S

93202A

932021

SUPERVISOR'S USE ONLY

SCHOLARSHIP EXEMPLAR

NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

QUALIFY FOR THE FUTURE WORLD
KIA NOHO TAKATŪ KI TŌ ĀMUA AO!

Scholarship 2019
Calculus

9.30 a.m. Friday 8 November 2019
Time allowed: Three hours
Total score: 40

ANSWER BOOKLET

There are five questions in this examination. Answer ALL FIVE questions.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

Write ALL your answers in this booklet.

Make sure that you have Formulae and Tables Booklet S–CALCF.

Show ALL working. Start your answer to each question on a new page. Carefully number each question.

Answers developed using a CAS calculator require **ALL commands to be shown**. Correct answers only will not be sufficient.

Check that this booklet has pages 2–27 in the correct order and that none of these pages is blank.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Question	Score
ONE	
TWO	
THREE	
FOUR	
FIVE	
TOTAL	/40

ASSESSOR'S USE ONLY

QUESTION NUMBER

ASSESSOR'S USE ONLY

1a. $\frac{(x-1)^2}{x-1} + \frac{(x^2+1)^2}{x+1} = \cancel{x}\ 2x$

~~x ... diffe~~

$f(x)$ - is differentiable for all x except $x=1$ and $x=-1$

1b. $\ln(-x^2+4x-4)$

$f(x)$ is real when $-(x-2)^2 > 0$ $\quad (x-2)^2 < 0$

~~$(x-2)^2$ ... always be zero when $x=2$~~

~~f(x) is real for all~~ no values ~~of x except $x=2$~~

$f(x)$ is never real, $(x-2)^2$ is always ≥ 0

Rev.

~~1d. $\sqrt{(x+1)^2} - \sqrt{(x-4)^2} \geq 1$~~

~~$\sqrt{(x+1)^2} \geq 1 + \sqrt{(x-4)^2}$~~

~~$x+1 \geq 1 + x - 4$~~

1ci. ~~...~~

A = 6 combos of 1 of 6 books

B = 10 combos of 2 of 5 remaining books

C = 1 combo of 3 of 3 remaining books

Total combos = $6 \times 10 \times 1 = 60$ combinations

cii. A = 15 combos of 2 of 6 books

B = 6 combos of 2 of 4 remaining books

C = 1 combo of 2 of 2 remaining books

Total combinations = $15 \times 6 \times 1 = 90$ combinations

QUESTION NUMBER

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$\cancel{(x+1)-(x+4)\geq 1}$

$\cancel{-(x+1)-(x+4)\geq 1}$

$\cancel{(x+1)+(x+4)}$

$\cancel{-(x+1)+(x+4)}$

1d. $(x+1)-(x-4)\geq 1$ $\quad 5\geq 1$

$-(x+1)-(x-4)\geq 1$ $\quad -2x+3\geq 1$ $\quad x\leq 1$

$(x+1)+(x-4)\geq 1$ $\quad 2x-3\geq 1$ $\quad x\geq 2$

$-(x+1)+(x-4)\geq 1$ $\quad -5\geq 1$

1e. $\sin^4 A+\cos^4 A=\frac{2}{3}$

$(\sin^2 A+\cos^2 A)^2-2\sin^2 A\cos^2 A=\frac{2}{3}$

$1-2\sin^2 A\cos^2 A=\frac{2}{3}$

$4\sin^2 A\cos^2 A=\left(1-\frac{2}{3}\right)\times 2=\frac{2}{3}$

$(2\sin A\cos A)^2=\frac{2}{3}$

$(\sin 2A)^2=\frac{2}{3}$ $\qquad 180°<2A<360°$

$\sin 2A=\pm\sqrt{\frac{2}{3}}$

$\cancel{2A=n\pi+(-1)^n\sin^{-1}(\ldots)}$

$\sin^{-1}\left(\sqrt{\frac{2}{3}}\right)<180°$ $\quad \therefore$ not $\sqrt{\frac{2}{3}}$

