Fundamentals of Image Processing

Frequency Domain Filtering

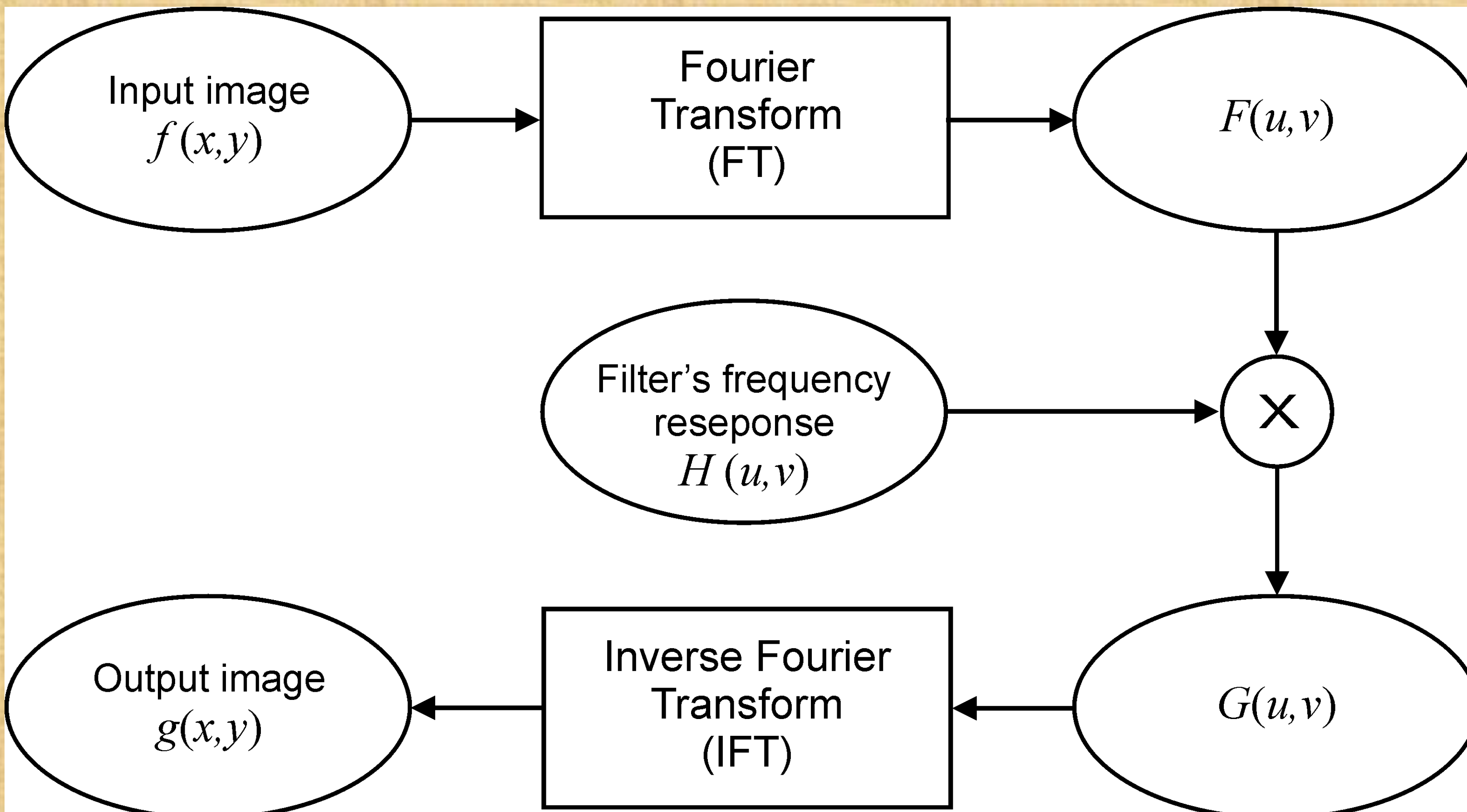
Objectives

- Which mathematical tools are used to represent an image's contents in the 2D frequency domain?
- What is the Fourier transform, what are its main properties, and how is it used in the context of frequency-domain filtering?
- What are the differences between low-pass and high-pass filters?
- What are the differences among ideal, Butterworth, and Gaussian filters

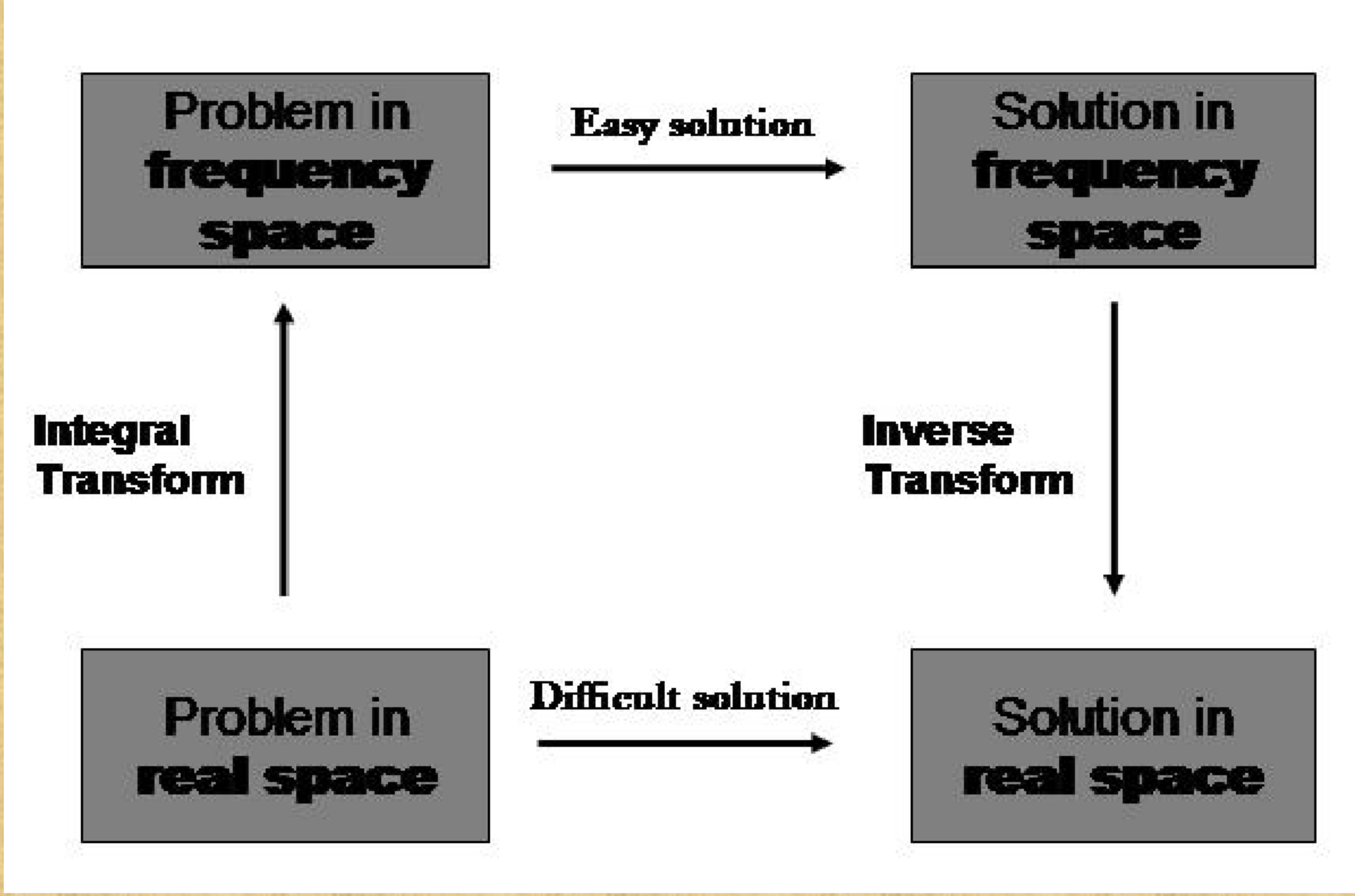
Frequency Domain

- Analysis of mathematical functions or signals with respect to frequency, rather than time
- A time-domain graph shows how a signal changes over time, whereas a frequency-domain graph shows how much of the signal lies within each given frequency band over a range of frequencies.
- A frequency-domain representation can also include information on the phase shift that must be applied to each sinusoid in order to be able to recombine the frequency components to recover the original time signal.
- In DIP: Analysis of the image in another domain rather than in spatial domain.

Frequency-domain operations



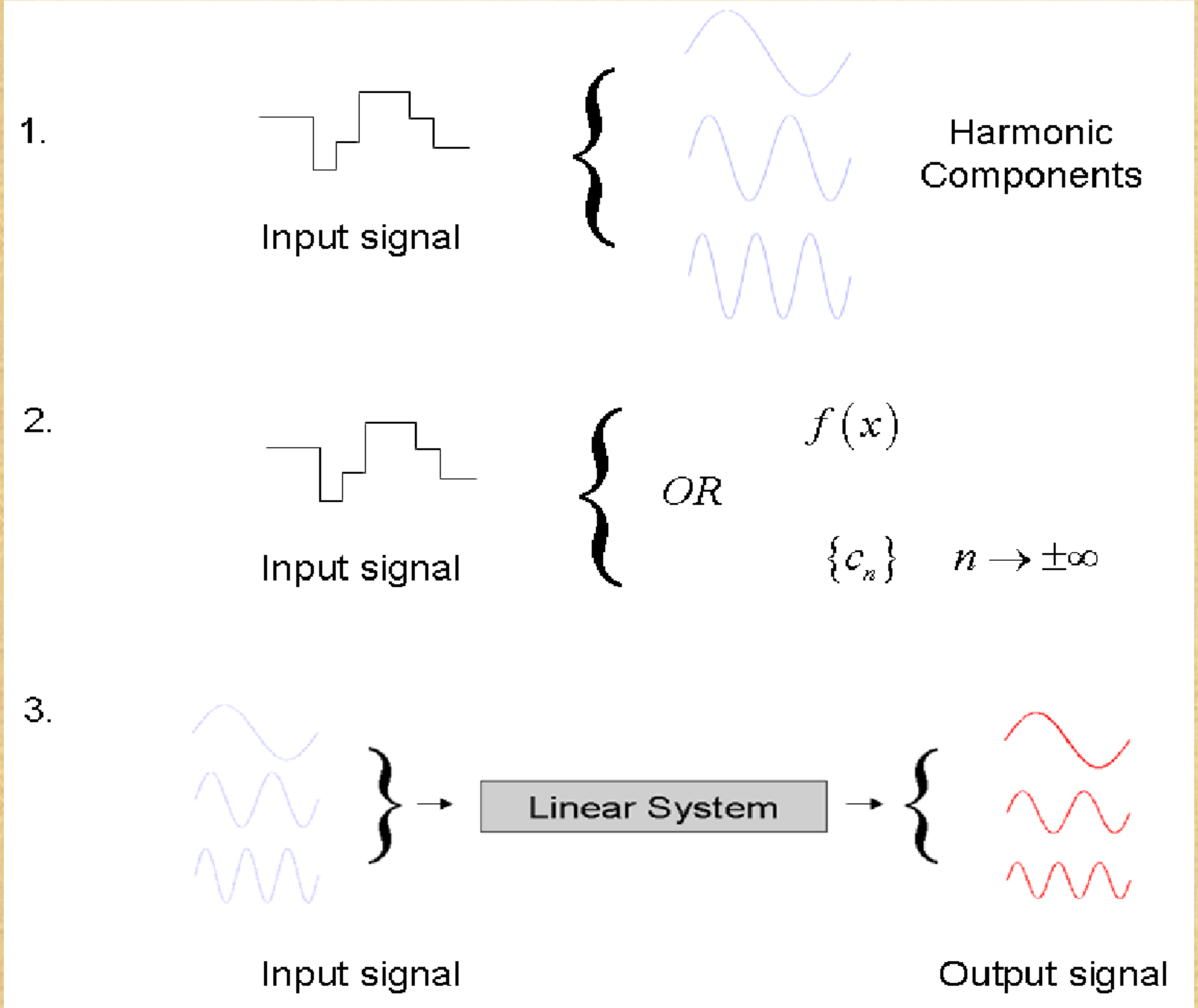
Easier


Problem in frequency space
Easy solution
Solution in frequency space
Integral Transform
Inverse Transform
Problem in real space
Difficult solution
Solution in real space

Background

- Jean Baptiste Joseph Fourier (21 March 1768 – 16 May 1830): French mathematician and physician
- 1822: *Théorie analytique de la chaleur* (The Analytic Theory of Heat)
- Main idea: Every periodic function can be expressed as a sum of sines/cosines (Fourier Series) → Harmonic analysis

