Matrices

Matrix is a two diamentional array.

Element represent করলে small letter, কোন Matrix কে represent করলে Capital letter.

কোন Matrix-এর Diagonal element ব্যতীত সকল element zero হলে থাকে বলে diagonal matrix.

Diagonal element-এর সকল value একই হলে তা scalar matrix.

Scalar matrix-এর value টি 1 হলে তা unit matrix.

কোন square matrix কে নিজের সাথে গুণ করলে যদি তা অপরিবর্তিত থাকে তবে তাকে Idompotent matrix বলে।

$A \cdot A = A$

কোন square matrix কে নিজের সাথে কয়েকবার গুণ করলে যদি তা zero হয়ে যায় তাহলে তা Nilpotent matrix.

যদি n সংখ্যক বার গুণ করার পর zero হয় তাহলে বলা হয় A is a nilpotent matrix of order n.

Matrix Multiplication : $A_{m\times n} B_{p\times q} = C_{m\times q}$

Same হতে হবে।

কোন Matrix কে কোন scalar দিয়ে গুণ করলে তা Scalar multiplication.

A rectangular matrix is called upper triangular matrix when the value of a_{ij} is zero where $i>j$

Example: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 3 \\ 0 & 0 & 9 \end{bmatrix}$

(ii) Minors of order 3×3 are:

$$\begin{vmatrix} 1 & 3 & 4 \\ 1 & 2 & 6 \\ 1 & 5 & 0 \end{vmatrix} = 1(-30) - 3(6) + 4(5-2)$$

$$= -30 + 18 + 12$$

$$= 0$$

Since there is a non-zero minor of order 3×3 and there is no minor of order 4×4, therefore the rank of the give matrix is 3

Elementary Row and Column Operators

Elementary Row operators:

(i) $R_{ij} \rightarrow$ Interchanging i-th and j-th row

(ii) $R_i(k) \rightarrow$ Multiplying each element of i-th row by a non zero number k.

(iii) $R_{ij}(k) \rightarrow$ Multiplying each element of j-th row by a non-zero number k. and then adding with the corresponding elements of i-th row.

Rank of Matrix

A non-zero matrix A of order m×n is said to have rank r if at least one of its r×r square minor is different from zero while all the other minors (if any) of order (r+1)(r+1) are zero.

Ex-1: Using minor test find the rank of the following matrices

(i) $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ -1 & -3 & -4 & -3 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 6 & 7 \\ 1 & 5 & 0 & 1 \end{bmatrix}$

(i) Minors of order 3×3 are:

$$\begin{vmatrix} 1 & 3 & 4 \\ 3 & 9 & 12 \\ -1 & -3 & -4 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 3 & 3 \\ 3 & 9 & 9 \\ -1 & -3 & -3 \end{vmatrix} = 0$$

$$\begin{vmatrix} 3 & 4 & 3 \\ 9 & 12 & 9 \\ -3 & -4 & -3 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 4 & 3 \\ 3 & 12 & 9 \\ -1 & -4 & -3 \end{vmatrix} = 0$$

Minors of order 2×2 are

$$\begin{vmatrix} 1 & 3 \\ 3 & 9 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 3 \\ -1 & -3 \end{vmatrix} = 0, \quad \begin{vmatrix} 12 & 9 \\ -4 & -3 \end{vmatrix} = 0$$

Similarly it can be shown that all the other minors of order 2×2 are 0.

~~Th~~ Since the given matrix as a non-zero minor of order 1 and all the other minors of order 2×2 and 3×3 are zero, therefore the rank of the given matrix is 1.

A square matrix is called Lower triangular matrix when a_{ij} is zero where $i<j$

Example: $\begin{bmatrix} 1 & 0 & 0 \\ 5 & 6 & 0 \\ 4 & 8 & 9 \end{bmatrix}$

So diagonal matrix can also defined as a square matrix which is a upper triangular matrix as well as lower triangular matrix.

Symmetric Matrix : $A \rightarrow$ square matrix
$A' = A$

Skew-Symmetric Matrix : $A \rightarrow$ Square matrix
$A' = -A$

Solve the following system of Linear Equations

$2x + 4y - z = 9$

$3x - y + 5z = 5$

$8x + 2y + 9z = 12$

The augmented matrix for the given system is

$$C = \begin{bmatrix} 2 & 4 & -1 & 9 \\ 3 & -1 & 5 & 5 \\ 8 & 2 & 9 & 12 \end{bmatrix} \overset{R_1(\frac{1}{2})}{\sim} \begin{bmatrix} 1 & 2 & -\frac{1}{2} & \frac{9}{2} \\ 3 & -1 & 5 & 5 \\ 8 & 2 & 9 & 12 \end{bmatrix}$$

$$\underset{R_{31}(-8)}{\overset{R_{21}(-3)}{\sim}} \begin{bmatrix} 1 & 2 & -\frac{1}{2} & \frac{9}{2} \\ 0 & -7 & \frac{13}{2} & -17/2 \\ 0 & -14 & 13 & -24 \end{bmatrix} \overset{R_{32}(-2)}{\sim} \begin{bmatrix} 1 & 2 & -\frac{1}{2} & \frac{9}{2} \\ 0 & -7 & \frac{13}{2} & -\frac{17}{2} \\ 0 & 0 & 0 & -7 \end{bmatrix}$$

Since
which implies that $\rho(A) = 2$ and $\rho(C) = 3$
Since $\rho(A) \neq \rho(C)$, therefore the given system has no solution. i.e. the system is inconsistent

