Heaven's Light is Our Guide

Computer Science & Engineering
Rajshahi University of Engineering & Technology

Lab Manual

Module- 02

Course Title: Sessional based on CSE 2101

Course No. : CSE 2102

Experiment No. 2

Name of the Experiment: Basic Structures: Sets, Functions, Sequences and Sum

Duration: 2 Cycles

Experiments/Problems:

[1] Given subsets A and B of a set with n elements, use bit strings to find A, A ∪ B, A ∩ B, A − B and A ⊕ B.
[2] Given two finite sets, list all elements in the Cartesian product of these two sets.
[3] Given a finite set, list all elements of its power set.
[4] Given a function f from {1, 2,...,n} to the set of integers, determine whether i) $f(x) = x^2$ ii) $f(x) = x + 1$ iii) $f(x) = x^3 + x^2 + x + 1$ are one-to-one.
[5] Given a function f from {1, 2,...,n} to itself, determine whether i) $f(x) = x^2$ ii) $f(x) = x + 1$ iii) $f(x) = x^3 + x^2 + x + 1$ is onto.
[6] Check $\lfloor 2x \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor$ is true for integer number x = [-100 100] [Note: Check wide range of set if possible]
[7] Find the following summation:
i) $\sum_{n=L}^{U} (a + nd)$ Where L<U, given L, U, a and d.
ii) $\sum_{j=L}^{U} ar^j$ Where L<U, given L, U, a and r.
iii) $\sum_{i=L}^{U} \sum_{j=L}^{U} (i + j)$ Where L<U, given L, U.
[8] Find the value of the following series: $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ...\ ......$
[9] Calculate the value of Pi (π)
[10] Calculate the value of golden ratio

Explain:

[9] **Introduction of PI**

(a) The Gregory-Leibniz Series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \cdots$$

Proof: Start with the Taylor series:

$$\frac{1}{1 + y} = 1 + y + y^2 + y^3 + \ldots$$

Apply the variable substitution $y = -x^2$

$$\frac{1}{1 + x^2} = 1 - x^2 + x^4 - x^6 + \ldots$$

Now since $\frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2}$, by integrating, we find that the Taylor expansion of $\tan^{-1} x$ is

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} ....$$

and the formula is obtained by substituting x=1

$$\tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} ....$$

$$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} ....$$

$$\pi = 4 (1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} .... \quad .... \quad)$$

$$\boxed{\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)}}$$

Variations:

The Gregory-Leibniz Series converges very slowly. One way to improve it is to use

$$\pi = 4 (\frac{2}{1.3} + \frac{2}{5.7} .... \quad .... \quad) = 8 (\frac{1}{1.3} + \frac{1}{5.7} .... \quad .... \quad)$$

$$\boxed{\pi = 8\left(\frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11} + \dots\right) = 8 \sum_{n=0}^{\infty} \frac{1}{(4k+1)(4k+3)}}$$

Even better is

$$\tan^{-1} \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} (1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3} ....)$$

$$\boxed{\frac{\pi}{6} = \frac{1}{\sqrt{3}} (1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3} ....)}$$

(b) Machin's Formula:

$$\pi = 16 \tan^{-1} (\frac{1}{5}) - 4\, an^{-1} (\frac{1}{239})$$

Proof:

First, start with: $\tan \alpha = \frac{1}{5}$

for some angle α that we need not be concerned with since it will disappear from the final result (For the curious, α ≈ 11° 18' 35.8").
Using the trig formula for double angles:

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha} = \frac{\frac{2}{5}}{1 - \frac{1}{25}} = \frac{5}{12}$$

Applying the double angle formula once more:

$$\tan 4\alpha = \frac{2\tan 2\alpha}{1-\tan^2 2\alpha} = \frac{\frac{10}{12}}{1-\frac{25}{144}} = \frac{120}{119}$$

This differs from 1 by only the small value 1/119, but tan π/4 = 1; therefore, 4α differs from π/4 by a small value so that tan (4α − π/4) is small. Indeed, it is just:

$$\tan(4\alpha - \frac{\pi}{4}) = \frac{\tan 4\alpha - \tan(\frac{\pi}{4})}{\tan 4\alpha + \tan(\frac{\pi}{4})} = \frac{\frac{120}{119}-1}{\frac{120}{119}+1} = \frac{1}{239}$$