The main idea



Mathematical Background: Complex Numbers

- A complex number **x** is of the form:

$$x = a + jb, \text{ where } j = \sqrt{-1}$$

α: **real part**, b: **imaginary** part

- Addition:

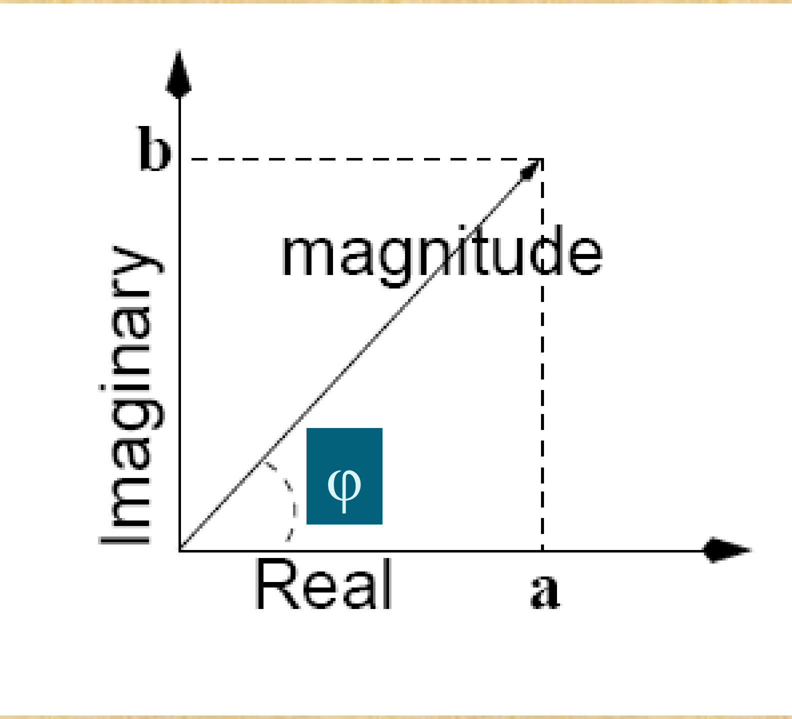
$$(a + jb) + (c + jd) = (a + c) + j(b + d)$$

- Multiplication:

$$(a + jb) \,.\, (c + jd) = (ac - bd) + j(ad + bc)$$

Mathematical Background: Complex Numbers (cont.)

- Magnitude-Phase (i.e.,vector) representation



Magnitude:

$$|x| = \sqrt{a^2 + b^2}$$

Phase:

$$\phi(x) = \tan^{-1}(b/a)$$

Magnitude-Phase notation:

$$x = |x| e^{j\phi(x)}$$

Mathematical Background: Complex Numbers (cont.)

- Multiplication using magnitude-phase representation

$$xy = |x|e^{j\phi(x)} \,.\, |y|e^{j\phi(y)} = |x|\,|y|\; e^{j(\phi(x)+\phi(y))}$$

- Complex conjugate

$$x^* = a - jb$$

- Properties

$$\begin{aligned} |x| &= |x^*| \\ \phi(x) &= -\phi(x^*) \\ xx^* &= |x|^2 \end{aligned}$$

Mathematical Background: Complex Numbers (cont.)

- Euler's formula

$$e^{\pm j\theta} = cos(\theta) \pm jsin(\theta)$$

- Properties

$$|e^{\pm j\theta}| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1$$

$$\phi(e^{\pm j\theta}) = \tan^{-1}(\pm \frac{sin(\theta)}{cos(\theta)}) = \tan^{-1}(\pm tan(\theta)) = \pm\theta$$

$$sin(\theta) = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$$

$$cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$$

Mathematical Background: Sine and Cosine Functions

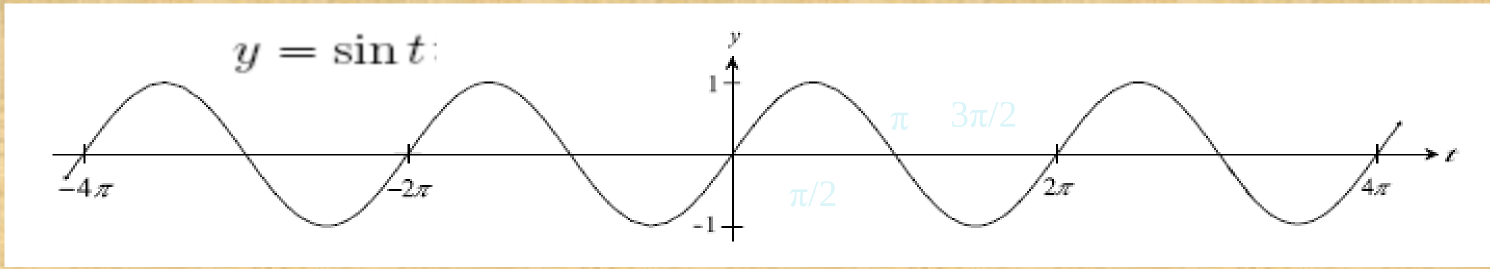
- Periodic functions
- General form of sine and cosine functions:

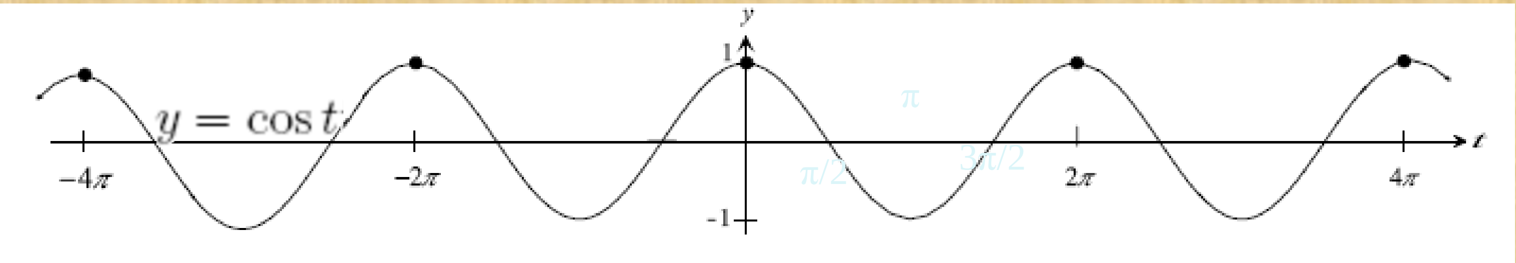
$$y\,(t) = A \sin[a(t+b)] \qquad y\,(t) = A \cos[a(t+b)]$$

$\lvert A \rvert$	amplitude
$\frac{2\pi}{\lvert a \rvert}$	period
b	phase shift

Mathematical Background: Sine and Cosine Functions (cont.)

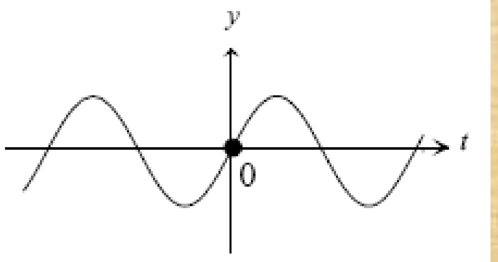
Special case: A=1, b=0, α=1



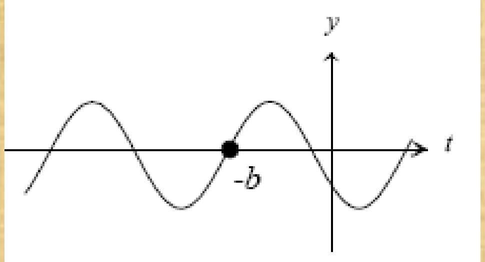


Mathematical Background: Sine and Cosine Functions (cont.)

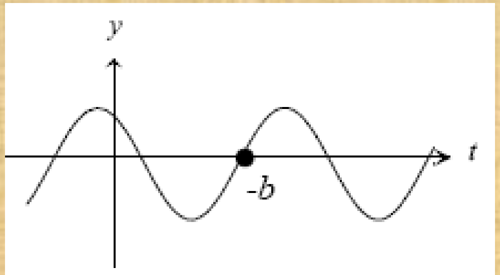
- Shifting or translating the sine function by a const b



(a) $y = \sin t$



(b) $y = \sin(t + b)$, $b > 0$



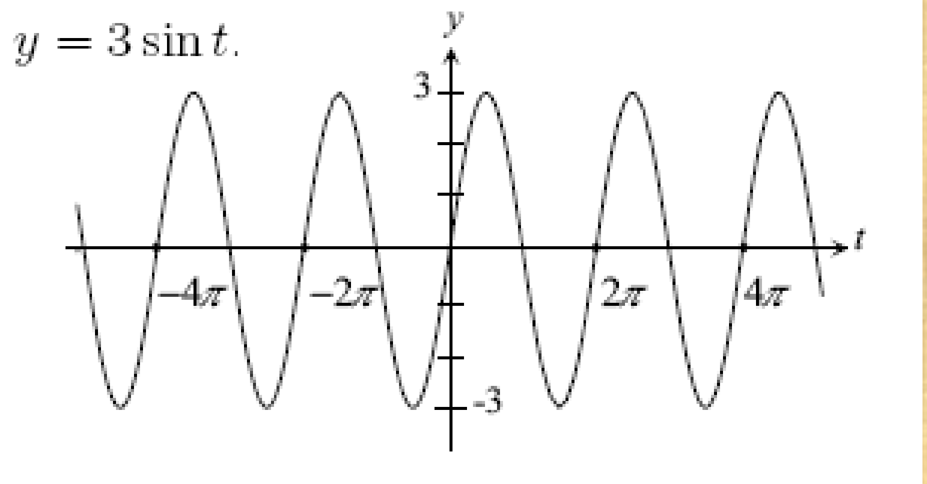
(c) $y = \sin(t + b)$, $b < 0$

Remember: cosine is a shifted sine function:

$$\cos(t) = \sin(t + \frac{\pi}{2})$$

Mathematical Background: Sine and Cosine Functions (cont.)

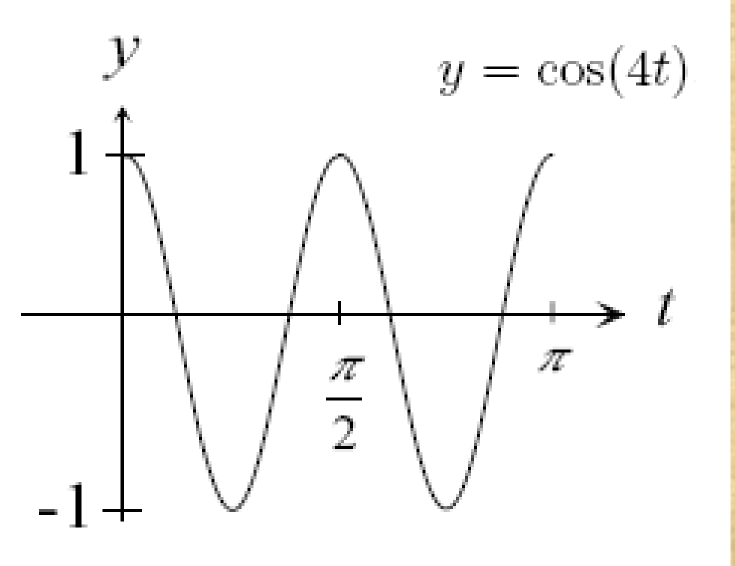
- Changing the amplitude A



Mathematical Background: Sine and Cosine Functions (cont.)

- Changing the period $T=2\pi/|\alpha|$

  consider A=1, b=0: $y=\cos(\alpha t)$



$\alpha = 4$

period $2\pi/4=\pi/2$

shorter period
higher frequency
(i.e., oscillates faster)

<u>Frequency</u> is defined as $f=1/T$

<u>Alternative notation</u>: $\sin(\alpha t)=\sin(2\pi t/T)=\sin(2\pi f t)$

Basis Functions

- Given a vector space of functions, S, then if any f(t) ϵ S can be expressed as

$$f(t) = \sum_k a_k \varphi_k(t)$$

the set of functions $\varphi_k(t)$ are called the expansion set of S.

- If the expansion is unique, the set $\varphi_k(t)$ is a basis.

