Mathematics

Course: Math 2113

Content: (i) Matrices

(ii) Linear Algebra

Referenced Books:

(i) PN Chatterjee } Matrices
(ii) M.L Khanna }

(iii) Seymour Lipschutz → Linear Algebra

→ Suzan Ahamed (SA)

Savar, Dhaka.

Matrix

$(1,2,3) \longrightarrow \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

↓

3D coordinate

$A_{m \times n}$ — m ↓ rows, n ↓ columns

$a_{ij} \rightarrow$ element

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \text{Unit/Identity matrix}$$

$I \cdot A = A$

Transpose matrix:

If column becomes row and row becomes column, then the new matrix is called the transpose matrix of the first one.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 9 \end{bmatrix} \qquad A' = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 9 \end{bmatrix}$$

If $A = A' \rightarrow$ Symmetric matrix

If $A' = -A \rightarrow$ skew-symmetric matrix

$$A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 9 \\ 3 & -9 & 0 \end{bmatrix} \qquad A' = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & -9 \\ -3 & 9 & 0 \end{bmatrix} = -1 \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 9 \\ 3 & -9 & 0 \end{bmatrix}$$

Here A is a skew-symmetric matrix.

Inverse Matrix:

$$A^{-1} = \frac{\text{adj } A}{|A|} \qquad AA^{-1} = A^{-1}A = I$$

If A and B are such two matrices where $AB = BA = I$, then B is the inverse matrix of A or vice versa.

$BA \rightarrow$ pre-multiplication

$AB \rightarrow$ post-multiplication

$|A| \neq 0 \longrightarrow$ A non-singular matrix

$|A| = 0 \longrightarrow$ A singular matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 1 \end{bmatrix}$$

Cofactor of 1, $\alpha_{11} = (-1)^{1+1} \begin{vmatrix} 4 & 6 \\ 2 & 1 \end{vmatrix} = -8$

... ...

Cofactor of 2, $\alpha_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 2$

... ...

$$\text{adj } A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}'$$

Date 4/4/2016

1(E)-Day

Rank of a Matrix

$2x+3y=13$

$x-y=9$

$$\Rightarrow \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ 9 \end{bmatrix}$$

Rank:

A non-zero matrix A of order $m \times n$ is said to have rank r if at least one point of its $r \times r$-square minor is different from zero while all the other minors (if any) of order $(r+1) \times (r+1)$ are zero.

Minor $\rightarrow$ Determinant of the square sub-matrix of a matrix

Ex-(i) Using minor test procedure, find the rank of the following matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 2 \\ 1 & 2 & 3 & 2 \end{bmatrix}_{3\times 4}$$

$-1(4-4)-2(2-2)+1(0)$

$=0$

Minors of order 3×3 are:

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{vmatrix} = 0 \qquad \begin{vmatrix} 2 & 3 & 1 \\ 4 & 6 & 2 \\ 2 & 3 & 2 \end{vmatrix} = 0$$

Minors of order 2x2 are:

$$\begin{vmatrix} 6 & 2 \\ 3 & 2 \end{vmatrix} = 12-6 = 6 \neq 0$$

Since, there is a minor of order 2 is non-zero and all the other minors of order 3x3 are equal to zero, therefore, the rank of the given matrix will be 2.

2(C) - Day

Date:

H

Vector Analysis

$OP = r$

$x = r\cos\theta$

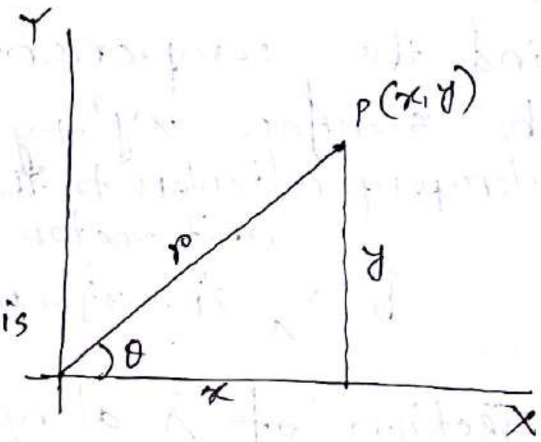
$y = r\sin\theta$

for 3D

$\overline{OP} = x\hat{i} + y\hat{j} + z\hat{k}$

Projection of op along x axis

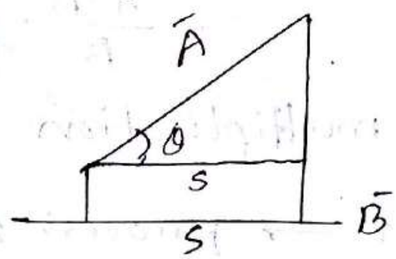
" " op " y "



* Projection ~~is alway~~ of $\bar{A}$ along $\bar{B}$ is s

$\cos\theta = \frac{s}{A}$

$s = A\cos\theta$



$\bar{A}\cdot\bar{B} = AB\cos\theta$

$s = A\cos\theta = \frac{\bar{A}\cdot\bar{B}}{B}$

Projection is scalar (s)
Component is vector ($s\hat{b}$) → Unit vector

$\vec{s} = A\cos\theta = \frac{\bar{A}\cdot\bar{B}}{B}\cdot\hat{b}$

$= \frac{\bar{A}\cdot\bar{B}}{B}\cdot\frac{\bar{B}}{B}$

$|s\hat{b}| = s$

$\frac{\bar{s}}{s} = \hat{b}$

Position vector

$x = \cos t$

$y = t^2$

$z = \sin t$

parametric equ^n of position

$\vec{r} = \cos t\,\hat{i} + t^2\hat{j} + \sin t\,\hat{k}$

- position vector at time t.

$x^2 + y^2 = a^2$ | $\left.\begin{array}{l} x = a\cos\theta \\ y = a\sin\theta \end{array}\right\}$ parametric equ^n of circle.

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$\bar{r} = \cos t\,\hat{i} + t^2\hat{j} + \sin t\,\hat{k}$ $\qquad \left|\frac{d\bar{r}}{dt}\right| = \sqrt{(-\sin t)^2 + (2t)^2 + (\cos t)^2}$

$\bar{v} = \frac{d\bar{r}}{dt} = -\sin t\,\hat{i} + 2t\,\hat{j} + \cos t\,\hat{k}$ at $t=5$

Find the component of velocity along-normal to the surface $xy^2 + yz + z^3 = 0$ / along $2\hat{i} - 3\hat{j} + 5\hat{k}$ / the vector perpendicular to the plane of vectors $2\hat{i} - 3\hat{j} + 10\hat{k}$ and $\hat{i} + 2\hat{j} - 5\hat{k}$

2nd

unit vector of

$\hat{b} = \widehat{2\hat{i} - 3\hat{j} + 5\hat{k}}$

Projection of $\bar{A}$ along $\bar{B}$ ⊗ into unit vector of $\bar{B}$

= component of $\bar{A}$ along $\bar{B}$

$= \frac{\bar{A}\cdot\bar{B}}{B}\cdot\hat{b}$

3rd

Cross multiplication of $2\hat{i} - 3\hat{j} + 10\hat{k}$ and $\hat{i} + 2\hat{j} - 5\hat{k}$

next → process 2nd

5

2(D)-Day $\quad i = i + jk$ $\qquad$ Date - 10/4/2016

Elementary row and column operators:

Row Operators:

(i) $R_{ij} \rightarrow$ Interchanging ith and jth rows

(ii) $R_i(K) \rightarrow$ Multiplying each element of ith row by a non-zero real number K.

(iii) $R_{ij}(k) \rightarrow$ Multiplying each element of jth row by a non-zero real number k and adding with the corresponding elements of ith row.

* Order and Rank will not be changed $\rightarrow$ equivalent matrix

$$A = \begin{bmatrix} 1 & 3 & 4 & 6 \\ 3 & 9 & 1 & 2 \\ 1 & 2 & 4 & 1 \end{bmatrix}$$

(i) $\rightarrow R_{12}$ $\qquad$ (ii) $\rightarrow R_1(5)$ $\qquad$ (iii) $\rightarrow R_{13}(5)$

3 নম্বর Row কে 5 দ্বারা গুণ করে 1 নম্বর Row এর সাথে যোগ করে 1 এই রাখব

Column Operators:

(i) $C_{ij} \rightarrow$ ... ... columns

(ii) $C_i(K) \rightarrow$ ... ... ith column by ... ..

(iii) $C_{ij}(K) \rightarrow$ ... ... jth column ... ... ith column.

(i) $\rightarrow C_{24}$ $\qquad$ (ii) $\rightarrow C_4(10)$ $\qquad$ (iii) $\rightarrow C_{13}(5)$

Normal form of matrix :-

[10 – 15 marks]

Any non-zero matrix A of order $m \times n$ can be reduced by using elementary row and column operators, to one of the four following forms –

(i) $[I_r]$, (ii) $\left[\begin{array}{c} I_r \\ \hline \bar{O} \end{array}\right]$, (iii) $[I_r \ \bar{O}]$, (iv) $\begin{bmatrix} I_r & \bar{O} \\ \bar{O} & \bar{O} \end{bmatrix}$

The above four forms are called the normal form of the matrix A. In this case, rank of the matrix A will be r i.e. $\rho(A) = r$.

$$A = \begin{bmatrix} 1 & 3 & 4 & 6 \\ 3 & 9 & 1 & 2 \\ 1 & 2 & 4 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \qquad \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]$$

$[I_3 \ \bar{O}]$

1 বানাতে হবে (i)

0 বানাতে হবে

$$\begin{bmatrix} I_2 & \bar{O} \\ \bar{O} & \bar{O} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 & 4 & 6 \\ 3 & 9 & 1 & 2 \\ 1 & 2 & 4 & 1 \end{bmatrix}$$

(i) → 0

(ii)

Ex-1 Find the normal form of the following matrix and hence find its rank. [একই সাথে row and column operation করা যাবে]

$$A = \begin{bmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & -4 \\ 5 & 3 & 3 & 11 \end{bmatrix}_{3\times 4} \underset{C_{31}(-1),\ C_{41}(-6)}{\overset{C_{21}(1)}{\sim}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 4 & -6 & -10 \\ 5 & 8 & -12 & -19 \end{bmatrix}$$

$$\underset{R_{31}(-5)}{\overset{R_{21}(-1)}{\sim}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & -6 & -10 \\ 0 & 8 & -12 & -19 \end{bmatrix} \overset{C_2\left(\frac{1}{4}\right)}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -10 \\ 0 & 2 & -12 & -19 \end{bmatrix}$$

$$\underset{C_{42}(10)}{\overset{C_{32}(6)}{\sim}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \overset{R_{32}(-2)}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\overset{C_{34}}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \left[I_3 \ \bar{0} \right]$$

which is the required normal form of the matrix and rank of the matrix is 3.

5

2(E)-Day

*Ex-1 Find the normal form of the following matrix and hence find its rank.

$$A = \begin{bmatrix} 3 & -3 & 1 & 1 \\ 2 & -1 & 1 & 0 \\ 1 & -2 & 0 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix} \overset{R_{13}}{\sim} \begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ 3 & -3 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$

$$\underset{C_{41}(-1)}{\overset{C_{21}(2)}{\sim}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 1 & -2 \\ 3 & 3 & 1 & -2 \\ -1 & -3 & -1 & 2 \end{bmatrix} \underset{R_{41}(1)}{\overset{R_{21}(-2)}{\underset{R_{31}(-3)}{\sim}}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & -2 \\ 0 & 3 & 1 & -2 \\ 0 & -3 & -1 & 2 \end{bmatrix}$$

$$\overset{C_2(\frac{1}{3})}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 \\ 0 & 1 & 1 & -2 \\ 0 & -1 & -1 & 2 \end{bmatrix} \underset{C_{41}(2)}{\overset{C_{31}(-1)}{\sim}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$\underset{R_{42}(1)}{\overset{R_{32}(-1)}{\sim}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} I_2 & \bar{0} \\ \bar{0} & \bar{0} \end{bmatrix}.$$

which is the required normal form of the given matrix and hence the rank of the matrix is 2.

* Find the Echelon form of the following matrix and hence find its rank.

** Echelon form use করার সময় column operator use করা যাবে

$$A = \begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

* Number of non-zero row will be the rank of the matrix.

1 বানানো হলো

$$\underset{\sim}{R_{12}} \begin{bmatrix} 1 & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \underset{R_{31}(-1)}{\overset{R_{21}(2)}{\sim}} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

0 বানানো হলো

$$\underset{\sim}{R_2\left(\frac{1}{3}\right)} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix} \underset{R_{42}(-1)}{\overset{R_{32}(2)}{\sim}} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is the required Echelon form of the given matrix and since there are two non-zero rows, therefore, the rank of the given matrix will be 2.

* Consider the matrix $A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ 3 & -3 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$

(a) Using minor test procedure find the rank of A.

(b) Reduce the matrix A to the normal form and hence find its rank.

(c) Reduce the matrix A to the Echelon form and hence find its rank.

(c)
$$A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ 3 & -3 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix} \underset{R_{41}(1)}{\overset{R_{21}(-2)}{\underset{R_{31}(-3)}{\sim}}} \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 3 & 1 & -2 \\ 0 & 3 & 1 & -2 \\ 0 & -3 & -1 & 2 \end{bmatrix}$$

$$\overset{R_2(\frac{1}{3})}{\sim} \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & \frac{1}{3} & -\frac{2}{3} \\ 0 & 3 & 1 & -2 \\ 0 & -3 & -1 & 2 \end{bmatrix} \underset{R_{42}(3)}{\overset{R_{32}(-3)}{\sim}} \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & \frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is the required echelon form of the given matrix and the number of non-zero rows is 2.

Hence, the rank of the matrix is 2.

Helal Sir

3(C) - Day

Date: 16/4/2016

Vector Analysis

Book by M.R. Sprieggel

Dot or Scalar product:

The dot or scalar product of two vectors $\vec{A}$ and $\vec{B}$ is denoted by $\vec{A}\cdot\vec{B}$ and defined by $\vec{A}\cdot\vec{B} = AB\cos\theta$ where θ is the angle between $\vec{A}$ & $\vec{B}$.

If $\vec{A}\cdot\vec{B} = 0$ then $\vec{A}$ & $\vec{B}$ are perpendicular.

Cross or Vector product:

The cross or vector product of two vectors $\vec{A}$ and $\vec{B}$ is denoted by $\vec{A}\times\vec{B}$ and defined by $\vec{A}\times\vec{B} = AB\sin\theta\,\hat{n}$ where θ is the angle between $\vec{A}$ & $\vec{B}$ and $\hat{n}$ is a unit vector.

If $\vec{A}\times\vec{B} = 0$ then $\vec{A}$ & $\vec{B}$ are parallel.

* Find the angle between $\vec{a} = \hat{i}+2\hat{j}-\hat{k}$ and $\vec{b} = -\hat{i}+\hat{j}-2\hat{k}$

Soln.

Given, $\vec{a} = \hat{i}+2\hat{j}-\hat{k}$

$\vec{b} = -\hat{i}+\hat{j}-2\hat{k}$

$\vec{a}\cdot\vec{b} = ab\cos\theta$

$\Rightarrow \cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{ab}$

$= \dfrac{-1+2+2}{(\sqrt{6})(\sqrt{6})} = \dfrac{1}{2}$

$\therefore \theta = \cos^{-1}\frac{1}{2} = \cancel{60}\ \square$ (Ans).

$a = |\vec{a}| = \sqrt{(1)^2+1^2+(-1)^2}$

$= \sqrt{6}$

$b = |\vec{b}| = \sqrt{6}$

* Using dot product find the unit vector perpendicular to both $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$

<u>Sol^n</u>: Given, $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$

Let, $\vec{c} = x\hat{i} + y\hat{j} + z\hat{k}$ be the required unit vector.

$$\vec{c} \cdot \vec{c} = 1$$

$$\Rightarrow x^2 + y^2 + z^2 = 1 \quad \cdots \cdots (i)$$

since, $\vec{a} \perp \vec{c}$

$$\therefore \vec{a} \cdot \vec{c} = 0$$

$$\Rightarrow (2\hat{i} + 3\hat{j} - \hat{k}) \cdot (x\hat{i} + y\hat{j} + z\hat{k}) = 0$$

$$\Rightarrow 2x + 3y - z = 0 \quad \cdots \cdots (ii)$$

and since $\vec{b} \perp \vec{c}$

$$\therefore \vec{b} \cdot \vec{c} = 0$$

$$\Rightarrow (3\hat{i} - 2\hat{j} + \hat{k}) \cdot (x\hat{i} + y\hat{j} + z\hat{k}) = 0$$

$$\Rightarrow 3x - 2y + z = 0 \quad \cdots \cdots (iii)$$

Solving (ii) & (iii), $\frac{x}{1} = \frac{y}{-5} = \frac{z}{-13}$

$$\therefore y = -5x, \; z = -13x$$

Using these values in (i)

$$x^2 + 25x^2 + 169x^2 = 1$$

$$\therefore x = \pm \frac{1}{\sqrt{195}}$$

$$\therefore y = \pm \frac{-5}{\sqrt{195}} \text{ and } z = \pm \frac{-13}{\sqrt{195}}$$

<u>Note</u>

Perpendicular
Normal
Orthogonal

For every case dot product = 0

Thus the required unit vector is

$$\vec{c} = \pm \left(\frac{\hat{i} - 5\hat{j} - 13\hat{k}}{\sqrt{195}} \right)$$

(Ans)

* Show that $\hat{i}+\hat{j}+\hat{k}$, $\hat{i}-\hat{k}$ and $\hat{i}-2\hat{j}+\hat{k}$ are mutually orthogonal.

* Find x, y & z if $\hat{i}+\hat{j}+2\hat{k}$, $-\hat{i}+z\hat{k}$ and $2\hat{i}+x\hat{j}+y\hat{k}$ are mutually orthogonal.

