S

93202MQ

932027

Karahipi 2017
Te Tuanaki

9.30 i te ata Rāmere 10 Whiringa-ā-rangi 2017
Te wā: Toru haora
Whiwhinga tapeke: 40

TE PUKAPUKA TŪMAHI

E rima ngā tūmahi i tēnei pukapuka. Me whakatutuki ngā tūmahi E RIMA KATOA.

Tuhia ō tuhinga ki te Pukapuka Tuhinga 93202MA.

Tangohia te Pukapuka Tikanga Tātai me ngā Tūtohi S–CALCMF i waenga o tēnei pukapuka.

Tuhia ō mahinga KATOA. Me tīmata tō whakautu ki ia tūmahi ki tētahi whārangi hou. Āta tuhia ā-tautia ia tūmahi.

Ko ngā whakautu i oti mā tētahi tātaitai CAS me mātua **whakaatu ngā tono KATOA**. Kāore e rawaka ngā whakautu tika anake.

Tirohia mēnā e tika ana te raupapatanga o ngā whārangi 2–11 kei roto i tēnei pukapuka, ka mutu, kāore tētahi o aua whārangi i te takoto kau.

KA TAEA TĒNEI PUKAPUKA TE PUPURI HEI TE MUTUNGA O TE WHAKAMĀTAUTAU.

TŪMAHI TUATAHI

(a) Me whakatau i ngā tau tōpū KATOA x me y kia noho ko $x^4 - y^2 = 71$.

Me whakaatu mā ngā tikanga taurangi kua kitea e koe ngā otinga KATOA.

(b) E rua ngā pūtake tūturu rarahi ōrite o te whārite $\dfrac{x^2 - bx}{p-1} = \dfrac{ax+c}{p+1}$ engari he tauaro te tohu.

Hāponotia ko $\dfrac{bc}{a} < 0$.

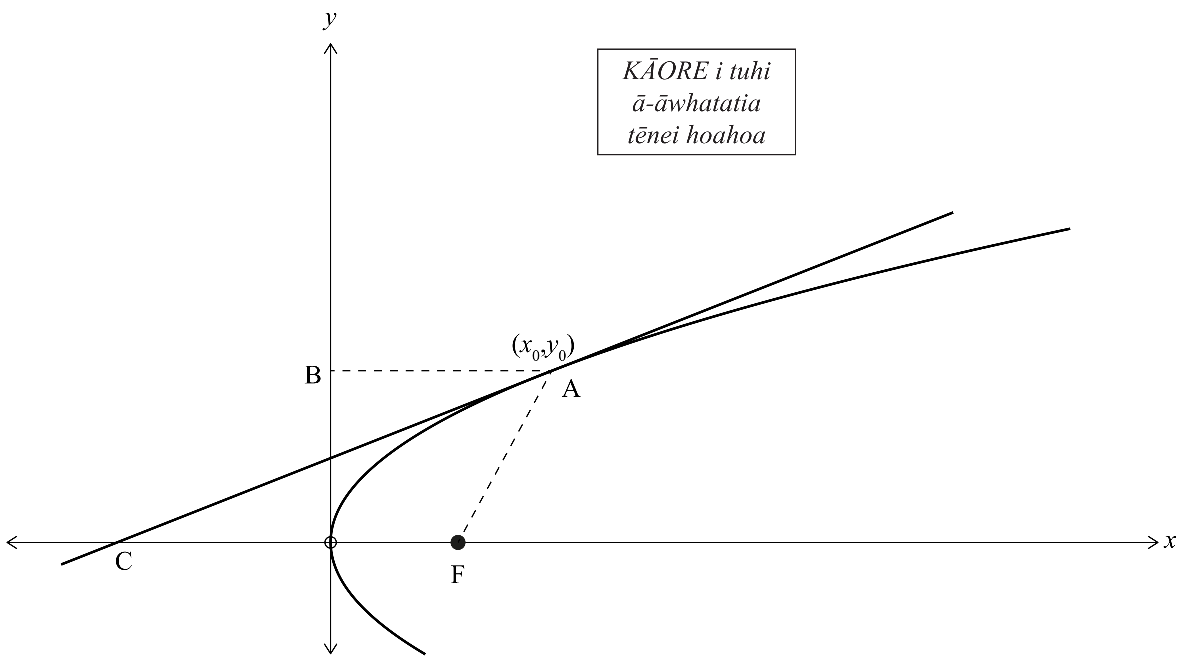
(c) Ko A (x_0, y_0) tētahi pūwāhi o te unahi $y^2 = 4ax$, $a > 0$.