For ~~the~~ what values of λ and μ, the following system

$x + y + z = 6$

$x + 2y + 3z = 10$ has

$x + 2y + \lambda z = \mu$

(i) Unique Solution
(ii) No Solution
(iii) Infinitely Many Solution

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The augmented matrix for the given system is

$$C = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ x & 2 & \lambda & \mu \end{bmatrix} \underset{R_{31}(-1)}{\overset{R_{21}(-1)}{\sim}} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{bmatrix}$$

$$\overset{R_{32}(-1)}{\sim} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{bmatrix}$$

(i) Unique solution : $\lambda \neq 3$

(ii) No Solution : $\lambda = 3$ and $\mu \neq 10$

(iii) Infinitely many Solution : $\lambda = 3$ and $\mu = 10$

$\boxed{\text{1st class test} \rightarrow 10}$

Ex-1 For which values of λ the following system has solution and solve completely in each case:

$$x+y+z=1$$
$$x+2y+4z=\lambda$$
$$x+4y+10z=\lambda^2$$

The augmented matrix for the given system is:

$$C=\begin{bmatrix}1&1&1&1\\1&2&4&\lambda\\1&4&10&\lambda^2\end{bmatrix}\underset{R_{31}(-1)}{\overset{R_{21}(-1)}{\sim}}\begin{bmatrix}1&1&1&1\\0&1&3&\lambda-1\\0&3&9&\lambda^2-1\end{bmatrix}$$

$$\overset{R_{32}(-3)}{\sim}\begin{bmatrix}1&1&1&1\\0&1&3&\lambda-1\\0&0&0&\lambda^2-3\lambda+2\end{bmatrix}$$

Here, we see that $\rho(A)=2$. The above system has a solution if $\rho(C)=2$ that is if $\lambda^2-3\lambda+2=0$ i.e. $\lambda=1$ or 2

For $\lambda=1$, the given system becomes

$$x+y+z=1$$
$$x+2y+4z=1$$
$$x+4y+10z=1$$

Now the corresponding Echelon form of the above system is

$$\begin{bmatrix}1&1&1&1\\0&1&3&0\\0&0&0&0\end{bmatrix}$$

The equivalent system is -

i.e. $(A-8I)X=\bar{0}$

$$\Rightarrow \begin{bmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence we get

$$-2x_1-2x_2+2x_3=0$$
$$-2x_1-5x-x_3=0$$
$$2x_1-x_2-5x_3=0$$

Here the coefficient matrix,

$$A_1 = \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \overset{R_1(-\frac{1}{2})}{\sim} \begin{bmatrix} 1 & 1 & -1 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \underset{R_{31}(-2)}{\overset{R_{21}(2)}{\sim}} \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix}$$

$$\overset{R_2(-\frac{1}{3})}{\sim} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -3 & -3 \end{bmatrix} \overset{R_{32}(3)}{\sim} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, the equivalent system is

$$x_1+x_2-x_3=0$$
$$x_2+x_3=0$$

Let x_3 be the free variable

Also let $x_3=k$ where k is any real number

Then $x_2=-k$

$x_1=2k$

Ex-1: Find the Eigen values and Eigen vectors of the following matrix.

(i) $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}$

(i) The ch. equation for the given matrix is

$|A-\lambda I| = 0$

$\Rightarrow \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow (6-\lambda)\{(3-\lambda)^2 - 1\} + 2\{-2(3-\lambda)+2\} + 2\{2-2(3-\lambda)\} = 0$

$\Rightarrow (6-\lambda)(9-6\lambda+\lambda^2-1) + 2(-6+2\lambda+2) + 2(2-6+2\lambda) = 0$

$\Rightarrow (6-\lambda)(\lambda^2-6\lambda+8) + 2(2\lambda-4) + 2(2\lambda-4) = 0$

$\Rightarrow 6\lambda^2 - 36\lambda + 48 + 4\lambda - 8 + 4\lambda - 8 - \lambda^3 + 6\lambda^2 - 8\lambda = 0$

$= -\lambda^3 + 12\lambda - 36\lambda + 32 = 0$

$\therefore \lambda = 2, 8, 2$

The eigen vector corresponding to the eigen values $\lambda = 8$ satisfies the following equation

$(A-\lambda I)X = \bar{0}$ where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$x+y+z=1$$
$$y+3z=0$$

Since $\rho(A)=\rho(C)=2<n$, therefore above system has infinitely many solutions.

In this case, free variable $= n-\rho(A) = 3-2 = 1$

Let,
z be the free variable

Also let, $z=k$ where k is any real number.

Then we have

$$y=-3k$$
$$x=1+2k$$

Hence the solution of the above system is

$$x=1+2k$$
$$y=-3k$$
$$z=k$$

$\lambda=2$ এর জন্য-ও করতে হবে।

II

Thus the solution is

$x_1 = 2k$

$x_2 = -k$

$x_3 = k$

Hence the eigen vectors corresponding to the eigen value $\lambda = 8$ is

$$X_{\lambda=8} = \begin{bmatrix} 2k \\ -k \\ k \end{bmatrix}$$

SA

Cayley-Hamilton Theorem:

Theorem: Every square matrix satisfies its characteristic equation

Consider the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 4 \\ 5 & 6 & 9 \end{bmatrix}$

Characteristic equation for A is

Theorem-এর প্রমাণ- 3 আছে।

$|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 3-\lambda & 4 \\ 5 & 6 & 9-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)\{(3-\lambda)(9-\lambda)-24\} + 0(18-2\lambda-20) + 0(12-15+5\lambda) = 0$

$\Rightarrow (1-\lambda)(27-3\lambda-9\lambda+\lambda^2-24) = 0$

$\Rightarrow (1-\lambda)(\lambda^2-12\lambda+3) = 0$

$\Rightarrow \lambda^2-12\lambda+3-\lambda^3+12\lambda^2-3\lambda = 0$

$\Rightarrow -\lambda^3+13\lambda^2-15\lambda+3 = 0$

$\therefore \lambda^3-13\lambda^2+15\lambda-3 = 0$

According to the theorem

$A^3 - 13A^2 + 15A - 3I = \bar{O}$

Using this theorem we can determine inverse matrix.