Soln:

Let, $\vec{a} = \hat{i}+\hat{j}+\hat{k}$

$\vec{b} = \hat{i}-\hat{k}$

$\vec{c} = \hat{i}-2\hat{j}+\hat{k}$

$\vec{a}\cdot\vec{b} = (\hat{i}+\hat{j}+\hat{k})\cdot(\hat{i}-\hat{k}) = 0$

$\vec{b}\cdot\vec{c} = (\hat{i}-\hat{k})\cdot(\hat{i}-2\hat{j}+\hat{k}) = 0$

$\vec{c}\cdot\vec{a} = (\hat{i}-2\hat{j}+\hat{k})\cdot(\hat{i}+\hat{j}+\hat{k}) = 0$

Since $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = 0$

So, they are mutually orthogonal.

* Soln:

Let $\vec{A} = \hat{i}+\hat{j}+2\hat{k}$

$\vec{B} = -\hat{i}+z\hat{k}$

$\vec{C} = 2\hat{i}+x\hat{j}+y\hat{k}$

Since $\vec{A}$, $\vec{B}$ & $\vec{C}$ are mutually orthogonal,

So, $\vec{A}\cdot\vec{B} = 0$

$\Rightarrow -1+0+2z = 0$

$\Rightarrow z = \frac{1}{2}$

$$\vec{B} \cdot \vec{C} = 0$$

$$\Rightarrow (-\hat{i} + z\hat{k}) \cdot (2\hat{i} + x\hat{j} + y\hat{k}) = 0$$

$$\Rightarrow -2 + 0 + zy = 0$$

$$\Rightarrow \frac{1}{2}y = 2$$

$$\therefore y = 4$$

Again

$$\vec{C} \cdot \vec{A} = 0$$

$$\Rightarrow 2 + x + 2y = 0$$

$$\Rightarrow 2 + x + 8 = 0$$

$$\Rightarrow x = -10$$

Suzan Sir

Date: 17/4/2016

3(D)-Day

System of Linear Equation

coefficient matrix → Var. matrix → column matrix

$x+y=6$
$x-y=4$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{\text{coefficient matrix}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{Var. matrix}} = \underbrace{\begin{bmatrix} 6 \\ 4 \end{bmatrix}}_{\text{column matrix}}$$

Matrix representation of the system of linear equation.

System of Linear Equation
- Homogeneous → const. term → 0 (i.e. $x+y=0$, $x-y=0$)
- Non-homogeneous

* Homogeneous system always has a solution (0, 0)

$x+y=9$
$x-y=10$

$$C = \begin{bmatrix} 1 & 1 & 9 \\ 1 & -1 & 10 \end{bmatrix}$$

→ Augmented matrix

<u>Ex-1</u> : Solve the following system.

$x+y+4z=6$
$3x+2y-2z=9$
$5x+y+2z=13$

System of Linear Equation

- Homogeneous System $AX = \bar{O}$ ($\bar{O} \to$ 0 matrix)
  - $\rho(A) = n$: Unique Solution
  - $\rho(A) < n$: Infinitely Many solutions
- Non-homogeneous system $AX = B$
  - $\rho(A) \neq \rho(C)$: No solutions
  - $\rho(A) = \rho(C) = n$: Unique solutions
  - $\rho(A) = \rho(C) < n$: Infinitely Many Solutions

where (i) $\rho(A) \to$ Rank of A

(ii) $n \to$ Number of variables / unknowns

(iii) $C \to$ Augmented matrix

Ex-1

Sol^n. The given system can be written as

$AX = B$ where $A = \begin{bmatrix} 1 & 1 & 4 \\ 3 & 2 & -2 \\ 5 & 1 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 6 \\ 9 \\ 13 \end{bmatrix}$

The augmented matrix for the given system is

$$C = \begin{bmatrix} 1 & 1 & 4 & 6 \\ 3 & 2 & -2 & 9 \\ 5 & 1 & 2 & 13 \end{bmatrix} \underset{R_{31}(-5)}{\overset{R_{21}(-3)}{\sim}} \begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & -1 & -14 & -9 \\ 0 & -4 & -18 & -17 \end{bmatrix}$$

$$\underset{\sim}{R_2(-1)}\begin{bmatrix}1 & 1 & 4 & 6\\ 0 & 1 & 14 & 9\\ 0 & -4 & -18 & -17\end{bmatrix}\underset{\sim}{R_{32}(4)}\begin{bmatrix}1 & 1 & 4 & 6\\ 0 & 1 & 14 & 9\\ 0 & 0 & 38 & 19\end{bmatrix}\underset{\sim}{R_3\left(\frac{1}{38}\right)}\begin{bmatrix}1 & 1 & 4 & 6\\ 0 & 1 & 14 & 9\\ 0 & 0 & 1 & 2\end{bmatrix}$$

$$\rho(A) = 3 = \rho(C) = n$$

$$x + y + 4z = 6$$

$$y + 14z = 9$$

$$z = \frac{1}{2}$$

$$\therefore y = 2$$

$$x = 2$$

Coefficient matrix এর ডানপাশে const. term গুলোকে অতিরিক্ত additional column matrix হিসাবে যোগ করলে যে নতুন matrix হয় তাকে Augmented matrix বলে।

System এর solution থাকলে → consistent
" " " না " → inconsistent

<u>3(E)-Day</u> Date: 18/4/2016

Ex(i) Solve the following system

$$2x+4y-z=9$$
$$3x-y+5z=5$$
$$8x+2y+9z=19$$

~~The given system can be written as~~

The augmented matrix for the given system is

$$C=\begin{bmatrix}2 & 4 & -1 & 9\\ 3 & -1 & 5 & 5\\ 8 & 2 & 9 & 19\end{bmatrix} \underset{\sim}{R_1\left(\tfrac{1}{2}\right)} \begin{bmatrix}1 & 2 & -\frac{1}{2} & \frac{9}{2}\\ 3 & -1 & 5 & 5\\ 8 & 2 & 9 & 19\end{bmatrix}$$

$$\underset{\sim}{\substack{R_{21}(-3)\\ R_{31}(-8)}} \begin{bmatrix}1 & 2 & -\frac{1}{2} & \frac{9}{2}\\ 0 & -7 & \frac{13}{2} & -\frac{17}{2}\\ 0 & -14 & 13 & -17\end{bmatrix} \underset{\sim}{R_2\left(-\tfrac{1}{7}\right)} \begin{bmatrix}1 & 2 & -\frac{1}{2} & \frac{9}{2}\\ 0 & 1 & -\frac{13}{14} & \frac{17}{14}\\ 0 & -14 & 13 & -17\end{bmatrix}$$

$$\underset{\sim}{R_{32}(14)} \begin{bmatrix}1 & 2 & -\frac{1}{2} & \frac{9}{2}\\ 0 & 1 & -\frac{13}{14} & \frac{17}{14}\\ 0 & 0 & 0 & 0\end{bmatrix}$$

which implies that $\rho(A)=\rho(C)=2$ ~~on~~

Since $\rho(A)=\rho(C)=2<n$, therefore the given system has infinitely many solutions.

free variable → যে variable যে কোন value নিতে পারে
Example: $x=0 \Rightarrow 0 \cdot y = x+0$, here y is the free variable
↓
no Restriction

free variable $= n - \rho(A)$
$= 3-2$
$= 1$

$x^2+y^2=9$
↳ 3D → cylinder

In this case, no. of free variable $= n - \rho(A)$
$= 3-2$
$= 1$

Let, z be the free variable. Also let $z=k$, where k is any real number.

Now the equivalent system is

$$x + 2y - \frac{1}{2}z = \frac{9}{2}$$
$$y - \frac{13}{14}z = \frac{17}{14}$$
$$z = k$$

$$\therefore y = \frac{17}{14} + \frac{13k}{14} = \frac{17+13k}{14}$$

$$\therefore x = \frac{9}{2} - \frac{34+26k}{14} + \frac{k}{2}$$

$$= \frac{63 - 34 - 26k + 7k}{14}$$

$$= \frac{29 - 19k}{14}$$

Therefore the solution of the given system is

$$x = \frac{29-19k}{14}$$
$$y = \frac{17+13k}{14}$$
$$z = k$$

Date - 23/4/2016

4(C) - Day

H

* Find the ~~value~~ vector of perpendicular to $\hat{i}+2\hat{k}$ and $\hat{i}+\hat{j}-\hat{k}$ and hence the area of the triangle with this two vectors as adjacent sides.

Soln: Let, $\vec{A} = \hat{i}+2\hat{k}$

$\vec{B} = \hat{i}+\hat{j}-\hat{k}$

$\vec{C} =$

Now, any vector perpendicular to $\vec{A}$ & $\vec{B}$ is $\vec{A}\times\vec{B}$

$$\therefore \vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2 \\ 1 & 1 & -1 \end{vmatrix} = -2\hat{i}+3\hat{j}+\hat{k} \quad \text{(Ans)}.$$

The required area is $\frac{1}{2}|\vec{A}\times\vec{B}|$

$$= \frac{1}{2}\sqrt{4+9+1}$$

$$= \frac{1}{2}\sqrt{14} \quad \text{(Ans)}$$

* Find the volume of ~~the~~ a tetrahedron whose one vector is at origin and other three vertices are (3, 2, 1), (2, 3, -1) and (-1, 2, 3)

Soln. Let, $\vec{a} = 3\hat{i}+2\hat{j}+\hat{k}$

$\vec{b} = 2\hat{i}+3\hat{j}-\hat{k}$

$\vec{c} = -\hat{i}+2\hat{j}+3\hat{k}$

We know, the volume of a tetrahedron is, $v = \frac{1}{6}[abc]$

$= \frac{1}{6}\vec{a}\cdot\vec{b}\times\vec{c}$ ....(i)

$=$

Ans. 5.

* Find the volume of a tetrahedron whose vertices are P(3,4,5), A(2,1,1), B(2,1,5) and C(1,4,2)

Soln:

Let, $\vec{P} = 3\hat{i} + 4\hat{j} + 5\hat{k}$

$\vec{A} = 2\hat{i} + \hat{j} + \hat{k}$

$\vec{B} = 2\hat{i} + \hat{j} + 5\hat{k}$

$\vec{C} = \hat{i} + 4\hat{j} + 2\hat{k}$

Now, $\vec{PA} = (\vec{A} - \vec{P})$

$= -\hat{i} - 3\hat{j} - 4\hat{k}$

$\vec{PB} = (\vec{B} - \vec{P})$

$\vec{PC} = (\vec{C} - \vec{P})$

We know the volume of a tetrahedron is $\frac{1}{6}\vec{PA}\cdot\vec{PB}\times\vec{PC}$

Ans. 4

* Show that the vector $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ form the sides of a right angled triangle.

Soln: Let, $\vec{a} = 2\hat{i}-\hat{j}+\hat{k}$

$\vec{b} = \hat{i}-3\hat{j}-5\hat{k}$

$\vec{c} = 3\hat{i}-4\hat{j}-4\hat{k}$

Now, $|\vec{a}| = \sqrt{4+1+1} = \sqrt{6}$

$|\vec{b}| = \sqrt{1+9+25} = \sqrt{35}$

$|\vec{c}| = \sqrt{9+16+16} = \sqrt{41}$

here

$|\vec{a}|^2+|\vec{b}|^2 = 6+35 = 41 = |\vec{c}|^2$

hence the required triangle is a right angled triangle.

* Show that the vector $\vec{a} = 3\hat{i}-2\hat{j}+\hat{k}$, $\vec{b} = \hat{i}-3\hat{j}+5\hat{k}$, $\vec{c} = 2\hat{i}+\hat{j}-4\hat{k}$ form the sides of a right angled triangle.

Soln.

Given,

$\vec{a} =$

$\vec{b} =$

$\vec{c} =$

$|\vec{a}| = \sqrt{9+4+1} = \sqrt{14}$

$|\vec{b}| = \sqrt{1+9+25} = \sqrt{35}$

$|\vec{c}| = \sqrt{4+1+16} = \sqrt{21}$

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* Prove that the vectors $\vec{A} = 3\hat{i}+\hat{j}-2\hat{k}$, $\vec{B} = -\hat{i}+3\hat{j}+4\hat{k}$, $\vec{C} = 4\hat{i}-2\hat{j}-6\hat{k}$ can form the sides of a triangle. Also, find the length of the medians of the triangle.

<u>Soln.</u>

The vectors can form a triangle if the sum of the two vectors equal to other vectors. here,

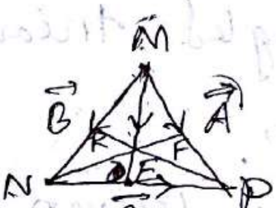
$$\vec{B}+\vec{C} = (-\hat{i}+3\hat{j}+4\hat{k})+(4\hat{i}-2\hat{j}-6\hat{k})$$

$$= 3\hat{i}+\hat{j}-2\hat{k}$$

$$= \vec{A}$$

Hence, the vectors can form the sides of a triangle.

H.W <u>2nd part</u>



$$\vec{ME} = \frac{1}{2}(\vec{MN}+\vec{MP})$$

$$= \frac{1}{2}(\vec{B}+\vec{A})$$

$$= \cancel{\frac{3}{2}\hat{i}+\frac{1}{2}\hat{j}-\hat{k}}\ \frac{1}{2}(2\hat{i}+4\hat{j}+2\hat{k}) = \hat{i}+2\hat{j}+\hat{k}$$

$$|\vec{ME}| = \sqrt{1+4+1} = \sqrt{6}$$

$$\vec{NF} = \frac{1}{2}(\vec{NM}+\vec{NP})$$

$$= \frac{1}{2}(-\vec{B}+\vec{C})$$

$$= \frac{1}{2}(\hat{i}-3\hat{j}-4\hat{k}+4\hat{i}-2\hat{j}-6\hat{k}) = \frac{5}{2}\hat{i}-\frac{5}{2}\hat{j}-5\hat{k}$$

$$\sqrt{\frac{25}{4}+\frac{25}{4}+25} = \sqrt{\frac{150}{4}} = \frac{\sqrt{150}}{2}$$

* An automobile travels 3 Km due north then ~~five~~ 5 Km northeast. Represent this displacement graphically and determine the resultant displacement analytically.

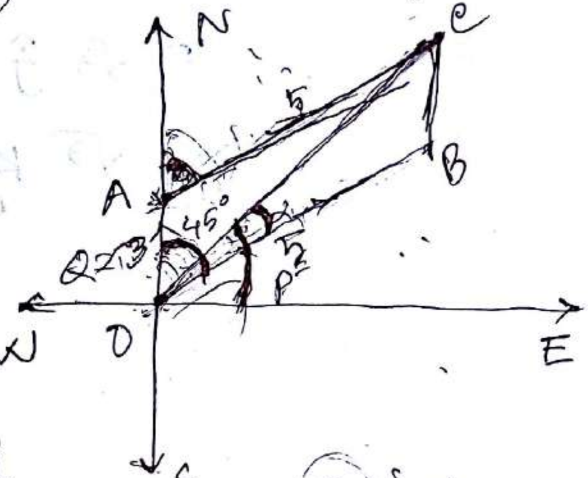
H.W

Ans: 7.43 Km and 61°35′ northeast.

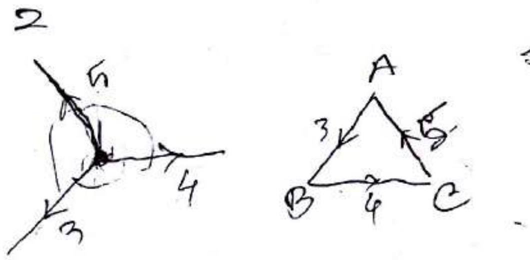
$OC^2 = Q^2 + P^2 + 2PQ\cos\theta$

$OC = \sqrt{3^2+5^2+30\cos 45^\circ}$

$= 7.43$ Km



$\alpha = \tan^{-1}\left(\dfrac{3\sin 45^\circ}{5+3\cos 45^\circ}\right)$



$\tan\alpha = \dfrac{Q\sin\alpha}{P+Q\cos\alpha}$

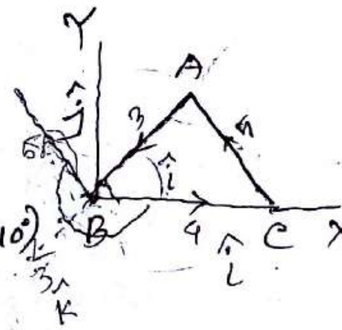
* A particle is subjected to forces 3 kg-wt, 4 kg-wt, 5 kg-wt respectively acting in directions parallel to $\vec{AB}, \vec{BC}, \vec{CA}$ of an equilateral $\triangle ABC$. Find the resultant force acting on the particle.

Soln: ~~Let~~

Let, $\vec{BC}$ direction be parallel to $\hat{i}$ and $\hat{j} \perp$ to $\vec{BC}$ at B.

3 kg-wt forces along $\vec{AB} = 3(\hat{i}\cos 120^\circ + \hat{j}\cos 210^\circ)$

$= -\frac{3}{2}\hat{i} - \frac{3\sqrt{3}}{2}\hat{j}$



4 Kg-wt forces along $\vec{BC} = 4(\hat{i}\cos 0^\circ + \hat{j}\cos 90^\circ)$
$= 4\hat{i}$

5 Kg-wt forces along $\vec{CA} = 5(\hat{i}\cos 240^\circ + \hat{j}\cos 330^\circ)$

$$= -\frac{5\hat{i}}{2} + \frac{5\sqrt{3}}{2}\hat{j}$$

$\therefore$ Total force $= (\cdots) + (\cdots) + (\cdots)$

$$= \sqrt{3}\,\hat{j}.$$

$= \sqrt{3}$ Kg-wt $\perp$ to $\vec{BC}$.

(Ans).

4(D)-Day

Date: 24/4/2016

5A

<u>Ex-1</u> Solve the following system
$$x+y+2z+w=5$$
$$2x+3y-z-2w=2$$
$$4x+5y+3z=7$$

<u>Sol^n.</u>

The augmented matrix for the given system is

$$C = \begin{bmatrix} 1 & 1 & 2 & 1 & 5 \\ 2 & 3 & -1 & -2 & 2 \\ 4 & 5 & 3 & 0 & 7 \end{bmatrix} \underset{R_{31}(-4)}{\overset{R_{21}(-2)}{\sim}} \begin{bmatrix} 1 & 1 & 2 & 1 & 5 \\ 0 & 1 & -5 & -4 & -8 \\ 0 & 1 & -5 & -4 & -13 \end{bmatrix}$$

$$\overset{R_{32}(-1)}{\sim} \begin{bmatrix} 1 & 1 & 2 & 1 & 5 \\ 0 & 1 & -5 & -4 & -8 \\ 0 & 0 & 0 & 0 & -5 \end{bmatrix}$$

Here, since, $\rho(A) \neq \rho(C)$, therefore the given system is <u>inconsistent</u>.