Taking the $\tan^{-1}$ of both sides results in:

$$4\alpha - \frac{\pi}{4} = \tan^{-1}(\frac{1}{239})$$

$$\frac{\pi}{4} = 4\alpha - \tan^{-1}(\frac{1}{239})$$

$$\frac{\pi}{4} = 4\tan^{-1}(\frac{1}{5}) - \tan^{-1}(\frac{1}{239})$$

$$\boxed{\pi = 16\tan^{-1}(\frac{1}{5}) - 4\tan^{-1}(\frac{1}{239})}$$

Example:

We know, $\tan^{-1}\frac{1}{5} = 1/5 - \frac{(1/5)^3}{3} + \frac{(1/5)^5}{5} - \frac{(1/5)^7}{7}$.......

1/5 = 0·200000000000000
1/375 =0·002666666666666
1/15625 = 0·000064000000000
1/546875 =0·000001828571428
1/17578125 = 0·000000056888889
1/537109375 =0·000000001861818
1/15869140625 = 0·000000000063015
1/457763671875 =0·000000000002184
1/12969970703125 = 0·000000000000077
1/362396240234375=0·000000000000002

So $\tan^{-1}\frac{1}{5}$ = 0·1973955598498807

And
1/239 = 0·004184100418410
1/40955757 =-0·000000024416591
1/3899056325995= 0·000000000000256

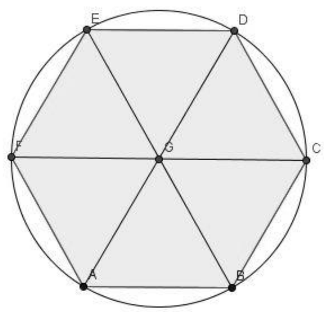
So $\tan^{-1}\frac{1}{239}$ =0·004184076002074

$$\pi = 16 \tan^{-1}\left(\frac{1}{5}\right) - 4 \tan^{-1}\left(\frac{1}{239}\right)$$

= 16 x 0·1973955598498807 – 4 x 0·004184076002074
= **3·1415926535897922**

(c) Archimedes Formula:

Pi has been known as early as 250 B.C. The first method of finding the value of π was made by Archimedes of Syracuse, Italy. He first draws a circle with an inscribed regular hexagon like as shown in the figure.

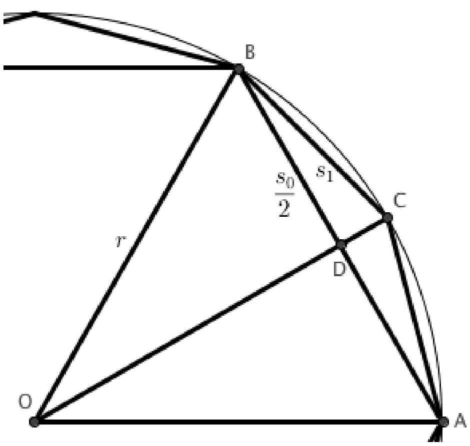


$$\pi = \frac{C}{D}$$

The radius of the circle is 1. Therefore, the length of each side of the hexagon is also 1. He approximated the value of π by the following formula,

$$\pi = \frac{\text{Perimeter of Hexagon}}{\text{diameter of Circle}}$$

$$\pi = \frac{N_0 \times s_0}{2 \text{ x Radius of Circle}} = \frac{6 \times 1}{2 \times 1} = 3$$



Hence s_0 =AB =1, N_0 = 6.
This was the first approximation of the value of π.

From Figure,
$AB \perp OD$, $BD = AD = s_0/2 = ½$, BC = s_1
From ΔODB,
$OB^2 = OD^2 + BD^2$

➔ $OD = a = \sqrt{1 - BD^2} = \sqrt{1 - \left(\frac{s_0}{2}\right)^2}$

➔ CD= b = 1 – a
From ΔBDC,
$BC^2 = BD^2 + CD^2$

➔ BC = s_1 = $\sqrt{\left(\frac{s_0}{2}\right)^2 + b^2}$

Number of Side, $N_1 = 2 \text{ x } N_0$

$s_1 = \sqrt{\left(\frac{s_0}{2}\right)^2 + b^2}$

$N_1 = 2 \text{ x } N_0$

$$\pi = \frac{N_1 \times s_1}{2 \text{ x Radius of Circle}}$$

a = $\sqrt{1 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}$

b = 1-a = $1 - \frac{\sqrt{3}}{2} \cong 0.133974596$

$s_1 \cong 0.51763809$
$N_1 = 12$

$$\pi = \frac{N_1 \times s_1}{2 \text{ x Radius of Circle}} \cong \frac{6.211657082}{2} \cong 3.105828541$$

