RAJSHAHI UNIVERSITY OF ENGINEERING AND TECHNOLOGY

RUET

ASSIGNMENT

Submitted to:
Shyla Afroge
Assistant Professor,
Department of Computer Science and Engineering
Rajshahi University of Engineering and Technology

Submitted by:
Riyad Morshed Shoeb
Roll No: 1603013
Section: A
Department of Computer Science and Engineering
Rajshahi University of Engineering and Technology

Chapter-2

Solution to Algebraic and Transcendental Equation

Problem:-2.1: Explain the bisection method for finding a real root of the equation $f(x)=0$ and write an algorithm for its implementation with a test for relative accuracy of the approximation.

Solⁿ: For the given function

$$f(x)=0$$

if there are any two points for which $f(a).f(b)<0$ or, $f(a)$ and $f(b)$ has opposite signs, then there is at least one root between a and b.

Let, $f(a)>0$ and

$f(b)<0$

so, the approximate root of the given function will be,

$$x_0 = \frac{a+b}{2}$$

Algorithm:

But x_0 might not be exact root of the function. For that, some conditions may arise. Using these conditions, most approximate root can be found. The algorithm for this -

(i) if $f(x)<0$, then set $b=x$ and continue calculating.

(ii) if $f(x)>0$, then set $a=x$ and continue calculating

(iii) if $f(x)=0$, then x is the root.

But fore most of the time, $f(x)$ will nevere be exactly zero. So, to find the root $|x_n - x_{n-1}|$ has to be relatively low.

Obtain a root, correct to three decimal places, using bisection method:

Problem 2.2: $x^3-4x-9=0$

Soln: let, $f(x)=x^3-4x-9$

$$a=3 \quad \text{and } b=2$$

$$\therefore f(a)=6 \text{ and } f(b)=-9$$

Hence,

n	a	b	$x_n=\frac{a+b}{2}$	$f(x_n)$
1	3	2	2.5	-3.375
2	3	2.5	2.75	0.796875
3	2.75	2.5	2.625	-1.41211
4	2.75	2.625	2.6875	-0.33911

5	2.75	2.6875	2.71875	0.2209167
6	2.71875	2.6875	2.703125	-0.061077
7	2.71875	2.703125	2.7109375	0.079423
8	2.7109375	2.703125	2.70703125	0.0090492
9	2.70703125	2.703125	2.705078125	-0.0260449
10	2.70703125	2.705078125	2.706054688	-0.00850211
11	2.70703125	2.706054688	2.706543594	0.0002811
12	2.706543594	2.706054688	2.706299141	-0.0041127

So, root of the equation is 2.706.

Problem-2.3: $x^3+x^2-1=0$

Sol^n: Let, $f(x)=x^3+x^2-1$

$$a=1 \text{ and } b=0$$

$$\therefore f(a)=1 \text{ and } f(b)=-1$$

hence,

n	a	b	$x_n=\frac{a+b}{2}$	$f(x_n)$
1	1	0	0.5	-0.625
2	1	0.5	0.75	-0.015625
3	1	0.75	0.875	0.43554
4	0.875	0.75	0.8125	0.196533
5	0.8125	0.75	0.78125	0.087189

6	0.78125	0.75	0.765625	0.034977
7	0.765625	0.75	0.7578125	0.009476
8	0.7578125	0.75	0.7539063	-0.003124
9	0.7578125	0.7539063	0.7558594	0.00316362
10	0.7558594	0.7539063	0.75488285	0.00001669
11	0.75488285	0.7539063	0.754394575	0.00155444

So, root of the given equation is 0.754.

Problem-2.4: $5x\log_{10} x - 6 = 0$

solution: let, $f(x) = 5x\log_{10} x - 6$

$a = 3$ and $b = 2$

$\therefore f(a) = 1.156819$ and $f(b) = -2.9897$

hence,

n	a	b	$x_n = \frac{a+b}{2}$	$f(x_n)$
1	3	2	2.5	-1.02575
2	3	2.5	2.75	0.090825
3	~~0.01~~ 2.75	2.5	2.625	-0.498928
4	2.75	2.625	2.6875	-0.23063
5	2.75	2.6875	2.71875	-0.095293
6	2.75	2.71875	2.734375	-0.027331
7	2.75	2.734375	2.7421875	0.006723

8	2.7421875	2.734375	2.73828125	-0.01031026
9	2.7421875	2.73828125	2.740234375	-0.00179534

Hence, the root is 2.7402.

Problem-2.5: $x^2+x-\cos x=0$

Solⁿ: Let $f(x)=x^2+x-\cos x$

$a=1$ and $b=0$

$\therefore f(a)=1.000152$ and $f(b)=-1$

So,

n	a	b	$x_n=\frac{a+b}{2}$	$f(x_n)$
1	1	0	0.5	-0.249962
2	1	0.5	0.75	0.312586
3	0.75	0.5	0.625	0.015684
4	0.625	0.5	0.5625	-0.121096
5	0.625	0.5625	0.59375	-0.08657
6	0.625	0.59375	0.609375	-0.019231
7	0.625	0.609375	0.6171875	0.0018341

hence, the root is 0.6172.

Problem – 2.6 : Give the sequence of steps in the regula-falsi method for determining a real root of the equation $f(x)=0$.

Sol^n:

Let. a and b are two real numbers for which $f(a)>0$ and $f(b)<0$.

Then. x is the approximate root of the function. where,

$$x = \frac{af(b) - bf(a)}{f(b)-f(a)}$$

if $f(x)$ is less than 0. then set $b=x$ and continue the process.

If $f(x)$ is greater than 0. then set $a=x$ and continue the process.

If $f(x)=0$. then. x is the root of the given function.

Use the method of false-position to find a real root correct to three decimal places, of the following equation-

Problem-2.7: $x^3+x^2+x+7=0$

Sol^n: Let, $f(x)=x^3+x^2+x+7$

$a=-2$ and $b=-3$

$f(a)=1$ and $f(b)=-14$

No.	a	b	$x_n=\frac{af(b)-bf(a)}{f(b)-f(a)}$	$f(x_n)$
1	-2	-3	-2.066667	0.377478
2	-2.066667	-3	-2.091171	0.151142
3	-2.091171	-3	2.099987	0.049130
4	-2.099987	-3	-2.103134	0.017513
5	-2.103134	-3	-2.104254	0.006221
6	-2.104254	-3	-2.104652	0.002220
7	-2.104652	-3	-2.104793	0.000794

Hence, the root is -2.104~~793~~

Problem-2.8: $x^3-x-4=0$

Sol^n: Let, $f(x)=x^3-x-4$

$a=1.6$ and $b=1.8$

$f(a)=-1.504$ and $f(b)=0.032$

n	a	b	$x_n = \frac{af(b)-bf(a)}{f(b)-f(a)}$	$f(x_n)$
1	1.6	1.8	1.795833	-0.00[illegible]
2	1.795833	1.8	1.796320	-0.000010

Hence, the root is 1.796

Problem-2.9: $x = 3e^{-x}$

Sol^n: Let, $f(x) = x - 3e^{-x}$

$a = 1$ and $b = 1.1$

$f(a) = -0.103638$ and $f(b) = 0.101386$

n	a	b	$x_n = \frac{af(b)-bf(a)}{f(b)-f(a)}$	$f(x_n)$
1	1	1.1	1.050549	0.001312
2	1	1.050549	1.049917	0.000016

$\therefore$ Hence the root is 1.05.

Problem-2.10: $x\tan x + 1 = 0$

Sol^n: Let, $f(x) = x\tan x + 1$

$a = 1.5$ and $b = 1.6$

$\therefore f(a) = 22.15213$ and $f(b) = -53.77205$

n	a	b	$x_n = \frac{af(b)-bf(a)}{f(b)-f(a)}$	$f(x_n)$
1	1.5	1.6	1.52654	33.47062
2	1.52654	1.6	1.55409	92.02272
3	1.55409	1.6	1.58267	-134.23626

$\therefore$ Hence, the root is 1.57705.

Problem-2.11: Find the real root, which lies between 2 and 3, of the equation

$$x\log_{10} x - 1.2 = 0$$

using the methods of bi-section and false-position to a tolerance of 0.5%

Soln: $0.5\% = \frac{0.05}{100} = 0.005$

Let, $f(x) = x\log_{10} x - 1.2$

$a = 2.8$ and $b = 2.7$

$\therefore f(a) = 0.05204$ and $f(b) = -0.03532$

using Bisection method:

n	a	b	$x_n = \frac{a+b}{2}$	$f(x_n)$
1	2.8	2.7	2.75	0.00816
2	2.75	2.7	2.725	-0.01363
3	2.75	2.725	2.7375	-0.00274
4	2.75	2.7375	2.74375	0.00271
5	2.74375	2.7375	2.740625	-0.000018
6	2.74375	2.740625	2.7421875	0.00134
7	2.7421875	2.740625	2.74140625	0.000663
8	2.74140625	2.740625	2.741015625	0.000322

Hence the root is 2.741.

using False-position method:

n	a	b	$x_n = \frac{a f(b) - b f(a)}{f(b) - f(a)}$	$f(x_n)$
1	2.8	2.7	2.74043	-0.00019
2	2.8	2.74043	2.74064	-0.000001
3	2.8	2.74064	2.74065	0

Hence, the root is 2.74065.

[Each time, $|x_n - x_{n+1}| \leq 0.005$]

Problem- 2.13: Use the method of Iteration to compute a real root of the following equation-

$$e^x = 3x$$

Solm: ~~Let,~~

~~$f_1(x) = e^x$~~

Given that, $e^x = 3x$

$$\Rightarrow x = \frac{1}{3} e^x = \varphi_1(x)$$

Again, $x = \log_e 3x = \ln 3x = \varphi_2(x)$

n	x_n	$x_{n+1} = \varphi_2(x)$	$\lvert x_{n+1} - x_n \rvert$
1	3.2	2.261763	0.938237
2	2.261763	1.914756	0.347007
3	1.914756	1.748202	0.166554
4	1.748202	1.657200	0.091002
5	1.657200	1.603742	0.053458
6	1.603742	1.570952	0.03279
7	1.570952	1.550294	0.020658
8	1.550294	1.537056	0.013238
9	1.537056	1.528481	0.008575
10	1.528481	1.522887	0.005594
11	1.522887	1.519220	0.003667

12	1.517220	1.516809	0.002411
13	1.516809	1.515221	0.001588
14	1.515221	1.514173	0.001048
15	1.514173	1.513482	0.000691
16	1.513482	1.513025	0.000457
17	1.513025	1.512723	0.000302
18	1.512723	1.512523	0.0002
19	1.512523	1.512392	0.000131

Hence, the root is 1.512.

Problem - 2.14: $x = \frac{1}{(x+1)^2}$

Sol^n: Given that,

$$x = \frac{1}{(x+1)^2} = \varphi(x)$$

n	x_n	$x_{n+1} = \varphi(x)$	$\lvert x_{n+1} - x_n \rvert$
1	0.5	0.444444	0.55556
2	0.444444	0.479289	0.034845
3	0.479289	0.456976	0.022313
4	0.456976	0.471080	0.014104
5	0.471080	0.462091	0.008989
6	0.462091	0.467791	0.0057
7	0.467791	0.464164	0.003627

8	0.46464	0.466466	0.002302
9	0.466466	0.465003	0.001463
10	0.465003	0.465932	0.000929
11	0.465932	0.465342	0.00058
12	0.465342	0.465717	0.000365
13	0.465717	0.465478	0.000239

∴ Hence, the root is 0.465.

Problem-2.15: $1+x^2=x^3$

Solⁿ: Given that,

$$x^3 = x^2+1$$

$$\Rightarrow x^2 = x+\frac{1}{x}$$

$$\Rightarrow x = \sqrt{x+\frac{1}{x}} = \varphi_1(x)$$

again,

$$x^2 = x^3-1$$

$$\Rightarrow x = x^2-\frac{1}{x} = \varphi_2(x)$$

again,

$$x = \sqrt{x^3-1} = \varphi_3(x)$$

again,

$$x^3-x^2=1$$

$$\Rightarrow x^2(x-1) = 1$$

$$\Rightarrow x = \frac{1}{\sqrt{x-1}} = \varphi_4(x)$$

n	x_n	$x_{n+1} = \varphi_1(x)$	$\lvert x_{n+1} - x_n \rvert$
1	1.5	1.471960	0.02804
2	1.471960	1.466740	0.00522
3	1.466740	1.465784	0.000956
4	1.465784	1.465610	0.000174
5	1.465610	1.465578	0.000032

$\therefore$ Hence, the root is 1.466.

Problem-2.16: $x - \sin x = \frac{1}{2}$

Soln: Given that,

$$x - \sin x = \frac{1}{2}$$

$$\Rightarrow x = \frac{1}{2} + \sin x = \varphi_1(x)$$

again,

$$\sin x = x - \frac{1}{2}$$

$$\Rightarrow x = \sin^{-1}\left(x - \frac{1}{2}\right) = \varphi_2(x)$$

n	x_n	$x_{n+1} = \varphi_1(x)$	$\lvert x_{n+1} - x_n \rvert$
1	0.5	0.508726	0.00872
2	0.508726	0.508878	0.000152
3	0.508878	0.508881	0.000003
4	0.508881	0.508881	0

$\therefore$ Hence, the root is 0.509.