→ কোনো sol^n নেই

Ex-ii Solve the following system
$$2x-y+z=0$$
$$3x+2y+z=0$$
$$x-3y+5z=0$$

<u>Sol^n.</u>

The coefficient matrix is

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 2 & 1 \\ 1 & -3 & 5 \end{bmatrix} \overset{R_{13}}{\sim} \begin{bmatrix} 1 & -3 & 5 \\ 3 & 2 & 1 \\ 2 & -1 & 1 \end{bmatrix} \underset{R_{31}(-2)}{\overset{R_{21}(-3)}{\sim}} \begin{bmatrix} 1 & -3 & 5 \\ 0 & 11 & -14 \\ 0 & 5 & -9 \end{bmatrix}$$

$$\overset{R_2(\frac{1}{11})}{\sim} \begin{bmatrix} 1 & -3 & 5 \\ 0 & 1 & -\frac{14}{11} \\ 0 & 5 & -9 \end{bmatrix} \overset{R_{32}(-5)}{\sim} \begin{bmatrix} 1 & -3 & 5 \\ 0 & 1 & -\frac{14}{11} \\ 0 & 0 & -\frac{29}{11} \end{bmatrix}$$

Therefore, $\rho(A) = 3$

since, $\rho(A) = n$, therefore the given system has unique solution and which is $x=0, y=0, t=0$.

Ex-3 Solve the following system

$$2x - 2y + 5z + 3w = 0$$
$$4x - y + z + w = 0$$
$$3x - 2y + 3z + 4w = 0$$
$$x - 3y + 7z + 6w = 0$$

Soln

The coefficient matrix for the given system is

$$A = \begin{bmatrix} 2 & -2 & 5 & 3 \\ 4 & -1 & 1 & 1 \\ 3 & -2 & 3 & 4 \\ 1 & -3 & 7 & 6 \end{bmatrix} \overset{R_{14}}{\sim} \begin{bmatrix} 1 & -3 & 7 & 6 \\ 4 & -1 & 1 & 1 \\ 3 & -2 & 3 & 4 \\ 2 & -2 & 5 & 3 \end{bmatrix}$$

$$\overset{\substack{R_{21}(-4) \\ R_{31}(-3) \\ R_{41}(-2)}}{\sim} \begin{bmatrix} 1 & -3 & 7 & 6 \\ 0 & 11 & -27 & -23 \\ 0 & 7 & -18 & -14 \\ 0 & 4 & -9 & -9 \end{bmatrix} \overset{R_{23}}{\sim} \begin{bmatrix} 1 & -3 & 7 & 6 \\ 0 & 7 & -18 & -14 \\ 0 & 11 & -27 & -23 \\ 0 & 4 & -9 & -9 \end{bmatrix}$$

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$\underset{\sim}{R_3\left(\frac{1}{7}\right)} \begin{bmatrix} 1 & -3 & 7 & 6 \\ 0 & 1 & -\frac{18}{7} & -2 \\ 0 & 11 & -27 & -23 \\ 0 & 4 & -9 & -9 \end{bmatrix} \underset{R_{42}(-4)}{\overset{R_{32}(-11)}{\sim}} \begin{bmatrix} 1 & -3 & 7 & 6 \\ 0 & 1 & -\frac{18}{7} & -2 \\ 0 & 0 & \frac{9}{7} & -1 \\ 0 & 0 & \frac{9}{7} & -1 \end{bmatrix}$

$\underset{\sim}{R_3\left(\frac{7}{9}\right)} \begin{bmatrix} 1 & -3 & 7 & 6 \\ 0 & 1 & -\frac{18}{7} & -2 \\ 0 & 0 & 1 & -\frac{7}{9} \\ 0 & 0 & \frac{9}{7} & -1 \end{bmatrix} \underset{\sim}{R_{43}\left(-\frac{9}{7}\right)} \begin{bmatrix} 1 & -3 & 7 & 6 \\ 0 & 1 & -\frac{18}{7} & -2 \\ 0 & 0 & 1 & -\frac{7}{9} \\ 0 & 0 & 0 & 0 \end{bmatrix}$

which implies that $\rho(A)=3$

Since $\rho(A) < n$, therefore the given system has infinitely many solutions.

In this case no. of free variable $= n-\rho(A)$
$= 4-3$
$= 1$

Let, ω be the free variable. Also let $\omega = K$, where K is any real number.

Now the equivalent system is

$$x - 3y + 7z + 6\omega = 0$$
$$y - \frac{18}{7}z - 2\omega = 0$$
$$z - \frac{7}{9}\omega = 0$$
$$\omega = K$$

Therefore, the soln of the given system is

$x = \frac{5}{9}K$

$y = 4K$

$z = \frac{7}{9}K$

$\omega = K$

Here, $z = \frac{7}{9}K$

$y = \frac{18}{7}z + 2\omega$

$= \frac{18}{7} \times \frac{7}{9}K + 2K$

$= 4K$

and $x = 3y - 7z - 6\omega$

$= 12K - \frac{49}{9}K - 6K$

$= \frac{108K - 49K - 54K}{9}$

$= \frac{5}{9}K$

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H.W Solve the following system

Ans: Many Sol^n

$$x-2y+z-\omega=0$$
$$x+y-2z+3\omega=0$$
$$4x+y-5z+8\omega=0$$
$$5x-7y+2z-\omega=0$$

C.T

* Characteristic roots and characteristic vectors

Or

Eigen Values or Eigen Vectors

Let, A be a non-zero matrix of order $n \times n$. Then

a) The matrix $A-\lambda I$ is called the characteristic matrix of A.

b) The determinant $|A-\lambda I|$ is called the characteristic polynomial of A

c) The equation $|A-\lambda I|=0$ is called the characteristic equation of A.

d) The roots of the equation $|A-\lambda I|=0$ are called the characteristic roots or eigen values or latent roots of A.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A-\lambda I = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix} \rightarrow \text{characteristic matrix of } A$$

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda)-6 = \lambda^2-5\lambda-2$$

$\hookrightarrow$ polynomial of A

$\lambda^2-5\lambda-2=0 \longrightarrow$ Characteristic equation

$\Rightarrow \lambda = \frac{5 \pm \sqrt{25+8}}{2} \rightarrow$ characteristic roots/ eigen values

Eigen vectors or Characteristic vectors:-

Corresponding to each value of λ there is a non-zero vector x which satisfies the equation $(A-\lambda I)x = \bar{0}$. These non-zero vectors are called the eigen vectors or characteristic vectors.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\lambda^2 - 5\lambda - 2 = 0$$

$$\Rightarrow \lambda = \frac{5 \pm \sqrt{25+8}}{2}$$

$$= \frac{5+\sqrt{33}}{2}, \frac{5-\sqrt{33}}{2}$$

$$x = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$(A-\lambda I)x = \bar{0}$$

$$\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \frac{5+\sqrt{33}}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} \frac{5+\sqrt{33}}{2} & 0 \\ 0 & \frac{5+\sqrt{33}}{2} \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{3+\sqrt{33}}{2} & 2 \\ 3 & \frac{3-\sqrt{33}}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(x, y) = (\ , \ )$ $\quad -\frac{3+\sqrt{33}}{2}x + 2y = 0, \quad 3x + \frac{3-\sqrt{33}}{2}y = 0$

Ex-1 Find the characteristic roots or eigen values and corresponding eigen vectors of the following matrices.

(i) $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$, (iv) $A = \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}$ (v) $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{bmatrix}$

(i) The ch. equation of the matrix A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\{(3-\lambda)(2-\lambda)-2\} - 2\{1(2-\lambda)-1\} + 1\{2-1(3-\lambda)\} = 0$$

$$\Rightarrow (2-\lambda)(6-5\lambda+\lambda^2-2) - 2(1-\lambda) + 1(\lambda-1) = 0$$

$$\Rightarrow (2-\lambda)(\lambda^2-5\lambda+4) - 2 + 2\lambda + \lambda - 1 = 0$$

$$\Rightarrow 2\lambda^2 - 10\lambda + 8 - \lambda^3 + 5\lambda^2 - 4\lambda + 3\lambda - 3 = 0$$

$$\Rightarrow -\lambda^3 + 7\lambda^2 - 11\lambda + 5 = 0$$

$$\Rightarrow \lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

$$\therefore \lambda = 1, 1, 5$$

Now, for $\lambda = 1$,

$(A - \lambda I)X = \bar{0}$, where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\Rightarrow \left(\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x + 2y + z \\ x + 2y + z \\ x + 2y + z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This implies $\quad x + 2y + z = 0$

$x + 2y + z = 0$

$x + 2y + z = 0$

Coefficent matrix,

$$A_1 = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \underset{R_{31}(-1)}{\overset{R_{21}(-1)}{\sim}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 1 < n$, So, there are many solutions.

No. of free variables $= n - \rho(A)$

$= 3 - 1$

$= 2$

Let, y and z be the free variables and

$y = k$ and $z = m$

Now the equivalent system is

$$x + 2y + z = 0$$
$$\therefore x = -2y - z.$$
$$= -2k - m$$

Thus, $x = -2k - m$

$y = k$

$z = m$

Therefore, the eigen vector corresponding to the eigen values $\lambda = 1$, is

$$X_{\lambda=1} = \begin{bmatrix} -2k-m \\ k \\ m \end{bmatrix}$$

For $\lambda = 5$, $\downarrow$

6

Ex-1 Find the eigen values and eigen vectors of the matrix,

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

The characteristic equation for the given matrix

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 5-\lambda & 1 \\ 1 & 1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)\{(5-\lambda)(3-\lambda)-1\} - 1(3-\lambda-1) + 1(1-5+\lambda) = 0$$

$$\Rightarrow (3-\lambda)(14-8\lambda+\lambda^2) - 2 + \lambda + \lambda - 4 = 0$$

$$\Rightarrow 42 - 24\lambda + 3\lambda^2 - 14\lambda + 8\lambda^2 - \lambda^3 + 2\lambda - 6 = 0$$

$$\Rightarrow \lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

$$\Rightarrow \lambda = 2, 3, 6$$

For $\lambda = 2$

$$(A - \lambda I)X = \bar{0}$$

$$\therefore (A - 2I)X = \bar{0}$$

$$\Rightarrow \left(\begin{bmatrix} 3 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

7

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which implies that

$$x+y+z=0$$
$$x+3y+z=0$$
$$x+y+z=0$$

Coefficient matrix of the above system is,

$$A_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix} \underset{R_{31}(-1)}{\overset{R_{21}(-1)}{\sim}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A_1) = 2$. Since $\rho(A_1) < n$, therefore the above system has infinitely many solutions.
In this case, free variables $= n - \rho(A_1)$
$= 3-2$
$= 1$

Let z be the free variable and $z = K$ where K is any real number.
Now, the equivalent system is

$$x+y+z=0 \qquad \therefore x = -K$$
$$2y = 0 \qquad y = 0$$
$$z = K \qquad z = K$$

Therefore, the eigen vector corresponding to the eigen value $\lambda = 2$,

$$X_{\lambda=2} = \begin{bmatrix} -K \\ 0 \\ K \end{bmatrix}$$, where k is any real number.

For $\lambda = 3$,

$$(A - \lambda I)X = 0$$

$$(A - 3I)X = 0$$

$$\Rightarrow \left(\begin{bmatrix} 3 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which implies

$$y + z = 0$$

$$x + 2y + z = 0$$

$$x + y = 0$$

Coefficient matrix of the above system is

$$A_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \overset{R_{12}}{\sim} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \overset{R_{31}(-1)}{\sim} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \overset{R_{32}(1)}{\sim} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A_1) = 2 < n$. Therefore, the above system has infinitely many solution.

No. of free variables $= n - \rho(A_1)$

$= 3 - 2$

$= 1$

Let z be the free variable and $z = K$ where K is any real number.

Now, the equivalent system is

$$x + 2y + z = 0$$
$$y + z = 0$$
$$z = K$$

$\therefore y = -K$ and $x = K$

$X_{\lambda=3} = \begin{bmatrix} K \\ -K \\ K \end{bmatrix}$, where $K \in \mathbb{R}$

For, $\lambda = 5$ &

Ex-ii Investigate for what values of λ and μ, the following system

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + \lambda z = \mu$$

have (i) No solution

(ii) Unique solution

(iii) Infinitely many solution

The augmented matrix for the given system is

$$C = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix} \xrightarrow[R_{31}(-1)]{R_{21}(-1)} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{bmatrix}$$

$$\xrightarrow{R_{32}(-1)} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{bmatrix}$$

(i) No solution: $\lambda = 3$ and $\mu \neq 10$

(ii) Unique solution: $\lambda \neq 3$

(iii) Many solution: $\lambda = 3$ and $\mu = 10$

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Cremer Solution

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5(E)-Day 11 →   **Date:** 3/4/2016

~~Cayley~~

Cayley – Hamilton Theorem:

Statement: Every square matrix satisfies its characteristic equation.

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix}$$

$$= (1-\lambda)(3-\lambda) - 2$$

$$= \lambda^2 - 4\lambda + 1$$

Ch. eq.

$$\lambda^2 - 4\lambda + 1 = 0$$

According to Cayley-Hamilton Theorem,

$$\boxed{A^2 - 4A + I = \bar{O}}$$

$$A^{-1}A^2 - 4A^{-1}A + A^{-1}I = A^{-1}\bar{O}$$

$$A - 4I + A^{-1} = \bar{O}$$

$$\Rightarrow \boxed{A^{-1} = 4I - A}$$

Ex-1 Verify Cayley-Hamilton theorem for the following matrix and hence find its inverse.

(i) $A = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (iii) $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

iii) The characteristic equation for the matrix A,

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\{(2-\lambda)^2 - 1\} + 1\{-1(2-\lambda) + 1\} + 1\{1 - 1(2-\lambda)\} = 0$$

$x = 3\sqrt{x+1}$

$\Rightarrow (2-\lambda)(3-4\lambda+\lambda^2)+\lambda-1+\lambda-1=0$

$\Rightarrow 6-8\lambda+2\lambda^2-3\lambda+4\lambda^2-\lambda^3+2\lambda-2=0$

$\Rightarrow \boxed{\lambda^3-6\lambda^2+9\lambda-4=0}$

According to the cayley-Hamilton theorem, we have to show that $A^3-6A^2+9A-4I=\bar{0}$

Here, $L.H.S. = A^3-6A^2+9A-4I$

$$= \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}^3 - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}^2 + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, from eq(1),

$A^3-6A^2+9A-4I=\bar{0}$

Multiplying both sides by A^{-1}, we get

$A^2-6A+9I-4A^{-1}=\bar{0}$

$\Rightarrow 4A^{-1} = A^2-6A+9I$

$\therefore A^{-1} = \frac{1}{4}(A^2-6A+9I)$.

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Imp

Theorem: If a square matrix has an inverse, then it is unique.

Proof: Let A be any square matrix. Also let A has two inverses B and C. Then by the definition

$$AB = BA = I$$

and $$AC = CA = I$$

Now $$B = B \cdot I = B \cdot AC = (BA)C = I \cdot C = C$$

$\therefore$ $B = C$ [Proved].

Prove → $\boxed{(AB)^{-1} = B^{-1}A^{-1}}$

Proof:

Linearly dependent and independent:

The vectors $\vec{A}, \vec{B}, \vec{C}, \ldots\ldots$ are called Linearly dependent if we can find a set of scalars $a, b, c, \ldots\ldots$ not all zero such that $a\vec{A}+b\vec{B}+c\vec{C}+\ldots\ldots=0$, otherwise they are Linearly independent.

If $\vec{a}, \vec{b}, \vec{c}$ are non-coplaner vectors, determine whether the vectors $\vec{r_1} = 2\vec{a}-3\vec{b}+\vec{c}$, $\vec{r_2} = 3\vec{a}-5\vec{b}+2\vec{c}$ and $\vec{r_3} = 4\vec{a}-5\vec{b}+\vec{c}$ are Linearly independent or dependent.

Soln.

Given,

$$\vec{r_1} = 2\vec{a}-3\vec{b}+\vec{c}$$
$$\vec{r_2} = 3\vec{a}-5\vec{b}+2\vec{c}$$
$$\vec{r_3} = 4\vec{a}-5\vec{b}+\vec{c}$$

Let, $x\vec{r_1}+y\vec{r_2}+z\vec{r_3}=0$ where x, y, z are scalars.

$$\Rightarrow x(2\vec{a}-3\vec{b}+\vec{c})+y(3\vec{a}-5\vec{b}+2\vec{c})+z(4\vec{a}-5\vec{b}+\vec{c})=0$$

$$\Rightarrow (2x+3y+4z)\vec{a}-(3x+5y+5z)\vec{b}+(x+2y+z)\vec{c}=0$$

Since $\vec{a}, \vec{b}$ & $\vec{c}$ are non coplaner

So,

$$2x+3y+4z=0 \quad \text{---(i)}$$
$$3x+5y+5z=0 \quad \text{---(ii)}$$
$$x+2y+z=0 \quad \text{---(iii)}$$

non coplaner হলে coefficients 0

Solving (i) & (ii) $x = -5z$

$y = 2z$

Using these values in (iii)

$$-10z + 6z + 4z = 0$$

$$\Rightarrow 0 = 0$$

So, $\vec{r_1}, \vec{r_2}, \vec{r_3}$ are linearly dependent.

In each case determine whether the vectors are linearly independent or dependent.