$180°<\sin^{-1}\left(-\sqrt{\frac{2}{3}}\right)<360°$

$\therefore \sin 2A=-\sqrt{\frac{2}{3}}$

QUESTION NUMBER

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2a. $\dfrac{\sqrt{x-2}}{\sqrt{x}} - \dfrac{\sqrt{x}}{\sqrt{x-2}} = \dfrac{x-2-x}{\sqrt{x(x-2)}} = \dfrac{-2}{\sqrt{x(x-2)}} = \dfrac{k}{4}$

$\therefore k\sqrt{x(x-2)} = -8$

$k^2x(x-2) = -8$

$k^2x^2 - 2k^2x + 8 = 0$

imaginary roots when $\Delta < 0$

$b^2 - 4ac = 4k^4 - 32k^2 < 0$

$k^2(k^2-8) < 0$

$k^2(k+\sqrt{8})(k-\sqrt{8}) < 0$

$-\sqrt{8} < k < \sqrt{8}$

Imaginary roots when $-\sqrt{8} < k < \sqrt{8}$

2b. $\log_{10}(x^2+y^2) = 1 + \log_{10} 13 = \log_{10} 130$

$\log_{10}(x+y) - \log_{10}(x-y) = 3\log_{10} 2$

$\log_{10}\left(\dfrac{x+y}{x-y}\right) = \log_{10} 8$

$x^2+y^2 = 130$ $\qquad \dfrac{x+y}{x-y} = 8$

$x+y = 8x - 8y$

$\cancel{x = -y}$ $\quad 9y = 7x \qquad y = \dfrac{7}{9}x$

$\cancel{x^2+(-x)^2 = 2x^2 = 130}$ $\qquad x^2 + \left(\dfrac{7}{9}x\right)^2 = 130$

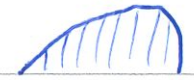
$\cancel{x^2 = 65}$ $\qquad x^2 + \dfrac{49}{81}x^2 = \dfrac{130}{81}x^2 = 130$

$\cancel{x = \pm\sqrt{65}}$

$\cancel{y = \pm\sqrt{65}}$ $\qquad x^2 = 81 \qquad x = \pm 9$

$y = \pm 7$

QUESTION NUMBER | ASSESSOR'S USE ONLY



$= y = \sqrt{x^2 - x^4}$

2c. $y^2 = x^2 - x^4$

when $0 < x < 1$, $\sqrt{x^2 - x^4} > 0$

$\therefore$ top right section: $y = \sqrt{x^2 - x^4}$ (positive)

Area of top right section (quarter): $\int_0^1 \sqrt{x^2 - x^4}\, dx$

$= \int_0^1 x\sqrt{1 - x^2}\, dx$

$\sqrt{1-x^2}$

$(1-x^2)^{\frac{3}{2}}$

$\frac{3}{2}(1-x^2)^{\frac{1}{2}} \cdot -2x$

$= -3x(1-x^2)^{\frac{1}{2}}$

$\int_0^1 x\sqrt{1-x^2} = \int_0^1 x(1-x^2)^{\frac{1}{2}}$

$= \left[\frac{-1}{3}(1-x^2)^{\frac{3}{2}}\right]_0^1 = \frac{1}{3}$

$\therefore$ Total area $= 4 \times \frac{1}{3} = \frac{4}{3}$ units

5

QUESTION NUMBER | ASSESSOR'S USE ONLY

3a. $\frac{f(4+h)-f(4)}{h} = \frac{((4+h)^2-4(4+h)+3)^2-9}{h}$

$= \frac{(16+8h+h^2-16-4h+3)^2-9}{h}$

$= \frac{(h^2+4h+3)^2-9}{h} = \frac{h^4+4h^3+3h^2+4h^3+16h^2+12h+3h^2+12h+9-9}{h}$