Problem- 2.18: Explain Newton-Raphson Method to compute a real root of the equation $f(x)=0$ and find the condition of convergence. Hence, find a non-zero root of the equation $x^2+4\sin x=0$.

Sol^n:

Let x_0 be an approximate root of

$f(x_0)=0$ and

~~$f(x+h)$~~ $x_1=x_0+h$

be the correct root so that $f(x_1)=0$

Expanding $f(x_0+h)$ by Taylor's series, we obtain,

$$f(x_0)+hf'(x_0)+\frac{h^2}{2!}f''(x_0)+\cdots=0$$

Neglecting the second and higher order derivatives, we have,

$$f(x_0)+hf'(x_0)=0$$

$$\Rightarrow h=-\frac{f(x_0)}{f'(x_0)}$$

hence,

$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}$$

For successive approximations,

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$$

condition of convergence:

When $|x_{n+1} - x_n|$ is sufficiently small, x_{n+1} is the root of $f(x) = 0$.

Let, $f(x) = x^2 + 4\sin x$

$\therefore f'(x) = 2x + 4\cos x$

$$\therefore x_{n+1} = x_n - \frac{x^2 + 4\sin x}{2x + 4\cos x}$$

Let, $x_0 = \pi = 3.14159$

n	x_n	x_{n+1}	$f(x_n)$	$f(x_{n+1})$
1	3.14159	-1.18115	9.869598	2.28318
2	-1.18115	-3.91593	-2.30506	-0.84287
3	-3.91593	-2.22004	18.13151	-10.691395
4	-2.22004	-1.969585	1.742238	-6.85839
5	-1.969585	-1.934296	0.173397	-5.471897
6	-1.934296	-1.933754	0.002866	-5.290779
7	-1.933754	-1.933754	0.0000.	-5.287671

Since, $|x_8 - x_7| \leq 0.00001$

Hence, the root is -1.933754.

Use the ~~Rap~~ Newton-Raphson method to obtain a root correct to three decimal places—

Problem – 2.20: $x^{\sin 2} - 4 = 0$

Sol^n:

Let, $f(x) = x^{\sin 2} - 4$

$\therefore f'(x) = \cancel{(\cos 2)}\, x^{(\sin 2)-1} \sin 2$

$$\therefore x_{n+1} = x_n - \frac{x_n^{\sin 2} - 4}{x_n^{(\sin 2)-1}\,\cancel{\cos 2}\,\sin 2}$$

Let, $x_0 = 2$

n	x_n	x_{n+1}
0	2	4.48495
1	4.48495	4.59308
2	4.59308	4.59320

Since, $|x_3 - x_2| < 0.001$,

Hence, the root is 4.5932

Problem – 2.21: $e^x = 4x$

Sol^n: Let, $f(x) = e^x - 4x$

$\therefore f'(x) = e^x - 4$

$$\therefore x_{n+1} = x_n - \frac{e^x - 4x}{e^x - 4}$$

Let, $x_0 = 2$

n	x_n	x_{n+1}
0	2	2.18027
1	2.18027	2.15395
2	2.15395	2.15329

Since, $|x_3 - x_2| < 0.001$

hence, the root is 2.15329.

Problem-2.22: $x^3 - 5x + 3 = 0$

Sol^n: Let $f(x) = x^3 - 5x + 3$

$$\therefore f'(x) = 3x^2 - 5$$

$$\therefore x_{n+1} = x_n - \frac{x^3 - 5x + 3}{3x^2 - 5}$$

Let, $x_0 = 2$

n	x_n	x_{n+1}
0	2	1.8571
1	1.8571	1.83479
2	1.83479	1.83424

Since, $|x_3 - x_2| < 0.001$

Hence, the root is 1.83424

Problem-2.23: $xe^x = \cos x$

Soln: Let, $f(x) = xe^x - \cos x$

$$\therefore f'(x) = e^x + xe^x + \sin x = (1+x)e^x + \sin x$$

$$\therefore x_{n+1} = x_n - \frac{xe^x - \cos x}{e^x(1+x) + \sin x}$$

Let, $x_0 = 2$

n	x_n	x_{n+1}
0	2	1.34157
1	1.34157	0.8477
2	0.8477	0.58756
3	0.58756	0.52158
4	0.52158	0.51777
5	0.51777	0.51776

Since $|x_6 - x_5| < 0.001$,

hence, the root is 0.51776.

Problem-2.24: $x = \frac{1+\cos x}{3}$

Soln: Let, $f(x) = \cancel{\cos x - 3x + 1}$ $3x - \cos x - 1$

$$\therefore f'(x) = 3 + \sin x$$

$$\therefore x_{n+1} = x_n - \frac{3x - \cos x - 1}{3 + \sin x}$$

Let, $x_0 = 2$

n	x_n	x_{n+1}
0	2	0.61455
1	0.61455	0.60711
2	0.60711	0.6071
3	0.6071	0.607102

Since, $|x_4 - x_3| < 0.0001$,

hence, the root is 0.6071.

Problem - 2.25: $\cot x = -x$

Soln: Let, $f(x) = x + \cot x$

$\therefore f'(x) = 1 - \operatorname{cosec}^2 x$

$$\therefore x_{n+1} = x_n - \frac{x + \cot x}{1 - \operatorname{cosec}^2 x}$$

Let, $x_0 = 2$

n	x_n	x_{n+1}
0	2	9.36376
1	9.36376	9.33761
2	9.33761	9.32153
3	9.32153	9.31799
4	9.31799	9.31787

Since, $|x_5 - x_4| < 0.001$,

hence, the root is 9.31787.

Problem - 2.27: Compute, to four decimal places, the root between 1 and 2 of the equation

$$x^3 - 2x^2 + 3x - 5 = 0$$

by (a) Method of False Position and

(b) Newton-Raphson Method

Soln:

(a) Let, $f(x) = x^3 - 2x^2 + 3x - 5$

$a = 2$ and $b = 1$

$\therefore f(a) = 1$ and $f(b) = -3$

n	a	b	x_n	$f(x_n)$
1	2	1	~~1.75~~ 0.875	~~[illegible]~~ 3.235313
2	~~1.75~~ 2	1.875	~~2.50704~~ 1.892103	~~[illegible]~~ -3.20541
3	2 ~~2.50704~~	1.892103	1.89001	-3.20921
4	2	1.89001	1.890272	-3.20874
5	2	1.890272	1.89024	-3.2088

Since, $|x_6 - x_5| < 0.0001$.

hence, the root is 1.89024.

(b) Let, $f(x) = x^3 - 2x^2 + 3x - 5$

$$\therefore f'(x) = 3x^2 - 4x + 3$$

$$\therefore x_{n+1} = x_n - \frac{x^3 - 2x^2 + 3x - 5}{3x^2 - 4x + 3}$$

let, $x_0 = 2$

n	x_n	x_{n+1}
0	2	1.85714
1	1.85714	1.84384
2	1.84384	1.84373
3	1.84373	1.843734

Since $|x_4 - x_3| \leq 0.0001$,

hence the root is ~~1.84.3~~ 1.843734.

Problem – 2.30: Using Ramanujan's Method find the smallest root of the following equation-

$$\sin x + x - 1 = 0$$

Sol^n: Let, $f(x) = \sin x + x - 1 = 0$

$$\Rightarrow \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right] + x - 1 = 0$$

$$\Rightarrow 1 \cancel{+2} - 2x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots = 0$$

$$\Rightarrow 1 - \left[2x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots\right] = 0$$

In case of higher order, the term converges to zero. So, they can be ignored.

$$\therefore f(x) \Rightarrow 1 - \left[2x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + 0\right] = 0$$

$\therefore a_1 = 2 \quad a_2 = 0 \quad a_3 = \frac{-1}{6} \quad a_4 = 0$

$a_5 = \frac{1}{120} \quad a_6 = 0 \quad a_7 = \frac{-1}{5040} \quad a_8 = 0$

$\therefore b_1 = 1$

$b_2 = a_1 b_1 = 2$

$b_3 = a_1 b_2 + a_2 b_1 = 4$

$b_4 = a_1 b_3 + a_2 b_2 + a_3 b_1 = 8 + 0 + \frac{-1}{6} = \frac{47}{6}$

n	b_n	$\frac{b_{n-1}}{b_n}$
4	$\frac{49}{6} = \cancel{8.166667}$ 7.833333	$\cancel{0.489796}$ 0.510638
5	$\cancel{16.666667}$ 15.333333	$\cancel{0.49}$ 0.51087
6	$\cancel{34.008334}$ 30.008333	$\cancel{0.490076}$ 0.510969

Since, $\left| \frac{b_5}{b_6} - \frac{b_4}{b_5} \right| < 0.001$

hence, the root is 0.510969

Ans.

Problem – 2.31: Use the iteration method to determine the root, between 5 and 8, of the equation $x^{2.2} = 69$

Soln: ~~let, $f(x) = x^{2.2} - 69$~~

Given that,

$$x^{2.2} = 69$$

$$\cancel{\Rightarrow 2.2 \ln x = 4.23411}$$

$$\Rightarrow \cancel{\ln x}\; x^{2+0.2} = 69 \Rightarrow x^2 = \frac{69}{x^{0.2}}$$

$$\Rightarrow x = \sqrt{\frac{69}{x^{0.2}}} = \phi(x)$$

let, $x_0 = 5$

n	x_{n-1}	x_n
1	5	7.071761
2	7.071761	6.830803
3	6.830803	6.854525
4	6.854525	6.852149
5	6.852149	6.852387

since, $|x_5 - x_4| < 0.001$, which is very small, hence, the root is 6.852387.

Problem-2.33: Point out the difference between regula-falsi and secant methods for finding a real root of $f(x)=0$. Apply both the methods to find a real root of the equation $x^3-2x-5=0$ to an accuracy of 4 decimal places.

Soln: In case of regula-falsi or false-position method, it needs two initial guess and they bracket the root of the equation. Which means, the root is bin between the two initial guesses. But, in case of secant method there is no such boundary. The initial guesses can boundary the root or both of them can be greater or smaller. And also, in case false position method, if the guesses are a and b then, $f(a)f(b)<0$. But secant method does not have such condition.

Solution regula-falsi:

Let, $a=2.4$ and $b=2$

$\therefore f(a)=4.024$ and $f(b)=-1$

Since, $f(a)f(b)<0$

n	a	b	$x_n=\frac{af(b)-bf(a)}{f(b)-f(a)}$	$f(x_n)$
1	2.4	2	2.079618	-0.165281
2	2.4	2.079618	2.092258	-0.025564
3	2.4	2.092258	2.094201	-0.003913

4	2.4	2.094201	2.094498	-0.000598
5	2.4	2.094498	2.094543	-0.000091
6	2.4	2.094543	2.09455	-0.000014

Since $|f(x_6)| < 0.0001$

hence, the root is 2.09455

Solution using secant method:

let, $x_{n-1} = 1.9$ and $x_n = 2$

$$\therefore x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

n	x_{n-1}	x_n	x_{n+1}
1	1.9	2	2.10627
2	2	2.10627	2.093906
3	2.10627	2.093906	2.094547
4	2.093906	2.094547	2.094551

Since, $|x_5 - x_4| < 0.0001$

hence, the root is 2.094551.

Ans.

Problem-2.21: Use the Newton-Raphson method to obtain a root, correct to three decimal places, of the following equation-

$$e^x = 4x$$

Soln: let $f(x) = e^x - 4x$

$\therefore f'(x) = e^x - 4$

let, $x_0 = 3.2$

$$\therefore x_n = x_{n-1} - \frac{f(x)}{f'(x)}$$

$$= x_{n-1} - \frac{e^x - 4x}{e^x - 4}$$

n	x_{n-1}	x_n
1	3.2	2.628588
2	2.628588	2.289662
3	2.289662	2.168238
4	2.168238	2.153497
5	2.153497	2.153292

Since, $|x_5 - x_4| < 0.001$

hence, the root is 2.153 Ans.

Problem-2.28: Using Ramanujan's Method find the smallest root of the following equation.

$$x^3 - 6x^2 + 11x - 6 = 0$$

Soln: Let, $f(x) = x^3 - 6x^2 + 11x - 6 = 0$

$$\Rightarrow 1 - \frac{11}{6}x + x^2 - \frac{1}{6}x^3 = 0$$

$$\Rightarrow 1 - \left[\frac{11}{6}x - x^2 + \frac{1}{6}x^3\right] = 0$$

$\therefore a_1 = \frac{11}{6}$, $a_2 = -1$, $a_3 = \frac{1}{6}$

and the othere terems of a is 0.