H.W. a) $\vec{A} = 2\hat{i} + \hat{j} - 3\hat{k}$, $\vec{B} = \hat{i} - 4\hat{k}$, $\vec{C} = 4\hat{i} + 3\hat{j} - \hat{k}$

b) $\vec{A} = \hat{i} - 3\hat{j} + 2\hat{k}$, $\vec{B} = 2\hat{i} - 4\hat{j} - \hat{k}$, $\vec{C} = 3\hat{i} + 2\hat{j} - \hat{k}$

(b)

Given, $\vec{A} = \hat{i} - 3\hat{j} + 2\hat{k}$, $\vec{B} = 2\hat{i} - 4\hat{j} - \hat{k}$, $\vec{C} = 3\hat{i} + 2\hat{j} - \hat{k}$

Let $x\vec{A} + y\vec{B} + z\vec{C} = 0$ where x, y, z are scalars.

$$\Rightarrow x(\hat{i} - 3\hat{j} + 2\hat{k}) + y(2\hat{i} - 4\hat{j} - \hat{k}) + z(3\hat{i} + 2\hat{j} - \hat{k}) = 0$$

$$\Rightarrow (x + 2y + 3z)\hat{i} - (3x + 4y - 2z)\hat{j} + (2x - y - z)\hat{k} = 0$$

~~Since~~, $\therefore$

$x + 2y + 3z = 0$ ... (i)

$3x + 4y - 2z = 0$ ... (ii)

$2x - y - z = 0$ ... (iii)

Solving (ii) & (iii) $x = \frac{6z}{11}$, $y = \frac{z}{11}$

Using these values in (i) $\frac{6z}{11} + \frac{z}{11} + 3z \neq 0$

Hence the vectors are linearly independent.

6(D)-Day 14 Date: 9/5/2016

Theorem: Let A and B are two square matrices of order $n \times n$; then $adj(AB) = (adj B)\cdot(adj A)$

Proof:

We know that

$$A^{-1} = \frac{adj\ A}{|A|}$$

$$|A|\,A^{-1} = adj\ A$$

Multiplying both sides by A,

$$\therefore |A|\,AA^{-1} = A\ adj\ A$$

$$\Rightarrow |A|\,I = A\ adj\ A$$

$$\therefore A\ adj\ A = |A|\,I$$

$$|A|\,I = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$$

Hence, we have

$$AB\ adj(AB) = |AB|\,I \quad \cdots\cdots (i)$$

Again we have

$$AB\cdot(adj B)\cdot(adj A)$$

$$= A\,|B|\,I\ adj\ A$$

$$= |B|\,(A\ adj\ A)\,I$$

$$= |B|\,|A|\,I$$

$$= |A|\,|B|\,I$$

$$= |AB|\,I$$

Note

$|AB| = |A|\,|B|$

$\therefore\ AB\,(\text{adj}B)(\text{adj}A) = |AB|\,I$ —— (ii)

Comparing Equ^n (i) & (ii), we have,

$$AB\,(\text{adj}\,AB) = AB\,(\text{adj}\,B)(\text{adj}\,A)$$

$$\therefore\ \text{adj}\,(AB) = (\text{adj}\,B)(\text{adj}\,A)$$

<u>Theorem</u>: If A and B are two non-singular square matrices of order $n\times n$, then prove that $(AB)^{-1} = B^{-1}A^{-1}$

<u>Proof</u>:

Given that A and B are non-singular matrices i.e. $|A| \neq 0$ and $|B| \neq 0$

We know that $|AB| = |A|\,|B|$

This implies $|AB| \neq 0$

We have,

$$AB\,B^{-1}A^{-1} = AIA^{-1}$$
$$= IAA^{-1}$$
$$= I$$

and $B^{-1}A^{-1}AB = B^{-1}IB = IB^{-1}B = I$

which implies that

$$(AB)^{-1} = B^{-1}A^{-1} \quad \Big|\ \because AB = BA = I$$

<u>Theorem:</u> Every square matrix can be uniquely expressed as the sum of a symmetric and a skew-symmetric matrix.

Let, A be any square matrix of order $n \times n$.
Then,

$$A = \frac{1}{2}A + \frac{1}{2}A' + \frac{1}{2}A - \frac{1}{2}A'$$

$$= \frac{1}{2}(A+A') + \frac{1}{2}(A-A')$$

$\therefore A = B + C$ where $B = \frac{1}{2}(A+A')$, $C = \frac{1}{2}(A-A')$

Now we have to show that B is symmetric and C is skew-symmetric.

$$B' = \left\{\frac{1}{2}(A+A')\right\}'$$

$$= \frac{1}{2}(A+A')' \qquad \because (A+B)' = A' + B'$$

$$= \frac{1}{2}(A' + (A')')$$

$$= \frac{1}{2}(A'+A)$$

$$= \frac{1}{2}(A+A')$$

$$= B$$

which implies that B is a symmetric matrix.

Again $C' = \left\{\frac{1}{2}(A-A')\right\}'$

$$= \frac{1}{2}(A' - (A')')$$

$$= \frac{1}{2}(A' - A)$$

$$= -\frac{1}{2}(A - A')$$

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$= \frac{1}{2}(A - A')'$

$= \frac{1}{2}(A + (-1)A')'$

$= \frac{1}{2}[A' + \{(-1)A'\}']$

$= \frac{1}{2}(A' - (A')')$

$= \frac{1}{2}(A' - A)$

$= -\frac{1}{2}(A - A')$

$= -C$

which implies that C is skew-symmetric.

Suppose, A has another representation.

Let, $A = A_1 + A_2$ — (i) where A_1 is symmetric and A_2 is skew-symmetric. i.e. $A_1' = A_1$ and $A_2' = -A_2$.

Now, $A' = (A_1 + A_2)'$

$\Rightarrow A' = A_1' + A_2'$

$\therefore A' = A_1 - A_2$ — (ii)

Adding Equ^n (i) and (ii), we get.

$A_1 = \frac{1}{2}(A + A')$

Subtracting Equ^n (ii) from (i), we get,

$A_2 = \frac{1}{2}(A - A')$

which implies that the representation is unique.

6(E)-Day Date: 10/5/2016

A particle is acted on by constant forces $3\hat{i}+2\hat{j}+5\hat{k}$, and $2\hat{i}+\hat{j}-3\hat{k}$ and is displaced from a point whose position vector is $2\hat{i}-\hat{j}-3\hat{k}$ to a point whose position vector is $4\hat{i}-3\hat{j}+7\hat{k}$. Calculate the work done.

Soln: Total force $= (3\hat{i}+2\hat{j}+5\hat{k}) + (2\hat{i}+\hat{j}-3\hat{k}) = 5\hat{i}+3\hat{j}+2\hat{k}$

Displacement $= (4\hat{i}-3\hat{j}+7\hat{k}) - (2\hat{i}-\hat{j}-3\hat{k}) = 2\hat{i}-2\hat{j}+10\hat{k}$

Work done $= \vec{F}\cdot\vec{D}$

$= (\quad)\cdot(\quad)$

$= 24$ (Ans)

Force of magnitudes 5 and 3 units acting in the directions $6\hat{i}+2\hat{j}+3\hat{k}$ and $3\hat{i}-2\hat{j}+6\hat{k}$ respectively act on a particle which is displaced from $(2,2,-1)$ to $(4,3,1)$. Find work done.

Soln.

First force of magnitude 5 unit acting in the direction $6\hat{i}+2\hat{j}+3\hat{k} = 5\left(\frac{6\hat{i}+2\hat{j}+3\hat{k}}{\sqrt{36+4+9}}\right)$

$= \frac{5}{7}(6\hat{i}+2\hat{j}+3\hat{k})$

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2nd force of magnitude 3 unit acting in the direction

$3\hat{i}-2\hat{j}+6\hat{k} = 3\left(\frac{3\hat{i}-2\hat{j}+6\hat{k}}{\sqrt{9+4+36}}\right) = \frac{3}{7}(3\hat{i}-2\hat{j}+6\hat{k})$

So, resultant force $= \frac{5}{7}(6\hat{i}+2\hat{j}+3\hat{k}) + \frac{3}{7}(3\hat{i}-2\hat{j}+6\hat{k})$

$= \frac{1}{7}(39\hat{i}+4\hat{j}+33\hat{k})$

Displacement $= (4\hat{i}+3\hat{j}+\hat{k}) - (2\hat{i}+2\hat{j}-\hat{k}) = 2\hat{i}+\hat{j}+2\hat{k}$

Workdone $= \vec{F}\cdot\vec{D} = (\quad)\cdot(\quad) = \frac{148}{7}$ (Ans).

Find a vector of magnitude 5 parallel to yz plane and perpendicular to $2\hat{i}+3\hat{j}+\hat{k}$.

Sol^n.

Let, $\vec{A} = b\hat{j}+c\hat{k}$ be any vector parallel to yz plane.

and $\vec{B} = 2\hat{i}+3\hat{j}+\hat{k}$

Since $\vec{A}\perp\vec{B}$

so $\vec{A}\cdot\vec{B} = 0$

or, $(b\hat{j}+c\hat{k})\cdot(2\hat{i}+3\hat{j}+\hat{k}) = 0$

or, $c = -3b$

$\therefore \vec{A} = b\hat{j}-3b\hat{k}$

Also $|\vec{A}| = \sqrt{b^2+9b^2} = b\sqrt{10}$

According to question $b\sqrt{10} = 5$

$b = \sqrt{\frac{5}{2}}$

and $c = -3\sqrt{\frac{5}{2}}$

Hence the required vector $\vec{A} = \sqrt{\frac{5}{2}}(\hat{j}-3\hat{k})$

(Ans).

Find the projection of $4\hat{i}-3\hat{j}+\hat{k}$ on the line passing through the points $(2,3,-1)$ & $(-2,-4,3)$.

Soln. Let, $\vec{A} = 4\hat{i}-3\hat{j}+\hat{k}$

Any vector passing through given points is

$$\vec{B} = (-2-2)\hat{i} + (-4-3)\hat{j} + (3+1)\hat{k}$$

$$= -4\hat{i}-7\hat{j}+4\hat{k}$$

Hence the projection of $\vec{A}$ on $\vec{B} = \dfrac{\vec{A}\cdot\vec{B}}{|\vec{B}|}$

$=$

Find the angles which the vector $\vec{A} = 3\hat{i}-6\hat{j}+2\hat{k}$ makes with coordinate axes.

Soln. Let, α, β, γ be the angles which $\vec{A}$ makes with, x, y, z axes respectively.

$\because \vec{A}\cdot\hat{i} = |\vec{A}|\,|\hat{i}|\cos\alpha$

$\Rightarrow (3\hat{i}-6\hat{j}+2\hat{k})\cdot\hat{i} = 7\cos\alpha$

$\Rightarrow \cos\alpha = \dfrac{3}{7}$

$\therefore \alpha = \cos^{-1}\left(\frac{3}{7}\right)$

$= 64\cdot6^\circ$ (Ans.)

similarly,

$\beta = \cos^{-1}\left(\frac{-6}{7}\right)$

$= 149^\circ$

$\gamma = \cos^{-1}\left(\frac{2}{7}\right)$

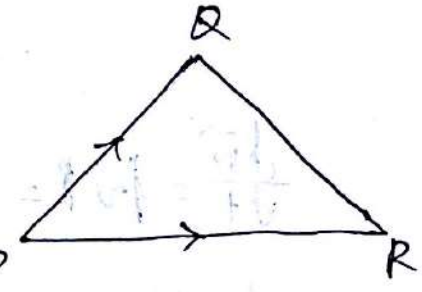
$= 73\cdot4^\circ$

Find the area of the triangle having vertices at $P(1,3,2)$, $Q(2,-1,1)$ & $R(-1,2,3)$.

Soln.

$\vec{PQ} = (2-1)\hat{i} + (-1-3)\hat{j} + (1-2)\hat{k} = \hat{i} - 4\hat{j} - \hat{k}$

$\vec{QR} = -2\hat{i} - \hat{j} + \hat{k}$



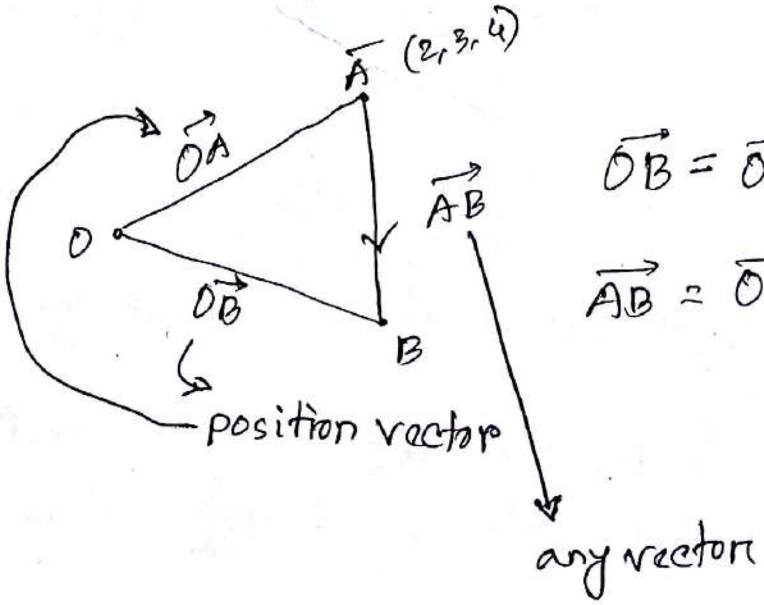
We know, the area of ΔPQR

$= \frac{1}{2} \left| \vec{PQ} \times \vec{QR} \right|$

$= \frac{1}{2} \left| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -4 & -1 \\ -2 & -1 & +1 \end{vmatrix} \right| = \frac{1}{2} \left| -5\hat{i} + \hat{j} - 9\hat{k} \right|$

$= \frac{1}{2} \sqrt{25+1+81}$

$= \frac{1}{2} \sqrt{107}$ (Ans).



$\vec{OB} = \vec{OA} + \vec{AB}$

$\vec{AB} = \vec{OB} - \vec{OA}$

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Proxy

7(D) - Day   Date :- 25/5/2016

$x = a\cos t$, $y = a\sin^2 t$, $z = t^3$

$$\boxed{\vec{r} = a\cos t\,\hat{i} + a\sin^2 t\,\hat{j} + t^3\,\hat{k}}$$

Position vector with respect to origin

$$\frac{d\vec{r}}{dt} = \vec{v} = -a\sin t\,\hat{i} + 2a\sin t\cos t\,\hat{j} + 3t^2\,\hat{k}$$

Find the component of velocity along x-axis.

Or along $2\hat{i} + 2\hat{j} - 4\hat{k}$

Suppose, at $t = 5$

$$\vec{v} = 4\hat{i} - \hat{j} + 3\hat{k}$$

Cayley-Hamilton Theorem: Every square matrix satisfies its characteristic equation.

i.e. if A be any non-zero square matrix of order $n \times n$ and if $|A-\lambda I| = (-1)^n(\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots\cdots + a_n)$ then according to the cayley-Hamilton theorem, we have to show that $A^n + a_1A^{n-1} + \cdots\cdots + a_nI = \bar{O}$

$A_{n\times n}$
$|A-\lambda I| = (-1)^n\lambda^n$
λ Highest

Proof:

Since the elements of $(A-\lambda I)$ are at the most of first degree in λ, therefore the elements of adj $(A-\lambda I)$ are ~~the~~ at the most of degree $(n-1)$ in λ.

Hence the matrix adj$(A-\lambda I)$ can be written as

$$\text{adj}(A-\lambda I) = B_0\lambda^{n-1} + B_1\lambda^{n-2} + \cdots\cdots + B_{n-1} \quad \cdots\cdots (i)$$

where $B_0, B_1, B_2, \cdots\cdots, B_{n-1}$ are some square matrices of order $n \times n$

For any square matrix A, we know that

$$A \text{ adj } A = (\text{adj} A)A = |A|I$$

Similarly for the matrix adj$(A-\lambda I)$, we have

$$(A-\lambda I)\text{ adj}(A-\lambda I) = |A-\lambda I|.I$$

$$\Rightarrow (A-\lambda I)(B_0\lambda^{n-1} + B_1\lambda^{n-2} + \cdots B_2\lambda^{n-3}\cdots + B_{n-1}) = (-1)^n(\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots\cdots + a_n)$$

Equating the coefficients of ~~Light~~ liked powers of λ of the above equation, we get.

λ^n : $-IB_0 = (-1)^n I$

λ^{n-1} : $AB_0 - IB_1 = (-1)^n a_1 I$

λ^{n-2} : $AB_1 - IB_2 = (-1)^n a_2 I$

... ...

... ...

λ^0 : $AB_{n-1} = (-1)^n a_n I$

Linear Algebra
(i) Vector space
(ii) Linear Mapping
(iii) Matrix linear Mapping
বাহাদুর দেখে আসতে হবে

Now, pre-multiplying both sides of the above equation by $A^n, A^{n-1}, \ldots\ldots, A, I$ respectively, we get:

$$-A^n B_0 = (-1)^n A^n$$

$$A^n B_0 - A^{n-1} B_1 = (-1)^n a_1 A^{n-1}$$

$$A^{n-1} B_1 - A^{n-2} B_2 = (-1)^n a_2 A^{n-2}$$

... ...

... ...

$$AB_{n-1} = (-1)^n a_n I$$

Now adding the above equations we get,

$$\bar{O} = (-1)^n \left(A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots \ldots + a_n I\right)$$

$\therefore (A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots \ldots + a_n I) = \bar{O}$. [multiplying with $(-1)^n$]

H

8(D)-Day Date - 11/7/2016

Vector Differentiation

Let, $\vec{A}$ be any vector and $\vec{A}$ is a function of u and if we can write $\vec{A}(u) = A_1(u)\hat{i} + A_2(u)\hat{j} + A_3(u)\hat{k}$

then $\frac{d(\vec{A})}{du} = \frac{d(A_1)}{du}\hat{i} + \frac{d(A_2)}{du}\hat{j} + \frac{d(A_3)}{du}\hat{k}$

Vector Differentiation operator 'del' $\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

Gradient of a scalar function ϕ is

$$\text{grad } \phi = \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Divergence of a vector $\vec{A}$ is $\nabla\cdot\vec{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\vec{A}$

Curl of a vector $\vec{A}$ is $\nabla\times\vec{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\vec{A}$

Gradient of a vector is always undefined.

—o—

Find the velocity and acceleration of a particle which moves along the curve $x = 2\sin 3t$, $y = 2\cos 3t$, $z = 8t$ at any time $t > 0$, also find the magnitude of them.