Ko AC te pātapa ki te unahi i te pūwāhi A, ina ko C kei te tuaka-x.

Ko F te arotahi o te unahi.

Ko B te pūwāhi $(0, y_0)$ kei te tuaka-y. Ka hangaia te koki BAF mā te tūhono i ngā pūwāhi B me F ki te pūwāhi A.

Me whakaatu e weherua ana a AC i te koki BAF.



QUESTION ONE

(a) Determine ALL integers x and y such that $x^4 - y^2 = 71$.

Show algebraically that you have found ALL of the solutions.

(b) The equation $\dfrac{x^2 - bx}{p-1} = \dfrac{ax+c}{p+1}$ has two real roots of equal magnitude but opposite in sign.

Prove that $\dfrac{bc}{a} < 0$.

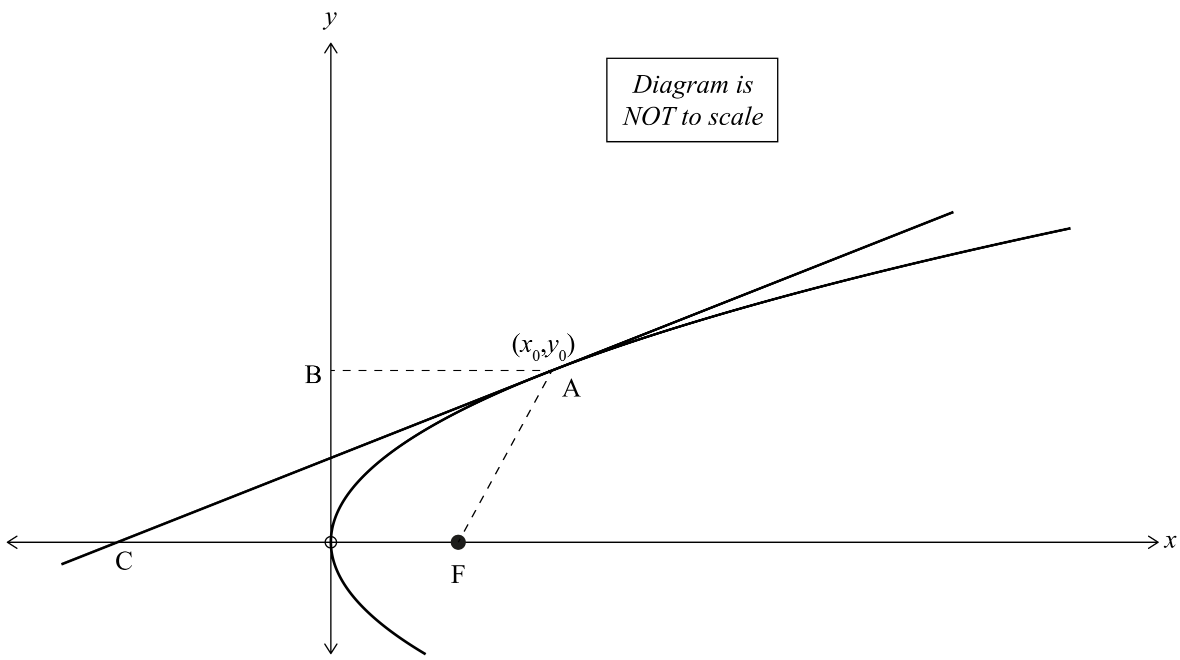
(c) A (x_0, y_0) is a point on the parabola $y^2 = 4ax$, $a > 0$.

AC is the tangent to the parabola at point A, where C is on the x-axis.

F is the focus of the parabola.

B is the point $(0, y_0)$ on the y-axis. Angle BAF is formed by joining points B and F to the given point A.

Show that AC bisects angle BAF.



