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SUPERVISOR'S USE ONLY

TOP SCHOLAR

NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

QUALIFY FOR THE FUTURE WORLD
KIA NOHO TAKATŪ KI TŌ ĀMUA AO!

Scholarship 2015
Calculus

2.00 p.m. Tuesday 17 November 2015
Time allowed: Three hours
Total marks: 40

ANSWER BOOKLET

There are five questions in this examination. Answer ALL FIVE questions, choosing ONE option from part (b) of Question Four.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

Write ALL your answers in this booklet.

Make sure that you have Formulae and Tables Booklet S–CALCF.

Show ALL working. Start your answer to each question on a new page. Carefully number each question.

Answers developed using a CAS calculator require **ALL commands to be shown**. Correct answers only will not be sufficient.

Check that this booklet has pages 2–27 in the correct order and that none of these pages is blank.

The diagram for Question Four (b) Option 2 is on page 27 of this booklet.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

This examination consists of five questions.
Answer all FIVE questions, choosing ONE option from part (b) of Question Four.

QUESTION NUMBER		ASSESSOR'S USE ONLY

ONE

a.

$$f(x) = x^3 + \frac{1}{12x}$$

$$f'(x) = 3x^2 + (-1)(12x)^{-2} \cdot 12$$

$$= 3x^2 - \frac{12}{12^2 x^2}$$

$$= 3x^2 - \frac{1}{12x^2}$$

$$[f'(x)]^2 = \left(3x^2 - \frac{1}{12x^2}\right)^2$$

$$= 9x^4 - \frac{6x^2}{12x^2} + \frac{1}{144x^4}$$

$$= 9x^4 - \frac{1}{2} + \frac{1}{144x^4}$$

$$[f'(x)]^2 + 1 = 9x^4 - \frac{1}{2} + \frac{1}{144x^4} + 1$$

$$= 9x^4 + \frac{1}{2} + \frac{1}{144x^4}$$

$$= (3x^2)^2 + \frac{2 \times 3x^2}{12x^2} + \left(\frac{1}{12x^2}\right)^2$$

$$= \left(3x^2 + \frac{1}{12x^2}\right)^2$$

Hence surface area of revolution $= \int_1^3 (2\pi)\left(x^3 + \frac{1}{12x}\right)\left(3x^2 + \frac{1}{12x^2}\right)dx$

$$= 2\pi \int_1^3 \left(3x^5 + \frac{x}{12} + \frac{x}{4} + \frac{1}{144x^3}\right)dx$$

$$= 2\pi \int_1^3 \left(3x^5 + \frac{1}{3}x + \cancel{\frac{1}{144x^3}} \frac{1}{144}x^{-3}\right)dx$$

$$= 2\pi \left[\frac{x^6}{2} + \frac{x^2}{6} - \frac{1}{576}x^{-4}\right]_1^3$$

$$= 2\pi \left[\frac{729}{2} + \frac{3}{2} - \frac{1}{46656}\right] - 2\pi\left[\frac{1}{2} + \frac{1}{6} - \frac{1}{576}\right]$$

$$= 2\pi \times 365.335098\ldots$$

$$= \cancel{229}\ 2300 \text{ unit}^2 \text{ (3 sf)}$$

3

QUESTION NUMBER		ASSESSOR'S USE ONLY

b. Let $y = f(x)$, then

$$\frac{dy}{dx} = y^3$$

Thus $\frac{dy}{y^3} = dx$

Integrate both sides:

$$\frac{y^{-2}}{-2} = x + C$$

$$\frac{1}{y^2} = -2x + C'$$

Where $C' = -2C$

When $x = 0$, $y = 2$, so

$$\frac{1}{4} = -2 \times 0 + C'$$

$$C' = \frac{1}{4}$$

Hence $\frac{1}{y^2} = -2x + \frac{1}{4}$

$$y^2 = \frac{1}{-2x + \frac{1}{4}}$$

$$y^2 = \frac{4}{1-8x}$$

$$y = \pm\sqrt{\frac{4}{1-8x}} = \pm 2(1-8x)^{-\frac{1}{2}}$$

~~If $f(x) = \sqrt{\frac{4}{1-8x}}$, then $f'(x) =$~~

If $f(x) = 2(1-8x)^{-\frac{1}{2}}$, then $f'(x) = 2 \times \left(-\frac{1}{2}\right)(1-8x)^{-\frac{3}{2}} \cdot (-8)$
$= 8(1-8x)^{-\frac{3}{2}} = \left(2(1-8x)^{-\frac{1}{2}}\right)^3 = [f(x)]^3$, so $f(x) = \sqrt{\frac{4}{1-8x}}$
is a solution.