$= h^3+8h^2+22h+24$

$= 24$ as $h \to 0$ $\therefore f'(4) = 24$

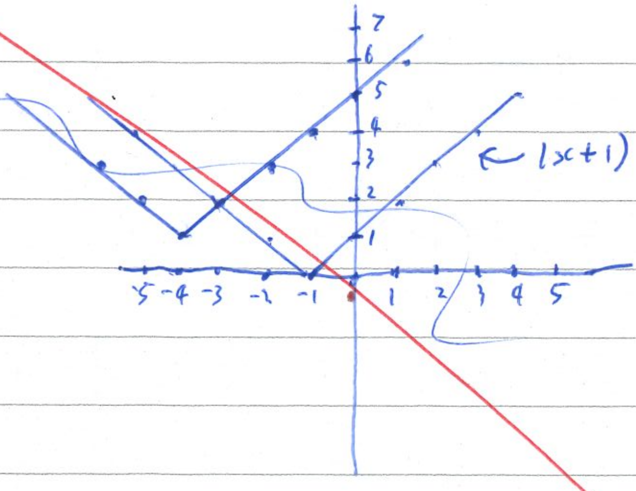
3b. $x^2+y^2=5^2$

$y = \sqrt{25-x}$ (for positive y-values)

$\frac{dy}{dx} = \frac{-1}{2\sqrt{25-x}}$

$\frac{dx}{dt} = \frac{dx}{dy} \times \frac{dy}{dt} = (-2\sqrt{25-x})(-2) = 4\sqrt{25-x}$

1d. $|x+1| \geq 1 + |x+4|$



QUESTION NUMBER

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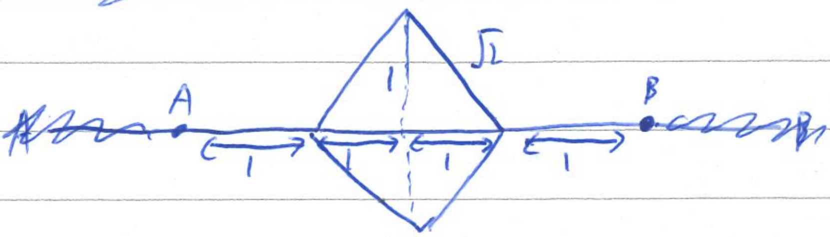
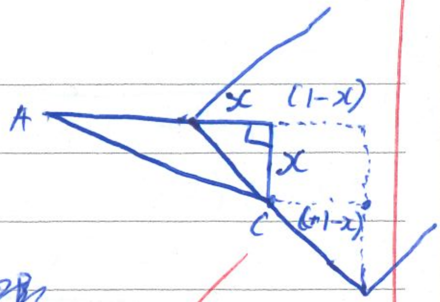
~~3b. $x = r\cos\theta$, $y = r\sin\theta$~~

3c.





$$AC = \sqrt{x^2 + (1+x)^2} = \sqrt{2x^2+2x+1}$$

$$BD = \sqrt{2x^2+2x+1}$$

$$CD = 2(1-x) = 2-2x$$

$$T = 3\left(2\sqrt{2x^2+2x+1}\right) + 2.5(2-2x) = 6\sqrt{2x^2+2x+1} - 5x + 5$$

$$\frac{dT}{dx} = \frac{6(4x+2)}{2\sqrt{2x^2+2x+1}} - 5 = \frac{12x+6}{\sqrt{2x^2+2x+1}} - 5 = 0 \text{ (min.)}$$

$$= \frac{12x+6-5\sqrt{2x^2+2x+1}}{\sqrt{2x^2+2x+1}} = 0$$

$$12x+6 = 5\sqrt{2x^2+2x+1}$$

$$144x^2 + 144x + 36 = 10x^2 + 10x + 5$$

$$\cancel{136x^2 + 136x + 32 = 0}$$

$$\cancel{17x^2 + 17x + 4 = 0}$$

$$134x^2 + 134x + 31 = 0$$

$$x = \frac{-17 \pm \sqrt{289-272}}{34}$$

$$= \frac{-17 \pm \sqrt{17}}{34}$$

QUESTION NUMBER

ASSESSOR'S USE ONLY

T = total time

3c. $AC = \sqrt{x^2 + (x+1)^2} = \sqrt{2x^2+2x+1}$ $\quad 3\,km\,h^{-1} = \frac{1}{3}\,h\,km^{-1}$