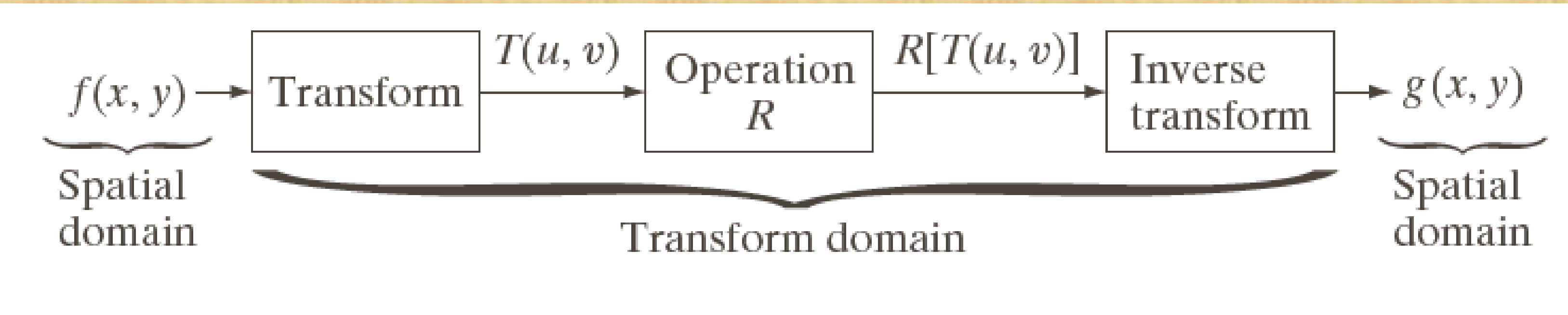
Image Transforms

- Many times, image processing tasks are best performed in a domain other than the *spatial domain*.
- Key steps:

  (1) Transform the image

  (2) Carry the task(s) in the *transformed domain*.

  (3) Apply *inverse transform* to return to the spatial domain.



Transformation Kernels

- Forward Transformation

forward transformation kernel

$$T(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)r(x,y,u,v) \qquad u = 0,1,\ldots, M-1,\ v = 0,1,\ldots, N-1$$

- Inverse Transformation

inverse transformation kernel

$$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} T(u,v)s(x,y,u,v) \qquad x = 0,1,\ldots, M-1,\ y = 0,1,\ldots, N-1$$

Kernel Properties

- A kernel is said to be *separable* if:

$$r(x,y,u,v) = r_1(x,u)r_2(y,v)$$

- A kernel is said to be *symmetric* if:

$$r(x,y,u,v) = r_1(x,u)r_1(y,v)$$

Notation

- Continuous Fourier Transform (FT)
- Discrete Fourier Transform (DFT)
- Fast Fourier Transform (FFT)

Fourier Series Theorem

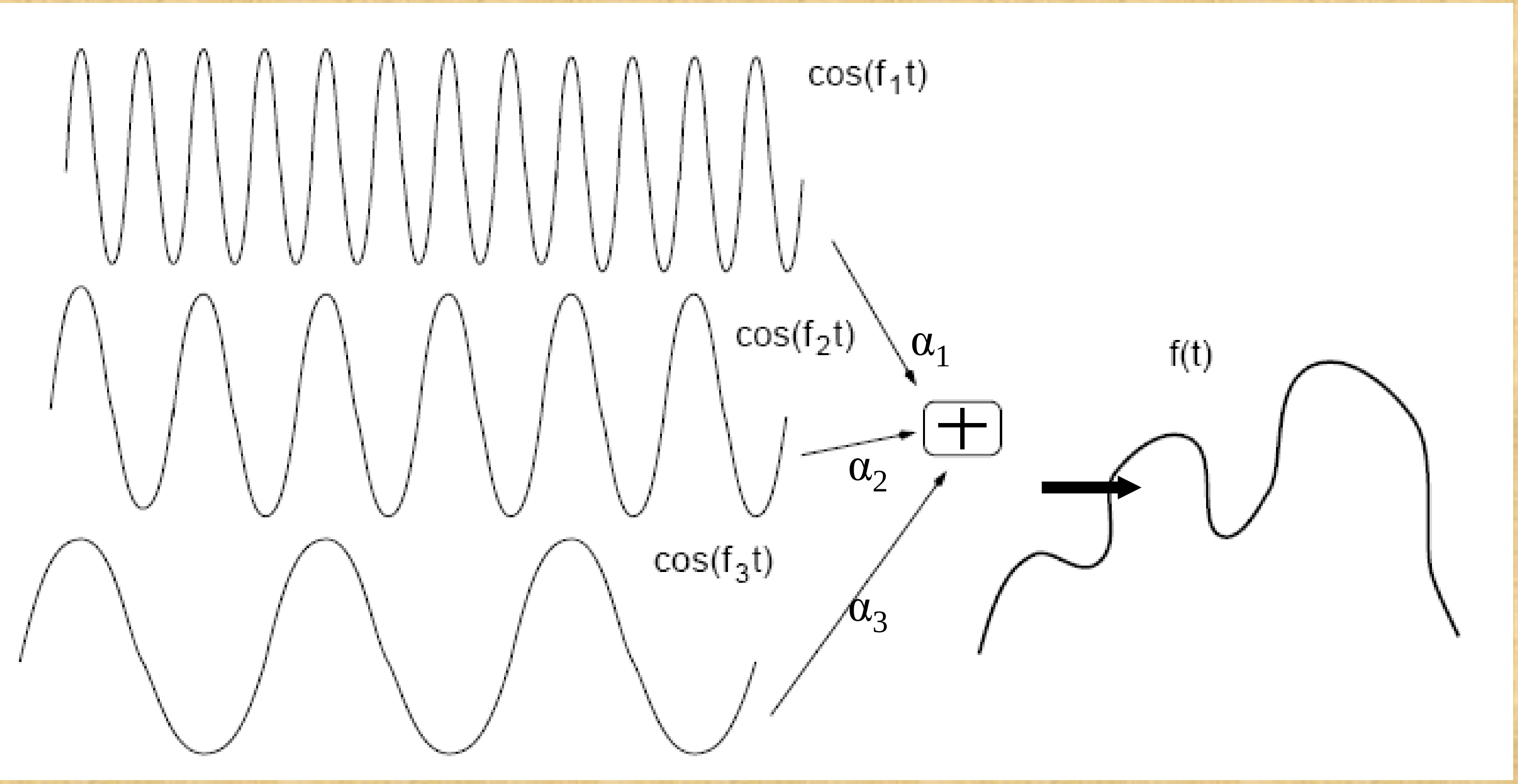
- Any periodic function f(t) can be expressed as a weighted sum (infinite) of sine and cosine functions of varying frequency:

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n cos(nf_0t) + \sum_{n=1}^{\infty} b_n sin(nf_0t)$$

f_0 is called the "fundamental frequency"

$$a_n = {}^1\!/_\pi \int_{-\pi}^{\pi} f(x)\ \cos(nf_0t)\ dt \qquad b_n = {}^1\!/_\pi \int_{-\pi}^{\pi} f(x)\ sin(nf_0t)\ dt$$

Fourier Series (cont.)

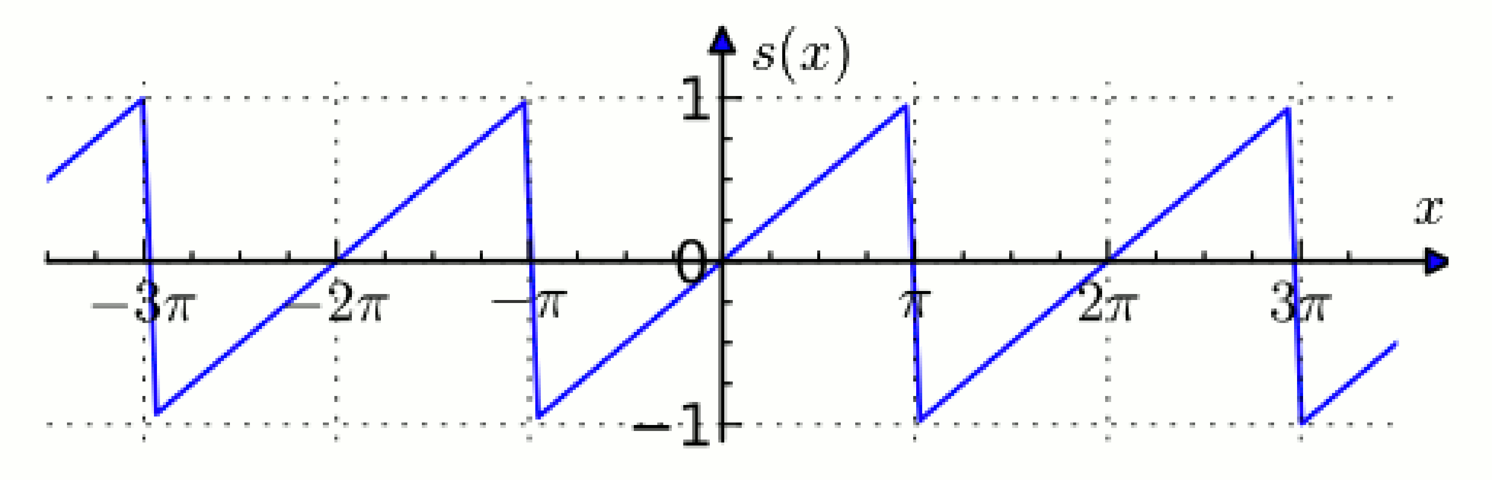


Fourier Series (cont.)

- Example:
  - Sawtoothwave:

$$s(x) = \frac{x}{\pi}, \qquad for \ -\pi < x < \pi$$

$$s(x + 2\pi k) = s(x), for \ -\infty < x < \infty, k \in \mathbb{Z}$$



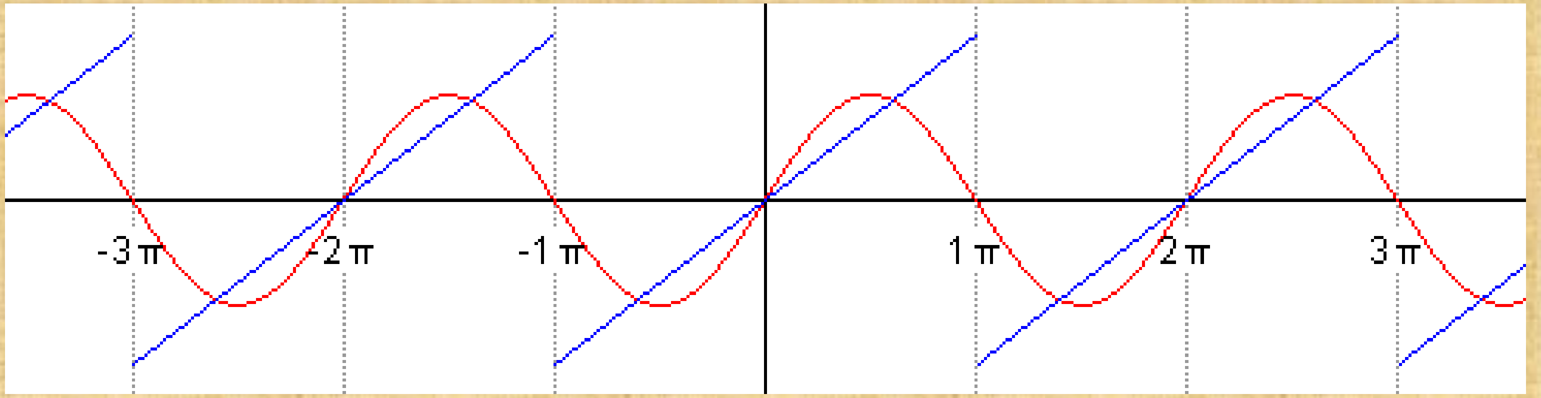
Fourier Series (cont.)

- Example:
  - The Fourier coefficients:

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} s(x)\cos(nx)\,dx = 0, \quad n \ge 0. \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} s(x)\sin(nx)\,dx \\
&= -\frac{2}{\pi n}\cos(n\pi) + \frac{2}{\pi^2 n^2}\sin(n\pi) \\
&= \frac{2\,(-1)^{n+1}}{\pi n}, \quad n \ge 1.
\end{aligned}
$$

Fourier Series (cont.)

- Example:
  - Animated plot of the first five successive partial Fourier series:



The Fourier transform of functions of one variable

The Fourier Transform of a function $f(t)$ is defined by

$$F(u) = \int_{-\infty}^{\infty} f(t) e^{-2\pi jut} dt,$$

and the inverse transform is defined by

$$f(t) = \int_{-\infty}^{\infty} F(u) e^{-2\pi jut} du,$$

The Fourier transform of functions of one variable (cont.)

Example:

Suppose that $f(t)$ is given as

$$f(t) = \begin{cases} a & -w/2 \le t \le w/2 \\ 0 & \text{otherwise} \end{cases}$$

The Fourier transform of this function is

The Fourier transform of functions of one variable (cont.)

