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Differential equation

Non-homogeneous DE:- An equation of the form $a_0 y^n + a_1 y^{n-1} + a_2 y^{n-2} + \cdots + a_{n-1} y^{(1)} + a_n y = F(x)$ --- (i) with $F(x) \neq 0$ is called a non-homogeneous DE.

In this case the solution of the corresponding homogeneous equation denoted by y_c, is called the complementary function (C.F) and a sol^n of the non-homogeneous eq^n is denoted by y_p is called particular integral. Hence the solution of (i) is the form $y = y_c + y_p$

(a) y_c can be obtained as in homogeneous equation.

(b) y_p can be defined by the three different method:-

(i) operator method

(ii) Method of undetermined co-efficients.

(iii) Method of variation of parameter.

(i) **operator method:-**

$D = d/dx \longrightarrow D(x^2) = 2x$

$\therefore 1/D = \int dx \longrightarrow 1/D(x^2) = \frac{x^3}{3}$

Algebric function:- Binomial theorem.

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \cdots + x^n$

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Q1:- Solve $(D^2-3D+2)y = x^2$

Solution:- Given that, $(D^2-3D+2)y = x^2$

let, $y = e^{mx}$ be the sol^n of the corresponding homogeneous part $(D^2-3D+2)y=0$ then the auxiliary eq^n is,

$m^2-3m+2=0$

$\Rightarrow m^2-2m-m+2=0$

$\Rightarrow m(m-2)-1(m-2)=0$

$\Rightarrow (m-2)(m-1)=0$

$\therefore m = 1, 2$

Hence the complementary function is,

$y_c = c_1e^x + c_2e^{2x}$ here c_1 and c_2 are arbitrary constant.

The particular sol^n is,

$$y_p = \frac{1}{D^2-3D+2} \cdot x^2$$

$$= \frac{1}{2(1-\frac{3}{2}D+\frac{D^2}{2})} \cdot x^2$$

$$= \frac{1}{2}\left\{1-\left(\frac{3}{2}D-\frac{1}{2}D^2\right)\right\}^{-1} x^2$$

$$= \frac{1}{2}\left\{1+\left(\frac{3}{2}D-\frac{1}{2}D^2\right)+\left(\frac{3}{2}D-\frac{1}{2}D^2\right)^2+\cdots\right\} x^2$$

$$= \frac{1}{2}\left[x^2+\frac{3}{2}\cdot 2x-\frac{1}{2}\times 2+\frac{9}{4}\cdot 2\right]$$

$$= \frac{1}{2}\left(x^2+3x+\frac{7}{2}\right)$$

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$\therefore$ The general solution is,

$y = y_c + y_p$

$= c_1 e^{x} + c_2 e^{2x} + \frac{1}{2}(x^2 + 3x + \frac{7}{2})$ (Ans:)

$(1-x)^{-1} = 1 + x + x^2 + \cdots$

$(1+x)^{-1} = 1 - x + x^2 - \cdots$

Wrong!!!

Q2:- Solve $(D^2 - D + 1)y = x^3 - 3x^2 + 1$

Solution:- Given that, $(D^2 - D + 1)y = x^3 - 3x^2 + 1$

let $y = e^{mx}$ be the solⁿ of the corresponding homogeneous part $(D^2 - D + 1)y = 0$, the auxiliary equation is,

$m^2 - m + 1 = 0$

$$\therefore m = \frac{-(-1) \pm \sqrt{1 - 4 \cdot 1 \cdot 1}}{2 \cdot 1}$$

$$= \frac{1 \pm \sqrt{-3}}{2}$$

$$= \frac{1 \pm \sqrt{3}i}{2}$$

$\therefore y_c = e^{x/2}\left(c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x\right)$

The particular solⁿ is,

$$y_p = \frac{1}{(D^2 - D + 1)} \cdot (x^3 - 3x + 1)$$

$$= \frac{1}{1 + (D^2 - D)} (x^3 - 3x + 1)$$

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$= \{1 + (D^2 - D)\}^{-1} (x^3 - 3x + 1)$

$= \{1 - (D^2 - D) + (D^2 - D)^2 - (D^2 - D)^3 + \cdots\} \cdot (x^3 - 3x + 1)$

$= (1 - D^2 + D + D^4 - 2D^3 + D^2 - D^6 + 3D^3 - 3D^3 + D^3)(x^3 - 3x + 1)$

$= (1 + D - D^3)(x^3 - 3x + 1)$

$= (x^3 - 3x + 1 + 3x^2 - 3 - 6)$

$= (x^3 + 3x^2 - 3x - 8)$

$\therefore$ The general soln is,

$y = y_c + y_p$

$= e^{x/2} (c_1 \cos \frac{\sqrt{3}}{2} x + \sin \frac{\sqrt{3}}{2} x \cdot c_2) + (x^3 + 3x^2 - 3x - 8)$

Exponential function:-

particular solns of exponential functions are given by,

(i) $\frac{1}{\phi(D)} e^{mx} = e^{mx} \frac{1}{\phi(m)}$ if $\phi(m) \neq 0$

(ii) $\frac{1}{\phi(D)} e^{mx} = e^{mx} \frac{1}{\phi(D+m)} \cdot 1$ if $\phi(m) = 0$

(iii) $\frac{1}{\phi(D)} e^{mx} \cdot F(x) = e^{mx} \frac{1}{\phi(D+m)} \cdot F(x)$

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Q3: Solve $(D^2+4)y = e^{3x}$

Soln:- Given that, $(D^2+4)y = e^{3x}$

let $y = e^{mx}$ be a soln of $(D^2+4)y = 0$

The auxiliary soln is, $m^2+4=0$

$\therefore m = \pm 2i$

The complementary soln is $y_c = c_1\cos 2x + c_2 \sin 2x$

Again the particular soln is $y_p = \frac{1}{D^2+4} e^{3x}$

$= e^{3x} \frac{1}{3^2+4}$

$= \frac{1}{13} e^{3x}$

Hence the general soln is, $y = y_c + y_p$

$= c_1\cos 2x + c_2 \sin 2x + \frac{1}{13} e^{3x}$ (Ans:)

Q4:- Solve $(D^2-9)y = e^{-3x}$

Soln:- Given that, $(D^2-9)y = e^{-3x}$

let $y = e^{mx}$ be the soln of $(D^2-9)y = 0$

Hence the auxiliary soln is, $m^2-9=0$

$\therefore m = \pm 3$

$\therefore$ The complementary soln is $y_c = c_1 e^{3x} + c_2 e^{-3x}$

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$\therefore y_p = \frac{1}{D^2-9} e^{-3x}$

$= \frac{1}{(D-3)^2-9} e^{-3x}$

$= e^{-3x} \cdot \frac{1}{-6D(1-D/6)}$

$= e^{-3x} \cdot \frac{1}{-6D} (1-D/6)^{-1}$

$= e^{-3x} \cdot \frac{1}{-6D} (1+D/6+ - -)$

$= e^{-3x} \cdot \frac{1}{-6D}$

$= -\frac{e^{-3x} \cdot x}{6}$

$= -\frac{xe^{-3x}}{6}$

$\therefore y = y_c + y_p$

$= c_1 e^{3x} + c_2 e^{-3x} - \frac{xe^{-3x}}{6}$ (Ans:)

05: Solve $(D^2+4D+4)y = x^3 e^{-2x}$

solⁿ:- Given that, $(D^2+4D+4)\ y = x^3 e^{-2x}$

let, $y = e^{mx}$ be the solⁿ of the $(D^2+4D+4)\ y = 0$

The auxiliary solⁿ is, $m^2+4m+4=0$

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$\Rightarrow m^2 + 2m + 2m + 4 = 0$

$\Rightarrow (m+2)(m+2) = 0$

$\therefore m = -2, -2$

$\therefore y_c = c_1 e^{-2x} + x c_2 e^{-2x}$

Again the particular solⁿ is,

$y_p = \dfrac{1}{D^2+4D+4} x^3 e^{-2x}$

$= e^{-2x} \dfrac{1}{(D-2)^2 + 4(D-2) + 4} x^3$

$= e^{-2x} \dfrac{1}{D^2-4D+4+4D-8+4} x^3$

$= e^{-2x} \dfrac{1}{D^2} x^3$

$= e^{-2x} \dfrac{1}{D} \cdot x^4/4$

$= e^{-2x} \cdot \dfrac{x^5}{20}$

$= 1/20 \cdot x^5 \cdot e^{-2x}$

$\therefore$ The general solution is $y = y_c + y_p$

$= (c_1 + c_2 x) e^{-2x} + 1/20 \cdot x^5 e^{-2x}$ (Ans)

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06:- Solve $(D^2+4D+4)y = x^3e^{-3x}$

Sol^n:- Given that. $(D^2+4D+4)y = x^3e^{-3x}$

Let $y = e^{mx}$ be the sol^n of $(D^2+4D+4)y=0$

The auxiliary sol^n is. $m^2+4m+4=0$

$\Rightarrow (m+2)(m+2)=0$

$\therefore m=-2,-2$

$\therefore y_c = (c_1+c_2x)e^{-2x}$

$\therefore$ The particular sol^n is, $y_p = \frac{1}{D^2+4D+4}x^3e^{-3x}$

$= e^{-3x}\frac{1}{(D-3)^2+4(D-3)+4}\cdot x^3$

$= e^{-3x}\frac{1}{D^2-6D+9+4D-12+4}\cdot x^3$

$= e^{-3x}\frac{1}{D^2-2D+1}\cdot x^3$

$= e^{-3x}\{1+(D^2-2D)\}^{-1}x^3$

$= e^{-3x}\{1-(D^2-2D)+(D^2-2D)^2-\cdots\}x^3$

$= e^{-3x}\{1-D^2+2D-4D^3+4D^2-D^6+6D^3+8D^3\}x^3$

$= e^{-3x}(1+3D^2+2D+4D^3)x^3$

$= e^{-3x}(x^3+18x+6x^2+24)$

$\therefore y = y_c+y_p = (c_1+c_2x)e^{-3x}+e^{-3x}(x^3+18x+6x^2+24)$ (Ans:)

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Trigonometric function:-

(i) $\frac{1}{\phi(D^2)}\begin{pmatrix}\sin ax\\ \text{or}\\ \cos ax\end{pmatrix} = \frac{1}{\phi(-a^2)}\begin{pmatrix}\sin ax\\ \text{or}\\ \cos ax\end{pmatrix}$ if $\phi(-a^2) \neq 0$

Q7:- Solve $(D^2+1)y = \sin 3x$

Sol^n:- Given that, $(D^2+1)y = \sin 3x$

let $y = e^{mx}$ be the sol^n of the $(D^2+1)y = 0$

The auxiliary equation is $m^2+1 = 0$

$\therefore m = \pm i$

$\therefore y_c = c_1\cos x + c_2\sin x$

The particular sol^n is, $y_p = \frac{1}{D^2+1}\sin 3x$

$= \frac{1}{-3^2+1}\sin 3x$

$= -\frac{1}{8}\sin 3x$

$\therefore$ The general solution is,

$y = y_c + y_p$

$= c_1\cos x + c_2\sin x - \frac{1}{8}\sin 3x$ (Ans:)

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Q8:- solve $(D^2+4)y = \cos 2x$

Sol^n:- Given that $(D^2+4)y = \cos 2x$

Let $y = e^{mx}$ be the sol^n of the $(D^2+4)y = 0$

The auxiliary equation is $m^2+4=0$

$\therefore m = \pm 2i$

The particular sol^n is $y_p = \frac{1}{D^2+4}\cos 2x$

$= \frac{x \cos 2x}{2D}$

$= \frac{x \sin 2x}{4}$

$\therefore y = y_c + y_p$

$= c_1\cos 2x + c_2 \sin 2x + \frac{1}{4} x \sin 2x$ (Ans:)

Q9:- solve $(D^2-3D+2)y = \sin 3x$

Sol^n:- Given that, $(D^2-3D+2)y = \sin 3x$

let $y = e^{mx}$ be the sol^n of the $(D^2-3D+2) = 0$

$m^2 - 3m + 2 = 0$

$\Rightarrow m^2 - 2m - m + 2 = 0$

$\Rightarrow m(m-2) - 1(m-2) = 0$

$\therefore m = 1, 2$

$\therefore y_c = c_1 e^{x} + c_2 e^{2x}$

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The particular solution is $y_p = \frac{1}{D^2 - 3D + 2} \sin 3x$

$= \frac{1}{-3^2 - 3D + 2} \sin 3x$

$= \frac{1}{-(3D + 7)} \sin 3x$

$= -\frac{(3D-7)}{(3D+7)(3D-7)} \sin 3x$

$= -\frac{(3D-7)}{9D^2 - 49} \sin 3x$

$= -\frac{3D-7}{9 \cdot -3^2 - 49} \sin 3x$

$= -\frac{3D-7}{-81-49} \sin 3x$

$= \frac{1}{130} (3D-7) \sin 3x$

$= 1/130 \; (9 \cos 3x - 7 \sin 3x)$

$\therefore y = y_c + y_p$

$= c_1 e^x + c_2 e^{2x} + \frac{1}{130} (9 \cos 3x - 7 \sin 3x)$

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10:- solve $(D-2)^2 y = 8x^2 e^{2x} \sin 2x$

sol^n: Given that, $(D-2)^2 y = 8x^2 e^{2x} \sin 2x$

Let $y = e^{mx}$ be the sol^n of $(D-2)^2 y = 0$

The auxiliary eq^n is $(m-2)^2 = 0$

$\therefore m = 2, 2$

$\therefore y_c = (c_1 + x c_2) e^{2x}$

The particular integral is, $y_p = \frac{1}{(D-2)^2} 8x^2 e^{2x} \sin 2x$

$= 8e^{2x} \frac{1}{(D+2-2)^2} \sin 2x$

$= 8e^{2x} \frac{1}{D^2} \sin 2x$

let, $Y = \frac{1}{D^2} x^2 \sin 2x$

$X = \frac{1}{D^2} x^2 \cos 2x$

$\therefore X + iY = \cos x + i \sin x$

$= \frac{1}{D^2} x^2 (\cos 2x + i \sin 2x)$

$= 1/D^2 \; x^2 e^{2ix}$

$= e^{2ix} \frac{1}{(D+2i)^2} x^2$

$= e^{2ix} \frac{1}{D^2 + 4Di - 4} \cdot x^2$

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$$= e^{2ix} \frac{1}{-4\{1-(iD+D^2/4)\}} x^2$$

$$= \frac{e^{2ix}}{-4} \{1 + (iD + D^2/4) + (iD + D^2/4)^2 + \cdots\} x^2$$

$$= \frac{e^{2ix}}{-4} (x^2 + 2xi + 1/2 - 2)$$

$$= -1/4 (\cos 2x + i \sin 2x)(x^2 + 2xi - 3/2)$$

$$\therefore Y = -1/4 [2x \cos 2x + \sin 2x (x^2 - 3/2)]$$

Hence, The required particular integral is,

$$y_p = 8e^{2x}(-1/4)[2x \cos 2x + (x^2 - 3/2) \sin 2x]$$

The general solution is, $y = y_c + y_p$

$$= (c_1 + c_2 x)e^{2x} - 2e^{2x}[2x \cos 2x + (x^2 - 3/2) \sin 2x] \text{ (Ans:)}$$

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Math from Advance Engr. Math

Q. $\frac{d^3y}{dx^3} + 3\, d^2y/dx^2 + 5\, dy/dx + 3y = 0$

Soln:- let $y = e^{mx}$ be the trial solution and the auxiliary equation are.

$m^3 + 3m^2 + 5m + 3 = 0$

$\Rightarrow m^3 + m^2 + 2m^2 + 2m + 3m + 3 = 0$

$\Rightarrow m^2(m+1) + 2m(m+1) + 3(m+1) = 0$

$\Rightarrow (m+1)(m^2+2m+3) = 0$

$\Rightarrow m+1 = 0 \qquad m^2+2m+3 = 0$

$\therefore m = -1$

$\Rightarrow \cancel{m^2+3m-2m+3 = 0}$

$\Rightarrow m = \frac{-2 \pm \sqrt{4-12}}{2}$

$= \frac{-2 \pm \sqrt{-8}}{2}$

$\therefore m = -1 \pm i\sqrt{2}$

$\therefore y = c_1 e^{-x} + e^{-x}\{c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x\}$

$\therefore y = c_1 e^{-x} + e^{-x}(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x)$ (Ans:-)

Q. $(D-1)^2 y = 0$

Soln:- let $y = e^{mx}$ be the trial solution and the auxili

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any equation are, $(m-1)^2 = 0$

$\Rightarrow (m-1)(m-1) = 0$

$\therefore m = 1, 1$

$\therefore y = c_1 e^x + x c_2 e^x$ (Ans:-)

Q. solve $(D^3 - 2D^2 - 5D + 6) y = 0$

Soln:- Let, $y = e^{mx}$ be the trial soln and the auxiliary equation are,

$m^3 - 2m^2 - 5m + 6 = 0$

$\Rightarrow m^3 - m^2 - m^2 + m - 6m + 6 = 0$

$\Rightarrow m^2 (m-1) - m(m-1) - 6(m-1) = 0$

$\Rightarrow (m-1)(m^2 - m - 6) = 0$

$\Rightarrow (m-1)(m^2 - 3m + 2m - 6) = 0$

$\Rightarrow (m-1) \{ m(m-3) + 2(m-3) \} = 0$

$\therefore m = 1, -2, 3$

$\therefore y = c_1 e^x + c_2 e^{-2x} + c_3 e^{3x}$ (Ans:-)

Q. solve $(D^2 + 4) y = 0$

Soln:- Let $y = e^{mx}$ be the trial soln and the auxiliary eqn are, $m^2 + 4 = 0$

$\therefore m = \pm 2i$

$\therefore y = c_1 \cos 2x + c_2 \sin 2x$

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Q. Solve $(D^2 - D + 1)y = 0$

Solⁿ:- Let $y = e^{mx}$ be the trial solⁿ and the auxiliary eqⁿ are,

$m^2 - m + 1 = 0$

$\Rightarrow m = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4\cdot1\cdot1}}{2\cdot1}$

$= \dfrac{1 \pm \sqrt{-3}}{2}$

$\therefore m = \dfrac{1 \pm i\sqrt{3}}{2}$

$\therefore y = e^{\frac{1}{2}x}\{c_1 \cos \sqrt{3}/_2 x + c_2 \sin \sqrt{3}/_2 x\}$ (Ans:-)

Q. Solve $(D^4 + 2D^3 - 3D^2)y = 0$

Solⁿ:- Let $y = e^{mx}$ be the trial solⁿ then the auxiliary eqⁿ are,

$m^4 + 2m^3 - 3m^2 = 0$

$\Rightarrow m^2(m^2 + 2m - 3) = 0$

$\Rightarrow m^2\{m^2 + 3m - m - 3\} = 0$

$\Rightarrow m^2(m+3)(m-1) = 0$

$\therefore m = 0, 0, -3, 1$

$\therefore y = c_1 + c_2 x + c_3 e^{x} + c_4 e^{-3x}$ (Ans:-)

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Q. Solve $(D^3-2D^2-5D+6)\,y = e^{4x}$

Sol^n:- $y_c = c_1e^{x} + c_2e^{-2x} + c_3e^{3x}$

$\therefore y_p = \frac{1}{D^3-2D^2-5D+6}e^{4x}$

$= \frac{1}{4^3-32-20+6}e^{4x}$

$= \frac{1}{18}e^{4x}$

$\therefore y = y_c + y_p$

$= c_1e^{x} + c_2e^{-2x} + c_3e^{3x} + \frac{1}{18}e^{4x}$

Q. Solve $(D^3-2D^2-5D+6)y = (e^{2x}+3)^2$

Sol^n:- Given that, $(D^3-2D^2-5D+6)y = (e^{2x}+3)^2$

Let, $y = e^{mx}$ be the trial sol^n then the complementary function, $y_c = c_1e^{x} + c_2e^{3x} + c_3e^{-2x}$

particular integral, $y_p = \frac{1}{D^3-2D^2-5D+6}(e^{2x}+3)^2$

$= \frac{1}{D^3-2D^2-5D+6}e^{4x} + \frac{6}{D^3-2D^2-5D+6}e^{2x} + \frac{9\,xe^{0x}}{D^3-2D^2-5D+6}$

$= \frac{1}{18}e^{4x} - \frac{6}{4}e^{2x} + \frac{9}{6}$

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$\therefore y = y_c + y_p$

$= c_1 e^{x} + c_2 e^{3x} + c_3 e^{-2x} + \frac{e^{4x}}{18} - 3/2\, e^{2x} + 3/2$ (Ans:-)

Q. solve $(D^3 - 2D^2 - 5D + 6)y = e^{3x}$

solⁿ:- Let, $y = e^{mx}$ be the trial solⁿ and the complementary function are. $y_c = c_1 e^{x} + c_2 e^{3x} + c_3 e^{-2x}$.

particular integral, $y_p = \frac{1}{D^3 - 2D^2 - 5D + 6} e^{3x}$

$= x \frac{1}{3D^2 - 4D - 5} e^{3x}$

$= x \frac{1}{10} e^{3x}$

$\therefore y = y_c + y_p$

$= c_1 e^{x} + c_2 e^{3x} + c_3 e^{-2x} + x \frac{1}{10} e^{3x}$ (Ans:-)

Q. solve $(D^3 - 5D^2 + 8D - 4)y = e^{2x} + 2e^{x} + 3e^{-x}$

solⁿ:- Let, $y = e^{mx}$ be the trial solⁿ and the complementary function is, $y_c = c_1 e^{x} + c_2 e^{2x} + c_3 x e^{2x}$

particular integral, $y_p = \frac{1}{D^3 - 5D^2 + 8D - 4} (e^{2x} + 2e^{x} + 3e^{-x})$

$= \frac{1}{3D^2 - 10D + 8} x e^{2x} + \frac{1}{3D^2 - 10D + 8} 2x e^{x} - \frac{3e^{-x}}{18}$

$= \frac{1}{6D - 10} x^2 e^{2x} + 2x e^{x} - \frac{1}{6} e^{-x}$

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$= \frac{1}{2}x^2e^{2x} + 2xe^x - \frac{1}{6}e^{-x}$

$\therefore y = y_c + y_p$

$= c_1e^x + c_2e^{2x} + c_3xe^{2x} + \frac{1}{2}x^2e^{2x} + 2xe^x - \frac{1}{6}e^{-x}$ (Ans:-)

Q. Solve $(D^4 + 10D^2 + 9)y = \cos(2x+3)$

Soln:- Let, $y = e^{mx}$ be the trial soln and the complementary function is, $y_c = c_1\cos x + c_2\sin x + c_3\cos 3x + c_4\sin 3x$

particular integral, $y_p = \frac{1}{D^4 + 10D^2 + 9}\cos(2x+3)$

$= \frac{1}{(D^2)^2 + 10D^2 + 9}\cos(2x+3)$

$= \frac{1}{16 - 40 + 9}\cos(2x+3)$

$= -\frac{1}{15}\cos(2x+3)$

$\therefore y = y_c + y_p$

$= c_1\cos x + c_2\sin x + c_3\cos 3x + c_4\sin 3x - \frac{1}{15}\cos(2x+3)$ (Ans)

