Generalizing,

$$|A| = |\bigcup_{i=1}^{n} A_i| = \sum_{i=1}^{n}|A_i| - \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n}|A_i \cap A_j|$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n} \sum_{\substack{k=1 \\ i \neq j \neq k}}^{n} |A_i \cap A_j \cap A_k| \ldots (-i)^{n+1}\left|\bigcap_{i=1}^{n} A_i\right| \tag{6.21}$$

Example 6.4

Given $|E| = 100$. where E indicates a set of students who have chosen subjects from different streams in the computer science discipline, it is found that 32 study subjects chosen from the Computer Networks (CN) stream, 20 from the Multimedia Technology (MMT) stream, and 45 from the Systems Software (SS) stream. Also, 15 study subjects from both CN and SS streams, 7 from both MMT and SS streams, and 30 do not study any subjects chosen from either of the three streams.

Find the number of students who study subjects belonging to all three streams.

Solution

Let A, B, C indicate students who study subjects chosen from CN, MMT, and SS streams respectively. The problem is to find $|A \cap B \cap C|$.

The no. of students who do not study any subject chosen from either of the three streams = 30.

i.e. $$|A^c \cap B^c \cap C^c| = 30$$

$\Rightarrow$ $$|(A \cup B \cup C)^c| = 30 \quad \text{(using De Morgan's laws)}$$

$\Rightarrow$ $$|E| - |A \cup B \cup C| = 30$$

$\Rightarrow$ $$|A \cup B \cup C| = |E| - 30$$
$$= 100 - 30 = 70$$

From the principle of inclusion and exclusion,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

$\Rightarrow$ $$|A \cap B \cap C| = |A \cup B \cup C| - |A| - |B| - |C| + |A \cap B| + |B \cap C| + |A \cap C|$$
$$= 70 - 32 - 20 - 45 + 15 + 7 + 10$$
$$= 5$$

Hence, the no. of students who study subjects chosen from all the three streams is 5.

6.3 FUZZY SETS

Fuzzy sets support a flexible sense of membership of elements to a set. While in crisp set theory, an element either belongs to or does not belong to a set, in fuzzy set theory many degrees of membership (between 0 and 1) are allowed. Thus, a membership function $\mu_A(x)$ is associated with a fuzzy set $\tilde{A}$ such that the function maps every element of the universe of discourse X (or the reference set) to the interval [0, 1].

Formally, the mapping is written as $\mu_{\tilde{A}}(x) : X \to [0, 1]$

A *fuzzy set* is defined as follows:

If X is a universe of discourse and x is a particular element of X, then a fuzzy set A defined on X may be written as a collection of ordered pairs

$$A = \{(x, \mu_{\tilde{A}}(x)), x \in X\} \tag{6.23}$$

where each pair $(x, \mu_{\tilde{A}}(x))$ is called a singleton. In crisp sets, $\mu_{\tilde{A}}(x)$ is dropped.

An alternative definition which indicates a fuzzy set as a union of all $\mu_{\tilde{A}}(x)/x$ singletons is given by

$$A = \sum_{x_i \in X} \mu_{\tilde{A}}(x_i)/x_i \quad \text{in the discrete case} \tag{6.24}$$

and

$$A = \int_X \mu_{\tilde{A}}(x)/x \quad \text{in the continuous case} \tag{6.25}$$

Here, the summation and integration signs indicate the union of all $\mu_{\tilde{A}}(x)/x$ singletons.

Example

Let $X = \{g_1, g_2, g_3, g_4, g_5\}$ be the reference set of students. Let $\tilde{A}$ be the fuzzy set of "smart" students, where "smart" is a fuzzy linguistic term.

$$\tilde{A} = \{(g_1, 0.4)\ (g_2, 0.5)\ (g_3, 1)\ (g_4, 0.9)\ (g_5, 0.8)\}$$

Here $\tilde{A}$ indicates that the smartness of g_1 is 0.4, g_2 is 0.5 and so on when graded over a scale of 0–1.

Though fuzzy sets model vagueness, it needs to be realized that the definition of the sets varies according to the context in which it is used. Thus, the fuzzy linguistic term "tall" could have one kind of fuzzy set while referring to the height of a building and another kind of fuzzy set while referring to the height of human beings.

6.3.1 Membership Function

The membership function values need not always be described by discrete values. Quite often, these turn out to be as described by a continuous function.

The fuzzy membership function for the fuzzy linguistic term "cool" relating to temperature may turn out to be as illustrated in Fig. 6.10.

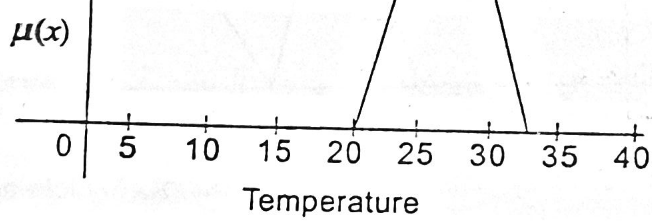


Fig. 6.10 Continuous membership function for "cool".

A membership function can also be given mathematically as

$$\mu_{\tilde{A}}(x) = \frac{1}{(1+x)^2}$$

The graph is as shown in Fig. 6.11.

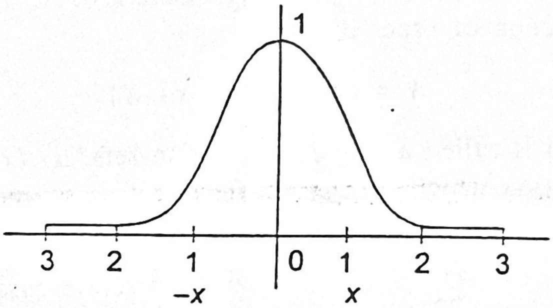


Fig. 6.11 Continuous membership function dictated by a mathematical function.