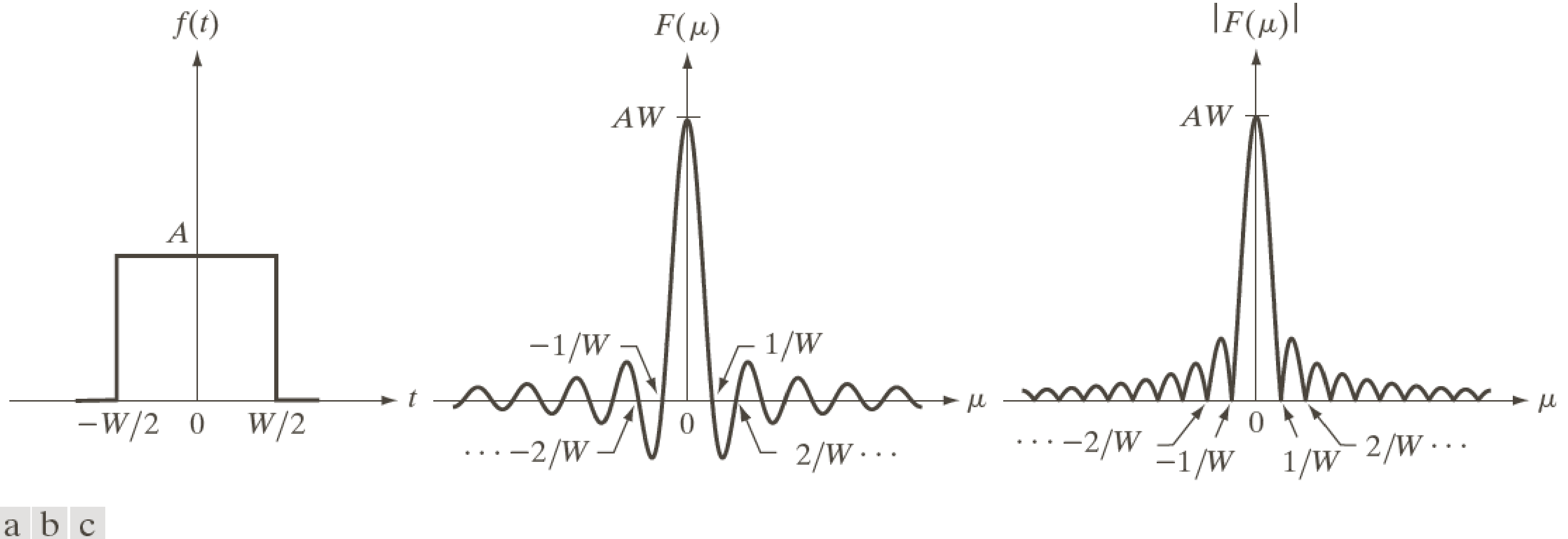


FIGURE 4.4 (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.

Mathematical foundation

- Convolution theorem

$$g(x,y) = f(x,y) * h(x,y)$$

FT

$$G(u,v) = F(u,v)H(u,v)$$

$$g(x,y) = \mathfrak{F}^{-1}\left[F(u,v)H(u,v)\right]$$

Inverse 2D Fourier Transform (FT)

Designing frequency-domain filters with MATLAB and the IPT

- 2 options:

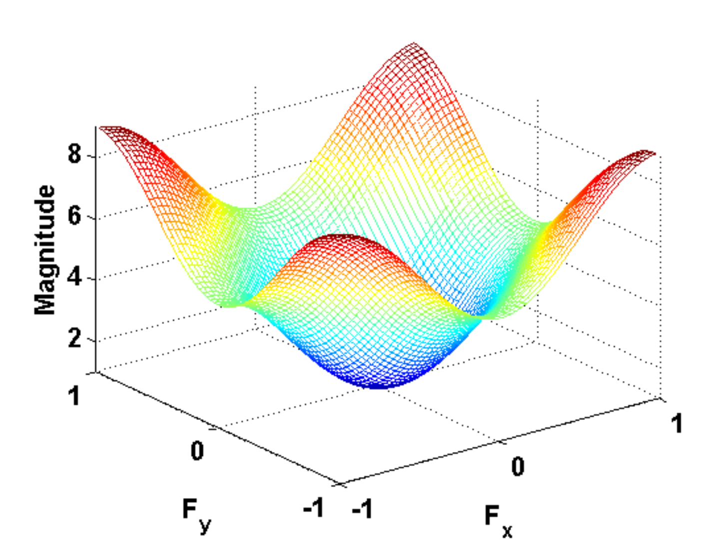
1. Obtain the frequency domain filter response function from spatial filter convolution mask. The IPT has a function that does exactly that: **`freqz2`**.

2. Generate filters directly in the frequency domain. In this case a meshgrid array (of the same size as the image) is created using the MATLAB function **`meshgrid`**.

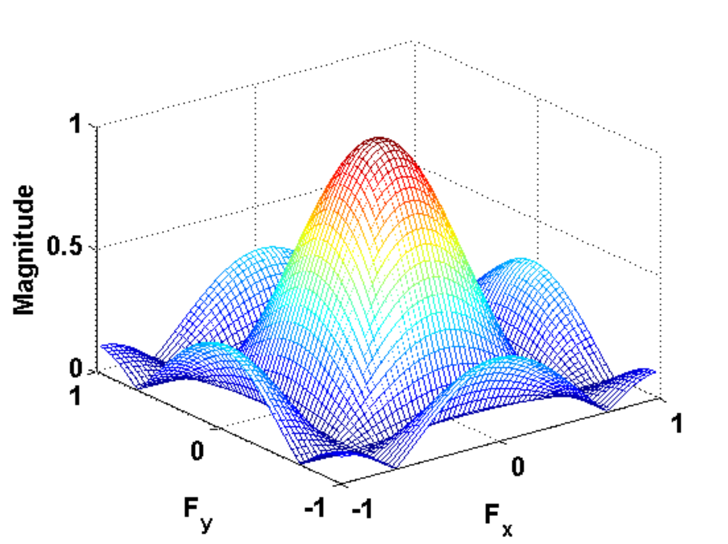
Examples of response functions for frequency-domain filters

- Example (PIVPUM):

```
h2 = ones(3,3) / 9
freqz2(h2) %(a)
h3 = [0 -1 0; -1 5 -1; 0 -1 0]
freqz2(h3) %(b)
```



(a)



(b)

Mathematical foundation (again ☺)

- *Transform*: a mathematical tool that allows the conversion of a set of values to another set of values, creating, therefore, a new way of representing the same information.
- In image processing:
  - original domain = spatial domain
  - results of frequency-domain operations lie in the transform domain.
- Motivation for using mathematical transforms in image processing: some tasks (e.g., frequency-domain filtering, image compression) are best performed by applying selected algorithms in the transform domain. transformation to the result

Mathematical foundation (cont.)

- 2D Fourier transform (in MATLAB: **`fft2`** and **`ifft2`**)

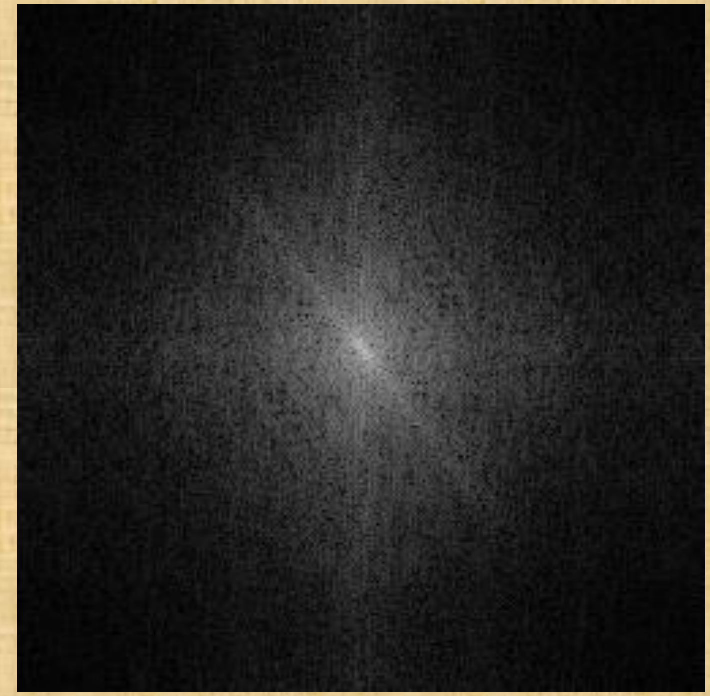
$$F(u,v) = T(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) \cdot \exp[-j2\pi(ux/M + vy/N)]$$

$$f(x,y) = \mathfrak{F}^{-1}[F(u,v)] = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} T(u,v) \cdot \exp[j2\pi(ux/M + vy/N)]$$

$$|F(u,v)| = \sqrt{[R(u,v)]^2 + [I(u,v)]^2}$$

$$\phi(u,v) = \arctan\left[\frac{I(u,v)}{R(u,v)}\right]$$

Example of an image and its FT



```
I = imread('Figure11_04_a.png');
Id = im2double(I);
ft = fft2(Id);
ft_shift = fftshift(ft);
imshow(log(1 + abs(ft_shift)), [])
```

Fourier Transform: properties

- Linearity

$$\mathfrak{F}[a \cdot f1(x,y) + b \cdot f2(x,y)] = a \cdot F1(u,v) + b \cdot F2(u,v)$$

and

$$a \cdot f1(x,y) + b \cdot f2(x,y) = \mathfrak{F}^{-1}[a \cdot F1(u,v) + b \cdot F2(u,v)]$$

Fourier Transform: properties (cont.)

- Translation

$$\mathfrak{F}[f(x-x_0,y-y_0)] = F(u,v)\cdot\exp[-j2\pi(ux_0/M+vy_0/N)]$$

and

$$f(x-x_0,y-y_0) = \mathfrak{F}^{-1}[F(u,v)\cdot\exp[j2\pi(ux_0/M+vy_0/N)]$$

Fourier Transform: properties (cont.)

- Conjugate symmetry

$$F(u,v) = F^*(-u,-v)$$

$F^*(u,v)$ is the conjugate of $F(u,v)$

where:

$$F(u,v) = R(u,v) + jI(u,v)$$

i.e., if:

$$F^*(u,v) = R(u,v) - jI(u,v)$$

then:

$$|F(u,v)| = |F(-u,-v)|$$

Fourier Transform: properties (cont.)

- Periodicity

$$F(u,v) = F(u+M, v+N)$$

and

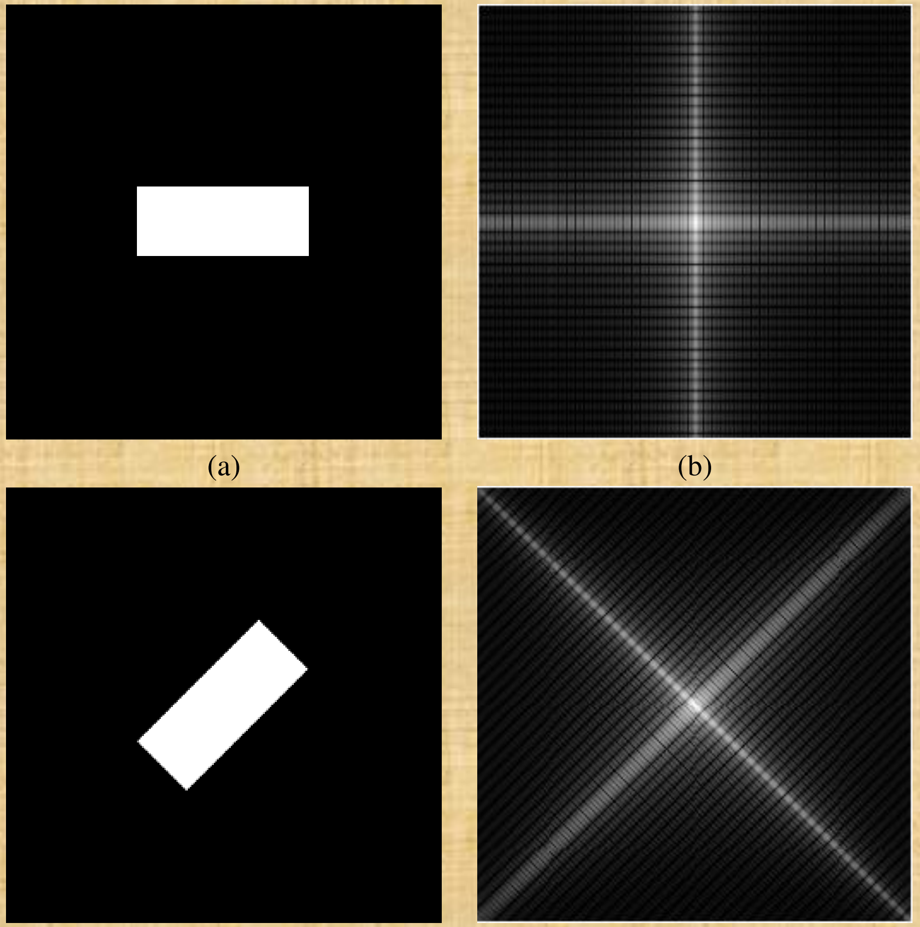
$$f(x,y) = f(x+M, y+N)$$

Fourier Transform: properties (cont.)