$A^2 - 13A + 15\cancel{A}I - 3A^{-1} = \bar{O}$ $[A * A^{-1} = I]$

$\Rightarrow A^{-1} = \frac{1}{3}(A^2 - 13A + 15I)$

Verify Cayley-Hamilton theorem for the following matrix A and hence find A^{-1}

$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 4 \\ 5 & 6 & 9 \end{bmatrix}$

Linear Algebra

Vector Space

1C

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x,y) : x \in \mathbb{R}, y \in \mathbb{R}\}$

$\mathbb{R}^3 = \{(x,y,z) \mid x,y,z \in \mathbb{R}\}$

এদের vector-ও বলা চলে।

Let, V be any non-empty set with the following property:

(i) Vector addition: for any $u, v \in V$, $u+v \in V$.

(ii) Scaler Multiplication: for any $v \in V$ and $k \in K$, $kv \in V$

Then, (iii) the set V is called a vector space over K if the following axioms hold in V for all $u, v, w \in V$:

(A1) $u+v = v+u$

(A2) $(u+v)+w = u+(v+w)$ [association]

(A3) There is an element $0 \in V$ called 0 vector such that $0+v = v$ for all v [additive identity]

(A4) for every $v \in V$ there is an $-v \in V$ such that $v+(-v) = 0$

M1) : We have

$k((x_1, y_1, z_1) + (x_2, y_2, z_2))$

$= k(x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$= (k(x_1 + x_2), k(y_1 + y_2), k(z_1 + z_2))$

$= (kx_1 + kx_2, ky_1 + ky_2, kz_1 + kz_2)$

$= (kx_1, ky_1, kz_1) + (kx_2 + ky_2 + kz_2)$

$= k(x_1, y_1, z_1) + k(x_2, y_2, z_2)$

SA

(M1) $k(u+v) = ku+kv$, $\forall u,v \in V$ and $k \in K$

(M2) $(k_1+k_2)v = k_1 v + k_2 v$ $\forall v \in V$ and $k_1, k_2 \in K$

(M3) $(ab)v = a(bv)$ $\forall v \in V$ and $a,b \in K$

(M4) $\exists$ an $1 \in K$ such that $1.v = v.1 = v$ for all $v \in V$

\# Show that $\mathbb{R}^3$ is a vector space over $\mathbb{R}$.

At first, we define two operations in $\mathbb{R}^3$ as

(i) vector addition :

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1+x_2, y_1+y_2, z_1+z_2)$$

for all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^3$

(ii) vector Multiplication :

$$k(x,y,z) = (kx, ky, kz) \quad \forall (x,y,z) \in \mathbb{R}^3 \text{ and } k \in \mathbb{R}$$

Let, $(x,y,z), (x_1,y_1,z_1), (x_2,y_2,z_2), (x_3,y_3,z_3) \in \mathbb{R}^3$
and $k, k_1, k_2 \in \mathbb{R}$

Then,

(A1) : We have $(x_1, y_1, z_1) + (x_2+y_2+z_2)$

$$= (x_1+x_2, y_1+y_2, z_1+z_2)$$
$$= (x_2+x_1, y_2+y_1, z_2+z_1)$$ [Real number এর property অনুযায়ী লিখলাম]
$$= (x_2, y_2, z_2) + (x_1, y_1, z_1)$$

(A2) : Do Yourself :(

(A3) : $\exists (0,0,0) \in \mathbb{R}$ such that $(x,y,z) + (0,0,0)$
$$= (0,0,0) + (x,y,z)$$
$$= (x,y,z)$$

(A4) for every $(x,y,z) \in \mathbb{R}^3$ there is an $(-x,-y,-z) \in \mathbb{R}^3$ such that

$$(x,y,z) + (-x,-y,-z) = (x-x, y-y, z-z)$$
$$= (0,0,0)$$

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Divergence Theorem:

Statement: If V is the volume bounded by a closed surface S and $\vec{A}$ is a vector function of position with continuous derivative then —

$$\iiint_V \nabla \cdot \vec{A}\, dv = \iint_S \vec{A} \cdot \hat{n}\, ds$$

বই দেখে করতে হবে।

SA

Linear Combination of Vectors:

Let, V be a vector space over a scalar field K. Then a vector $v \in V$ is called a linear combination of vectors $u_1, u_2, u_3, \ldots, u_n \in V$ if there exists some scalars $k_1, k_2, \ldots, k_n \in K$ such that

$$v = k_1 u_1 + k_2 u_2 + k_3 u_3 + \cdots + k_n u_n$$

Consider the vector space $\mathbb{R}^3$ in $\mathbb{R}$. Express $(3,7,-4) \in \mathbb{R}^3$ as a linear combination of vectors $u_1 = (1,2,3)$, $u_2 = (2,3,7)$, $u_3 = (3,5,6)$

