Entanglement As A Measure of Semi-classicality In The Jaynes Cummings Model

Progress Report PHYS6018

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January 2025

Abstract

For quantum mechanics to be a fundamental theory, it should be able to reproduce classical physics. Recent work has shown that this true for the Jaynes-Cummings model, where taking the correct limit reproduces the dynamics of a classical field interacting with a qubit. This report aims to study the physics of field-qubit systems as the limit is approached with a focus on entanglement. The average Von Neumann entropy $\tilde{S}$ as a function of the field dynamics is the entanglement measure used. The two timescales averaged over are the collapse time and Rabi period. It is found that for a qubit initially in an excited state $|+z\rangle$ and a field in a displaced fock state $|\alpha, n\rangle$, all states' $\tilde{S} \to 0$ in the semi-classical limit, with smaller n entangling less.

Contents

1 Introduction

The revered Jaynes Cummings model is the simplest exactly solvable light-matter model in Quantum Optics. It describes a quantized field interacting with a two-level system, or qubit [1]. The system's Hamiltonian is given by

$$\hat{H}_Q = \frac{1}{2}\Omega\hat{\sigma}_z + \omega_0\hat{a}^\dagger\hat{a} + \lambda(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-) \tag{1}$$

where Ω is the qubit frequency, ω_0 is the field frequency, and λ is the single photon coupling. The remaining terms are the operators $\hat{a}^\dagger$, $\hat{a}$, $\hat{\sigma}_+$, $\hat{\sigma}_-$, and $\hat{\sigma}_z$, which are the creation, annihilation, raising, lowering, and Pauli Z operators respectively. The operators $\hat{\sigma}_\pm$ can be expressed as $\hat{\sigma}_\pm = \hat{\sigma}_x \pm i\hat{\sigma}_y$. An alternative semi-classical model can be derived by assuming the field to be classical in the field-qubit system [2]. This takes the form

$$\hat{H}_{SC} = \frac{1}{2}\Omega\hat{\sigma}_z + A(e^{i\omega_0 t}\hat{\sigma}_- + e^{-i\omega_0 t}\hat{\sigma}_+) \tag{2}$$

where t is time and A is the semi-classical driving amplitude. Recent work has shown that Eq (2) can be derived from Eq (1) [3]. By expressing Eq (1) in the orthonormal basis of displaced Fock states $D(\alpha)|n\rangle$, and assuming $\omega_0 = \Omega = 1$ (unit resonance), the joint limit $\lambda \to 0$ and $|\alpha| \to \infty$ may be taken whilst keeping the product $A = \lambda|\alpha|$ fixed. This is considered to be the limit of semi-classicality, where the field behaves classically whilst the qubit remains in a quantum state.

This report aims to understand the physical behaviour of systems modeled by the Jaynes Cummings model as they tend towards the semi-classical limit. An overview of the field and qubit dynamics is presented in Sections 2.1 and 2.2 A particular focus is placed upon the entanglement, which is measured by averaging the Von Neumann entropy in Section 2.3. Two different timescales, the Rabi period and the collapse time, are considered in Section 3.

The Jaynes Cummings model, and its extension the quantum Rabi model, are central to the emerging area of quantum technologies to model field-qubit interactions due to their applicability and simplicity. As the semi-classical limit is a mathematical limit and therefore physically unreachable (∞ is not a measurable quantity), all systems traditionally modeled semi-classically should display previously unpredicted quantum behaviour. This could lead to novel quantum error correction in areas like circuit and cavity QED where semi-classical models are often assumed. This is of significant experimental relevance to technologies such as quantum computing.