Q. Solve $(D^2 + 3D - 4)y = \sin 2x$

Soln:- Let, $y = e^{mx}$ be the trial soln and the complementary function, $y_c = c_1e^x + c_2e^{-4x}$

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particular integral, $y_p = \frac{1}{D^2+3D-4} \sin 2x$

$= \frac{1}{-4+3D-4} \sin 2x$

$= \frac{1}{3D-8} \sin 2x$

$= \frac{(3D+8)}{9D^2-64} \sin 2x$

$= \frac{(3D+8)}{-100} \sin 2x$

$= \frac{-1}{100} \left(\frac{3x \cos 2x}{1/2} + 8.\sin 2x \right)$

$= \frac{-1}{100} (6\cos 2x + 8 \sin 2x)$

$\therefore y = y_c + y_p$

$= c_1 e^{x} + c_2 e^{-4x} \frac{1}{100} (6\cos 2x + 8 \sin 2x)$ (Ans:)

Q. solve $(D^3+D^2+D+1) y = \sin 2x + \cos 3x$

Solⁿ:- Let, $y = e^{mx}$ be the trial solⁿ, hence the complementary function, $y_c = c_1 \cos x + c_2 \sin x + c_3 e^{-x}$

particular integral, $y_p = \frac{1}{D^3+D^2+D+1} (\sin 2x + \cos 3x)$

$= \frac{1}{D^2 \cdot D+D^2+D+1} \sin 2x + \frac{1}{D^2 \cdot D + D^2+D+1} \cos 3x$

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$$= \frac{1}{-4D-4+D+1} \sin 2x + \frac{1}{-9D-9+D+1} \cos 3x$$

$$= \frac{1}{-3D+3} \sin 2x + \frac{1}{-8D-8} \cos 3x$$

$$= -\frac{1}{3} \frac{1}{D+1} \sin 2x + \left(-\frac{1}{8}\right) \frac{1}{D+1} \cos 3x$$

$$= -\frac{1}{3} \frac{D-1}{D^2-1} \sin 2x - \frac{1}{8} \frac{D-1}{D^2-1} \cos 3x$$

$$= -\frac{1}{3} \frac{D-1}{-5} \sin 2x - \frac{1}{8} \frac{D-1}{-10} \cos 3x$$

$$= \frac{1}{15} (D-1) \sin 2x + \frac{1}{80} (D-1) \cos 3x$$

$$= \frac{1}{15} (2\cos 2x - \sin 2x) + \frac{1}{80} (-3\sin 3x - \cos 3x)$$

$\therefore y = y_c + y_p$

$= c_1 \cos x + c_2 \sin x + c_3 e^{-x} + \frac{1}{15} (2\cos 2x - \sin 2x) - \frac{1}{80} (3\sin 3x$
$+ \cos 3x)$ (Ans:)

Q. Solve $(D^2+4)y = \cos 2x + \cos 4x$

Soln:- Let, $y = e^{mx}$ be the trial sol^n hence the complementary function, $y_c = c_1 \cos 2x + c_2 \sin 2x$

particular integral. $y_p = \frac{1}{D^2+4} (\cos 2x + \cos 4x)$

$$= \frac{1}{2D} x \cos 2x + \frac{1}{-16+4} \cos 4x$$

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$= \frac{1}{4} x \sin 2x - \frac{1}{12} \cos 4x$

$\therefore y = y_c + y_p$

$= c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4} x \sin 2x - \frac{1}{12} \cos 4x$ (Ans:-)

Q. solve $(2D^2 + 2D + 3) y = x^2 + 2x - 1$

solⁿ:- Let, $y = e^{mx}$ be the trial solⁿ hence the complementary function $y_c = e^{-1/2 x} (c_1 \cos \sqrt{5}/2 x + c_2 \sin \sqrt{5}/2 x)$

particular integral, $y_p = \frac{1}{2D^2 + 2D + 3} (x^2 + 2x - 1)$

$= \frac{1}{3(1 + \frac{2}{3} D^2 + 2/3 D)} (x^2 + 2x - 1)$

$= \frac{1}{3} \cdot \{1 + (2/3 D^2 + 2/3 D)\}^{-1} (x^2 + 2x - 1)$

$= \frac{1}{3} \{1 - 2/3 D^2 - 2/3 D + \frac{4}{9} D^4 + \frac{4}{9} D^2\} (x^2 + 2x - 1)$

$= (\frac{1}{3} - \frac{2}{9} D^2 - 2/9 D + \frac{4}{27} D^2) (x^2 + 2x - 1)$

$= \frac{1}{3} (x^2 + 2x - 1) - 2/27 D^2 (x^2 + 2x - 1) - 2/9 D (x^2 + 2x - 1)$

$= \frac{1}{3} (x^2 + 2x - 1) - \frac{2}{27} (2) - 2/9 (2x + 2)$

$\therefore y = y_c + y_p$

$= e^{-1/2 x} (c_1 \cos \sqrt{5}/2 x + c_2 \sin \sqrt{5}/2 x) + \frac{1}{3} (x^2 + 2x - 1) - 4/27 - 2/9 (2x + 2)$ (Ans:-)

Q. Solve, $(D^3 - 2D + 4)y = x^4 + 3x^2 - 5x + 2$

Solⁿ:- Let, $y = e^{mx}$ be the trial solⁿ and the complementary function. $y_c = c_1 e^{-2x} + e^{x}(c_2 \cos x + c_3 \sin x)$

particular integral. $y_p = \frac{1}{D^3 - 2D + 4}(x^4 + 3x^2 - 5x + 2)$

$= \frac{1}{4(1 + \frac{D^3}{4} - D/2)}(x^4 + 3x^2 - 5x + 2)$

$= \cancel{\frac{1}{4}\left(1 - \frac{D^3}{4} + D/2 + \left(\frac{D^6}{16} - D^2/4\right)^2 - \frac{D^9}{64} + \frac{D^3}{8}\right)(x^4 + 3x^2 - 5x + 2)}$

$= \cancel{\left(\frac{1}{4} - D^3/16 + D/8 - D^2/16 + D^3/32\right)(x^4 + 3x^2 - 5x + 2)}$

$= \frac{1}{4}\left\{1 + \left(\frac{D^3}{4} - D/2\right)\right\}^{-1}(x^4 + 3x^2 - 5x + 2)$

$= \left(\frac{1}{4} + 1/8\,D + \frac{1}{18}D^2 - 1/32\,D^3 - 3/64\,D^4\right)(x^4 + 3x^2 - 5x + 2)$

$= \frac{1}{4}x^4 + 1/2\,x^3 + 3/2\,x^2 - 5/4\,x - 7/8$

$\therefore y = y_c + y_p$

$= c_1 e^{-2x} + e^{x}(c_2 \cos x + c_3 \sin x) + 1/4\,x^4 + 1/2\,x^3 + 3/2\,x^2 - 5/4\,x - 7/8$ (Ans)

Q. Solve, $(D^3 - 4D^2 + 3)y = x^2$

Solⁿ:- Let, $y = e^{mx}$ be the trial solⁿ, hence the complementary function, $y_c = c_1 + c_2 e^{x} + c_3 e^{3x}$

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particular integral, $y_p = \frac{1}{D^3 - 4D^2 + 3D} x^2$

$= \frac{1}{3D(1 + D^2/3 - 4/3 D)} x^2$

$= \frac{1}{3D} (1 + \frac{D^2}{3} - \frac{4}{3}D) x^2$

$= \frac{1}{3D} \{1 + (\frac{D^2}{3} - 4/3 D)\}^{-1} x^2$

$= \frac{1}{3D} \{1 - D^2/3 + 4/3 D + (\frac{D^2}{3} - 4/3 D)^2 - \} x^2$

$= \frac{1}{3D} \{1 - D^2/3 + 4/3 D + D^4/9 + \frac{16}{9} D^2\} x^2$

$= \frac{1}{D} (1/3 + 4/9 D + \frac{13}{27} D^2) x^2$

$= \frac{1}{D} (\frac{1}{3}x^2 + \frac{8}{9}x + \frac{26}{27})$

$= \frac{1}{9}x^3 + \frac{8}{9} x^2/2 + 26/27 x$

$\therefore\ y = y_c + y_p$

$= c_1 + c_2 e^x + c_3 e^{3x} + \frac{1}{9}x^3 + 4/9 x^2 + \frac{26}{27} x$ (Ans.)

Q. Solve $(D^4 + 2D^3 - 3D^2) y = x^2 + 3e^{2x} + 4\sin x$

Solⁿ:- Let, $y = e^{mx}$ be the trial solⁿ and hence the complementary function, $y_c = c_1 + c_2 x + c_3 e^x + c_4 e^{-3x}$

particular integral, $y_p = \frac{1}{D^4 + 2D^3 - 3D^2} (x^2 + 3e^{2x} + 4\sin x)$

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$$= \frac{1}{D^4+2D^3-3D^2} x^2 + \frac{1}{D^4+2D^3-3D^2} 3e^{2x} + 4\frac{1}{D^4+2D^3-3D^2} \sin x$$

$$= \frac{1}{D^2(D^2+2D-3)} x^2 + 3/20\, e^{2x} + 4\frac{1}{(D^2)^2+2D^2\cdot D-3D^2} \sin x$$

$$= \frac{1}{D^2} \times \frac{1}{-3(1-\frac{D^2}{3}-2/3D)} x^2 + \frac{3}{20}e^{2x} + \frac{1}{4/4} \times \frac{1}{1-2D+3} \sin x$$

$$= -\frac{1}{3D^2}\left(1+\left(-\frac{D^2}{3}-2/3D\right)\right)^{-1} x^2 + 3/20\, e^{2x} + 4\frac{1}{-4-2D} \sin x$$

$$= -\frac{1}{3D^2}\left\{1-(D^2/3+2/3D)\right\}^{-1} x^2 + 3/20\, e^{2x} - \frac{2}{D-2} \sin x$$

$$= -\frac{1}{3D^2}\left\{1+D^2/3+2/3D+\frac{4}{9}D^2\right\} x^2 + \frac{3}{20}e^{2x} - \frac{2(D+2)}{D^2-4} \sin x$$

$$= \left(-\frac{1}{3D^2} - \frac{1}{9} - \frac{2}{9}D/D^2 - \frac{4}{27}\right)x^2 + 3/20\, e^{2x} + \frac{2(D+2)}{5} \sin x$$

$$= \left(-\frac{1}{3D^2} - \frac{1}{9} - \frac{2}{9D} - \frac{4}{27}\right)x^2 + 3/20\, e^{2x} + \frac{2}{5}(\cos x + 2\sin x)$$

$$= -\frac{1}{3}\left(\frac{x^4}{12}\right) - \frac{1}{9}x^2 - 2/9\cdot\frac{x^3}{3} - \frac{4}{27}x^2 + \frac{3}{20}e^{2x} + \frac{2}{5}(\cos x + 2\sin x)$$

$$= -\frac{x^4}{36} - \frac{1}{9}x^2 - \frac{2x^3}{27} - 4/27\, x^2 + 3/20\, e^{2x} + \frac{2}{5}(\cos x + 2\sin x)$$

$\therefore y = y_c + y_p$

$$= c_1 + c_2 x + c_3 e^{x} + c_4 e^{-3x} - \frac{x^4}{36} - \frac{1}{9}x^2 - \frac{2x^3}{27} - \frac{4}{27}x^2 + \frac{3}{20}e^{2x}$$

$$+ \frac{2}{5}(\cos x + 2\sin x)$$ (Ans:-)

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Q. Solve $(D^2-4)y = x^2 e^{3x}$

SOL^n:- Let, $y = e^{mx}$ be the trial sol^n and the complementary function, $y_c = c_1 e^{2x} + c_2 e^{-2x}$

particular integral, $y_p = \frac{1}{D^2-4} x^2 e^{3x}$

$= e^{3x} \cdot \frac{1}{(D+3)^2-4} x^2$

$= e^{3x} \cdot \frac{1}{D^2+6D+5} x^2$

$= e^{3x} \cdot \frac{1}{5(1+\frac{D^2}{5}+\frac{6}{5}D)} x^2$

$= \frac{e^{3x}}{5} \left\{ 1+\left(\frac{D^2}{5}+\frac{6}{5}D\right)\right\}^{-1} x^2$

$= \frac{e^{3x}}{5} \left\{ 1-\frac{D^2}{5}-\frac{6}{5}D+\frac{36}{25}D^2 \right\} x^2$

$= \frac{e^{3x}}{5} \left\{ 1-\frac{6}{5}D+\frac{31}{25}D^2 \right\} x^2$

$= \frac{e^{3x}}{5} \left(x^2-\frac{6}{5}2x+\frac{31}{25}\times 2 \right)$

$= e^{3x} \left(\frac{x^2}{5}-\frac{12x}{25}+\frac{62}{125} \right)$

$\therefore y = y_c + y_p$

$= c_1 e^{2x} + c_2 e^{-2x} + e^{3x} \left(\frac{x^2}{5}-\frac{12x}{25}+\frac{62}{125} \right)$ (Ans:)

Date :

Ex. Solve $(D^2+2D+4)y = e^x \sin 2x$

SolN:- Let, $y = e^{mx}$ be the trial solN and the complement function is $y_c = e^{-x}(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x)$

particular integral, $y_p = \frac{1}{D^2+2D+4} e^x \sin 2x$

$= \frac{1}{D^2+4D+7} e^x \sin 2x$

$= e^x \frac{1}{4D+3} \sin 2x$

$= e^x \frac{(4D-3)}{16D^2-9} \sin 2x$

$= e^x \frac{(4D-3)}{-73} \sin 2x$

$= \frac{-e^x}{73}(8\cos 2x - 3\sin 2x)$

$\therefore y = y_c + y_p$

$= e^{-x}(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x) - e^x/73\,(8\cos 2x - 3\sin 2x)$ (Ans:)

Ex. Solve $(D^2-4D+3)y = 2xe^{3x} + 3e^x \cos 2x$ ---- (i)

SolN:- Let, $y = e^{mx}$ be the trial solN of (i) and hence the complementary function, $y_c = c_1e^x + c_2e^{3x}$

particular integral, $y_p = \frac{1}{D^2-4D+3}(2xe^{3x} + 3e^x \cos 2x)$

$= 2e^{3x} \frac{1}{\cancel{(D+3)^2-4D+3}} x + 3e^x \frac{1}{\cancel{(D+1)^2-4D+3}} \cos 2x$

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$$= 2e^{3x}\frac{1}{D^2+6D+9-4D+3}\,x + 3e^{x}\frac{1}{D^2+2D+1-4D+3}\cos 2x$$

$$= 2e^{3x}\frac{1}{D^2+2D+12}\,x + 3e^{x}\frac{1}{D^2-2D+4}\cos 2x$$

$$= 2e^{3x}\frac{1}{12\left(1+\frac{D^2}{12}+\frac{D}{6}\right)} + 3e^{x}\frac{1}{-4-2D+4}$$

$$= \frac{1}{D^2-4D+3}\,2xe^{3x} + \frac{1}{D^2-4D+3}\,3e^{x}\cos 2x$$

$$= 2e^{3x}\frac{1}{D^2+6D+9-4D-12+3}\,x + \frac{1}{D^2+2D+1-4D-4+3}\cos 2x \cdot 3e^{x}$$

$$= 2e^{3x}\frac{1}{D^2+2D}\,x + 3e^{x}\frac{1}{D^2-2D}\cos 2x$$

$$= 2e^{3x}\frac{1}{2D\,(1+D/2)}\,x + 3e^{x}\frac{1}{-4-2D}\cos 2x$$

$$= e^{3x}\cdot\frac{1}{D}\left(1-D/2+D^2/2^2\right)x - 3/2\,e^{x}\frac{(D-2)}{D^2-4}\cos 2x$$

$$= e^{3x}\left(\frac{1}{D}-\frac{1}{2}\right)x - 3/2\,e^{x}\frac{(D-2)}{-8}\cos 2x$$

$$= e^{3x}\left(\frac{x^2}{2}-\frac{1}{2}x\right) + 3/16\,e^{x}\left(-2\sin 2x-2\cos 2x\right)$$

$$= e^{3x}\left(\frac{x^2}{2}-\frac{x}{2}\right) - \frac{3}{8}\,e^{x}\left(\sin 2x+\cos 2x\right)$$

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$\therefore y = y_c + y_p$

$= c_1 e^{x} + c_2 e^{3x} + \frac{1}{2} e^{3x}(x^2 - x) - \frac{3}{8} e^{x}(\cos 2x + \sin 2x)$ (Ans:-)

Q. Solve $(D^2 + 3D + 2)y = x \sin 2x$

Solⁿ:- Let, $y = e^{mx}$ be the trial solⁿ and hence the complementary function is $y_c = c_1 e^{-x} + c_2 e^{-2x}$

particular integral, $y_p = \frac{1}{D^2 + 3D + 2} x \sin 2x$

$= x \frac{\sin 2x}{D^2 + 3D + 2} - \frac{2D + 3}{(D^2 + 3D + 2)^2} \sin 2x$

$= x \frac{\sin 2x}{-4 + 3D + 2} - \frac{2D + 3}{(-4 + 3D + 2)^2} \sin 2x$

$= x \frac{1}{3D - 2} \sin 2x - \frac{2D + 3}{(3D - 2)^2} \sin 2x$

$= x \frac{(3D + 2)}{9D^2 - 4} \sin 2x - \frac{2D + 3}{9D^2 - 12D + 4} \sin 2x$

$= \frac{-x}{40} (3D + 2) \sin 2x - \frac{2D + 3}{-12D - 32} \sin 2x$

$= \frac{-x}{40} (6\cos 2x + 2\sin 2x) + \frac{1}{4} \frac{(2D + 3)}{3D + 8} \sin 2x$

$= \frac{-x}{40} (6\cos 2x + 2\sin 2x) + \frac{1}{4} \frac{(2D + 3)(3D - 8)}{9D^2 - 64} \sin 2x$

$= -\frac{x}{40} (6\cos 2x + 2\sin 2x) - \frac{1}{400} (6D^2 - 7D - 24) \sin 2x$

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$= \frac{-x}{40}(6\cos 2x + 2\sin 2x) - \frac{1}{400}(-24\sin 2x - 14\cos 2x - 24\sin 2x)$

$= \frac{-x}{40}(6\cos 2x + 2\sin 2x) + \frac{1}{400}(48\sin 2x + 14\cos 2x)$ (Ans:-)

Q. Solve $(D^2-1)y = x^2 \sin 3x$

Soln:- Let, $y = e^{mx}$ be the trial soln and the complementary function is $y_c = c_1 e^x + c_2 e^{-x}$

particular integral, $y_p = \frac{1}{D^2-1} x^2 \sin 3x$

$= x \frac{1}{D^2-1} x \sin 3x - \frac{2D}{(D^2-1)^2} x \sin 3x$

$= x^2 \frac{1}{D^2-1} \sin 3x - \frac{2D}{(D^2-1)^2} x \sin 3x - \frac{2D\, x \sin 3x}{D^4 - 2D^2 + 1} + \frac{2D(4D^3-4D)}{(D^4-2D^2+1)^2} \sin 3x$

$= x^2 \frac{1}{D^2-1} \sin 3x - x\frac{2D}{(D^2-1)^2} \sin 3x - 2D\left\{x \frac{1}{(D^2-1)^2} \sin 3x\right\}$

$+ \qquad + \frac{8D^2}{(D^2-1)^3} \sin 3x$

$= -\frac{1}{10} x^2 \sin 3x - \frac{3}{50} x \cos 3x - \frac{1}{50} D(x \sin 3x) + \frac{9}{125} \sin 3x$

$= \frac{-1}{10} x^2 \sin 3x - \frac{3}{25} x \cos 3x + \frac{9}{125} \sin 3x$

$\therefore y = y_c + y_p$

Ans: $= c_1 e^x + c_2 e^{-x} + \frac{9}{125} \sin 3x - \frac{1}{10} x^2 \sin 3x - \frac{3}{25} x \cos 3x$

(Ans:-)

Date :

Q. Solve $(D^3 - 3D^2 - 6D + 8)y = xe^{-3x}$

Sol^n:- Let, $y = e^{mx}$ be the trial sol^n and the complement function, $y_c = c_1x + c_2e^{4x} + c_3e^{-2x}$

particular integral. $y_p = \frac{1}{D^3 - 3D^2 - 6D + 8} xe^{-3x}$

$= e^{-3x} \frac{1}{(D-3)^3 - 3(D-3)^2 - 6(D-3) + 8} x$

$= e^{-3x} \frac{1}{D^3 - 3\times D^2 \times 3 + 3\times D \times 3^2 - 27 - 3(D^2 - 6D + 9) - 6D + 18 + 8} x$

$= e^{-3x} \frac{1}{D^3 - 9D^2 + 27D - 27 - 3D^2 + 18D - 27 - 6D + 18 + 8} x$

$= e^{-3x} \frac{1}{D^3 - 12D^2 + 39D - 28} x$

$= e^{-3x} \frac{1}{-28\left(1 - \frac{D^3}{28} + \frac{12}{28}D^2 - \frac{39}{28}D\right)} x$

$= \frac{e^{-3x}}{-28} \frac{1}{1 + \left(\frac{12D^2}{28} - \frac{D^3}{28} - \frac{39}{28}D\right)} x$

$= \frac{-e^{-3x}}{28} \left\{1 + \frac{39}{28}\right\} x$

$= e^{-3x}\left(\frac{-1}{28}x - \frac{39}{784}\right)$

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$\therefore\ y = y_c + y_p$

$= c_1e^{x} + c_2e^{4x} + c_3e^{-2x} + e^{-3x}\left(-\frac{1}{28}x - \frac{39}{784}\right)$ (Ans:)

Date :

partial differential equation

Def^n:- An equation involving one or more partial derivatives is called partial differential equation.

$$x\, \delta z/\delta x + y\, \delta z/\delta y = z \quad \text{--- (1)}$$

Linear PDE:- A PDE is said to be linear if it is of the 1st degree in the partial derivatives.

$$\delta u/dt = \left(\frac{\delta^2 u}{\delta x^2}\right)^1 + \frac{\delta u}{\delta x} \cdot \frac{\delta u}{\delta y}$$

Q. solve $(D^2 - D + 1)y = x^3 - 3x + 1$

sol^n:- Given that, $(D^2 - D + 1)y = x^3 - 3x + 1$ --- (i)

let $y = e^{mx}$ be the trial sol^n of $(D^2 - D + 1)y = 0$

Then the auxiliary equation is,

$m^2 - m + 1 = 0$

$\therefore m = \frac{-(-1) \pm \sqrt{1 - 4 \cdot 1 \cdot 1}}{2 \cdot 1}$

$= \frac{1 \pm \sqrt{-3}}{2}$

$= \frac{1 \pm \sqrt{3}i}{2}$

$\therefore$ The complementary solution is, $y_c = e^{x/2}\left(c_1 \cos \sqrt{3}/2\, x + c_2 \sin \sqrt{3}/2\, x\right)$.

