$$\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4....}}}}$$

$$1-1+1-1+1........=?$$

$$\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4....}}}}$$

Discrete mathematics

Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

$$\exists_{x\in\Re}\exists_{y\in\Re}(x=y)$$

$$\forall_{x}(\Re \mid x)$$

$$\sum_{x=1}^{\infty}\frac{1}{x}=?$$

$$\sum_{x=1}^{\infty}x=?$$

Chapter 2

RIZOAN TOUFIQ

ASSISTANT PROFESSOR

DEPARTMENT OF COMPUTER SCIENCE & ENGINEERING

RAJSHAHI UNIVERSITY OF ENGINEERING & TECHNOLOGY

Cardinality of Sets

Section 2.5

Section Summary

- Cardinality
- Countable Sets
- Computability

Cardinality

Definition: The *cardinality* of a set A is equal to the cardinality of a set B, denoted

$$|A| = |B|,$$

if and only if there is a one-to-one correspondence (*i.e.*, a bijection) from A to B.

- If there is a one-to-one function (*i.e.*, an injection) from A to B, the cardinality of A is less than or the same as the cardinality of B and we write $|A| \leq |B|$.
- When $|A| \leq |B|$ and A and B have different cardinality, we say that the cardinality of A is less than the cardinality of B and write $|A| < |B|$.

Cardinality

- **Definition**: A set that is either finite or has the same cardinality as the set of positive integers (**Z**$^+$) is called *countable*. A set that is not countable is *uncountable*.
- The set of real numbers **R** is an uncountable set.
- When an infinite set is countable (*countably infinite*) its cardinality is $\aleph_0$ (where $\aleph$ is aleph, the 1st letter of the Hebrew alphabet). We write $|S| = \aleph_0$ and say that S has cardinality "aleph null."

Showing that a Set is Countable

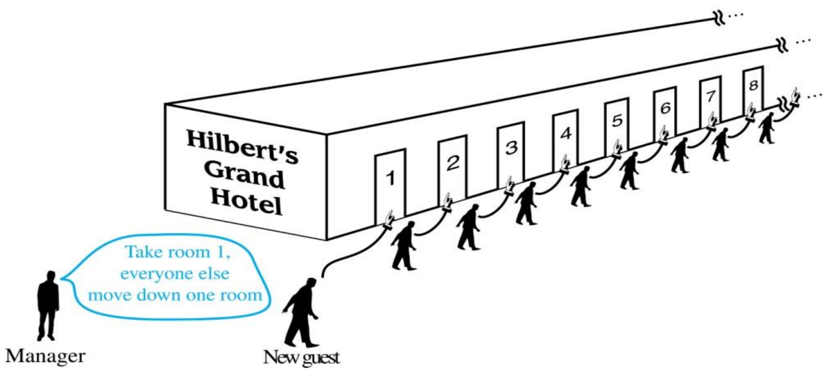
- An infinite set is countable if and only if it is possible to list the elements of the set in a sequence (indexed by the positive integers).
- The reason for this is that a one-to-one correspondence f from the set of positive integers to a set S can be expressed in terms of a sequence $a_1, a_2, \ldots, a_n, \ldots$ where $a_1 = f(1),\ a_2 = f(2), \ldots,\ a_n = f(n), \ldots$

Hilbert’s Grand Hotel

David Hilbert

The Grand Hotel (example due to David Hilbert) has countably infinite number of rooms, each occupied by a guest. We can always accommodate a new guest at this hotel. How is this possible?

Explanation: Because the rooms of Grand Hotel are countable, we can list them as Room 1, Room 2, Room 3, and so on. When a new guest arrives, we move the guest in Room 1 to Room 2, the guest in Room 2 to Room 3, and in general the guest in Room n to Room $n + 1$, for all positive integers n. This frees up Room 1, which we assign to the new guest, and all the current guests still have rooms.



The hotel can also accommodate a countable number of new guests, all the guests on a countable number of buses where each bus contains a countable number of guests (see exercises).