$\therefore b_1 = a_1 = \frac{11}{6} = 1.833333$

$b_2 = a_1 b_1 = \frac{121}{36} = 3.361111$

$b_3 = a_1 b_2 + a_2 b_1 = \frac{935}{216} = 4.328704$

n	b_n	$\frac{b_{n-1}}{b_n}$
3	4.328704	0.776471
4	4.880401	0.886957
5	5.178883	0.942366
6	5.335668	0.970616
7	5.416575	0.985063
8	5.457867	0.992434
9	5.478793	0.996181
10	5.489349	0.998077
11	5.494658	0.999034
12	5.497323	0.999515

Since, $\left|\frac{b_{10}}{b_{11}} - \frac{b_{11}}{b_{12}}\right| < 0.001$

hence, the smallest root is 0.999515

≈ 1

Ans.

Problem-2.29: Using Ramanujan's method find the smallest root of the following equation-

$$x + x^3 - 1 = 0$$

Solⁿ: Given that,

$$x + x^3 - 1 = 0$$

$$\Rightarrow 1 - [x + x^3] = 0$$

$a_1 = 1$, $a_2 = 0$, $a_3 = 1$ and other higher terms of a is 0.

$b_1 = a_1 = 1$

$b_2 = a_1 b_1 = 1$

$b_3 = a_1 b_2 + a_2 b_1 = 1$

$b_4 = a_1 b_3 + a_2 b_2 + a_3 b_1 = 2$

n	b_n	$\frac{b_{n-1}}{b_n}$
4	2	0.5
5	3	0.666667

6	4	0.75
7	6	0.666667
8	9	0.666667
9	13	0.692308
10	19	0.684211
11	28	0.678571
12	41	0.682927
13	60	0.683333
14	88	0.681818
15	129	0.682171
16	189	0.682540
17	277	0.682310
18	406	0.682266

Since, $\left|\frac{b_{16}}{b_{17}} - \frac{b_{17}}{b_{18}}\right| < 0.001$

hence, the root is 0.682

Ans.

Solve the following systemm by Newton-Raphson method,

Problem - 2.43: $x^2 - y^2 = 4$

$x^2 + y^2 = 16$

Soln: Let, $f(x,y) = x^2 - y^2 - 4 = 0$

and, $g(x,y) = x^2 + y^2 - 16 = 0$

$\therefore \frac{df}{dx} = 2x \qquad \frac{df}{dy} = -2y$

$\frac{dg}{dx} = 2x \qquad \frac{dg}{dy} = 2y$

Let, $x_0 = 3$ and $y_0 = 2$

$\therefore \frac{df}{dx_0} = 6 \qquad \frac{df}{dy_0} = -4 \qquad f_0 = 1$

$\frac{dg}{dx_0} = 6 \qquad \frac{dg}{dy_0} = 4 \qquad g_0 = -3$

$\therefore D = \begin{vmatrix} 6 & -4 \\ 6 & 4 \end{vmatrix} = 24 + 24 = 48$

$\therefore h = \frac{1}{D} \begin{vmatrix} -1 & -4 \\ 3 & 4 \end{vmatrix} = \frac{1}{48}(-4 + 12) = \frac{8}{48} = \frac{1}{6}$

$= 0.16667$

$\therefore k = \frac{1}{D} \begin{vmatrix} 6 & -1 \\ 6 & 3 \end{vmatrix} = \frac{1}{48}(18 + 6) = \frac{24}{48} = \frac{1}{2}$

$= 0.5$

$$x_1 = 3 + 0.16667 = 3.16667$$

$$y_1 = 2 + 0.5 = 2.5$$

$$\frac{df}{x_1} = 6.3333 \qquad \frac{df}{dy_1} = -5 \qquad f_1 = -0.2222$$

$$\frac{dg_1}{dx_1} = 6.3333 \qquad \frac{dg}{dy_1} = 5 \qquad g_1 = 0.27778$$

$$D = \begin{vmatrix} 6.3333 & -5 \\ 6.3333 & 5 \end{vmatrix} = 63.3333$$

$$h = \frac{1}{D} \begin{vmatrix} -0.2222 & -5 \\ 0.27778 & 5 \end{vmatrix} = \frac{0.2779}{63.3333}$$

$$= 0.00439$$

$$k = \frac{1}{D} \begin{vmatrix} 6.3333 & -0.2222 \\ 6.3333 & 0.27778 \end{vmatrix} = \frac{3.16652}{63.3333}$$

$$= 0.045$$

$$x_2 = 3.16667 + 0.00439 = 3.17106$$

$$y_2 = 2.5 + 0.045 = 2.545$$

Ans.

Problem-2.45: To find the smallest root of the equation $f(x) = x^3 - x - 1 = 0$ by the iteration method, $f(x) = 0$ should be rewritten as

(a) $x = x^3 - 1$ (b) $x = (x+1)^{1/3}$

(c) $x = \dfrac{1}{x^2 - 1}$ (d) $x = \dfrac{x+1}{x^2}$

Find the correct choice in the above.

Soln: Let, $x_0 = 0.5$

(a) $x = x^3 - 1 = \varphi(x)$

$\therefore \varphi'(x) = 3x^2$

$|\varphi'(x_0)| = |3 \cdot (0.5)^2| = 0.75 < 1$

(b) $x = (x+1)^{1/3} = \varphi(x)$

$$\therefore \varphi'(x) = \frac{1}{3\sqrt[3]{(x+1)^2}}$$

$$\therefore |\varphi'(x_0)| = \left| \frac{1}{3\sqrt[3]{(0.5+1)^2}} \right| \approx 0.254381 < 1$$

(c) $x = \dfrac{1}{x^2 - 1} = \varphi(x)$

$$\therefore \varphi'(x) = \frac{-2x}{(x^2-1)^2}$$

$$\therefore |\varphi'(x_0)| = \left| \frac{-2(0.5)}{[(0.5)^2-1]^2} \right| = 1.777778 > 1$$

(d) $x = \frac{x+1}{x^2} = \varphi(x)$

$$\therefore \varphi'(x) = \frac{d}{dx}\left[(x+1) \cdot x^{-2}\right]$$

$$= x^{-2} + (x+1)(-2x^{-3})$$

$$= \frac{1}{x^2} - \frac{2(x+1)}{x^3}$$

$$\therefore |\varphi'(x_0)| = \left| \frac{1}{(0.5)^2} - \frac{2(0.5+1)}{(0.5)^3} \right| = 20 > 1$$

There are two choices [(a) and (b)] that are right for iteration method procedure for the given equation.

But, in case of (b), $|\varphi'(x_0)|$ is much smaller. So, using this choice, root of the equation will be found quicker.

Hence, the ~~most~~ correct choice is option (b).

Ans.

Problem-3.7: Find the polynomial which approximates the following values—

x	3	4	5	6	7	8	9
y	13	21	31	43	57	73	91

if the number 31 is the fifth term of the series, find the first and tenth term of the series.

Soln: The Difference table—

x_n	y_n	Δy_n	$\Delta^2 y_n$	$\Delta^3 y_n$	$\Delta^4 y_n$	$\Delta^5 y_n$	$\Delta^6 y_n$
3	13						
4	21	8					
5	31	10	2				
6	43	12	2	0			
7	57	14	2	0	0		
8	73	16	2	0	0	0	
9	91	18	2	0	0	0	0

using Newton's foreward difference interpolation formula—

$$y_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0$$

where, $p = \frac{x - x_0}{h}$ and $h = 1$

when $x=1$, $p=-2$

$$\therefore y(1)=13+(-2)\times 8+\frac{(-2)(-2-1)}{2!}\times 2$$

$$=13-16+\frac{2\times 3\times 2}{2}$$

$$=-3+6$$

$$=3$$

Ans.

using Newton's backward difference interpolation formula -

$$y_n(x)=y_n+p\nabla y_n+\frac{p(p+1)}{2!}\nabla^2 y_n$$

where, $p=\frac{x-x_n}{h}$ and $h=1$

when, $x=10$, $p=1$

$$\therefore y_n(10)=91+1\times 18+\frac{1\times(1+2)}{2!}\times 2$$

$$=91+18+2$$

$$=111$$

Ans.

Problem-3.8: Find $f(0.23)$ and $f(0.29)$ from the following table-

x	0.20	0.22	0.24	0.26	0.28	0.30
$f(x)$	1.6596	1.6698	1.6804	1.6912	1.7024	1.7139

Solⁿ: The difference table-

Let, $y = f(x)$

x_n	y_n	Δy_n	$\Delta^2 y_n$	$\Delta^3 y_n$	$\Delta^4 y_n$	$\Delta^5 y_n$
0.20	1.6596					
0.22	1.6698	0.0102				
0.24	1.6804	0.0106	0.0004			
0.26	1.6912	0.0108	0.0002	-0.0002		
0.28	1.7024	0.0112	0.0004	0.0002	0.0004	
0.30	1.7139	0.0115	0.0003	-0.0001	-0.0003	-0.0007

here, $h = 0.02$

from, Gauss' Central Difference forward interpolation method,

$$y_p = y_0 + G_1 \Delta y_0 + G_2 \Delta^2 y_{-1} + G_3 \Delta^3 y_{-1}$$

where, $G_1 = P$

$$G_2 = \frac{P(P-1)}{2!}$$

$$G_3 = \frac{(P+1)P(P-1)}{3!}$$

fore $f(0.23)$

$$p = \frac{0.23-0.26}{0.02} = -1.5$$

$$G_1 = -1.5$$

$$G_2 = \frac{-1.5\times(-1.5-1)}{2!} = 1.875$$

$$G_3 = \frac{(-1.5+1)(-1.5)(-1.5-1)}{3!} = -0.3125$$

$$\therefore f(0.23) = 1.6912 + (-1.5)(0.0108) + (1.875)(0.0002) + (-0.3125)(-0.0002)$$

$$= 1.6912 + -0.0162 + 0.000375 + 0.0000625$$

$$= 1.6754375$$

Ans.

fore $f(0.29)$

$$p = \frac{0.29-0.26}{0.02} = 1.5$$

$$G_1 = 1.5$$

$$G_2 = \frac{(1.5)(1.5-1)}{2!} = 0.375$$

$$G_3 = \frac{(1.5+1)(1.5)(1.5-1)}{3!} = 0.3125$$

$$\therefore f(0.29) = 1.6912 + (1.5)(0.0108) + (0.375)(0.0002) + (0.3125)(-0.0002)$$

$$= 1.6912 + 0.0162 + 0.000075 \mp 0.0000625$$

$$= 1.7074125$$

Ans.

Problem - 3.10: From the table of cubes given below, find $(6.36)^3$ and $(6.61)^3$.

x	6.1	6.2	6.3	6.4	6.5	6.6	6.7
x^3	226.981	238.328	250.047	262.144	274.625	287.496	300.763

Sol^n: The difference table:

x_n	$y_n = x_n^3$	Δy_n	$\Delta^2 y_n$	$\Delta^3 y_n$	$\Delta^4 y_n$	$\Delta^5 y_n$	$\Delta^6 y_n$
6.1	226.981						
6.2	238.328	11.347					
6.3	250.647	12.319	0.972				
6.4	262.144 y_0	11.497 Δy_0	-0.822	-1.794			
6.5	274.625	12.481	0.989 $\Delta^2 y_0$	1.806	3.6		
6.6	287.496	12.871	0.39	-0.594 $\Delta^3 y_0$	-2.4 $\Delta^4 y_0$	-6	
6.7	300.763	13.267	0.396	0.006	0.6	3 $\Delta^5 y_0$	9 $\Delta^6 y_0$

here, $h = 0.1$ and $x_0 = 6.4$

from Gauss's central Difference backward interpolation -

$$y_p = y_0 + G_1 \Delta y_{-1} + G_2 \Delta^2 y_{-1} + G_3 \Delta^3 y_{-2}$$

where, $G_1 = p$

$$G_2 = \frac{p(p+1)}{2!}$$

$$G_3 = \frac{(p-1)p(p+1)}{3!}$$

when $x = 6.36$

$$p = \frac{6.36 - 6.4}{0.1} = -0.4$$

$$G_1 = -0.4$$

$$G_2 = \frac{(-0.4)(-0.4+1)}{2!} = -0.12$$

$$G_3 = \frac{(-0.4-1)(-0.4)(-0.4+1)}{3!} = 0.056$$

$\therefore (6.36)^3 = 262.144 + (-0.4)(11.497) + (-0.12)(-0.822) + (0.056)(-1.794)$

$= 262.144 - 4.5988 + 0.09864 - 0.100464$

$= 257.543376$

Ans.

when $x = 6.61$

$$p = \frac{6.61 - 6.4}{0.1} = 2.1$$

$$G_1 = 2.1$$

$$G_2 = \frac{(2.1)(2.1+1)}{2!} = 3.255$$

$$G_3 = \frac{(2.1-1)(2.1)(2.1+1)}{3!} = 1.1935$$

$\therefore (6.61)^3 = 262.144 + (2.1)(11.497) + (3.255)(-0.822) + (1.1935)(-1.794)$

$= 262.144 + 24.1437 - 2.67561 - 2.141139$

$= 281.470951$

Ans.