If $f(x) = -2(1-8x)^{-\frac{1}{2}}$, then $f'(x) = (-2) \times \left(-\frac{1}{2}\right)(1-8x)^{-\frac{3}{2}} \times (-8) = -8(1-8x)^{-\frac{3}{2}}$
$= \left(-2(1-8x)^{-\frac{1}{2}}\right)^3$, so $f(x) = -\sqrt{\frac{4}{1-8x}}$ is a solution.

QUESTION NUMBER

ASSESSOR'S USE ONLY

Therefore all possible solutions are $f(x) = \pm\sqrt{\frac{4}{1-8x}}$.

3

3

c. Suppose that there are M kg of salt at the t-th minute.
Take an extremely short time of dt. ~~dt~~ Since dt is very short, the concentration of brine in the solution is constant in this period.
During this period, $0.8 \times 6 \times dt$ kg of salt is added, and at the same time, $6 \times \frac{M}{200} dt$ of salt is running out. $\frac{M}{200}$ is the concentration of salt in the tank, and since water is added and ran out at a constant rate, volume is constant at 200 litres.

Therefore, change in mass = salt inflow − salt outflow

$$dm = 4.8\,dt - \frac{6m}{200}\,dt$$

$$\cancel{dm = 4.8 - \frac{3}{100}}$$

$$dm = (4.8 - 0.03M)\,dt$$

$$\frac{dm}{4.8 - 0.03M} = dt$$

Integrate both sides:

$$\frac{\ln|4.8 - 0.03M|}{-0.03} = t + c$$

$$|4.8 - 0.03M| = Ae^{-0.03t}$$

Where $A = e^{-0.03C}$.

When $t = 0$ (initially), $M = 200 \times 0.5 = 100$ kg, so

$$|4.8 - 0.03 \times 100| = A$$

$$A = \cancel{\pm 6}\ 1.8$$

hence $|4.8 - 0.03M| = 1.8e^{-0.03t}$

When $M = 130$ kg,

$$1.8e^{-0.03t} = |4.8 - 0.03 \times 130| = 0.9$$

QUESTION NUMBER

ASSESSOR'S USE ONLY

$$e^{-0.03t} = \frac{1}{2}$$

$$-0.03t = \ln\frac{1}{2} = -\ln 2$$

$$t = \frac{\ln 2}{0.03}$$

$$= \frac{100\ln 2}{3}.$$

3

After $\frac{100\ln 2}{3}$ minutes, the tank will contain 130 kg of salt.

3

QUESTION NUMBER | ASSESSOR'S USE ONLY

TWO

a. From $9^{2x+y} - 9^x \times 3^y = 6$

$\Rightarrow 9^{2x} \times 9^y - 9^x \times 3^y = 6$

$(9^x)^2 \times (3^y)^2 - 9^x \times 3^y = 6$

$(9^x \times 3^y)^2 - (9^x 3^y) - 6 = 0$

$(9^x \times 3^y - 3)(9^x 3^y + 2) = 0$

Since $9^x 3^y$ is a product of powers, which must be positive, $9^x 3^y$ must be positive. Hence $9^x \times 3^y = 3$ only.

so $3^{2x} \times 3^y = 3$

$\Rightarrow 3^{2x+y} = 3$

$2x + y = 1$

$\Rightarrow y = 1 - 2x \quad (1)$

~~substitute into log(x)~~ From $\log_{(x+1)}(y+3) + \log_{x+1}(y+x+4) = 3$

$\Rightarrow \log_{x+1}(y+3)(y+x+4) = 3$

$(y+3)(y+x+4) = (x+1)^3 \quad (2)$

Substitude (1) into (2): $(1-2x+3)(1-2x+x+4) = (x+1)^3$

$(4-2x)(5-x) = (x+1)^3$

$20 - 10x - 4x + 2x^2 = x^3 + 3x^2 + 3x + 1$

$x^3 + x^2 + 17x - 19 = 0$

Let $p(x) = x^3 + x^2 + 17x - 19$

Notice $p(1) = 1 + 1 + 17 - 19 = 0 \Rightarrow x - 1$ is a factor of $p(x)$

By long division,

$$
\begin{array}{r}
x^2 + 2x - 19 \\
x - 1 \,\big)\, x^3 + x^2 + 17x - 19 \\
x^3 - x^2 \\
\hline
2x^2 + 17x \\
2x^2 - 2x \\
\hline
19x - 19 \\
19x - 19 \\
\hline
0
\end{array}
$$