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Soln

Let the position vector of the particle is $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$= 2\sin 3t\,\hat{i} + 2\cos 3t\,\hat{j} + 8t\,\hat{k}$

$\therefore$ velocity $\vec{v} = \frac{d\vec{r}}{dt} = 6\cos 3t\,\hat{i} - 6\sin 3t\,\hat{j} + 8\hat{k}$ Ans.

acceleration $\vec{a} = \frac{d\vec{v}}{dt} = -18\sin 3t\,\hat{i} - 18\cos 3t\,\hat{j}$ Ans.

magnitude of $\vec{v} = |\vec{v}| = \sqrt{6^2\cos^2 3t + 6^2\sin^2 3t + 8^2}$

$= 10$ Ans.

magnitude of $\vec{a} = |\vec{a}| = \sqrt{18^2\sin^2 3t + 18^2\cos^2 3t}$

$= 18$ Ans.

Find a unit tangent vector to any point on the curve $x = a\cos\omega t$, $y = a\sin\omega t$, $z = bt$ where a, b, ω are constant.

Soln Let the position vector of the point is

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$= a\cos\omega t\,\hat{i} + a\sin\omega t\,\hat{j} + bt\,\hat{k}$

Now, the tangent vector ~~of the point~~ is

$$\vec{T} = \frac{d\vec{r}}{dt} = -a\omega \sin\omega t\,\hat{i} + a\omega \cos\omega t\,\hat{j} + b\hat{k}$$

$$\therefore |\vec{T}| = \sqrt{a^2\omega^2 \sin^2\omega t + a^2\omega^2 \cos^2\omega t + b^2}$$

$$= \sqrt{a^2\omega^2 + b^2}$$

$$\therefore \text{unit tangent vector} = \frac{\vec{T}}{|\vec{T}|}$$

$$= \frac{-a\omega \sin\omega t\,\hat{i} + a\omega \cos\omega t\,\hat{j} + b\hat{k}}{\sqrt{a^2\omega^2 + b^2}}$$

(Ans.)

A particle moves along the curve $x = t^3 + 1$, $y = t^2$, $z = 2t + 5$ where t is the time. Find the Components of its velocity & acceleration at $t = 1$ in the direction $2\hat{i} + 3\hat{j} + 6\hat{k}$.

Solⁿ. Let the position vector of the particle is

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$= (t^3+1)\hat{i} + t^2\hat{j} + (2t+5)\hat{k}$$

Velocity, $\vec{v} = \frac{d\vec{r}}{dt} = 3t^2\hat{i} + 2t\hat{j} + 2\hat{k}$

when $t = 1$, we have $\frac{d\vec{r}}{dt} = 3\hat{i} + 2\hat{j} + 2\hat{k}$

~~unit ve~~

unit vector along $2\hat{i}+3\hat{j}+6\hat{k} = \dfrac{2\hat{i}+3\hat{j}+6\hat{k}}{\sqrt{4+9+36}}$

$= \dfrac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k})$

$\therefore$ Component of velocity $\vec{v}$ ~~along~~ in the direction

$2\hat{i}+3\hat{j}+6\hat{k} = (3\hat{i}+2\hat{j}+2\hat{k})\cdot\dfrac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k})$

$= \dfrac{24}{7}$ Ans.

Now, acceleration, $\vec{a} = \dfrac{d\vec{v}}{dt} = 6t\,\hat{i} + 2\hat{j}$

at $t=1$, $\vec{a} = 6\hat{i}+2\hat{j}$

$\therefore$ Component of $\vec{a}$ along in the direction

$2\hat{i}+3\hat{j}+6\hat{k}$ is $= (6\hat{i}+2\hat{j})\cdot\dfrac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k})$

$= \dfrac{18}{7}$ Ans.)

H.W

A particle moves along the curve $x = 2t^2$, $y = t^2 - 4t$ and $z = 3t - 5$ where t is the time. Find the components of its velocity and acceleration at $t=1$ in the direction $\hat{i} - 3\hat{j} + 2\hat{k}$.

Ans: $\frac{8\sqrt{14}}{7}$, $-\frac{\sqrt{14}}{7}$

H.W

The position vector of a particle at t is $\vec{r} = \cos(t-1)\,\hat{i} + \sinh(t-1)\,\hat{j} + \alpha t^3 \hat{k}$. Find the condition imposed on α by requiring that at $t=1$, the acceleration is normal to the position.

→ hyperbolic (always +)

$$\boxed{\begin{aligned} \frac{d}{d\theta}(\sinh\theta) &= \cosh\theta \\ \frac{d}{d\theta}(\cosh\theta) &= \sinh\theta \end{aligned}}$$

Ans: $\alpha = \pm \frac{1}{\sqrt{6}}$

Soln.

$\vec{r} = \cos(t-1)\hat{i} + \sinh(t-1)\hat{j} + \alpha t^3 \hat{k}$

$\vec{v} = \frac{d\vec{r}}{dt} = -\sin(t-1)\hat{i} + \cosh(t-1)\hat{j} + 3\alpha t^2 \hat{k}$

$\vec{a} = \frac{d\vec{v}}{dt} = -\cos(t-1)\hat{i} + \sinh(t-1)\hat{j} + 6\alpha t \hat{k}$

at $t=1$,

$\vec{a} = -\cos 0\,\hat{i} + \sin 0\,\hat{j} + 6\alpha\hat{k}$

$= -\hat{i} + 6\alpha\hat{k}$

at $t=1$,

$\vec{r} = \cos 0\,\hat{i} + \sinh 0\,\hat{k} + \alpha\hat{k}$
$= \hat{i} + \alpha\hat{k}$

Since the acceleration is normal to the position,

$\vec{a}\cdot\vec{r} = 0$

$\Rightarrow (-\hat{i} + 6\alpha\hat{k}).(\hat{i} + \alpha\hat{k}) = 0$

$\Rightarrow -1 + 6\alpha^2 = 0$

$\Rightarrow \alpha^2 = \frac{1}{6}$

$\therefore \alpha = \pm\frac{1}{\sqrt{6}}$

[Ans.]

5

Date: 12/7/2016

8(E)-Day

Linear Algebra

Vector Space

$x, y \in \mathbb{R}$

$x+y \in \mathbb{R}$

$x \cdot y \in \mathbb{R}$

$0+x = x+0 = x$ [Additive identity]

$1 \cdot x = x \cdot 1 = x$ [Multiplicative identity]

$x \in \mathbb{R}$, $-x \in \mathbb{R}$

$x+(-x) = 0$ [~~Negative~~ Additive inverse]

[Additive identity]

$x \in \mathbb{R}$, $x^{-1} \in \mathbb{R}$ [Multiplicative inverse]

$x \cdot x^{-1} = 1$ [Multiplicative identity]

$x, y \in \mathbb{R}$

$x+y = y+x$ → Additive

Commutative Law

$xy = yx$ → Multiplicative

$x, y, z \in \mathbb{R}$

$x(y+z) = xy + xz$ → distributive law

$(x+y)+z = x+(y+z)$ → Associative Law

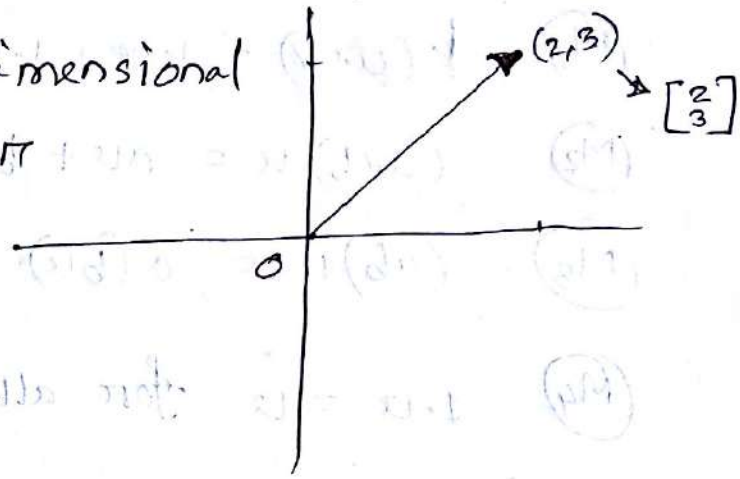
$(xy)z = x(yz)$

$\boxed{\{R, +, \cdot\}}$ → Real space

$\{M_{m,n}, +, \cdot, \text{Scalar Multiplication}\} \rightarrow$ Space

$3A \rightarrow$ Scalar mutiplication

Any point of a two dimensional coordinate is a vector



$\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^K \rightarrow$ Space

$\mathbb{R}^2 \downarrow$ 2D, $\mathbb{R}^3 \downarrow$ 3D, $\mathbb{R}^K \downarrow$ KD

Matrix $\rightarrow$ a vector / collection of vectors

<u>Vector Space:</u>

Let V be any non empty set and K be any scalar field. ($K = \mathbb{R}$) Now we define two operations on V as:

i) Vector addition: For any $u, v \in V$, $u+v \in V$

ii) Scalar multiplication: For any $u \in V$ and $k \in K$, $ku \in V$

Then V is called a vector space over K if the following axioms hold for any $u, v, w \in V$:

(A1) - $(u+v)+w = u+(v+w)$

(A2) there exists an $0 \in V$ such that $0+u = u+0 = u$ for all $u \in V$

(A_3) For each $u \in V$, there exists a $-u \in V$ such that $u + (-u) = 0$

(A_4) $u + v = v + u$ for all $u, v \in V$

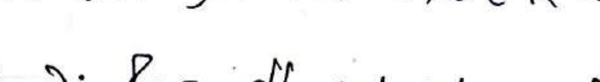
(M_1) $k(u+v) = ku + kv$ for all $k \in K$ and $u, v \in V$.

(M_2) $(a+b)u = au + bu$ for all $a, b \in K$ and $u \in V$

(M_3) $(ab)u = a(bu)$ for all $a, b \in K$, $u \in V$

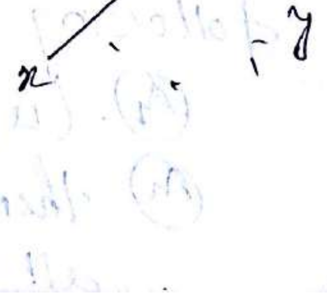
(M_4) $1 \cdot u = u$ for all $u \in V$ and the unit scalar $1 \in K$

R



$R^2 = R \times R = \{(a,b) \mid a, b \in R\}$

$R^3 = R \times R \times R = \{(a,b,c) \mid a,b,c \in R\}$



Date: 18/7/2016

9 (D) - Day

Directional ~~Differential~~ derivative:

The component of $\nabla \phi$ in the direction of a vector $\vec{d}$ is equal to $\nabla\phi \cdot \vec{d}$ and is called the directional derivative of ϕ in the direction of $\vec{d}$.

* Find a unit vector normal to the surface $x^2+3y^2+2z^2=6$ at $P(2,0,1)$

Soln:

Let, $\phi = x^2+3y^2+2z^2-6$

$\therefore$ $\nabla\phi$ is normal vector.

$$\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left(x^2+3y^2+2z^2-6\right)=2x\hat{i}+6y\hat{j}+4z\hat{k}$$

Normal vector at $(2,0,1)$ is $4\hat{i}+4\hat{k}$.

$\therefore$ Unit normal vector is $\frac{4\hat{i}+4\hat{k}}{\sqrt{16+16}}=\frac{1}{\sqrt{2}}(\hat{i}+\hat{k})$

(Ans).

* The temperature at any point in space is given by $T=xy+yz+zx$.
Determine the derivative of T in the direction of $3\hat{i}-4\hat{k}$ at $(1,1,1)$

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Sol^n:

Given $T = xy + yz + zx$

$$\nabla T = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xy + yz + zx)$$

$$= \hat{i}(y+z) + \hat{j}(x+z) + \hat{k}(y+x)$$

Directional derivative at $(1,1,1) = 2\hat{i} + 2\hat{j} + 2\hat{k}$

Directional derivative at $(1,1,1)$ in the direction

$$(3\hat{i} - 4\hat{k}) = (2\hat{i} + 2\hat{j} + 2\hat{k}) \cdot \frac{(3\hat{i} - 4\hat{k})}{\sqrt{9+16}}$$

$$= \frac{1}{5}(6-8)$$

$$= -\frac{2}{5} \text{ (Ans.)}$$

Find the directional derivative of $\operatorname{div}(\vec{u})$ at $(1,2,2)$ in the direction of outer normal of $x^2+y^2+z^2=9$ for $\vec{u} = x^4\hat{i} + y^4\hat{j} + z^4\hat{k}$.

Sol^n:

Given, $\vec{u} = x^4\hat{i} + y^4\hat{j} + z^4\hat{k}$

$$\operatorname{div}(\vec{u}) = \nabla \cdot \vec{u} = ($$

$$= 4x^3 + 4y^3 + 4z^3$$

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Outer normal $= \nabla(x^2+y^2+z^2-9)$

$= ($

$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

Outer normal at $(1,2,2) = 2\hat{i} + 4\hat{j} + 4\hat{k}$

Directional derivative $= \nabla(4x^3+4y^3+4z^3)$

$=$

$= 12x^2\hat{i} + 12y^2\hat{j} + 12z^2\hat{k}$

Directional derivative at $(1,2,2) = 12\hat{i} + 48\hat{j} + 48\hat{k}$

Directional derivative in the direction of outer normal (of ~~$x^2+y^2+z^2=9$ the sphere~~ the sphere

$= (12\hat{i} + 48\hat{j} + 48\hat{k}) \cdot \left(\frac{2\hat{i}+4\hat{j}+4\hat{k}}{\sqrt{4+16+16}}\right)$

$= 68$ (Ans)

1

H.W

Find the direction derivative of the scalar function $f(x,y,z) = x^2+xy+z^2$ at $A(1,-1,-1)$ in the direction of the line $\vec{AB}$ where $B(3,2,1)$

Note: $\vec{A} = \hat{i} - \hat{j} - \hat{k}$ $\quad \vec{AB} = \vec{B} - \vec{A}$ $\quad$ Ans: $\frac{1}{\sqrt{17}}$

$\vec{B} = 3\hat{i} + 2\hat{j} + \hat{k}$ $\quad =$

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(1) Soln

$\vec{A} = \hat{i} - \hat{j} - \hat{k}$

$\vec{B} = 3\hat{i} + 2\hat{j} + \hat{k}$

$\therefore \vec{AB} = \vec{B} - \vec{A}$

$= 3\hat{i} + 2\hat{j} + \hat{k} - \hat{i} + \hat{j} + \hat{k}$

$= 2\hat{i} + 3\hat{j} + 2\hat{k}$

Now,

Let, $\phi = x^2 + xy + z^2$

$\therefore \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + xy + z^2)$

$= (2x + y)\hat{i} + x\hat{j} + 2z\hat{k}$

at $(-1, -1, -1)$

$\nabla\phi = \hat{i} + \hat{j} - 2\hat{k}$

Directional derivative in the direction of the line $\vec{AB} = (\hat{i} + \hat{j} - 2\hat{k}) \cdot \frac{2\hat{i} + 3\hat{j} + 2\hat{k}}{\sqrt{4+9+4}}$

$= \frac{1}{\sqrt{17}}$

[Ans]

(2) Divergence of $f(x, y, z)$ is

$\nabla \cdot f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (xy\,\hat{i} + xy^2\hat{j} + z^2\hat{k})$

$= \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(xy^2) + \frac{\partial}{\partial z}(z^2)$

$= y + 2xy + 2z$

Directional derivative.

$= \nabla(y + 2xy + 2z)$

$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(y + 2xy + 2z)$

$= 2y\hat{i} + (1 + 2x)\hat{j} + 2\hat{k}$

Directional derivative at $(2, 1, 2) = 2\hat{i} + 5\hat{j} + 2\hat{k}$

Outer normal $= \nabla(x^2 + y^2 + z^2 - 9)$

$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

Outer normal at $(2, 1, 2) = 4\hat{i} + 2\hat{j} + 4\hat{k}$

Directional derivative in the direction of the sphere) $= (2\hat{i} + 5\hat{j} + 2\hat{k}) \cdot \frac{4\hat{i} + 2\hat{j} + 4\hat{k}}{\sqrt{16+4+16}}$

$= \frac{8 + 10 + 8}{6}$

$= \frac{26}{6} = \frac{13}{3}$ (Ans).

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(2)

Find the directional derivative of the divergence of $f(x,y,z) = xy\,\hat{i} + xy^2\,\hat{j} + z^2\,\hat{k}$ at $(2,1,2)$ in the direction of the outer normal to the sphere $x^2+y^2+z^2=9$.

Ans: $\frac{13}{3}$

Show that the gradient field describing a motion is irrotational.

<u>Note</u> vector এর curl 0 হলে vector টি irrotational

<u>Sol^n</u>: Let, a field be $f(x,y,z)$

gradient of $f(x,y,z) = \nabla f$

$$= \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

Curl of $\nabla f = \nabla \times \nabla f$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial^2 f}{\partial y\,\partial z} - \frac{\partial^2 f}{\partial z\,\partial y}\right) - j\left(\quad \right) + k\left(\quad \right.$$

$= 0$

Since curl of gradient field is zero. So its motion is irrotational. (Proved)

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5

Date: 19/7/2016

9(E)-Day

Ex-1

Prove that $\mathbb{R}^2$ and $\mathbb{R}^3$ are vector spaces over $\mathbb{R}$.

$$\mathbb{R}^2 = \{(a,b) \mid a,b \in \mathbb{R}\}$$

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$$

$$K(a,b) = (Ka, Kb)$$

A1 $(u+v)+w = u+(v+w)$

$$((a_1, b_1) + (a_2, b_2)) + (a_3, b_3)$$

$$= (a_1+a_2, b_1+b_2) + (a_3, b_3)$$

$$= (a_1+a_2+a_3, b_1+b_2+b_3)$$

Now, $(a_1, b_1) + ((a_2, b_2) + (a_3, b_3))$

A2 $0 \in V$, $0+u = u$

$$(0,0) + (a,b) = (0+a, 0+b)$$

$$= (a,b)$$

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A_3 $u \in V$, $-u \in V$

$(a,b) \in \mathbb{R}^2$, $(-a,-b) \in \mathbb{R}^2$

A_4 $u+v = v+u$

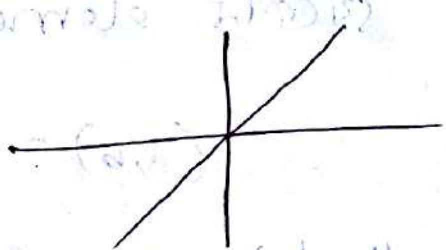
M_1 $k(u+v) = ku+kv$

$u = (a,b)$ $v = (c,d)$

$\downarrow$

$X = \{(a,b,c) \mid a=b=c\}$

$Y = \{(a,b) \mid a=b\}$, $y=x$



A_1 ✓ M_1 ✓

A_2 ✓ M_2 ✓

A_3 ✓ M_3 ✓

A_4 ✓ M_4 ✓

X and Y are vector spaces.