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Now let $y_p = Ax^3 + Bx^2 + Cx + D$

$\therefore y_p' = 3Ax^2 + 2Bx + C$

$\therefore y_p'' = 6Ax + 2B$

From equation (i) we get,

$6Ax + 2B - 3Ax^2 - 2Bx - C + Ax^3 + Bx^2 + Cx + D = x^3 - 3x + 1$

$\Rightarrow Ax^3 + x^2(B - 3A) + x(6A - 2B + C) + 2B - C + D = x^3 - 3x + 1$

Equating the coefficients of both sides for like power of x we get, $A = 1$

$B - 3A = -3$

$\Rightarrow B - 3\cdot 1 = -3$

$\therefore B = 0$

$6A - 2B + C = 0$

$\Rightarrow C = 2B - 6A$

$\therefore C = -6$

$2B - C + D = 01$

$\Rightarrow D = 1 - 2B + C$

$\therefore D = -5$

Hence the particular sol^n, $y_p = x^3 - 6x - 5$

$\therefore y = y_c + y_p$

$= e^{x/2}\left(c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x\right) + x^3 - 6x - 5$

Date :

Lagrange's method:- If $u(x,y,z)=c_1$ and $v(x,y,z)=c_2$ be any two independent solⁿ of $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$, where P, Q, R are the function of x,y,z then $\varphi(u,v)=0$ or $v=\varphi(u)$ is a general solⁿ of the lagrange's linear equation $Pp+Qq=R$.

proof:- let $u=u(x,y,z)$ and $v=v(x,y,z)$ be any two function of x,y,z connected by $\varphi(u,v)=0$ --- (i)

partially differentiating (i) with respect x and y,

$$\frac{\delta\varphi}{\delta u}\left(\frac{\delta u}{\delta x}+\frac{\delta u}{\delta z}\cdot\frac{\delta z}{\delta x}\right)+\frac{\delta\varphi}{\delta v}\left(\frac{\delta v}{\delta x}+\frac{\delta v}{\delta z}\cdot\frac{\delta z}{\delta x}\right)=0 \quad \text{--- (ii)}$$

$$\frac{\delta\varphi}{\delta u}\left(\frac{\delta u}{\delta y}+\frac{\delta u}{\delta z}\cdot\frac{\delta z}{\delta y}\right)+\frac{\delta\varphi}{\delta v}\left(\frac{\delta v}{\delta y}+\frac{\delta v}{\delta z}\cdot\delta z/\delta y\right)=0 \quad \text{--- (iii)}$$

Now eliminating (ii) and (iii) $\delta\varphi/\delta u$ and $\delta\varphi/\delta v$

$$\begin{vmatrix} \delta u/\delta x+p\,\delta u/\delta z & \delta v/\delta x+p\,\delta v/\delta z \\ \delta u/\delta y+q\,\delta u/\delta z & \delta v/\delta y+q\,\delta v/\delta z \end{vmatrix}=0$$

where, $p=\frac{\delta z}{\delta x}$ and $q=\delta z/\delta y$

$$\Rightarrow(\delta u/\delta x+p\,\delta u/\delta z)(\delta v/\delta y+q\,\delta v/\delta z)-(\delta u/\delta y+p\,\delta u/\delta z)(\delta v/\delta x+q\,\delta v/\delta z)=0$$

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$\Rightarrow p\left(\frac{\delta u}{\delta z}\frac{\delta v}{\delta y} - \frac{\delta u}{\delta y}\frac{\delta v}{\delta z}\right) + q\left(\frac{\delta u}{\delta x}\frac{\delta v}{\delta z} - \frac{\delta u}{\delta z}\frac{\delta v}{\delta x}\right)$

$= \frac{\delta u}{\delta y}\frac{\delta v}{\delta x} - \frac{\delta u}{\delta x}\cdot\frac{\delta v}{\delta y}$

let. $\lambda P = \frac{\delta u}{\delta z}\frac{\delta v}{\delta y} - \frac{\delta u}{\delta y}\frac{\delta v}{\delta z}$

$\lambda Q = \frac{\delta u}{\delta x}\cdot\frac{\delta v}{\delta z} - \frac{\delta u}{\delta z}\cdot\frac{\delta v}{\delta x}$

$\lambda R = \frac{\delta u}{\delta y}\cdot\frac{\delta v}{\delta x} - \frac{\delta u}{\delta x}\cdot\frac{\delta v}{\delta y}$

From equation (i) $\lambda Pp + \lambda Qq = \lambda R$

$\therefore Pp + Qq = R$ ------(v)

which is partial differential equation in p and q and free from the arbitrary function $\phi(u,v)$. Also it is a lagranges linear equation.

let $u(x,y,z) = c_1$ and $v(x,y,z) = c_2$ be two integrals of (v).

Differentiating these with respect to x, y, z we get,

$\frac{\delta u}{\delta x}dx + \frac{\delta u}{\delta y}dy + \frac{\delta u}{\delta z}dz = 0$ ----(vi)

and $\frac{\delta v}{\delta x}dx + \frac{\delta v}{\delta y}dy + \frac{\delta v}{\delta z}dz = 0$ ----(vii)

Solving (vi) and (vii) we get,

$$\frac{dx}{\left(\frac{\delta u}{\delta y}\frac{\delta v}{\delta z} - \frac{\delta u}{\delta z}\frac{\delta v}{\delta y}\right)} = \frac{dy}{\left(\frac{\delta u}{\delta z}\cdot\frac{\delta v}{\delta x} - \frac{\delta v}{\delta z}\cdot\frac{\delta u}{\delta x}\right)}$$

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$= dz / (\delta u/\delta x \, \delta v/\delta y - \delta v/\delta x \cdot \delta u/\delta y)$

$\Rightarrow \frac{dx}{\lambda P} = \frac{dy}{\lambda Q} = \frac{dz}{\lambda R}$

$\Rightarrow \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

which is required auxiliary equation.

solve the PDE $\frac{y^2 z}{x} p + xzq = y^2$

Given that $\frac{y^2 z}{x} p + xzq = y^2$ ---- (i)

The lagrange's auxiliary eq^n is,

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$\Rightarrow \frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{xz} = \frac{dz}{y^2}$

Taking 1st and 2nd ratio,

$\frac{x\,dx}{y^2 z} = \frac{dy}{xz}$

$\Rightarrow x^2 dx = y^2 dy$

$\Rightarrow x^2 dx - y^2 dy = 0$

$\Rightarrow x^3/3 - y^3/3 = c$ [$\because$ by integration]

$\Rightarrow x^3 - y^3 = c_1$

Taking 1st and 3rd ratio,

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$$\frac{x\,dx}{y^2 z} = \frac{dz}{y^2}$$

$\Rightarrow x\,dx - z\,dz = 0$

$\Rightarrow x^2/2 - z^2/2 = c_2$ [$\because$ by integration]

$\therefore x^2 - z^2 = c_2$

Here the general solⁿ is $\phi(x^3-y^3, x^2-z^2)=0$ where ϕ is an arbitrary function.

Q. Solve the PDE $(3x+y-z)p + (x+y-z)q = 2(z-y)$

Given that $(3x+y-z)p + (x+y-z)q = 2(z-y)$ — (i)

The Lagrange's auxiliary equation is,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{3x+y-z} = \frac{dy}{x+y-z} = \frac{dz}{2(z-y)}$$

choosing $1,-3,-1$ as multiplier,

$$\frac{dx}{3x+y-z} = \frac{dy}{x+y-z} = \frac{dz}{2(z-y)} = \frac{dx-3dy-dz}{0}$$

$\therefore dx - 3dy - dz = 0$

$\Rightarrow x - 3y - z = c_1$ [$\because$ by integration]

Again choosing multiplier as $1,1,-1$ and $1,-1,1$

$$\frac{dx+dy-dz}{3x+y-z+x+y-z-2(z-y)} = \frac{dx-dy+dz}{3x+y-z-(x+y-z)+(z-y)2}$$

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$$\Rightarrow \frac{dx+dy-dz}{4(x+y-z)} = \frac{dx-dy+dz}{2(x-y+z)}$$

$$\Rightarrow \tfrac{1}{2}\log(x+y-z) = \log(x-y+z) + \log c_2$$

$$\Rightarrow \frac{\sqrt{x+y-z}}{x-y+z} = c_2$$

$\therefore$ Here the general sol^n is $\phi\left(x-3y-z, \frac{\sqrt{x+y-z}}{x-y+z}\right) = 0$ where ϕ is an arbitrary function.

Solve the PDE $(y+zx)p - (x+yz)q = x^2-y^2$

Given that $(y+zx)p - (x+yz)q = x^2-y^2$ --- (i)

The lagrange's auxiliary equation is,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{y+zx} = \frac{dy}{-(x+yz)} = \frac{dz}{x^2-y^2}$$

choosing $y, x, 1$ as multiplier,

$$\frac{dx}{y+zx} = \frac{dy}{-x-yz} = \frac{dz}{x^2-y^2} = \frac{ydx+xdy+dz}{0}$$

$\Rightarrow ydx + xdy + dz = 0$

$\Rightarrow d(xy) + dz = 0$

$\Rightarrow xy + z = c_1$ (by integration)

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Again choosing $x, y, -z$ as multiplier,

$$\frac{dx}{y+zx} = \frac{dy}{-x-yz} = \frac{dz}{x^2-y^2} = \frac{xdx+ydy-zdz}{0}$$

$\Rightarrow xdx + ydy - zdz = 0$

$\Rightarrow x^2 + y^2 - z^2 = c_2$ [$\because$ by integration]

Hence the general soln is $\varphi(xy+z,\ x^2+y^2-z^2) = 0$ (Ans:)

Solve the PDE $(y-z)p + (x-y)q = z-x$

Given that, $(y-z)p + (x-y)q = z-x$ - - - (i)

The lagrange's auxiliary equn are,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{y-z} = \frac{dy}{x-y} = \frac{dz}{z-x}$$

choosing multiplier as 1, 1, 1 we get,

$$\frac{dx}{y-z} = \frac{dy}{x-y} = \frac{dz}{z-x} = \frac{dx+dy+dz}{0}$$

$\Rightarrow dx + dy + dz = 0$

$\therefore x+y+z = c_1$ [$\because$ by integration]

Choosing multiplier as x, y, z and we get,

$$\frac{dx}{y-z} = \frac{dy}{x-y} = \frac{dz}{z-x} = \frac{xdx+ydy+zdz}{0}$$

$\Rightarrow xdx + ydy + zdz = 0$

$\therefore x^2+y^2+z^2 = c_2$ [$\because$ by integration]

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Hence the general soln is $\phi(x+y+z,\ x^2+y^2+z^2)=0$ where ϕ is an arbitrary function.

Find the integral surface of the linear PDE $x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$ which contain the line $x+y=0, z=1$

The lagrange's auxiliary eqn is,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{x^2-y^2}$$

choosing the multiplier, $x, y, -1$

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{x^2-y^2} = \frac{xdx+ydy-dz}{0}$$

$\Rightarrow xdx+ydy-dz=0$

$\Rightarrow x^2+y^2-2z=c_1$ [$\because$ by integration] ---- (i)

Again choosing the divider x, y, z

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)} = \frac{dx/x + dy/y + dz/z}{0}$$

$$\Rightarrow \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

$\therefore xyz = c_2$ [by integration] ------ (ii)

Since (i) and (ii) passes through $x+y=0, z=1$

$xy = c_2$ and $x^2+y^2-2 = c_2$

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$\Rightarrow (x+y)^2 - 2xy - z = c_1$

$\Rightarrow 0 - 2xy - z - c_1 = 0$

$\Rightarrow -2c_2 - z - c_1 = 0$

$\Rightarrow c_1 + 2c_2 + z = 0$

$\Rightarrow x^2 + y^2 - 2z + 2xyz + z = 0$

which is the required surface.

Find the eqn of the integral surface of the PDE $2y(z-3)p + (2x-z)q = y(2x-3)$ which passes through the circle $z=0$, $x^2+y^2=2x$.

The lagrange's auxiliary equation is,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Taking $1, 2y, -2$ as multiplier,

$$\frac{dx}{2y(z-3)} = \frac{dy}{2x-z} = \frac{dz}{y(2x-3)} = \frac{dx + 2y\,dy - 2dz}{0}$$

$\Rightarrow dx + 2y\,dy - 2dz = 0$

$\Rightarrow x^2 + y^2 - 2z = c_1$ (By integration) - - - (i)

Taking 1st and 3rd ratio,

$$\frac{dx}{2y(z-3)} = \frac{dz}{y(2x-3)}$$

$\Rightarrow (2x-3)\,dx = (z-3)\,dz$

$\Rightarrow x^2 - 3x - z^2 + 6z = c_2$ - - - - - (ii)

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Adding (i) and (ii),

$x^2+y^2-2x+4z-z^2 = c_1+c_2$

$\Rightarrow 2x-2x = c_1+c_2$

$\therefore c_1+c_2 = 0$

$\therefore x^2+y^2-2x+4z-z^2 = 0$ (Ans:-)

Find the equation of the integral surface of the PDE $(x-y)y^2p + (y-x)x^2q = (x^2+y^2)z$ which passes through as $xz = a^3,\ y=0$.

The lagrange's auxiliary equation is,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Taking $x^2, y^2, 0$ and $1, -1, 0$ as multiplier,

$$\frac{dx}{y^2(x-y)} = \frac{dy}{x^2(y-x)} = \frac{dz}{z(x^2+y^2)} = \frac{x^2dx+y^2dy}{0} = \frac{dx-dy}{(x-y)(x^2+y^2)}$$

$\therefore x^2dx + y^2dy = 0$

$\Rightarrow x^3+y^3 = c_1$ --- (i) [$\because$ by integration]

Taking 3rd and 5th ratio,

$$\frac{dz}{z} = \frac{dx-dy}{x-y}$$

$\Rightarrow \log z = \log(x-y) + \log c_2$

$\therefore z/x-y = c_2$ --- (ii) [$\because$ by integration]

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If it passes through $xz = a^3$, $y = 0$

from (i) $x^3 = c_1$ and

from (ii) $z/x = c_2$

$$\Rightarrow \frac{xz}{x^2} = c_2$$

$$\Rightarrow \frac{a^3}{x^2} = c_2$$

$$\Rightarrow \frac{a^9}{(x^3)^2} = c_2^3$$

$$\Rightarrow \frac{a^9}{c_1^2} = c_2^3$$

$$\Rightarrow c_1^2 c_2^3 = a^9$$

$$\Rightarrow (x^3+y^3) \frac{z^3}{(x-y)^3} = a^9$$

$\therefore$ $z^3 (x^3+y^3) = a^9 (x-y)^3$ (Ans:-)

. Find the eq^n of the integral surface of the PDE $4yzp + q + 2y = 0$ which passes through $y^2+z^2=1$, $x+z=2$.

The Lagrange's auxiliary eq^n is,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{4yz} = \frac{dy}{1} = \frac{dz}{-2y}$$

Taking 1st and 3rd ratio,

$$\frac{dx}{2z} = -\frac{dz}{1}$$

$\Rightarrow x + z^2 = c_1$ - - (i) [$\because$ by integration]

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Taking 2nd and 3rd ratio,

$$\frac{dy}{1} = \frac{dz}{-2y}$$

$\Rightarrow 2y\,dy + dz = 0$

$\Rightarrow y^2 + z = c_2$ - - (ii) [$\because$ by integration]

Adding (i) and (ii)

$x+z+z^2+y^2 = c_1+c_2$

$\Rightarrow 2+1 = c_1+c_2$

$\Rightarrow c_1+c_2 = 3$

$\therefore x+z^2+y^2+z = 3$ (Ans:-)

Non-linear PDE:-

Charpits method:- If $F(x, y, z, p, q) = 0$ then the charpits AE are,

$$\frac{dp}{\frac{\delta F}{\delta x} + p\frac{\delta F}{\delta z}} = \frac{dq}{\frac{\delta F}{\delta y} + q\frac{\delta F}{\delta z}} = \frac{dz}{-p\frac{\delta F}{\delta p} - q\,\delta F/\delta q} = \frac{dx}{-\delta F/\delta p} = \frac{dy}{-\delta F/\delta q}$$

Or, $$\frac{dp}{F_x + pF_z} = \frac{dq}{F_y + qF_z} = \frac{dz}{pF_p - qF_q} = \frac{dx}{-F_p} = \frac{dy}{-F_q}$$

Find the complete integral of the given partial differential equation by charpits method, $p^2 - y^2 q = y^2 - x^2$

Let $F(x, y, z, p, q) = p^2 - y^2 q - y^2 + x^2$ - - - - (i)

charpits AE are,

$$\frac{dp}{F_x + pF_z} = \frac{dq}{F_y + pF_z} = \frac{dz}{-pF_p - qF_q} = \frac{dx}{-F_p} = \frac{dy}{-F_q}$$

$$\Rightarrow \frac{dp}{2x + 0} = \frac{dq}{-2yq - 2y + 0} = \frac{dz}{-2p^2 + y^2 q} = \frac{dx}{-2p} = dy/y^2$$

Taking 1st and 4th ratio,

$$\frac{dp}{2x} = \frac{dx}{-2p}$$

$\Rightarrow p^2 + x^2 = c_1$ -- (ii)

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solving (i) and (ii). $p = \sqrt{c_1 - x^2}$

$q = c_1/y^2 - 1$

We know. $dz = p\,dx + q\,dy$

$= \sqrt{c_1 - x^2}\,dx + (c_1/y^2 - 1)\,dy$

$\therefore z = \dfrac{x\sqrt{c_1 - x^2}}{2} + c_1/2 \sin^{-1}\dfrac{x}{\sqrt{c_1}} - \dfrac{c_1}{y} - y + k$ (Ans:)

Find the complete integral surface of PDE by charpits method. $z^2(p^2 z^2 + q^2) = 1$

Let $F(x, y, z, p, q) = p^2 z^4 + z^2 q^2 - 1$ - - - (i)

charpits AE are,

$$\frac{dp}{F_x + pF_z} = \frac{dq}{F_y + qF_z} = \frac{dz}{-pF_p - qF_q} = \frac{dx}{-F_p} = \frac{dy}{-F_q}$$

$$\Rightarrow \frac{dp}{0 + p(4p^2 z^3 + 2zq^2)} = \frac{dq}{0 + q(4p^2 z^3 + 2zq^2)}$$

$$= \frac{dz}{-p\cdot 2pz^4 - q\cdot 2qz^2} = \frac{dx}{-2pz^4} = \frac{dy}{-2qz^2}$$

Taking 1st and 2nd ratio,

$dp/p = dq/q$

$\therefore p = cq$ - - - - (ii)

Solving (i) and (ii) $p = \frac{c}{z\sqrt{c^2z^2+1}}$

$$q = \frac{1}{z\sqrt{c^2z^2+1}}$$

We know, $dz = p\,dx + q\,dy$

$$\Rightarrow dz = \frac{c\,dx}{z\sqrt{c^2z^2+1}} + \frac{1\,dy}{z\sqrt{c^2z^2+1}}$$

$$\Rightarrow \cancel{z=}\, z\sqrt{c^2z^2+1}\,dz = c\,dx + dy$$

$$\Rightarrow \frac{1}{3c^2}(c^2z^2+1)^{3/2} = cx + y + k \quad [\because \text{by integration}] \text{ (Ans:-)}$$

Find the complete integral of the PDE by Charpits method. $pxy + pq + qy = yz$

Let, $F(x, y, z, p, q) = pxy + pq + qy - yz = 0$ ---(i)

Charpits AE are,

$$\frac{dp}{F_x + pF_z} = \frac{dq}{F_y + qF_z} = \frac{dz}{-pF_p - qF_q} = \frac{dx}{-F_p} = \frac{dy}{-F_q}$$

$$\Rightarrow \frac{dp}{py + p(-y)} = \frac{dq}{q + px - qy - \cancel{z}} = \frac{dz}{-pxy - pq - pq - qy} = \frac{dx}{-xy - q} = \frac{dy}{-(p+y)}$$

Taking 1st ratio. $dp = 0$

$\therefore p = c$ ---(ii)

Solving (i) and (ii) $p = c$

$$q = \frac{yz - cxy}{c + y}$$

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We know, $dz = p\,dx + q\,dy$

$\Rightarrow dz = c\,dx + \frac{y(z-cx)}{c+y}\,dy$

$\Rightarrow \frac{dz - c\,dx}{z-cx} = (1 - \frac{c}{c+y})dy$

$\Rightarrow \log(z-cx) = y - c\log(c+y) + \log K$

$\Rightarrow z - cx = e^{y}K(c+y)^{-c}$

$\therefore z = cx + e^{y}K(c+y)^{-c}$ (Ans:-)

Solve $16p^2z^2 + 9q^2z^2 + 4z^2 - 4 = 0$ by charpits method and identity the surface.

let, $F(x,y,z,p,q) = 16p^2z^2 + 9q^2z^2 + 4z^2 - 4 = 0$ --- (i)

The charpits AE are,

$$\frac{dp}{F_x + pF_z} = \frac{dq}{F_y + qF_z} = \frac{dz}{-pF_p - qF_q} = \frac{dx}{-F_p} = \frac{dy}{-F_q}$$

$$\Rightarrow \frac{dp}{32p^3z + 18pq^2z + 8pz} = \frac{dq}{32p^2qz + 18q^3z + 8qz} = \frac{dz}{-32p^2z^2 - 18q^2z^2}$$

$$= \frac{dx}{-32pz^2} = \frac{dy}{-18qz^2} = \frac{dx + 4p\,dz + 4z\,dp}{0}$$

$\therefore dx + 4p\,dz + 4z\,dp = 0$

$\Rightarrow dx + 4d(pz) = 0$

$\therefore x + 4pz = a$ [$\because$ by integration]

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$\therefore p = \frac{a-x}{4z}$ --- (ii)

solving (i) and (ii)

$\cancel{16} \frac{(a-x)^2}{\cancel{16} z^2} \cdot \cancel{z^2} + 9q^2 z^2 + 4z^2 - 4 = 0$

$\Rightarrow 9q^2 z^2 = 4 - 4z^2 - (x-a)^2$

$\Rightarrow q = \frac{2}{3z} \sqrt{1 - z^2 - 1/4 (a-x)^2}$

Now we know :- $dz = p\,dx + q\,dy$

$\Rightarrow dz = \frac{a-x}{4z} dx + \frac{2}{3z} \sqrt{1 - z^2 - 1/4 (a-x)^2}\, dy$

$\Rightarrow 3/2 \; \frac{z\,dz - 1/4 (a-x)\, dx}{\sqrt{1 - z^2 - 1/4 (a-x)^2}} = dy$

$\Rightarrow -3/2 \sqrt{1 - z^2 - 1/4 (a-x)^2} = y + c$

which is the required soln and a and c constant.

$\Rightarrow 9/4 \; (1 - z^2 - 1/4 (a-x)^2) = (y+c)^2$

$\therefore 4/9 (y+c)^2 + 1/4 (a-x)^2 + z^2 = 1$

which represents an ellipsoid.