TŪMAHI TUARUA

(a) He tapawhā hāngai a ABCD me

$AB = 3\sqrt{3}$ ngā wae
AD = 3 ngā wae

He tapawhā a PQRS me

$\angle BRQ = \angle CRS$

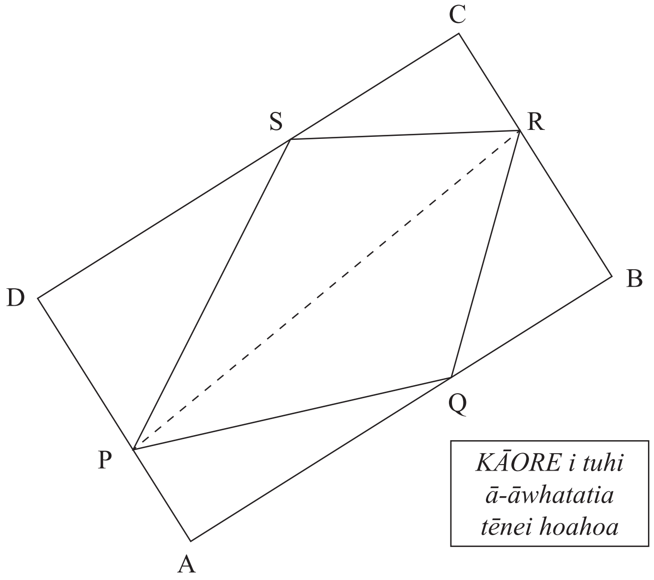
$\angle CSR = \angle DSP$

$\angle SPD = \angle APQ$

$\angle AQP = \angle BQR$

$\angle APQ = \theta$ me $AP = x$.

Kua honoa a P ki a R.



(i) Whakaaturia mai kāore te paenga PQRS i te whirinaki ki x.

(ii) Mēnā ko $\theta = \dfrac{\pi}{3}$ tātoru me $x = 2$, tātaihia te roanga pū o PR.

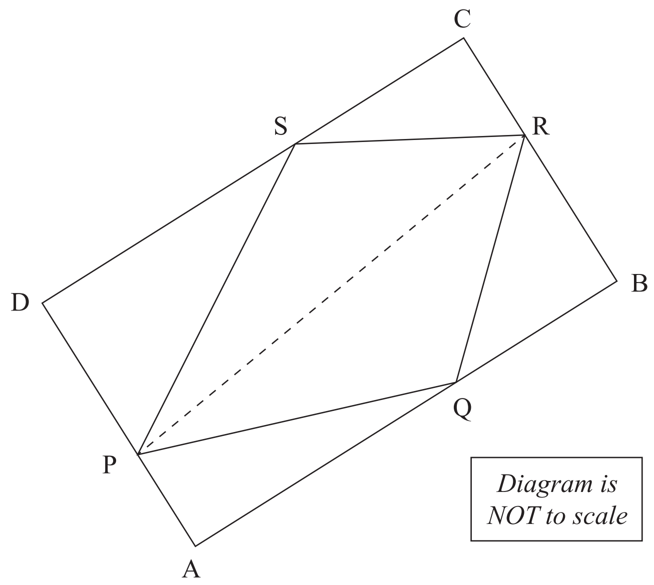
(b) Whakaotia te pūnaha whārite nei mā ngā tikanga taurangi:

$x + y - z = 1 \qquad (1)$

$x^2 + y^2 - z^2 = 5 - 2xy \qquad (2)$

$x^3 + y^3 - z^3 = 43 - 3xy \qquad (3)$

QUESTION TWO



(a) ABCD is a rectangle with

$AB = 3\sqrt{3}$ units

$AD = 3$ units

PQRS is a quadrilateral with

$\angle BRQ = \angle CRS$

$\angle CSR = \angle DSP$

$\angle SPD = \angle APQ$

$\angle AQP = \angle BQR$

$\angle APQ = \theta$ and $AP = x$.

P is joined to R.

(i) Show that the perimeter of PQRS is not dependent on x.

(ii) If $\theta = \dfrac{\pi}{3}$ radians and $x = 2$, calculate the exact length of PR.

(b) Solve algebraically the system of equations:

$x + y - z = 1$ (1)

$x^2 + y^2 - z^2 = 5 - 2xy$ (2)

$x^3 + y^3 - z^3 = 43 - 3xy$ (3)

TŪMAHI TUATORU

(a) Mō $y = x^{(x^x)}$, whiriwhiria $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ ina ko $x = 2$.

Ka taea e koe ngā tikanga taurangi tika te whakamahi pērā i te pārōnaki huna, taupū kōaro hoki / rānei.