Problem – 3.13: Find the missing terms in the following –

x	0	5	10	15	20	25	30
y	1	3	?	73	225	?	1153

Soln: Using Lagrange's interpolation formula –

$$y(10) = \frac{(10-5)(10-15)(10-20)\cancel{(10-25)}(10-30)}{(0-5)(0-15)(0-20)\cancel{(0-25)}(0-30)} \times 1 +$$

$$\frac{(10-0)(10-15)(10-20)\cancel{(10-25)}(10-30)}{(5-0)(5-15)(5-20)\cancel{(5-25)}(5-30)} \times 3 +$$

$$\frac{(10-0)(10-5)(10-20)\cancel{(10-25)}(10-30)}{(15-0)(15-5)(15-20)\cancel{(15-25)}(15-30)} \times 73 +$$

$$\frac{(10-0)(10-5)(10-15)\cancel{(10-25)}(10-30)}{(20-0)(20-5)(20-15)\cancel{(20-25)}(20-30)} \times 225 +$$

$$\frac{(10-0)(10-5)(10-15)(10-20)}{(30-0)(30-5)(30-15)(30-20)} \times 1153$$

$$= \frac{-5000}{45000} \times 1 + \frac{-10000}{-18750} \times 3 + \frac{10000}{11250} \times 73 +$$

$$\frac{5000}{-15000} \times 225 + \frac{2500}{112500} \times 1153$$

$$= \frac{-1}{9} + \frac{8}{5} + \frac{584}{9} - 75 + \frac{1153}{45}$$

$$= 17$$

Ans.

$$y(25) = \frac{(25-5)(25-15)(25-20)(25-30)}{(0-5)(0-15)(0-20)(0-30)} \times 1 +$$

$$\frac{(25-0)(25-15)(25-20)(25-30)}{(5-0)(5-15)(5-20)(5-30)} \times 3 +$$

$$\frac{(25-0)(25-5)(25-20)(25-30)}{(15-0)(15-5)(15-20)(15-30)} \times 73 +$$

$$\frac{(25-0)(25-5)(25-15)(25-30)}{(20-0)(20-5)(20-15)(20-30)} \times 225 +$$

$$\frac{(25-0)(25-5)(25-15)(25-20)}{(30-0)(30-5)(30-15)(30-20)} \times 1153$$

$$= \frac{-5000}{45000} \times 1 + \frac{-6250}{-18750} \times 3 + \frac{-12500}{11250} \times 73 +$$

$$\frac{-25000}{-15000} \times 225 + \frac{25000}{112500} \times 1153$$

$$\cancel{= \frac{-1}{9} + 1 - \frac{730}{9} + \frac{1125}{4} + \frac{1153}{4}}$$

$$= \frac{-1}{9} + 1 - \frac{730}{9} + 375 + \frac{2306}{9}$$

$$= 551$$

Ans.

Problem-3.15: certain values of x and $f(x)$ are given below. Find $f(1.235)$.

x	1.00	1.05	1.10	1.15	1.20	1.25
$f(x)$	0.682689	0.706282	0.728668	0.749856	0.769861	0.788700

Soln: Let, $y = f(x)$

The difference table-

x_n	y_n	Δy_n	$\Delta^2 y_n$	$\Delta^3 y_n$	$\Delta^4 y_n$	$\Delta^5 y_n$
1.00	0.682689					
1.05	0.706282	0.023593				
1.10	0.728668	0.022386	-0.001207			
1.15	0.749856	0.021188	-0.001198	0.000009		
1.20	0.769861	0.020005	-0.001183	0.000015	0.000006	
1.25	0.788700	0.018839	-0.001166	0.000017	0.000002	-0.000004

here, $h = 0.05$

From, Gauss' central Difference forward interpolation formula,

$$y(x) = y_0 + G_1 \Delta y_0 + G_2 \Delta^2 y_{-1} + G_3 \Delta^3 y_{-1} \quad \text{and,}$$

$$G_1 = P$$

$$G_2 = \frac{P(P-1)}{2!}$$

$$G_3 = \frac{P(P+1)(P-1)}{3!}$$

for $f(1.235)$, $p = \dfrac{1.235 - 1.15}{0.05} = 1.7$

$\therefore G_1 = 1.7$

$G_2 = 0.595$

$G_3 = 0.5355$

$$\therefore f(1.235) = \overset{0.749856}{\cancel{1.15}} + (1.7) \times (0.021188) + (0.595)(-0.001198) + (0.5355)(0.000009)$$

$$= 0.749856 + \cancel{0.012607}\ 0.0360196 - 0.000713 + 0.0000048$$

$$= 0.7851674$$

Ans.

Problem-3.18: State Gauss's backward formula and use it to find the value of $\sqrt{12525}$ given that $\sqrt{12500} = 111.8034$, $\sqrt{12510} = 111.8481$, $\sqrt{12520} = 111.8928$, $\sqrt{12530} = 111.9375$ and $\sqrt{12540} = 111.9822$.

Soln: Gauss's central difference backward interpolation formula—

$$y(x) = y_0 + G_1 \Delta y_{-1} + G_2 \Delta^2 y_{-1} + G_3 \Delta^3 y_{-2} + G_4 \Delta^4 y_{-2} + \cdots$$

where, $G_1 = p$

$$G_2 = \frac{p(p+1)}{2!}$$

$$G_3 = \frac{p(p+1)(p-1)}{3!}$$

$$G_4 = \frac{p(p+1)(p-1)(p+2)}{4!}$$

and, $p = \frac{x - x_0}{h}$, $h = x_n - x_{n-1}$

The difference table for given data—

x_n	y_n	Δy_n	$\Delta^2 y_n$	$\Delta^3 y_n$	$\Delta^4 y_n$
12500	111.8034	0.0447			
12510	111.8481	0.0447	0	0	
12520	111.8928	0.0447	0	0	0
12530	111.9375	0.0447	0		
12540	111.9822				

assuming, $y = \sqrt{x}$

$h = 10$

for $\sqrt{12525}$, $p = \dfrac{12525 - 12520}{10} = 0.5$

$$\sqrt{12525} = 111.8928 + (0.5)(0.0447)$$
$$= 111.8928 + 0.02235$$
$$= 111.91515$$

Ans.

Problem – 3.19: State Stirling's formula for interpolation at the middle of a table of values and find $e^{1.91}$ from the following table–

x	1.7	1.8	1.9	2.0	2.1	2.22
e^x	5.4739	6.0496	6.6859	7.3891	8.1662	9.0250

Sol^n: Stirling's formula for interpolation –

$$y_p = y_0 + \frac{\Delta y_{-1} + \Delta y_0}{2} + \frac{p^2 \Delta^2 y_{-1}}{2} + \frac{p(p^2-1)}{3!}\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}$$
$$+ \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

where, $p = \dfrac{x - x_0}{h}$ and $h = x_n - x_{n-1}$

The difference table for the given data–

x_n	$y_n = e^{x_n}$	Δy_n	$\Delta^2 y_n$	$\Delta^3 y_n$	$\Delta^4 y_n$	$\Delta^5 y_n$
1.7	5.4739					
1.8	6.0496	0.5757				
1.9	6.6859	0.6363	0.0606			
2.0	7.3891	0.7032	0.0669	0.0063		

2.1	8.1662	0.7771	0.0739	0.0070	0.0007	
2.2	9.0250	0.8588	0.0817	0.0078	0.0008	0.0001

for $e^{1.91}$ $\quad p = \dfrac{1.91 - 2.0}{0.1} = -0.9$ as $h = 0.1$

$$\therefore e^{1.91} = 7.3891 + \frac{0.6363 + 0.7032}{2} + \frac{(-0.9)^2}{2}(0.0669) + \cancel{= 7.3891 + 0.66} + \frac{(-0.9)(0.9^2-1)}{6}\;\frac{0.0063+0}{2}$$

$$= 7.3891 + 0.66975 + 0.0270945 + 0.00008 9775$$

$$= 8.08603$$

Ans.

Problem – 3.20: Using Stirling's formula find $\cos(0.17)$, given that $\cos(0) = 1$, $\cos(0.05) = 0.9988$, $\cos(0.10) = 0.9950$, $\cos(0.15) = 0.9888$, $\cos(0.20) = 0.9801$, $\cos(0.25) = 0.9689$ and $\cos(0.30) = 0.9553$.

Soln: The difference table for given data – assuming, $y = \cos(x)$

x_n	y_n	Δy_n	$\Delta^2 y_n$	$\Delta^3 y_n$	$\Delta^4 y_n$	$\Delta^5 y_n$	$\Delta^6 y_n$
0	1						
0.05	0.9988	-0.0012					
0.10	0.9950	-0.0038	-0.0026				
0.15	0.9888	-0.0062	-0.0024	0.0002			
0.20	0.9801	-0.0087	-0.0025	-0.0001	-0.0003		

0.25	0.9689	-0.0112	-0.0025	0	0.0001	0.0004	
0.30	0.9553	-0.0136	-0.0024	0.0001	0.0001	0	-0.0004

here, $h = 0.05$

for, $\cos(0.17)$, $p = \dfrac{0.17 - 0.15}{0.05} = 0.4$

$$\therefore \cos(0.17) = 0.9888 + \frac{(-0.0062 - 0.0087)}{2} + \frac{(0.4)^2}{2}(-0.0024) + \frac{(0.4)(0.4^2 - 1)}{6}\,\frac{(-0.0001 + 0.0002)}{2}$$

$$= 0.9888 - 0.00745 - 0.000192 - 0.0000028$$

$$= 0.9811552$$

Ans.

Problem- 3.22: The complete eliptic integral of the second kind is defined as,

$$K(m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}}$$

Find $K(0.25)$ given that,

$K(0.20) = 1.6596$, $K(0.22) = 1.6698$, $K(0.24) = 1.6806$

$K(0.26) = 1.6912$, $K(0.28) = 1.7624$, $K(0.30) = 1.7139$

Soln: Let, $y = K(m)$

The difference table for given data

m_n	y_n	Δy_n	$\Delta^2 y_n$	$\Delta^3 y_n$	$\Delta^4 y_n$	$\Delta^5 y_n$
0.20	1.6596					
0.22	1.6698	0.0102				
0.24	1.6806	0.0108	0.0006			
0.26	1.6912	0.0106	-0.0002	-0.0008		
0.28	1.7024	0.0112	0.0006	0.0008	0.0016	
0.30	1.7139	0.0115	0.0003	-0.0003	-0.0011	-0.0027

here, $h = 0.02$

for $K(0.25)$, $p = \dfrac{0.25 - 0.20}{0.02} = 2.5$

using Newton's forward difference interpolation,

$$K(0.25) = 1.6596 + (2.5)(0.0102) + \frac{(2.5)(1.5)(0.0006)}{2!} +$$

$$\frac{(2.5)(1.5)(0.5)(-0.0008)}{3!} + \frac{(2.5)(1.5)(0.5)(-0.5)(0.0016)}{4!}$$

$$+ \frac{(2.5)(1.5)(0.5)(-0.5)(-1.5)(-0.0027)}{5!}$$

$$= 1.6596 + 0.0255 + 0.001125 - 0.00025 - 0.0000625$$

$$- 0.00003164062$$

$$= 1.685881$$

Ans.

Problem-3.25: find $\cos(12.5°)$ given that,
$\cos(0°)=1$, $\cos(5°)=0.9962$, $\cos(10°)=0.9848$,
$\cos(15°)=0.9658$, $\cos(20°)=0.9397$

Soln: Let $y=\cos(x)$

The difference table for the given data-

x_n	y_n	Δy_n	$\Delta^2 y_n$	$\Delta^3 y_n$	$\Delta^4 y_n$
0°	1				
5°	0.9962	-0.0038			
10°	0.9848	-0.0114	-0.0076		
15°	0.9658	-0.0190	-0.0076	0	
20°	0.9397	-0.0261	-0.0071	0.0005	0.0005

here, $h=5°$, $x_0=10°$

for $\cos(12.5°)$, $p=\frac{12.5°-10°}{5°}=0.5$

from Gauss' central difference foreward interpolation,

$$y(x)=y_0+G_1\Delta y_0+G_2\Delta^2 y_{-1}+G_3\Delta^3 y_{-1}$$

where, $G_1=p=0.5$

$$G_2=\frac{p(p-1)}{2!}=\frac{(0.5)(-0.5)}{2}=-0.125$$

$$G_3=\frac{p(p+1)(p-1)}{3!}=-0.0625$$

$\therefore \cos(12.5°)=0.9848+(0.5)(-0.0190)+(-0.125)(-0.0076)$
$+(-0.0625)\times 0$

$= 0.9848 - 0.0095 + 0.00095 - 0$

$= 0.97625$

Ans.

Problem–3.27: State Lagrange's interpolation formula and find a bound for the error in linear interpolation.