2 System Dynamics

2.1 Field Dynamics

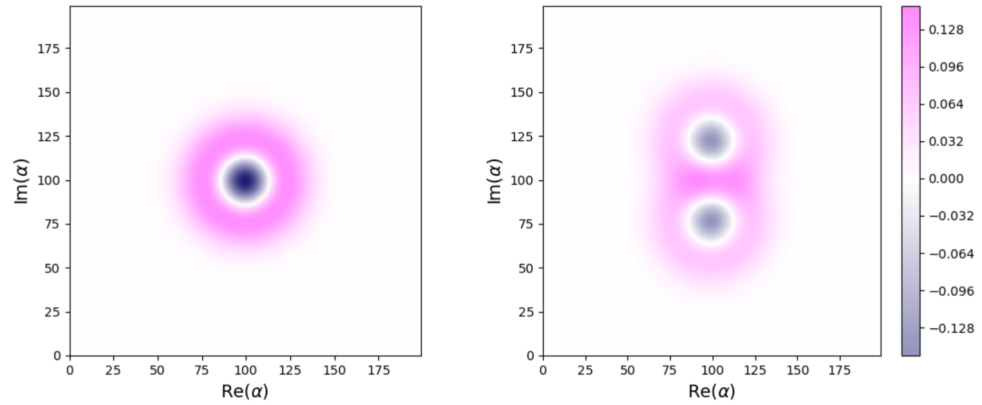
The category of field state studied in this report is the displaced Fock state. It is given by $D(\alpha)\,|n\rangle = |\alpha, n\rangle$ where $|n\rangle$ is a Fock state (eigenstate of the number operator) and $D(\alpha)$ is known as the displacement operator. Physically, the displacement operator can be understood as displacing the expectation values of the field's position and momentum by a value dependent on α. Assuming the field is initialised in the state $|\alpha, n\rangle$ and the two-level system is initialised in the excited state $|{+z}\rangle$, the time-dependent state vector for the full system can be written as

$$|\Psi(t)\rangle = k_1\left[k_2(|{+z}\rangle + k_3\,|{-z}\rangle) \otimes |\alpha_+, n\rangle + k_2^*(|{+z}\rangle - k_3\,|{-z}\rangle) \otimes |\alpha_-, n\rangle\right] \tag{3}$$

where $k_1 = \frac{1}{2}e^{-i\omega_0 t/2}$, $k_2 = e^{-i\lambda|\alpha|t/2}$, $k_3 = e^{i\omega_0 t}$, and t is time [4]. The states $|\alpha_\pm, n\rangle = D(\alpha_\pm)\,|n\rangle$ where

$$\alpha_\pm(t) = \alpha e^{-i\omega_0 t}\left[1 \mp \frac{i\lambda t}{2|\alpha|}\right] \tag{4}$$

This demonstrates that for any time $t > 0$, the field splits into two subsystems $|\alpha_\pm, n\rangle$, which in turn causes the bipartite subsystems to become entangled with each other, as explored further in Section 2.3. To represent the dynamics of the field, the reduced density matrix of the field ρ_f is visualised



(a) T_0 (b) $20T_{Rabi}$

Figure 1: Wigner functions for the state $|\alpha, 1\rangle$ initially (T_0) and after 20 Rabi periods T_{Rabi}, where $T_{Rabi} \propto \frac{1}{\lambda|\alpha|}$.

using a Wigner function. This converts ρ_f into a distribution in phase space, where the axes are position x and momentum p. Wigner functions can behave similarly to probability distributions but have purely quantum mechanical features such as negative values, and are therefore considered quasiprobability distributions. A visualisation of the field dynamics for the state $|\alpha, 1\rangle$ using the Wigner representation can be seen in Figure 1 (note that the values $A = 0.01$ and $\lambda = 10^{-3}$ are used for all simulations in Section 2). The magnitude of overlap between the two field substates, $\langle \alpha_+, n|\alpha_-, n\rangle$, is given by

$$|\gamma(t)| = L_n(\lambda^2 t^2)e^{-\lambda^2 t^2/2} \tag{5}$$

where L_n represents a Laguerre polynomial, and n is the quantum number of states $|\alpha, n\rangle$. The Laguerre polynomial accounts for both the overlap of negative and positive regions as well as the increased uncertainty product of higher n (the distribution has a larger width in phase space).

As $\lambda \to 0$ $|\gamma(t)| \to 1$, resulting in the field substates being equal. Additionally, as $\alpha \to \infty$, the ratio between the expectation value of an observable $\hat{O}$ and uncertainty product $\frac{\langle O\rangle}{\Delta O} \to \infty$, so the distribution appears point-like. These observations define a physical interpretation for the classical limit for the field, where the observables are localized for all time.