QUESTION NUMBER | ASSESSOR'S USE ONLY

so $p(x) = (x-1)(x^2 + 2x - 19)$

so $x = 1$ or $x = \dfrac{-2 \pm \sqrt{4^2 - 4 \times 1 \times (-19)}}{2}$

$= -1 \pm \sqrt{20}$

If ~~y~~ $x = 1$, then $y = 1 - 2 \times 1 = -1$

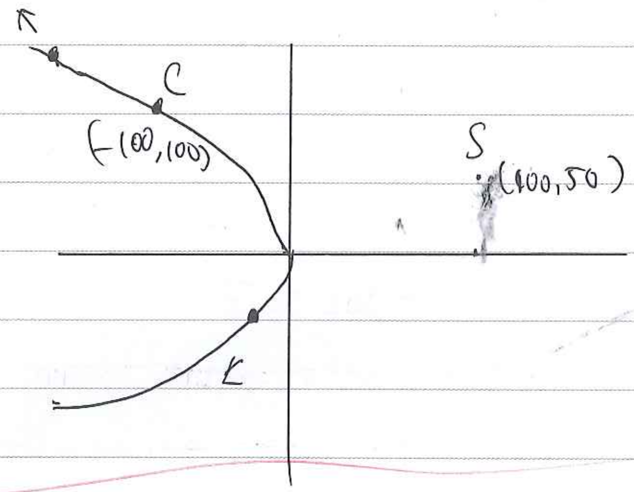
If $x = -1 - \sqrt{20}$, then $1 + x = -\sqrt{20} < 0$. The base of logarithms cannot be negative, so reject.

if $x = -1 + \sqrt{20}$, $y = 1 - (2 \times (-1) + 2\sqrt{20}) = -3 - 2\sqrt{20}$.

But $y + 3 = -\sqrt{20} < 0$, so $\log_{x+1}(y+3)$ will be undefined.

Hence $x = 1, y = -1$ is the only solution.

b.



Let the parabola be $y^2 = 4kx$ for some k. ~~the~~ The car was at $(-100,100)$, which is on the road, so $100^2 = 4k(-100) \Rightarrow 4k = 100$.

so $y^2 = -100x$ is the parabola.

The headlight of the car hits the statue if the tangent at C passes through. ~~Or in another words the gradient of line CS is equal to the gradient of tangent~~

$y^2 = -100x \Rightarrow 2y\dfrac{dy}{dx} = -100 \Rightarrow \dfrac{dy}{dx} = \dfrac{-100}{2y} = \dfrac{-50}{y}$.

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At a point $\left(-\frac{y_0^2}{100}, y_0\right)$ on the parabola, the gradient of tangent is $-\frac{50}{y_0}$. So the equation of tangent is

$$y - y_0 = -\frac{50}{y_0}\left(x + \frac{y_0^2}{100}\right)$$

$$\Rightarrow y_0 y - y_0^2 = -50\left(x + \frac{y_0^2}{100}\right)$$

$(100, 50)$ lie on this line, so

$$\cancel{y_0 y}\ 50y_0 - y_0^2 = -50\left(100 + \frac{y_0^2}{100}\right)$$

$$50y_0 - y_0^2 = -5000 - \frac{y_0^2}{2}$$

$$\frac{y_0^2}{2} - 50y_0 - 5000 = 0$$

$$y_0^2 - 100y_0 - 10000 = 0$$

$$y_0 = \frac{100 \pm \sqrt{100^2 - 4\times(-10000)}}{2}$$

$$= 50 \pm \frac{\sqrt{50000}}{2}$$

$$= 50 \pm 50\sqrt{5}$$

~~However, at $y_0 = 50 - 50\sqrt{5}$, the car is at the traveling away from the statue, so the headlight would not shine on the~~ statue.

Hence, at $y_0 = 50 \pm 50\sqrt{5}$ and $x = -\frac{y_0^2}{100} = -\frac{(50(1+\sqrt{5}))^2}{100} = -\frac{2500(1+2\sqrt{5}+5)}{100}$

$= 25(6+2\sqrt{5}) = -(150 + 50\sqrt{5})$, the car's headlight is shining at the statue.

~~That means, the car's headlight shine at the statue when it is $150 + 50\sqrt{5}$~~

Or, at $y = 50 - 50\sqrt{5}$, $x = -\frac{y_0^2}{100} = \frac{-2500}{100} \times (1-\sqrt{5})^2 = -25(1-2\sqrt{5}+5)$

$= -25(6 - 2\sqrt{5}) = -150 + 50\sqrt{5}$.

QUESTION NUMBER | ASSESSOR'S USE ONLY

That means, when the car is $150 + 50\sqrt{5}$ m to the west and $50 + 50\sqrt{5}$ m to the north of the origin, or is $150 - 50\sqrt{5}$ m to the west and ~~$50 - 50\sqrt{5}$~~ $50\sqrt{5} - 50$ m to the south of the origin, the car's backlight will illuminate the statue.