Problem–3.26: Evaluate $f(25)$ from the set of values-

$f(20) = 2859,\ f(24) = 3162,\ f(28) = 3544$

$f(32) = 3992$

Soln: From Lagrange's interpolation formula-

$$f(25) = \frac{(1)(-3)(-7)}{(-4)(-8)(-12)} \times 2854 + \frac{(5)(-3)(-7)}{(4)(-4)(-8)} \times 3162 +$$

$$\frac{(5)(1)(-7)}{(8)(4)(-4)} \times 3544 + \frac{(5)(1)(-3)}{(12)(8)(4)} \times 3992$$

$$= -\frac{9989}{64} + \frac{166005}{64} + \frac{15505}{16} - \frac{2495}{16}$$

$= 3251$

Ans.

Problem-3.29: Find $y(2)$ from the following data using Lagrange's formula-

x	0	1	3	4	5
y	0	1	81	256	625

Soln: From Lagrange's interpolation formula-

$$y(2) = \frac{(2-1)(2-3)(2-4)(2-5)}{(-1)(-3)(-4)(-5)} \times 0 + \frac{(2)(-1)(-2)(-3)}{(1)(-2)(-3)(-4)} \times 1 +$$

$$\frac{(2)(1)(-2)(-3)}{(3)(2)(-1)(-2)} \times 81 + \frac{(2)(1)(-1)(-2)}{(4)(3)(1)(-1)} \times 256 +$$

$$\frac{(2)(1)(-1)(-2)}{(5)(4)(2)(1)} \times 625$$

$$= 0 + 0.5 + 81 - \frac{256}{3} + \frac{125}{2}$$

$$= 58.6667$$

Ans.

Problem-3.30: Let the values of the function $y = \sin x$ be tabulated at the abscissae 0, $\pi/4$ and $\pi/2$. If the Lagrange polynomial $L_2(x)$ is fitted to this data, find a bound for the error in the interpolated value.

Problem: Find a cubic polynomial which fits the data

$(-2,-12), (-1,-8), (2,3)$ and $(3,5)$

Sol^n: Using Lagrange's interpolation formula

$$y(x) = \frac{(x+1)(x-2)(x-3)}{(-1)(-4)(-5)}(-12) + \frac{(x+2)(x-2)(x-3)}{(1)(-3)(-4)}(-8) +$$

$$\frac{(x+2)(x+1)(x-3)}{(4)(3)(-1)}(3) +$$

$$\frac{(x+2)(x+1)(x-2)}{(5)(4)(1)}5$$

$$= \frac{(x+1)(x^2-5x+6)}{(-20)}(-12) + \frac{(x^2-4)(x-3)}{12}(-8) +$$

$$\frac{(x^2+3x+2)(x-3)}{-12}3 + \frac{(x+1)(x^2-4)}{20}5$$

$$= \frac{3}{5}(x^3-5x^2+6x+x^2-5x+6) -$$

$$\frac{2}{3}(x^3-3x^2-4x+12) +$$

$$-\frac{1}{4}(x^3+3x^2+2x-3x^2-9x-6) +$$

$$\frac{1}{4}(x^3-4x+x^2-4)$$

$$= \frac{3}{5}(x^3-4x^2+x+6) - \frac{2}{3}(x^3-3x^2-4x+12)$$

$$= \frac{1}{4}(x^3 + x^2 - \ldots - 7)$$

= [illegible]

$$= \frac{26x^3 - 9x^2 + 16x - 369}{60}$$

Ans.

Problem-3.44: Values of x and $\sqrt[3]{x}$ are given below-

(51, 3.708), (55, 3.803), (57, 3.848) Find x when $\sqrt[3]{x} = 3.780$

Soln: from Lagrange's interpolation formula

$$y(x) = \frac{(x-55)(x-57)}{(-4)(-6)} \times 3.708 + \frac{(x-51)(x-57)}{(4)(-2)} \times 3.803 + \frac{(x-51)(x-55)}{6 \times 2} \times 3.848$$

$$= \frac{x^2 - 57x - 55x + 3135}{24} \times 3.708 - \frac{x^2 - 51x - 57x + 2907}{8} \times 3.803 + \frac{x^2 - 55x - 51x + 2805}{12} \times 3.848$$

$$= (x^2 - 112x + 3135) \times 0.1545 - (x^2 - 108x + 2907) \times 0.475375 + (x^2 - 106x + 2805) \times 0.360667$$

$$= (0.1545 - 0.475375 + 0.360667)x^2 + (-17.304 + 51.3405 - 38.230702)x +$$

$(484.3575 - 1381.915125 + 1011.670935)$

$\Rightarrow y(x) = \sqrt[3]{x} = 0.039792x^2 - 4.194202x + 114.11331$

$\Rightarrow 3.780 = 0.039792x^2 - 4.194202x + 114.11331$

$\Rightarrow 0.039792x^2 - 4.194202x + 110.33331 = 0$

$$\Rightarrow x = \frac{4.194202 \pm \sqrt{17.591330 - 17.561532}}{0.079584}$$

$$\Rightarrow x = \frac{4.194202 \pm 0.172621}{0.079584}$$

$\Rightarrow x = 54.870615$ or, 50.532532

Since, two different real numbers cannot have same cubic root and

$\sqrt[3]{54.870615} \approx 3.780$

$\therefore x = 54.80615$

Ans.

Chapter-4

Problem-4.1: Explain the method of least squares to fit a straight line of the form $Y = a_0 + a_1 x$ to the data (x_i, y_i):-

x	1	2	3	4	5	6
y	2.4	3.1	3.5	4.2	5.0	6.0

Solution: Let $Y = a_0\cancel{x} + a_1 x$ is to be fitted in a straight line for the given data.

$$\therefore S = \left[y_1 - (a_0\cancel{x_1} + a_1 x_1)\right]^2 + \left[y_2 - (a_0\cancel{x_2} + a_1 x_2)\right]^2 + \cdots$$

for S to be minimum, we have,

$$\frac{dS}{da_0} = 0 \quad \text{and} \quad \frac{dS}{da_1} = 0$$

$$\frac{dS}{da_0} = 2\left[y_1 - (a_0\cancel{x_1} + a_1 x_1)\right](-\cancel{x_1}) + 2\left[y_2 - (a_0\cancel{x_2} + a_1 x_2)\right](-\cancel{x_2}) + \cdots$$

$$\Rightarrow -2\cancel{x_1}\left[y_1 - (a_0\cancel{x_1} + a_1 x_1)\right] - 2\cancel{x_2}\left[y_2 - (a_0\cancel{x_2} + a_1 x_2)\right] - \cdots = 0$$

$$\Rightarrow \cancel{x_1}\left[y_1 - (a_0\cancel{x_1} + a_1 x_1)\right] + \cancel{x_2}\left[y_2 - (a_0\cancel{x_2} + a_1 x_2)\right] + \cdots = 0$$

$$\Rightarrow \cancel{x_1}y_1 + \cancel{x_2}y_2 + \cdots = (a_0 + a_0 + \cdots) + a_1 x_1 + a_1 x_2 + \cdots$$

$$\Rightarrow \sum_{i=1}^{m} y_1 = m a_0 + a_1 \sum_{i=1}^{m} x_i$$

again $\frac{\partial S}{\partial a_1} = 0$

$\Rightarrow 2[y_1 - (a_0 + a_1 x_1)](-x_1) + 2[y_2 - (a_0 + a_1 x_2)](-x_2) + \cdots = 0$

$\Rightarrow x_1[y_1 - (a_0 + a_1 x_1)] + x_2[y_2 - (a_0 + a_1 x_2)] + \cdots = 0$

$\Rightarrow x_1 y_1 + x_2 y_2 + \cdots = (a_0 x_1 + a_0 x_2 + \cdots) + (a_1 x_1^2 + a_1 x_2^2 + \cdots)$

$\Rightarrow \sum_{i=1}^{m} x_i y_i = a_0 \sum_{i=1}^{m} x_i + a_1 \sum_{i=1}^{m} x_i^2$

For the given values —

x	y	xy	x^2
1	2.4	2.4	1
2	3.1	6.2	4
3	3.5	10.5	9
4	4.2	16.8	16
5	5.0	25	25
6	6.0	36	36
$\Sigma x = 21$	$\Sigma y = 24.2$	$\Sigma xy = 96.9$	$\Sigma x^2 = 91$

here, $m = 6$

$\therefore 6a_0 + 21a_1 = 24.2$

$21a_0 + 91a_1 = 96.9$

using calculator,

$a_0 = 1.5933 \quad a_1 = 0.6971$ Ans.

Problem-4.2: Find the values of a_0 and a_1 so that $Y = a_0 + a_1 x$ fits the data given in the table—

x	0	1	2	3	4
y	1.0	2.9	4.8	6.7	8.6

Soln: For the given data—

x	y	xy	x^2
0	1.0	0	0
1	2.9	2.9	1
2	4.8	9.6	4
3	6.7	20.1	9
4	8.6	34.4	16
$\Sigma x = 10$	$\Sigma y = 24$	$\Sigma xy = 67$	$\Sigma x^2 = 30$

here, $m = 5$

$\therefore 5a_0 + 10a_1 = 24$

$10a_0 + 30a_1 = 67$

using calculator,

$a_0 = 1$ $a_1 = 1.9$

Ans.

Problem - 4.4: Use the method of Least squares to fit the straight line $Y = a + bx$ to the data -

x	0	1	2	3
y	2	5	8	11
W	1	1	1	1

Soln: For the given data -

x	y	xy	x^2
0	2	0	0
1	5	5	1
2	8	16	4
3	11	33	9
$\Sigma x = 6$	$\Sigma y = 26$	$\Sigma xy = 54$	$\Sigma x^2 = 14$

here, $m = 4$

$\therefore 4a_0 + 6b_1 = 26$

$6a_0 + 14b_1 = 54$

using calculator,

$a_0 = 2$ $\quad$ $b_1 = 3$

Ans.

Problem – 4.5: Find the values of a, b, c so that $Y = a + bx + cx^2$ is the best fit to the data –

x	0	1	2	3	4
y	1	0	3	10	21

Soln: For the given data –

x	y	x^2	x^3	x^4	xy	x^2y
0	1	0	0	0	0	0
1	0	1	1	1	0	0
2	3	4	8	16	6	12
3	10	9	27	81	30	90
4	21	16	64	256	84	336

$\Sigma x = 10$ $\Sigma y = 35$ $\Sigma x^2 = 30$ $\Sigma x^3 = 100$ $\Sigma x^4 = 354$ $\Sigma xy = 120$ $\Sigma x^2y = 438$

here, $m = 5$

$\therefore 5a_0 + 10b_1 + 30c_2 = 35$

$10a_0 + 30b_1 + 100c_2 = 120$

$30a_0 + 100b_1 + 354c_2 = 438$

using calculator,

~~$a_0 = 0.22857$~~ $a = 1$

~~$b_1 = 1.75714$~~ $b = -3$

~~$c_2 = 1.71429$~~ $c = 2$

~~Ans.~~ Ans.

Problem- 4.6: Fit a least squares parabola $Y = a + bx + cx^2$ to the data-

x	0	1	2	3	4	5	6
y	71	89	67	43	31	18	9

Soln: For the given data:

x	y	x^2	x^3	x^4	xy	x^2y
0	71	0	0	0	0	0
1	89	1	1	1	89	89
2	67	4	8	16	134	268
3	43	9	27	81	129	387
4	31	16	64	216	124	496
5	18	25	125	625	90	450
6	9	36	216	1296	54	324
$\Sigma x = 21$	$\Sigma y = 328$	$\Sigma x^2 = 91$	$\Sigma x^3 = 441$	$\Sigma x^4 = 2235$	$\Sigma xy = 620$	$\Sigma x^2y = 2014$

here, $n = 7$

$\therefore$ $7a + 21b + 91c = 328$

$21a + 91b + 441c = 620$

$91a + 441b + 2235c = 2014$

using calculator,

$a = 78.3571$

$b = -4$

$c = -1.5$ Ans.

Problem- 4.7: Determine the normal equations if the cubic polynomial $Y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ is fitted to the data (x_i, y_i), $i = 1, 2, 3, \ldots, m$.