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Solve $p^2+q^2 = py - qx$ by charpits method

let $F(x, y, z, p, q) = p^2+q^2 - py + qx = 0$ ---(i)

charpits auxiliary Equation are,

$$\frac{dp}{q} = \frac{dq}{-p} = \frac{dz}{-2p^2+py-2q^2-qx} = \frac{dx}{-2q+y} = \frac{dy}{-2q-x}$$

Taking 1st and 2nd ratio,

$pdp + qdq = 0$

$\Rightarrow p^2+q^2 = a$

$\therefore p = \sqrt{a-q^2}$

From(i) we get,

$a-\cancel{q^2}+\cancel{q^2} - \sqrt{a-q^2}\cdot y + qx = 0$

$\Rightarrow (a+qx)^2 = (a-q^2)y^2$

$\Rightarrow a^2+2aqx + q^2x^2 = ay^2 - q^2y^2$

$\Rightarrow q^2(x^2+y^2) + 2axq + a^2 - ay^2 = 0$

$$\therefore q = \frac{-2ax \pm \sqrt{4a^2x^2 - 4(x^2+y^2)\cdot(a^2-ay^2)}}{2(x^2+y^2)}$$

$$= \frac{-ax \pm \sqrt{\cancel{a^2x^2} - \cancel{a^2x^2} + ay^2x^2 - a^2y^2 + ay^4}}{(x^2+y^2)}$$

$$= \frac{-ax \pm \sqrt{ay^2(x^2+y^2-a)}}{(x^2+y^2)}$$

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$$p = \left\{ a - \left(\frac{-ax \pm \sqrt{ay^2(x^2+y^2-a)}}{(x^2+y^2)} \right)^2 \right\}^{1/2}$$

$\therefore\ dz = p\,dx + q\,dy$

$$= \left\{ a - \left(\frac{-ax \pm \sqrt{ay^2(x^2+y^2-a)}}{(x^2+y^2)} \right)^2 \right\}^{1/2} dx + \frac{-ax \pm \sqrt{ay^2(x^2+y^2-a)}}{(x^2+y^2)}\, dy$$

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Find the complete singular integral $(p^2+q^2)y = qz$.

let $F(x,y,z,p,q) = (p^2+q^2)y - qz = 0$ ---- (i)

The charpits AE are,

$$\frac{dp}{F_x + pF_z} = \frac{dq}{F_y + qF_z} = \frac{dz}{-pF_p - qF_q} = \frac{dx}{-F_p} = \frac{dy}{-F_q}$$

$$\Rightarrow \frac{dp}{-pq} = \frac{dq}{p^2+q^2-q^2} = \frac{dz}{-2p^2y - 2q^2y + qz} = \frac{dx}{-2py} = \frac{dy}{-2qy+z}$$

Taking 1st and 2nd ratio,

$$\Rightarrow \frac{dp}{-pq} = \frac{dq}{p^2}$$

$\Rightarrow pdp + qdq = 0$

$\therefore p^2+q^2 = a$ [$\because$ by integration]

Solving (i) and (ii)

$$q = \frac{ay}{z} \text{ and } p = \frac{\sqrt{az^2 - a^2y^2}}{z}$$

we know. $dz = pdx + qdy$

$$\Rightarrow dz = \frac{\sqrt{az^2-a^2y^2}}{z}dx + \frac{ay}{z}dy$$

$$\Rightarrow \frac{azdz - a^2ydy}{\sqrt{az^2-a^2y^2}} = adx$$

$\Rightarrow \sqrt{az^2 - a^2y^2} = ax + b$ [$\because$ by integration]

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$\therefore a^2z^2 - a^2y^2 = (ax+by)^2$ - - - (iii)

Differenting (iii) with respect to a and b we get,

$z^2 - 2ay = 2x(ax+b)$ - - - (iv)

$0 = 2(ax+b)$ - - - (v)

solving (iv) and (v)

$a = z^2/2y^2$

$b = -\frac{xz^2}{2y^2}$

putting the value of a and b in (iii),

$\frac{z^4}{2y^2} - \frac{z^4}{4y^2} = 0$ (Ans:-)

4. Solve $2z + p^2 + qy + 2y^2 = 0$

let $F(x.y.z.p.q) = 2z + p^2 + qy + 2y^2 = 0$ - - (i)

The charpits auxillary AE are.

$$\frac{dp}{F_x + pF_z} = \frac{dq}{F_y + qF_z} = \frac{dz}{-pF_p - qF_q} = \frac{dx}{-F_p} = dy/_-$$

$$\Rightarrow \frac{dx}{-2p} = \frac{dy}{-y} = \frac{dz}{-2p^2 - qy} = \frac{dp}{2p} = \frac{dq}{q+4y+2q}$$

Taking 1st and 4th ratio, $dx + dp = 0$

$\Rightarrow x + p = a$

$\therefore p = a - x$ [$\because$ by integra

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putting these values in (i)

$$q = \frac{-2z-(a-x)^2-2y^2}{y}$$

we have, $dz = p\,dx + q\,dy$

$$\Rightarrow dz = (a-x)\,dx - \left\{\frac{2z+(a-x)^2+2y^2}{y}\right\}dy$$

$$\Rightarrow y\,dz = y(a-x)\,dx - \{2z+(a-x)^2+2y^2\}\,dy$$

$$\Rightarrow 2y^2 dz = 2y^2(a-x)\,dx - 4yz\,dy - 2y(a-x)^2 dy - 4y^3 dy$$

$$\Rightarrow 2y^2 dz + 4yz\,dy + 2y^2(x-a)\,dx + 2y(x-a)^2 dy + 4y^3 dy = 0$$

$$\therefore 2y^2 z + y^2(x-a)^2 + y^4 = c \quad [\because \text{ by integration}]$$

4. Find a complete integral of $q^2 = z^2p^2(1-p^2)$ by charpits method.

Let $F(x,y,z,p,q) = q^2 - z^2p^2 + z^2p^4 = 0$ ---(i)

$$\frac{dp}{0+p\times(2zp^4-2zp^2)} = \frac{dq}{0+q(2zp^4-2zp^2)} = \frac{dz}{2p^2z^2-\cancel{2q^2}-4p^4z^2}$$

$$= \frac{dx}{-4p^3z^2+2pz^2} = \frac{dy}{-2q}$$

Taking 1st and 2nd ratio,

$$\frac{dp}{p} = \frac{dq}{q} \quad \therefore p = aq \quad \text{--- (ii)}$$

$q^2 = z^2 p^2 (1 - p^2)$

$\Rightarrow q^2 = z^2 a^2 q^2 (1 - a^2 q^2)$

$\Rightarrow q^2 = z^2 a^2 q^2 - z^2 a^4 q^4$

$\Rightarrow 1 = z^2 a^2 - z^2 a^4 q^2$

$\Rightarrow z^2 a^4 q^2 = z^2 a^2 - 1$

$\Rightarrow q^2 = \dfrac{z^2 a^2 - 1}{z^2 a^4}$

$\therefore q = \dfrac{\sqrt{z^2 a^2 - 1}}{z \cdot a^2}$

$\therefore p = \dfrac{a\sqrt{z^2 a^2 - 1}}{z a^2}$

Hence, $dz = p\,dx + q\,dy$

$\Rightarrow dz = \dfrac{a\sqrt{z^2 a^2 - 1}}{z a^2}\,dx + \dfrac{\sqrt{z^2 a^2 - 1}}{z a^2}\,dy$

$\Rightarrow \dfrac{z a^2\,dz}{\sqrt{z^2 a^2 - 1}} = a\,dx + dy$

$\Rightarrow z^2 a^2 - 1 = c$

$\Rightarrow 2 z a^2 dz = dc$

$\Rightarrow \dfrac{1}{2} \times \dfrac{dc}{\sqrt{c}} = ax + y + k$

$\therefore \sqrt{z^2 a^2 - 1} = ax + y + k$ . . (Ans:-)

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Find the complete and singular solⁿ of $2xz - px^2 - 2qxy + pq = 0$ by charpits method.

Let $F(x,y,z,p,q) = 2zx - px^2 - 2qxy + pq = 0$ - - - (i)

$$\frac{dp}{2z - 2px^2 - 2qy + 2px} = \frac{dq}{-2qx + 2qx} = \frac{dz}{x^2 - pq + 2qxy - pq}$$

$$= \frac{dx}{x^2 - q} = \frac{dy}{2xy - p}$$

Taking 2nd ratio, $dq = 0$

$\therefore q = a$ - - (ii)

From (i) we get, $2xz - px^2 - 2axy + ap = 0$

$\Rightarrow p(a - x^2) = 2axy - 2xz$

$$\Rightarrow p = \frac{2x(ay - z)}{(a - x^2)}$$

$\therefore dz = pdx + qdy$

$$\Rightarrow dz = \frac{2x(ay - z)}{a - x^2} dx + a\,dy$$

$$\Rightarrow \frac{dz - a\,dy}{ay - z} = \frac{2x\,dx}{a - x^2}$$

$\therefore ay - z = c$

$\Rightarrow a\,dy - dz = dc$

$\Rightarrow dz - a\,dy = -dc$

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$\therefore\ a - x^2 = c$

$\Rightarrow -2x\,dx = dc$

$\Rightarrow 2x\,dx = -dc$

$\Rightarrow \dfrac{-dc}{c} = \dfrac{-dc}{c}$

$\Rightarrow \log c = \log c + \log k$

$\Rightarrow \log(ay - z) = \log(a - x^2) + k$

$\Rightarrow ay - z = (a - x^2)k$

$\therefore ay - k(a - x^2) = z$ ---- (iii)

Differentiating (iii) with respect to a and k

$0 = y - ak + kx^2$

$\Rightarrow 0 = y - k + 0$

$\therefore y = k$

$0 = 0 - a + x^2$

$\therefore x^2 = a$

$\therefore$ (iii) we get, $\cancel{ak}\ x^2y - y(\cancel{x^2} - \cancel{x^2}) = z$

$\therefore\ z = x^2y$ (Ans:-)

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Boundary value problem

Q. Find a solⁿ of one dimensional heat conducting equation $\frac{\delta v}{\delta t} = k\frac{\delta^2 v}{\delta x^2}$ with the condition $v \neq \alpha,\ t = \alpha$

$v = 0,\ x = 0$ or π

$v = 100,\ t = 0,\ 0 < x < \pi$

Solⁿ:-

Given that, $\frac{\delta v}{\delta t} = k\frac{\delta^2 v}{\delta x^2}$ --- (i)

let $v = 100 - v'$

$\therefore \frac{\delta v}{\delta t} = -\frac{\delta v'}{\delta t}$ and $\frac{\delta^2 v}{\delta x^2} = -\frac{\delta^2 v'}{\delta x^2}$

From (i) we get,

$\frac{\delta v'}{\delta t} = k\frac{\delta^2 v'}{\delta x^2}$ --- (ii)

$v' \neq \alpha,\ t = \alpha$

$v' = 0,\ x = 0$ or π

$v' = 100,\ t = 0,\ 0 < x < \pi$

let, $v'(x,t) = F(x), f(t)$ be a trial solⁿ of $\frac{\delta v'}{\delta t} = F(x)f'(t)$

$\therefore \frac{\delta^2 v'}{\delta x^2} = F''(x)f(t)$

putting these values in (ii)

$F(x)f'(t) = kF''(x)f(t)$

$\Rightarrow \frac{f'(t)}{kf(t)} = \frac{F''(x)}{F(x)} = -p^2$ (say)

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$\therefore f'(t) + kp^2 f(t) = 0$

$\therefore f(t) = Ae^{-kp^2 t}$

and $F''(x) + p^2 F(x) = 0$

$\therefore F(x) = B\cos px + c\sin px$

$\therefore v' = Ae^{-kp^2 t}(B\cos px + c\sin px)$

$= e^{-kp^2 t}(L\cos px + M\sin px)$ - - - (iii)

where $L = AB$, $M = AC$

For the condition. $v = 0$ if $x = 0$

From eqn (iii) $0 = Le^{-kp^2 t}$

$\therefore L = 0$

$\therefore v(x,t) = Me^{-kp^2 t}\sin px$ - - - (iv)

For $v' = 0$ if $x = \pi$

$0 = Me^{-kp^2 t}\sin p\pi$

$\Rightarrow \sin p\pi = 0 = \sin n\pi$

$\therefore p = n$

From (iv) we get,

$$v(x,t) = \sum_{n=1}^{\alpha} M_n e^{-kn^2 t}\sin n\pi \quad - - - (v)$$

For $v' = 100$, if $t = 0$, $0 < x < \pi$

$$\therefore 100 = \sum_{n=1}^{\alpha} M_n \sin nx$$

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$\Rightarrow M_n = 2/\pi \int_0^{\pi} 100 \sin nx \, dx$

$= \frac{200}{\pi}\left[-\frac{\cos n\pi}{n}\right]_0^{\pi}$

$= \frac{200}{n\pi}\left[1-(-1)^n\right]$

From equation (v) we get,

$v'(x,t) = \sum_{n=1}^{\alpha} \frac{200}{n\pi}\left[1-(-1)^n\right] e^{-Kn^2 t} \sin nx$

$\Rightarrow 100 - v = \sum_{n=1}^{\alpha} \frac{200}{n\pi}\left[1-(-1)^n\right] e^{-Kn^2 t} \sin nx$

$\therefore v = 100 - \sum_{n=1}^{\alpha} \frac{200}{n\pi}\left[1-(-1)^n\right] e^{-Kn^2 t} \sin nx$ (Ans:-)

$\boxed{4}$. Solve the boundary value problem,

$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y} = 0$ for $0 < x < \pi$

$u(0,y) = u(\pi,y) = u(x,\pi) = 0$

$u(x,0) = \sin^2 x$

sol^n:- Given that, $\delta^2 u/\delta x^2 + \frac{\delta^2 u}{\delta y^2} = 0$ - - (1)

let $u(x,y) = F(x) f(y)$ be trial sol^n of (1)

$\frac{\delta^2 u}{\delta x^2} = F''(x) f(y)$

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$$\frac{\delta^2 u}{\delta y^2} = F(x) f''(y)$$

putting these values in eqn (i) we get,

$F''(x) f(y) + F(x) f''(y) = 0$

$\Rightarrow \frac{F''(x)}{F(x)} = -\frac{f''(y)}{f(y)} = -k^2$ (say)

$\therefore F''(x) + k^2 F(x) = 0$

$\therefore F(x) = A\cos kx + B\sin kx$

and $f''(y) - k^2 f(y) = 0$

$\Rightarrow f(y) = Ce^{ky} + De^{-ky}$

$\therefore u(x,y) = (A\cos kx + B\sin kx)(Ce^{ky} + De^{-ky})$ - - (ii)

For the condition $u(0,y) = 0$

from (ii) $0 = A(Ce^{ky} + De^{-ky})$

$\therefore A = 0$

$\therefore u(x,y) = B\sin kx \cdot (Ce^{ky} + De^{-ky})$

for $u(\pi, y) = 0$, $0 = B\sin k\pi \, (Ce^{ky} + De^{-ky})$

$\Rightarrow \sin k\pi = 0 = \sin n\pi$

$\therefore k = n$

$\therefore u(x,y) = \sum_{n=1}^{\infty} \sin nx \left[L_n e^{ny} + M_n e^{-ny}\right]$ where $L_n = BC$, $M_n = BD$

for $u(x, \pi) = 0$

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$\therefore 0 = \sum_{n=1}^{\infty} \sin n\pi \left[L_n e^{n\pi} + M_n e^{-n\pi} \right]$

$\Rightarrow L_n e^{n\pi} + M_n e^{-n\pi} = 0$

$\Rightarrow M_n = -L_n e^{2n\pi}$

$\therefore u(x,y) = \sum_{n=1}^{\infty} L_n \sin n\pi (e^{ny} - e^{-ny} \cdot e^{2\pi n}) \quad ----\text{(iii)}$

For the condition, $u(x,0) = \sin^2 x$

$\sin^2 x = \sum_{n=1}^{\infty} L_n \sin n\pi (1 - e^{2n\pi})$

$\therefore L_n (1 - e^{2n\pi}) = 2/\pi \int_0^{\pi} \sin^2 x \cdot \sin nx \, dx$

$= 2/\pi \int_0^{\pi} 1/2 (1 - \cos 2x) \cdot \sin nx \, dx$

$= \frac{1}{\pi} \int_0^{\pi} \sin nx \, dx - 1/\pi \int_0^{\pi} 1/2 \left[\sin (n+2)x + \sin (n-2)x \right] dx$

$= 1/n\pi \left[1 - (-1)^n \right] + \frac{1}{2\pi} \left[\frac{\cos (n+2)\pi}{n+2} + \frac{\cos (n-2)\pi}{n-2} - 1/n+2 - \frac{1}{n-2} \right]$

$= \frac{1}{n\pi} \left[1 - (-1)^n \right] + \frac{1}{2\pi} \left[\frac{(-1)^n}{n+2} + \frac{(-1)^n}{n-2} - \frac{2n}{n^2-4} \right]$

$= \frac{1}{n\pi} \left[1 - (-1)^n \right] + 1/2\pi \left[\frac{2n(-1)^n}{n^2-4} - 2n/n^2-4 \right]$

$= 1/\pi \left[1 - (-1)^n \right] \left[\frac{1}{n} - \frac{n}{n^2-4} \right]$

$\therefore L_n = \frac{1}{\pi (1 - e^{2n\pi})} \left[1 - (-1)^n \right] \left[\frac{-4}{n(n^2-4)} \right]$

$\therefore u(x,y) = \sum_{n=1}^{\infty} \frac{1}{\pi (1 - e^{2n\pi})} \frac{4}{n(n^2-4)} \left[(-1)^n - 1 \right] \sin nx \left[e^{ny} - e^{-ny} e^{2n\pi} \right]$

(Ans:-)

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Series Solution

power series:- A series of the form $\sum_{n=0}^{\infty} c_n x^n$ is called power series.

□ Define an ordinary point, a singular point and a regular singlarpoint and a irregular singular point of $a_0(x) y'' + a_1(x) y' + a_2(x) y = 0$

soln:- consider the differential equation,

$a_0(x) y'' + a_1(x) y' + a_2(x) y = 0$ ---- (i)

It's normalized differential eqn is $y'' + P_1 x y' + P_2(x) y = 0$

where $P_1 = \frac{a_1(x)}{a_0(x)}$, $P_2(x) = \frac{a_2(x)}{a_0(x)}$

Ordinary point:- The point $x = x_0$ is called an ordinary point of the DE(i) if both of the function P_1 and P_2 are analytic at $x = x_0$.

Singular point:- The point $x = x_0$ is called a singular point of the DE(i) if either one or both of these function is not analytic at $x = x_0$.

Regular singular point:- The singular point $x = x_0$ is called a regular singular point of DE(i) if both $(x - x_0) P_1(x)$ and $(x - x_0)^2 P_2(x)$ are analytic at $x = x_0$.

Irregular singular point:- The singular point $x = x_0$ is called an irregular singular point of DE(i) if both $(x-x_0) P_1(x)$ and $(x-x_0)^2 P_2(x)$ are not analytic at $x = x_0$.

Frobenious method

(1) solve $2x^2y'' - xy' + (x-5)y = 0$ by frobenious method.

soln:- Given that, $2x^2y'' - xy' + (x-5)y = 0$ --- (i)

Here, $\left.\begin{aligned} P_1(x) &= -\frac{x}{2x^2} \\ P_2(x) &= \frac{x-5}{2x^2} \end{aligned}\right\}$ are not analytic at $x_0 = 0$

x_0 is a singular point.

Now $\left.\begin{aligned} xP_1(x) &= -\frac{x}{2x} = -\frac{1}{2} \\ \text{and } x^2P_2(x) &= \frac{x-5}{2} \end{aligned}\right\}$ are both analytic at $x = 0$

So $x = 0$ is a regular singular point for (i).

According to Frobenious method the DE (i) has the solution of the form,

$$y = \sum_{n=0}^{\infty} c_n x^{n+r} \quad \text{---- (ii)}$$

$$= x^r (c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots) \quad (A)$$

$$\left.\begin{aligned} y' &= \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} \quad \text{---- (iii)} \\ y'' &= \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2} \end{aligned}\right\}$$

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putting (ii) and (iii) in eq^n (i) we get,

$$\sum_{n=0}^{\alpha} \left[2(n+r)(n+r-1)c_n x^{n+r} - (n+r)c_n x^{n+r} + c_n x^{n+r+1} - 5c_n x^{n+r}\right] = 0$$

$$\Rightarrow \sum_{n=0}^{\alpha} \left[2(n+r)(n+r-1)c_n x^{n+r} - (n+r)c_n x^{n+r} - 5c_n x^{n+r}\right] + \sum_{n=1}^{\alpha} c_{n-1} x^{n+r} = 0$$

$$\Rightarrow \left[2r(r-1) - r - 5\right] c_0 x^r + \sum_{n=1}^{\alpha} \left[\{2(n+r)(n+r-1) - (n+r) - 5\} c_n + c_{n-1}\right] x^{n+r} = 0 \quad ------(iv)$$

$$\because \sum_{n=0}^{\alpha} c_n x^{n+r+1} = \sum_{m=1}^{\alpha} c_m x^{m+r}$$

put $m = n+1$

$n = 0$ to α

$m = 1$ to α

Equating the coefficient of x^r equal to zero we get the indicial eq^n is, $\{2r(r-1) - r - 5\} c_0 = 0$, $c_0 \neq 0$

$$\therefore 2r^2 - 2r - r - 5 = 0$$

$$\therefore r = 5/2, -1$$

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Again from eqⁿ (iv) the recurrence solⁿ is,

$$c_n = -\frac{c_{n-1}}{2(n+r)(n+r-1) - (n+r) - 5} \quad \text{--- (v)}$$

Case-1:- when $r = 5/2$ then

$$c_n = -\frac{c_{n-1}}{(n+5/2)(n+5/2-1) - (n+5/2) - 5}$$

$$\therefore c_n = -\frac{c_{n-1}}{n(2n+7)} ; \; n \geq 1$$

putting $n = 1, 2, 3 - - -$ (vi)

$$c_1 = \frac{-c_0}{9}$$

$$c_2 = -\frac{c_1}{2(4+7)} = -\frac{c_1}{22} = \frac{c_0}{22 \times 9}$$

$$c_3 = -c_2/39 = \frac{-c_0}{39 \times 22 \times 9}$$

putting these in (A) we get,

$$y_1 = c_0 x^{5/2}\left(1 - \tfrac{1}{9}x + \frac{1}{22 \times 9}x^2 - \frac{1}{39 \times 22 \times 9}x^3 + \cdots\right) \quad \text{--- (vi)}$$

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Case-II:- when $r = -1$ then,

$$c_n = -\frac{c_{n-1}}{2(n-2)(n-1)-(n-1)-5} = -\frac{c_{n-1}}{2n^2-7n} ; \quad n \geq 1$$

when $n = 1, 2, 3 \ldots$ we get,

$$c_1 = \frac{-c_0}{-5} = \frac{1}{5} c_0 , \quad c_2 = -\frac{c_1}{-6} = \frac{1}{6} \cdot \frac{1}{5} c_0 = \frac{1}{30} c_0$$

$$c_3 = -\frac{c_2}{-3} = \frac{1}{3} \cdot \frac{1}{30} \cdot c_0 = \frac{1}{90} c_0$$

putting these values in eq^n (A) we get,

$$y_2 = c_0 \frac{1}{x} \left(1 + \frac{1}{5}x + \frac{1}{30}x^2 + \frac{1}{90}x^3 + \ldots\right) \ldots \text{(vii)}$$

Hence the general sol^n is,

$$y = k_1 y_1 + k_2 y_2$$

$$= k_1 c_0 x^{5/2} \left(1 - \frac{1}{9}x + \frac{1}{22\times 9}x^2 - \frac{1}{39\times 22 \times 9}x^3 + \ldots\right)$$

$$+ k_2 c_0 \frac{1}{x} \left(1 + \frac{1}{5}x + \frac{1}{30}x^2 + \frac{1}{90}x^3 + \ldots\right)$$

$$= A x^{5/2} \left(1 - \frac{1}{9}x + \frac{1}{22\times 9}x^2 - \frac{1}{39\times 22\times 9}x^3 + \ldots\right) + B \frac{1}{x} \left(1 + \frac{1}{5}x + \frac{1}{30}x^2 + \frac{1}{90}x^3 + \ldots\right) \text{ (Ans:)}$$

where $A = k_1 c_0$, $B = k_2 c_0$ are arbitrary constant.