$CD = 2(1-x)$ $\quad 2.5\,km\,h^{-1} = 0.4\,h\,km^{-1}$

$T = \frac{1}{3}\left(2\sqrt{2x^2+2x+1}\right) + 0.4\left(2(1-x)\right)$

$= \frac{2}{3}\sqrt{2x^2+2x+1} + \frac{4}{5}(1-x)$

$\frac{dT}{dx} = \left(\frac{2}{3}\right)\frac{1}{2\sqrt{2x^2+2x+1}}(4x+2) + \frac{4}{5}(-1)$

$= \frac{4x+2}{3\sqrt{2x^2+2x+1}} - \frac{4}{5} = 0$ (min.)

1.449443826

94442

94452

$4x+2 - \frac{12}{5}\sqrt{2x^2+2x+1} = 0$

$2x+1 = \frac{6}{5}\sqrt{2x^2+2x+1}$

$4x^2+4x+1 = \left(\frac{36}{25}\right)(2x^2+2x+1)$

$100x^2+100x+25 = 72x^2+72x+36$

$28x^2+28x-11=0$

$x = \frac{-28 \pm \sqrt{28^2+44\times 28}}{56} = \frac{-28 \pm \sqrt{2016}}{56}$

~~Prove minimum: $f(0.3018) = 1.4494438$~~

~~$f(0.3018) = 2.448$ hrs~~

~~$f(0.3013) =$~~

~~$f(0.3023) =$~~

$= \frac{-7 \pm \sqrt{126}}{14} = \frac{-7+\sqrt{126}}{14}$

(no neg x)

$= 0.3018$

Q3

6

∴ boat should be positioned 0.3018 km below the AB line

QUESTION NUMBER

ASSESSOR'S USE ONLY

3c. Prove minimum: ~~f(0.30188)~~ $f(0.3018) = 1.4494438$

$f(0.3) = 1.4494442$

$f(0.305) = 1.4494452$

4a. $T = \frac{1}{t_p}\int_0^{t_p} \left(a - \frac{a-b}{t_p}t\right)^2 + \left(p(t)^2\right)\, dt$

$p(t) = 1 - a + \frac{a-b}{t_p}t$

$T = \frac{1}{t_p}\int_0^{t_p} \left(a - \frac{a-b}{t_p}t\right)^2 + \left(1 - a + \frac{a-b}{t_p}t\right)^2\, dt$

~~$= \frac{1}{t_p}\int_0^{t_p} a^2 - \frac{2a(a-b)}{t_p}t + \frac{(a-b)^2}{t_p}t^2 + 1 - a + \frac{a-b}{t_p}$~~

~~$1 - a + a^2 - \frac{a(a-b)}{t_1}t + \frac{a-b}{t_p}t - \frac{a(a-b)}{t_p}t + \frac{(a-b)^2}{t_p^2}t^2$~~

~~$= \frac{1}{t_p}\int_0^{t_p} 2a^2 - 2a + 1$~~

$= \frac{1}{t_p}\int_0^{t_p} a^2 - \frac{a(a-b)}{t_p}t + \frac{(a-b)^2}{t_p^2}t^2 + (1-a)^2 + \frac{2(1-a)(a-b)}{t_p}t$