$M_{2,2}$ = set of all 2×2 matrices over $\mathbb{R}$,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} p & q \\ r & s \end{bmatrix} \in M_{2,2}$$

It's also a vector space.

—x—

$(1,0), (0,1) \in \mathbb{R}^2$

$(-3,5) \in \mathbb{R}^2$

$(-3,5) = \boxed{-3}\,(1,0) + \boxed{5}\,(0,1)$

যেকোন element generate করার সম্ভব.

$(a,b) = a(1,0) + b(0,1)$

Linear Combination of vectors:

Let V be a vector space over K. Then a vector $v \in V$ is said to be the linear combination of vectors $u_1, u_2, \dots \dots, u_n \in V$ if there ~~is~~ exists some scalars $k_1, k_2, \dots, k_n \in K$ such that

$$v = k_1u_1 + k_2u_2 + \dots \dots + k_nu_n$$

Note $v = (5, 9)$, $u_1 = (1, 0)$, $u_2(0, 1)$

$k_1 = 5$, $k_2 = 9$

$$v = k_1 u_1 + k_2 u_2$$

$$(5, 9) = 5(1, 0) + 9(0, 1)$$

$(5, 9)$ is a linear combination of $(1, 0)$ and $(0, 1)$

Ex-1: Express $v = (3, 7, 4) \in \mathbb{R}^3$ as a linear combination of the vectors $u_1 = (1, 2, 3)$, $u_2 = (2, 3, 7)$, $u_3 = (3, 5, 6)$

Sol^n

Let, $(3, 7, 4) = x(1, 2, 3) + y(2, 3, 7) + z(3, 5, 6)$

$$= (x + 2x + 3x) + (2y + 3y + 7y) + (3z + 5z + 6z)$$

$$\Rightarrow (3, 7, 4) = (x + 2y + 3z,\ 2x + 3y + 5z,\ 3x + 7y + 6z)$$

Thus.

$$x + 2y + 3z = 3$$
$$2x + 3y + 5z = 7$$
$$3x + 7y + 6z = 4$$

No solution $\longrightarrow$ not possible

Unique " $\longrightarrow$ once

Many " $\longrightarrow$ Many times

Augmented matrix of the given system is

$$C = \begin{bmatrix} 1 & 2 & 3 & 3 \\ 2 & 3 & 5 & 7 \\ 3 & 7 & 6 & 4 \end{bmatrix} \underset{R_{31}(-3)}{\overset{R_{21}(-2)}{\sim}} \begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & -3 & -5 \end{bmatrix}$$

$$\overset{R_{23}}{\sim} \begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & 1 & -3 & -5 \\ 0 & -1 & -1 & 1 \end{bmatrix} \overset{R_{32}(1)}{\sim} \begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & -4 & -4 \end{bmatrix}$$

$\rho(C) = \rho(A) = 3 = n$

So, it has unique solution

$$x + 2y + 3z = 3$$
$$y - 3z = -5$$
$$-4z = -4$$
$$\therefore z = 1$$
$$y = -2$$
$$x = 1$$

10(c) – Day

Date: 24/7/2016

H

Prove that the vector $\vec{A} = 3y^4z^2\hat{i} + 4x^3z^2\hat{j} - 3x^2y^2\hat{k}$ is solenoidal.

⇒ The vector will be solenoidal if its divergent is zero i.e. $\nabla\cdot\vec{A} = 0$.

$$\therefore \nabla\cdot\vec{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(3y^4z^2\hat{i} + 4x^3z^2\hat{j} - 3x^2y^2\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial z}(-3x^2y^2)$$

$$= 0$$

Hence, $\vec{A}$ is solenoidal.

Show that $\vec{A} = (2x^2 + 8xy^2z)\hat{i} + (3x^3y - 3xy)\hat{j} - (4y^2z^2 + 2x^3z)\hat{k}$ is not solenoidal but $\vec{B} = xyz^2\vec{A}$ is solenoidal.

H.W

Soln

$$\nabla\cdot\vec{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left((2x^2+8xy^2z)\hat{i} + (3x^3y - 3xy)\hat{j} - (4y^2z^2+2x^3z)\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(2x^2+8xy^2z) + \frac{\partial}{\partial y}(3x^3y-3xy) - \frac{\partial}{\partial z}(4y^2z^2+2x^3z)$$

$$= 4x + 8y^2z + 3x^3 - 3x - 8y^2z - 2x^3$$

$$= x + x^3$$

$$\neq 0$$

So, $\vec{A}$ is not solenoidal.

$$\vec{B} = (2x^3yz^2 + 8x^2y^3z^3)\hat{i} + (3x^4y^2z^2 - 3x^2y^2z^2)\hat{j} - (4xy^3z^4 + 2x^4yz^3)\hat{k}$$

$$\nabla\cdot\vec{B} = 6x^2yz^2 + 16xy^3z^3 + 6x^4yz^2 - 6x^2yz^2 - 16xy^3z^3 - 6x^4yz^2$$

$$= 0$$

So, $\vec{B}$ is solenoidal.

Show that $\vec{E} = \frac{\vec{r}}{r^2}$ is irrotational.

$\Rightarrow$ $\vec{E}$ ~~is~~ will be irrotational if $\nabla \times \vec{E} = 0$

Let, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\therefore \quad r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$

$\therefore \vec{E} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{x^2+y^2+z^2}$

$$\therefore \nabla \times \vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{x^2+y^2+z^2} & \frac{y}{x^2+y^2+z^2} & \frac{z}{x^2+y^2+z^2} \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}\left(\frac{z}{x^2+y^2+z^2}\right) - \frac{\partial}{\partial z}\left(\frac{y}{x^2+y^2+z^2}\right)\right\} + \hat{j}\left\{ \quad \right\} + \hat{k}\left\{ \quad \right\}$$

$$= \hat{i}\left\{\frac{-2yz}{(x^2+y^2+z^2)^2} + \frac{2yz}{(x^2+y^2+z^2)^2}\right\} + \hat{j}\{\cdots\} + \hat{k}\{\cdots\}$$

$= 0$

Hence $\vec{E}$ is irrotational.

Show that the vector field $\vec{v} = \frac{-x\hat{i}-y\hat{j}}{\sqrt{x^2+y^2}}$ is a sink field.

$\Rightarrow$ $\vec{v}$ will be sink field if $\nabla \times \vec{v} = 0$

Now, $\vec{r} = x\hat{i} + y\hat{j}$

$$\therefore \vec{v} = \frac{-x\hat{i}-y\hat{j}}{\sqrt{x^2+y^2}} = -\frac{\vec{r}}{r}$$

$$\therefore \nabla \times v = \nabla \times \frac{-\vec{r}}{r}$$

$$= -(\nabla r^{-1} \times \vec{r} + r^{-1} \nabla \times \vec{r})$$

$$= -r^{-2} \frac{\vec{r} \times \vec{r}}{r} + r^{-1} \cdot 0$$

$$= 0$$

Hence, $\vec{v}$ is a sink field.

Vector Integration:

Surface integration:

$$\iint_S \vec{A}\cdot\hat{n}\, ds = \iint_R \vec{A}\cdot\hat{n}\, \frac{dx\, dy}{|\hat{n}\cdot\hat{k}|}$$

$$= \iint_R \vec{A}\cdot\hat{n}\, \frac{dx\, dz}{|\hat{n}\cdot\hat{j}|} = \iint_R \vec{A}\cdot\hat{n}\, \frac{dy\, dz}{|\hat{n}\cdot\hat{i}|}$$

Divergence theorem:

volume integration

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\vec{F}\, dv$$

surface integration

\# Evaluate :- $\iint_S \vec{A}\cdot\hat{n}\, ds$ where $\vec{A} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$ and s is that part of the plane $2x + 3y + 6z = 12$ which is located in first octant.

Ans: 24

Date: 25/7/2016

10(D)-Day

H

Soln:

Normal to the surface $2x+3y+6z=12$ is

$$\nabla(2x+3y+6z-12) = 2\hat{i}+3\hat{j}+6\hat{k}$$

$\therefore$ Unit normal, $\hat{n} = \frac{2\hat{i}+3\hat{j}+6\hat{k}}{7}$

$$\therefore \vec{A}\cdot\hat{n} = (18z\,\hat{i}-12\hat{j}+3y\,\hat{k})\cdot\left(\frac{2\hat{i}+3\hat{j}+6\hat{k}}{7}\right)$$

$$= \frac{36z-36+18y}{7}$$

$$\iint_S \vec{A}\cdot\hat{n}\,ds = \iint_R \vec{A}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$= \iint_R \frac{36z-36+18y}{7}\,\frac{dx\,dy}{6/7}$$

$$= \frac{1}{6}\iint_{R'}(72-12x-18y-36+18y)\,dx\,dy$$

$$= \int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}}(6-2x)\,dy\,dx$$

$$= \int_{x=0}^{6}\left[(6-2x)y\right]_0^{\frac{12-2x}{3}}dx$$

$$= \frac{4}{3}\int_{x=0}^{6}(18-9x+x^2)\,dx = 24 \quad \text{Ans.}$$

Evaluate $\iint_S \vec{A}\cdot\hat{n}\,ds$ where $\vec{A} = z\hat{i}+x\hat{j}-3y^2z\hat{k}$ and s is the surface of the cylinder $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$.

H.W As prev

$\boxed{\text{Ans: } 90 \quad \hat{n} = \frac{x\hat{i}+y\hat{j}}{4}}$

Soln. We know,

$$\iint_S \vec{A}\cdot\hat{n}\,ds = \iint_R \vec{A}\cdot\hat{n}\,\frac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$$

Let, $\phi = x^2+y^2-16$

Normal to the surface $= \nabla\phi$

$$= \nabla(x^2+y^2-16)$$

$$= 2x\hat{i}+2y\hat{j}$$

Unit normal vector, $\hat{n} = \dfrac{2(x\hat{i}+y\hat{j})}{\sqrt{4(x^2+y^2)}}$

$$= \frac{x\hat{i}+y\hat{j}}{\sqrt{16}}$$

$$= \frac{x\hat{i}+y\hat{j}}{4}$$

$$\vec{A}\cdot\hat{n} = (z\hat{i}+x\hat{j}-3y^2z\hat{k})\cdot\frac{x\hat{i}+y\hat{j}}{4}$$

$$= \frac{1}{4}(xz+xy) = \frac{1}{4}x(y+z)$$

$$|\hat{n}\cdot\hat{i}| = \frac{x}{4}$$

$$\iint_S \vec{A}\cdot\hat{n}\,ds = \iint_R \vec{A}\cdot\hat{n}\,\frac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$$

$$= \iint_R \frac{1}{4}x(y+z)\left(\frac{4}{x}\right)dy\,dz$$

$$= \iint_R (y+z)\,dy\,dz$$

$$= \int_0^5\int_0^4 (y+z)\,dy\,dz$$

$$= \int_0^5 \left[\frac{y^2}{2} + zy\right]_0^4 dy$$

$$= \int_0^5 (8+4z)\,dz \qquad \cancel{8+4z}$$

$$= \left[8z + \frac{4z^2}{2}\right]_0^5$$

$$= 90 \quad \text{(Ans.)}$$

Evaluate $\iint_S \phi\,\hat{n}\,ds$ where $\phi = \frac{3}{8}xyz$ and s is the surface of the cylinder $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$

Soln. We know, $\iint_S \phi\,\hat{n}\,ds = \iint_R \vec{A}\cdot\hat{n}\,\frac{dx\,dz}{|\hat{n}\cdot\hat{j}|}$

Normal to the surface of $x^2+y^2=16$ is

$$\nabla(x^2+y^2-16) = 2x\,\hat{i} + 2y\,\hat{j}$$

Unit normal, $\hat{n} = \dfrac{2x\,\hat{i}+2y\,\hat{j}}{\sqrt{4x^2+4y^2}}$

$$= \frac{x\hat{i}+y\hat{j}}{\sqrt{x^2+y^2}}$$

$$= \frac{x\hat{i}+y\hat{j}}{4}$$

$$\phi\,\hat{n} = \frac{3}{8}xyz\left(\frac{x\hat{i}+y\hat{j}}{4}\right)$$

$$= \frac{3}{8}y\left(\frac{x^2z\,\hat{i}+xyz\,\hat{j}}{4}\right)$$

$$\therefore \iint_S \phi \hat{n}\, ds = \iint_R \frac{3}{8} y \frac{(x^2 z \hat{i} + xyz \hat{j})}{4} \frac{dx\, dz}{y/4}$$

$$= \frac{3}{8} \int_{x=0}^{4} \int_{z=0}^{5} \left(x^2 z \hat{i} + xz \sqrt{16 - x^2}\, \hat{j} \right) dz\, dx$$

$$= \frac{3}{8} \int_{x=0}^{4} \left[\frac{x^2 z^2}{2} \hat{i} + \frac{x z^2}{2} \sqrt{16 - x^2}\, \hat{j} \right]_0^5 dx$$

$$= \frac{3}{8} \frac{25}{2} \hat{i} \int_0^4 x^2 dx + \frac{3}{8} \frac{25}{2} \hat{j} \int_0^4 x \sqrt{16 - x^2}\, dx$$

$$\downarrow$$

$$= 100 \hat{i} + 100 \hat{j} \text{ (Ans).}$$

\# If $\vec{F} = y\hat{i} + (x - 2xz)\hat{j} - xy\hat{k}$, evaluate $\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds$, where s is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy plane.

Soln. We know $\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \iint_R (\nabla \times \vec{F}) \hat{n} \frac{dx dy}{|\hat{n} \cdot \hat{k}|}$

Normal ....

$$\hat{n} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{a}$$

Now, $\nabla \times \vec{F} = \ldots$

$$= x\hat{i} + y\hat{j} - 2z\hat{k}$$

$$\therefore \iint_{S} (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \iint_{R} (x\hat{i} + y\hat{j} - 2z\hat{k}) \cdot \left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{a}\right) \frac{dx\,dy}{z/a}$$

$$= \iint_{R} \frac{x^2 + y^2 - 2z^2}{z}\, dx\,dy$$

$$= \iint_{R} \frac{x^2 + y^2 - 2a^2 + 2x^2 + 2y^2}{\sqrt{a^2 - x^2 - y^2}}\, dx\,dy$$

$$= \iint_{R} \frac{3x^2 + 3y^2 - 2a^2}{\sqrt{a^2 - x^2 - y^2}}\, dx\,dy$$

$$= \int_{x=0}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{3(x^2 + y^2) - 2a^2}{\sqrt{a^2 - x^2 - y^2}}\, dx\,dy$$

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Let, $x = r\cos\theta$, and $y = r\sin\theta$

$\therefore r = \sqrt{x^2+y^2}$

and $dx\,dy = |K|\,dr\,d\theta$

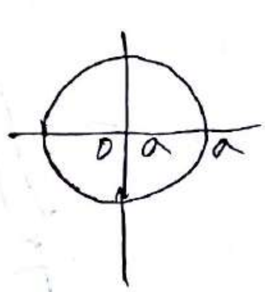
here $|K| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

$= r$

$\therefore dx\,dy = r\,dr\,d\theta$

$\downarrow$

Next class



10(E)-Day

Date: 26/7/2016

Spanning Set:

Let V be a vector space over K. A set $\{u_1, u_2, \ldots, u_n\}$ of vectors in V is said to form a spanning set of V if every $v \in V$ can be expressed as a linear combination of these vectors, that is there exist scalars $k_1, k_2, \ldots \ldots, k_n$ such that

$$v = k_1 u_1 + k_2 u_2 + \cdots \cdots + k_n u_n$$

$(3,9,2) \in \mathbb{R}^3$, $\{\overset{u_1}{(1,0,0)}, \overset{u_2}{(0,1,0)}, \overset{u_3}{(0,0,1)}\}$ → spanning set

$(3,9,2) = \{3(1,0,0), 9(0,1,0), 2(0,0,1)\}$

Ex-1

Test whether the following vectors in $\mathbb{R}^3$ form a spanning set or not.

① $\{(1,1,1), (1,1,0), (1,0,0)\}$

② $\{(1,2,3), (1,3,5), (1,5,9)\}$

② Let, $(a,b,c) \in \mathbb{R}^3$

Also let, $(a,b,c) = x(1,2,3) + y(1,3,5) + z(1,5,9)$

$$\Rightarrow (a,b,c) = (x+y+z) + (2x+3y+5z) + (3x+5y+9z)$$

Which implies that

$$x+y+z=a$$
$$2x+3y+5z=b$$
$$3x+5y+9z=c$$

The augmented matrix for the above system is

$$C = \begin{bmatrix} 1 & 1 & 1 & a \\ 2 & 3 & 5 & b \\ 3 & 5 & 9 & c \end{bmatrix} \underset{R_{31}(-3)}{\overset{R_{21}(-2)}{\sim}} \begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 3 & b-2a \\ 0 & 2 & 6 & c-3a \end{bmatrix}$$

$$\underset{\sim}{R_{31}(-2)} \begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 3 & b-2a \\ 0 & 0 & 0 & a-2b+c \end{bmatrix}$$

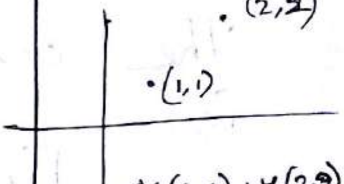
Since, $\rho(C) \neq \rho(A)$

Therefore, the above system has no solution. Hence, the given set doesn't form a spanning set.