Different shapes of membership functions exist. The shapes could be triangular, trapezoidal, curved or their variations as shown in Fig. 6.12.

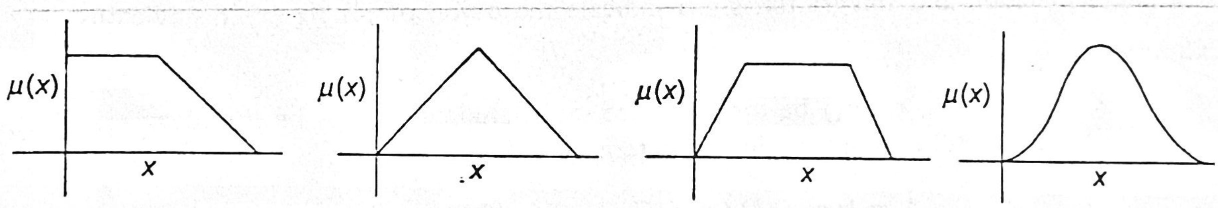


Fig. 6.12 Different shapes of membership function graphs.

Example

Consider the set of people in the following age groups

0–10	40–50
10–20	50–60
20–30	60–70
30–40	70 and above

The fuzzy sets "young", "middle-aged", and "old" are represented by the membership function graphs as illustrated in Fig. 6.13.

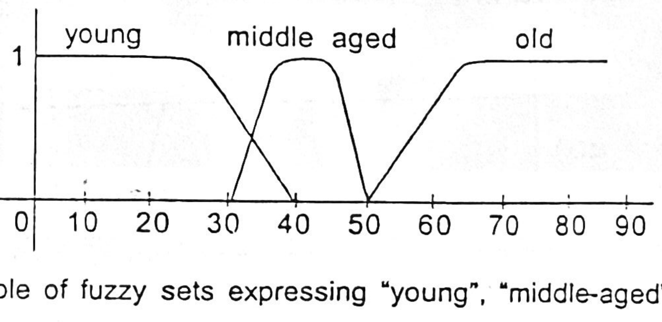


Fig. 6.13 Example of fuzzy sets expressing "young", "middle-aged", and "old".

6.3.2 Basic Fuzzy Set Operations

Given X to be the universe of discourse and $\tilde{A}$ and $\tilde{B}$ to be fuzzy sets with $\mu_A(x)$ and $\mu_B(x)$ as their respective membership functions, the basic fuzzy set operations are as follows:

Union

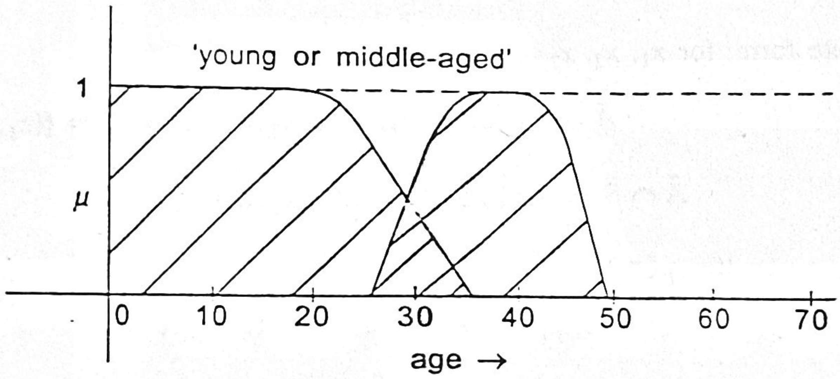
The union of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is a new fuzzy set $\tilde{A} \cup \tilde{B}$ also on X with a membership function defined as

$$\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x)) \tag{6.26}$$

Example

Let $\tilde{A}$ be the fuzzy set of young people and $\tilde{B}$ be the fuzzy set of middle-aged people as illustrated in Fig. 6.13. Now $\tilde{A} \cup \tilde{B}$, the fuzzy set of "young or middle-aged" will be given by

$\tilde{A} \cup \tilde{B}$:



In its discrete form, for x_1, x_2, x_3

if $A = \{(x_1,0.5), (x_2,0.7), (x_3,0)\}$ and $\tilde{B} = \{(x_1,0.8), (x_2,0.2), (x_3,1)\}$

$$\tilde{A} \cup \tilde{B} = \{(x_1,0.8), (x_2,0.7), (x_3,1)\}$$

since,

$$\mu_{\tilde{A} \cup \tilde{B}}(x_1) = \max(\mu_{\tilde{A}}(x_1), \mu_{\tilde{B}}(x_1))$$
$$= \max(0.5, 0.8)$$
$$= 0.8$$

$$\mu_{\tilde{A} \cup \tilde{B}}(x_2) = \max(\mu_{\tilde{A}}(x_2), \mu_{\tilde{B}}(x_2)) = \max(0.2, 0.7) = 0.7$$

$$\mu_{\tilde{A} \cup \tilde{B}}(x_3) = \max(\mu_{\tilde{A}}(x_3), \mu_{\tilde{B}}(x_3)) = \max(0,1) = 1$$