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Using the frobenious method solve $x^2y'' - xy' + (x^2+1)y = 0$

Soln:- Given that, $x^2y'' - xy' + (x^2+1)y = 0$ ----- (i)

Here, $P_1(x) = -x/x^2$, $P_2(x) = \dfrac{x^2+1}{x}$ } are not analytic at $x=0$, so $x=0$ is a singular point. (i)

Now, $xP_1(x) = -1$, $x^2P_2(x) = x^2+1$ } are both analytic at $x=0$. so $x=0$ is a regular singular point of (i)

Hence by frobenious method the soln of the form,

$$y = \sum_{n=0}^{\alpha} c_n x^{n+r} \;;\; c_0 \neq 0, \; c_n = 0, \; n > 0 \quad \text{--- (ii)}$$

$$= x^r (c_0 + c_1 x + c_2 x^2 + \cdots) \quad \text{--- (A)}$$

$$\left.\begin{aligned} y' &= \sum_{n=0}^{\alpha} (n+r) c_n x^{n+r-1} \\ y'' &= \sum_{n=0}^{\alpha} (n+r)(n+r-1) c_n x^{n+r-2} \end{aligned}\right\} \quad \text{--- (iii)}$$

putting (ii) and (iii) in eqn (i) we get,

$$\Rightarrow \sum_{n=0}^{\alpha} (n+r)(n+r-1) c_n x^{n+r} - \sum_{n=0}^{\alpha} (n+r) c_n x^{n+r} + \sum_{n=0}^{\alpha} c_n x^{n+r+2} + \sum_{n=0}^{\alpha} c_n x^{n+r} = 0$$

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$$\Rightarrow \sum_{n=0}^{\alpha} (n+r)(n+r-1) c_n x^{n+r} - \sum_{n=0}^{\alpha} (n+r) c_n x^{n+r} + \sum_{n=2}^{\alpha} c_{n-2} x^{n+r} + \sum_{n=0}^{\alpha} c_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=0}^{\alpha} \{(n+r)(n+r-1-1 \cancel{+1}) + 1\} c_n x^{n+r} + \sum_{n=2}^{\alpha} c_{n-2} x^{n+r} = 0$$

$$\Rightarrow \{r(r-2)+1\} c_0 x^r + \{(r+1)(r-1)+1\} c_1 x^{r+1} + \sum_{n=2}^{\alpha} \{(n+r)(n+r-2)+1\} c_n x^{n+r} + \sum_{n=2}^{\alpha} c_{n-2} x^{n+r} = 0$$

Equating the coefficient of x^r and x^{r+1} to zero,

$\{r(r-2)+1\} c_0 = 0$

$\therefore r = 1, 1 \ [\because c_0 \neq 0]$

Again $\{(r+1)(r-1)+1\} c_1 = 0$

since $(r+1)(r-1)+1 \neq 0$, $c_1 = 0$

Equating the coefficient of x^{n+r} to zero,

$$c_n = \frac{-c_{n-2}}{(n+r)(n+r-2)+1} ; n \geq 2$$

putting $n = 2, 3, 4 \cdots \quad c_2 = \frac{-c_0}{(2+r)r+1} = \frac{-c_0}{r(r+2)+1} = -c_0/(r+1)^2$

since $c_1 = 0$, so $c_3 = c_5 = 0 \cdots$

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$c_4 = -\dfrac{c_2}{(r+2)(r+4)+1} = \dfrac{c_0}{(r+1)^2(r+3)^2}$

putting these values in eqⁿ (A),

$y = x^r(c_0 + c_1x + c_2x^2 + \cdots)$

$= x^r\left(c_0 - \dfrac{c_0x^2}{(r+1)^2} + \dfrac{c_0x^4}{(r+1)^2(r+3)^2} - \cdots\right)$

$= c_0x^r\left[1 - \dfrac{x^2}{(r+1)^2} + \dfrac{x^4}{(r+1)^2(r+3)^2} - \cdots\right]$

$\therefore \delta y/\delta r = c_0x^r\log x\left[1 - \dfrac{x^2}{(r+1)^2} + \dfrac{x^4}{(r+1)^2(r+3)^2} - \cdots\right] + c_0x$

$\left[\dfrac{2x^2}{(r+1)^3} - \dfrac{2x^4}{(r+1)^2(r+3)^2}\left(\dfrac{1}{r+1} + \dfrac{1}{r+3}\right)\right]$

Now $y_1 = [y]_{r=1} = c_0x[1 - x^2/4 + x^4/64 - \cdots]$

$y_2 = [\delta y/\delta r]_{r=1} = y_1\log x + c_0x\left[\dfrac{2x^2}{8} - \dfrac{2x^4}{4\cdot16}(1/2+1/4) - \cdots\right]$

$= c_0\log x\cdot x[1 - x^2/4 + x^4/64 - \cdots] + c_0x\left[2x^2/8 - \dfrac{3x^4}{128} - \cdots\right]$

$\therefore y = k_1y_1 + k_2y_2$

$= k_1c_0x[1 - x^2/4 + x^4/64 - \cdots] + k_2\{xc_0\log x\,(1 - x^2/4 + \dfrac{x^4}{64} -$

$+ c_0x\,(2x^2/8 - 3x^4/128 - \cdots)\}$

(Ans:-)

Date :

Boundary value problem

Q1. Find the one dimentional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \text{ for } u(0,t) = u(l,t) = 0$$

and $u(x,0) = f(x)$, $\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$, $0 \le x \le l$

Soln:-

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Series solution

Find a series solⁿ by frobenious method of $(2x+x^2)y''+y'-2y=0$.

Solution:- The given equation is $(2x+x^2)y''+y'-2y=0$ --- (i)

Here, $P_1(x) = \dfrac{1}{2x+x^2}$

$P_2(x) = \dfrac{-2}{2x+x^2}$ } are not analytic at $x=0$

so that $x=0$ is a singular point of the DE (i).

$xP_1(x) = \dfrac{x}{2x+x^2} = \dfrac{1}{2+x}$

$x^2P_2(x) = \dfrac{-2x^2}{2x+x^2}$ } are both analytic at $x=0$,

so $x=0$ is regular singular point of DE (i).

By frobenious method the equation has the form

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$= x^r(c_0+c_1x+c_2x^2+c_3x^3+--) \quad --- (A)$$

$$y' = \sum_{n=0}^{\infty} (n+r)\, c_n x^{n+r-1} \quad --- (ii)$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)\, c_n x^{n+r-2} \quad --- (iii)$$

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putting (ii) and (iii) in equation (i) we get,

$(2x+x^2)y'' + y' - 2y = 0$

$$\Rightarrow \sum_{n=0}^{\infty} 2x(n+r)(n+r-1)c_n x^{n+r-2} + \sum_{n=0}^{\infty} x^2 \cdot (n+r)(n+r-1) c_n x^{n+r-2} + \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} - 2\cdot \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} 2(n+r)(n+r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)(n+r-1) c_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} - 2\sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}\left[2(n+r)(n+r-1)c_n x^{n+r-1} + (n+r)c_n x^{n+r-1}\right] + \sum_{n=1}^{\infty}\left[(n+r-1)(n+r-2)c_{n-1}x^{n+r-1} - 2c_{n-1}\cdot x^{n+r-1}\right] = 0$$

$$\Rightarrow [2r(r-1)+r]\cdot c_0 x^{r-1} + \sum_{n=1}^{\infty}\{2(n+r)(n+r-1)+(n+r)\} c_n \cdot x^{n+r-1}\} + \sum_{n=1}^{\infty}\{(n+r-1)(n+r-2)-2\} c_{n-1}\cdot x^{n+r-1} = 0$$

$$\Rightarrow [2r(r-1)+r]\, c_0 x^{r-1} + \sum_{n=1}^{\infty}\left[\{2(n+r)(n+r-1)+(n+r)\}c_n + \{(n+r-1)(n+r-2)-2\}c_{n-1}\right]x^{n+r-1} = 0 \quad \text{--- (iv)}$$

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Equating the co-efficient of x^{r-1} to zero.

$2r(r-1) + r c_0 = 0$, $c_0 \neq 0$

$\Rightarrow 2r^2 - 2r + r = 0$

$\Rightarrow 2r^2 - r = 0$

$\Rightarrow r(2r-1) = 0$

$\therefore r = 0, \frac{1}{2}$

Again from (iv) the recurrence sol^n is,

$$c_n = -\frac{((n+r-1)(n+r-2)-2)\, c_{n-1}}{2(n+r)(n+r-1) + (n+r)}$$

$$= -\frac{c_{n-1}\{(n+r-1)(n+r-2)-2\}}{(n+r)(2n+2r-1)} \quad \text{- - - (v)}$$

Case-I:- when $r=0$ then.

$$c_n = -\frac{c_{n-1}\{(n-1)(n-2)-2\}}{n(2n-1)}$$

$\therefore n = 1, 2, 3$ - - we get,

$$c_1 = -\frac{c_0 \times (1-1)(1-2)-2}{1(2-1)} = \frac{2c_0}{1} = 2c_0$$

$$c_2 = -\frac{c_1\{(2-1)(2-2)-2\}}{2(4-1)} = \frac{2c_1}{2\times 3} = \frac{2\times 2c_0}{6} = \frac{c_0}{3}$$

- - - -

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$y_1 = c_0\,(1 + 2x + x^2/3 + \cdots)$

Case-II:- when $r = 1/2$ then.

$$c_n = -\frac{c_{n-1}\{(n-1/2)(n-3/2)-2\}}{(n+1/2)\,2n}$$

when $n = 1, 2, 3 \cdots$ we get,

$$c_1 = \frac{9/4\, c_0}{3} = 3/4\, c_0$$

$$c_2 = -\frac{(3/2\cdot\frac{1}{2}-2)\,c_1}{5/2\cdot 4} = \frac{5}{4\cdot 5\cdot 2}\cdot 3/4\, c_0$$

$$= \frac{3}{32}\, c_0$$

$\cdots$

$y_2 = c_0\, x^{1/2}\,(1 + 3/4\, x + 3/32\, x^2 + \cdots)$

Hence the general sol^n is. $y = k_1 y_1 + k_2 y_2$

$= k_1 c_0 (1 + 2x + \frac{x^2}{3} + \cdots) + k_2\, c_0\, x^{1/2}(1 + 3/4\, x + 3/32\, x^2 + \cdots$

$= A\,(1 + 2x + \frac{x^2}{3} + \cdots) + B\, x^{\frac{1}{2}}\,(1 + \frac{3}{4}x + \frac{3}{32}x^2 + \cdots)$

where, $A = k_1 c_0$ (Ans:-)

$B = k_2 c_0$.

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Find the series solⁿ of $x^2y'' - xy' + (x^2 + 8/9)y = 0$ by Frobenious method.

Solution:- The given solⁿ is, $x^2y'' - xy' + (x^2 + 8/9)y = 0$ --- (i)

$\therefore P_1(x) = \frac{-x}{x^2} \quad P_2(x) = \frac{x^2 + 8/9}{x^2}$ } are not analytic at $x=0$.

$xP_1(x) = -1$ and $x^2P_2(x) = x^2 + 8/9$ } are analytic at $x=0$, so $x=0$ is a regular singular point of DE (i).

By Frobenious method the solⁿ is,

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}, \quad c \neq 0 \quad \text{--- (ii)}$$

$$= (c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots) \cdot x^r \quad \text{---- (A)}$$

$$y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} \quad \text{--- (iii)}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2} \quad \text{--- (iv)}$$

Hence from (ii), (iii) and (iv) we get,

$x^2y'' - xy' + (x^2 + 8/9)y = 0$

$$\Rightarrow \sum_{n=0}^{\infty} \left[(n+r)(n+r-1) c_n x^{n+r} - (n+r) c_n x^{n+r} + c_n x^{n+r+2} + 8/9\, c_n x^{n+r} \right] = 0$$

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$$\Rightarrow \sum_{n=0}^{\alpha} \{(n+r)(n+r-1)-(n+r)+8/9\} c_n x^{n+r} + \sum_{n=2}^{\alpha} c_{n-2} x^{n+r} = 0$$

$$\Rightarrow \sum_{n=0}^{\alpha} \{(n+r)(n+r-1)-(n+r)+8/9\} c_n x^{n+r} + \sum_{n=2}^{\alpha} [\{(n+r)(n+r-1) -(n+r)+8/9\} c_n + c_{n-2}] x^{n+r} = 0$$

$$\Rightarrow \{r(r-1)-r+8/9\} c_0 \cdot x^r + \{(r+1)r-(r+1)+8/9\} c_1 x^{r+1} + \sum_{n=2}^{\alpha} [\{(n+r)(n+r-1)-(n+r)+\tfrac{8}{9}\} c_n + c_{n-2}] x^{n+r} = 0$$

$- - - -$ (v)

Equating the co-efficient of x^r and x^{r+1} equal to zero,

$\{r(r-1)-r+\frac{8}{9}\} c_0 = 0$

$\{(r+1)r-(r+1)+8/9\} c_1 = 0$

$c_0 \neq 0 \quad r^2-r-r+8/9 = 0$

$\Rightarrow r^2-2r+8/9 = 0$

$\Rightarrow 9r^2-18r+8 = 0$

$\Rightarrow r^2-2r+8/9 = 0$

$\Rightarrow r(r-2)+8/9 = 0$

$\therefore r^2+8/9 = 0, \; -2r+8/9 = 0$

$\Rightarrow r = \frac{4}{3}, \quad r = 2/3.$

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Here, $q = 0$

Equating the co-efficient of x^{n+r} to zero,

$$c_n = -\frac{c_{n-2}}{(n+r)(n+r-2)+8/9} \quad \because n \geq 2$$

Case-I :- when $r = 4/3$ then $c_n = -\dfrac{c_{n-2}}{(n+4/3)(n+4/3-2)+8/9}$

putting $n = 2, 3, 4 \ldots$

$$c_2 = -\frac{c_0}{\frac{10}{3}\cdot 4/3 + \frac{8}{9}} = -\frac{3}{16}c_0$$

$c_3 = 0$, $c_5 = 0$, $c_7 = 0 \ldots$

$$c_4 = \frac{-c_2}{(4+4/3)(2+4/3)+8/9} = \frac{-3c_0}{\left(16\times\frac{16}{3}\cdot\frac{10}{3}+8/9\right)} = \frac{9c_0}{896}$$

Now $r = 4/3$ then,

$$y_1 = c_0 \cdot x^{4/3}\left(1 - \frac{3x^2}{16} + \frac{9x^4}{896} - \ldots\right)$$

Case-II :- when $r = 2/3$ then, $c_n = -\dfrac{c_{n-2}}{(n+2/3)(n+2/3-2)+8/9}$

$n = 2, 3, 4 \ldots$ $c_2 = -3/8\, c_0$

$c_1 = 0$, $c_3 = 0$, $c_5 = 0 \ldots$

$c_4 = 9/320\, c_0$

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$\therefore y_2 = c_0 x^{2/3}(1 + (-3/8 x^2) + \frac{9}{320}x^4 - - -)$

The general solution is,

$y = k_1 y_1 + k_2 y_2$

$= k_1 c_0 x^{4/3}(1 - \frac{3x^2}{16} + \frac{9x^4}{896} - - -) + k_2 c_0 x^{2/3}(1 - \frac{3x^2}{8} + \frac{9x^4}{320} - -)$

$= A x^{4/3}(1 - \frac{3x^2}{16} + \frac{9x^4}{896} - - -) + B x^{2/3}(1 - \frac{3x^2}{8} + \frac{9x^4}{320} - -)$

(Ans:-)

Ex. Find the series solution by Frobenious method,

$2x^2 y'' + xy' + (x^2 - 3)y = 0$

Hints:- $r = -1, 3/2$

$y_1 = c_0(x^{-1} + x/2 - x^3/4 + \cdot -)$

$y_2 = c_0(x^{3/2} - 1/18 x^{7/2} + \frac{1}{936}x^{11/2} - \cdot -)$

Ex. Bessel zero order এর series solution?

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Q. Find a series solution by Frobenious method.

$4xy'' + 2y' + y = 0$

Hints:- $r = 0, \frac{1}{2}$

$$c_n = -\frac{c_{n-1}}{2(n+r)(2n+2r-1)} \quad n \geq 1$$

Solution:- Given that, $4xy'' + 2y' + y = 0$ - - (i)

Here, $P_1(x) = \frac{2}{4x}$, $P_2(x) = \frac{1}{4x}$ } are not analytic at $x = 0$ so $x = 0$ is a singular point.

$xP_1(x) = \frac{1}{2}$ $x^2 P(x) = \frac{x}{4}$ } are analytic at $x = 0$ so that $x = 0$ is a regular singular point.

Now by frobenious method equation (i) can be written as,

$$y = \sum_{n=0}^{\alpha} c_n x^{n+r} \quad - - (ii)$$

$$= x^n (c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots)$$

$$y' = \sum_{n=0}^{\alpha} (n+r) c_n x^{n+r-1} \quad - - - (iii)$$

$$y'' = \sum_{n=0}^{\alpha} (n+r)(n+r-1) c_n x^{n+r-2} \quad - - (iv)$$

From equation (i), (ii), (iii) and (iv) we get,

$$\sum_{n=0}^{\alpha} 4(n+r)(n+r-1) c_n x^{n+r-1} + \sum_{n=0}^{\alpha} 2(n+r) c_n x^{n+r-1} + \sum_{n=0}^{\alpha} c_n x^{n+r} = 0$$

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$$\Rightarrow \sum_{n=0}^{\infty}\{4(n+r)(n+r-1)+2(n+r)\}c_n x^{n+r-1}+\sum_{n=1}^{\infty}c_{n-1}x^{n+r-1}=0$$

$$\Rightarrow \sum_{n=0}^{\infty}\{2(n+r)(2n+2r-1)\}c_n x^{n+r-1}+\sum_{n=1}^{\infty}c_{n-1}x^{n+r-1}=0$$

$$\Rightarrow 2r(2r-1)\cdot c_0 x^{r-1}+\sum_{n=1}^{\infty}\left[\{2(n+r)(2n+2r-1)\}c_n+c_{n-1}\right]x^{n+r-1}=0$$

$\therefore$ Equating the co-efficient of x^{r-1} equal to zero,

$2r(2r-1)=0$

$\Rightarrow r=0,\ r=1/2$

The recurrence solution is,

$$c_n=\frac{-c_{n-1}}{2(n+r)(2n+2r-1)} \quad n\geq 1$$

Case-I:- when $r=0$.

$$c_n=\frac{-c_{n-1}}{2n(2n-1)}$$

$n=1,2,3\ldots$

$$c_1=\frac{-c_0}{2(2-1)}=\frac{-c_0}{2},\quad c_2=\frac{-c_1}{4(4-1)}=\frac{-c_1}{12}=\frac{c_0}{24}$$

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$y_1 = 1 \cdot c_0 \left(1 - \frac{1}{2}x + \frac{1}{24}x^2 - \cdots\right)$

Case-II :- when $r_1 = \frac{1}{2}$

$$c_n = \frac{-c_{n-1}}{2(n+\frac{1}{2})(2n+2\cdot\frac{1}{2}-1)}$$

$$= \frac{-c_{n-1}}{(2n+1)(2n)}$$

$n = 0, 1, 2, 3 \cdots$ we get.

$$c_1 = \frac{-c_0}{(2+1)\times 2} = \frac{-c_0}{6}$$

$$c_2 = \frac{-c_1}{(4+1)4} = \frac{c_0}{120}$$

$$\therefore y_2 = c_0 x^{1/2}\left(1 - \frac{1}{6}x + \frac{1}{120}x^2 - \cdots\right)$$

$$\therefore y = K_1 y_1 + y_2 K_2$$

$$= K_1 c_0 \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^2 - \cdots\right) + K_2\left(1 - \frac{1}{6}x + \frac{1}{120}x^2 - \cdots\right) c_0 x^{1/2}$$

$$= A c_0 \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^2 - \cdots\right) + B c_0 x^{1/2}\left(1 - \frac{1}{6}x + \frac{1}{120}x^2 - \cdots\right)$$

where. $A = K_1 c_0$ and $B = K_2 c_0$ are constant.

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Bessel' DE

Bessel's DE :- The equation of the form $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-n^2) = 0$ is called Bessel's differential equation where n is a real constant and it's sol^n $y = c_1 J_n(x) + c_2 Y_n(x)$ is called Bessel's function of order n where c_1 and c_2 is an arbitrary constant and the function $J_n(x)$ is called the Bessel's func^n of order n of the first kind and $Y_n(x)$ is called the Bessel's function of order n of the 2nd kind.