2.2 Qubit Dynamics

The qubit dynamics of the bipartite system can be understood through the reduced density operator ρ_q, given by

$$\begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix} \tag{6}$$

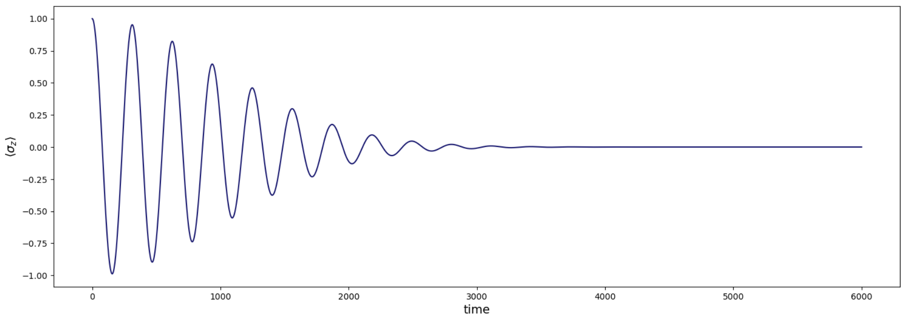
where the diagonal elements are known as populations and the off-diagonal elements are known as coherences. A frequently used representation of the bipartite system's dynamics is the population dynamics. A convenient method to calculate this quantity is to calculate the atomic inversion, given by

$$\langle \sigma_z \rangle = \langle \Psi| \sigma_z |\Psi\rangle \tag{7}$$

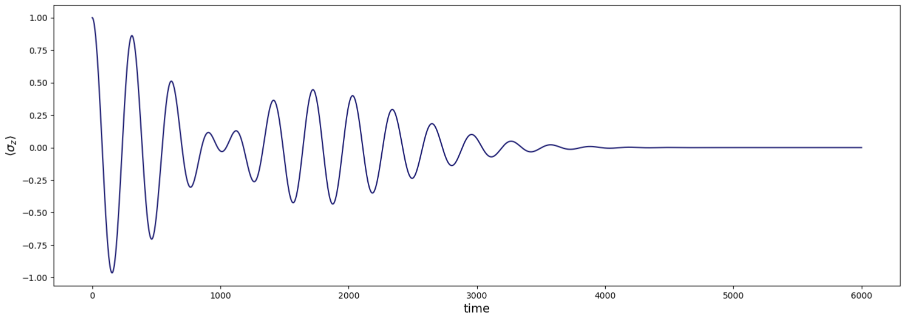
where Ψ is given by Eq (3). $\langle \sigma_z \rangle$ for (Eq 3) can be written as

$$\langle \sigma_z \rangle = |\gamma| cos(2\lambda|\alpha|t) \tag{8}$$

which is displayed in Figure 2. This is equivalent to calculating $P_e - P_g$, where P_e and P_g are the probabilities of the qubit being in the excited and ground state respectively. The state $|\alpha, 1\rangle$ has a dip at $t \approx 1000$ in Figure 2b, referred to as nodal collapse. It corresponds to positive and



(a) $|\alpha, 0\rangle$



(b) $|\alpha, 1\rangle$

Figure 2: Atomic inversion $\langle\sigma_z\rangle$ for different field states over 20 Rabi periods T_{Rabi}. $\lambda = 10^{-3}$.

negative regions between the two field substates destructively interfering. The persistent flattening of oscillations present in Figures 2a and 2b is due to the substates becoming increasingly far apart in phase space, referred to as total collapse. Higher n results in longer total collapse times as the widths of the substates increase in phase space, thus requiring more time for separation to occur. It also increases the frequency of nodal collapse.

To describe the behaviour of the coherences π_{12} and π_{21}, which represent the degree of superposition between the two qubit states, the expectation values $\langle\sigma_+\rangle$ and $\langle\sigma_-\rangle$ are calculated respectively. These are given by

$$\langle\sigma_+\rangle = \frac{k_3}{2}|\gamma|sin(2\lambda|\alpha|t) \quad \text{and} \quad \langle\sigma_-\rangle = \frac{k_3^*}{2}|\gamma|sin(2\lambda|\alpha|t) \tag{9}$$

where k_3 is the phase factor in Eq (3).

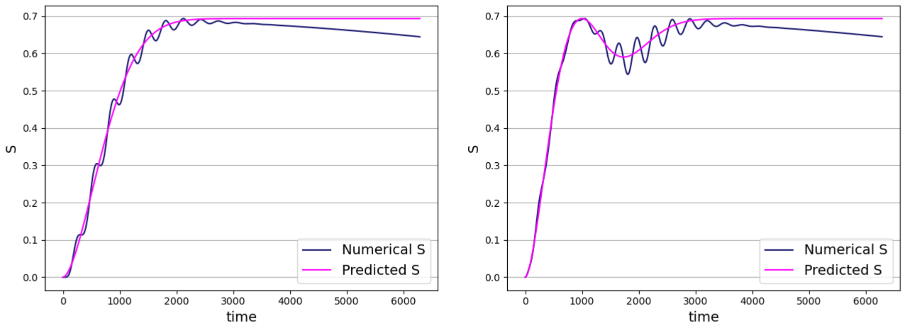
As $\lambda \to 0$, $|\gamma(t)| \to 1$ for all t. This means the total and nodal collapses take an infinitely long time, leaving the atomic inversion and coherences sinusoidal. These are the signature behaviours of qubit dynamics in the semi-classical limit.