Linear Dependence and Independence:

Let, V be a vector space over K. ~~A set of~~ The vectors $u_1, u_2, \ldots\ldots, u_n$ in V are linearly dependent if there exists scalars $k_1, k_2, \ldots\ldots, k_n$ not all of them are equal to zero such that

$$K_1 u_1 + K_2 u_2 + \ldots\ldots + K_n u_n = 0$$



$x(1,1)+y(2,2) = (0,0)$

x and y are both not equal to zero

$\boxed{\text{Ex-9}}$

~~The~~ Test whether or not the following vectors are linearly dependent:

1. $\{(1,1,2), (2,3,1), (4,5,5)\}$
2. $\{(1,2,5), (2,5,1), (1,5,2)\}$

(1) Let, $x(1,1,2) + y(2,3,1) + z(4,5,5) = (0,0,0)$

$$x + 2y + 4z = 0$$
$$x + 3y + 5z = 0$$
$$2x + y + 5z = 0$$

Coefficient matrix of the given system is

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 2 & 1 & 5 \end{bmatrix} \underset{R_{31}(-2)}{\overset{R_{21}(-1)}{\sim}} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & -3 & -3 \end{bmatrix} \overset{R_{32}(3)}{\sim} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since $\rho(A) < n$.

Hence, the vectors are linearly dependent.

(2)

Let, $x(1,2,5) + y(2,5,1) + z(1,5,2) = (0,0,0)$
which implies that

$$x + 2y + z = 0$$
$$2x + 5y + 5z = 0$$
$$5x + y + 2z = 0$$

The coefficient matrix of the above system is

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 5 \\ 5 & 1 & 2 \end{bmatrix} \underset{R_{31}(-5)}{\xrightarrow{R_{21}(-2)}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -9 & -3 \end{bmatrix}$$

$$\xrightarrow{R_{32}(9)} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 24 \end{bmatrix}$$

Since, $\rho(A) = n$, therefore, the system has unique solution. ~~and~~
Hence, the vectors are linearly independent.

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11(D)-Day

Date : 01/8/2016

Prev class

$$\therefore \iint_S (\nabla\times\vec{F})\cdot\hat{n}\, ds = \int_{r=0}^{a}\int_{\theta=0}^{2\pi} \frac{3(r^2\cos^2\theta + r^2\sin^2\theta)-2a^2}{\sqrt{a^2-r^2\cos^2\theta - r^2\sin^2\theta}}\, r\, dr\, d\theta$$

$$= \int_{r=0}^{a}\int_{\theta=0}^{2\pi} \frac{3r^2-2a^2}{\sqrt{a^2-r^2}}\, r\, dr\, d\theta$$

$$= \int_{r=0}^{a}\left[\frac{3r^2-2a^2}{\sqrt{a^2-r^2}}\,\theta\right]_0^{2\pi} r\, dr$$

$$= 2\pi\int_0^a \frac{3r^2}{\sqrt{a^2-r^2}}\, r\, dr - 2\pi\int_0^a \frac{2a^2}{\sqrt{a^2-r^2}}\, r\, dr$$

Let,
$t = a^2 - r^2$
$dt = -2r\,dr$
when,
$r\to 0$ then $t\to a^2$
$r\to a$ then $t\to 0$

$$= 2\pi\int_{a^2}^{0} \frac{3(a^2-t)}{\sqrt{t}}\left(-\frac{1}{2}\right)dt + 2\pi\cdot\frac{1}{2}\int_{a^2}^{0}\frac{2a^2}{\sqrt{t}}\, dt$$

$$= -3\pi\left[a^2\frac{t^{\frac{1}{2}}}{\frac{1}{2}} - \frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_{a^2}^{0} + \pi\cdot 2a^2\cdot\left[\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right]_{a^2}^{0}$$

$= 0$ [Ans.]

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If $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$, evaluate $\iint \vec{F}\cdot\hat{n}\, ds$

H.W

where s is the surface of the cube s bounded by $x=0,\ x=1,\ y=0,\ y=1,\ z=0,\ z=1$.

hints: $\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint \nabla\cdot\vec{F}\, dv \longrightarrow$

Ans. $\frac{3}{2}$

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Evaluate $\iint_S \vec{A}\cdot\hat{n}\, ds$ over the surface S of the region bounded by the cylinder $x^2+z^2=9$. $x=y=z=0$ and $y=8$ if $\vec{A} = 6z\,\hat{i} + (2x+y)\hat{j} - x\,\hat{k}$

Solⁿ

By divergence theorem

$$\iint_S \vec{A}\cdot\hat{n}\, ds = \iiint \nabla\cdot\vec{A}\, dv$$

here $\nabla\cdot\vec{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(6z\,\hat{i} + (2x+y)\hat{j} - x\,\hat{k}\right)$

$= 1$

$$\iint_S \vec{A}\cdot\hat{n}\, ds = \iiint \nabla\cdot\vec{A}\, dv$$

$1 + \cos 2\theta =$
$\cos 2\theta = 1 - 2\cos^2\theta$
$1 + \cos 2\theta = 2\cos^2\theta$

$$= \int_{x=0}^{3}\int_{y=0}^{8}\int_{z=0}^{\sqrt{9-x^2}} 1\, dx\, dy\, dz$$

$$= \int_{x=0}^{3}\int_{y=0}^{8} \sqrt{9-x^2}\, dx\, dy$$

$$= 8\int_{x=0}^{3} \sqrt{9-x^2}\, dx$$

$\boxed{\text{Ans: } 18\pi}$

$$= 8\int_{\theta=0}^{\frac{\pi}{2}} \sqrt{9(1-\sin^2\theta)}\; 3\cos\theta\, d\theta$$

Let,
$x = 3\sin\theta$
$dx = 3\cos\theta\, d\theta$
when $x \to 0,\ \theta \to 0$
$x = 3,\ \theta \to \frac{\pi}{2}$

$$= 24\int_{\theta=0}^{\frac{\pi}{2}} 3\cos^2\theta\, d\theta$$

$$= \frac{24\times 3}{2}\int_{\theta=0}^{\frac{\pi}{2}} (\cos 2\theta + 1)\, d\theta$$

$$= 36 \times \left[\frac{1}{2}\sin 2\theta + \theta\right]_0^{\frac{\pi}{2}} = 18\pi$$

(Ans.

Find the surface area of the plane $x+2y+2z=12$ cut-off by a) $x=0,\ y=0,\ x=1,\ y=1$

H.W b) $x=y=0$ and $x^2+y^2=16$

Ans: 6π

Soln

(a) We have the surface area of the plane is

$$\iint_S ds = \iint_R \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

normal to the surface $= \nabla(x+2y+2z-12)$

$= \hat{i}+2\hat{j}+2\hat{k}$

Unit normal $\hat{n} = \dfrac{\hat{i}+2\hat{j}+2\hat{k}}{3}$

$$|\hat{n}\cdot\hat{k}| = \frac{2}{3}$$

$$\iint_S ds = \int_{x=0}^{1}\int_{y=0}^{1} \frac{dx\,dy}{\frac{2}{3}} = \frac{3}{2}\int_0^1 dx$$

$$= \frac{3}{2} \quad \text{(Ans.)}$$

Evaluate $\iiint (2x+y)\,dv$ where v is the closed region bounded by the cylinder $z=4-x^2$ and the planes $x=0$, $y=0$, $y=2$ and $z=0$

Ans: $\frac{80}{3}$

<u>Soln</u>

$$\iiint_V (2x+y)\,dv = \int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{4-x^2} (2x+y)\,dx\,dy\,dz$$

11(E)-Day Date: 2/8/2016

Basis and Dimension of a vector space:

Basis:

Let V be a vector space over K. A set $S = \{u_1, u_2, u_3, \dots \dots, u_n\}$ of vectors in V is a basis if S has the following properties:

(i) S is linearly independent

(ii) S ~~pp~~ spans V

Dimension:

A vector space V is said to be n-dimensional and denoted by $\dim V = n$. if V has a basis of exactly n elements.

Ex-1

Test whether the following set form a basis of R^3 or R^4:

(i) $S = \{(1,1,1), (1,2,3), (2,-1,1)\}$

(ii) $S = \{(1,1,2), (1,2,5) (5,3,4)\}$

(iii) $S = \{(1,1,1,1), (0,1,1,1), (0,0,1,1), (0,0,0,1)\}$

(i) Let, $x(1,1,1)+y(1,2,3)+z(2,-1,1)=(0,0,0)$

$$x+y+2z=0$$
$$x+2y-z=0$$
$$x+3y+z=0$$

* determinant 0 হলে Rank 3 এর চেয়ে কম। তাই dependent হবে।

→ Rank বের করে দেখাতে হবে।

Let, $(a,b,c)\in \mathbb{R}^3$

Also let, $(a,b,c)=x(1,1,1)+y(1,2,3)+z(2,-1,1)$

$$x+y+2z=a$$
$$x+2y-z=b$$
$$x+3y+z=c$$

↓

Diagonalization: (একটা Problem থাকবেই)

Q(ii) Consider the matrix $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$

a) Find all eigen values and eigen vectors of A.

b) Find a non-singular matrix P and P^{-1} such that $D = P^{-1}AP$ is diagonal

c) Compute A^{50} using diagonal factorization.

d) Find the positive square root of A i.e, a matrix B such that $B^2 = A$.

$$D = P^{-1}AP$$

$$PD = IAP$$

$$PDP^{-1} = IAI$$

$$PDP^{-1} = A$$

$$\therefore A = PDP^{-1}$$

$$A^n = PD^nP^{-1}$$

a) $|A-\lambda I|=0$

$\Rightarrow \begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow (2-\lambda)(3-\lambda)-2=0$

$\Rightarrow \lambda^2-5\lambda+4=0$

$\therefore \lambda = 1, 4$

$\lambda = 1,$

$$\left(\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} - 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x+2y=0$

$x+2y=0$

$y=K$

$x=-2K$

$\therefore X_{\lambda=1} = \begin{bmatrix} -2K \\ K \end{bmatrix}$

12(c) - Day Date: 7/8/2016

One of the following three theorems will be set in the exam.

Green's theorem in a plane:

Statement:

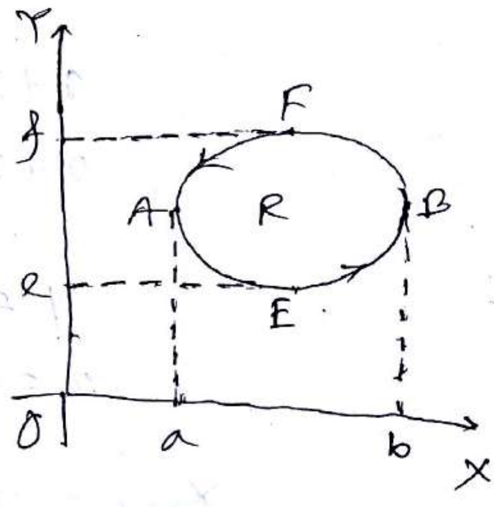
If R is a closed region of the xy plane bounded by a simple closed curve C and if M and N are continuous function of x & y having continuous derivative in R.

Then $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Proof

Let C be a closed curve which has property that straight line parallel to the coordinate axes cuts C atmost two points.

Let the equation of the curve AEB and AFB be $y = Y_1(x)$ and $y = Y_2(x)$ respectively.



If R is the region bounded by C, we have

$$\iint_R \frac{\partial M}{\partial y}\,dx\,dy = \int_{x=a}^{b} \int_{y=Y_1(x)}^{Y_2(x)} \frac{\partial M}{\partial y}\,dx\,dy$$

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$$\Rightarrow \iint_R \frac{\partial M}{\partial y}\,dx\,dy = \int_{x=a}^{b} \left[M(x,y)\right]_{Y_1}^{Y_2} dx$$

$$= \int_a^b \left[M(x,y_2) - M(x,y_1)\right] dx$$

$$= -\int_b^a M(x,y_2)\,dx - \int_a^b M(x,Y_1)\,dx$$

$$(ii) = -\left[\int_a^b M(x,Y_1) + \int_b^a M(x,Y_2)\right] dx$$

$$\therefore \iint_R \frac{\partial M}{\partial y}\,dx\,dy = -\oint M\,dx \quad \ldots\ldots (i)$$

[$\oint \rightarrow$ closed integration]

Again let the equation of EAF and EBF be $x = X_1(y)$, $x = X_2(y)$ respectively.

Then $$\iint_R \frac{\partial N}{\partial x}\,dx\,dy = \int_{y=e}^{f} \int_{x=X_1}^{X_2} \frac{\partial N}{\partial x}\,dx\,dy$$

$$= \int_e^f \left[N(x,y)\right]_{X_1}^{X_2} dy$$

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$$= \int_{e}^{f} [N(x_2, y) - N(x_1, y)]\, dy$$

$$= \int_{e}^{f} N(x_2, y)\, dy + \int_{f}^{e} N(x_1, y)\, dy$$

$$= \int_{f}^{e} N(x_1, y)\, dy + \int_{e}^{f} N(x_2, y)\, dy$$

$$= \oint N\, dy \quad \ldots \quad \ldots \text{(ii)}$$

Adding (i) & (ii), we obtain,

$$\oint_C M\, dx + N\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$$

[Proved]

Divergence Theorem:

→ volume integration ↔ surface integration

H.W (from book)

$$\iiint_V \nabla \cdot \vec{A}\, dv = \iint_S \vec{A} \cdot \hat{n}\, ds$$

Stokes Theorem:

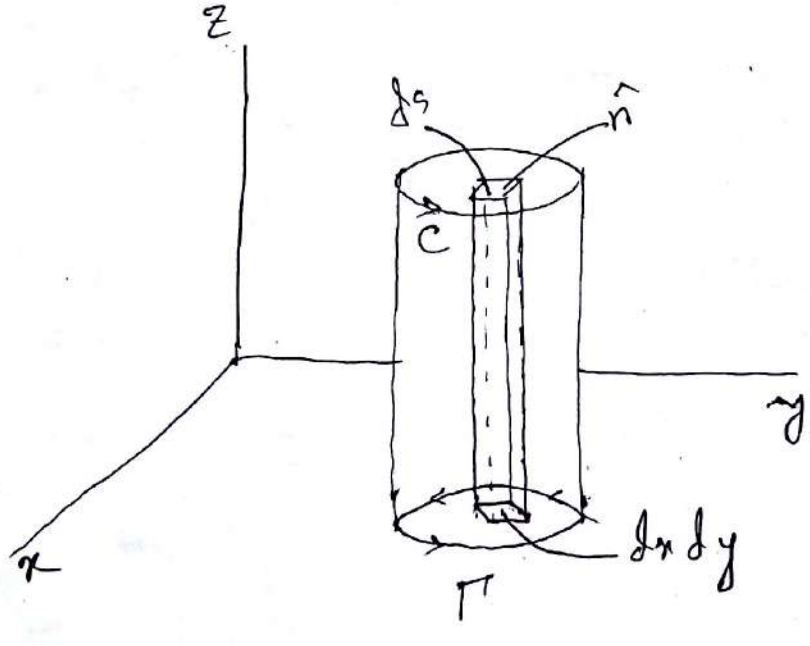
Statement:

The ~~in~~ line integral of the tangential component of a vector $\vec{A}$ taken around a simple closed curve C is equal to the surface integral of the normal component of the ~~curve~~ curl of $\vec{A}$. i.e.

$$\oint_C \vec{A} \cdot d\vec{r} = \iint_S (\nabla \times \vec{A}) \cdot \hat{n} \, ds$$

where n is normal to S.

Proof



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Let, S be a surface which is such that its projection on the xy, yz and zx planes are regions bounded by simple closed curves as shown in a figure.

Assume S to have representations $z = f(x,y)$ or $x = g(y,z)$ or $y = h(x,z)$

We have to show that

$$\iint_S (\nabla \times \vec{A}) \cdot \hat{n} \, ds = \iint_S \left[\nabla \times [A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}]\right] \cdot \hat{n} \, ds$$

$$= \oint_C \vec{A} \cdot d\vec{r} .$$

Consider first $\iint_S [\nabla \times A_1 \hat{i}] \cdot \hat{n} \, ds$

Since $\nabla \times A_1 \hat{i} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & 0 & 0 \end{vmatrix} = \frac{\partial A_1}{\partial z} \hat{j} - \frac{\partial A_1}{\partial y} \hat{k}$

$$\therefore [\nabla \times A_1 \hat{i}] \cdot \hat{n} \, ds = \left(\frac{\partial A_1}{\partial z} \hat{n} \cdot \hat{j} - \frac{\partial A_1}{\partial y} \hat{n} \cdot \hat{k}\right) ds \quad \ldots\ldots (1)$$

If $z = f(x,y)$ is taken as the equation of S then the position vector of any point of S is

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\Rightarrow \vec{r} = x\hat{i} + y\hat{j} + f(x,y)\hat{k}$$

$$\therefore \frac{\partial \vec{r}}{\partial y} = \hat{j} + \frac{\partial f}{\partial y} \hat{k}$$

But, $\frac{\partial \vec{r}}{\partial y}$ is a vector tangent to S and perpendicular with $\hat{n}$.