Let, $(3,7,-4) = x(1,2,3) + y(2,3,7) + z(3,5,6)$

$\Rightarrow (3,7,-4) = (x+2y+3z, 2x+3y+5z, 3x+7y+6z)$

which implies that

$$x+2y+3z = 3$$
$$2x+3y+5z = 7$$
$$3x+7y+6z = -4$$

The augmented matrix for the given system is

$$C = \begin{bmatrix} 1 & 2 & 3 & 3 \\ 2 & 3 & 5 & 7 \\ 3 & 7 & 6 & -4 \end{bmatrix} \underset{R_{31}(-3)}{\overset{R_{21}(-2)}{\sim}} \begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & -3 & -13 \end{bmatrix}$$

$$\overset{R_2(-1)}{\sim} \begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & -3 & -13 \end{bmatrix} \overset{R_{32}(-1)}{\sim} \begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -4 & -12 \end{bmatrix}$$

whic implies that $\rho(A) = \rho(C) = 3$

Since $\rho(A) = \rho(C) = 3 = n$, therefore the above system has unique solution. Now, the equivalent system is -

$$x + 2y + 3z = 3$$
$$y + z = -1$$
$$-4z = -12$$

Solving these, we get -

$$x = 2$$
$$y = -4$$
$$z = 3$$

Therefore

$(3,7,-4) = 2(1,2,3) - 4(2,3,7) + 3(3,5,6)$

\# Express $M = \begin{bmatrix} 4 & 7 \\ 7 & 9 \end{bmatrix}$ as a linear combination of the matrices $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$

\# Express $P = 3t^2 + 5t - 5$ as a linear combination of the polinomials $P_1 = t^2 + 2t + 1$, $P_2 = 2t^2 + 5t + 4$, $P_3 = t^2 + 3t + 6$

\# Express $v = (1,-2,5)$ in $\mathbb{R}^3$ as a linear combination of the vectors $u_1 = (1,1,1)$, $u_2 = (1,2,3)$, $u_3 = (2,-1,1)$

Suppose that,

$$\cancel{3t^2+5t}+1$$
$$3t^2 + 5t - 5 = x(t^2+2t+1) + y(2t^2+5t+4) + z(t^2+3t+6)$$

where, x, y, z are scalars is to be determined.

$\therefore 3t^2 + 5t - 5 = (x+2y+z)t^2 + (2x+5y+3z)t + (x+4y+6z)$

Equating the coefficients like powers of t -

$$x + 2y + z = 3$$
$$2x + 5y + 3z = 5$$
$$x + 4y + 6z = -5$$

The augmented matrix for the given system is

$$C = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 5 & 3 & 5 \\ 1 & 4 & 6 & -5 \end{bmatrix} \underset{R_{31}(-1)}{\overset{R_{21}(-2)}{\sim}} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & 5 & -8 \end{bmatrix} \overset{R_{32}(-2)}{\sim} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 3 & -6 \end{bmatrix}$$

which implies that $\rho(A) = \rho(C) = 3$

Since $\rho(A) = \rho(C) = 3 = n$, therefore the above system has unique solution.

Now, the equivalent system is

$$x + 2y + z = 3$$
$$y + z = -1$$
$$3z = -6$$

Solving these we get -

$$x = 3$$
$$y = 1$$
$$z = -2$$

System of linear equation, eigen value, eigen vector — 2nd CT

$$\iint_S (\nabla \times A_3 \hat{k}) \cdot \hat{n}\, ds = \oint_C A_3\, dz$$

adding these we get

$$\iint_S (\nabla \times \vec{A}) \cdot \hat{n}\, ds = \oint \vec{A} \cdot d\vec{r}$$

So that $\frac{\delta r}{\delta y} = \hat{j} + \frac{\delta f}{\delta y} \hat{k}$

$\therefore \hat{n} \cdot \frac{\delta r}{\delta y} = \hat{n} \cdot \hat{j} + \frac{\delta f}{\delta y} \hat{n} \cdot \hat{k} = 0$ $\left[\frac{\delta \vec{r}}{\delta y} \text{ is perpendicular to } \hat{n} \right]$

$\Rightarrow \hat{n} \cdot \hat{j} = - \frac{\delta z}{\delta y} \hat{n} \cdot \hat{k}$

Putting this in (i)

$$(\nabla \times A_1 \hat{i}) \cdot \hat{n}\, ds = -\left(\frac{\delta A_1}{\delta y} + \frac{\delta A_1}{\delta z} \cdot \frac{\delta z}{\delta y} \right) \hat{n} \cdot \hat{k}\, ds \quad \text{- - - - (ii)}$$

Now, on S, $A_1(x, y, z) = A_1(x, y, f(x, y)) = F(x, y)$

hence,

$$\frac{\delta A_1}{\delta y} + \frac{\delta A_1}{\delta z} \frac{\delta z}{\delta y} = \frac{\delta F}{\delta y}$$

from (ii)

$$(\nabla \times A_1 \hat{i}) \cdot \hat{n}\, ds = - \frac{\delta F}{\delta y} \hat{n} \cdot \hat{k}\, ds = - \frac{\delta F}{\delta y} dx dy$$

[$\hat{n} \cdot \hat{k} = 1$ হবে cause দুটোর Direction একই দিকে]

Then,

$$\iint_S (\nabla \times A_1 \hat{i}) \cdot \hat{n}\, ds = \iint_R - \frac{\delta F}{\delta y} dx dy$$