Intersection

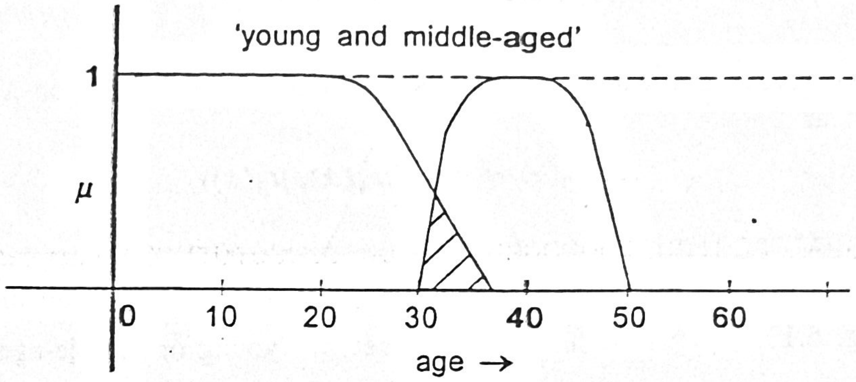
The intersection of fuzzy sets $\tilde{A}$ and $\tilde{B}$ is a new fuzzy set $\tilde{A} \cap \tilde{B}$ with membership function defined as

$$\mu_{\tilde{A} \cap \tilde{B}}(x) = \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) \tag{6.27}$$

Example

For $\tilde{A}$ and $\tilde{B}$ defined as "young" and "middle-aged" as illustrated in previous examples.

$\tilde{A} \cap \tilde{B}$:



In its discrete form, for x_1, x_2, x_3

if $\tilde{A} = \{(x_1, 0.5), (x_2, 0.7), (x_3, 0)\}$ and $\tilde{B} = \{(x_1, 0.8), (x_2, 0.2), (x_3, 1)\}$

$$\tilde{A} \cap \tilde{B} = \{(x_1, 0.5), (x_2, 0.2), (x_3, 0)\}$$

since,

$$\begin{aligned}
\mu_{\tilde{A} \cap \tilde{B}}(x_1) &= \min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{B}}(x_1)) \\
&= \min(0.5, 0.8) \\
&= 0.5 \\
\mu_{\tilde{A} \cap \tilde{B}}(x_2) &= \min(\mu_{\tilde{A}}(x_2), \mu_{\tilde{B}}(x_2)) \\
&= \min(0.7, 0.2) \\
&= 0.2 \\
\mu_{\tilde{A} \cap \tilde{B}}(x_3) &= \min(\mu_{\tilde{A}}(x_3), \mu_{\tilde{B}}(x_3)) \\
&= \min(0, 1) \\
&= 0
\end{aligned}$$

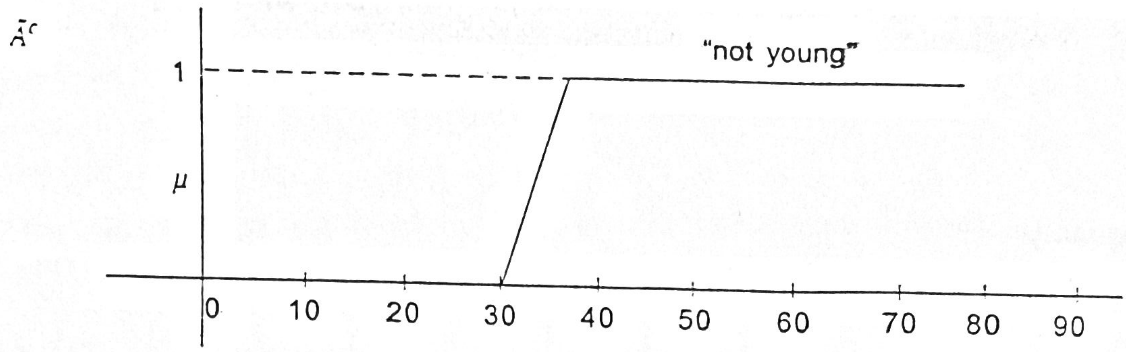
Complement

The complement of a fuzzy set $\tilde{A}$ is a new fuzzy set $\tilde{A}$ with a membership function

$$\mu_{\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x) \tag{6.28}$$

Example

For the fuzzy set $\tilde{A}$ defined as "young" the complement "not young" is given by $\tilde{A}^c$. In its discrete form, for x_1, x_2, and x_3



if
$$\tilde{A} = \{(x_1, 0.5)\ (x_2, 0.7)\ (x_3, 0)\}$$
then,
$$\tilde{A}^c = \{(x_1, 0.5)\ (x_2, 0.3)\ (x_3, 1)\}$$
since,
$$\mu_{\tilde{A}^c}(x_1) = 1 - \mu_{\tilde{A}}(x_1)$$
$$= 1 - 0.5$$
$$= 0.5$$
$$\mu_{\tilde{A}^c}(x_2) = 1 - \mu_{\tilde{A}}(x_2)$$
$$= 1 - 0.7$$
$$= 0.3$$
$$\mu_{\tilde{A}^c}(x_3) = 1 - \mu_{\tilde{A}}(x_3)$$
$$= 1 - 0$$
$$= 1$$

Other operations are,

Product of two fuzzy sets

The product of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is a new fuzzy set $\tilde{A} \cdot \tilde{B}$ whose membership function is defined as

$$\mu_{\tilde{A}\cdot\tilde{B}}(x) = \mu_{\tilde{A}}(x)\,\mu_{\tilde{B}}(x) \tag{6.29}$$

Example

$$\tilde{A} = \{(x_1, 0.2), (x_2, 0.8), (x_3, 0.4)\}$$
$$\tilde{B} = \{(x_1, 0.4)\ (x_2, 0), (x_3, 0.1)\}$$
$$\tilde{A} \cdot \tilde{B} = \{(x_1, 0.08)\ (x_2, 0)\ (x_3, 0.04)\}$$