2.3 Entanglement Dynamics

Entanglement within a system refers to an inability to express the system as a tensor product of the constituent states. The initial state of the field-qubit system can be expressed using their respective density matrices as $\rho_f \otimes \rho_q$. This is referred to as a separable system as it can be expressed in this way. For $t > 0$, the field and qubit mix with each other, or entangle. The system can no longer be expressed as a separable state once the field substates are no longer identical. In the semi-classical picture, the field will take an infinitely long time to separate, so the field and qubit should never be entangled.

For closed bipartite systems, the Von Neumann entropy is a suitable measure of entanglement [5]. It is defined as

$$S = -\sum_i \beta_i ln\beta_i \tag{10}$$



(a) $|\alpha, 0\rangle$ (b) $|\alpha, 1\rangle$

Figure 3: Von Neumann entropy over 20 Rabi periods for different displaced Fock states. The predictions significantly deviate after total collapse occurs because additional quantum processes, known as revivals, occur which are not taken into account by Eq (3).

where β_i represents an eigenvalue of either reduced density operator for the field and qubit subsystems [6]. The eigenvalues of the qubit density operator can be expressed as

$$\beta_{\pm} = \frac{1}{2}\left[1 \pm \sqrt{\langle\sigma_z\rangle^2 + 4\langle\sigma_+\rangle\langle\sigma_-\rangle}\right] \tag{11}$$

and so these are used (rather than the field's eigenvalues) as the expressions for the relevant terms were derived in Section 2.2 [7]. Substituting Eqs (8) and (9) into Eq (11), the eigenvalues can be expressed as

$$\beta_{\pm} = \frac{1 \pm |\gamma|}{2} \tag{12}$$

This equation does not fully describe the entanglement dynamics, as seen in Figure 3 where unpredicted oscillations arise. These oscillations are not at present understood. As $|\gamma| \to 1$ for all t, resulting in the expression $S = -ln(1) - 0ln(0)$. Both terms in this expression equal zero, hence $S = 0$ for all t which agrees with the semi-classical model [8]. Therefore, simply arguing that semi-classicality arises when $S = 0$ is insufficient. This is particularly relevant as certain quantum phenomena cause disentanglement naturally for short durations [9]. It is instead argued that the average entropy $\tilde{S} \to 0$ for classical fields. As semi-classicality is a limit, and therefore unreachable, it is argued that smaller values of $\tilde{S}$ correspond to a more semi-classical system. The choice of timescale to average over is discussed in Section 3.

3 Entanglement Timescales

3.1 Rabi Period

The first timescale used is the Rabi period $T_{Rabi} \propto \frac{1}{\lambda|\alpha|}$, and therefore inversely proportional to the period of Rabi oscillations (see Eq 8). It has meaning in both the semi-classical and quantum models, so it can be used to test different field states' convergence to classicality. The only changing term is the scale factor $|\gamma| \to 1$ as the semi-classical limit is approached which results in $\tilde{S} \to 0$.

Figure 4 demonstrates this, as $\tilde{S} \to 0$ when $\lambda \to 0$.

3.2 Collapse Time

The collapse time $T_c \propto \frac{1}{\lambda}$. Physically, it defines the approximate point at which the collapse of Rabi oscillations occurs. For simplicity, the collapse time of this report is written as $T_c = \frac{1}{\lambda}$. It removes the dependence of $|\gamma|$ on λ, as shown by writing the final value of γ to be

$$|\gamma_c| = L_n(1)e^{-1} \tag{13}$$

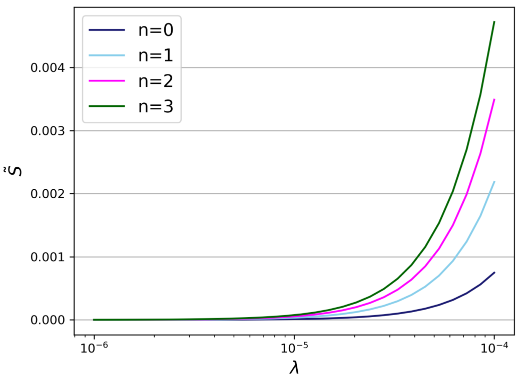


Figure 4: $\tilde{S}$ as a function of λ for different displaced Fock states $|\alpha, n\rangle$. The timescale used is the Rabi period $T_{Rabi} = \lambda|\alpha|$.