- Separability
  - The Fourier Transform is separable, i.e., the FT of a 2D image can be computed by two passes of the 1D FT algorithm, once along the rows (columns), followed by another pass along the columns (rows) of the result.

Fourier Transform: properties (cont.)

- Rotation
  - If an image is rotated by a certain angle θ, its 2D FT will be rotated by the same angle.



(a) (b) (c) (d)

Continuous Fourier Transform (FT)

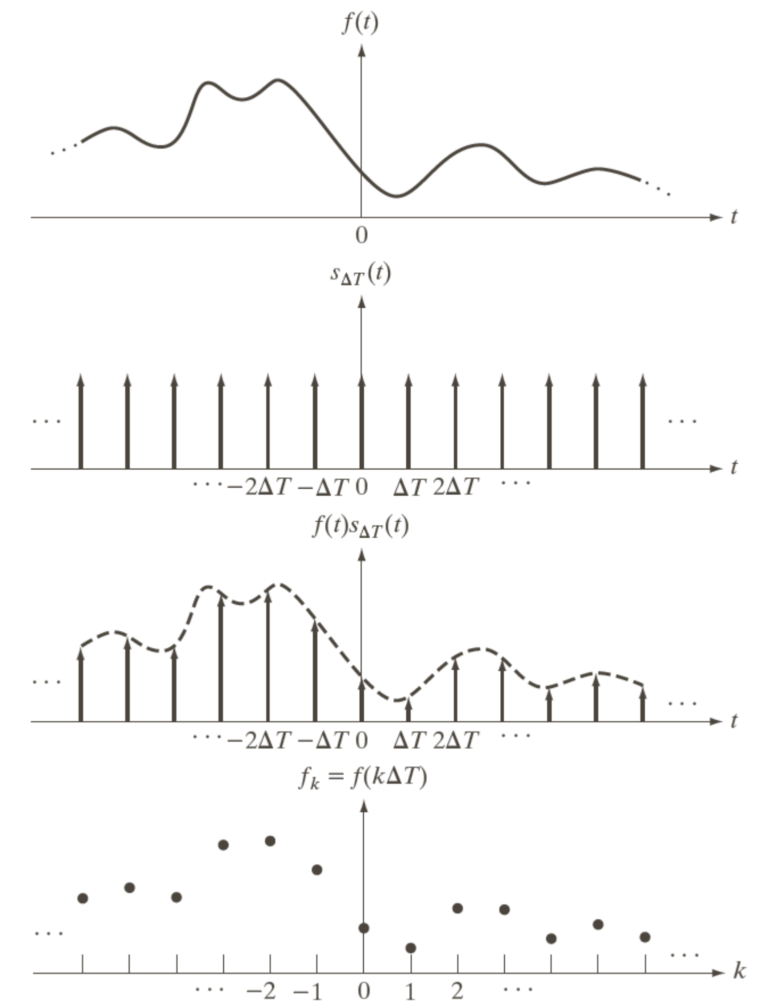
- Transforms a signal (i.e., function) from the **spatial** (x) domain to the **frequency** (u) domain.

Forward FT: $$F(f(x)) = F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} dx$$

Inverse FT: (IFT) $$F^{-1}(F(u)) = f(x) = \int_{-\infty}^{\infty} F(u) e^{j2\pi ux} du$$

where $$e^{\pm j\theta} = cos(\theta) \pm jsin(\theta)$$

Sampling



a
b
c
d

FIGURE 4.5
(a) A continuous function. (b) Train of impulses used to model the sampling process. (c) Sampled function formed as the product of (a) and (b). (d) Sample values obtained by integration and using the sifting property of the impulse. (The dashed line in (c) is shown for reference. It is not part of the data.)

The Discrete Fourier Transform(DFT) of one variable

Using M equally spaced sample values the DFT is defined as

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-2\pi jux/M}, \quad u = 0, 1, 2, \ldots, M-1$$

and the Inverse DFT (IDFT) is defined as

$$f(x) = \frac{1}{m} \sum_{u=0}^{M-1} F(u) e^{-2\pi jux/M}, \quad x = 0, 1, 2, \ldots, M-1$$

The Fourier transform of functions of two variables

The Fourier Transform of a 2D function $f(u,v)$ is defined by

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)e^{-2\pi j(ut+vz)}dtdz$$

and the inverse transform is defined by

$$f(t,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{-2\pi j(ut+vz)}dudv,$$

The Fourier transform of functions of two variables (cont.)

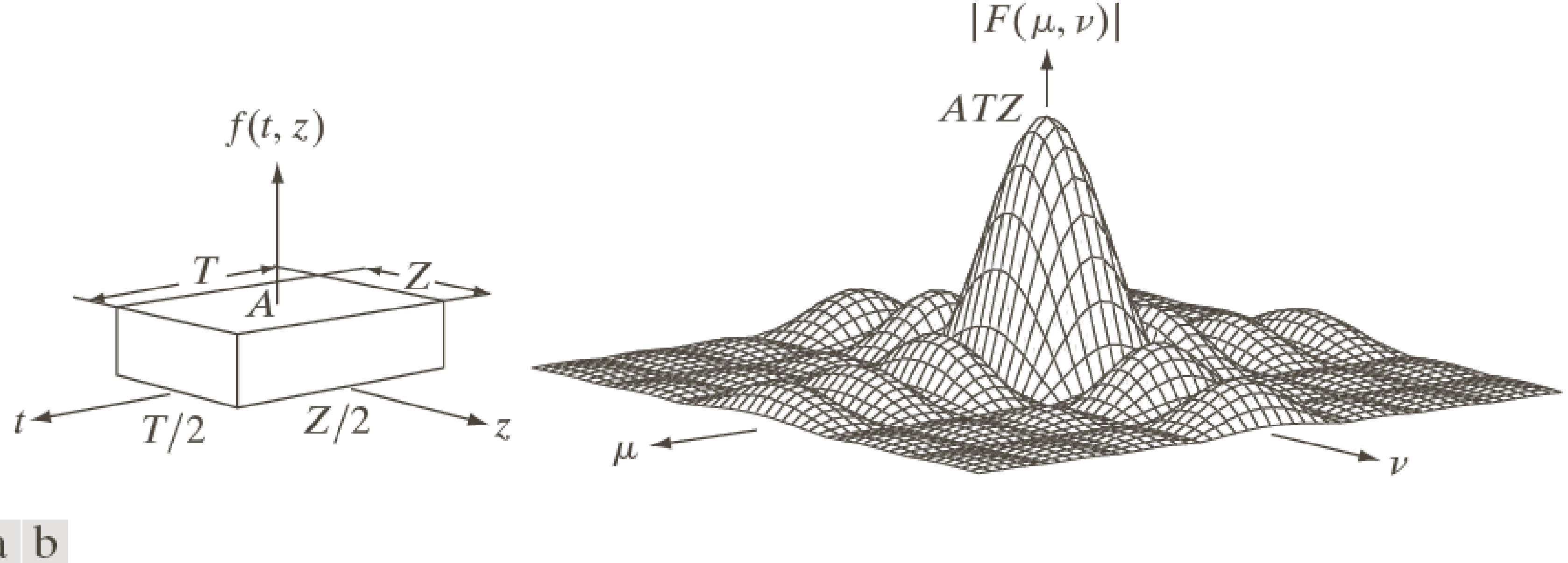


FIGURE 4.13 (a) A 2-D function, and (b) a section of its spectrum (not to scale). The block is longer along the t-axis, so the spectrum is more "contracted" along the μ-axis. Compare with Fig. 4.4.

The Fourier transform of functions of two variables (cont.)

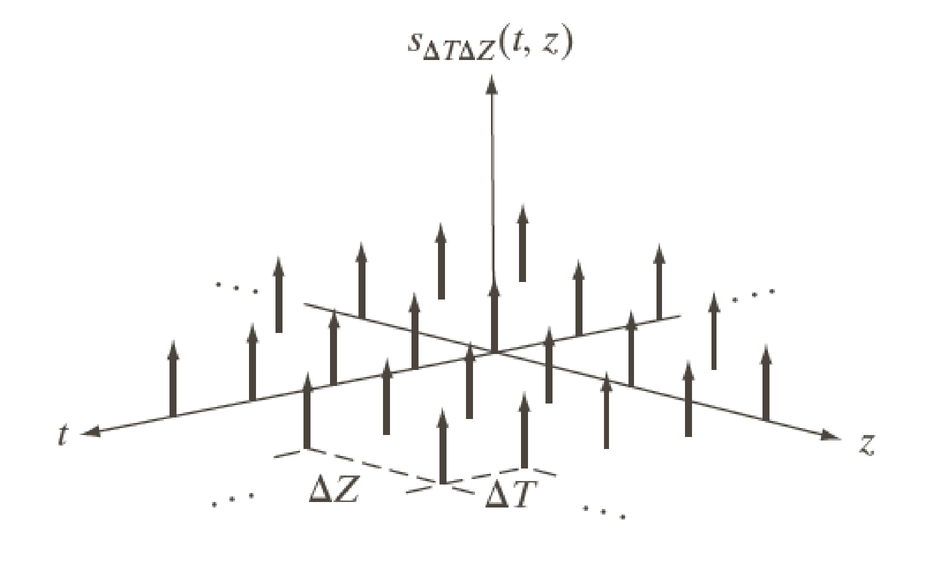
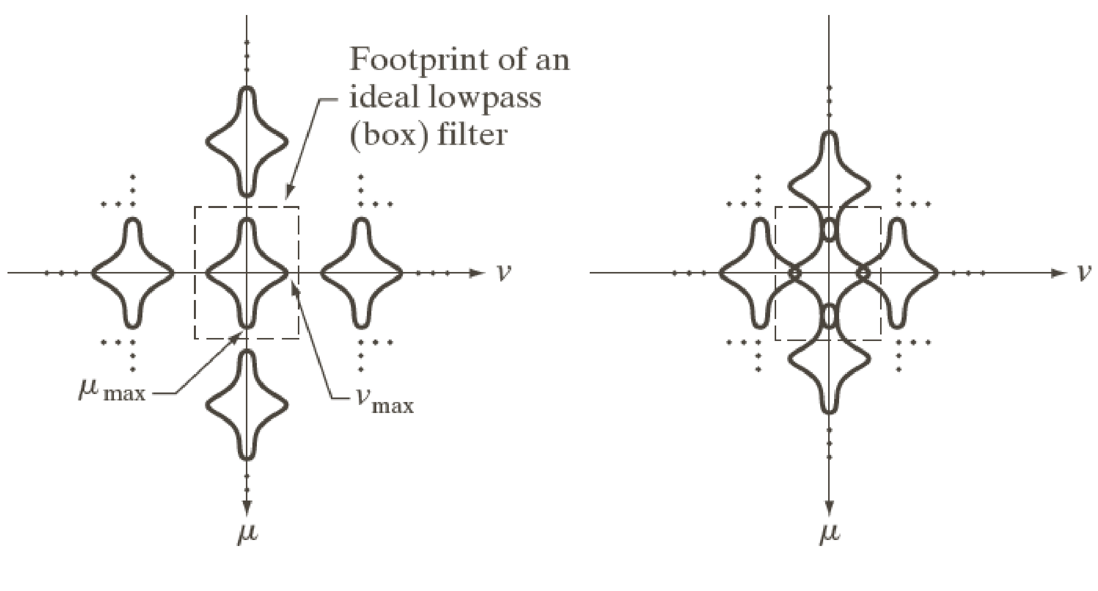


FIGURE 4.14 Two-dimensional impulse train.

The Fourier transform of functions of two variables (cont.)



a b

FIGURE 4.15 Two-dimensional Fourier transforms of (a) an over-sampled, and (b) under-sampled band-limited function.

The 2D discrete Fourier transform

The 2D Discrete Fourier Transform (DFT) for an $M \times N$ image whose intensity values are represented by $f(x, y)$, defined as

$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x, y)e^{-2\pi j(ux/M + vy/N)},$$

where $u = 1, 2 ..., M - 1,\ v = 1, 2, ..., N - 1.$

The frequency domain is the coordinate system based on the frequency variables u and v.

2D DFT (cont.)

The 2D inverse Discrete Fourier Transform (IDFT) for an $M \times N$ image is given by

$$f(x,y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) e^{2\pi j(ux/M + vy/N)},$$

where $x = 1, 2, ..., M-1$ and $y = 1, 2, ..., N-1$.
Given the DFT of an image we can obtain the original image by IDFT.

Application of DFT

Clearly, although $f(x, y)$ is real, its Fourier transform is complex. A visual analysis is based on its spectrum which is defined as

$$|F(u,v)| = [R^2(u,v) + I^2(u,v)]^{1/2}$$

The following function is defined to be the phase angle of the transformation

$$\phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$$

Application of DFT (cont.)