Since the car will be travelling away from the statue from $(100, 100)$ to either of $(150 + 50\sqrt{5}, 50 + 50\sqrt{5})$ or $(150 - 50\sqrt{5}, 50 - 50\sqrt{5})$, the car's headlight never illuminate the statue.

3

c. $$\frac{dS}{dt} = kS(N - S)$$

$$\frac{dS}{S(N-S)} = k\,dt$$

Notice that $\frac{1}{S} + \frac{1}{N-S} = \frac{N-S+S}{S(N-S)} = \frac{N}{S(N-S)}$

so $\frac{1}{S(N-S)} = \frac{1}{N}\left(\frac{1}{S} + \frac{1}{N-S}\right)$

Thus $\frac{1}{N}\left(\frac{1}{S} + \frac{1}{N-S}\right) dS = k\,dt$

Integrate both sides:

$$\frac{1}{N}\left(\ln|S| - \ln|N-S|\right) = kt + C$$

$$\ln\left|\frac{S}{N-S}\right| = Nkt + NC$$

$$\left|\frac{S}{N-S}\right| = Ae^{Nkt}$$

Since $S > 0$ and $S \leq N$ (number of students know the rumor is at most the total number of students), absolute value takes positive sign.

When $t = 0$, $S = 2$, so $\frac{2}{N-2} = A$

QUESTION NUMBER | ASSESSOR'S USE ONLY

so $\frac{S}{N-S} = \frac{2}{N-2} e^{Nkt}$

Let $P = \frac{2}{N-2} e^{Nkt}$, so

$\frac{S}{N-S} = P$

$S = PN - SP$

$S + SP = PN$

$S(1+P) = PN$

$S = \frac{PN}{1+P} \times \frac{\frac{1}{P}}{\frac{1}{P}}$

$S = \frac{N}{P^{-1}+1}$

Since $P^{-1} = \left(\frac{2}{N-2} e^{Nkt}\right)^{-1} = \frac{N-2}{2} e^{-Nkt}$

$S = \frac{N}{1 + \frac{1}{2}(N-2)e^{-Nkt}}$

as desired.

3

3

QUESTION NUMBER | ASSESSOR'S USE ONLY

THREE

a. $z = \cos\theta + i\sin\theta$

by de Moivre's theorem,

$z^n = \cos n\theta + i\sin n\theta$

$z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

Since $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$ for all θ,

$z^{-n} = \cos n\theta - i\sin n\theta$

$z^n + \frac{1}{z^n} = z^n + z^{-n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta = 2\cos n\theta$

Hence

$2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$

$= z^6 + \frac{1}{z^6} + 6(z^4 + \frac{1}{z^4}) + 15(z^2 + \frac{1}{z^2}) + 20$

$= z^6 + 6z^4 + 15z^2 + 20 + 15\frac{1}{z^2} + 6\frac{1}{z^4} + \frac{1}{z^6}$

$= \binom{6}{0}z^6 + \binom{6}{1}z^4 + \binom{6}{2}z^2 + \binom{6}{3}z^0 + \binom{6}{4}\frac{1}{z^2} + \binom{6}{5}\frac{1}{z^4} + \binom{6}{6}\frac{1}{z^6}$

$= \left(z + \frac{1}{z}\right)^6$

$= (\cos\theta + i\sin\theta + \cos\theta - i\sin\theta)^6$

$= (2\cos\theta)^6$

$= 64\cos^6\theta$

so $\cos^6\theta = \frac{2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20}{64}$

$= \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$

4

4

b. Since $\log 2$, $\log(2\sin x - 1)$, $\log(1-y)$ is arithmetic sequence,

$\log(2\sin x - 1) - \log 2 = \log(1-y) - \log(2\sin x - 1)$

$\Rightarrow 2\log(2\sin x - 1) = \log(1-y) + \log 2$

For logs to be defined, $1-y > 0 \Rightarrow y < 1$ and $2\sin x - 1 > 0$

QUESTION NUMBER | ASSESSOR'S USE ONLY

$\Rightarrow \sin x > \frac{1}{2}$.

So $\frac{1}{2} < \sin x \le 1$.

$\log(2\sin x - 1)^2 = \log(2-2y)$

$\Rightarrow (2\sin x - 1)^2 = 2 - 2y$

Consider real number $z = (2\sin x - 1)^2$. ~~The sin...~~
Vertex of z is at $\sin x = \frac{1}{2}$ and $z = 0$. Since $\frac{1}{2} < \sin x \le 1$, z is strictly increasing, so

$(2\cdot\frac{1}{2} - 1)^2 < z \le (2\times 1 - 1)^2$

$\Rightarrow 0 < z \le 1$

So since $z = 2 - 2y \Rightarrow y = \frac{2-z}{2}$

Maximum value of y occurs when z approaches 0, then y approaches 1. Minimum value is at when $z = 1$, then $y = \frac{1}{2}$.