, where R is the projection of S on xy plane

By Green's theorem last integral equals

$$\int_\Gamma F dx \quad , \quad \Gamma \text{ is boundary of } R$$

we have $\oint_\Gamma F dx = \oint A_1 dx$

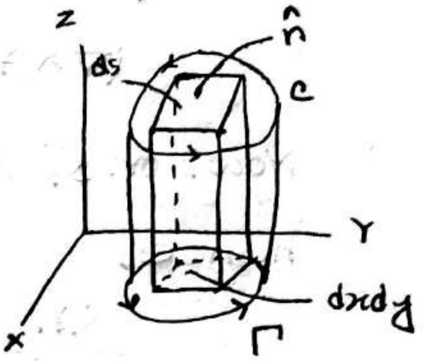
$$\Rightarrow \iint_S (\nabla \times A_1 \hat{i}) \cdot \hat{n}\, ds = \oint_C A_1 dx$$

Similarly $\iint_S (\nabla \times A_2 \hat{j}) \cdot \hat{n}\, ds = \oint_C A_2 dy$

Stokes Theorem

Statement: The line integral of the tangential component of a vector $\vec{A}$ taken around a simple closed curve C is equal to the surface integral of the normal component of the curl of $\vec{A}$ taken over any surface S having C as its boundary. i.e. $\oint \vec{A} \cdot d\vec{r} = \iint_S (\nabla \times \vec{A}) \cdot \hat{n}\, ds$.

Proof: Let, S be a surface and its projection on the xy, yz & zx planes are region bounded by simple closed curves as shown in the figure.



Let, S have representation

$z = f(x,y)$ or, $x = g(y,z)$ or $y = h(x,z)$

We must show that –

$$\iint_S (\nabla \times \vec{A}) \cdot \hat{n}\, ds = \iint_S [\nabla \times (A_1\hat{i} + A_2\hat{j} + A_3\hat{k})] \cdot \hat{n}\, ds$$

$$= \oint_C \vec{A} \cdot d\vec{r}, \text{ where C is boundary of S}$$

Consider first, $\iint_S (\nabla \times A_1\hat{i}) \cdot \hat{n}\, ds$

Here,

$$\nabla \times A_1\hat{i} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ A_1 & 0 & 0 \end{vmatrix} = \frac{\delta A_1}{\delta z}\hat{j} - \frac{\delta A_1}{\delta y}\hat{k}$$

$$\therefore (\nabla \times A_1\hat{i}) \cdot \hat{n}\, ds = \left(\frac{\delta A_1}{\delta z}\hat{n}\cdot\hat{j} - \frac{\delta A_1}{\delta y}\hat{n}\cdot\hat{k}\right) ds \quad \ldots\ldots (1)$$

If $z = f(x,y)$ is taken as the equation of S.

Then position vector to any point of S is $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$
$= x\hat{i} + y\hat{j} + f(x,y)\hat{k}$

Spanning set of vectors:

Let, V be a vector space over a scalar field K. Then a set of vectors $\{u_1, u_2, \dots, u_n\}$ in V is said to form a spanning set for V if every $v \in V$ can be expressed as a linear combination of vectors in $\{u_1, u_2, \dots, u_n\}$ i.e. there exists some scalars $k_1, k_2, \dots, k_n$ in K such that

$$v = k_1u_1 + k_2u_2 + \dots + k_nu_n$$

\# Determine whether following set of vectors in $\mathbb{R}^3$ form a spanning vector set or not.

(i) $u_1 = (1,1,1)$, $u_2 = (1,1,0)$, $u_3 = (1,0,0)$

(ii) $\{(1,2,3), (1,3,5), (1,5,9)\}$

Solution:

Let $(a,b,c) \in \mathbb{R}^3$

Also let, $(a,b,c) = x(1,1,1) + y(1,1,0) + z(1,0,0)$

i.e.
$$x+y+z = a$$
$$x+y = b$$
$$x = c$$

These implies
$$x = c$$
$$y = b-c$$
$$z = a-b$$

Therefore

$$(a,b,c) = c(1,1,1) + (b-c)(1,1,0) + (a-b)(1,0,0)$$

Linear Dependance and Independance of Vectors:

Let, V be a vector space over K. Then a set of vectors $\{u_1, u_2, \ldots, u_n\}$ in V is said to be linearly independent if

$$k_1u_1 + k_2u_2 + \ldots + k_nu_n = 0 \text{ implies } k_1 = k_2 = \ldots = k_n = 0$$
$$\text{where } k_1, k_2, \ldots k_n \in K$$

otherwise the vectors are called linearly dependent.

\# Test whether the following vectors are linearly dependent or not:

(i) $u = (1,1,2)$, $v = (2,3,1)$, $w = (4,5,5)$
(ii) $u = (1,2,5)$, $v = (2,5,1)$, $w = (1,5,2)$

Let $x(1,1,2) + y(2,3,1) + z(4,5,5) = (0,0,0)$

Then, $x + 2y + 4z = 0$
$x + 3y + 5z = 0$
$2x + y + 5z = 0$

Here.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 2 & 1 & 5 \end{bmatrix} \underset{R_{31}(-2)}{\overset{R_{21}(-1)}{\sim}} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & -3 & -3 \end{bmatrix} \overset{R_{32}(3)}{\sim} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Here, we can see that $\rho(A) = 2$ which is less than number of variable. So there is $(3-2) = 1$ free variable.

Let,
z be the free variable

Also let, $z = k$ where k is any real number.