The functions $|F(u,v)|$ and $\phi(u,v)$ can be used to express $F(u,v)$ in polar representation:

$$F(u,v) = |F(u,v)| e^{j\phi(u,v)}$$

The power spectrum is defined to be the square of magnitude

$$P(u,v) = |F(u,v)|^2 = R^2(u,v) + I^2(u,v)$$

Application of DFT (cont.)

- It can be shown that

$$F(u,v) = F(u+M,v) = F(u,v+N) = F(u+M,v+N)$$

- The DFT is infinitely periodic in both u and v directions, with the periodicity determined by M and N.

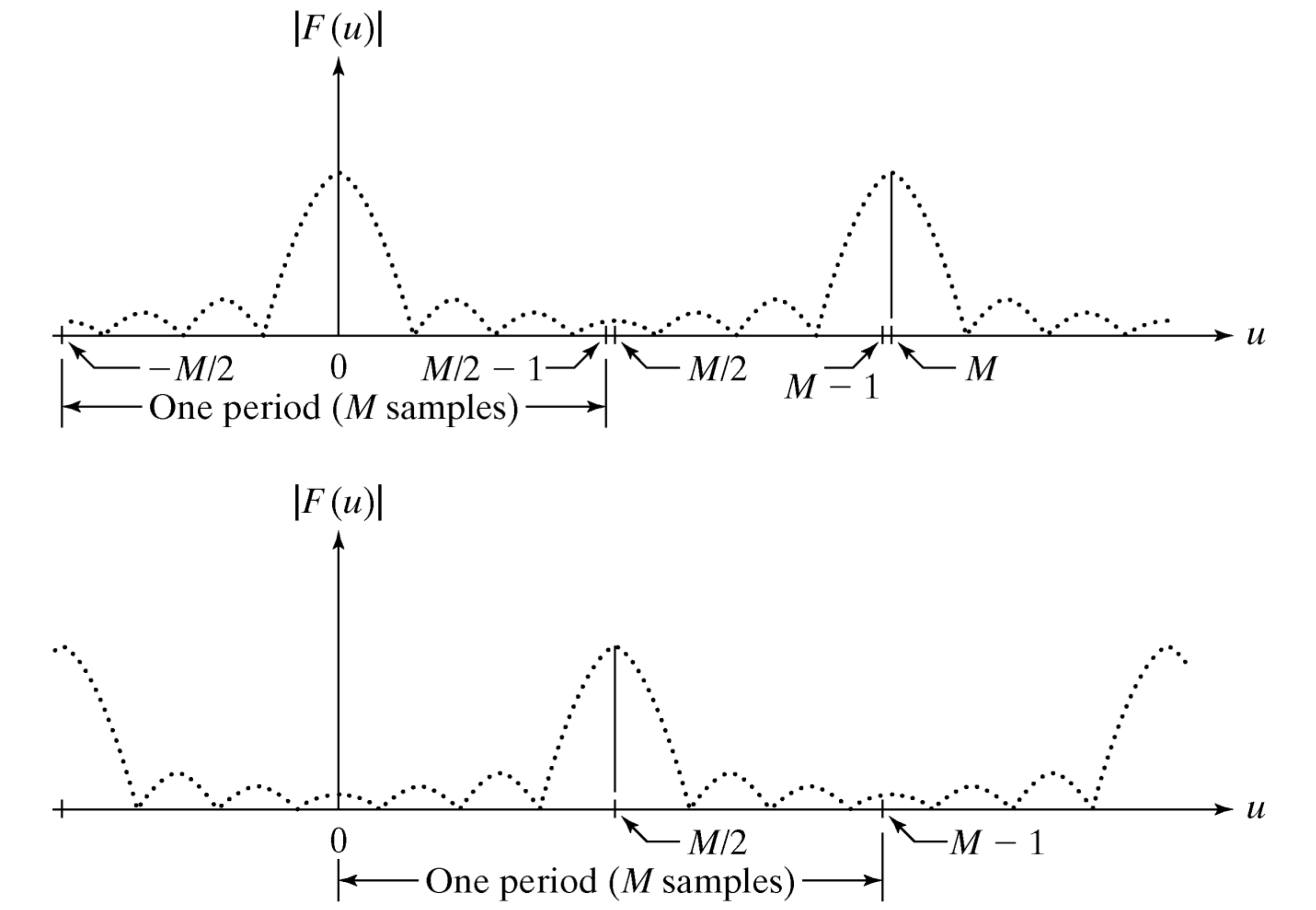
Application of DFT (cont.)

- Periodicity is also a property of the inverse DFT.

$$f(x, y) = f(x + M, y) = f(x, y + N) = f(x + M, y + N)$$

- An image obtained by taking the inverse DFT is also infinitely periodic in both *u* and *v* directions, with the periodicity determined by *M* and *N*.

Application of DFT (cont.)



a
b

FIGURE 4.1
(a) Fourier spectrum showing back-to-back half periods in the interval $[0, M - 1]$.
(b) Centered spectrum in the same interval, obtained by multiplying $f(x)$ by $(-1)^x$ prior to computing the Fourier transform.

Application of DFT (cont.)

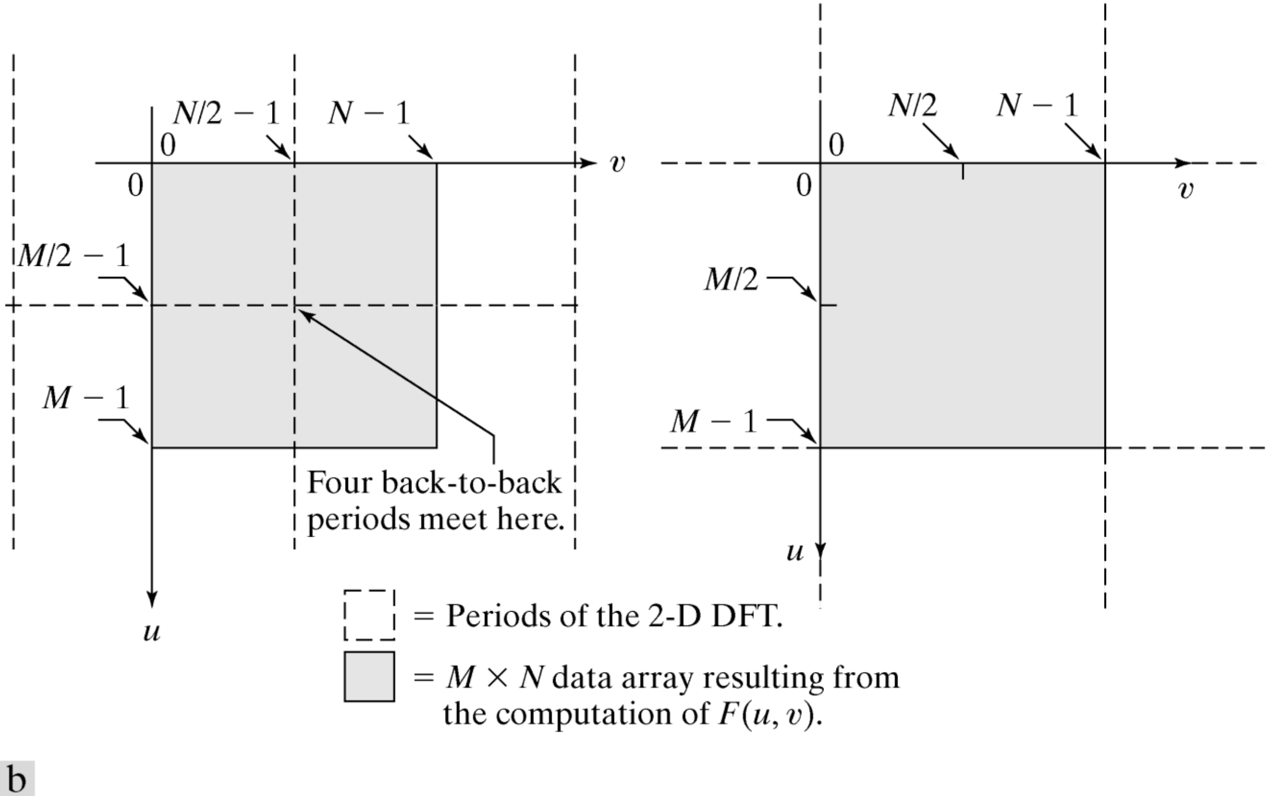
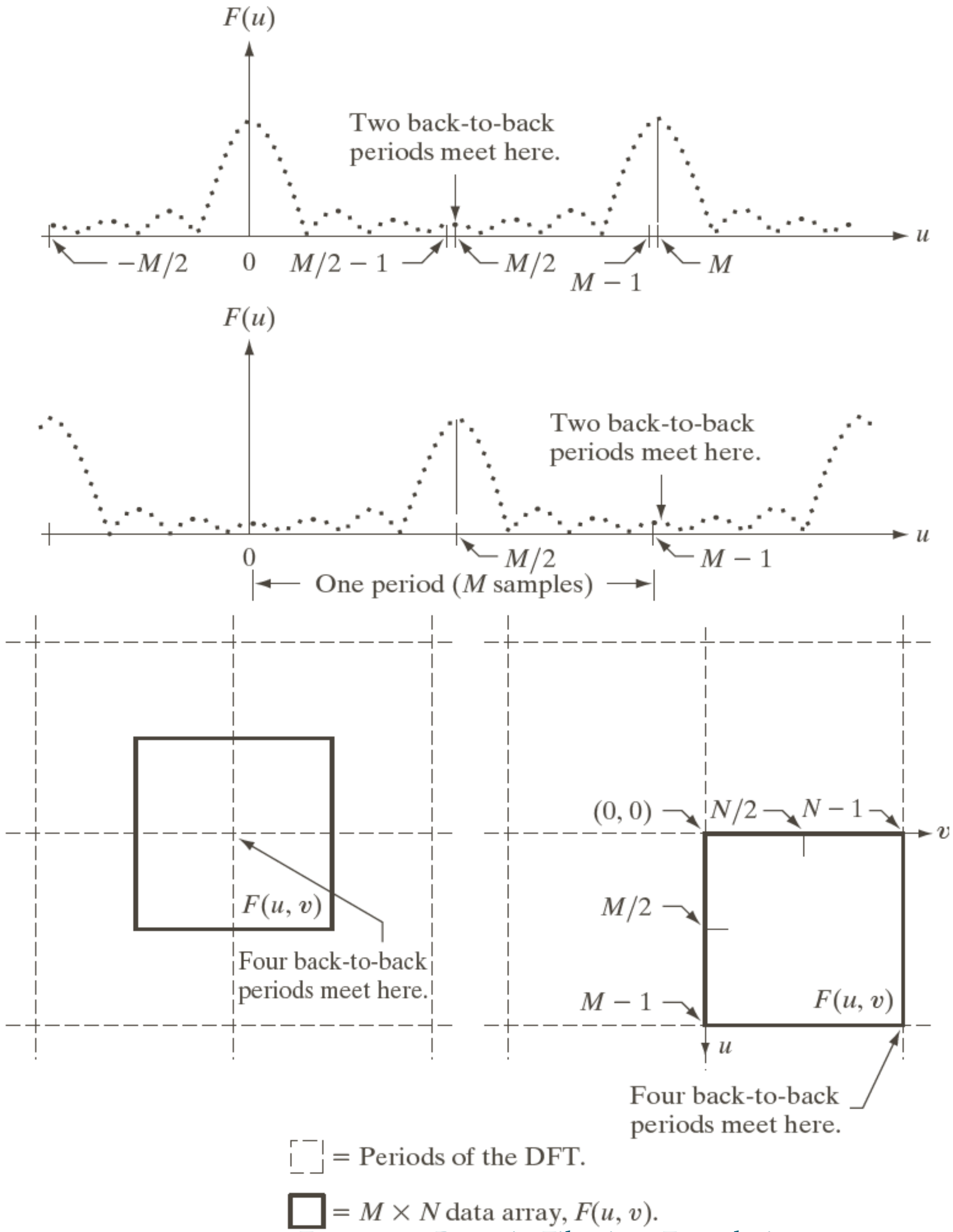


FIGURE 4.2 (a) $M \times N$ Fourier spectrum (shaded), showing four back-to-back quarter periods contained in the spectrum data. (b) Spectrum obtained by multiplying $f(x, y)$ by $(-1)^{x+y}$ prior to computing the Fourier transform. Only one period is shown shaded because this is the data that would be obtained by an implementation of the equation for $F(u, v)$.



a
b
c d

FIGURE 4.23 Centering the Fourier transform. (a) A 1-D DFT showing an infinite number of periods. (b) Shifted DFT obtained by multiplying $f(x)$ by $(-1)^x$ before computing $F(u)$. (c) A 2-D DFT showing an infinite number of periods. The solid area is the $M \times N$ data array, $F(u, v)$, obtained with Eq. (4.5-15). This array consists of four quarter periods. (d) A Shifted DFT obtained by multiplying $f(x, y)$ by $(-1)^{x+y}$ before computing $F(u, v)$. The data now contains one complete, centered period, as in (b).