Hence $\frac{1}{2} \le y < 1$.

3

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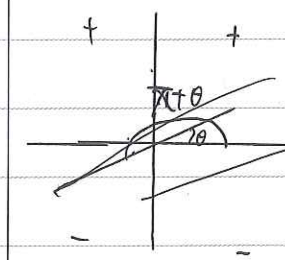
C. $\cos(\frac{5\pi}{2} - x) = \cos(\frac{\pi}{2} - x + 2\pi) = \cos(\frac{\pi}{2} - x) = \sin x$



~~$\sin(x-\pi) = \sin(x+\pi-2\pi) = \sin(x+\pi)$~~

~~From this graph diagram, $\sin(x+\pi) = -\sin x$. so $\sin(x-\pi) = -\sin x$.~~

So the denominator is $4\cos^2(\frac{5\pi}{2} - x) - \sin^2(2x-\pi)$

$= 4\sin^2 x - (\sin(2x - 2\pi))^2$

$= 4\sin^2 x - \sin^2 2x$

QUESTION NUMBER | ASSESSOR'S USE ONLY

Hence LHS $= \dfrac{4\cos^2 2x - 4\cos^2 x + 3\sin^2 x}{4\sin^2 x - \sin^2 2x}$

Now LHS $= \dfrac{4\cos^2 2x - 4\cos^2 x + 3\sin^2 x}{4\sin^2 x - \sin^2 2x} - 4 + 4$

$= \dfrac{4\cos^2 2x - 4\cos^2 x + 3\sin^2 x - 16\sin^2 x + 4\sin^2 2x}{4\sin^2 x - \sin^2 2x} + 4$

$= \dfrac{4(\cos^2 2x + \sin^2 2x) - 4\cos^2 x - 13\sin^2 x}{4\sin^2 x - \sin^2 2x} + 4$

$= \dfrac{4 - 4\cos^2 x - 4\sin^2 x - 9\sin^2 x}{4\sin^2 x - \sin^2 2x} + 4$

$= \dfrac{4 - 4(\sin^2 x + \cos^2 x) - 9\sin^2 x}{4\sin^2 x - \sin^2 2x} + 4$

$= \dfrac{4 - 4 - 9\sin^2 x}{4\sin^2 x - \sin^2 2x} + 4$

$= -\dfrac{9\sin^2 x}{4\sin^2 x - (2\sin x\cos x)^2} + 4$

$= -\dfrac{\sin^2 x}{\sin^2 x}\cdot\dfrac{9}{4 - 4\cos^2 x} + 4$

$= \dfrac{9}{4\cos^2 x - 4} + 4$

Since $\cos 2x + 1 = \cos^2 x - \sin^2 x + \cos^2 x + \sin^2 x = 2\cos^2 x$

$\Rightarrow 2\cos^2 x - 1 = \cos 2x$

So $\dfrac{9}{4\cos^2 x - 4} + 4$

$= \dfrac{9}{2(2\cos^2 x - 1) - 2} + 4$

$= \dfrac{9}{2\cos 2x - 2} + 4$

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$$= \frac{\cancel{9+8(\cos 2x-1)}}{2(\cos 2x-1)} + \frac{8(\cos 2x-1)}{2(\cos 2x-1)}$$

$$= \frac{9+8\cos 2x-8}{2(\cos 2x-1)}$$

$$= \frac{8\cos 2x+1}{2(\cos 2x-1)}$$

$$= \text{RHS}$$

As desired.