Showing that a Set is Countable

Example 1: Show that the set of positive even integers E is countable set.

Solution: Let $f(x) = 2x$.

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \ldots\ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 2 & 4 & 6 & 8 & 10 & 12 & \ldots\ldots \end{array}$$

Then f is a bijection from $\mathbf{N}$ to E since f is both one-to-one and onto. To show that it is one-to-one, suppose that $f(n) = f(m)$. Then $2n = 2m$, and so $n = m$. To see that it is onto, suppose that t is an even positive integer. Then $t = 2k$ for some positive integer k and $f(k) = t$.

Showing that a Set is Countable

Example 2: Show that the set of integers **Z** is countable.

Solution: Can list in a sequence:

$0, 1, -1, 2, -2, 3, -3, \ldots\ldots\ldots$

Or can define a bijection from **N** to **Z**:

- When n is even: $f(n) = n/2$
- When n is odd: $f(n) = -(n-1)/2$

The Positive Rational Numbers are Countable

- **Definition**: A *rational number* can be expressed as the ratio of two integers p and q such that $q \neq 0$.
  - ¾ is a rational number
  - $\sqrt{2}$ is not a rational number.

Example 3: Show that the positive rational numbers are countable.

Solution: The positive rational numbers are countable since they can be arranged in a sequence:

$$r_1, r_2, r_3, \ldots$$

The next slide shows how this is done. →

The Positive Rational Numbers are Countable

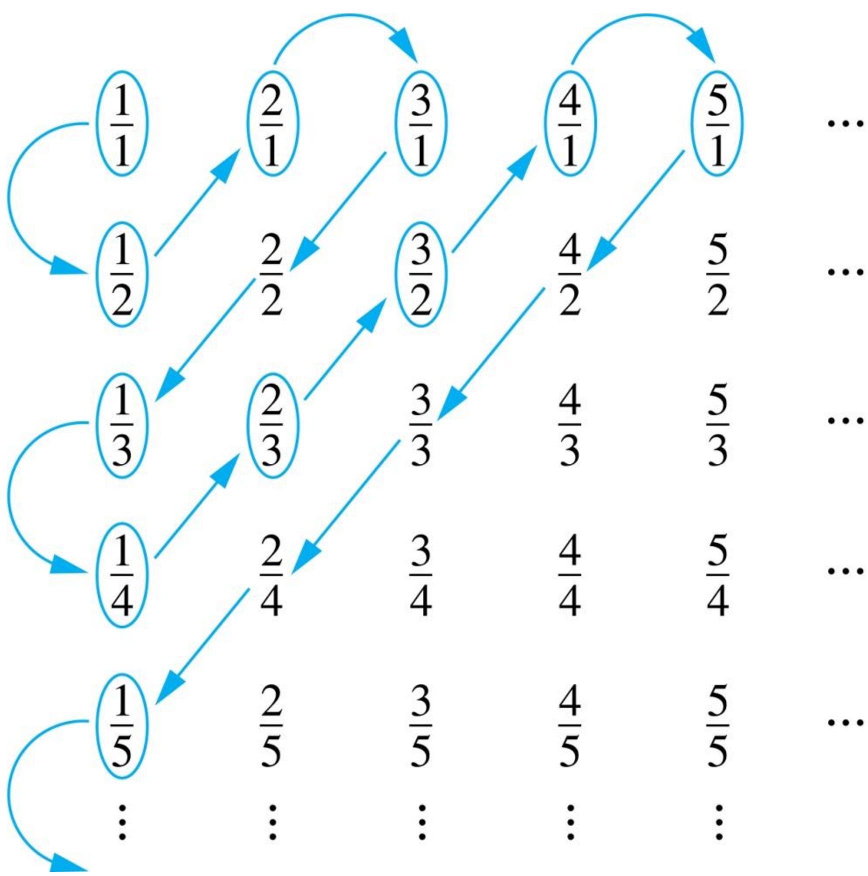
First row $q = 1$.
Second row $q = 2$.
etc.