From s_1 and N_1, we can calculater s_2 and N_2.

s_0	N_0	$\pi = \frac{N_0 \times s_0}{2}$	$a = \sqrt{1 - \left(\frac{s_0}{2}\right)^2}$	$b = 1 - a$	$s_1 = \sqrt{\left(\frac{s_0}{2}\right)^2 + b^2}$	$N_1 = 2 \text{ x } N_0$
1.0000000	6	**3.0000000**	0.8660254	0.1339746	**0.5176381**	6
0.5176381	**12**	**3.1058285**	0.9659258	0.0340742	0.2610524	**12**
0.2610524	24	**3.1326286**	0.9914449	0.0085551	0.1308063	24
0.1308063	48	**3.1393502**	0.9978589	0.0021411	0.0654382	48
0.0654382	96	**3.1410320**	0.9994646	0.0005354	0.0327235	96
0.0327235	192	**3.1414525**	0.9998661	0.0001339	0.0163623	192
0.0163623	384	**3.1415576**	0.9999665	0.0000335	0.0081812	384
0.0081812	768	**3.1415839**	0.9999916	0.0000084	0.0040906	768
0.0040906	1,536	**3.1415905**	0.9999979	0.0000021	0.0020453	1,536
0.0020453	3,072	**3.1415921**	0.9999995	0.0000005	0.0010227	3,072
0.0010227	6,144	**3.1415925**	0.9999999	0.0000001	0.0005113	6,144
0.0005113	12,288	**3.1415926**	1.0000000	0.0000000	0.0002557	12,288
0.0002557	24,576	**3.1415926**	1.0000000	0.0000000	0.0001278	24,576
0.0001278	49,152	**3.1415927**	1.0000000	0.0000000	0.0000639	49,152
0.0000639	98,304	**3.1415927**	1.0000000	0.0000000	0.0000320	98,304
0.0000320	196,608	**3.1415927**	1.0000000	0.0000000	0.0000160	196,608
0.0000160	393,216	**3.1415927**	1.0000000	0.0000000	0.0000080	393,216
0.0000080	786,432	**3.1415927**	1.0000000	0.0000000	0.0000040	786,432
0.0000040	1,572,864	**3.1415927**	1.0000000	0.0000000	0.0000020	1,572,864

(*using Microsoft Excel)

(d) Chudnovsky Brother Formula:
The current darling of the π world is the Chudnovsky algorithm b it converges much quicker. It is also rather complicated. The formula itself is derived from one by Ramanjuan who's work was extraordinary in the extreme. It isn't trivial to prove, so I won't! Here is Chudnovsky's formula for π as it is usually stated:

$$\frac{1}{\pi} = 12 \sum_{k=0}^{\infty} \frac{(-1)^k (6k)!(13591409 + 545140134k)}{(3k)!(k!)^3 640320^{3k+3/2}}$$

First let's get rid of that funny fractional power:

$$\begin{aligned}\frac{1}{\pi} &= 12\sum_{k=0}^{\infty}\frac{(-1)^k(6k)!(13591409+545140134k)}{(3k)!(k!)^3 640320^{3k+3/2}}\\ &= \frac{12}{640320\sqrt{640320}}\sum_{k=0}^{\infty}\frac{(-1)^k(6k)!(13591409+545140134k)}{(3k)!(k!)^3 640320^{3k}}\\ &= \frac{1}{426880\sqrt{10005}}\sum_{k=0}^{\infty}\frac{(-1)^k(6k)!(13591409+545140134k)}{(3k)!(k!)^3 640320^{3k}}\end{aligned}$$

Now let's split it into two independent parts. We can split it into two series which we will call `a` and `b` and work out what π is in terms of `a` and `b`:

$$\begin{aligned} a &= \sum_{k=0}^{\infty}\frac{(-1)^k(6k)!}{(3k)!(k!)^3 640320^{3k}}\\ &= 1-\frac{6\cdot5\cdot4}{(1)^3 640320^3}+\frac{12\cdot11\cdot10\cdot9\cdot8\cdot7}{(2\cdot1)^3 640320^6}-\frac{18\cdot17\cdots13}{(3\cdot2\cdot1)^3 640320^9}+\cdots\\ b &= \sum_{k=0}^{\infty}\frac{(-1)^k(6k)!k}{(3k)!(k!)^3 640320^{3k}}\\ \frac{1}{\pi} &= \frac{13591409a+545140134b}{426880\sqrt{10005}}\\ \pi &= \frac{426880\sqrt{10005}}{13591409a+545140134b}\end{aligned}$$