(b) Waiho ko $y = \mathrm{e}^x \sin x$.

(i) Me whakaatu ko $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2^{\frac{1}{2}} \mathrm{e}^x \sin\left(x + \dfrac{\pi}{4}\right)$.

(ii) Whakapuakitia a $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ e ai ki ngā pānga taupū me ngā pānga sine anake.

(iii) Whiriwhiria he kīanga mō $\dfrac{\mathrm{d}^n y}{\mathrm{d}x^n}$ me te aromātai i tēnei kīanga i $x = 0$.

(c) Ka pēnei te tautuhi i ngā pānga pūwerewere $\sinh x$ me $\cosh x$:

$$\sinh x = \frac{1}{2}\left(\mathrm{e}^x - \mathrm{e}^{-x}\right)$$

$$\cosh x = \frac{1}{2}\left(\mathrm{e}^x + \mathrm{e}^{-x}\right)$$

Ka tohua te pānga kōaro $\sinh x$ mā te $\sinh^{-1} x$.

Mā te pārōnaki huna, tētahi tikanga kē rānei, me whakaatu ko $\dfrac{\mathrm{d}\left(\sinh^{-1} x\right)}{\mathrm{d}x} = \dfrac{1}{\sqrt{x^2 + 1}}$.

QUESTION THREE

(a) For $y = x^{(x^x)}$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ where $x = 2$.

You may use any valid algebraic techniques such as implicit differentiation and/or logarithms.

(b) Let $y = \mathrm{e}^x \sin x$.

(i) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = 2^{\frac{1}{2}}\mathrm{e}^x \sin\left(x + \frac{\pi}{4}\right)$.

(ii) Express $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in terms of exponential and sine functions only.

(iii) Find an expression for $\frac{\mathrm{d}^ny}{\mathrm{d}x^n}$ and evaluate this expression at $x = 0$.

(c) The hyperbolic functions $\sinh x$ and $\cosh x$ are defined as follows:

$$\sinh x = \frac{1}{2}\left(\mathrm{e}^x - \mathrm{e}^{-x}\right)$$

$$\cosh x = \frac{1}{2}\left(\mathrm{e}^x + \mathrm{e}^{-x}\right)$$

The inverse function of $\sinh x$ is denoted by $\sinh^{-1} x$.

By implicit differentiation, or otherwise, show that $\frac{\mathrm{d}\left(\sinh^{-1} x\right)}{\mathrm{d}x} = \frac{1}{\sqrt{x^2+1}}$.

TŪMAHI TUAWHĀ

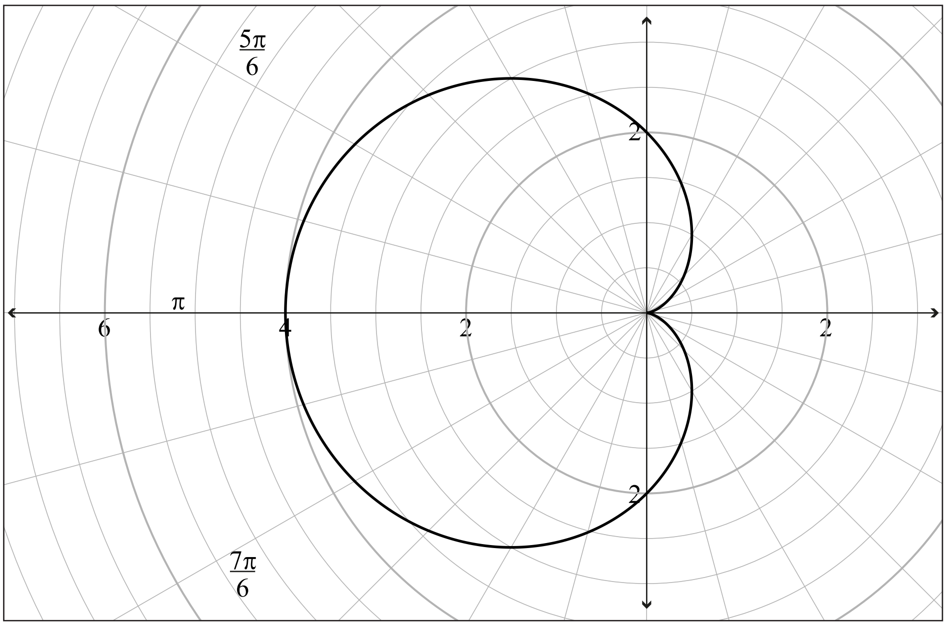
(a) Hāponotia ko $\int \tan x \tan(2x)\tan(3x)\,dx = \ln|\cos x| + \frac{1}{2}\ln|\cos 2x| - \frac{1}{3}\ln|\cos 3x| + \mathrm{c}$