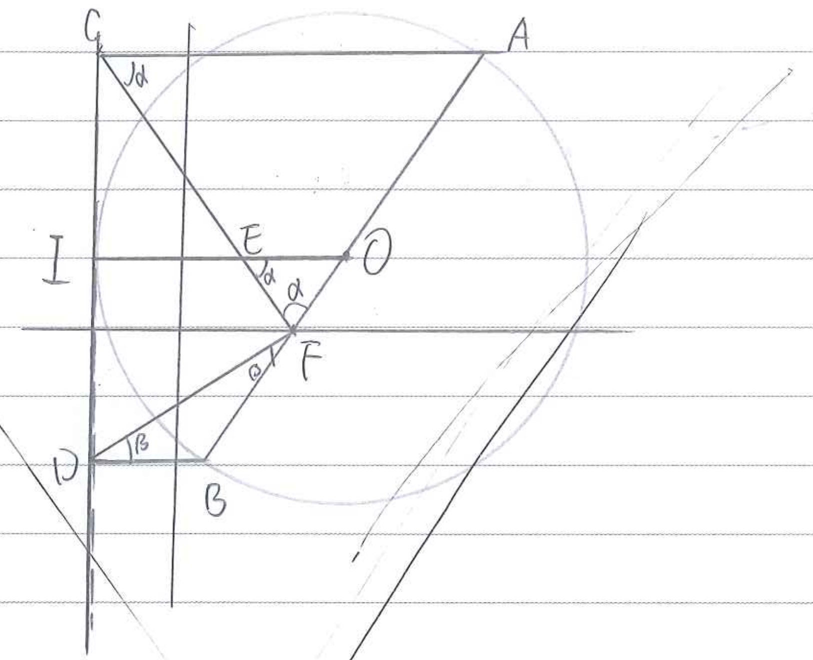
3

3

QUESTION NUMBER | ASSESSOR'S USE ONLY

Four

b.



In the above diagram, F is the focus, AB is the focal chord, CD is the directrex, and AC, BD are perpendicular to CD.

Since A, B are points on parabola, by definition, ~~A~~ AF=AC and BF=BD. So Triangles ACF and BDF are both ~~equi~~ isosceles. Let $\angle AFC = \alpha$, $\angle BFD = \beta$. Then $\angle ACF = \angle AFC = \alpha$ (isosceles triangle) and $\angle BFD = \angle FDB = \beta$. By angles in a triangle, $\angle CAF = 180° - 2\alpha$, $\angle FBD = 180° - 2\beta$. Since AC and BD are perpendicular to a common line AC and BD are parallel, so ~~∠CAB~~ $\angle CAF + \angle DBF = 180° \Rightarrow 180° - 2\alpha + 180° - 2\beta = 180° \Rightarrow 2\alpha + 2\beta = 180° \Rightarrow \alpha + \beta = 90°$.

Let O be the centre and I be an intersection of the circle and the directrix.

QUESTION NUMBER | ASSESSOR'S USE ONLY

Five (a) $y=kx^2$, $\frac{dy}{dx} = 2kx$, gradient of normal is $-\frac{1}{2kx}$.

At x_0, gradient is $-\frac{1}{2kx_0}$.

So the equation of normal is $y - kx_0^2 = -\frac{1}{2kx_0}(x - x_0)$

$$y = -\frac{1}{2kx_0}x + \frac{1}{2k} + kx_0^2.$$

$$\Rightarrow y = -\frac{1}{2kx_0}x + kx_0^2 + \frac{1}{2k} \quad (1)$$

2

b. $y = kx^2$ (2)

equate (1) and (2):

$$kx^2 = -\frac{1}{2kx_0}x + kx_0^2 + \frac{1}{2k}$$

$$\Rightarrow kx^2 + \frac{1}{2kx_0}x - kx_0^2 - \frac{1}{2k} = 0 \quad (3)$$

Since $x = x_0$ is a ~~solution~~ root to 3, (3) can be factorised into $(x - x_0)\left(kx + \frac{1}{2kx_0} + kx_0\right) = 0$

so the x-coordinate of intersection is

$$x = \frac{-\frac{1}{2kx_0} - kx_0}{k}$$

$$= -\frac{1}{2k^2x_0} - x_0$$

so y-intersection is $y = -\frac{1}{2kx_0}\left(-\frac{1}{2k^2x_0} - x_0\right) + kx_0^2 + \frac{1}{2k}$

$$= \frac{1}{4k^3x_0^2} + \frac{1}{2k} + kx_0^2 + \frac{1}{2k}$$

$$= kx_0^2 + \frac{1}{4k^3x_0^2} + \frac{1}{k}$$

~~$\frac{dy}{dx_0} = 2kx_0 + \frac{1}{2k^3x_0^3}$ $\left(\left((4k^3x_0^2)^{-1}\right)' = (4k^3)^{-1}x^{-3}\cdot -2 = -\frac{1}{4k^3x^3}x^{-2} = -\frac{1}{2k^3x^3}\right)$~~

~~for minimum, $\frac{dy}{dx_0} = 0 \Rightarrow 2kx_0 + \frac{1}{2k^3x_0^3} = 0 \Rightarrow$~~