Here $J_n(x) = \sum_{s=0}^{\alpha} (-1)^s \frac{\left(\frac{x}{2}\right)^{n+2s}}{(n+s)!\, s!}$

and $Y_n(x) = \frac{1}{\sin n\pi}\left[\cos n\pi\, J_n(x) - J_{-n}(x)\right]$

Q. prove that $\exp\left\{\frac{x}{2}(t - 1/t)\right\} = \sum_{n=-\alpha}^{\alpha} J_n(x)\, t^n$

sol^n:- L.H.S

$\exp\{x/2\,(t - 1/t)\}$

$= \exp\frac{xt}{2} \cdot \exp\frac{-x}{2t}$

$= \left[1 + \frac{tx/2}{1!} + \frac{(tx/2)^2}{2!} + \cdots + \frac{(tx/2)^n}{n!} + \frac{(tx/2)^{n+1}}{(n+1)!} + \frac{\left(\frac{tx}{2}\right)^{n+2}}{(n+2)!} + \cdots\right]\left[1 - \frac{x/2t}{1!} + \frac{(x/2t)^2}{2!} - \frac{\left(\frac{x}{2t}\right)^3}{3!} + \cdots + (-1)^n (x/2t)^n/n! + \cdots\right]$

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Equating the coefficient of t^n in this product, is given by,

$$= \frac{(x/2)^n}{n!} - \frac{\left(\frac{x}{2}\right)^{n+2}}{(n+1)!\,1!} + \frac{\left(\frac{x}{2}\right)^{n+4}}{(n+2)!\,2!} - \cdots \text{ to } \infty$$

$$= \sum_{s=0}^{\infty} (-1)^s \frac{\left(\frac{x}{2}\right)^{n+2s}}{(n+s)!\,s!} = J_n(x)$$

Hence we obtain $\exp\left\{\frac{x}{2}\left(t - \frac{1}{t}\right)\right\} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$

The function $\exp\left\{\frac{x}{2}\left(t - \frac{1}{t}\right)\right\}$ is called the generating function of Bessel's f^n, $J_n(x)$ for $n = 0, 1, 2 \cdots$

Q. Prove that $J_{-n}(x) = (-1)^n J_n(x)$

Sol^n:- We know, $J_n(x) = \sum_{s=0}^{\infty} (-1)^s \frac{\left(\frac{x}{2}\right)^{n+2s}}{(n+s)!\,s!}$

Replacing n by $-n$ and we get,

$$J_{-n}(x) = \sum_{s=0}^{\infty} (-1)^s \frac{(x/2)^{2s-n}}{(s-n)!\,s!}$$

Now putting $s-n = r$ we get,

$$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^{n+r} \frac{(x/2)^{2s-n}}{r!\,(n+r)!}$$

$$= (-1)^n \sum_{r=0}^{\infty} (-1)^r \frac{(x/2)^{n+2r}}{(n+r)!\,r!}$$

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$= (-1)^n J_n(x)$ (proved)

Q. Prove that $J_{1/2}(x) = \sqrt{2/\pi x}\,\sin x$

Proof:- We have, $J_n(x) = \sum_{s=0}^{\alpha} (-1)^s \frac{(x/2)^{n+2s}}{(n+s)!\,s!}$

$$= \frac{(x/2)^n}{n!} - \frac{(x/2)^{n+2}}{(n+1)!\,1!} + \frac{(x/2)^{n+4}}{(n+2)!\,2!} - \cdots$$

putting $n = 1/2$

$$J_{1/2}(x) = \frac{(x/2)^{1/2}}{(1/2)!} - \frac{(x/2)^{1/2+2}}{(1/2+1)!\,1!} + \frac{(x/2)^{1/2+4}}{(\frac{1}{2}+2)!\,2!} - \cdots$$

$$= \frac{(x/2)^{1/2}}{1/2!}\left[1 - \frac{(x/2)^2 \cdot \frac{1}{2}!}{1!\,3/2!\,\frac{1}{2}!} + \frac{(\frac{x}{2})^4 \cdot 1/2!}{2!\cdot 5/2!\,\cancel{3/2}\cdot 1/2!} - \cdots\right]$$

$$= \frac{(x/2)^{\frac{1}{2}}}{\frac{1}{2}\sqrt{1/2}}\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right]$$

$$= \frac{\overset{(x/2)^{1/2}}{\cancel{x^{1/2}\cdot(\frac{1}{2})\cdot\sqrt{2}\cdot 2}}}{\frac{1}{2}\sqrt{\pi}\cdot x}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]$$

$$= \frac{\sqrt{2}}{\sqrt{\pi x}}\sin x$$

$\therefore\ J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\,\sin x$ (proved)

$$= \frac{(x/2)^{1/2}}{1/2!}\left[1 - \frac{(x/2)^2}{3/2!\,1!} + \frac{(x/2)^4}{5/2\cdot 3/2!\,2!} - \cdots\right]$$

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Q. prove that, $J_{-\frac{1}{2}}(x) = \sqrt{2/\pi x}\cos x$

Proof:- We have, $J_{-n}(x) = \sum_{s=0}^{\alpha} \frac{(-1)^s (\frac{x}{2})^{2s-n}}{s!\,(s-n)!}$

$= \frac{(x/2)^{-n}}{(-n)!} - \frac{(x/2)^{2-n}}{1!\,(1-n)!} + \frac{(\frac{x}{2})^{4-n}}{2!\,(2-n)!} - - -$

putting $n = \frac{1}{2}$

$J_{-\frac{1}{2}}(x) = \frac{(x/2)^{-\frac{1}{2}}}{(-\frac{1}{2})!} - \frac{(x/2)^{2-\frac{1}{2}}}{1!\,(1-\frac{1}{2})!} + \frac{(x/2)^{4-\frac{1}{2}}}{2!\,(2-\frac{1}{2})!} - - -$

$= \frac{(\frac{x}{2})^{-\frac{1}{2}}}{(-\frac{1}{2})!}\left[1 - \frac{(x/2)^2 \cdot (-\frac{1}{2})!}{1!\,(1-\frac{1}{2})!} + \frac{(x/2)^4 (-\frac{1}{2})!}{2!\,(2-\frac{1}{2})!} - - -\right]$

$= \frac{(x/2)^{-\frac{1}{2}}}{\sqrt{\frac{1}{2}}}\left[1 - \frac{(\frac{x}{2})^2 \sqrt{\frac{1}{2}}}{1!\,\frac{1}{2}\sqrt{\frac{1}{2}}} + \frac{(x/2)^4 \sqrt{\frac{1}{2}}}{2!\,\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\frac{1}{2}}} - - -\right]$

$= \sqrt{\frac{2}{x}} \cdot \frac{1}{\sqrt{\pi}}\left[1 - x^2/2! + \frac{x^4}{4!} - - -\right]$

$= \sqrt{\frac{2}{\pi x}}\cos x$ (proved)

Q. Recurrence Relation:- We have, $\exp\{x/2(t - 1/t)\} = \sum_{n=-\alpha}^{\alpha} J_n(x)t^n - - - (i)$

Diff w.r.to x, $\frac{1}{2}(t - 1/t)\sum_{n=-\alpha}^{\alpha} J_n(x)t^n = \sum_{n=-\alpha}^{\alpha} J_n'(x)t^n$

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$$\Rightarrow \frac{1}{2}\sum_{n=-\alpha}^{\alpha} J_n(x)\,t^{n+1} - \frac{1}{2}\sum_{n=-\alpha}^{\alpha} J_n(x)\,t^{n-1} = \sum_{n=-\alpha}^{\alpha} J_n'(x)\,t^n$$

$$\Rightarrow \frac{1}{2}\sum_{n=-\alpha}^{\alpha} J_{n-1}(x)\,t^{n} - \frac{1}{2}\sum_{n=-\alpha}^{\alpha} J_{n+1}(x)\,t^{n} = \sum_{n=\alpha}^{\alpha} J_n'(x)\,t^n \qquad [\because n=n-1]$$

Equating the coefficient of t^n,

$$\frac{1}{2}J_{n-1}(x) - \frac{1}{2}J_{n+1}(x) = J_n'(x)$$

$$\Rightarrow 2J_n'(x) = J_{n-1}(x) - J_{n+1}(x) \quad \text{- - - - (ii)}$$

Differentiating w.r. to t in (i) we get,

$$\frac{x}{2}(1+1/t^2)\exp\{x/2\,(t-1/t)\} = \sum_{n=-\alpha}^{\alpha} J_n(x)\cdot n\cdot t^{n-1}$$

$$\Rightarrow \frac{x}{2}(1+1/t^2)\sum_{n=-\alpha}^{\alpha} J_n(x)\,t^n = \sum_{n=-\alpha}^{\alpha} n\,J_n(x)\,t^{n-1}$$

$$\Rightarrow \frac{x}{2}\sum_{n=-\alpha}^{\alpha} J_{n-1}(x)\,t^{n-1} + x/2\sum_{n=-\alpha}^{\alpha} J_{n+1}(x)\,t^{n-1} = \sum_{n=-\alpha}^{\alpha} n\,J_n(x)\,t^{n-1}$$

Equating the coefficient of t^{n-1} we get,

$$\frac{x}{2}J_{n-1}(x) + x/2\,J_{n+1}(x) = n\,J_n(x)$$

$$** \Rightarrow \frac{2n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x) \quad \text{- - - (iii)}$$

Adding (ii) and (iii) we get,

$$2J_n'(x) + \frac{2n}{x}J_n(x) = 2J_{n-1}(x)$$

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$\Rightarrow x J_n'(x) = x J_{n-1}(x) - n J_n(x) \quad \text{--- (iv)}$

subtracting (iii) from (ii) we get,

$2 J_n'(x) - 2n/x \; J_n(x) = -2 J_{n+1}(x)$

$\Rightarrow x J_n'(x) = n J_n(x) - x J_{n+1}(x) \quad \text{--- (v)}$

Q. prove that, $J_{3/2}(x) = \sqrt{2/\pi x} \left(\frac{\sin x}{x} - \cos x\right)$

proof:- We have, $\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$

$\Rightarrow J_n(x) = \frac{x}{2n} \left[J_{n-1}(x) + J_{n+1}(x) \right]$

putting $n = 1/2$ so,

$J_{1/2}(x) = \frac{x}{2 \cdot 1/2} \left[J_{-1/2}(x) + J_{3/2}(x) \right]$

$\Rightarrow J_{3/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x)$

$= \frac{1}{x} \sqrt{\frac{2}{\pi x}} \sin x - \sqrt{2/\pi x} \cos x$

$\therefore J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x\right)$ (proved)

Q. prove that $J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(-\frac{\cos x}{x} - \sin x\right)$

proof:- we have,

$$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$

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$\Rightarrow J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$

putting $n = -\frac{1}{2}$ we get,

$\Rightarrow J_{-1/2}(x) = \frac{x}{2\cdot -\frac{1}{2}}\left[J_{-3/2}(x) + J_{1/2}(x)\right]$

$\Rightarrow x J_{-3/2}(x) = -x J_{1/2}(x) - J_{-1/2}(x)$

$\Rightarrow J_{-3/2}(x) = \frac{-J_{-1/2}(x)}{x} - J_{1/2}(x)$

$= -\frac{1}{x}\sqrt{2/\pi x}\cos x - \sqrt{\frac{2}{\pi x}}\sin x$

$\therefore J_{-3/2}(x) = \sqrt{2/\pi x}\left(\frac{-\cos x}{x} - \sin x\right)$ (proved)

Ex. If a, b, c are differential root of $J_n(x) = 0$ then prove that

$\int_0^1 x J_n(ax) J_n(bx)\,dx = 0$ [orthogonality eq^n]

Proof:- We know that, $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$

Bessel's eq^n can be written as,

$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (a^2x^2 - n^2)y = 0$ - - - - (i)

$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (b^2x^2 - n^2)y = 0$ - - - (ii)

Let $J_n(ax) = u$ and $v = J_n(bx)$ be the sol^n of (i) and (ii)

we get, $x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} + (a^2x^2 - n^2)u = 0$ - - - (iii)

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and $x^2 \frac{d^2v}{dx^2} + x\frac{dv}{dx} + (b^2x^2 - n^2)v = 0$ --- (iv)

multiplying (iii) by $\frac{v}{x}$ and (iv) $\frac{u}{x}$ we have,

$$xv\frac{d^2u}{dx^2} + v\frac{du}{dx} + (a^2x^2 - n^2)\frac{uv}{x} = 0 \quad \text{--- (v)}$$

and $xu\frac{d^2v}{dx^2} + u\frac{dv}{dx} + (b^2x^2 - n^2)\frac{uv}{x} = 0$ --- (vi)

subtracting (6) from (5) we get,

$$\Rightarrow \left(xv\frac{d^2u}{dx^2} - xu\frac{d^2v}{dx^2}\right) + \left(v\frac{du}{dx} - u\frac{dv}{dx}\right) + (a^2 - b^2)x^2\frac{uv}{x} = 0$$

$$\Rightarrow \frac{d}{dx}\left[x\left\{v\frac{du}{dx} - u\frac{dv}{dx}\right\}\right] = (b^2 - a^2)\,xuv$$

Integrating w.r to x over $(0,1)$

$$(b^2 - a^2)\int_0^1 xuv\,dx = \left[x\left\{v\frac{du}{dx} - u\frac{dv}{dx}\right\}\right]_0^1$$

$$= \left[x\left\{J_n(bx)\cdot a J_n'(ax) - J_n(ax)\cdot bJ_n'(bx)\right\}\right]_0^1$$

$$= aJ_n(b)\cdot J_n'(a) - bJ_n(a)J_n'(b)$$

since a, b are root of $J_n(x) = 0$ so,

$$J_n(a) = J_n'(a) = J_n(b) = J_n'(b) = 0$$

$$\therefore (b^2 - a^2)\int_0^1 xuv\,dx = 0$$

$$\therefore \int_0^1 x J_n(ax) J_n(bx)\,dx = 0$$ (proved)

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Legendre DE

Legendre DE:- The DE of the form $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$ is known as Legendre DE of degree n where n is a positive integer or zero.

Legendre's polynomial, $P_n(x) = \sum_{r=0}^{N} (-1)^r \frac{(2n-2r)!\, x^{n-2r}}{2^n r!\,(n-r)!\,(n-2r)!}$

where $N = n/2$ when n is even

$N = \frac{n-1}{2}$ when n is odd.

Rodrigues formula:-

prove that, $P_n(x) = \frac{1}{2^n \cdot n!} \frac{d^n}{dx^n}(x^2-1)^n$

proof:- $\frac{1}{2^n n!} \frac{d^n}{dx^n}\{(x^2-1)^n\}$

$$= \frac{1}{2^n \cdot n!} \frac{d^n}{dx^n}\{ x^{2n} - {}^nC_1 x^{2(n-1)} + {}^nC_2 x^{2(n-2)} - \cdots + {}^nC_n(-1)^n x^{2(n-n)}\}$$

$$= \frac{1}{2^n \cdot n!} \cdot \frac{d^n}{dx^n}\{ \sum_{r=0}^{n} (-1)^r\, {}^nC_r\, x^{(2n-2r)}$$

$$= \frac{1}{2^n \cdot n!} \sum_{r=0}^{N} (-1)^r \frac{n!}{r!(n-r)!} \cdot \frac{(2n-2r)!}{(2n-2r-n)!} \cdot x^{2n-2r-n}$$

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$$= \sum_{r=0}^{N} \frac{(-1)^r (2n-2r)!}{2^n \cdot r!(n-r)!(n-2r)!} x^{n-2r}$$

$= P_n(x)$ [From Legendre formula] (proved)

Q. Establish the generating function of legendre eqn.

Or, Use generating f^n $\phi(x,h) = (1-2xh+h^2)^{-1/2}$ to prove $P_n(x)$ satisfies legendre eqn.

Solⁿ:- If $|h|<1$ then $\phi(x,h) = (1-2xh+h^2)^{-1/2}$ is called the generating f^n of legendre polynomial.

We know the rodrigue's formula is,

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

$$P_0(x) = 1$$

$$P_1(x) = \frac{1}{2} \cdot 2x = x$$

$$P_2(x) = \frac{1}{2}(3x^2-1)$$

$$P_3(x) = \frac{1}{2^3 \cdot 3!} \frac{d^3}{dx^3} (x^2-1)^3$$

$$= \frac{1}{2}(5x^3 - 3x)$$

Now, $(1-2xh+h^2)^{-1/2} = \{1-h(2x-h)\}^{-\frac{1}{2}}$

$$= 1 - \frac{(-\frac{1}{2}) h(2x-h)}{1!} + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2!} h^2 (2x-h)^2 - \cdots$$

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$= 1 + hx + \frac{1}{2}(3x^2-1)h^2 + \frac{1}{2}(5x^3-2x)h^3 + \cdots$

$= P_0(x) + P_1(x)h + P_2(x)h^2 + P_3(x)h^3 + \cdots$

$= \sum_{n=0}^{\alpha} P_n(x)h^n$

$\therefore \phi(x,h) = (1-2xh+h^2)^{-1/2}$

$= \sum_{n=0}^{\alpha} P_n(x)h^n$ which is called generating func^n

Q. Establish the orthogonal property,

(i) $\int_{-1}^{1} P_m(x) P_n(x)\,dx = 0$ if $m \neq n$

(ii) $\int_{-1}^{1} P_m(x) P_n(x)\,dx = \frac{2}{2n+1}$ if $m = n$

Wrong!!

Sol^n (i) :- we know legendre DE,

$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$

satisfying this eq^n by $P_n(x)$ $(1-x^2)\frac{d^2P_n(x)}{dx^2} - 2x\frac{dP_n(x)}{dx} +$

$n(n+1)P_n(x) = 0$

Multiplying both sides by $P_m(x)$ and integrating over the limit -1 to 1 wr. to x,

$\Rightarrow \int_{-1}^{1} P_m(x)\frac{d}{dx}\left\{(1-x^2)\frac{dP_n(x)}{dx}\right\}dx + n(n+1)\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0$

$\Rightarrow \left[P_m(x)\left\{(1-x^2)\frac{dP_n(x)}{dx}\right\}\right]_{-1}^{1} - \int_{-1}^{1}\frac{dP_m(x)}{dx}\left\{(1-x^2)\frac{dP_n(x)}{dx}\right\}dx$

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$$+ n(n+1)\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0$$

$$\Rightarrow -\int_{-1}^{1} \frac{dP_m(x)}{dx}\left\{(1-x^2)\frac{dP_n(x)}{dx}\right\} dx + n(n+1)\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0 \quad \text{------(i)}$$

Interchanging m and n in eq^n (i) we get,

$$\Rightarrow -\int_{-1}^{1} \frac{dP_n(x)}{dx}\left\{(1-x^2)\frac{dP_m(x)}{dx}\right\} dx + m(m+1)\int_{-1}^{1} P_n(x) P_m(x)\, dx = 0 \quad \text{------(ii)}$$

substracting equation (i) and (ii),

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0 \quad \text{if } m-n \neq 0$$

$$\therefore m \neq n$$

$$\therefore \int_{-1}^{1} P_m(x) P_n(x)\, dx = 0 \quad \text{(proved)}$$

soL^n(ii) :- when $m = n$,

$$\int_{-1}^{1} [P_n(x)]^2 dx = \frac{(-1)^n (2n)!}{2^{n+n}(n)!\,(n)!}\int_{-1}^{1} \frac{d^{n-n}}{dx^{n-n}}(x^2-1)^n dx$$

$$= \frac{(2n)!}{2^n (n!)^2}\int_{-1}^{1}(1-x^2)^n dx$$

putting $x = \sin\theta \quad \therefore dx = \cos\theta\, d\theta$

$$= \frac{(2n)!}{2^{2n}(n!)^2}\, 2\int_0^{\pi/2} \cos^{2n}\theta \cdot \cos\theta\, d\theta$$

$$= \frac{(2n)!}{2^{2n}(n!)^2}\, 2 \cdot \frac{\Gamma(n+1)\,\Gamma(\frac{1}{2})}{2\Gamma\left(\frac{2n+3}{2}\right)}$$

$$= \frac{2}{2n+1} \quad \text{which proves the result.}$$

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Soln (1) :- $\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = 0$ if $n \neq m$

The legendre equation can be written as,

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\} + n(n+1)\, y = 0$$

$$\therefore \frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\} + n(n+1)\, P_n = 0 \quad \text{--- (i)}$$

$$\frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\} + m(m+1)\, P_m = 0 \quad \text{--- (ii)}$$

Multiplying (i) by P_m and (ii) by P_n and subtracting,

$$P_m \frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\} - P_n \frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\} + P_m P_n\left[n(n+1) - m(m+1)\right] = 0 \quad \text{--- (iii)}$$

Integrating by parts,

$$\int_{-1}^{1} P_m \frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\} dx = -\int_{-1}^{1} \frac{dP_n}{dx}\cdot\frac{dP_m}{dx}\,(1-x^2)\, dx$$

$$\int_{-1}^{1} P_n \frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\} dx = -\int_{-1}^{1} \frac{dP_m}{dx}\,\frac{dP_n}{dx}\,(1-x^2)\, dx$$

Integrating (iii) w.r. to x we get,

$$\cancel{-\int_{-1}^{1} \frac{dP_m}{dx}\,\frac{dP_n}{dx}\,(1-x^2)\, dx} + \cancel{\int_{-1}^{1} \frac{dP_m}{dx}\cdot\frac{dP_n}{dx}\,(1-x^2)\, dx} + (n-m)(n+m+1)\int_{-1}^{1} P_m P_n\, dx = 0$$

$$\Rightarrow (n-m)(n+m+1)\int_{-1}^{1} P_m P_n\, dx = 0$$

$$\therefore \int_{-1}^{1} P_m P_n\, dx = 0 \quad \text{(proved)}$$

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Laplace transformation:-

Defn:- let $f(t)$ be a function defined for all positive value of t then $F(s) = \int_0^\alpha e^{-st} f(t)\,dt$ provided the integral exists is called the Laplace transformation of $f(t)$.

It is denoted by $Lf(t) = F(s) = \int_0^\alpha e^{-st} f(t)\,dt$

Find the Laplace transformation of constant K?

Soln:- We have, $Lf(x) = \int_0^\alpha e^{-sx} f(x)\,dx$

$$= \int_0^\alpha e^{-sx} K\,dx$$

$$= K\left[\frac{e^{-sx}}{-s}\right]_0^\alpha$$

$$= \frac{K}{s} \text{ (Ans.)}$$

Find the laplace transformation of t^n?

Soln:- We have, $Lf(t) = \int_0^\alpha e^{-st} f(t)\,dt$

$$= \int_0^\alpha e^{-st} t^n\,dt$$

let, $st = x$

$\Rightarrow t = \frac{x}{s}$

$\therefore dt = \frac{1}{s}\,dx$

$$\therefore Lf(t) = \int_0^\alpha e^{-x}\left(\frac{x}{s}\right)^n \frac{dx}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\alpha e^{-x} x^n\,dx$$

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$= \frac{1}{s^{n+1}} \sqrt{n+1}$

$= \frac{n!}{s^{n+1}}$ (Ans:-)

Find the laplace transformation of e^{at}?

Solⁿ:- we have, $Lf(t) = \int_0^\alpha e^{-st} f(t)\, dt$

$\Rightarrow L(e^{at}) = \int_0^\alpha e^{-st} e^{at}\, dt$

$= \int_0^\alpha e^{-t(s-a)}\, dt$

$= \int_0^\alpha e^{-(s-a)t}\, dt$

$= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^\alpha$

$= \frac{1}{s-a}$ (Ans:)

Find the laplace transformation of sinat?

Solⁿ:- $L(\sin at) = L\left[\frac{e^{iat} - e^{-iat}}{2i} \right]$

$= \frac{1}{2i} \left[L(e^{iat}) - L(e^{-iat}) \right]$

$= \frac{1}{2i} \left[\frac{1}{s-ia} - \frac{1}{s+ia} \right]$

$= \frac{1}{\cancel{2i}} \times \frac{\cancel{2i}a}{s^2+a^2} = \frac{a}{s^2+a^2}$ (Ans:)

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Find the laplace transformation of $\cos at$?