Finally note that we can calculate the next `a` term from the previous one, and the `b` terms from the terms which simplifies the calculations rather a lot.

$$\begin{aligned} a_k &= \frac{(-1)^k(6k)!}{(3k)!(k!)^3 640320^{3k}}\\ b_k &= k\cdot a_k\\ \frac{a_k}{a_{k-1}} &= -\frac{(6k-5)(6k-4)(6k-3)(6k-2)(6k-1)6k}{3k(3k-1)(3k-2)k^3 640320^3}\\ &= -\frac{24(6k-5)(2k-1)(6k-1)}{k^3 640320^3}\end{aligned}$$

[Ref: http://www.craig-wood.com/nick/articles/pi-chudnovsky/)

(d) David Bailey, Peter Borwein and Simon Plouffe (BBP) Formula:
An interesting new method was recently proposed by David Bailey, Peter Borwein and Simon Plouffe. It can compute the *N*th **hexadecimal** digit of Pi efficiently without the previous *N-1* digits. The method is based on the formula:

$$\pi=\sum_{k=0}^{\infty}\frac{1}{16^k}\left(\frac{4}{8k+1}-\frac{2}{8k+4}-\frac{1}{8k+5}-\frac{1}{8k+6}\right)$$

in *O(N)* time and *O(log N)* space. The following 160 character C program, written by Dik T. Winter at CWI, computes pi to 800 decimal digits.

```
int a=10000,b,c=2800,d,e,f[2801],g;main(){for(;b-c;)f[b++]=a/5;
for(;d=0,g=c*2;c-=14,printf("%.4d",e+d/a),e=d%a)for(b=c;d+=f[b]*a,
f[b]=d%--g,d/=g--,--b;d*=b);}
```

can be rewritten as:

```
#include <stdio.h>

int main() {
    int r[2800 + 1];
    int i, k;
    int b, d;
    int c = 0;

    for (i = 0; i < 2800; i++) {
        r[i] = 2000;
    }

    for (k = 2800; k > 0; k -= 14) {
        d = 0;

        i = k;
        for (;;) {
            d += r[i] * 10000;
            b = 2 * i - 1;

            r[i] = d % b;
            d /= b;
            i--;
            if (i == 0) break;
            d *= i;
        }
        printf("%.4d", c + d / 10000);
        c = d % 10000;
    }

    return 0;
}
```

[Ref: https://crypto.stanford.edu/pbc/notes/pi/code.html]

[10] **Calculate the value of golden ratio (ϕ or φ)**
Two quantities *a* and *b* are said to be in the *golden ratio* φ if

$$\frac{a+b}{a} = \frac{a}{b} = \varphi.$$

One method for finding the value of φ is to start with the left fraction. Through simplifying the fraction and substituting in b/a = 1/φ,

$$\frac{a+b}{a} = 1 + \frac{b}{a} = 1 + \frac{1}{\varphi}.$$

Therefore,

$$1 + \frac{1}{\varphi} = \varphi.$$

Multiplying by φ gives

$$\varphi + 1 = \varphi^2$$

which can be rearranged to

$$\varphi^2 - \varphi - 1 = 0.$$

Using the quadratic formula, two solutions are obtained:

$$\varphi = \frac{1 + \sqrt{5}}{2} = 1.6180339887\ldots$$

and

$$\varphi = \frac{1 - \sqrt{5}}{2} = -0.6180339887\ldots$$

Because φ is the ratio between positive or negative quantities φ is necessarily positive:

$$\varphi = \frac{1 + \sqrt{5}}{2} = \frac{-1 - \sqrt{5}}{-2} = 1.6180339887\ldots.$$

Report:

Your completed work must be submitted through a LAB REPORT.

Read:

[1] Kenneth H. Rosen, "Discrete Mathematics and its Application" , 7th Edition: Chapter 2 (Basic Structures: Sets, Functions, Sequences and Sum).