Since
$$\mu_{\tilde{A}\cdot\tilde{B}}(x_1) = \mu_{\tilde{A}}(x_1) \cdot \mu_{\tilde{A}}(x_2)$$
$$= 0.2 \cdot 0.4 = 0.08$$
$$\mu_{\tilde{A}\cdot\tilde{B}}(x_2) = \mu_{\tilde{A}}(x_2) \cdot \mu_{\tilde{A}}(x_2)$$
$$= 0.8 \cdot 0 = 0$$

$$\mu_{\tilde{A}\cdot\tilde{B}}(x_3) = \mu_{\tilde{A}}(x_3)\cdot\mu_{\tilde{A}}(x_3)$$
$$= 0.4 \cdot 0.1$$
$$= 0.04$$

Equality

Two fuzzy sets $\tilde{A}$ and $\tilde{B}$ are said to be equal ($\tilde{A} = \tilde{B}$) if $\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x)$ (6.30)

Example

$$\tilde{A} = \{(x_1, 0.2)(x_2, 0.8)\}$$
$$\tilde{B} = \{(x_1, 0.6)(x_2, 0.8)\}$$
$$\tilde{C} = \{(x_1, 0.2)(x_2, 0.8)\}$$
$$\tilde{A} \neq \tilde{B}$$

since $\mu_{\tilde{A}}(x_1) \neq \mu_{\tilde{B}}(x_1)$ although

$$\mu_{\tilde{A}}(x_2) = \mu_{\tilde{B}}(x_2)$$

but $\tilde{A} = \tilde{C}$

since $\mu_{\tilde{A}}(x_1) = \mu_{\tilde{C}}(x_1) = 0.2$

and $\mu_{\tilde{A}}(x_2) = \mu_{\tilde{C}}(x_2) = 0.8$

Product of a fuzzy set with a crisp number

Multiplying a fuzzy set $\tilde{A}$ by a crisp number a results in a new fuzzy set product a. $\tilde{A}$ with the membership function

$$\mu_{a\cdot\tilde{A}}(x) = a\cdot\mu_{\tilde{A}}(x) \tag{6.31}$$

Example

$$\tilde{A} = \{(x_1, 0.4), (x_2, 0.6), (x_3, 0.8)\}$$

For $a = 0.3$

$$a\cdot\tilde{A} = \{(x_1, 0.12), (x_2, 0.18), (x_3, 0.24)\}$$

since, $\mu_{a\cdot\tilde{A}}(x_1) = a\cdot\mu_{\tilde{A}}(x_1)$

$$= 0.3 \cdot 0.4$$
$$= 0.12$$

$$\mu_{a\cdot\tilde{A}}(x_2) = a\cdot\mu_{\tilde{A}}(x_2)$$
$$= 0.3 \cdot 0.6$$
$$= 0.18$$

$$\mu_{a\cdot\tilde{A}}(x_3) = a \cdot \mu_{\tilde{A}}(x_3)$$
$$= 0.3 \cdot 0.8$$
$$= 0.24$$

Power of a fuzzy set

The α power of a fuzzy set $\tilde{A}$ is a new fuzzy set A^{α} whose membership function is given by

$$\mu_{A^{\alpha}}(x) = (\mu_{\tilde{A}}(x))^{\alpha} \tag{6.32}$$

Raising a fuzzy set to its second power is called *Concentration* (CON) and taking the square root is called *Dilation* (DIL).

Example

$$\tilde{A} = \{(x_1, 0.4), (x_2, 0.2), (x_3, 0.7)\}$$

For $\alpha = 2$

$$\mu_{\tilde{A}^2}(x) = (\mu_{\tilde{A}}(x))^2$$

Hence, $(\tilde{A})^2 = \{(x_1, 0.16), (x_2, 0.04), (x_3, 0.49)\}$

Since $\mu_{\tilde{A}^2}(x_1) = (\mu_{\tilde{A}}(x_1))^2 = (0.4)^2 = 0.16$

$$\mu_{\tilde{A}^2}(x_2) = (\mu_{\tilde{A}}(x_2))^2 = (0.2)^2 = 0.04$$

$$\mu_{\tilde{A}^2}(x_3) = (\mu_{\tilde{A}}(x_3))^2 = (0.7)^2 = 0.49$$

Difference

The difference of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is a new fuzzy set $\tilde{A} - \tilde{B}$ defined as

$$\tilde{A} - \tilde{B} = (\tilde{A} \cap \tilde{B}^c) \tag{6.33}$$

Example

$$\tilde{A} = \{(x_1, 0.2), (x_2, 0.5), (x_3, 0.6)\}; \ \tilde{B} = \{(x_1, 0.1), (x_2, 0.4), (x_3, 0.5)\}$$

$$\tilde{B}^c = \{(x_1, 0.9), (x_2, 0.6), (x_3, 0.5)\}$$

$$\tilde{A} - \tilde{B} = \tilde{A} \cap \tilde{B}^c$$

$$= \{(x_1, 0.2)(x_2, 0.5)(x_3, 0.5)\}$$

Disjunctive sum

The disjunctive sum of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is a new fuzzy set $\tilde{A} \oplus \tilde{B}$ defined as

$$\tilde{A} \oplus \tilde{B} = (\tilde{A}^c \cap \tilde{B}) \cup (\tilde{A} \cap \tilde{B}^c) \tag{6.34}$$

Example

$$\tilde{A} = \{(x_1,0.4)(x_2,0.8)(x_3,0.6)\}$$

$$\tilde{B} = \{(x_1,0.2)(x_2,0.6)(x_3,0.9)\}$$

Now,

$$\tilde{A}^c = \{(x_1,0.6)(x_2,0.2)(x_3,0.4)\}$$

$$\tilde{B}^c = \{(x_1,0.8)(x_2,0.4)(x_3,0.1)\}$$

$$\tilde{A}^c \cap \tilde{B} = \{(x_1,0.2)(x_2,0.2)(x_3,0.4)\}$$

$$\tilde{A} \cap \tilde{B}^c = \{(x_1,0.4)(x_2,0.4)(x_3,0.1)\}$$

$$\tilde{A} \oplus \tilde{B} = \{(x_1,0.4)(x_2,0.4)(x_3,0.4)\}$$

6.3.3 Properties of Fuzzy Sets

Fuzzy sets follow some of the properties satisfied by crisp sets. In fact, crisp sets can be thought of as special instances of fuzzy sets. Any fuzzy set $\tilde{A}$ is a subset of the reference set X. Also, the membership of any element belonging to the null set $\varnothing$ is 0 and the membership of any element belonging to the reference set is 1.