SOL^n:- we have $L(\cos at) = L\left(\frac{e^{iat}+e^{-iat}}{2}\right)$

$$= \frac{1}{2}\left\{L(e^{iat}) + L(e^{-iat})\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-ia} + \frac{1}{s+ia}\right\}$$

$$= \frac{s}{s^2+a^2} \quad \text{(Ans:)}$$

Find the laplace transformation of t^2e^{at}?

SOL^n:- we have, $L\, f(t) = \int_0^\alpha e^{-st} f(t)\,dt$

$$= \int_0^\alpha e^{-st} t^2 e^{at}\,dt$$

$$= \int_0^\alpha t^2 e^{-(s-a)t}\,dt$$

$$= \left[t^2 \frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\alpha + \frac{1}{s-a}\int_0^\alpha 2t\, e^{-(s-a)t}\,dt$$

$$= 0 + \frac{2}{s-a}\left\{\left[\left\{t\cdot\frac{e^{-(s-a)t}}{-(s-a)}\right\}\right]_0^\alpha + \frac{1}{s-a}\int_0^\alpha e^{-(s-a)t}\,dt\right\}$$

$$= \frac{2}{s-a}\left[\frac{1}{s-a}\cdot\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\alpha\right]$$

$$= \frac{2}{(s-a)^3} \quad \text{(Ans)}$$

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Some necessary rules:-

$f(x)$	$\longrightarrow$ $Lf(x)$	$f(x)$	$\longrightarrow$ $Lf(x)$
1	$\longrightarrow \frac{1}{s}$	$\sin ax$	$\longrightarrow \frac{a}{s^2+a^2}$
a	$\longrightarrow \frac{a}{s}$	$\cos ax$	$\longrightarrow \frac{s}{s^2+a^2}$
e^{ax}	$\longrightarrow \frac{1}{s-a}$	x^n	$\longrightarrow \frac{n!}{s^{n+1}}$
e^{-ax}	$\longrightarrow \frac{1}{s+a}$	$e^{-bx}\cos ax$	$\longrightarrow \frac{s+b}{(s+b)^2+a^2}$
xe^{-ax}	$\longrightarrow \frac{1}{(s+a)^2}$	$e^{-bx}\sin ax$	$\longrightarrow \frac{a}{(s+b)^2+a^2}$
xe^{+ax}	$\longrightarrow \frac{1}{(s-a)^2}$	x^2e^{ax}	$\longrightarrow \frac{2}{(s-a)^3}$

Prove that (i) $L\{\sinh at\} = \frac{a}{s^2-a^2}$

(ii) $L\{\cosh at\} = \frac{s}{s^2-a^2}$ if $s > |a|$

Sol^n (i): (i) $L(\sinh at) = L\left\{\frac{e^{at}-e^{-at}}{2}\right\}$

$$= \int_0^{\alpha} e^{-st}\left(\frac{e^{at}-e^{-at}}{2}\right)dt$$

$$= \frac{1}{2}\int_0^{\alpha} e^{-st}e^{at}\,dt - \tfrac{1}{2}\int_0^{\alpha} e^{-st}\cdot e^{-at}\,dt$$

$$= \left(\frac{1}{2}\,\frac{1}{s-a} - \tfrac{1}{2}\cdot\frac{1}{s+a}\right)$$

$$= \frac{1}{2}\left(\frac{\cancel{s}+a-\cancel{s}+a}{s^2-a^2}\right) = \frac{a}{s^2-a^2} \quad \text{(proved)}$$

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(iii) $L(\cosh at) = L\left(\frac{e^{at}+e^{-at}}{2}\right)$

$= \int_0^\alpha e^{-st}\left(\frac{e^{at}+e^{-at}}{2}\right) dt$

$= \int_0^\alpha \frac{1}{2} e^{-st} e^{at} dt + \int_0^\alpha \frac{1}{2} e^{-st} . e^{-at} dt$

$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$

$= \frac{s}{s^2-a^2}$ (Ans:-)

Q. Find $L\{F(t)\}$, if $F(t) = \begin{cases} 5 & 0<t<3 \\ 0 & t>3 \end{cases}$

SOL^n:- We know. $Lf(t) = \int_0^\alpha e^{-st} f(t)\, dt$

$= \int_0^3 e^{-st}\, 5\, dt + \int_3^\alpha e^{-st}(0)\, dt$

$= 5\left[\frac{e^{-st}}{-s}\right]_0^3$

$= 5/-s \left\{ e^{-3s} - e^0 \right\}$

$= \frac{5}{-s}\left(e^{-3s}-1\right)$

$= \frac{5}{s}\left(1-e^{-3s}\right)$ (Ans:)

Q. Find the Laplace transformetion,

$L\{4e^{5t} + 6t^3 - 3\sin 4t + 2\cos 2t\}$

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Solⁿ:- $L\,f(t) = L(4e^{5t}) + L(6t^3) - L(3\sin 4t) + L(2\cos 2t)$

$= 4\cdot\frac{1}{s-5} + 6\cdot\frac{3!}{s^4} - 3\times\frac{4}{s^2+16} + 2\frac{s}{s^2+4}$

$= \frac{4}{s-5} + \frac{36}{s^4} - \frac{12}{s^2+16} + \frac{2s}{s^2+4}$ (Ans:-)

Qo. Find $L(e^{4t}\cosh 5t)$

We know, $L(\cosh 5t) = \frac{s}{s^2-25}$

$\therefore L(e^{4t}\cosh 5t) = \frac{(s-4)}{(s-4)^2-25}$

$= \frac{s-4}{s^2-8s+16-25} = \frac{s-4}{s^2-8s-9}$ (Ans:-)

Qo. $L\{e^{-2t}(3\cos 6t - 5\sin 6t)\}$

$\therefore L(3\cos 6t - 5\sin 6t) = L(3\cos 6t) - L(5\sin 6t)$

$= 3\times\frac{s}{s^2+36} - 5\times\frac{6}{s^2+36}$

$= \frac{3s-30}{s^2+36}$

$\therefore L\{e^{-2t}(3\cos 6t - 5\sin 6t)\} = \frac{3(s+2)-30}{(s+2)^2+36}$

$= \frac{3s+6-30}{s^2+4s+4+36}$

$= 3s-24/s^2+4s+40$ (Ans

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Q. Prove the change of scale property if $L\{f(t)\} = f(s)$ then $L\{f(at)\} = \frac{1}{a} f(s/a)$

Proof:- We have, $L f(at) = \int_0^\alpha e^{-st} f(at)\, dt$

putting $t = \frac{u}{a}$

$dt = d(\frac{u}{a})$

$= \int_0^\alpha e^{-s(u/a)} \cdot f(u)\, d(\frac{u}{a})$

$= \frac{1}{a} \int_0^\alpha e^{-s(u/a)} \cdot f(u)\, du$

$= \frac{1}{a} \left[\frac{e^{-s(\frac{u}{a})}}{-s/a} \right]_0^\alpha$

$= \frac{1}{a} f(s/a)$ (proved)

Q. Find (i) $L\{t \sin at\}$

(ii) $L\{t^2 \cos at\}$

Sol^n(i):- According to the multiplication of t^n theorem,

$L(\sin at) = \frac{a}{s^2+a^2}$

$\therefore L(t \sin at) = (-1)^1 \frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)$

$= -\frac{(s^2+a^2)\cdot\frac{da}{ds} - a \times \frac{d}{ds}(s^2+a^2)}{(s^2+a^2)^2}$

$= \frac{2as}{(s^2+a^2)^2}$

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(ii) According to the multiplication of t^n theorem,

$L(\cos at) = \frac{s}{s^2+a^2}$

$\therefore L(t^2 \cos at) = (-1)^2 \frac{d^2}{ds^2}\left(\frac{s}{s^2+a^2}\right)$

$= (-1)^2 \frac{d}{ds}\left(\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right)$

$= \frac{d}{ds}\left(\frac{a^2-s^2}{(s^2+a^2)^2}\right)$

$= \frac{d}{ds}\left(\frac{a^2-s^2}{s^4+2s^2a^2+a^4}\right)$

$$= \frac{(s^2+a^2)^2 \cdot -2s - (a^2-s^2)\{4s^3+4sa^2\}}{(s^2+a^2)^4}$$

$$= \frac{(s^2+a^2)^2 \cdot -2s - (a^2-s^2)\cdot 4s(s^2+a^2)}{(s^2+a^2)^4}$$

$$= \frac{(s^2+a^2)\{(s^2+a^2)\cdot -2s - (a^2-s^2)\cdot 4s\}}{(s^2+a^2)^4}$$

$$= \frac{-2s^3 - 2a^2s - 4a^2s + 4s^3}{(s^2+a^2)^4}$$

$$= \frac{2s^3 - 6a^2s}{(s^2+a^2)^4}$$ (Ans:-)

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Ex. prove that, $L\{\sin t\} = L\left\{\int_0^t \frac{\sin u}{u} du\right\} = \frac{1}{s}\tan^{-1}\frac{1}{s}$

SOL^{n}:- putting, $u = tv$ $\quad u=0, v=0$

$du = t\,dv \quad u=t, v=1$

$$\int_0^t \frac{\sin u}{u} du = \int_0^1 \frac{\sin tv}{\cancel{t}v} \cancel{t}dv = \int_0^1 \frac{\sin tv}{v} dv$$

$$\Rightarrow L\left\{\int_0^t \frac{\sin u}{u} du = \int_0^\alpha e^{-st}\left\{\int_0^1 \frac{\sin tv}{v} dv\right\} dt\right.$$

$$= \int_0^1 \frac{1}{v}\left\{\int_0^\alpha e^{-st}\cdot \sin tv\, dt\right\} dv$$

$$= \int_0^1 \frac{1}{\cancel{v}} \cdot \frac{\cancel{v}}{s^2+v^2} dv$$

$$= \int_0^1 \frac{dv}{s^2+v^2}$$

$$= \left[\frac{1}{s}\tan^{-1}\frac{v}{s}\right]_0^1$$

$$= \frac{1}{s}\tan^{-1}\frac{1}{s} \text{ (proved)}$$

Ex. prove that,

(i) $L(2t^2 - e^{-t})$

(ii) $L(10\sin 6t)$

(iii) $L(6\sin 2t - 5\cos 2t)$

(iv) $L(\sin t - \cos t)^2$

(v) $L(3\cosh 5t - 4\sinh 5t)$

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(vi) $L(3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t)$

(vii) $L(t^3 e^{-3t})$

(viii) $L(e^{-t}\cos 2t)$

(ix) $L(2e^{3t}\sin 4t)$

(x) $L(e^{-t}\sin^2 t)$

(xi) $L(1+te^{-t})^3$

Solution:-

(i) $L(2t^2 - e^{-t})$

$= L(2t^2) - L(e^{-t})$

$= 2\cdot\frac{2!}{s^3} - \frac{1}{s+1}$

$= \frac{4}{s^3} - \frac{1}{s+1}$

$= \frac{4s+4-s^3}{s^3(s+1)}$

(ii) $L(10\sin 6t)$

$= 10 \times \frac{6}{s^2+36}$

$= \frac{60}{s^2+36}$

(iii) $L(6\sin 2t - 5\cos 2t)$

$= L(6\sin 2t) - L(5\cos 2t)$

$= 6 \times \frac{2}{s^2+4} - 5 \times \frac{s}{s^2+4}$

$= \frac{12-5s}{s^2+4}$

(iv) $L(\sin t - \cos t)^2$

$= L(1 - 2\sin t\cos t)$

$= L(1 - \sin 2t)$

$= \frac{1}{s} - \frac{2}{s^2+4}$

$= \frac{s^2+4-2s}{s(s^2+4)}$

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(v) $L(3\cosh 5t - 4\sinh 5t)$

$= \frac{3\times s}{s^2-25} - \frac{4\times 5}{s^2-25}$

$= \frac{3s-20}{s^2-25}$

(vi) $L(3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t)$

$= \frac{3\times 4!}{s^5} - \frac{2\times 3!}{s^4} + 4\cdot\frac{1}{s+3} - \frac{2\cdot 5}{s^2+25} + 3\cdot\frac{s}{s^2+4}$

$= \frac{72}{s^5} - \frac{12}{s^4} + \frac{4}{s+3} - \frac{10}{s^2+25} + \frac{3s}{s^2+4}$

(vii) $L(t^3 e^{-3t})$

For $L(t^3) = \frac{3!}{s^4}$

$\therefore L(t^3 e^{-3t}) = \frac{3!}{(s+3)^4}$

(viii) $L(e^{-t}\cos 2t)$

$= \frac{(s+1)}{(s+1)^2+4}$

$= \frac{s+1}{s^2+2s+5}$

(x) $L(e^{-t}\sin^2 t)$

$= \frac{2}{(s+1)^2+4}$

(ix) $L(2e^{3t}\sin 4t)$

$= 2\times\frac{4}{(s-3)^2+16}$

$= \frac{8}{s^2-6s+25}$

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(x) $L(e^{-t}\sin^2 t)$

$= L\frac{1}{2}(e^{-t}(1-\cos 2t))$

$= \frac{1}{2} L(e^{-t} - e^{-t}\cos 2t)$

$= \frac{1}{2} \times \left\{ \frac{1}{s+1} - \frac{(s+1)}{(s+1)^2+4} \right\}$

$= \frac{1}{2} \left\{ \frac{1}{s+1} - \frac{s+1}{s^2+2s+5} \right\}$

$= \frac{1}{2} \left\{ \frac{\cancel{s^2}+\cancel{2s}+5-\cancel{s^2}-\cancel{2s}-1}{(s+1)(s^2+2s+5)} \right\}$

$= \frac{2}{(s+1)(s^2+2s+5)}$ (Ans:)

(xi) $L(1+te^{-t})^3$

$= L(1+3te^{-t}+3t^2e^{-2t}+t^3e^{-3t})$

$= \frac{1}{s} + 3\cdot\frac{\lfloor 1}{(s+1)^2} + 3\cdot\frac{2!}{(s+2)^3} + \frac{3!}{(s+3)^4}$

Q: Find $L\{f(t)\}$ if $f(t) = \begin{cases} 0 & 0<t<2 \\ 4 & t>2 \end{cases}$

$\therefore L f(t) = \int_0^{\alpha} e^{-st} f(t)\,dt$

$= \int_0^2 e^{-st}(0)\,dt + \int_2^{\alpha} e^{-st}\,4\,dt$

$= \int_2^{\alpha} e^{-st}\cdot 4\,dt = 4\left[\frac{e^{-st}}{-s}\right]_2^{\alpha} = \frac{4}{s}e^{-2s}$ (Ans:)

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Qn. Find $Lf(t)$, $f(t) = \begin{cases} 2t & 0 \le t \le 5 \\ 1 & t > 5 \end{cases}$

$\therefore \; L f(t) = \int_0^{\alpha} e^{-st} f(t)\, dt$

$= \int_0^{5} e^{-st} (2t)\, dt + \int_5^{\alpha} e^{-st} (1)\, dt$

$= 2\left[t \int_0^5 e^{-st} dt - \int\left\{\frac{dt}{dt} \int_0^5 e^{-st} dt\right\} dt\right] + \left[\frac{e^{-st}}{-s}\right]_5^{\alpha}$

$= 2\left\{\left[t \times \frac{e^{-st}}{-s}\right]_0^5 - \int_0^5 \frac{e^{-st}}{-s} dt\right\} + \frac{e^{-5s}}{s}$

$= \cancel{2\times 0} + \frac{2}{s}\int_0^5 e^{-st} dt + \frac{e^{-5s}}{s} + 2\left\{\frac{5e^{-5s}}{-s}\right\}$

$= \frac{2}{s}\left[\frac{e^{-st}}{-s}\right]_0^5 + \frac{e^{-5s}}{s} - \frac{10e^{-5s}}{s}$

$= \frac{2}{s}\left[\frac{e^{-5s}}{-s} + \frac{e^{-0}}{s}\right] + \frac{e^{-5s}}{s} - \frac{10e^{-5s}}{s}$

$= \frac{2}{s}\left(\frac{e^{-5s}}{-s} + \frac{1}{s}\right) - \frac{9e^{-5s}}{s}$

$= \frac{2}{s^2}(1 - e^{-5s}) - \frac{9e^{-5s}}{s}$ (Ans:-)

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Inverse laplace transformation

Defn:- If $Lf(x) = F(x)$ is defined as the laplace transformation of $f(x)$ then $f(x) = L^{-1} F(x)$ is known as inverse Laplace transformation.

Convolution theorem:-

Statement:- If $F(s) = Lf(x) = \int_0^\infty e^{-sx} f(x)\,dx$ and $G(s) = Lg(y) = \int_0^\infty e^{-sy} g(y)\,dy$ then $F(s)\,G(s) = \int_0^x L f(u)\, g(x-u)\,du$

$$= L\int_0^x f(x-u)\,g(u)\,du$$

proof:- We get, $F(s)\,G(s) = \int_0^\infty e^{-sx} f(x)\,dx \int_0^\infty e^{-sy} g(y)\,dy$

$= \int_0^\infty \int_0^\infty e^{-s(x+y)} f(x)\,g(y)\,dx\,dy$ when $x>0, y>0$

let $t = x+y$ and $u = y$

$\therefore x = t-u$ $\quad \therefore y = u$

Then $\iint e^{-s(x+y)} f(x)\,g(y)\,dx\,dy = \iint e^{-st} f(t-u)\,g(u)\,J\,dt\,du$

where J is Jaccobian given by. $J = \frac{\delta(x,y)}{\delta(t,u)}$

$$= \begin{vmatrix} \frac{\delta x}{\delta t} & \frac{\delta x}{\delta u} \\ \frac{\delta y}{\delta t} & \frac{\delta y}{\delta u} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}$$

$$= 1$$

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2nd shifting theorem:- If $L\{f(t)\} = F(s)$ then $L\{e^{-at} f(t)\} = F(s+a)$

proof:- we have, $L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\,dt$

$$\Rightarrow L\{e^{-at} f(t)\} = \int_0^{\infty} e^{-st} \cdot e^{-at} f(t)\,dt$$

$$= \int_0^{\infty} e^{-t(s+a)} f(t)\,dt$$

$$= \int_0^{\infty} e^{-ut} f(u)\,dt$$

$$= F(u) = F(s+a) \quad \text{[proved]}$$

Necessary formula:- (i) $L^{-1}\left(\frac{1}{s}\right) = 1$

(ii) $L^{-1}\left(\frac{a}{s}\right) = a$

(iii) $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

(iv) $L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$

(v) $L^{-1}\left(\frac{s}{s^2+a^2}\right) = \sin at$

(vi) $L^{-1}\left(\frac{a}{a^2+s^2}\right) = \cos at$

(vii) $L^{-1}\left(\frac{n!}{s^{n+1}}\right) = t^n$

(viii) $L^{-1}\left\{\frac{s+b}{(s+b)^2+a^2}\right\} = \cos at\, e^{-bt}$

(ix) $L^{-1}\left\{\frac{a}{(s+b)^2+a^2}\right\} = e^{-bt} \sin at$

1:- Find $L^{-1}\left(\frac{s+1}{s^2+2s}\right)$

we have, $\frac{s+1}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2}$

$$\therefore (s+1) = A(s+2) + Bs$$

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For $s=0$, $2A=1$, $A=\frac{1}{2}$

For $s=-2$, $-2B=-1$, $B=\frac{1}{2}$

Hence we get, $L^{-1}\left(\frac{s+1}{s^2+2s}\right) = \frac{1}{2}L^{-1}(1/s) + \frac{1}{2}L^{-1}\left(\frac{1}{s+2}\right)$

$= \frac{1}{2} + \frac{1}{2}e^{-2t}$ (Ans:)

2:- Find $L^{-1}\left(\frac{a^2}{s(s+a)^2}\right)$

we have, $\frac{a^2}{s(s+a)^2} = \frac{A}{s} + \frac{B}{s+a} + \frac{C}{(s+a)^2}$

$\therefore a^2 = A(s+a)^2 + B\cdot s(s+a) + Cs$

when $s=0$, $A=1$

" $s=-a$, $a^2 = 0 - Ca$ $\therefore C=-a$ and $B=-1$

$\therefore L^{-1}\left(\frac{a^2}{s(s+a)^2}\right) = \left(\frac{1}{s} - \frac{1}{s+a} - \frac{a}{(s+a)^2}\right)L^{-1}$

$= 1 - e^{-at} - a\,t\,e^{-at}$ (Ans:)

3:- Find $L^{-1}\left\{\frac{K^2}{s(s^2+K^2)}\right\}$

we have, $\frac{K^2}{s(s^2+K^2)} = \frac{A}{s} + \frac{(Bs+C)}{s^2+K^2}$

$\Rightarrow K^2 = A(s^2+K^2) + (Bs+C)s$

when, $s=0$, $A=1$ and $C=0$

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For $k^2 = -s^2$

$\therefore -s^2 = (Bs+c)s$

$\therefore (Bs+c) = -s$

$$\therefore L^{-1}\left\{\frac{k^2}{s(s^2+k^2)}\right\} = L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{s}{s^2+k^2}\right)$$

$$= 1 - \cos kt \quad \text{(Ans)}$$

4. prove that, $L^{-1}\left(\frac{6s-4}{s^2-4s+20}\right) = 2e^{2t}(3\cos 4t + \sin 4t)$

$$\therefore L^{-1}\left(\frac{6s-4}{s^2-4s+20}\right)$$

$$= L^{-1}\left(\frac{6s-4}{s^2-4s+4+4^2}\right)$$

$$= L^{-1}\left(\frac{6s-4}{(s-2)^2+4^2}\right)$$

$$= L^{-1}\left\{\frac{6s+8-12}{(s-2)^2+4^2}\right\}$$

$$= L^{-1}\left\{\frac{6(s-2)+8}{(s-2)^2+4^2}\right\}$$

$$= L^{-1}\left\{\frac{6(s-2)}{(s-2)^2+4^2}\right\} + L^{-1}\left\{\frac{8}{(s-2)^2+4^2}\right\}$$

$$= 6\cdot e^{2t}\cos 4t + 2e^{2t}\sin 4t$$

$$= 2e^{2t}(3\cos 4t + \sin 4t) \quad \text{(proved)}$$

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Q. Find $L^{-1}\{(s^2-1)^{-1}\}$

We have, $\frac{1}{s^2-1} = \frac{1}{(s+1)(s-1)} = \frac{A}{s+1} + \frac{B}{s-1}$

$\Rightarrow 1 = A(s-1) + B(s+1)$

For, $s=1$, $B = 1/2$

" $s=-1$, $A = -1/2$

$\therefore L^{-1}\{(s^2-1)^{-1}\} = L^{-1} -\frac{1}{2}\left(\frac{1}{s+1}\right) + L^{-1}\frac{1}{2}\left(\frac{1}{s-1}\right)$

$= -\frac{1}{2}e^{-t} + \frac{1}{2}e^{t}$

$= \frac{1}{2}(e^{t} - e^{-t})$ (Ans:-)

