. There were a few very short and ingenious methods but on the other hand quite a few candidates chose to work with $x = e^{\sin^{-1} a}$ $\quad y = e^{\cos^{-1}\left(\frac{1}{b}\right)}$, and were generally unsuccessful. Those candidates who recognised the difference of two squares in the expansion of tan(ln *xy*) . tan(ln *x/y*) often went on to the final solution with relative ease. There were also candidates who did not use correct thinking and showed some misconceptions, such as not knowing that:

$$\tan\left(\ln\left(\frac{x}{y}\right)\right) \neq \frac{\sin\ln x}{\cos\ln y} \qquad \sin\left(\sin^{-1} a + \cos^{-1}\left(\frac{1}{b}\right)\right) \neq a + \sin\cos^{-1}\left(\frac{1}{b}\right),$$

$$\tan[\ln(xy)] \neq \tan(\ln x) + \tan(\ln y), \qquad \tan(\ln xy) \neq \frac{\sin}{\cos}(\ln xy) \qquad \text{and}$$

$\tan(\ln xy) \neq \tan(\ln x) + \tan(\ln y)$, etc.

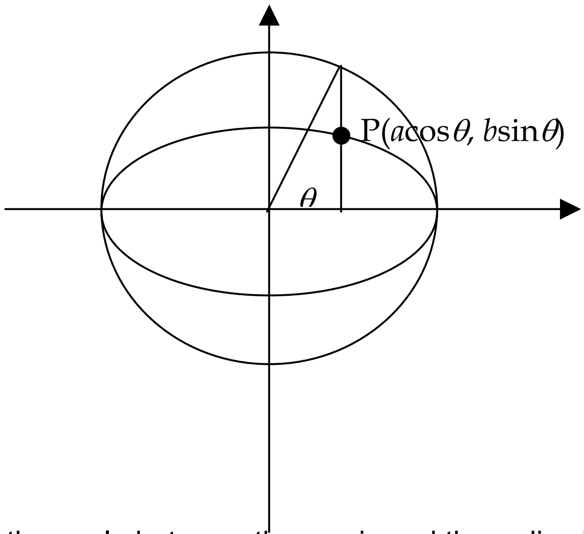
Q4(a) It seems that most candidates have forgotten how to reason geometrically, and so this proved to be a difficult question for many. Successful candidates usually used the cosine rule or the sine rule in triangle OPT to find the relationship between *R* and *r*. A number used triangles PQT and OQP and the cosine rule to equate the length of QP. It was disappointing that candidates did not often avoid the use of the cosine or sine rule by the prudent use of the symmetry of an isosceles triangle. Candidates should realize that a good diagram is essential in a problem of this type; some wrongly thought that QT or PT were diameters of the circle of radius *r,* while others wrongly simplified the problem by making *R* = 2*r*. The use of degrees was accepted here, but at this level one does not expect to see $C = r\left(25\tfrac{5}{7}\text{Þ}\right)\bullet\frac{360}{25\tfrac{5}{7}\text{Þ}}$ instead of *C*=2π*r* for the circumference of a circle!

Q4(b) This was a difficult question and few candidates managed to get past the successful derivation of the initial differential equation since dealing with more than 2 variables proved beyond the majority. Those who managed to complete the question realized the need to use the substitution to reduce the problem to 2 variables and then separate these variables and integrate. A small number of the very best students showed great insight in using results such as $\int\left(y + x\frac{dy}{dx}\right)dx = \int\left(x + y\frac{dx}{dy}\right)dy = xy$, and obtained the required answer speedily, such as in

$$\frac{dy}{dx} = \frac{-y}{2y + x}, \quad (2y + x)\frac{dy}{dx} + y = 0, \quad 2y\frac{dy}{dx} + x\frac{dy}{dx} + y = 0, \quad \text{integrating wrt } x, \quad y^2 + xy = k.$$

Q5(a) This proved accessible to virtually all candidates, with many successfully substituting $\theta = \frac{\pi}{4}$ early on and hence simplifying the working. Generally *y=mx+c* proved better than $y - y_1 = m(x - x_1)$ as a strategy here. The result that the product of the gradient of perpendicular lines is –1 appeared to be not well known. Some of the outstanding candidates found a short method here, using only the co-ordinates of *P* and the gradient of the normal in a right-angled triangle. This again pointed to the value of a good diagram. It is surprising to see a few candidates who cannot differentiate implicitly sitting this paper. They wrote things like $y = \left(b^2 - \frac{b^2x^2}{a^2}\right)^{\frac{1}{2}}$ $\frac{dy}{dx} = \frac{1}{2}\left(b^2 - \frac{b^2x^2}{a^2}\right)^{-\frac{1}{2}} \bullet -\frac{2b^2x}{a^2}$, which may be correct but is not very useful here!

Q5(b) Many candidates thought they had found a short solution here but they did not understand the definition of the parameter θ for an ellipse. They had not seen the use of the director circle to define θ, as below:



Thus the angle θ is the angle between the *x*-axis and the radius joining the origin O to a point on the circle $x^2 + y^2 = a^2$, **not** the ellipse. The point where the perpendicular from the point on the circle to the *x*-axis meets the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is P $(a\cos\theta, b\sin\theta)$, and so θ is not the angle between OP and the *x*-axis. The few who got beyond this initial point did quite well on the question, with most finding the equation of OQ and substituting for *y* in the equation of the ellipse.