(b) Ko te roanga o te ānau S e whakaaturia ana mā ngā taunga ahuroa ka tukuna mā te

$$S = \int_{\theta_1}^{\theta_2} \sqrt{\left\{ r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 \right\}}\,\mathrm{d}\theta$$

Kimihia te roa o te ānau whānui $r = a(1 - \cos\theta)$ e ai ki te aumou a.

Hei whakaatu i te hanga, kua tātuhia te ānau mō $a = 2$ i raro nei.



(c) E ai ki te Ture Tō ā-Papa Whānui a Newton, ko te tōpana tō ā-papa F ki tētahi ahanoa o te papatipu m kua whakawhanatia ake poutūhia mai i te mata o Papatūānuku ka tohua mā te

$$F = \frac{-mg\mathrm{R}^2}{(x+\mathrm{R})^2}$$

ina ko $x = x(t)$ te tawhiti o te ahanoa i runga ake i te mata o Papatūānuku i te wā t, ko R te pūtoro o Papatūānuku, ā, ko g te whakaterenga nā te tō ā-papa. Me te aha, mā te Ture tuarua a Newton, ko $F = ma$, nō reira

$$ma = -\frac{mg\mathrm{R}^2}{(x+\mathrm{R})^2}$$

Me kī ka whakarewa poutūhia he tākirirangi mā te tere tuatahi o te v_0. Waiho ko h te teitei mōrahi i runga ake i te mata o Papatūānuku ka taea e te ahanoa.

Me whakaatu ko $v_0 = \sqrt{\frac{2g\mathrm{R}h}{\mathrm{R}+h}}$

QUESTION FOUR

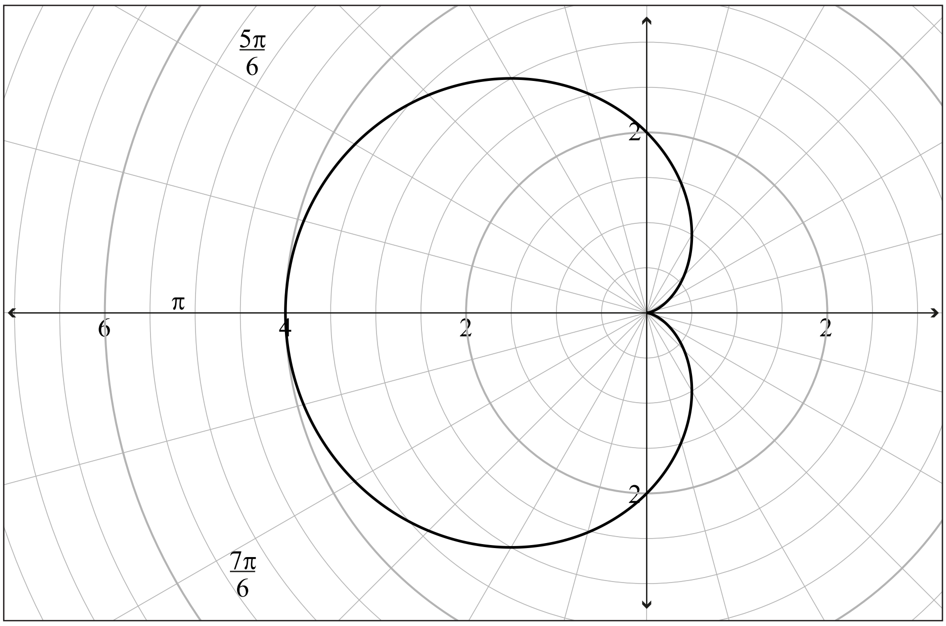
(a) Prove that $\int \tan x \tan(2x) \tan(3x)\, dx = \ln|\cos x| + \frac{1}{2}\ln|\cos 2x| - \frac{1}{3}\ln|\cos 3x| + c$

(b) The length of a curve S expressed in polar coordinates is given by

$$S = \int_{\theta_1}^{\theta_2} \sqrt{\left\{ r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta} \right)^2 \right\}}\, \mathrm{d}\theta$$

Find the length of the entire curve $r = a(1 - \cos\theta)$ in terms of the constant a.