$+ \frac{(a-b)^2}{t_p^2}t^2$

$= \frac{1}{t_p}\int_0^{t_p} \frac{2(a-b)^2}{t_p^2}t^2 + \frac{2(1-a)(a-b) - a(a-b)}{t_p}t + \left(a^2 + (1-a)^2\right)$

$= \left[\frac{2(a-b)^2}{3t_p^3}t^3 + \frac{(1-a)(a-b) - a(a-b)}{t_p^2}t^2 + \frac{a^2 + (1-a)^2}{t_p}t\right]_0^{t_p}$

NCON

$= \frac{2(a-b)^2}{3} + (a-b)(1-2a) + a^2 + (1-a)^2$

$= \frac{2}{3}(a-b)^2 + b(2a-1) + a - 2a^2 + 2a^2 - 2a + 1$

$= 1 - a + b(2a-1) + \frac{2}{3}(a-b)^2$

QUESTION NUMBER

ASSESSOR'S USE ONLY

4b. $4x^2\frac{dy}{dx} = y^2 - 2xy$

$\frac{dy}{dx} = \frac{y^2-2xy}{4x^2}$

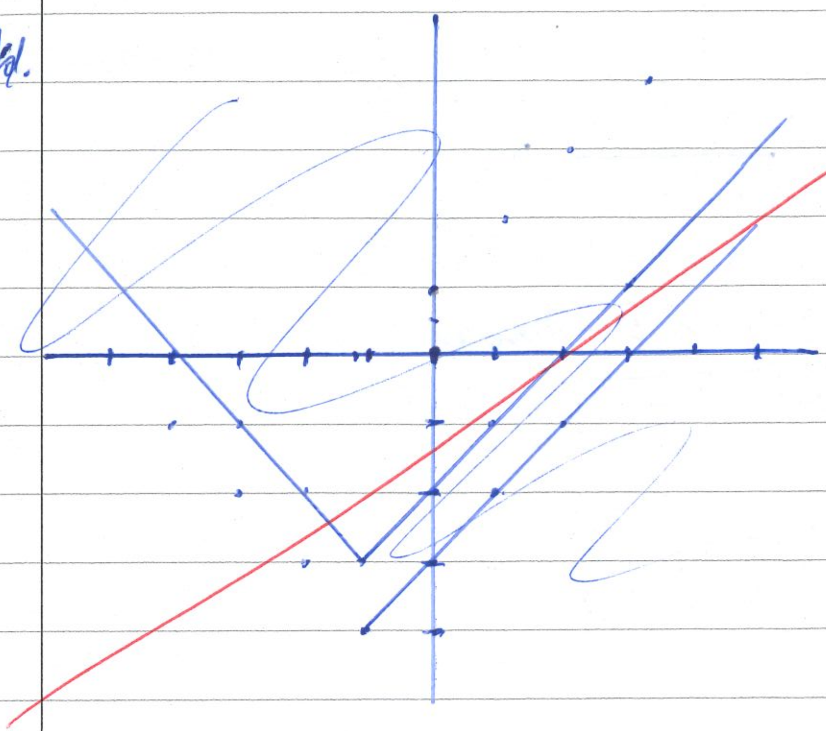
$\cancel{\frac{4x^2}{y^2+2xy}}$ $\frac{4x^2}{y(y+2x)}$ $\frac{dy}{dx} = \frac{\Lambda}{y} + \frac{\Lambda}{y+2x}$

P[3]