Soln: We have got,

$$S = [y_1 - (a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3)]^2 + [y_2 - (a_0 + a_1 x_2 + a_2 x_2^2 + a_3 x_2^3)]^2 + \cdots$$

for S to be minimum,

$$\frac{dS}{da_0} = 0 \quad \frac{dS}{da_1} = 0 \quad \frac{dS}{da_2} = 0 \quad \frac{dS}{da_3} = 0$$

$$\frac{dS}{da_0} = 0$$

$$\Rightarrow -2[y_1 - (a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3)] - 2[y_2 - (a_0 + a_1 x_2 + a_2 x_2^2 + a_3 x_3^3)] - \cdots = 0$$

$$\Rightarrow [y_1 - (a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3)] + [y_2 - (a_0 + a_1 x_2 + a_2 x_2^2 + a_3 x_2^3)] + \cdots = 0$$

$$\Rightarrow y_1 + y_2 + \cdots = (a_0 + a_0 + \cdots) + a_1(x_1 + x_2 + \cdots) + a_2 + (x_1^2 + x_2^2 + \cdots) + a_3(x_1^3 + x_2^3 + \cdots)$$

$$\Rightarrow \sum_{i=1}^{m} y_i = m a_0 + a_1 \sum_{i=1}^{m} x_i + a_2 \sum_{i=1}^{m} x_i^2 + a_3 \sum_{i=1}^{m} x_i^3$$

$$\frac{dS}{da_1} = 0$$

$$\Rightarrow -2x_1[y_1 - (a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3)] - 2x_2[y_2 - (a_0 + a_1 x_2 +$$

$a_2x_2^2+a_3x_2^3)]-\cdots=0$

$\Rightarrow x_1[y_1-(a_0+a_1x_1+a_2x_1^2+a_3x_1^3)]+x_2[y_2-(a_0+a_1x_2+a_2x_2^2+a_3x_2^3)]+\cdots=0$

$\Rightarrow x_1y_1+x_2y_2+\cdots=a_0(x_1+x_2+\cdots)+a_1(x_1^2+x_2^2+\cdots)+a_2(x_1^3+x_2^3+\cdots)+a_3(x_1^4+x_2^4+\cdots)$

$$\Rightarrow \sum_{i=1}^{m}x_iy_i=a_0\sum_{i=1}^{m}x_i+a_1\sum_{i=1}^{m}x_i^2+a_2\sum_{i=1}^{m}x_i^3+a_3\sum_{i=1}^{m}x_i^4$$

$\cancel{\frac{dS}{da_2}=0}$

Similarly,

$$\sum_{i=1}^{m}x_i^2y_i=a_0\sum_{i=1}^{m}x_i^2+a_1\sum_{i=1}^{m}x_i^3+a_2\sum_{i=1}^{m}x_i^4+a_3\sum_{i=1}^{m}x_i^5$$

$$\sum_{i=1}^{m}x_i^3y_i=a_0\sum_{i=1}^{m}x_i^3+a_1\sum_{i=1}^{m}x_i^4+a_2\sum_{i=1}^{m}x_i^5+a_3\sum_{i=1}^{m}x_i^6$$

Problem – 48: Determine the constants a and b, by the method of least squares, such that the curve $y=ae^{bx}$ fits the data –

x	2	4	6	8	10
y	4.077	11.084	30.128	81.897	222.62

Soln: Given that,

$$y=ae^{bx}$$

$$\Rightarrow \ln y=\ln a+bx$$

x	y	$\ln y$	$x\ln y$	x^2
2	4.077	1.4054	2.8108	4
4	11.084	2.4055	9.622	16
6	30.128	3.4055	20.433	36
8	81.897	4.4055	35.244	64
10	222.62	5.4055	54.055	100
$\Sigma x = 30$		$\Sigma \ln y = 17.0274$	$\Sigma x\ln y = 122.1648$	$\Sigma x^2 = 220$

here, $m = 5$

$\therefore 5\ln a + 30b = 17.0274$

$30\ln a + 220b = 122.1648$

using calculator,

$\ln a = 0.40542 \quad b = 0.5$

$\therefore a = 1.5$

Ans.

Problem-4.9: Fit a function of the form $y = ax^b$ for the following data—

x	61	26	7	2.6
y	350	400	500	600

Soln: Given that,

$y = ax^b$

$\Rightarrow \ln y = \ln a + b\ln x$

for the given data-

x	y	$\ln x$	$\ln y$	$(\ln x)(\ln y)$	$(\ln x)^2$
61	350	4.1109	5.8579	24.0812	16.8995
26	400	3.2581	5.9915	19.5209	10.6152
7	500	1.9459	6.2146	12.093	3.7865
2.6	600	0.9555	6.3969	6.1122	0.913
$\Sigma=96.6$	$\Sigma=1850$	$\Sigma=10.2704$	$\Sigma=24.4609$	$\Sigma=61.8073$	$\Sigma=32.2142$

here, $m=4$

$\therefore 4\ln a + 10.2704b = 24.4609$

$10.2704\ln a + 32.2142b = 61.8073$

using calculator,

$\ln a = 6.5539 \quad b = -0.1709$

$\therefore a = 702$

$\therefore$ the function will be, $y = 702x^{-0.1709}$

Ans.

Problem-4.10: Using the method of least squares, fit a curve of the form $Y = \frac{a}{x} + bx$ to the following data (x, y):

$(1, 5.43), (2, 6.28), (4, 10.32), (6, 14.86), (8, 19.51)$

$$\sum_{i=1}^{m} \frac{y_i}{x_i} = a\sum_{i=1}^{1} \frac{1}{x^2} + mb$$

$$\sum_{i=1}^{m} x_i y_i = ma + b\sum_{i=1}^{m} x_i^2$$

	x	y	$\frac{1}{x}$	$\frac{1}{x^2}$	x^2	$\frac{y}{x}$	xy
	1	5.43	1	1	1	5.43	5.43
	2	6.28	0.5	0.25	4	3.14	12.56
	4	10.32	0.25	0.0625	16	2.58	41.28
	6	14.86	0.167	0.028	36	2.4767	89.16
	8	19.51	0.125	0.0156	64	2.4388	156.08
$\Sigma =$	21	56.4	2.042	1.3561	121	16.0655	304.51

$16.0655 = 1.3561a + 5b$

$304.51 = 5a + 121b$

using calculatore,

$a = 3.0296$

$b = 2.3914$

Ans.

Chapter-6

Numerical Differentiation and Integration

Problem-6.1: Find $\frac{d}{dx}J_0(x)$ at $x=0.1$ from the following table -

(0,1.0), (0.1, 0.9975), (0.2, 0.9900), (0.3, 0.9776), (0.4, 0.9604)

Sol^n: The forward difference table for the given data -

x_n	y_n	Δy_n	$\Delta^2 y_n$	$\Delta^3 y_n$	$\Delta^4 y_n$
0	1.0	-0.0025	-0.0050	0.0001	0
0.1	0.9975	-0.0075	-0.0049	0.0001	
0.2	0.9900	-0.0124	-0.0048		
0.3	0.9776	-0.0172			
0.4	0.9604				

Since, $x=0.1$, is which is a tabular value.

Let, $x_0 = 0.1$

$$\therefore \frac{d}{dx}J_0(0.1) = \frac{1}{0.1}\left[-0.0075 - \frac{1}{2}(-0.0049) + \frac{1}{3}(0.0001)\right]$$

$$= 10\left[-0.0075 + 0.00245 + 0.00003\right]$$

$$= \cancel{-10\times(-0.00992)}\ 10\times(-0.00502)$$

$$= \cancel{-0.0992}\ -0.0502$$

Ans.

Problem-6.2: The following table gives angular displacements in θ (radians) at different times t (in seconds):

(0, 0.052), (0.02, 0.105), (0.04, 0.168), (0.06, 0.242), (0.08, 0.327), (0.10, 0.408), (0.12, 0.489).

calculate the angular velocity at $t=0.06$

Soln: Forward difference table for the given data-

t_n	θ_n	$\Delta \theta_n$	$\Delta^2 \theta_n$	$\Delta^3 \theta_n$	$\Delta^4 \theta_n$	$\Delta^5 \theta_n$
0	0.052	0.053	0.010	0.001	-0.001	-0.014
0.02	0.105	0.063	0.011	0	-0.015	0.034
0.04	0.168	0.074	0.011	-0.015	0.019	
0.06	0.242	0.085	-0.004	0.004		
0.08	0.327	0.081	0.0			
0.10	0.408	0.081				
0.12	0.489					

Since, $t=0.06$, which is a tabular value.

let, $x_0=0.06$

$\therefore$ angular velocity $=\frac{d\theta}{dt}$

$=\frac{1}{0.02}\left[0.085-\frac{1}{2}(-0.004)+\frac{1}{3}(0.004)\right]$

$=50\times 0.08833$

$=4.41667$ (Ans.)

Problem-6.3: From the following values of x and y, find $\frac{dy}{dx}$ at $x = 0.6$.

(0.4, 1.5836), (0.5, 1.7974), (0.6, 2.0442), (0.7, 2.3275), (0.8, 2.6511)

Soln: Forward difference table for the given data—

x_n	y_n	Δy_n	$\Delta^2 y_n$	$\Delta^3 y_n$	$\Delta^4 y_n$
0.4	1.5836	0.2138	0.113	-0.1566	0.2404
0.5	1.7974	0.3268	-0.0435	0.0838	
0.6	2.0442	0.2833	0.0403		
0.7	2.3275	0.3236			
0.8	2.6511				

since, $x = 0.6$, which is a tabular value,

Let, $x_0 = 0.6$

$$\therefore \frac{dy}{dx} = \frac{1}{0.1}\left[0.2833 - \frac{1}{2}(0.0403)\right]$$

$$= 10 \times 0.26315$$

$$= 2.6315$$

Ans.

Problem – 64: The distances (x cm) traversed by a particle at different times (t seconds) are given below –

t	0.0	0.1	0.2	0.3	0.4	0.5	0.6
x	3.01	3.16	3.29	3.36	3.40	3.38	3.32

~~sol~~ Find the velocity of the particle at $t = 0.3$ seconds.

Sol^n: Forward difference table for the given data –

t_n	x_n	Δx_n	$\Delta^2 x_n$	$\Delta^3 x_n$	$\Delta^4 x_n$	$\Delta^5 x_n$
0.0	3.01	0.15	-0.02	-0.02	0.03	-0.07
0.1	3.16	0.13	-0.04	0.01	-0.04	0.09
0.2	3.29	0.07	-0.03	-0.03	0.05	
0.3	3.36	0.04	-0.06	0.02		
0.4	3.40	-0.02	-0.04			
0.5	3.38	-0.06				
0.6	3.32					

Since, $t = 0.3$ is a tabular value,

then, $\frac{dt}{dx} = \frac{1}{0.1}\left[0.04 - \frac{1}{2}(0.06) + \frac{1}{3}(0.02)\right]$

$= 10 \times 0.076667$

$= 0.76667$

Ans.

Problem-6.5: From the following values of x and y, find $\frac{dy}{dx}$ when, (a) $x=1$, (b) $x=3$, (c) $x=6$ and (d) $\frac{d^2y}{dx^2}$ at $x=3$

x	0	1	2	3	4	5	6
y	6.9897	7.4036	7.7815	8.1291	8.4510	8.7506	9.0309

Soln: The forward difference table fo the given data—

x_n	y_n	Δy_n	$\Delta^2 y_n$	$\Delta^3 y_n$	$\Delta^4 y_n$	$\Delta^5 y_n$	$\Delta^6 y_n$
0	6.9897	0.4139	-0.036	0.0057	-0.0011	-0.0001	0.0009
1	7.4036	0.3779	-0.0303	0.0046	-0.0012	0.0008	
2	7.7815	0.3476	-0.0257	0.0034	-0.0004		
3	8.1291	0.3219	-0.0223	0.003			
4	8.4510	0.2996	-0.0193				
5	8.7506	0.2803					
6	9.0309						

(a) at $x=1$

$$\frac{dy}{dx} = \frac{1}{1}\left[0.3779 - \frac{1}{2}(-0.0303) + \frac{1}{3}(0.0046) - \frac{1}{4}(-0.0012) + \frac{1}{5}(0.0008)\right]$$

$$= 0.3779 + 0.01515 + 0.00153 + 0.0003 + 0.00016$$

$$= 0.39504 \text{ Ans.}$$

(b) at $x = 3$,

$$\frac{dy}{dx} = \frac{1}{1}\left[0.3219 - \frac{1}{2}(-0.0223) + \frac{1}{3}(0.003)\right]$$

$$= 0.3219 + 0.01115 + 0.001$$

$$= 0.33405$$

Ans.

(c) at $x = 6$

$$\frac{dy}{dx} = \frac{1}{1}\left[0.2803 + \frac{1}{2}(-0.0193) + \frac{1}{3}(0.003) + \frac{1}{4}(-0.0004) + \frac{1}{5}(0.0002) + \frac{1}{6}(0.0009)\right]$$

$$= 0.2803 + 0.00965 + 0.001 + 0.0001 + 0.00016 + 0.00015$$

$$= \cancel{0.29106}\ 0.27156$$

Ans.

(d) at $x = 3$

$$\frac{d^2y}{dx^2} = \frac{1}{1}\left[-0.0223 - 0.0003\right]$$

$$= \cancel{-0.0223}\ -0.0226$$

Ans.

Problem-6.6: A rod is rotating in a plane about one of its ends. The angle θ (in radians) at different times t (seconds) are given below—

t	0	0.2	0.4	0.6	0.8	1.0
θ	0.0	0.15	0.50	1.15	2.0	3.20

Find its anglar velocity and anglar acceleration when $t = 0.6$ seconds.