The properties satisfied by fuzzy sets are

Commutativity:

$$\tilde{A} \cup \tilde{B} = \tilde{B} \cup \tilde{A}$$

$$\tilde{A} \cap \tilde{B} = \tilde{B} \cap \tilde{A} \tag{6.35}$$

Associativity:

$$\tilde{A} \cup (\tilde{B} \cup \tilde{C}) = (\tilde{A} \cup \tilde{B}) \cup \tilde{C}$$

$$\tilde{A} \cap (\tilde{B} \cap \tilde{C}) = (\tilde{A} \cap \tilde{B}) \cap \tilde{C} \tag{6.36}$$

Distributivity:

$$\tilde{A} \cup (\tilde{B} \cap \tilde{C}) = (\tilde{A} \cup \tilde{B}) \cap (\tilde{A} \cup \tilde{C})$$

$$\tilde{A} \cap (\tilde{B} \cup \tilde{C}) = (\tilde{A} \cap \tilde{B}) \cup (\tilde{A} \cap \tilde{C}) \tag{6.37}$$

Idempotence:

$$\tilde{A} \cup \tilde{A} = \tilde{A}$$

$$\tilde{A} \cap \tilde{A} = \tilde{A} \tag{6.38}$$

Identity:

$$\tilde{A} \cup \varnothing = \tilde{A}$$

$$\tilde{A} \cup X = \tilde{A}$$

$$\tilde{A} \cap \varnothing = \varnothing$$

$$\tilde{A} \cup X = X \tag{6.39}$$

Transitivity: If $\tilde{A} \subseteq \tilde{B} \subseteq \tilde{C}$, then $\tilde{A} \subseteq \tilde{C}$ (6.40)

Involution:

$$(\tilde{A}^c)^c = \tilde{A} \tag{6.41}$$

De Morgan's laws:

$$(\tilde{A} \cap \tilde{B})^c = (\tilde{A}^c \cup \tilde{B}^c)$$

$$(\tilde{A} \cup \tilde{B})^c = (\tilde{A}^c \cap \tilde{B}^c) \tag{6.42}$$

Since fuzzy sets can overlap, the laws of excluded middle do not hold good. Thus,

$$\tilde{A} \cup \tilde{A}^c \neq X \tag{6.43}$$

$$\tilde{A} \cap \tilde{A}^c \neq \varnothing \tag{6.44}$$

Example 6.5

The task is to recognize English alphabetical characters (*F, E, X, Y, I, T*) in an image processing system.

Define two fuzzy sets $\tilde{I}$ and $\tilde{F}$ to represent the identification of characters *I* and *F*.

$$\tilde{I} = \{(F, 0.4), (E, 0.3), (X, 0.1), (Y, 0.1), (I, 0.9), (T, 0.8)\}$$

$$\tilde{F} = \{(F, 0.99), (E, 0.8), (X, 0.1), (Y, 0.2), (I, 0.5), (T, 0.5)\}$$

Find the following.

(a) (i) $\tilde{I} \cup \tilde{F}$ (ii) $(\tilde{I} - \tilde{F})$ (iii) $\tilde{F} \cup \tilde{F}^c$

(b) Verify De Morgan's Law, $(\tilde{I} \cup \tilde{F})^c = \tilde{I}^c \cap \tilde{F}^c$

Solution

(a) (i) $\tilde{I} \cup \tilde{F} = \{(F,0.99), (E,0.8), (X,0.1), (Y,0.2), (I,0.9), (T,0.8)\}$

(ii) $\tilde{I} - \tilde{F} = (\tilde{I} \cap \tilde{F}^c)$

$= \{(F,0.01), (E,0.2), (X,0.1), (Y,0.1), (I,0.5), (T,0.5)\}$

(iii) $\tilde{F} \cup \tilde{F}^c = \{(F,0.99), (E,0.8), (X,0.9), (Y,0.8), (I,0.5), (T,0.5)\}$

(b) De Morgan's Law

$$(\tilde{I} \cup \tilde{F})^c = \tilde{I}^c \cap \tilde{F}^c$$

$$\tilde{I} \cup \tilde{F} = \{(F,0.99), (E,0.8), (X,0.1), (Y,0.2), (I,0.9), (T,0.8)\}$$

$$(\tilde{I} \cup \tilde{F})^c = \{(F,0.01), (E,0.2), (X,0.9), (Y,0.8), (I,0.1), (T,0.2)\}$$

$$\tilde{I}^c = \{(F,0.6), (E,0.7), (X,0.9), (Y,0.9), (I,0.1), (T,0.2)\}$$

$$\tilde{F}^c = \{(F,0.01), (E,0.2), (X,0.9), (Y,0.8), (I,0.5), (T,0.5)\}$$

and

$$\tilde{I}^c \cap \tilde{F}^c = \{(F,0.01), (E,0.2), (X,0.9), (Y,0.8), (I,0.1), (T,0.2)\}$$