In order to demonstrate shape, the curve for $a = 2$ is drawn below.



(c) According to Newton's Law of Universal Gravitation, the gravitational force F on an object of mass m that has been projected vertically upward from the Earth's surface is given by

$$F = \frac{-mg\mathrm{R}^2}{(x + \mathrm{R})^2}$$

where $x = x(t)$ is the object's distance above the Earth's surface at time t, R is the earth's radius, and g is the acceleration due to gravity. Also, by Newton's second Law, $F = ma$, and so

$$ma = -\frac{mg\mathrm{R}^2}{(x + \mathrm{R})^2}$$

Suppose a rocket is fired vertically upward with an initial velocity of v_0. Let h be the maximum height above the Earth's surface reached by the object.

Show that $v_0 = \sqrt{\dfrac{2g\mathrm{R}h}{\mathrm{R} + h}}$

TŪMAHI TUARIMA

(a) (i) Hāponotia te tuakiri: $\cos(5\theta) = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

(ii) Whiriwhiria he pūrau, e ai ki $\cos\theta$, me ngā pūtake $\cos\frac{2\pi}{9}$, $\cos\frac{4\pi}{9}$, me $\cos\frac{8\pi}{9}$.

He āwhina: Me whiriwhiri te otinga whānui i te tuatahi mō $\cos(5\theta) = \cos(4\theta)$.

(b) Ka pēnei te tautuhi i te raupapa $\{a_n\}$:

$\{a_1\} = 2$

$\{a_2\} = 7$

$\{a_{n+1}\} = \frac{1}{2}\left(a_n + a_{n-1}\right)$ ina $n \geq 2$

(i) Whiriwhiria he tātai pū mō te kīanga tua-n o te raupapa.

(ii) He aha te tepenga o a_n ina $n \to +\infty$?

Tērā pea ka whai hua ēnei tikanga tātai ki a koe:

$$T_n = T_1 + (n-1)d$$

$$T_n = T_1 r^{n-1}$$

$$S_n = \frac{n}{2}\left[2T_1 + (n-1)d\right]$$

$$S_n = \frac{T_1(1-r^n)}{1-r}$$

— *Kua mutu te whakamātautau* —

QUESTION FIVE

(a) (i) Prove the identity: $\cos(5\theta) = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

(ii) Find a polynomial, in terms of $\cos\theta$, with roots $\cos\frac{2\pi}{9}$, $\cos\frac{4\pi}{9}$, and $\cos\frac{8\pi}{9}$.

Hint: First find the general solution for $\cos(5\theta) = \cos(4\theta)$.

(b) The sequence $\{a_n\}$ is defined as follows:

$\{a_1\} = 2$

$\{a_2\} = 7$

$\{a_{n+1}\} = \frac{1}{2}(a_n + a_{n-1})$ where $n \geq 2$

(i) Find an exact formula for the nth term of the sequence.

(ii) What is the limit of a_n as $n \to +\infty$?

You may find the following formulae useful:

$$T_n = T_1 + (n-1)d$$
$$T_n = T_1 r^{n-1}$$
$$S_n = \frac{n}{2}\left[2T_1 + (n-1)d\right]$$
$$S_n = \frac{T_1(1-r^n)}{1-r}$$

— *End of examination* —

English translation of the wording on the front cover

93202MQ

Scholarship 2017
Calculus

9.30 a.m. Friday 10 November 2017
Time allowed: Three hours
Total marks: 40

QUESTION BOOKLET

There are five questions in this booklet. Answer ALL FIVE questions.

Write your answers in Answer Booklet 93202A.

Pull out Formulae and Tables Booklet S–CALCF from the centre of this booklet.

Show ALL working. Start your answer to each question on a new page. Carefully number each question.

Answers developed using a CAS calculator require **ALL commands to be shown**. Correct answers only will not be sufficient.

Check that this booklet has pages 2–11 in the correct order and that none of these pages is blank.

YOU MAY KEEP THIS BOOKLET AT THE END OF THE EXAMINATION.