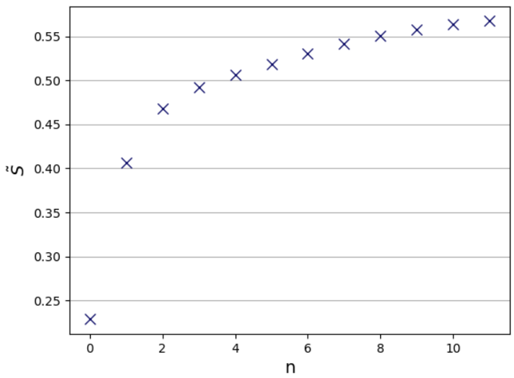


Figure 5: $\tilde{S}$ as a function of n for different displaced Fock states $|\alpha, n\rangle$. The timescale used is $T_c = \frac{1}{\lambda}$.

This time domain therefore resists the semi-classical limit and allows $\tilde{S}$ to be measured directly as a function of n, the quantum number of the displaced Fock state $|\alpha, n\rangle$. Orthogonality for coherent states occurs only when the two field states fully separate. This suggests that $\tilde{S}$ will increase as n increases because the frequency of nodal collapses in a given time period will increase.

This is confirmed in Figure 5. Like Figure 4, the average entanglement increases as n increases but at a slowing rate, which is considered further in Section 3.3.

3.3 Discussion

In Figure 5, $\frac{d\tilde{S}}{dn}$ decreases as n increases. This is due to the increased uncertainty product (phase space width) of the field states being proportional to n, resulting in a larger overlap between the field substates. If the field states did not have positive and negative regions, the average entanglement would instead decrease as n increased, suggesting that inherently quantum properties of the field could manifest as classical when using $\tilde{S}$ as a semi-classicality measure. Whilst this did not happen, it represents an important risk of achieving false positives by assuming smaller $\tilde{S}$ corresponds to a more classical field state. A potential example of this is tue squeezed field state, where the uncertainty product is squeezed to cause a slower total collapse. An additional complication is assuming that all timescales will predict that as the quantum nature of the field increases, $\tilde{S}$ increases. This has been successful over two timescales but it is reasonable to assume an additional timescale could exist that predicts the opposite.

Even if $\tilde{S}$ were a reliable metric of system semi-classicality, this does not mean it is a useful metric for field classicality. As an example, if the qubit's initial state were to be changed such that the value of $\tilde{S}$ decreases, our metric implies the field is in a more classical state despite remaining fixed. This subtle difference in result interpretation is not yet fully understood by the author and so will be an area of future research.

4 Conclusion and Outlook

This study aims to develop an understanding of entanglement behaviour in the Jaynes-Cummings model as it tends to the semi-classical limit. The metric used to quantify the behaviour is the average Von Neumann entropy $\tilde{S}$, where a smaller $\tilde{S}$ is assumed to be a more semi-classical system. This was successfully expressed in terms of field dynamics. The metric agrees with current literature that coherent states are the most classical among displaced Fock states, and that $\tilde{S} \to 0$ in the semi-classical limit for all displaced Fock states [10]. It was found that increased uncertainty widths in phase space result in slower total collapse, an indication that less semi-classical systems could have a lower $\tilde{S}$. The first solution to this is to simultaneously consider field, qubit, and entanglement dynamics to prevent false positives through a single metric. The second solution is to consider more robust entanglement measures. These are not mutually exclusive and a combination of approaches will be used.

The significance of a more classical field not necessarily equalling a more semi-classical system must be further consolidated, and so understanding the entanglement dynamics as the initial qubit state changes is an important next step. An alternative measure of entanglement convergence is the Shannon entropy. Shannon entropy is often considered the classical analogue of Von Neumann entropy, and comparing how the two behave in the semi-classical limit could be a novel way of measuring how quantum the system is [11]. Although more research must be done, it may be a more reliable entanglement measure than considering the Von Neumann entropy alone.

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