QUESTION NUMBER | ASSESSOR'S USE ONLY

so $y = kx_0^2 + \frac{1}{4k^3x_0^2} + \frac{1}{k}$

Since $(\sqrt{a} - \sqrt{b})^2 \geq 0$ for all positive numbers a and b

$\Rightarrow a + b - 2\sqrt{ab} \geq 0$

$\Rightarrow a + b \geq 2\sqrt{ab}$

let $a = kx_0^2$, $b = \frac{1}{4k^3x_0^2}$, then

$$kx_0^2 + \frac{1}{4k^3x_0^2} \geq 2\sqrt{kx_0^2 \times \frac{1}{4k^3x_0^2}} = 2\cdot\sqrt{\frac{1}{4k^2}} = \frac{2}{2k} = \frac{1}{k}$$

with equality if and only if $a=b \Rightarrow kx_0^2 = \frac{1}{4k^3x_0^2}$

$\Rightarrow x_0^4 = \frac{1}{4k^4}$

$\Rightarrow x_0 = \frac{1}{\sqrt{2}}\cdot\frac{1}{k}$

so equation is $y = -\frac{1}{2k\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{k}}x + k\cdot\frac{1}{2}\cdot\frac{1}{k^2} + \frac{1}{2k}$

$$= -\frac{1}{\frac{2}{\sqrt{2}}}x + \frac{1}{2k} + \frac{1}{2k}$$

$$y = -\frac{\sqrt{2}}{2}x + \frac{1}{k}$$

3

3

c. From (3) $\Rightarrow (x - x_0)\left(kx + \frac{1}{2kx_0} + kx_0\right) = 0$, 2 intersections are $x_1 = x_0$ and $x_2 = -\frac{1}{2k^2x_0} - x_0$

so area $= \int_{x_2}^{x_1}\left(-\frac{1}{2kx_0}x + kx_0^2 + \frac{1}{2k} - kx^2\right)dx$

$$= \left[-\frac{1}{4kx_0}x^2 + \left(kx_0^2 + \frac{1}{2k}\right)x - \frac{kx^3}{3}\right]_{x_2}^{x_1}$$

$$= \left[-\frac{1}{4kx_0}\cdot x_0^2 + \left(kx_0^2 + \frac{1}{2k}\right)x_0 - \frac{kx_0^3}{3}\right] -$$

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$$\left[-\frac{1}{4kx_0}x_2^2+k\left(x_0^2+\frac{1}{2k}\right)x_2-\frac{kx_2^3}{3}\right]$$

$$=\left[-\frac{x_0}{4k}+kx_0^3+\frac{x_0}{2k}-\frac{kx_0^3}{3}\right]+\frac{x_2^2}{4kx_0}+kx_0^2x_2+\frac{x_2}{2k}-\frac{kx_2^3}{3}$$

$x_2^2=\frac{1}{4k^4x_0^2}+\frac{1}{k^2}+x_0^2$, $x_0^2x_2=-\frac{1}{2k}-x_0^2$,

$$x_2^3=-\frac{1}{8k^6x_0^3}-\frac{3}{4k^4x_0}-\frac{3}{2k^2}x_0-x_0^3$$

so $\frac{x_2^2}{4kx_0}+kx_0^2x_2+\frac{x_2}{2k}-\frac{kx_2^3}{3}$

$$=\frac{1}{4k^5x_0^3}+\cancel{\frac{1}{4k^3x_0}}+\frac{x_0}{4k}-\frac{1}{2}-kx_0^2-\cancel{\frac{1}{4k^3x_0}}-\frac{x_0}{2k}$$

$$+\frac{1}{24k^5x_0^3}+\frac{1}{4k^5x_0}+\frac{1}{4k}x_0+\frac{kx_0^3}{3}$$

$$=\frac{1}{4k^4x_0^3}-\cancel{\frac{x_0}{4k}}-\frac{1}{2}-kx_0^2+\frac{1}{24k^5x_0^3}+\frac{1}{4k^3x_0}+\frac{1x_0}{4k}+kx_0^3$$

$$\text{Area}=-\cancel{\frac{x_0}{4k}}+kx_0^3+\cancel{\frac{x_0}{2k}}-\frac{kx_0^3}{3}+\frac{1}{4k^4x_0^3}-\cancel{\frac{x_0}{4k}}-\frac{1}{2}-kx_0^2+\frac{1}{24k^5x_0^3}+\frac{1}{4k^3x_0}$$

$$+\frac{1}{4k^3x_0}$$

$$=\frac{7}{3}x_0^3+\frac{1}{4k^6x_0^3}$$

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(b) ~~Focus is~~ let the parabola be $y^2=4ax$. Then focus is at $(a,0)$.