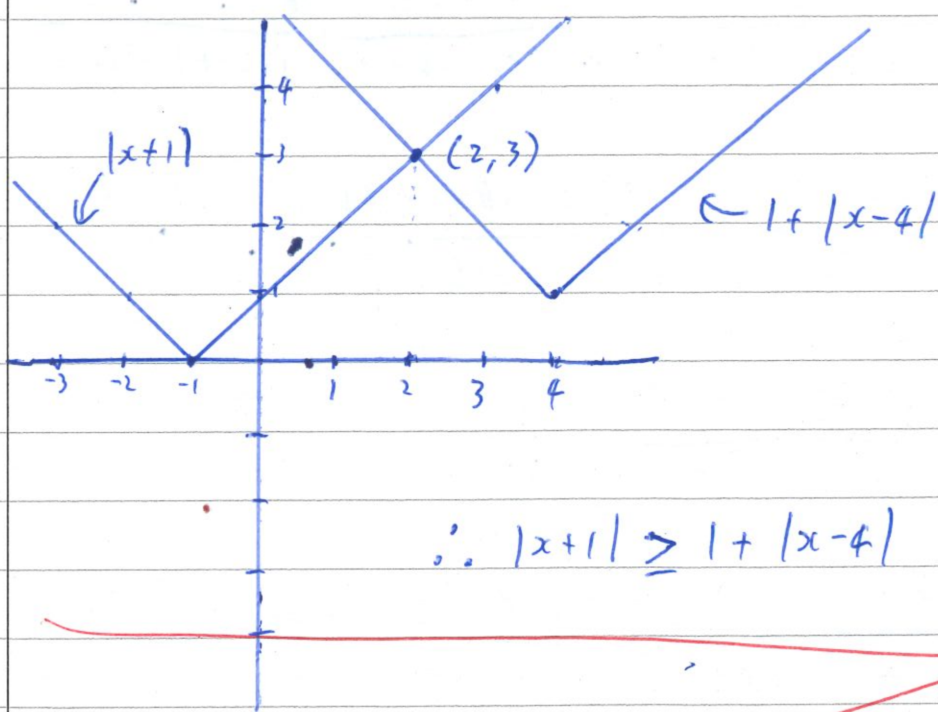
QUESTION NUMBER

ASSESSOR'S USE ONLY

1d.



1d. $|x+1| \geq 1 + \cancel{|x+4|}\ |x-4|$



$\therefore\ |x+1| \geq 1 + |x-4|$ when $x \geq 2$

Q1

8

QUESTION NUMBER | ASSESSOR USE ONLY

5a. $\cancel{\cos\theta + i\sin\theta}$

$$\cancel{\frac{w-1}{w+1} = \frac{(w-1)^2}{w^2-1} = \frac{w^2-2w+1}{w^2-1}} \qquad \cancel{w^2 = \cos^2\theta - \sin^2\theta - 2i\cos\theta\sin\theta}$$

$$= \frac{\cos^2\theta - \sin^2\theta - 2i\sin\theta\cos\theta - 2\cos\theta - 2i\sin\theta + 1}{\cos^2\theta - \sin^2\theta - 2i\sin\theta\cos\theta - 1}$$

$$\cancel{= \frac{2\cos^2\theta - 2i\sin 2\theta - 2\cos\theta - 2i\sin\theta}{-2\sin^2\theta - i\sin 2\theta}}$$

$$\cancel{i\tan\frac{\theta}{2} = \frac{i\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}$$

5a. $$\frac{w-1}{w+1} = \frac{\cos\theta + i\sin\theta - 1}{\cos\theta + i\sin\theta + 1} = \frac{\cos\theta + i\sin\theta - 1}{\cos\theta + i\sin\theta + 1} \times \frac{\cos\theta + 1 - i\sin\theta}{\cos\theta + 1 - i\sin\theta}$$

$$= \frac{\cos^2\theta + \cos\theta - i\sin\theta\cos\theta + i\sin\theta\cos\theta + i\sin\theta + \sin^2\theta - \cos\theta - 1 + i\sin\theta}{(\cos\theta + 1)^2 + \sin^2\theta}$$

$$= \frac{\cos^2\theta + \sin^2\theta + 2i\sin\theta - 1}{\cos^2\theta + \sin^2\theta + 2\cos\theta + 1} = \frac{2i\sin\theta}{2\cos\theta + 2} = \frac{i\sin\theta}{\cos\theta + 1}$$

$$= i\left(\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2} - 1 + 1}\right) = i\left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right) = i\tan\frac{\theta}{2}$$