Constructing the List

First list p/q with $p + q = 2$.
Next list p/q with $p + q = 3$

And so on.

Terms not circled are not listed because they repeat previously listed terms



1, ½, 2, 3, 1/3,1/4, 2/3, ...

Strings

Example 4: Show that the set of finite strings S over a finite alphabet A is countably infinite.

Assume an alphabetical ordering of symbols in A

Solution: Show that the strings can be listed in a sequence. First list

1. All the strings of length 0 in alphabetical order.
2. Then all the strings of length 1 in lexicographic (as in a dictionary) order.
3. Then all the strings of length 2 in lexicographic order.
4. And so on.

This implies a bijection from $\mathbf{N}$ to S and hence it is a countably infinite set.

The set of all Java programs is countable.

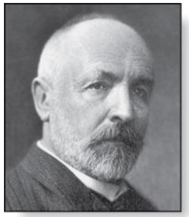
Example 5: Show that the set of all Java programs is countable.

Solution: Let S be the set of strings constructed from the characters which can appear in a Java program. Use the ordering from the previous example. Take each string in turn:

- Feed the string into a Java compiler. (A Java compiler will determine if the input program is a syntactically correct Java program.)
- If the compiler says YES, this is a syntactically correct Java program, we add the program to the list.
- We move on to the next string.

In this way we construct an implied bijection from $\mathbf{N}$ to the set of Java programs. Hence, the set of Java programs is countable. ◀

The Real Numbers are Uncountable



Georg Cantor
(1845-1918)

Example: Show that the set of real numbers is uncountable.

Solution: The method is called the Cantor diagnalization argument, and is a proof by contradiction.

1. Suppose **R** is countable. Then the real numbers between 0 and 1 are also countable (any subset of a countable set is countable - an exercise in the text).
2. The real numbers between 0 and 1 can be listed in order r_1, r_2, r_3,… .
3. Let the decimal representation of this listing be

$$\begin{aligned} r_1 &= 0.d_{11}d_{12}d_{13}d_{14}d_{15}d_{16}\ldots \\ r_2 &= 0.d_{21}d_{22}d_{23}d_{24}d_{25}d_{26}\ldots \\ r_3 &= 0.d_{31}d_{32}d_{33}d_{34}d_{35}d_{36}\ldots \\ &\vdots \end{aligned}$$

4. Form a new real number with the decimal expansion $r = .r_1r_2r_3r_4\ldots$
   where $r_i = 3$ if $d_{ii} \neq 3$ and $r_i = 4$ if $d_{ii} = 3$
5. r is not equal to any of the r_1, r_2, r_3,... Because it differs from r_i in its ith position after the decimal point. Therefore there is a real number between 0 and 1 that is not on the list since every real number has a unique decimal expansion. Hence, all the real numbers between 0 and 1 cannot be listed, so the set of real numbers between 0 and 1 is uncountable.
6. Since a set with an uncountable subset is uncountable (an exercise), the set of real numbers is uncountable.

Computability (Optional)

- **Definition**: We say that a function is **computable** if there is a computer program in some programming language that finds the values of this function. If a function is not computable we say it is **uncomputable**.
- There are uncomputable functions. We have shown that the set of Java programs is countable. Exercise 38 in the text shows that there are uncountably many different functions from a particular countably infinite set (i.e., the positive integers) to itself. Therefore (Exercise 39) there must be uncomputable functions.

Query???

$$\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4....}}}}$$

$$\exists_{x\in\Re}\exists_{y\in\Re}(\,x=y\,)=?$$

$$\sum_{x=1}^{\infty} x = ?$$

$$\sum_{x=1}^{\infty}\frac{1}{x}=?$$

$$\forall_{x}(\,\Re\,|\,x\,)=?$$

$$\exists_{x\in\Re}\exists_{y\in\Re}(\,x=y\,)=?$$

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$$1-1+1-1+1........=?$$

$$\sum_{x=1}^{\infty}\frac{1}{x}=?$$