Hence, $(\tilde{I} \cup \tilde{F})^c = \tilde{I}^c \cap \tilde{F}^c$

Example 6.6

Consider the fuzzy sets $\tilde{A}$ and $\tilde{B}$ defined on the interval $X = [0, 5]$ of real numbers, by the membership grade functions

$$\mu_{\tilde{A}}(x) = \frac{x}{x+1}, \ \mu_{\tilde{B}}(x) = 2^{-x}$$

Determine the mathematical formulae and graphs of the membership grade functions of each of the following sets

(a) A^c, B^c

(b) $A \cup B$

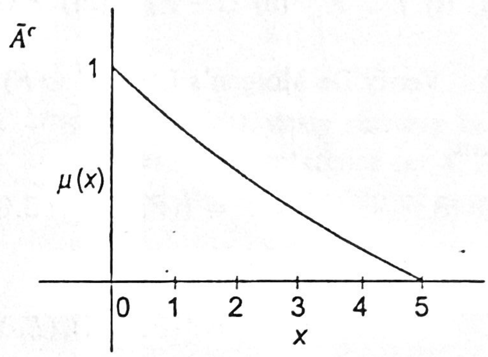
(c) $A \cap B$

(d) $(A \cup B)^c$

Solution

(a) $\mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(x) = 1 - \dfrac{x}{x+1}$

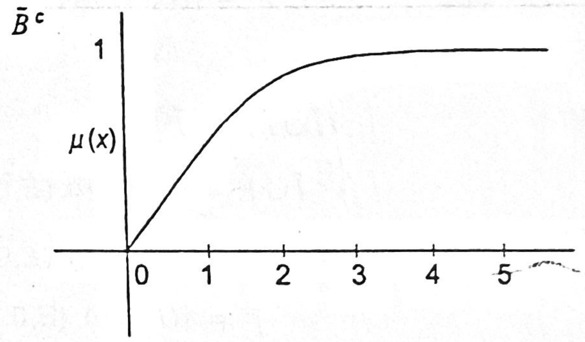
$$= \frac{1}{x+1}$$



$$\mu_{\tilde{B}^c}(x) = 1 - \mu_{\tilde{B}}(x)$$

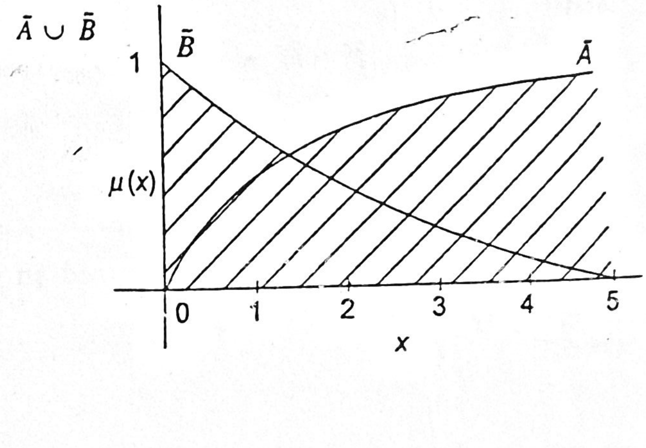
$$= 1 - 2^{-x}$$

$$= \frac{2^x - 1}{2^x}$$



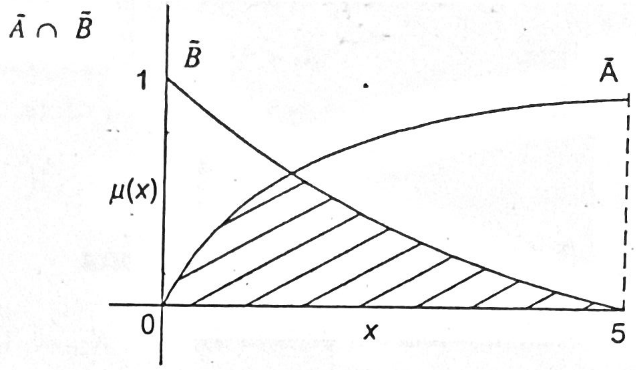
(b) $\mu_{\tilde{A} \cup \tilde{B}}(x) = \max(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$

$$= \max\left(\frac{x}{x+1}, 2^{-x}\right)$$



(c) $\mu_{\tilde{A} \cap \tilde{B}}(x) = \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$

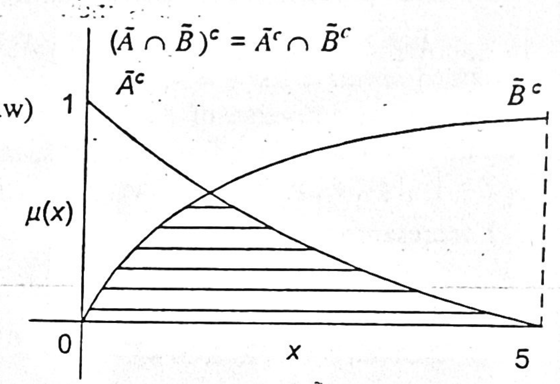
$$= \min\left(\frac{x}{x+1}, 2^{-x}\right)$$



(d) $\mu_{(\tilde{A} \cup \tilde{B})^c}(x) = \mu_{\tilde{A}^c \cap \tilde{B}^c}(x)$ (∵ De Morgan's law)

$$= \min(\mu_{\tilde{A}^c}(x), \mu_{\tilde{B}^c}(x))$$

$$= \min\left(\frac{1}{x+1}, \frac{2^x - 1}{2^x}\right)$$



6.4 CRISP RELATIONS

In this section, we review crisp relations as a prelude to fuzzy relations. The concept of relations between sets is built on the Cartesian product operator of sets.