Let the focal chord be $y=m(x-a)$. Then substitute in $y^2=4ax$:

$$m^2(x^2-2ax+a^2)=4ax$$

$$m^2x^2-2am^2x+m^2a^2-4ax=0$$

$$m^2x^2-(2am^2+4a)x+m^2a^2=0$$

If x_1, x_2 are roots, then $x_1+x_2=\frac{2am^2+4a}{m^2}=2a+\frac{4a}{m^2}$

~~$x_1x_2=\frac{m^2a^2}{m^2}=a^2$~~ $x_1x_2=\frac{m^2a^2}{m^2}=a^2$

~~Since the~~ Corresponding $y_1+y_2=m(x_1-a+x_2-a)$

$$=m(x_1+x_2-2a)$$

$$=m\left(2a+\frac{4a}{m^2}-2a\right)$$

$$=\frac{4a}{m}$$

Since centre is the midpoint of intersections, centre is at

$$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)=\left(a+\frac{2a}{m^2},\frac{2a}{m}\right).$$

$y_1y_2=m^2(x_1-a)(x_2-a)$

$=m^2(x_1x_2-a(x_1+x_2)+a^2)$

$=m^2\left(\frac{4a}{m}-2a^2-\frac{4a^2}{m^2}+a^2\right)$

$=m^2\left(\frac{4a}{m}-a^2-\frac{4a^2}{m^2}\right)+a^2$

so $(x_1-x_2)^2=(x_1+x_2)^2-4x_1x_2$

$=a^2\left(2+\frac{4}{m^2}\right)^2-4\left(\frac{4a}{m}\right)$

$=a^2\left(4+\frac{16}{m^2}+\frac{16}{m^4}\right)-\frac{4a}{m}$

$=4a^2+\frac{16a^2}{m^2}+\frac{16a^2}{m^4}-\frac{4a}{m}$

two $(y_1-y_2)^2=(y_1+y_2)^2-4y_1y_2$.

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$$\cancel{= \frac{4a}{m} - 4m^2\left(\frac{4a}{m} - a^2 - \frac{4a^2}{m}\right)}$$
$$\cancel{= \frac{4a}{m} - \frac{16a}{m} + 4m^2a^2 + 4ma^2}$$
$$\cancel{= 4m^2a^2 + 4ma^2 - \frac{12a}{m}}$$
~~distance of chord~~ $\cancel{^2 = (x_1-x_2)^2 + (y_1-y_2)^2}$
$$\cancel{= 4a^2 + \frac{16a^2}{m^2} + \frac{16a^2}{m^4} - \frac{4a}{m} + 4a^2m^2 + 4ma^2 - \frac{12a}{m}}$$

$$(x_1-x_2)^2 = (x_1+x_2)^2 - 4x_1x_2$$
$$= a^2\left(2+\frac{4}{m^2}\right)^2 - 4a^2$$
$$= a^2\left(4+\frac{16}{m^2}+\frac{16}{m^4}\right) - 4a^2$$
$$= \frac{16a^2}{m^2} + \frac{16a^2}{m^4}$$

$$y_1y_2 = m^2(x_1-a)(x_2-a)$$
$$= m^2(x_1x_2 - a(x_1+x_2) + a^2)$$
$$= m^2\left(a^2 - a^2\left(2+\frac{4}{m^2}\right) + a^2\right)$$
$$= m^2\left(a^2 - 2a^2 - \frac{4a^2}{m^2} + a^2\right)$$
$$= -4a^2$$

$$(y_1-y_2)^2 = (y_1+y_2)^2 - 4y_1y_2$$
$$= \left(\frac{4a}{m}\right)^2 - 4(-4a^2)$$
$$= \frac{16a^2}{m^2} + 16a^2$$

$$(x_1-x_2)^2 + (y_1-y_2)^2 = \frac{16a^2}{m^2} + \frac{16a^2}{m^4} + \frac{16a^2}{m^2} + 16a^2$$
$$= 16a^2\left(\frac{1}{m^4} + \frac{2}{m^2} + 1\right)$$
$$= 16a^2\left(\frac{1}{m^2}+1\right)^2$$

So ~~dista~~ diameter of circle $= \sqrt{16a^2\left(\frac{1}{m^2}+1\right)^2} = 4a\left(\frac{1}{m^2}+1\right)$

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Distance from centre to directrix is x coordinate + a $= a + \frac{2a}{m^2} + a = 2a + \frac{2a}{m^2}$

So Distance is $2a\left(1+\frac{1}{m^2}\right)$ 5

Notice that radius of circle $= 2a\left(1+\frac{1}{m^2}\right)$ = distance from centre to directrix, so the directrix must be tangent to the circle. 5

(a) If x is a real root, then

$$3x^3 + (2-3ai)x^2 + (6+2bi)x + 4 = 0$$
$$\Rightarrow (3x^3+2x^2+6x+4) + (2bx - 3ax^2)i = 0$$

So the imaginary part has to be zero, therefore

$$2bx - 3ax^2 = 0$$
$$x(2b - 3ax) = 0$$

$x=0$ is not a solution, so x can only be $\frac{2b}{3a}$.

That means that if this equation has real root, then it has only one.

Notice that the real part factorises to $(3x+2)(x^2+2)$, so the real root is $x = -\frac{2}{3}$ only. 3

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Since $x = -\frac{2}{3}$ is indeed a root, the equation has ~~only~~ exactly one root.