Then we have

$y = -k$

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Basis of a Vector Space:

Let V be a vector space over K. Then a set $S=\{u_1,u_2,\dots,u_n\}$ is called the basis of V if S satisfies the following two conditions:

(i) S is linearly independent
(ii) S span V

Dimension of Vector Space:

A vector space V is said to be n-dimensional if it has a basis of n elements.

Determine whether or not each of the following set form a basis of $\mathbb{R}^3$ or $\mathbb{R}^4$.

(i) $\{(1,1,1),(1,2,3),(2,-1,1)\}$
(ii) $\{(1,1,2),(1,2,5),(5,3,4)\}$
(iii) $\{(1,1,1,1),(0,1,1,1),(0,0,1,1),(0,0,0,1)\}$

(ii) Let, $S=\{(1,1,2),(1,2,5),(5,3,4)\}$
To show that S form a basis of $\mathbb{R}^3$, we have to show that

(a) S is linearly independent
(b) S span $\mathbb{R}^3$

(a) Let $x(1,1,2)+y(1,2,5)+z(5,3,4)=(0,0,0)$
which implies that

$$x+y+5z=0$$
$$x+2y+3z=0$$
$$2x+5y+4z=0$$

where

$$A=\begin{bmatrix}1&1&5\\1&2&3\\2&5&4\end{bmatrix}\underset{R_{31}(-2)}{\overset{R_{21}(-1)}{\sim}}\begin{bmatrix}1&1&5\\0&1&-2\\0&3&-6\end{bmatrix}\overset{R_{32}(-3)}{\sim}\begin{bmatrix}1&1&5\\0&1&-2\\0&0&0\end{bmatrix}$$

i.e. we can write

$$(a,b,c) = (a+b-c)(1,1,1) + \left(\frac{-2a-b+3c}{5}\right)(1,2,3) + \frac{a-2b+c}{5}(2,-1,1)$$

which implies that S span $\mathbb{R}^3$

Therefore S form a basis for $\mathbb{R}^3$

which implies that

$$x+y+2z = a$$
$$x+2y-z = b$$
$$x+3y+z = c$$

Here,

$$C = \begin{bmatrix} 1 & 1 & 2 & a \\ 1 & 2 & -1 & b \\ 1 & 3 & 1 & c \end{bmatrix} \underset{R_{31}(-1)}{\overset{R_{21}(-1)}{\sim}} \begin{bmatrix} 1 & 1 & 2 & a \\ 0 & 1 & -3 & b-a \\ 0 & 2 & -1 & c-a \end{bmatrix} \overset{R_{32}(-2)}{\sim} \begin{bmatrix} 1 & 1 & 2 & a \\ 0 & 1 & -3 & b-a \\ 0 & 0 & 5 & a-2b+c \end{bmatrix}$$

which implies that $\rho(A) = \rho(C)$

Now the equivalent system is –

$$x+y+2z = a$$
$$y-3z = b-a$$
$$5z = a-2b+c$$

$$\therefore z = \frac{a-2b+c}{5}$$

$$\therefore y = b-a+\frac{3(a-2b+c)}{5}$$
$$= \frac{5b-5a+3a-6b+3c}{5}$$
$$= \frac{-2a-b+3c}{5}$$

$$\therefore x = a-y-2z$$
$$= a-\frac{-2a-b+3c}{5}-\frac{2(a-2b+c)}{5}$$
$$= \frac{5a+2a+b-3c-2a+4b-2c}{5}$$
$$= \frac{5a+5b-5c}{5}$$
$$= a+b-c$$

which implies that $\rho(A)=2$

Since $\rho(A) < n$, ~~they are linearly dependent~~ there are infinitely many solution.

Hence, the vectors are linearly dependent.

Since S is linearly dependent, therefore S doesnot form a basis for $\mathbb{R}^3$.

① Let, $S=\{(1,1,1),(1,2,3),(2,-1,1)\}$

ⓐ Let, $x(1,1,1)+y(1,2,3)+z(2,-1,1)=(0,0,0)$

which implies that -

$$x+y+2z=0$$
$$x+2y-z=0$$
$$x+3y+z=0$$

where

$$A=\begin{bmatrix}1&1&2\\1&2&-1\\1&3&1\end{bmatrix}\underset{R_{31}(-1)}{\overset{R_{21}(-1)}{\sim}}\begin{bmatrix}1&1&2\\0&1&-3\\0&2&-1\end{bmatrix}\overset{R_{32}(-2)}{\sim}\begin{bmatrix}1&1&2\\0&1&-3\\0&0&5\end{bmatrix}$$

which Implies that $\rho(A)=3$

Since $\rho(A)=n$, therefore the above system has unique solution.

$\therefore x=0,\ y=0,\ z=0$

Thus S is linearly independent.

ⓑ Let $(a,b,c)\in\mathbb{R}^3$

Also let $(a,b,c)=x(1,1,1)+y(1,2,3)+z(2,-1,1)$

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Linear Mapping or Transformations

$f: A \to B$

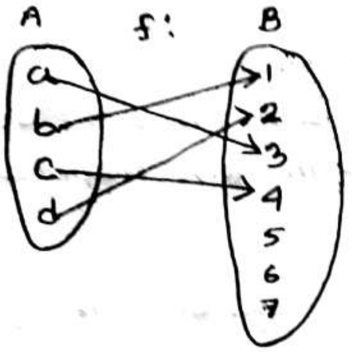
Image and Kernel of a Mapping:

Let, $f: A \to B$ be a mapping from A to B. Then the image of f denoted by $\mathrm{Im} f$ and defined by

$$\mathrm{Im} f = \{ b \in B : \text{There is an } a \in A \text{ where } f(a) = b \}$$

and the kernel of f is denoted by $\ker f$ and defined by

$$\ker f = \{ a \in A : f(a) = 0 \}$$



$f: \mathbb{R}^3 \to \mathbb{R}^3$

$f(x, y, z) = (x, y, 0)$

$\mathrm{Im} f$: xy-plane

$\ker f$ = z-axis

$\mathrm{Im} f = \{1, 2, 3, 4\}$

Let U and V are two vector spaces over the same scalar field K. Then a mapping / transforms $T: U \to V$ is called linear if T satisfies the following two properties:

(i) for every $u, v \in U$, $T(u+v) = T(u) + T(v)$

(ii) for every $u \in U$ and $k \in K$, $T(ku) = kT(u)$

\# Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ be a mapping defined by

$$F(x, y) = (3x + 2y, 5x + 9y)$$

Determine whether or not F is linear.

$F((x_1, y_1) + (x_2, y_2)) = F(x_1, y_1) + F(x_2, y_2)$

Do yourself.

Matrix Representation of a Linear Mapping:

Let T be a linear mapping/operator from V into itself. Also let $S=\{u_1, u_2, \ldots, u_n\}$ is a basis of V. Then the vectors $T(u_1), T(u_2), \ldots, T(u_n)$ are in V

Therefore the vector $T(u_1), T(u_2), \ldots, T(u_n)$ can be expressed as a linear combination of $u_1, u_2, \ldots, u_n$ That is

$$T(u_1) = a_{11}u_1 + a_{12}u_2 + \ldots + a_{1n}u_n$$
$$T(u_2) = a_{21}u_1 + a_{22}u_2 + \ldots + a_{2n}v_n$$
$$\ldots$$
$$T(u_n) = a_{n1}v_1 + a_{n2}u_2 + \ldots + a_{nn}u_n$$

The the transpose of the above matrix of co-efficients denoted by $[T]_s$ or $m_s(T)$ is called the matrix representation of T relative to the basis S.

\# Find the matrix representation of the linear operator $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x,y) = (2x+3y, 4x-5y)$ relative to the basis

(i) $S=\{u_1, u_2\} = \{(1,2), (2,5)\}$ (ii) $S=\{u_1, u_2\} = \{(1,0), (0,1)\}$

Given that,

$$T(x,y) = (2x+3y, 4x-5y)$$

Then, $T(u_1) = T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}8\\-6\end{bmatrix}$

Now Let $T(u_1) = \begin{bmatrix}8\\-6\end{bmatrix} = xu_1 + yu_2$

$$\Rightarrow \begin{bmatrix}8\\-6\end{bmatrix} = \begin{bmatrix}x\\2x\end{bmatrix} + \begin{bmatrix}2y\\5y\end{bmatrix}$$

which implies that
$$x+2y = 8$$
$$2x+5y = -6$$

That is $x = 52$, $y = -22$

Hence, we can write

$$T(u_1) = \begin{bmatrix} 8 \\ -6 \end{bmatrix} = 52u_1 - 22u_2 \quad \cdots\cdots \quad (i)$$

Again,

$$T(u_2) = T\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} 19 \\ -17 \end{bmatrix}$$

Now Let,

$$\begin{bmatrix} 19 \\ -17 \end{bmatrix} = xu_1 + yu_2 = x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

i.e. $x + 2y = 19$

$2x + 5y = -17$

$\therefore x = 129, \; y = -55$

Hence we can write

$$T(u_2) = \begin{bmatrix} 19 \\ -17 \end{bmatrix} = 129u_1 - 55u_2 \quad \cdots\cdots \quad (ii)$$

Therefore, we have

$T(u_1) = 52u_1 - 22u_2$

$T(u_2) = 129u_1 - 55u_2$

Here the co-efficient matrix is $\begin{bmatrix} 52 & -22 \\ 129 & -55 \end{bmatrix}$

Therefore the matrix representation of T relative to the basis S is

$$[T]_S = \begin{bmatrix} 52 & 129 \\ -22 & -55 \end{bmatrix}$$

\# Find the matrix representation of the linear operator $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x,y,z) = (x+2y-3z,\; 4x-5y-6z,\; 7x+8y+9z)$ relative to the basis $S = \{u_1, u_2, u_3\}$

$= \{(1,0,0),\; (0,1,0),\; (0,0,1)\}$

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Diagonalization

Consider the matrix $A=\begin{bmatrix}3 & 1\\ 2 & 2\end{bmatrix}$

(a) Find all Eige values and Eigen vectors of A

(b) find a non-singular matrix P and P^{-1} such that $D=P^{-1}AP$ is diagonal

(c) find the positive square root of A i.e. find a matrix B such that $B^2=A$

(d) compute A^{12} using diagonal factorization.

determinant non-zero হলে or non-singular.

The ch. equation for the matrix A is

$$\begin{vmatrix}3-\lambda & 1\\ 2 & 2-\lambda\end{vmatrix}=0$$

$\Rightarrow (3-\lambda)(2-\lambda)-2=0$

$\Rightarrow 6-3\lambda-2\lambda+\lambda^2-2=0$

$\Rightarrow \lambda^2-5\lambda+4=0$

$\Rightarrow \lambda^2-4\lambda-\lambda+4=0$

$\Rightarrow (\lambda-4)-1(\lambda-4)=0$

$\Rightarrow (\lambda-4)(\lambda-1)=0$

$\therefore \lambda=1,4$

For $\lambda=1$

$(A-\lambda I)X=\overline{0}$

i.e. $\begin{bmatrix}2 & 1\\ 2 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$

$\Rightarrow 2x+y=0$

$2x+y=0$

Here,

$A_1=\begin{bmatrix}2 & 1\\ 2 & 1\end{bmatrix}\overset{R_{21}(-1)}{\sim}\begin{bmatrix}2 & 1\\ 0 & 0\end{bmatrix}$

Therefore $\rho(A)=1<n$

Free variable $=n-\rho(A)=1$

Let,

y be the free variable.