6.4.1 Cartesian Product

The *Cartesian product* of two sets A and B denoted by $A \times B$ is the set of all ordered pairs such that the first element in the pair belongs to A and the second element belongs to B.

i.e.
$$A \times B = \{(a, b) / a \in A,\ b \in B\}$$

If $A \neq B$ and A and B are non-empty then $A \times B \neq B \times A$.

The Cartesian product could be extended to n number of sets

$$\mathop{\times}_{i=1}^{n} A_i = \{(a_1, a_2, a_3, \ldots, a_n) / a_i \in A_i \text{ for every } i = 1, 2, \ldots, n\} \quad (6.45)$$

Observe that
$$\left| \mathop{\times}_{i=1}^{n} A_i \right| = \prod_{i=1}^{n} |A_i| \quad (6.46)$$

Example

Given $A_1 = \{a, b\}, A_2 = \{1, 2\}, A_3 = \{\alpha\}$, since $\quad R \circ S\,(1, 1) = \max\{\min(0, 0), \min(1, 0), \min(0, 0)\}$

$= \max(0, 0, 0) = 0.$

$R \circ S\,(1, 3) = \max\{0, 0, 0\} = 0$

$R \circ S\,(1, 5) = \max\{0, 1, 0\} = 1.$

Similarly, $\quad R \circ S\,(3, 1) = 0.$

$R \circ S\,(3, 3) = R \circ S\,(3, 5) = R \circ S\,(5, 1) = R \circ S\,(5, 3) = R \circ S\,(5, 5) = 0$

$R \circ S$ from the relation matrix is $\{(1, 5)\}$.

Also,
$$S \circ R = \begin{array}{c} \\ 1 \\ 3 \\ 5 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

6.5 FUZZY RELATIONS

Fuzzy relation is a fuzzy set defined on the Cartesian product of crisp sets $X_1, X_2, \ldots, X_n$ where the n-tuples $(x_1, x_2, \ldots, x_n)$ may have varying degrees of membership within the relation. The membership values indicate the strength of the relation between the tuples.

Example

Let R be the fuzzy relation between two sets X_1 and X_2 where X_1 is the set of diseases and X_2 is the set of symptoms.

X_1 = {typhoid, viral fever, common cold}

X_2 = {running nose, high temperature, shivering}

The fuzzy relation R may be defined as

	Running nose	*High temperature*	*Shivering*
Typhoid	0.1	0.9	0.8
Viral fever	0.2	0.9	0.7
Common cold	0.9	0.4	0.6

6.5.1 Fuzzy Cartesian Product

Let $\tilde{A}$ be a fuzzy set defined on the universe X and $\tilde{B}$ be a fuzzy set defined on the universe Y, the Cartesian product between the fuzzy sets $\tilde{A}$ and $\tilde{B}$ indicated as $\tilde{A} \times \tilde{B}$ and resulting in a fuzzy relation $\tilde{R}$ is given by

$$\tilde{R} = \tilde{A} \times \tilde{B} \subset X \times Y \tag{6.52}$$

where $\tilde{R}$ has its membership function given by

$$\begin{aligned} \mu_{\tilde{R}}(x, y) &= \mu_{\tilde{A} \times \tilde{B}}(x, y) \\ &= \min (\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)) \end{aligned} \tag{6.53}$$

Example

Let $\tilde{A} = \{(x_1, 0.2), (x_2, 0.7), (x_3, 0.4)\}$ and $\tilde{B} = \{(y_1, 0.5), (y_2, 0.6)\}$ be two fuzzy sets defined on the universes of discourse $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$ respectively. Then the fuzzy relation $\tilde{R}$ resulting out of the fuzzy Cartesian product $\tilde{A} \times \tilde{B}$ is given by

$$\tilde{R} = \tilde{A} \times \tilde{B} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.2 & 0.2 \\ 0.5 & 0.6 \\ 0.4 & 0.4 \end{bmatrix} \end{array}$$

since,

$$\begin{aligned} \tilde{R}(x_1, y_1) &= \min (\mu_{\tilde{A}}(x_1), \mu_{\tilde{B}}(y_1)) = \min (0.2, 0.5) = 0.2 \\ \tilde{R}(x_1, y_2) &= \min (0.2, 0.6) = 0.2 \\ \tilde{R}(x_2, y_1) &= \min (0.7, 0.5) = 0.5 \\ \tilde{R}(x_2, y_2) &= \min (0.7, 0.6) = 0.6 \\ \tilde{R}(x_3, y_1) &= \min (0.4, 0.5) = 0.4 \\ \tilde{R}(x_3, y_2) &= \min (0.4, 0.6) = 0.4 \end{aligned}$$

6.5.2 Operations on Fuzzy Relations

Let $\tilde{R}$ and $\tilde{S}$ be fuzzy relations on $X \times Y$.