Then $\hat{n} \cdot \frac{\partial \vec{r}}{\partial y} = \hat{n} \cdot \hat{j} + \frac{\partial f}{\partial y} \hat{n} \cdot \hat{k} = 0$

$$\Rightarrow \hat{n} \cdot \hat{j} = -\frac{\partial z}{\partial y} \hat{n} \cdot \hat{k} \quad \Big| \because z = f(x, y)$$

Putting this in (1),

$$[\nabla \times \vec{A_1} \hat{i}] \cdot \hat{n} \, ds = \left[-\frac{\partial A_1}{\partial z} \frac{\partial z}{\partial y} \hat{n} \cdot \hat{k} - \frac{\partial A_1}{\partial y} \hat{n} \cdot \hat{k} \right] ds$$

$$= -\left(\frac{\partial A_1}{\partial y} + \frac{\partial A_1}{\partial z} \frac{\partial z}{\partial y} \right) \hat{n} \cdot \hat{k} \, ds \quad \cdots (2)$$

Now, on S. $A_1(x, y, z) = A_1(x, y, f(x, y)) = F(x, y)$ [say]

So. $\frac{\partial A_1}{\partial y} + \frac{\partial A_1}{\partial z} \cdot \frac{\partial z}{\partial y} = \frac{\partial F}{\partial y}$

Then from (2) $[\nabla \times A_1 \hat{i}] \cdot \hat{n} \, ds = -\frac{\partial F}{\partial y} dx \, dy$ [$\hat{n} \cdot \hat{k} = 1$ because they are parallel]

Then $\iint_S [\nabla \times A_1 \hat{i}] \cdot \hat{n} \, ds = \iint_R -\frac{\partial F}{\partial y} dx \, dy$

The projection of S on the xy plane ~~by Green's~~ ~~theorem~~, R

By Green's Theorem [$- \rightarrow$ direction so ignored],

$$\iint_R -\frac{\partial F}{\partial y} dx \, dy = \int_{\Gamma} F \, dx = \oint A_1 \, dx$$

Similarly $\iint_S (\nabla \times A_2 \hat{j}) \cdot \hat{n} \, ds = \oint_C A_2 \, dy$ and $\iint_S (\nabla \times A_3 \hat{k}) \cdot \hat{n} \, ds = \oint_C A_3 \, dz$

Adding these, $\iint_S (\nabla \times \vec{A}) \cdot \hat{n} \, ds = \oint_C \vec{A} \cdot d\vec{r}$

[Proved]

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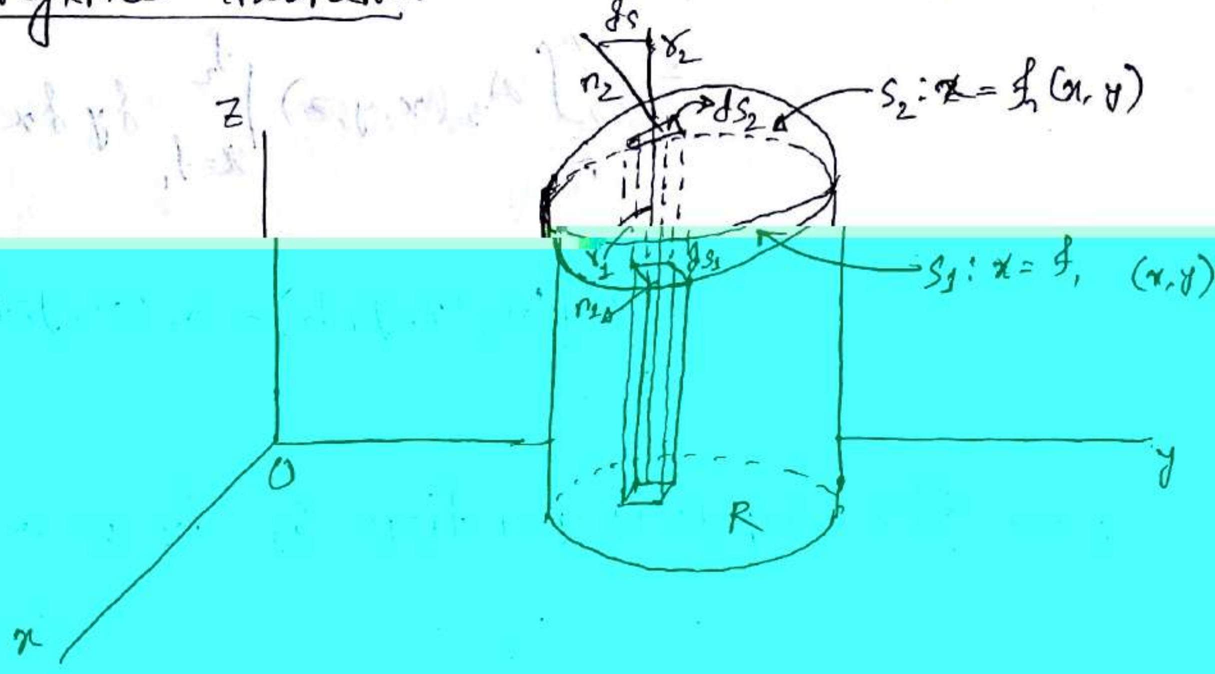
Prove that $\iiint_V \nabla \cdot \vec{A}\, dv = \iint_S \vec{A} \cdot \hat{n}\, ds$

Divergence Theorem:



Let S be ~~a~~ a closed surface which is such that any line parallel to the coordinate axes cuts S in at most two points. Assume the equations of the lower and upper portions, S_1 and S_2, to be $z = f_1(x, y)$ and $z = f_2(x, y)$ respectively. Denote the projection of the surface on the xy plane by R. Consider

$$\iiint_V \frac{\partial A_3}{\partial z}\, dv = \iiint_V \frac{\partial A_3}{\partial z}\, dz\, dy\, dx$$

$$= \iint_R \left[\int_{z = f_1(x,y)}^{f_2(x,y)} \frac{\partial A_3}{\partial z} \right] dy\, dx$$

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$$= \iint_R A_3(x,y,z) \Big|_{z=f_1}^{f_2} dy\, dx$$

$$= \iint_R \left[A_3(x,y,f_2) - A_3(x,y,f_1) \right] dy\, dx$$

For the upper portion S_2. $dy\, dx = \cos \gamma_2\, dS_2$

$= k \cdot n_2\, dS_2$

since the normal n_2 to S_2 makes an acute ange γ_2 with k.

For the lower portion S_1, $dy\, dx = -\cos \gamma_1\, dS_1$

$= -k \cdot n_1\, dS_1$

since the normal n_1 to S_1 makes an obtuse angle γ_1 with k.

Then,

$$\iint_R A_3(x,y,f_2)\, dy\, dx = \iint_{S_2} A_3 k \cdot n_2\, dS_2$$

$$\iint_R A_3(x,y,f_1)\, dy\, dx = -\iint_{S_1} A_3\, k \cdot n_1\, dS_1$$

and

$$\iint_R A_3(x,y,f_2)\,dy\,dx - \iint_R A_3(x,y,f_1)\,dy\,dx$$

$$= \iint_{S_2} A_3\, k\cdot n_2\, dS_2 + \iint_{S_1} A_3\, k\cdot n_1\, dS_1$$

$$= \iint_S A_3\, k\cdot n\, dS$$

So that (1) $\iiint_V \frac{\partial A_3}{\partial z}\,dv = \iint_S A_3 k\cdot n\, dS$

(2) $\iiint_V \frac{\partial A_1}{\partial x}\,dv = \iint_S A_1 i\cdot n\, dS$

(3) $\iiint_V \frac{\partial A_2}{\partial y}\,dv = \iint_S A_2 \hat{j}\cdot n\, dS$

Adding (1), (2) & (3),

$$\therefore \iiint_V \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) dv = \iint_S (A_1 i + A_2 j + A_3 k)\cdot n\, dS$$

$$\therefore \iiint_V \nabla\cdot A\, dv = \iint_S A\cdot n\, dS$$

Q.37:

Verify Green's theorem in the plane for $\oint_c (3x^2-8y^2)\,dx+(4y-6xy)\,dy$ where c is the boundary of the region defined by

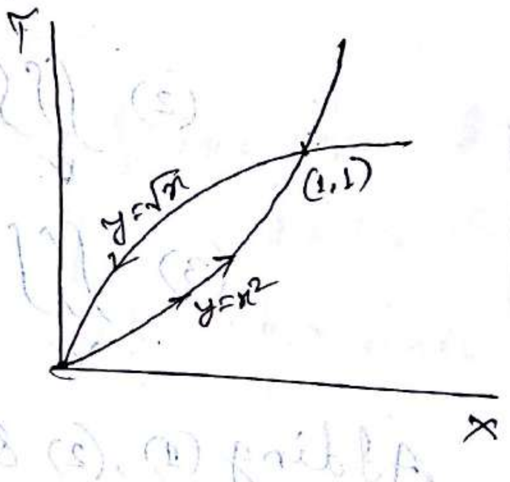
a) $y=\sqrt{x}$, $y=x^2$

b) $x=0$, $y=0$, ~~$x+y=1$~~ $x+y=1$

a) Soln:

For directly solving $y=\sqrt{x}$, $y=x^2$, $x=0$, $y=0$ and ~~$x+y=1$~~ $x=1$, $y=1$

~~Along $y=x^2$~~



Along $y=x^2$ $\therefore$ $dy=2x\,dx$

$$\oint (3x^2-8y^2)\,dx+(4y-6xy)\,dy=\int_0^1 (3x^2-\cancel{8y^2}\,8x^4)\,dx+(4x^2-6x^3)\,2x\,dx$$

$$=\int_0^1 (3x^2-8x^4+8x^3-12x^4)\,dx$$

$$=-1$$

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Along $y = \sqrt{x}$ $\therefore dy = \frac{1}{2}x^{-\frac{1}{2}}dx$

$$\oint (3x^2-8y^2)dx + (4y-6xy)dy = \int_1^0 (3x^2-8x)dx + (4\sqrt{x}-6x\sqrt{x})\cdot \frac{1}{2}x^{-\frac{1}{2}}dx$$

$$= \int_1^0 (3x^2-8x+2-3x)\,dx$$

$$= \frac{5}{2}$$

$\therefore \oint (3x^2-8y^2)dx + (4y-6xy)dy$ bounded by the region $y=\sqrt{x}$, $y = x^2$ is $\frac{5}{2} - 1 = \frac{3}{2}$

By Green's theorem,

$$\oint (3x^2-8y^2)dx + (4y-6xy)dy = \int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} \left\{\frac{\partial(4y-6xy)}{\partial x} - \frac{\partial(3x^2-8y^2)}{\partial y}\right\} dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} (-6y+16y)\,dx\,dy = \int_{x=0}^{1} \left[\frac{10y^2}{2}\right]_{x^2}^{\sqrt{x}} dx$$

$$= \frac{3}{2}$$

Hence, Green's theorem is verified.

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$$X_{\lambda=1} = \begin{bmatrix} -2K \\ K \end{bmatrix}$$

$$p = \left(\begin{bmatrix} \\ \\ \end{bmatrix} \begin{bmatrix} \\ \end{bmatrix} \right)$$

Let, $u_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$(1,0),(0,1)$

$(2,3) = 2\times1+2\cdot0, 3\cdot0+3\cdot1$
$= (2,3)$

For $\lambda = 4$,

$$\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-2x + 2y = 0$

$x - y = 0$

$$A_2 = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \overset{R_{12}}{\sim} \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \overset{R_{21}(2)}{\sim} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

Let $y = K$

$\therefore x = K$

$$X_{\lambda=4} = \begin{bmatrix} K \\ K \end{bmatrix}$$

Let, $u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Linear independent test:

Let $x u_1 + y u_2 = \bar{0}$

$$\Rightarrow x\begin{bmatrix}2\\-1\end{bmatrix} + y\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$\Rightarrow$ Thus

$$2x+y=0$$
$$-x+y=0$$

Coefficent matrix of the above system

$$A_3 = \begin{bmatrix}2 & 1\\-1 & 1\end{bmatrix} \overset{R_{12}}{\sim} \begin{bmatrix}-1 & 1\\2 & 1\end{bmatrix} \overset{R_1(-1)}{\sim} \begin{bmatrix}1 & -1\\2 & 1\end{bmatrix}$$

$$\overset{R_{21}(-2)}{\sim} \begin{bmatrix}1 & -1\\0 & 3\end{bmatrix}$$

Equivalent system

$$x-y=0$$
$$3y=0$$
$$\therefore y=0$$
$$\text{and } x=0$$

~~Now, we form~~

Now, we form P as

$$P = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

Here $|P| = 3$

$$\therefore P^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \therefore A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Now

$$D = P^{-1}AP$$

$$= \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1 & -3 \\ 4 & 8 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

Eigen values

ⓒ We have,

$$D = P^{-1}AP$$

$$\Rightarrow PD = AP$$

$$\Rightarrow PDP^{-1} = A$$

$$\therefore A = PDP^{-1}$$

$$A^{50} = PD^{50}P^{-1}$$

$$\Rightarrow A^{50} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}^{50}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1.26\times10^{30} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$\therefore A^{50} =$$

ⓓ $B = P\sqrt{D}\,P^{-1}$

$$= \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$=$$

Linear Operator :

$$\begin{cases} F: V \longrightarrow U \\ (1)\ F(u+v) = F(u) + F(v) \quad \forall\, u, v \in V \\ (2)\ F(ku) = k(F(u)) \quad \forall\, u \in V \text{ and } k \in K \end{cases}$$

$$f: R \rightarrow R$$

$$f(x) = x^2$$

$$f(x+y) = f(x) + f(y)$$

$$\Rightarrow (x+y)^2 \neq x^2 + y^2$$

So, it's not a linear function.

$$\frac{d}{dx} : R \longrightarrow R$$

$$\frac{d}{dx}\left(f(x) + g(x)\right) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

$$\frac{d}{dx}\left(c\, f(x)\right) = c \frac{d}{dx} f(x)$$

So, it's linear

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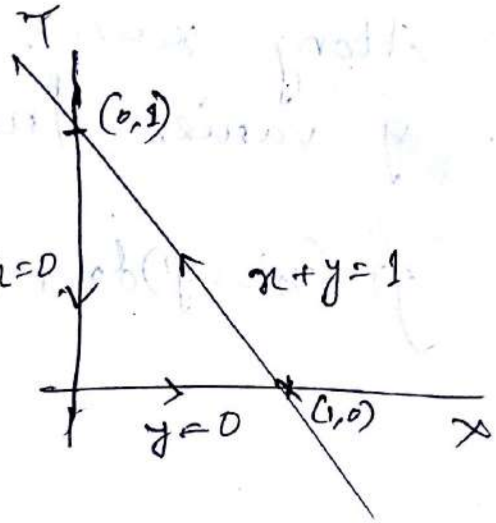
37. (b)



Along $y=0$, $\therefore dy=0$

x varies from 0 to 1.

$$\oint (3x^2-8y^2)\,dx + (4y-6xy)\,dy$$

$$= \int_0^1 3x^2\,dx \qquad [y=0,\ dy=0]$$

Along $x+y=1$, $y=1-x$, $dy=-dx$

x varies from 1 to 0.

$$\oint (3x^2-8y^2)\,dx + (4y-6xy)\,dy$$

$$= \int_{x=1}^{0} \{3x^2-8(1-x)^2\}\,dx - \{4(1-x)-6x(1-x)\}\,dx$$

$$= \int_1^0 (-11x^2+26x-12)\,dx$$

$$= \frac{8}{3}$$

Along $x=0$, $\therefore dx=0$

y varies from 1 to 0

$$\oint (3x^2-8y^2)dx+(4y-6xy)dy = \int_{y=1}^{0} 4y\,dy$$

$$= -2$$

$\therefore \oint (3x^2-8y^2)dx+(4y-6xy)dy$ bounded by

$x=0,\ y=0$ and $x+y=1$ is $1+\frac{8}{3}-2=\frac{5}{3}$

By Green's theorem.

$$\oint (3x^2-8y^2)dx+(4y-6xy)dy$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x} \left\{\frac{\partial}{\partial x}(4y-6xy) - \frac{\partial}{\partial y}(3x^2-8y^2)\right\} dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x} 10y\,dx\,dy$$

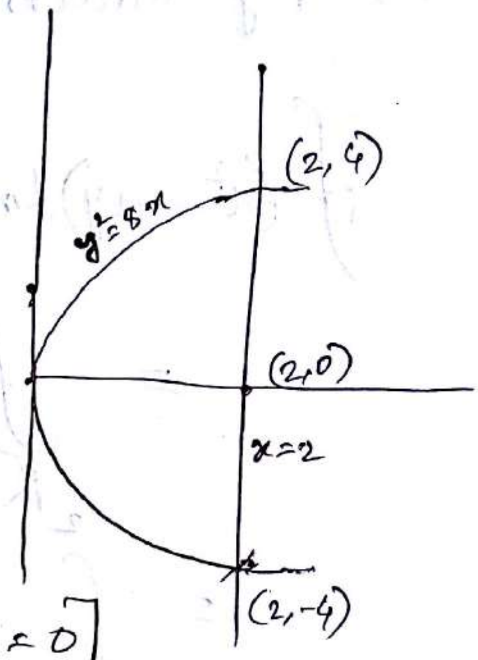
$$= \int_{x=0}^{1}\left[\frac{10y^2}{2}\right]_0^{1-x} dx = \int_0^1 (5-10x+5x^2)\,dx$$

$$= \frac{5}{3}$$

Hence, Green's Theorem is verified.

Q.40: Evaluate $\oint (x^2-2xy)\,dx + (x^2y+3)\,dy$ around the boundary of the region defined by $y^2=8x$ and $x=2$ (a) directly, (b) By Green's theorem.

Soln:



(a) Along $x=2$, $\therefore dx=0$

Solving $y=\pm 4$

$$\oint (x^2-2xy)\,dx + (x^2y+3)\,dy$$

$$= \int_{y=-4}^{4} (4y-3)\,dy = 24 \quad [\because dx=0]$$

Along $y^2=8x$ $\therefore dx = \frac{1}{4}y\,dy$

$$\oint (x^2-2xy)\,dx + (x^2y+3)\,dy$$

$$= \int_{4}^{-4} \left\{ \left(\frac{y^4}{64} - \frac{y^3}{4}\right)\frac{1}{4}y\,dy + \left(\frac{y^5}{64}+3\right)dy \right\}$$

$=$

$= \frac{128}{5} - 24$

$$\therefore \oint (x^2-2xy)\,dx + (x^2y+3)\,dy = \frac{128}{5} - 24 + 24$$

$$= \frac{128}{5} \quad \text{(Ans.)}$$

(b) By Green's theorem

$$\oint (x^2-2xy)\,dx + (x^2y+3)\,dy$$

$$= \iint_R \left\{ \frac{\partial}{\partial x}(x^2y+3) - \frac{\partial}{\partial y}(x^2-2xy) \right\} dx\,dy$$

$$= \int_{x=0}^{2} \int_{y=-\sqrt{8x}}^{\sqrt{8x}} (2xy+2x)\,dx\,dy$$

$$= \int_{x=0}^{2} \left[x^2y + 2xy \right]_{-\sqrt{8x}}^{\sqrt{8x}} dx$$

$$= \int_{x=0}^{2} 4x\sqrt{8x}\,dx$$

$$= \frac{128}{5} \quad \text{(Ans.)}$$