Also let $y=k$ where k is any real number.

The equivalent system

$2x+y=0$

i.e. $x=-\frac{k}{2}$

$y=k$

Again along $x^2+y^2=1$,

Let, $x=\cos\theta$, $y=\sin\theta$

$$\oint (2x-y^3)\,dx - xy\,dy$$

$$= \int_0^{2\pi} (2\cos\theta-\sin^3\theta)(-\sin\theta)d\theta - \cos^2\theta\sin\theta\,d\theta$$

$$= \int_0^{2\pi} (-\sin 2\theta)d\theta + \int_0^{2\pi} \sin^4\theta\,d\theta + \int_0^{2\pi} \cos^2\theta\sin\theta\,d\theta$$

$$= \frac{\cos 2\theta}{2}\Big|_0^{2\pi} + \cos^3\theta\Big|_0^{2\pi} + \frac{1}{4}\int_0^{2\pi} (1-\cos 2\theta)^2 d\theta$$

$$= \frac{\pi}{2} + \frac{1}{8}\left[0+\frac{\sin 4\theta}{4}\right]_0^{2\pi}$$

$$= \frac{\pi}{2}+\frac{\pi}{4}$$

$$= \frac{3\pi}{4}$$

Thus the line integral closed by the two circles is

$$\frac{243\pi}{4} - \frac{3\pi}{4} = \frac{240\pi}{4} = 60\pi$$

By Green's Theorem,

$$\iint_R \left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}\right) dx\,dy = \iint_R (-y+3y^2)\,dx\,dy$$

Along the circle $x^2+y^2=9$,

$$= \int_{x=-3}^{3}\int_{y=-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (-y+3y^2)\,dx\,dy$$

$$= \int_{x=-3}^{3} \left[-\frac{y^2}{2}+y^3\right]_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dx$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(\begin{bmatrix} r\cos\phi \\ r\sin\phi \end{bmatrix}\right) - \begin{bmatrix} r\cos(\phi+\theta) \\ r\cos(\phi+\theta) \end{bmatrix}$$

$r\cos\phi\cos\theta - r\sin\phi\sin\theta$

$= r\sin$

Therefore one of the eigen vector corresponding to the eigen value $\lambda=1$ is

$$X_{\lambda=1} = u_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Again for $\lambda=4$

$(A-\lambda I)X=\bar{O}$

i.e. $\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore$ $-x+y=0$

$2x-2y=0$

Here,

$$A_2 = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \overset{R_1(-1)}{\sim} \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \overset{R_{21}(-2)}{\sim} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

Equivalent system

$x-y=0$

i.e. $x=l$

$y=l$.

Therefore one of the eigen vector corresponding to the eigen value $\lambda=4$ is

$$X_{\lambda=4} = u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(b) Now we test the wether or not u_1 and u_2 are linearly independent.

Let,

$x u_1 + y u_2 = 0$

i.e. $x\begin{bmatrix} 1 \\ -2 \end{bmatrix} + y\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

i.e. $x+y=0$
$-2x+y=0$

Here,

$$A_3=\begin{bmatrix}1 & 1\\ -2 & 1\end{bmatrix} \overset{R_{21}(2)}{\sim} \begin{bmatrix}1 & 1\\ 0 & 3\end{bmatrix}$$

which implies that $x=0, y=0$

Hence u_1 and u_2 are linearly independent.

Now we form the non-singular matrix P as

$$P=\begin{bmatrix}1 & 1\\ -2 & 1\end{bmatrix}$$

2x2 order এর matrix-এর inverse

$$P^{-1}=\begin{bmatrix}d/|P| & -b/|P|\\ -c/|P| & a/|P|\end{bmatrix}$$

Here,

$$P^{-1}=\frac{1}{3}\begin{bmatrix}1 & -1\\ 2 & 1\end{bmatrix}$$

Now,

$$D=P^{-1}AP$$
$$=\frac{1}{3}\begin{bmatrix}1 & -1\\ 2 & 1\end{bmatrix}\begin{bmatrix}3 & 1\\ 2 & 2\end{bmatrix}\begin{bmatrix}1 & 1\\ -2 & 1\end{bmatrix}$$
$$=\frac{1}{3}\begin{bmatrix}1 & -1\\ 2 & 1\end{bmatrix}\begin{bmatrix}1 & 4\\ -2 & 4\end{bmatrix}$$
$$=\frac{1}{3}\begin{bmatrix}3 & 0\\ 0 & 12\end{bmatrix}$$
$$=\begin{bmatrix}1 & 0\\ 0 & 4\end{bmatrix}$$

(c) ~~Let~~ We have

$$D=P^{-1}AP$$
$$\Rightarrow PD=AP$$
$$\Rightarrow PDP^{-1}=A$$

i.e. $A=PDP^{-1}$

Now form B i.e. square root of A

$B = P\sqrt{D}\,P^{-1}$

$$= \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

$=$

(d) We have,

$$A = PDP^{-1}$$

Hence $A^{12} = PD^{12}P^{-1}$

$$= \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$