Union

$$\mu_{\tilde{R} \cup \tilde{S}}(x, y) = \max (\mu_{\tilde{R}}(x, y),\ \mu_{\tilde{S}}(x, y)) \tag{6.54}$$

Intersection

$$\mu_{\tilde{R} \cap \tilde{S}}(x, y) = \min (\mu_{\tilde{R}}(x, y),\ \mu_{\tilde{S}}(x, y)) \tag{6.55}$$

Complement

$$\mu_{\tilde{R}^c}(x, y) = 1 - \mu_{\tilde{R}}(x, y) \tag{6.56}$$

Composition of relations

The definition is similar to that of crisp relation. Suppose $\tilde{R}$ is a fuzzy relation defined on $X \times Y$, and $\tilde{S}$ is a fuzzy relation defined on $Y \times Z$, then $\tilde{R} \circ \tilde{S}$ is a fuzzy relation on $X \times Z$. The fuzzy max-min composition is defined as

$$\mu_{\tilde{R}\circ\tilde{S}}(x,z) = \max_{y \in Y}(\min(\mu_{\tilde{R}}(x,y),\ \mu_{\tilde{S}}(y,z))) \tag{6.57}$$

Example

$$X = \{x_1, x_2, x_3\}\ Y = \{y_1, y_2\}\ Z = \{z_1, z_2, z_3\} \tag{6.58}$$

Let $\tilde{R}$ be a fuzzy relation
$$\begin{array}{cc} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0.5 & 0.1 \\ 0.2 & 0.9 \\ 0.8 & 0.6 \end{bmatrix} \end{array}$$

Let $\tilde{S}$ be a fuzzy relation
$$\begin{array}{cc} & \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{array}{c} y_1 \\ y_2 \end{array} & \begin{bmatrix} 0.6 & 0.4 & 0.7 \\ 0.5 & 0.8 & 0.9 \end{bmatrix} \end{array}$$

Then $R \circ S$, by max-min composition yields,

$$R \circ S = \begin{array}{cc} & \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0.5 & 0.4 & 0.5 \\ 0.5 & 0.8 & 0.9 \\ 0.6 & 0.6 & 0.7 \end{bmatrix} \end{array}$$

$$\begin{aligned} \mu_{\tilde{R}\circ\tilde{S}}(x_1,z_1) &= \max(\min(0.5, 0.6), \min(0.1, 0.5)) \\ &= \max(0.5, 0.1) \\ &= 0.5 \end{aligned}$$

$$\begin{aligned} \mu_{\tilde{R}\circ\tilde{S}}(x_1,z_2) &= \max(\min(0.5, 0.4), \min(0.1, 0.8)) \\ &= \max(0.4, 0.1) \\ &= 0.4 \end{aligned}$$

Similarly,

$$\mu_{\tilde{R}\circ\tilde{S}}(x_1,z_3) = \max(0.5, 0.1) = 0.5$$

$$\mu_{\tilde{R}\circ\tilde{S}}(x_2,z_1) = \max(0.2, 0.5) = 0.5$$

$$\mu_{\tilde{R}\circ\tilde{S}}(x_2,z_2) = \max(0.2, 0.8) = 0.8$$

$$\mu_{\tilde{R}\circ\tilde{S}}(x_2,z_3) = \max(0.2, 0.9) = 0.9$$

$$\mu_{\tilde{R}\circ\tilde{S}}(x_3,z_1) = \max(0.6, 0.5) = 0.6$$

$$\mu_{\tilde{R}\circ\tilde{S}}(x_3,z_2) = \max(0.4, 0.6) = 0.6$$

$$\mu_{\tilde{R}\circ\tilde{S}}(x_3,z_3) = \max(0.7, 0.6) = 0.7$$

Example 6.7

Consider a set $P = \{P_1, P_2, P_3, P_4\}$ of four varieties of paddy plants, set $D = \{D_1, D_2, D_3, D_4\}$ of the various diseases affecting the plants and $S = \{S_1, S_2, S_3, S_4\}$ be the common symptoms of the diseases.

Let $\tilde{R}$ be a relation on $P \times D$ and $\tilde{S}$ be a relation on $D \times S$

For,

$$\tilde{R} = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \begin{array}{c} \begin{array}{cccc} D_1 & D_2 & D_3 & D_4 \end{array} \\ \begin{bmatrix} 0.6 & 0.6 & 0.9 & 0.8 \\ 0.1 & 0.2 & 0.9 & 0.8 \\ 0.9 & 0.3 & 0.4 & 0.8 \\ 0.9 & 0.8 & 0.1 & 0.2 \end{bmatrix} \end{array} \qquad \tilde{S} = \begin{array}{c} \\ D_1 \\ D_2 \\ D_3 \\ D_4 \end{array} \begin{array}{c} \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} \\ \begin{bmatrix} 0.1 & 0.2 & 0.7 & 0.9 \\ 1 & 1 & 0.4 & 0.6 \\ 0 & 0 & 0.5 & 0.9 \\ 0.9 & 1 & 0.8 & 0.2 \end{bmatrix} \end{array}$$

Obtain the association of the plants with the different symptoms of the diseases using max-min composition.

Solution

To obtain the association of the plants with the symptoms, $R \circ S$ which is a relation on the sets P and S is to be computed.

Using max-min composition,

$$R \circ S = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \begin{array}{c} \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} \\ \begin{bmatrix} 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.7 & 0.9 \end{bmatrix} \end{array}$$

SUMMARY

- Fuzzy set theory is an effective tool to tackle the problem of uncertainty.
- In crisp logic, an event can take on only two values, either a 1 or 0 depending on whether its occurrence is true or false respectively. However, in fuzzy logic, the event may take a range of values between 0 and 1.
- Crisp sets are fundamental to the study of fuzzy sets. The basic concepts include universal set, membership, cardinality of a set, family of sets, Venn diagrams, null set, singleton set, power set, subset, and super set. The basic operations on crisp sets are union, intersection, complement, and difference. A set of properties are satisfied by crisp sets. Also, the concept of partition and covering result in the two important rules, namely rule of addition and principle of inclusion and exclusion.
- Fuzzy sets support a flexible sense of membership and is defined to be the pair $(x, \mu_{\tilde{A}}(x))$ where $\mu_{\tilde{A}}(x)$ could be discrete or could be described by a continuous function. The membership functions could be triangular, trapezoidal, curved or its variations.