Soln: The difference table for the given data—

t_n	θ_n	$\Delta\theta_n$	$\Delta^2\theta_n$	$\Delta^3\theta_n$	$\Delta^4\theta_n$	$\Delta^5\theta_n$
0	0.0	0.15	0.20	0.10	-0.20	0.45
0.2	0.15	0.35	0.30	-0.10	0.25	
0.4	0.50	0.65	0.20	0.15		
0.6	1.15	0.85	0.35			
0.8	2.0	1.20				
1.0	3.20					

at $t = 0.6$

angular velocity $= \frac{d\theta}{dt}$

$= \frac{1}{0.2}\left[0.85 - \frac{1}{2}(0.35) + \frac{1}{3}(0.15) - \frac{1}{4}(0.25) + \frac{1}{5}(0.45)\right]$

$= 5 \times 0.7525 = 3.7625$ Ans.

angulare accelercation $= \frac{d^2\theta}{dt^2}$

$= \frac{1}{0.2}\left[0.35 - 0.15 + \frac{11}{12}(0.25) - \frac{5}{6}(0.45)\right]$

$= 5\left[0.35 - 0.15 + 0.23 - 0.375\right]$

$= 5 \times 0.055$

$= 0.275$

Ans.

Problem-69: A cubic function $y=f(x)$ satisfies the following data:

x	0	1	3	4
$f(x)$	1	4	40	85

Determine $f(x)$ and hence find $f'(2)$ and $f''(2)$.

Soln: $$f(x) = \frac{(x-1)(x-3)(x-4)}{(-1)(-3)(-4)}\times 1 + \frac{x(x-3)(x-4)}{(1)(-2)(-3)}\times 4 + \frac{x(x-1)(x-4)}{(3)(2)(-1)}\times 40 + \frac{x(x-1)(x-3)}{4\times 3\times 1}\times 85$$

$$= \frac{(x^2-3x-4x+12)(x-1)}{-12} + \frac{x(x^2-4x-3x+12)}{6}\times 4 + \frac{x(x^2-4x-x+4)}{-6}\times 40 + \frac{x(x^2-3x-x+3)}{12}\times 85$$

$$= \frac{(x^2-7x+12)(x-1)}{-12} + \frac{3}{2}x(x^2-7x+12) - \frac{20}{3}x(x^2-5x+4) + \frac{85}{12}x(x^2-4x+3)$$

$$= \frac{1}{12}\left[(x^3-4x^2-7x^2+28x+12x-48)+18(x^3-7x^2+12x)-80(x^3-5x^2+4x)+85(x^3-4x^2+3x)\right]$$

$$= \frac{1}{12}\left[-x^3+11x^2-40x+48+18x^3-126x^2+216x\right.$$

$$\left.-80x^3+400x^2-320x+85x^3-340x^2+255x\right]$$

$$= \frac{1}{12}\left[22x^3-55x^2+111x+48\right]$$

$$\therefore f(x) = \frac{1}{12}\left[22x^3-5x^2+111x+48\right]$$

$$f'(x) = \frac{1}{12}\left[66x^2-10x+111\right]$$

$$\therefore f'(2) = \frac{1}{12}\left[66\times4-10\times2+111\right]$$

$$= \frac{1}{12}\left[264-20+111\right]$$

$$= \frac{1}{12}\times355 = 29.583333$$ (Ans.)

$$f''(x) = \frac{1}{12}\left[132x-10\right]$$

$$\cancel{= \frac{1}{12}[12}$$

$$\therefore f''(2) = \frac{1}{12}\left[132\times2-10\right]$$

$$= \frac{1}{12}\left[264-10\right]$$

$$= \frac{1}{12}\times254 = 21.166667$$ (Ans.)

Problem – 6.11: From the following values of x and y. find $\frac{dy}{dx}$ at $x=2$.

(2, 11), (3, 49), (4, 123)

Solⁿ: Difference table for the given data-

x_n	y_n	Δy_n	$\Delta^2 y_n$
2	11	38	26
3	49	74	
4	123		

here, $h=1$

$\therefore \frac{dy}{dx}$ at $x=2$

$= \frac{1}{1}\left[38 - \frac{1}{2}(26)\right]$

$= 38-13$

$= 25$ Ans.

Problem – 6.12: From the following values of x and y. determine the value of $\frac{dy}{dx}$ at each of the points-

(1, 3), (2, 11), (4, 69), (5, 131)

Solⁿ:

Problem-6.13: Given the values of x and y
(1.2, 0.9320), (1.3, 0.9636), (1.4, 0.9855), (1.5, 0.9975), (1.6, 0.9996)
find x, correct to two decimal places, for which y is maximum and find this value of y.

Sol^n: Difference table for given data—

x_n	y_n	Δy_n	$\Delta^2 y_n$
1.2	0.9320	0.0316	-0.0097
1.3	0.9636	0.0219	-0.0099
1.4	0.09855	0.0120	-0.0039
1.5	0.9915	0.0081	
1.6	0.9996		

Let, $x_0 = 1.2$, then

$$0 = 0.0316 + \frac{2p-1}{2}(-0.0097)$$

$$\Rightarrow (2p-1)(-0.0097) = -0.0632$$

$$\Rightarrow 2p-1 = 6.52$$

$$\Rightarrow 2p = 7.52$$

$$\Rightarrow p = 3.76$$

$$\therefore x = 1.2 + 3.76 \times 0.1 = 1.576$$

for $x = 1.576$, y will be maximum.

using Newton's Forward Difference Interpolation method.

$$y = 0.9320 + 3.76 \times 0.0316 + \frac{3.76 \times 2.76}{2}(-0.0097)$$

$$= 0.9320 + 0.118816 - 0.050331$$

$$= 1$$

Ans.

Problem-6.13: Evaluate

(a) $\int_0^{\pi} x\sin x\,dx$ and (b) $\int_{-2}^{2} \frac{x}{5+2x}\,dx$

Sol^n:

(a)

x	$f(x)$
0	0
$\frac{\pi}{4}$	$\frac{\pi}{4\sqrt{2}}$
$\frac{\pi}{2}$	$\frac{\pi}{2}$
$\frac{3\pi}{4}$	$\frac{3\pi}{4\sqrt{2}}$
π	0

using trapezoidal rule,

$$\int_0^{\pi} x\sin x\,dx = \frac{\pi/4}{2}\left[0+2\left(\frac{\pi}{4\sqrt{2}}+\frac{\pi}{2}+\frac{3\pi}{4\sqrt{2}}\right)+0\right]$$

$$=\frac{\pi}{8}\times 2\times \pi \times \frac{1+\sqrt{2}}{2}$$

$$=2.9784166$$

(Ans.)

(b)

x	$f(x)$
-2	-2
-1	-0.333333
0	0
1	0.142857
2	0.222222

$$\int_{-2}^{2} \frac{x}{5+2x}\,dx = \frac{1}{2}\Big[-2+2(-0.333333+0+$$
$$0.142857)+0.222222\Big]$$
$$= \frac{1}{2}(-2.158730)$$
$$= -1.079365$$

Ans.

Problem – 8.17: Write an algorithm to evaluate the integral

$$I = \int_{x_0}^{x_n} y\,dx$$

by the trapezoidal rule with step size h.
Given the values of x and $y(x)$:

(0, 0.399), (0.5, 0.352), (1.0, 0.242), (1.5, 0.129),
(2.0, 0.054)

Find an approximate value of

$$\int_0^2 y(x)\,dx$$

Soln: Let, Given a set of data points (x_0, y_0), (x_1, y_1), ..., (x_n, y_n) of a function $y = f(x)$, where $f(x)$ is not known explicitely, it is required to compute the value of the definite integral,

$$I = \int_{x_0}^{x_n} y\,dx$$

Let the interval $[x_0, x_n]$ be divided into n equal sub-intervals such that $x_0 < x_1 < x_2 < \dots < x_n$. Clearly $x_n = x_0 + nh$.
Approximating y by Newton's forward difference formula, we obtain,

$$I = \int_{x_0}^{x_n} \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \cdots \right] dx$$

Since, $x_0 = x_0 + ph$, $dx = h\,dp$

$$\therefore I = h\int_{0}^{n} \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \cdots \right] dp$$

$$\Rightarrow \int_{x_0}^{x_n} y\,dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \cdots \right]$$

setting $n = 1$, all differences higher than the first will become 0 and we obtain,

$$\int_{x_0}^{x_1} y\,dx = h\left(y_0 + \frac{1}{2}\Delta y_0\right) = \frac{h}{2}(y_0 + y_1)$$

for the next intervals $[x_1 . x_2]$. similarly

$$\int_{x_1}^{x_2} y\,dx = \frac{h}{2}(y_1 + y_2)$$

and so on, For the last interval $[x_{n-1}, x_n]$

$$\int_{x_{n-1}}^{x_n} y\,dx = \frac{h}{2}(y_{n-1} + y_n)$$

Combining all these expressions, we obtain

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + \cdots + y_{n-1}) + y_n \right]$$

which is the Trapezoidal rule.

Part 2: here, $h = 0.5$

$$\int_0^2 y(x)\,dx = \frac{0.5}{2}\left[0.399 + 2(0.352 + 0.242 + 0.129) + 0.054\right]$$

$$= \frac{1}{4}\left[0.453 + 1.446\right]$$

$$= \frac{1}{4} \times 1.899$$

$$= 0.4748$$ Ans.

Problem-6.19: Evaluate

$$I = \int_0^{\pi/2} \sqrt{\sin x}\,dx$$

using Simpson's $\frac{1}{3}$ rule with $h = \frac{\pi}{12}$.

Sol^n: The data table will be.

x	$f(x)$
0	0
$\pi/12$	0.258819
$\pi/6$	0.5
$\pi/4$	0.707107
$\pi/3$	0.866025
$5\pi/12$	0.965926
$\pi/2$	1

$$\therefore I = \frac{\pi/12}{3}\Big[0+4(0.258819+0.707107+0.965926)$$
$$+2(0.5+0.866025)+1\Big]$$
$$= \frac{\pi}{36}\left[1+\cancel{4}\,7.727408+2.73205\right]$$
$$=1$$

Ans.

Problem- 6.18: Estimate the value of the integral

$$\int_1^3 \frac{1}{x} dx$$

by Simpson's rule with 4 strips and 8 strips respectively. Determine the error in each case.

Soln: Real value =

$$\int_1^3 \frac{1}{x} dx = [\ln x]_1^3 = [\ln 3 - \ln 1] = 1.098612$$

using 4 strips-

x	$f(x)$
1	1
$\frac{5}{3}$	0.6
$\frac{7}{3}$	0.428571
3	0.333333

$$\int_1^3 \frac{1}{x} dx = \frac{\frac{2}{3}}{3}\left[1 + 4\times 0.6 + 2\times 0.428571 + 0.333333\right]$$

$$= \frac{2}{9} \times 4.590476$$

$$= 1.020106$$

$$\text{error} = \frac{|1.098612 - 1.020106|}{1.098612} \times 100 = 7.145926\%$$

using 8 strips -

x	$f(x)$
1	1
$\frac{9}{7}$	$\frac{7}{9}$
$\frac{11}{7}$	$\frac{7}{11}$
$\frac{13}{7}$	$\frac{7}{13}$
$\frac{15}{7}$	$\frac{7}{15}$
$\frac{17}{7}$	$\frac{7}{17}$
$\frac{19}{7}$	$\frac{7}{19}$
3	$\frac{1}{3}$

$$\therefore \int_1^3 \frac{1}{x}dx = \frac{\frac{2}{7}}{3}\left[1+4\left(\frac{7}{9}+\frac{7}{13}+\frac{7}{17}\right)+2\left(\frac{7}{11}+\frac{7}{15}+\frac{7}{19}\right)+\frac{1}{3}\right]$$

$$= \frac{2}{21}\left[1+6.912016+2.942903+0.333333\right]$$

$$= \frac{2}{21} \times 11.188252$$

$$= 1.065548$$

$$\text{error} = \frac{|1.098612-1.065548|}{1.098612} \times 100$$

$$= 3.009616\%$$

Ans.

Problem-6.20: Using Simpson's 1/3 rule with $h=1$, evaluate the integral,

$$I = \int_3^7 x^2 \log x \, dx$$

Solⁿ: Data table for the given function-

x	$f(x)$
3	4.294091
4	9.632960
5	17.474250
6	28.013445
7	41.409804

$$\therefore I = \frac{1}{2}\left[4.294091 + 4(9.632960 + 28.013445) + 2\times 17.474250 + 41.409804\right]$$

$$= \frac{1}{2}\left[45.703895 + 150.58562 + 34.949\right]$$

$$= 115.6192575$$

Ans.

Problem-6.21: Write an algorithm to evaluate $\int_{x_0}^{x_{2n}} y\,dx$ using Simpson's 1/3 rule when $y(x)$ is given at $x_0, x_0+h, \ldots, x_0+2nh$.
Evaluate $\int_0^1 e^{-x^2} \sin x\,dx$ using Simpson's 1/3 rule with $h=0.1$.