Q5

5

QUESTION NUMBER

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5bi. $\theta = \tan^{-1}\left(\frac{b\sin\theta}{a\cos\theta}\right) = \tan^{-1}\left(\frac{b}{a}\tan\theta\right)$

$\tan \theta = \frac{b}{a} \tan\theta$

5bii. ~~$D = \theta - \theta = \theta - \tan^{-1}\left(\frac{b}{a}\tan\theta\right)$~~

5bii. Greatest difference when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

4b. $y(x) = u(x) \times x \qquad u(x) = \frac{y(x)}{x}$

$4x^2 \frac{dy}{dx} = y^2 - 2xy \qquad 4x^2 \frac{dy}{dx} = u^2x^2 - 2ux^2$

$4\frac{dy}{dx} = u^2 - 2u$

$\frac{du \times x}{dx} = u^2 - 2u$

$du = \frac{u^2-2u}{x} dx$

$u = (u^2-2u)\ln x + c$

$\frac{1}{u-2} = \ln x + c$

$u = \frac{1}{\ln x + c} + 2$

$y = \frac{1}{x(\ln x + c)} + \frac{2}{x} = \frac{2\ln x + 2c}{x(\ln x + c)}$

$-6 = \frac{2c}{c} =$

Q4

1

QUESTION NUMBER

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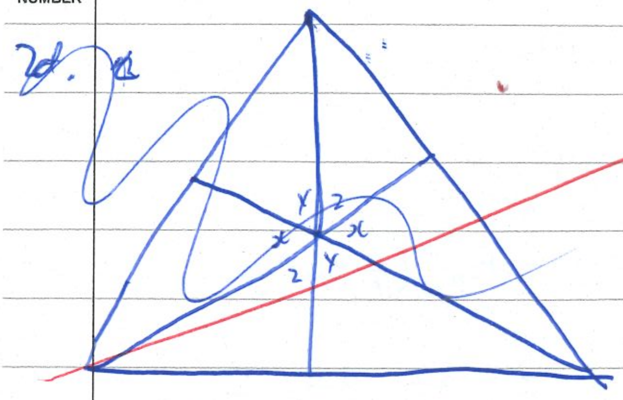
2d.



2c. $y = x\sqrt{1-x^2}$ (for top half of function)

$\int_0^1 x\sqrt{1-x^2}\,dx$ $\qquad x = \sin u$

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93202A

Annotated Exemplar for 93202 Calculus Scholarship

Total Score: 25

Question	Mark	Annotation
1	**8**	The candidate worked with clear setting out and explained the steps taken in **1b** and **1c**. The candidate showed ability in using trig identity appropriately, and recognised the correct domain to achieve the final answer. The candidate could have shown more details in solving **1d**.
2	**5**	The candidate successfully manipulated the logarithm expressions, and subsequently solved the simultaneous equations in **2b**, but failed to recognise the domain restriction and didn't check the validity of the solutions. The candidate could use reverse chain rule in integration in **2c**. The candidate didn't attempt **2d**.
3	**6**	The candidate substitute x = 4 value into the function, therefore simplified the working process successfully in **3a.** The candidate constructed a correct function in **3c,** and successfully used differentiation to optimise it.
4	**1**	In **4a**, the candidate made the first error in expanding a perfect square expression. Unfortunately, instead of working consistently to demonstrate the integration skills required by this question (a second mark would have been awarded), a second error occurred. This is a quite common scenario in marking this question: 'RAWW – right answer wrong working'.
5	**5**	The candidate correctly rationalised the complex number using conjugate, then successfully applied appropriate trigonometry identities in proving **5a**. In **5b**, the candidate recognised the difference between the 2 mentioned angles, and established a connection between them, but failed to minimise the difference of the 2 angles, mostly likely because of the difficulty in differentiating the 'arctan' function.