Computing and visualizing the 2D DFT

- The FFT of an $M \times N$ image array is obtained by the syntax

F=fft2(f)

- The Fourier spectrum is obtained as

S=abs(F)

Computing and visualizing the 2D DFT (cont.)

- The funcion fftshift moves the origin of transform to the center of the frequency rectangle.

  Fc=fftshift (F)

- Log transformation:

  S2=log(1+abs(Fc))

- Function ifftshift reverses the centering:

  F=ifftshift(Fc)

Computing and visualizing the 2D DFT (cont.)

- To compute inverse FFT

  f=ifft2 (F)

- If the imput used to compute F is real then the inverse should also be real. However fft2 often has small imaginary components resulting from round-off errors. It is good practice to extract the real part of the result:

  f=real(ifft2(F))

Computing and visualizing the 2D DFT (cont.)

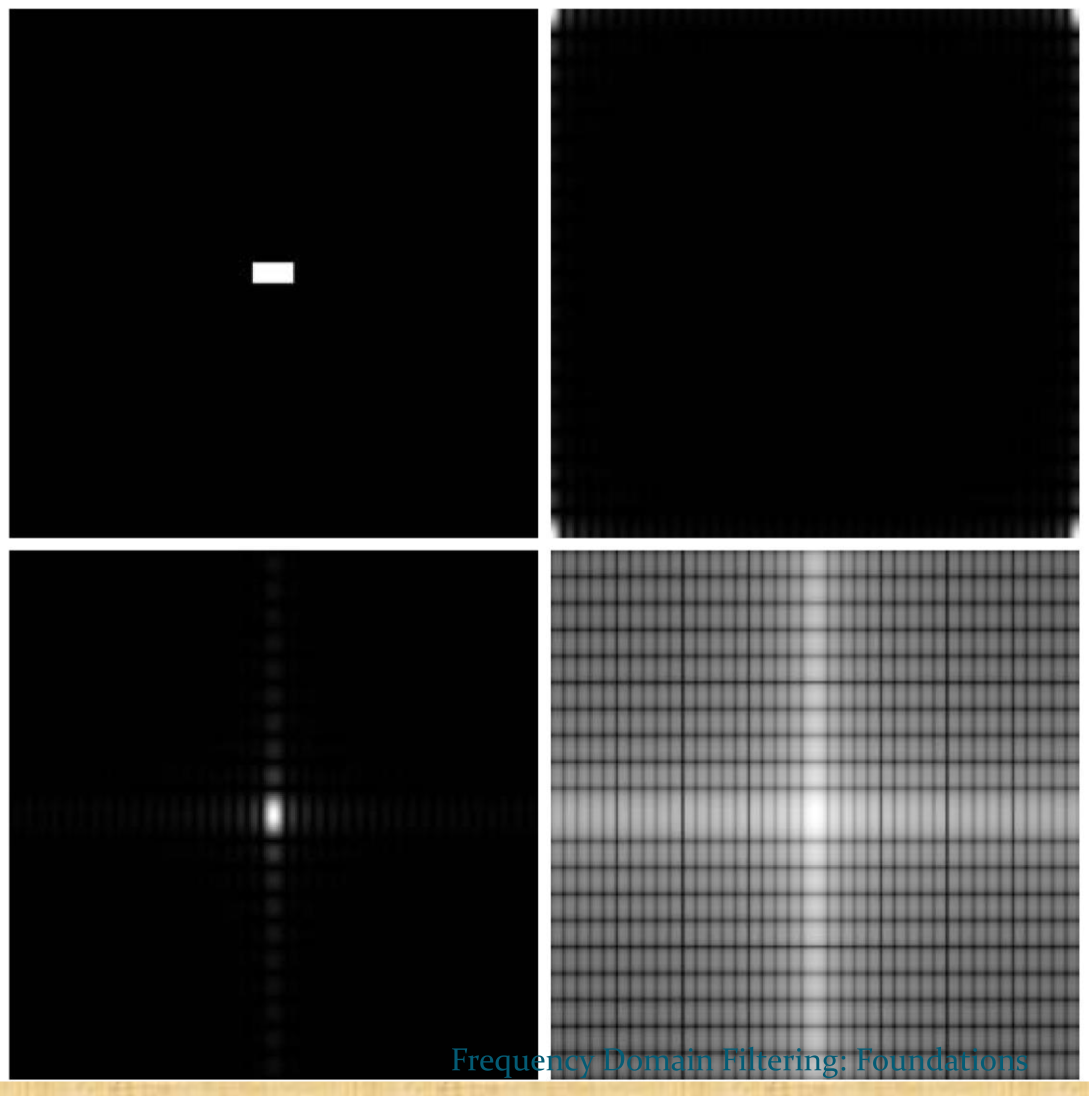
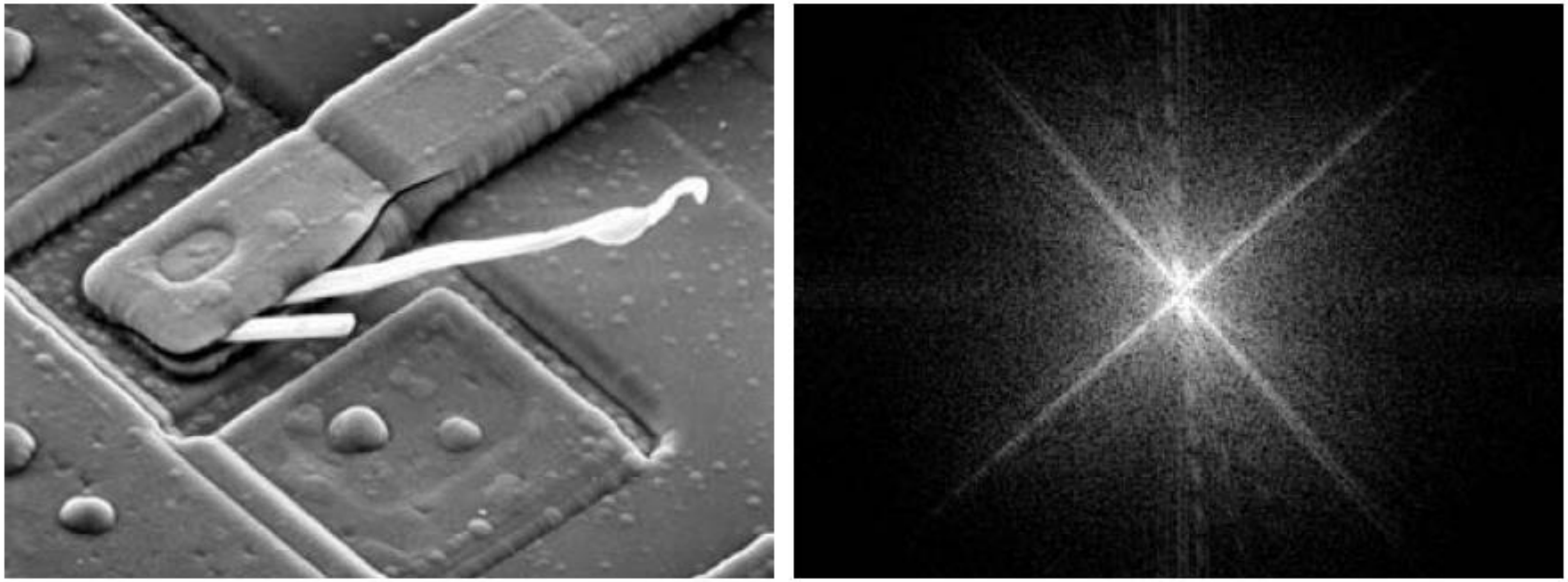


FIGURE 4.3
(a) A simple image.
(b) Fourier spectrum.
(c) Centered spectrum.
(d) Spectrum visually enhanced by a log transformation.

Computing and visualizing the 2D DFT (cont.)



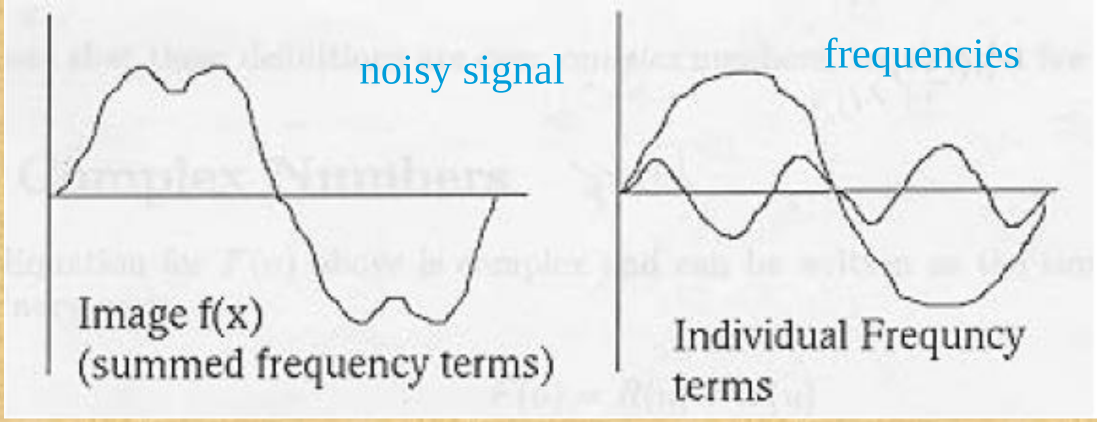
a b

FIGURE 4.29 (a) SEM image of a damaged integrated circuit. (b) Fourier spectrum of (a). (Original image courtesy of Dr. J. M. Hudak, Brockhouse Institute for Materials Research, McMaster University, Hamilton, Ontario, Canada.)

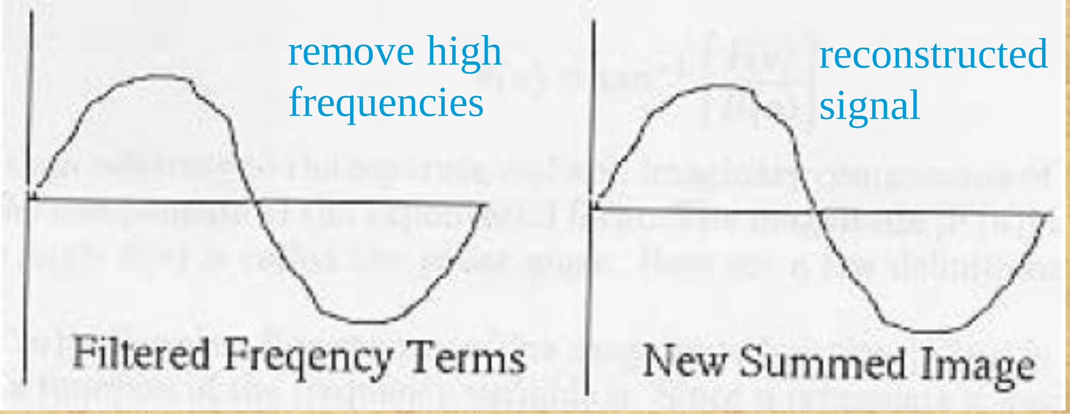
Why is FT Useful?

- Easier to remove undesirable frequencies.
- Faster perform certain operations in the **frequency** domain than in the **spatial** domain.

Example: Removing undesirable frequencies

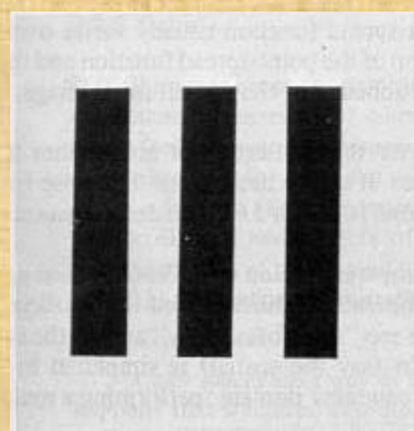


To remove certain frequencies, set their corresponding F(u) coefficients to zero!

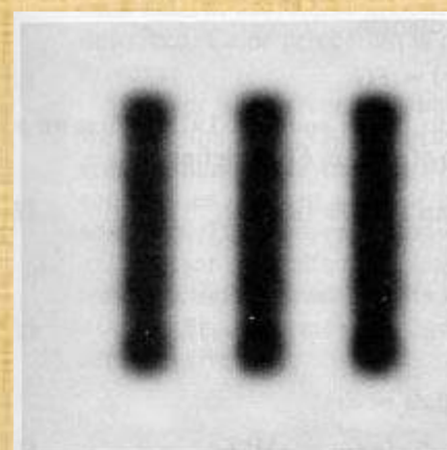


How do frequencies show up in an image?