Soln:

<u>Second Part:</u>

Given that,

$$I = \int_0^1 e^{-x^2} \sin x\,dx \quad \text{and } h = 0.1$$

$$\therefore f(x) = e^{-x^2} \sin x\,dx$$

. Data table for the given function -

x	$f(x)$
0	0
0.1	0.09884
0.2	0.190879
0.3	0.270085
0.4	0.33184
0.5	0.373377
0.6	0.393938

0.7	0.394665
0.8	0.378256
0.9	0.348469
1.0	0.30956

$$\therefore \int_0^1 e^{-x^2} \sin x \, dx$$

$$= \frac{0.1}{3}\Big[0 + 4\times(0.09884 + 0.27085 + 0.373377 + 0.394665 + 0.348469) + 2(0.190879 + 0.33184 + 0.393938 + 0.378256) + 0.30956\Big]$$

$$= \frac{1}{30}\Big[5.944804 + 2.589826 + 0.30956\Big]$$

$$= 0.294806$$

Problem - 6.22: Compute the values of

$$I = \int_0^1 \frac{dx}{1+x^2}$$

using the trapezoidal rule with $h = 0.5$, 0.25 and 0.125. ~~Co~~

Soln: When, $h = 0.5$

x	$f(x)$
0	1
0.5	0.8
1	0.5

$$\therefore I = \frac{0.5}{2}\left[1 + 2 \times 0.8 + 0.5\right]$$

$$= 0.775 \text{ Ans.}$$

When, $h = 0.25$

x	$f(x)$
0	1
0.25	0.941176
0.50	0.8
0.75	0.64
1.00	0.5

$$\therefore I = \frac{0.25}{2}\left[1 + 2(0.941175 + 0.8 + 0.64) + 0.5\right]$$

$$= 0.782794 \text{ (Ans.)}$$

When, $x = 0.125$

x	$f(x)$
0	1
0.125	0.984615
0.250	0.941176
0.375	0.876712
0.500	0.8
0.625	0.719101
0.750	0.64
0.875	0.566372
1.000	0.5

$$\therefore I = \frac{0.125}{2}\Big[1 + 2(0.984615 + 0.941176 + 0.876712 + 0.8 + 0.719101 + 0.64 + 0.566372) + 0.5\Big]$$

$$= 0.784747$$

Ans.

Problem-6.24: Estimate the value of the integral $I = \int_0^{\pi/2} \frac{dx}{\sqrt{x}\sqrt{1-x}}$ using trapezoidal rule. What is the exact value?

Soln: Let $h = \frac{\pi}{8}$

x	$f(x)$
0	$-$
$\frac{\pi}{8}$	3.023716
$\frac{\pi}{4}$	2.309401
$\frac{3\pi}{8}$	2.065591
$\frac{\pi}{2}$	2

$$\therefore I = \frac{1/8}{2}\left[2(3.023716 + 2.309401 + 2.065591) + 2\right]$$

$$= \frac{1}{16}\left[14.797416 + 2\right]$$

$$= 1.049839$$

Problem-6.25: Derive Simpson's 3/8 rule.

$$\int_{x_0}^{x_3} y\,dx = \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3)$$

using this rule, evaluate

$$\int_0^1 \frac{1}{1+x}\,dx \quad \text{with } h = \frac{1}{6}.$$

Evaluate the integral by Simpson's 1/3 rule and compare the results.

Sol^n: Given a set of data points (x_0, y_0), (x_1, y_1), ..., (x_n, y_n) of a function $y = f(x)$, where $f(x)$ is not known explicitely, it is required to compute the definite integral

$$I = \int_a^b y\,dx$$

Let the interval $[a, b]$ be divided into n equal subintervals such that $a = x_0 < x_1 < \dots < x_n = b$. Clearly, $x_n = x_0 + nh$. Hence, the integral becomes,

$$I = \int_{x_0}^{x_n} y\,dx$$

Approximating y by Newton's forward difference formula we obtain,

$$I = \int_{x_0}^{x_n} \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \cdots\right] dx$$

Since, $x = x_0 + ph$, $dx = h\,dp$ and hence the above integral becomes,

$$I = h\int_0^n \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \cdots\right] dp$$

which gives on simplification,

$$\int_{x_0}^{x_n} y\,dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \cdots\right]$$

Setting $n=3$ in the ~~gener~~ formula we observe that all the difference highere than the third will be 0 and thus,

$$\int_{x_0}^{x_3} y\,dx = 3h\left(y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right)$$

$$= \cancel{\frac{3h}{8}\left[8y_0 + 12(y_1 - y_0) + 6(\Delta y_1 - \Delta y_0) + (\Delta^2 y_1 - \Delta^2 y_0)\right]}$$

$$= \cancel{\frac{3h}{8}\left[8y_0 + 12y_1 - 12y_0\right.}$$

$$= \frac{3h}{8}\left[8y_0 + 12\Delta y_0 + 6\Delta^2 y_0 + (\Delta^2 y_1 - \Delta^2 y_0)\right]$$

$$= \frac{3h}{8}\left[8y_0 + 12\Delta y_0 + 5(\Delta y_1 - \Delta y_0) + \Delta y_2 - \Delta y_1\right]$$

$$= \frac{3h}{8}\left[8y_0 + 7(y_1 - 7y_0) + 4(y_2 - y_1) + y_3 - y_2\right]$$

$$= \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right]$$ (Derived)

Data table for the given function-

x	$f(x)$
0	1
$\frac{1}{6}$	0.857143
$\frac{1}{3}$	0.75
$\frac{1}{2}$	0.666667
$\frac{2}{3}$	0.6
$\frac{5}{6}$	0.545455
1	0.5

using Simpson's 3/8 rule-

$$I = \frac{3\times\frac{1}{6}}{8}\left[1+3(0.857143+0.75+0.6+0.545455)+2\times 0.666667+0.5\right]$$

$$= \frac{1}{16}\left[1+\cancel{3\times}\, 8.257793+1.333333+0.5\right]$$

$$= 0.693195$$

using Simpson's 1/3 rule-

$$I = \frac{\frac{1}{6}}{3}\left[1+4(0.857143+0.666667+0.545455)+2(0.75+0.6)+0.5\right]$$

$$= \frac{1}{18}\left[1+8.277059+2.7+0.5\right]$$

$$= 0.69317$$

(Ans.)

Problem-6.28: Evaluate

$$\int_0^2 \frac{dx}{x^3+x+1}$$

by Simpson's 1/3 rule with $h=0.25$.

Soln: Data table for the given function–

x	$f(x)$
0	1
0.25	0.790123
0.50	0.615385
0.75	0.460432
1.00	0.333333
1.25	0.237918
1.50	0.170213
1.75	0.123314
2.00	0.090909

$$\therefore I = \frac{0.25}{3}[1+4(0.790123+0.460432+0.237918+0.123314)+2(0.615385+0.333333+0.170213)+0.090909]$$

$$=\frac{1}{12}[1+6.447148+2.237862+0.090909]$$

$$=0.81466$$

(Ans)

Problem-6.32: Evaluate

$$I = \int_0^1 \frac{dx}{1+x}$$

by Simpson's 1/3 rule with $h = 0.125$

Soln: Data table for given function-

x	$f(x)$
0	~~0.5~~ 1
0.125	0.888889
0.250	0.8
0.375	0.727273
0.500	0.666667
0.625	0.615385
0.750	0.571429
0.875	0.533333
1.000	0.5

$$I = \frac{0.125}{3}\left[1 + 4(0.888889 + 0.727273 + 0.615385 + 0.533333) + 2(0.8 + 0.666667 + 0.571429) + 0.5\right]$$

$$= \frac{1}{24}\left[1 + 11.05952 + 4.076192 + 0.5\right]$$

$$= 0.693155$$

(Ans)

Problem-6.39: Use the trapezoidal rule to evaluate the double integral

$$\int_{-2}^{2}\int_{0}^{4}(x^2-xy+y^2)\,dx\,dy$$

Soln: Data table for the given function-

x	$f(x)$
0	y^2
1	$1-y+y^2$
2	$4-2y+y^2$
3	$9-3y+y^2$
4	$16-4y+y^2$

$$\int_0^4 (x^2-xy+y^2)\,dx = \frac{1}{2}\left[y^2+(1-y+y^2+4-2y+y^2+9-3y+y^2)+16-4y+y^2\right]$$

$$=\frac{1}{2}\left[5y^2-10y+30\right]$$

$$=\frac{5}{2}\left[y^2-2y+6\right]$$

$$\therefore \int_{-2}^{2}\int_{0}^{4}(x^2-xy+y^2)\,dx\,dy=\frac{5}{2}\int_{-2}^{2}(y^2-2y+6)\,dy$$

Now.

y	$f(y)$
-2	14
-1	9
0	6
1	5
2	6

$$\therefore I = \frac{5}{2} \times \frac{1}{2}\left[14+2(9+6+5)+6\right]$$

$$= \frac{5}{4}\left[14+40+6\right]$$

$$= 75$$

Ans.

Problem-6.40: Use simpson's $1/3$ rule to evaluate the double integral in Problem 6.39.

Solⁿ: Data table for the given function

x	$f(x)$
0	y^2
1	$1-y+y^2$
2	$4-2y+y^2$
3	$9-3y+y^2$
4	$16 \pm 4y+y^2$

$$\int_0^4 (x^2 - xy + y^2)\,dx = \frac{1}{3}\left[y^2 + 4(1 - y + y^2 + 9 - 3y + y^2) + 2(4 - 2y + y^2) + 16 - 4y + y^2\right]$$

$$= \frac{1}{3}\left[y^2 + 40 - 16y + 8y^2 + 8 - 4y + 2y^2 + 16 - 4y + y^2\right]$$

$$= \frac{1}{3}\left[10y^2 - 24y + 64\right]$$

$$= \frac{2}{3}\left[5y^2 - 12y + 32\right]$$

Now,

y	$f(y)$
-2	76
-1	49
0	32
1	25
2	28

$$\therefore I = \frac{2}{3} \times \frac{1}{3}\left[76 + 4(49 + 25) + 2 \times 32 + 28\right]$$

$$= \frac{2}{9}\left[76 + 296 + 64 + 28\right]$$

$$= 103.111$$

Ans.

Chapter-8
Numerical Solution of Ordinary Differential Equations

Problem-8.1: Given

$$\frac{dy}{dx} = 1+xy, \quad y(0)=1$$

obtain the taylor series for $y(x)$ and compute $y(0.1)$ correct to four decimal places.

Soln: Since, $x_0 = 0$, the taylor series will be

$$y(x) = y_0 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + \cdots$$

Given that,

$$\frac{dy}{dx} = y' = 1+xy, \quad y(0)=1$$

$$\Rightarrow y' = 1+0\times1 = 1$$

$$y'' = \cancel{1}+y+xy'$$

$$= 1+0\times1 = 1$$

$$y''' = y' + \cancel{y'} + xy'' = 2y' + xy''$$

$$= 2\times1 + 0\times1 = 2$$

hence, the taylor series for the given ordinary differential equation is,

$$y(x) = 1+x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots$$

$$\therefore y(0.1) = 1+0.1+\frac{(0.1)^2}{2}+\frac{(0.1)^3}{3}$$ [ignoring the higher order terms]

$$= 1+0.1+0.005+0.000333$$

$$= 1.105333 \text{ (Ans.)}$$

Problem-8.2: Show that the differential equation $\frac{d^2y}{dx^2} = -xy$, $y(0)=1$, and $y'(0)=0$

has the series solution.

$$y = 1 - \frac{x^3}{3!} + \frac{1\times 4}{6!}x^6 - \frac{1\times 4\times 7}{9!}x^9 + \cdots$$

Solⁿ: Given that,

$$\frac{d^2y}{dx^2} = y'' = -xy = -0.\times 1 = 0$$

Since, $x_0 = 0$, the series will be,

$$y(x) = y_0 + xy' + \frac{x^2}{2!}y'' + \frac{x^3}{3!}y''' + \cdots$$

$$y''' = -y - xy' = -1 - 0 = -1$$

$$y^{iv} = -y' - y' - xy'' = -2y' - xy'' = 0$$

$$y^{v} = -2y'' - y'' - xy''' = -3y''' - xy''' = 0 - 0(-1) = 0$$

$$y^{vi} = -3y''' - y''' - xy^{iv} = -4y''' - xy^{iv} = -4(-1) - 0 = 4$$

$$y^{vii} = -4y^{v} - y^{iv} - xy^{v} = -5y^{iv} - xy^{v} = 0$$

$$y^{viii} = -5y^{v} - y^{v} - xy^{vi} = -6y^{v} - xy^{vii} = 0$$

$$y^{ix} = -6y^{vi} - y^{vi} - xy^{vii} = -7y^{vi} - xy^{vii} = -7\times 4$$

$\therefore$ the series solution is,

$$y = 1 + 0 + 0 + \frac{x^3}{3!}\times(-1) + 0 + 0 + \frac{x^6}{6!}(4) + 0 + 0 + \frac{x^9}{9!}\times(-7\times 4)$$

$$= 1 - \frac{x^3}{3!} + \frac{1\times 4}{6!}x^6 - \frac{1\times 4\times 7}{9!}x^9 + \cdots$$

(Showed)