- Low frequencies correspond to slowly varying information (e.g., continuous surface).
- High frequencies correspond to quickly varying information (e.g., edges)

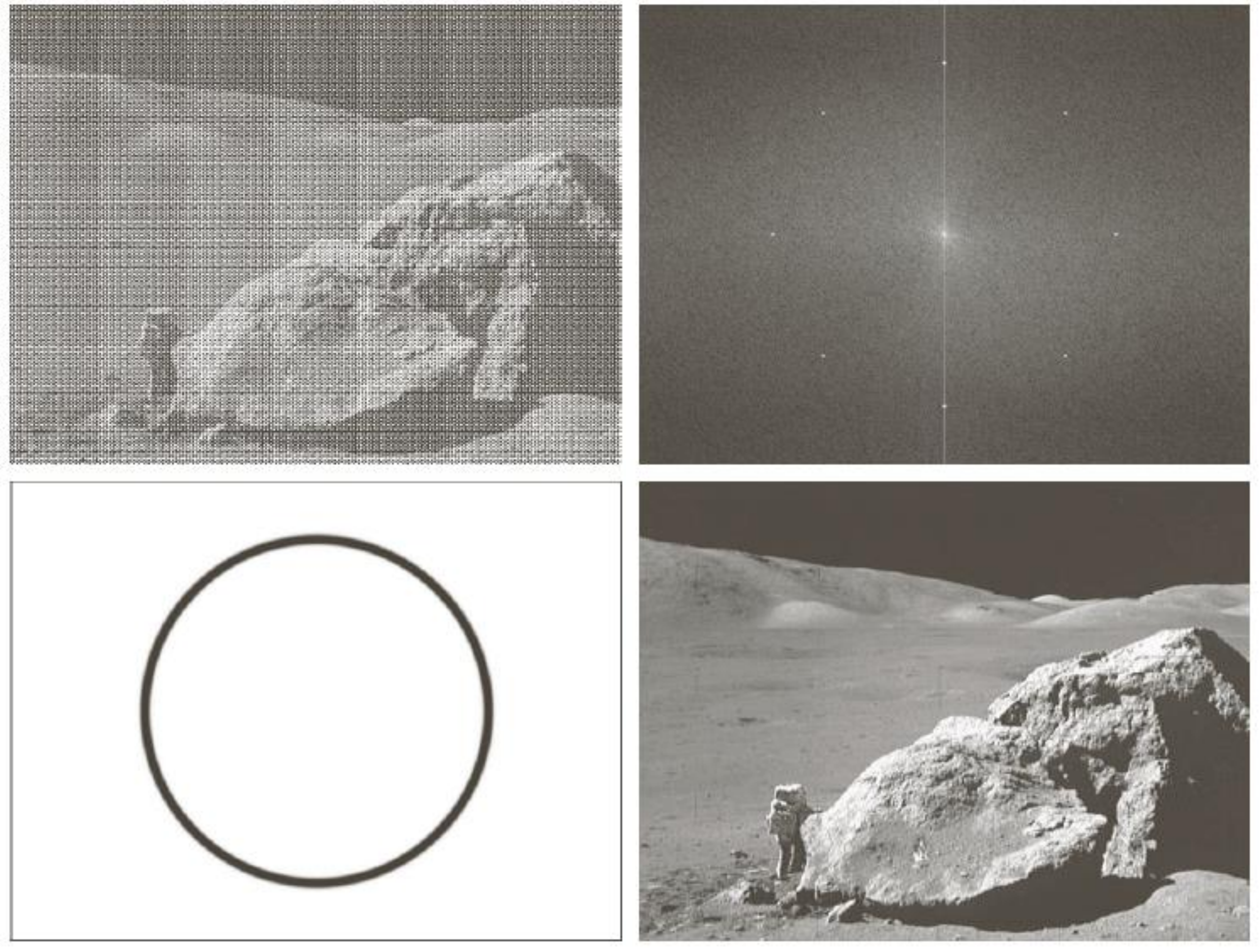


Original Image



Low-passed

Example of noise reduction using FT



Symmetry properties of 2D DFT

	Spatial Domain†		Frequency Domain†
1)	$f(x, y)$ real	$\Leftrightarrow$	$F^*(u, v) = F(-u, -v)$
2)	$f(x, y)$ imaginary	$\Leftrightarrow$	$F^*(-u, -v) = -F(u, v)$
3)	$f(x, y)$ real	$\Leftrightarrow$	$R(u, v)$ even; $I(u, v)$ odd
4)	$f(x, y)$ imaginary	$\Leftrightarrow$	$R(u, v)$ odd; $I(u, v)$ even
5)	$f(-x, -y)$ real	$\Leftrightarrow$	$F^*(u, v)$ complex
6)	$f(-x, -y)$ complex	$\Leftrightarrow$	$F(-u, -v)$ complex
7)	$f^*(x, y)$ complex	$\Leftrightarrow$	$F^*(-u - v)$ complex
8)	$f(x, y)$ real and even	$\Leftrightarrow$	$F(u, v)$ real and even
9)	$f(x, y)$ real and odd	$\Leftrightarrow$	$F(u, v)$ imaginary and odd
10)	$f(x, y)$ imaginary and even	$\Leftrightarrow$	$F(u, v)$ imaginary and even
11)	$f(x, y)$ imaginary and odd	$\Leftrightarrow$	$F(u, v)$ real and odd
12)	$f(x, y)$ complex and even	$\Leftrightarrow$	$F(u, v)$ complex and even
13)	$f(x, y)$ complex and odd	$\Leftrightarrow$	$F(u, v)$ complex and odd

†Recall that x, y, u, and v are *discrete* (integer) variables, with x and u in the range $[0, M - 1]$, and y, and v in the range $[0, N - 1]$. To say that a complex function is *even* means that its real *and* imaginary parts are even, and similarly for an odd complex function.

TABLE 4.1 Some symmetry properties of the 2-D DFT and its inverse. $R(u, v)$ and $I(u, v)$ are the real and imaginary parts of $F(u, v)$, respectively. The term *complex* indicates that a function has nonzero real and imaginary parts.

Summary of 2D DFT properties

Name	Expression(s)
1) Discrete Fourier transform (DFT) of $f(x, y)$	$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)}$
2) Inverse discrete Fourier transform (IDFT) of $F(u, v)$	$f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(ux/M + vy/N)}$
3) Polar representation	$F(u, v) = \|F(u, v)\| e^{j\phi(u,v)}$
4) Spectrum	$\|F(u, v)\| = \left[R^2(u, v) + I^2(u, v)\right]^{1/2}$ $R = \text{Real}(F); \quad I = \text{Imag}(F)$
5) Phase angle	$\phi(u, v) = \tan^{-1}\left[\frac{I(u, v)}{R(u, v)}\right]$
6) Power spectrum	$P(u, v) = \|F(u, v)\|^2$
7) Average value	$\overline{f}(x, y) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) = \frac{1}{MN} F(0, 0)$

(*Continued*)

TABLE 4.2
Summary of DFT definitions and corresponding expressions.

Summary of 2D DFT properties (cont.)

Name	Expression(s)
8) Periodicity (k_1 and k_2 are integers)	$F(u, v) = F(u + k_1M, v) = F(u, v + k_2N)$ $= F(u + k_1M, v + k_2N)$ $f(x, y) = f(x + k_1M, y) = f(x, y + k_2N)$ $= f(x + k_1M, y + k_2N)$
9) Convolution	$f(x, y) \bigstar h(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n)h(x - m, y - n)$
10) Correlation	$f(x, y) \star h(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f^*(m, n)h(x + m, y + n)$
11) Separability	The 2-D DFT can be computed by computing 1-D DFT transforms along the rows (columns) of the image, followed by 1-D transforms along the columns (rows) of the result. See Section 4.11.1.
12) Obtaining the inverse Fourier transform using a forward transform algorithm.	$MNf^*(x, y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F^*(u, v)e^{-j2\pi(ux/M + vy/N)}$ This equation indicates that inputting $F^*(u, v)$ into an algorithm that computes the forward transform (right side of above equation) yields $MNf^*(x, y)$. Taking the complex conjugate and dividing by MN gives the desired inverse. See Section 4.11.2.

Summary of 2D DFT pairs

Name	DFT Pairs
1) Symmetry properties	See Table 4.1
2) Linearity	$af_1(x, y) + bf_2(x, y) \Leftrightarrow aF_1(u, v) + bF_2(u, v)$
3) Translation (general)	$f(x, y)e^{j2\pi(u_0x/M + v_0y/N)} \Leftrightarrow F(u - u_0, v - v_0)$ $f(x - x_0, y - y_0) \Leftrightarrow F(u, v)e^{-j2\pi(ux_0/M + vy_0/N)}$
4) Translation to center of the frequency rectangle, $(M/2, N/2)$	$f(x, y)(-1)^{x+y} \Leftrightarrow F(u - M/2, v - N/2)$ $f(x - M/2, y - N/2) \Leftrightarrow F(u, v)(-1)^{u+v}$
5) Rotation	$f(r, \theta + \theta_0) \Leftrightarrow F(\omega, \varphi + \theta_0)$ $x = r\cos\theta \quad y = r\sin\theta \quad u = \omega\cos\varphi \quad v = \omega\sin\varphi$
6) Convolution theorem[†]	$f(x, y) \star h(x, y) \Leftrightarrow F(u, v)H(u, v)$ $f(x, y)h(x, y) \Leftrightarrow F(u, v) \star H(u, v)$

(Continued)

TABLE 4.3
Summary of DFT pairs. The closed-form expressions in 12 and 13 are valid only for continuous variables. They can be used with discrete variables by sampling the closed-form, continuous expressions.

Summary of 2D DFT pairs (cont.)

Name	DFT Pairs
7) Correlation theorem†	$f(x,y) \star h(x,y) \Leftrightarrow F^*(u,v)\,H(u,v)$ $f^*(x,y)h(x,y) \Leftrightarrow F(u,v) \star H(u,v)$
8) Discrete unit impulse	$\delta(x,y) \Leftrightarrow 1$
9) Rectangle	$\text{rect}[a,b] \Leftrightarrow ab\dfrac{\sin(\pi ua)}{(\pi ua)}\dfrac{\sin(\pi vb)}{(\pi vb)}e^{-j\pi(ua+vb)}$
10) Sine	$\sin(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow$ $j\dfrac{1}{2}\left[\delta(u+Mu_0, v+Nv_0) - \delta(u-Mu_0, v-Nv_0)\right]$
11) Cosine	$\cos(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow$ $\dfrac{1}{2}\left[\delta(u+Mu_0, v+Nv_0) + \delta(u-Mu_0, v-Nv_0)\right]$

The following Fourier transform pairs are derivable only for continuous variables, denoted as before by t and z for spatial variables and by μ and ν for frequency variables. These results can be used for DFT work by sampling the continuous forms.

Name	DFT Pairs
12) *Differentiation* (The expressions on the right assume that $f(\pm\infty, \pm\infty) = 0$.)	$\left(\dfrac{\partial}{\partial t}\right)^m\left(\dfrac{\partial}{\partial z}\right)^n f(t,z) \Leftrightarrow (j2\pi\mu)^m(j2\pi\nu)^n F(\mu,\nu)$ $\dfrac{\partial^m f(t,z)}{\partial t^m} \Leftrightarrow (j2\pi\mu)^m F(\mu,\nu);\ \dfrac{\partial^n f(t,z)}{\partial z^n} \Leftrightarrow (j2\pi\nu)^n F(\mu,\nu)$
13) *Gaussian*	$A2\pi\sigma^2 e^{-2\pi^2\sigma^2(t^2+z^2)} \Leftrightarrow Ae^{-(\mu^2+\nu^2)/2\sigma^2}$ (A is a constant)

†Assumes that the functions have been extended by zero padding. Convolution and correlation are associative, commutative, and distributive.

Summary

- You should now know
  - The Fundamentals of frequency domain filtering

References

- Rafael C. Gonzalez, Richard E. Woods, “Digital Image Processing, 4th Ed.”; Pearson, 2018, (DIP)
- Rafael C. Gonzalez, Richard E. Woods, Steven L. Eddins, “Digital Image Processing Using MATLAB”; Pearson, 2004, (DIPUM)
- Oge Marques, “Practical Image and Video Processing Using MATLAB”; Wiley, 2011, (PIVPUM)
- Ronald N. Bracewell, “The Fourier Transform and Its Applications, 3rd Ed.”; McGraw Hill, 2000
- Hüseyin Yalaz, Samsun M. Başarıcı, Erkan Özefe, “Systemtheoretische Untersuchung der Gauß-, LoG-, DoG- und Gabor-Funktionen”; M.Sc. Thesis, Universität Hamburg, 2000