Q. Find $L^{-1}\frac{s^2}{(s^2+a^2)(s^2+b^2)}$

$= L^{-1}\left\{\frac{1}{a^2-b^2}\left[\frac{a^2}{s^2+a^2} - \frac{b^2}{s^2+b^2}\right]\right\}$

$= \frac{1}{a^2-b^2}\left\{L^{-1}\left(\frac{a^2}{s^2+a^2}\right) - L^{-1}\left(\frac{b^2}{s^2+b^2}\right)\right\}$

$= \frac{1}{a^2-b^2}(a\sin at - b\sin bt)$ (Ans:)

Q.7:- $L^{-1}\left(\frac{s-1}{(s+3)(s^2+2s+2)}\right)$

We have, $\frac{s-1}{(s+3)(s^2+2s+2)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+2s+2}$

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$\Rightarrow s-1 = A(s^2+2s+2) + Bs(s+3) + C(s+3)$

Equatting the co-efficients,

$0 = A + B$

$1 = 2A + 3B + C$

$-1 = 2A + 3C$

solving $A = -4/5,\ B = 4/5,\ C = 1/5$

$\therefore L^{-1}\left(\frac{s-1}{(s+3)(s^2+2s+2)}\right) = -4/5\, L^{-1}\left(\frac{1}{s+3}\right) + L^{-1}\frac{4/5\, s + 1/5}{s^2+2s+2}$

$= -\frac{4}{5} e^{-3t} + L^{-1} \frac{4}{5}\left\{\frac{s}{s^2+2s+1+1}\right\} + L^{-1}\cdot\frac{1}{5}\left\{\frac{1}{s^2+2s+2}\right\}$

$= -\frac{4}{5} e^{-3t} + L^{-1} \frac{4}{5}\left\{\frac{s+1-1}{s^2+2s+1+1}\right\} + L^{-1}\cdot\frac{1}{5}\left\{\frac{1}{(s+1)^2+1}\right\}$

$= -\frac{4}{5} e^{-3t} + \frac{4}{5} L^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} + \frac{4}{5} L^{-1}\left\{\frac{-1}{(s+1)^2+1}\right\} + \frac{1}{5} L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$

$= -\frac{4}{5} e^{-3t} + 4/5\, e^{-t}\cos t - 4/5\, e^{-t}\sin t + \frac{1}{5}\sin t\, e^{-t}$

Ex. 8:- $L^{-1}\left\{\frac{27-12s}{(s+4)(s^2+9)}\right\}$

we have, $\frac{27-12s}{(s+4)(s^2+9)} = \frac{A}{s+4} + \frac{Bs+C}{s^2+9}$

$\therefore\ 27 - 12s = A(s^2+9) + Bs(s+4) + C(s+4)$

Equating co-efficients, $0 = A + B$

$-12 = 4B + C$

$27 = 9A + 4C$

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solving, $A = 3, B = -3, C = 0$

$$\Rightarrow L^{-1}\left\{\frac{27-12s}{(s+4)(s^2+9)}\right\} = \left(\frac{3}{s+4} + \frac{-3s+0}{s^2+9}\right)L^{-1}$$

$$= 3L^{-1}\left(\frac{1}{s+4}\right) + L^{-1}\cdot(-3)\left(\frac{s}{s^2+3^2}\right)$$

$$= 3e^{-4t} - 3\cos 3t$$

Q.9:- $L^{-1}\left\{\frac{11s^2-2s+5}{(s-2)(2s-1)(s+1)}\right\}$

we have, $\frac{11s^2-2s+5}{(s-2)(2s-1)(s+1)} = \frac{A}{s-2} + \frac{B}{2s-1} + \frac{C}{s+1}$

$$\therefore 11s^2 - 2s + 5 = A(2s-1)(s+1) + B(s-2)(s+1) + C(s-2)(2s-1)$$

$$= A(2s^2+2s-s-1) + B(s^2+s-2s-2) + C(2s^2-s-4s+2)$$

$$= A(2s^2+s-1) + B(s^2-s-2) + C(2s^2-5s+2)$$

Equating the co-efficients, $2A+B+2C = 11$

$A - B - 5C = -2$

$-A - 2B + 2C = 5$

Here $A = 5, B = -3, C = 2$

$$\therefore L^{-1}\{f(s)\} = L^{-1}5\frac{1}{s-2} - 3L^{-1}\frac{1}{2s-1} + 5L^{-1}\frac{1}{s+1}$$

$$= 5\times e^{2t} - 3L^{-1}\frac{1}{2(s-1/2)} + 5e^{-t}$$

$$= 5e^{2t} - 3/2\, e^{t/2} + 5e^{-t}$$

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10:- $L^{-1}\left(\frac{s+1}{6s^2+7s+2}\right)$

We have, $\frac{s+1}{6s^2+7s+2} = \frac{s+1}{6s^2+4s+3s+2} = \frac{s+1}{3s(1+2s)+2(1+2s)}$

$= \frac{s+1}{(2s+1)(3s+2)} = \frac{A}{2s+1} + \frac{B}{(3s+2)}$

$\therefore (s+1) = A(3s+2) + B(2s+1)$

Equating co-efficients. $3A+2B=1$

$2A+B=1$

$\therefore A=1,\ B=-1$.

$\therefore L^{-1}\left(\frac{s+1}{6s^2+7s+2}\right) = L^{-1}\left(\frac{1}{2s+1}\right) + L^{-1}\left(\frac{-1}{3s+2}\right)$

$= L^{-1}\left(\frac{1}{2(s+1/2)}\right) + L^{-1}\left(\frac{-1}{3(s+2/3)}\right)$

$= \frac{1}{2}e^{-t/2} - \frac{1}{3}e^{-2/3t}$

Ex. 11:- $L^{-1}\left\{\frac{3s+16}{s^2-s-6}\right\}$

we have, $\frac{3s+16}{s^2-3s+2s-6} = \frac{3s+16}{s(s-3)+2(s-3)} = \frac{3s+16}{(s-3)(s+2)}$

$= \frac{A}{s-3} + \frac{B}{s+2}$ $\therefore 3s+16 = A(s+2) + B(s-3)$

Equating co-efficients. $A+B=3$

$2A-3B=16$

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solving $A = 6,\ B = -2$

$\therefore L^{-1}\{f(s)\} = L^{-1}\left(\frac{6}{s-3}\right) - L^{-1}\frac{2}{(s+2)}$

$= 6e^{3t} - 2e^{-2t}$

Q.12 :- $L^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\}$

we have, $\frac{s^2}{(s^2+2^2)^2} = \frac{s}{s^2+2^2}\cdot\frac{s}{s^2+2^2}$

$\therefore L^{-1}\left(\frac{s}{s^2+2^2}\right) = \cos 2t$

By convolution theorem we get,

$L^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\} = \int_0^t \cos 2u \cdot \cos(2t - 2u)\, du$

$= \int_0^t \cos 2u \cdot (\cos 2u \cdot \cos 2t + \sin 2t \sin 2u)\, du$

$= \int_0^t (\cos^2 2u \cdot \cos 2t + \cos 2u \sin 2u \sin 2t)\, du$

$= \frac{\cos 2t}{2} \int_0^t (1 + \cos 4u)\, du + \int_0^t \frac{\sin 2t}{2} \cdot \sin 4u\, du$

$= \frac{\cos 2t}{2}\left[u + \frac{\sin 4u}{4}\right]_0^t + \frac{\sin 2t}{2}\left[-\frac{\cos 4u}{4}\right]_0^t$

$= \frac{\cos 2t}{2}\left[t + \frac{\sin 4t}{4}\right] + \frac{\sin 2t}{2}\left[-\frac{\cos 4t}{4} + \frac{\cos 0}{4}\right]$

$= \frac{t}{2}\cos 2t + \frac{1}{8}\sin 4t \cos 2t - \frac{1}{8}\cos 4t \sin 2t + \frac{\sin 2t}{8}$

$= t/2 \cos 2t + \frac{1}{8}(\sin(4t - 2t)) + \frac{\sin 2t}{8}$

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$= t/2 \cos 2t + \frac{1}{4} \sin 2t$

Q. 13:- $L^{-1}\left\{\frac{3s-8}{s^2+4} - \frac{4s-24}{s^2-16}\right\}$

$= L^{-1}\left\{\frac{3s}{s^2+2^2} - \frac{8}{s^2+2^2}\right\} - L^{-1}\left\{\frac{4s}{s^2-4^2} + \frac{24}{s^2-4^2}\right\}$

$= 3 \cdot \cos 2t - 4 \sin 2t - 4 \cosh 4t + 6 \sinh 4t$

Q 14:- $L^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\}$

We have, $\frac{3s+1}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$

$\therefore 3s+1 = A(s^2+1) + (s-1)Bs + C(s-1)$

solving $A = 2,\ B = -2,\ C = 1$

$\therefore L^{-1}\{f(s)\} = \left(\frac{2}{s-1} + \frac{-2s+1}{s^2+1}\right) L^{-1}$

$= L^{-1}\left(\frac{2}{s-1}\right) - L^{-1}\left(\frac{2s}{s^2+1}\right) + L^{-1}\left(\frac{1}{s^2+1}\right)$

$= 2e^{t} - 2\cos t + \sin t$

Q. 15:- $L^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}$

$\therefore \frac{2s^2-4}{(s+1)(s-2)(s-3)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s-3}$

$\therefore 2s^2-4 = A(s-2)(s-3) + B(s+1)(s-3) + C(s+1)(s-2)$

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Solving, $A = -1/6,\ B = -4/3,\ C = 7/2$

$\therefore L^{-1}\{f(s)\} = -\frac{1}{6}L^{-1}\frac{1}{(s+1)} - 4/3\, L^{-1}\frac{1}{s-2} + 7/2\, L^{-1}\frac{1}{s-3}$

$= -\frac{1}{6}e^{-t} - 4/3\, e^{2t} + 7/2\, e^{3t}$

Q.16:- $L^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\}$

Here, $L^{-1}\left(\frac{1}{s^2}\right) = t,\quad L^{-1}\left(\frac{1}{(s+1)^2}\right) = te^{-t}$

By convolution theorem,

$L^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\} = \int_0^t ue^{-u}(t-u)\,du$

$= \int_0^t e^{-u}(ut-u^2)\,du$

$= \left[(ut-u^2)\cdot(-e^{-u}) - (t-2u)\cdot(e^{-u}) + (-2)\cdot(-e^{-u})\right]_0^t$

$= te^{-t} + 2e^{-t} + t - 2$ (Ans:)

Q.17:- $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$

Here, $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at,\quad L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a}$

By convolution theorem we get,

$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \int_0^t \cos au \cdot \frac{1}{a}\sin(at-au)\,du$

$= \frac{1}{a}\int_0^t \cos au(\sin at\cos au - \cos at\sin au)\,du$

$= \frac{1}{a}\sin at\int_0^t \cos^2 au\,du - \frac{1}{a}\cos at\int_0^t \frac{1}{2}\times 2\sin au\cos au\,d$

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$= \frac{1}{a}\sin at \int_0^t \frac{1}{2}(1+\cos 2au)\,du - \frac{1}{a}\cos at \cdot \frac{1}{2}\int_0^t \sin 2au\,du$

$= \frac{1}{2a}\sin at \left[u + \frac{\sin 2au}{2a}\right]_0^t - \frac{1}{2a}\cos at \left[\frac{-\cos 2au}{2a}\right]_0^t$

$= \frac{1}{2a}\sin at \left[t + \frac{\sin 2at}{2a}\right] - \frac{1}{2a}\cos at \left[\frac{-\cos 2at}{2a} + \frac{1}{2a}\right]$

$= \frac{1}{2a}\sin at \left[t + \frac{\cancel{2}\sin at \cos at}{\cancel{2}a}\right] - \frac{1}{2a}\cos at \left[\frac{1-\cos 2at}{2a}\right]$

$= \frac{t\sin at}{2a} + \frac{\sin^2 at \cdot \cancel{\cos at}}{2a^2} - \frac{1}{2a}\cos at \cdot \frac{\cancel{2\sin^2 at}}{\cancel{2a}}$

$= \frac{t\sin at}{2a}$ (Ans:-)

Ex 18:- $L^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\}$

$= L^{-1}\left\{\frac{3s+7}{s^2-2s+1-4}\right\}$

$= L^{-1}\left\{\frac{3s+7}{(s-1)^2-2^2}\right\}$

$= L^{-1}\left\{\frac{3s-3+10}{(s-1)^2-2^2}\right\}$

$= L^{-1}\left\{\frac{3(s-1)}{(s-1)^2-2^2}\right\} + L^{-1}\left\{\frac{10}{(s-1)^2-2^2}\right\}$

$= 3\cdot e^t \cosh 2t + 5\sinh 2t$ (Ans:)

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Application of differential equations

1:- prove that, $L\{f'(t)\} = sLf(t) - f(0)$

Proof:- $L\{f'(t)\} = \int_0^\alpha e^{-st} f'(t)\,dt$

$= \left[e^{-st} f(t)\right]_0^\alpha - \int_0^\alpha -s e^{-st} f(t)\,dt$

$= -f(0) + s\int_0^\alpha e^{-st} f(t)\,dt$

$= sLf(t) - f(0)$ (proved)

2:- prove that, $L\{f''(t)\} = s^2 Lf(t) - sf(0) - f'(0)$

Proof:- we know that, $L\{f'(t)\} = sLf(t) - f(0)$ --(i)

Replacing $f'(t)$ and $f(t)$ by $f''(t)$ and $f'(t)$ in (i),

$L\{f''(t)\} = sLf'(t) - f'(0)$

$= s\{sLf(t) - f(0)\} - f'(0)$

$= s^2 Lf(t) - f(0) - f'(0)$ (proved)

3:- prove that, $L\{f'''(t)\} = s^3 Lf(t) - s^2 f(0) - sf'(0) -$

Proof:- we know that,

$L\{f''(t)\} = s^2 Lf(t) - f(0) - f'(0)$ --- (i)

and $L\{f'(t)\} = sL[f(t)] - f(0)$ --- (ii)

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Replacing $f'(t)$ and $f''(t)$ by $f''(t)$ and $f'''(t)$ in (i) we get,

$L\{f'''(t)\} = s^2 L f'(t) - sf'(0) - f''(0)$

$= s^2\{sL f(t) - f(0)\} - sf'(0) - f''(0)$

$= s^3\{L f(t)\} - s^2 f(0) - sf'(0) - f''(0)$ (proved)

Q 4:- prove that, $L\{f^{(iv)}(t)\} = s^4 L\{f(t)\} - s^3 f(0) - s^2 f'(0) - sf''(0) - f'''(0)$

proof:- we know that,

$L f'''(t) = s^3 L f(t) - s^2 f(0) - sf'(0) - f''(0)$ --- (i)

Replacing $f'''(t)$ by $f^{iv}(t)$ and $f''(t)$ by $f'''(t)$ and get,

$L^{iv}(t) = s^3 L f'(t) - s^2 f'(0) - sf''(0) - f'''(0)$

$= s^3\{sL f(t) - f(0)\} - s^2 f'(0) - sf''(0) - f'''(0)$

$= s^4 L f(t) - s^3 f(0) - s^2 f'(0) - sf''(0) - f'''(0)$ (proved)

Q 5:- Solve the DE $y' + y = \sin t$, $y(0) = 1$, by Laplace transformation.

Soln:- Given that, $y' + y = \sin t$ --- (i)

Taking Laplace transformation both sides,

$L(y') + L(y) = L(\sin t)$

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$\Rightarrow s L(y) - y(0) + L(y) = \frac{1}{s^2+1}$

$\Rightarrow L(y)\ (s+1) = y(0) + \frac{1}{s^2+1}$

$\Rightarrow L(y)\,(s+1) = 1 + \frac{1}{s^2+1}$

$\Rightarrow L(y) = \frac{1}{s+1} + \frac{1}{(s+1)(s^2+1)}$

$\therefore \frac{1}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{(s+1)(s^2+1)}$ (say)

$\therefore 1 = A(s^2+1) + (Bs+C)(s+1)$

$= As^2 + A + Bs^2 + Bs + Cs + C$

Equating the co-efficients and get, $A = 1/2$, $B = -1/2$, $C = +1/2$

$\therefore L(y) = \frac{1}{s+1} + \frac{1}{2}\cdot\frac{1}{s+1} + (-\frac{1}{2})\frac{s}{s^2+1} + \frac{1}{2}\frac{1}{s^2+1}$

$L(y) = \frac{3}{2}\cdot\frac{1}{s+1} - \frac{1}{2}\frac{s}{s^2+1} + \frac{1}{2}\frac{1}{s^2+1}$

$\therefore y = \frac{3}{2}e^{-t} - 1/2\cos t + 1/2\sin t$ (Ans:-)

6:- Solve the DE $y'' + 9y = \cos 2t$, $y(0) = 1$, $y(\pi/2) = -1$ by laplace transformation.

Sol^n:- Given that, $y'' + 9y = \cos 2t$

$\Rightarrow L(y'') + 9L(y) = L(\cos 2t)$

$\Rightarrow s^2 L(y) - s\,y(0) - y'(0) + 9L(y) = \frac{s}{s^2+4}$

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$\Rightarrow L(y)\{s^2+9\} = s+c+\frac{s}{s^2+4}$ $[\,y'(0)=c \text{ (say)}\,]$

$\Rightarrow L(y) = \frac{s}{s^2+9} + \frac{c}{s^2+9} + \frac{s}{(s^2+9)(s^2+4)}$

$= \frac{s}{s^2+9} + \frac{c}{s^2+9} + \left\{\frac{s}{5(s^2+4)} - \frac{s}{5(s^2+9)}\right\}$

$\Rightarrow y = \cos 3t + c/3 \sin 3t + \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t$

$\therefore y = 4/5 \cos 3t + \frac{c}{3}\sin 3t + \frac{1}{5}\cos 2t$ — (i)

$\therefore y(\pi/2) = 4/5 \cos 3\pi/2 + c/3 \sin 3\cdot\pi/2 + 1/5 \cos\cdot 2\pi/2$

$\Rightarrow -1 = 0 - c/3 - 1/5$

$\Rightarrow c = \frac{12}{5}$

$\therefore y = \cos 3t + \frac{12}{5}\cdot\frac{1}{3}\sin 3t + \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t$

$= \frac{4}{5}\cos 3t + 4/5 \sin 3t + \frac{1}{5}\cos 2t$ (Ans:)

Q 7:- Find the bounded solⁿ of $\frac{\delta u}{\delta t} = \frac{\delta^2 u}{\delta x^2}$, $x>0, t>0$ such that $u(0,t)=1$, $u(x,0)=0$ by laplace transformation

solⁿ:- Given that, $\frac{du}{dt} = \frac{\delta^2 u}{\delta x^2}$

$\Rightarrow L\left(\frac{du}{dt}\right) = L\left(\frac{d^2u}{dx^2}\right)$

$\Rightarrow su - u(x,0) = \frac{d^2u}{dx^2}$

$\Rightarrow \frac{d^2u}{dx^2} - su = 0$ — (i)

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and $u(0,s) = 1/s$ ---- ②

From (i) we get, $u = u(x,s) = c_1 e^{x\sqrt{s}} + c_2 e^{-\sqrt{s}x}$

since $u(x,t)$ must be bounded as $x \to \alpha$

$u(x,s) = L\{u(x,t)\}$ must be also bounded $x \to \alpha$

we must have, $c_1 = 0$ and $\sqrt{s} > 0$ so that,

$u(x,s) = c_2 e^{-\sqrt{s}x}$ ---- ③

From (i) and (ii) we get, $c_2 = \frac{1}{s}$ so that,

$u(x,s) = \frac{e^{-\sqrt{s}x}}{s}$

$\therefore u(x,t) = L^{-1}(u(x,s))$

$= L^{-1}\left\{\frac{e^{-x\sqrt{s}}}{s}\right\}$

$= L^{-1}\left\{\frac{e^{-\sqrt{s}x}}{s}\right\}$

$= \operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right)$

$= \frac{2}{\sqrt{\pi}} \int_{x/(2\sqrt{t})}^{\alpha} e^{-u^2} du$

*** 8:- Solve $ty'' + y' + 4ty = 0$, $y(0) = 3$, $y'(0) = 0$

sol^n:- Given that, $ty'' + y' + 4ty = 0$

$\Rightarrow L\,t(y'') + Ly' + 4L\,t(y) = 0$

$\Rightarrow -\frac{d}{ds}\{s^2 y - s\,y(0) - y'(0)\} + \{sy - y(0)\} - 4\frac{dy}{ds} = 0$

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$\Rightarrow -2sy - s^2\frac{dy}{ds} + 3 + sy - 3 - 4\frac{dy}{ds} = 0$

$\Rightarrow -(s^2+4)\frac{dy}{ds} - sy = 0$

$\Rightarrow (s^2+4)\frac{dy}{ds} + sy = 0$

$\Rightarrow \frac{dy}{y} + \frac{s\,ds}{s^2+4}$ [By variable seperation]

$\Rightarrow \log y + \frac{1}{2}\log(s^2+4) = \log c$ [∵ integration]

$\Rightarrow y = \frac{c}{\sqrt{s^2+4}}$

$\Rightarrow L(y(t)) = \frac{c}{\sqrt{s^2+4}}$

$\Rightarrow Y = c\,L^{-1}\left(\frac{1}{\sqrt{s^2+4}}\right) = c\,J_0\,2(t)$ [Bessel equation]

$\Rightarrow Y(0) = c\,J_0(0)$

$\therefore c = 3$

$\therefore y = 3J_0(2t)$ (Ans!)

Q. (i) solve $ty'' + 2y' + 2y = 0$, $y(0) = 1$, $y(\pi) = 0$

** (ii) solve $y''' - ty'' + y = 1$, $y(0) = 1$, $y'(0) = 2$

Q: solve $y'' - 3y' + 2y = 4e^{2t}$, $y(0) = -3$, $y'(0) = 5$

sol^n:- $y'' - 3y' + 2y = 4e^{2t}$

$\Rightarrow L(y'') - 3L(y') + 2L(y) = 4L(e^{2t})$

$\Rightarrow s^2 L(y) - s\,y(0) - y'(0) - 3\{s L(y) - y(0)\} + 2L(y) = \frac{4}{s-2}$

$\Rightarrow s^2 y + 3s - 5 - 3sy - 9 + 2y = \frac{4}{s-2}$ $[\,L(y) = y\,]$

$\Rightarrow (s^2 - 3s + 2)\,y + 3s - 14 = \frac{4}{s-2}$

$\Rightarrow y\,(s^2 - 3s + 2) = \frac{4}{s-2} + 14 - 3s$

$\Rightarrow y = \frac{4}{(s-2)(s^2-3s+2)} + \frac{14-3s}{s^2-3s+2}$

$= \frac{4 + 14s - 3s^2 - 28 + 6s}{(s-2)\,(s-1)\,(s-2)}$

$= \frac{-3s^2 + 20s - 24}{(s-1)\,(s-2)^2}$

$= \frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$

$\therefore\ y = -7e^{+t} + 4e^{+2t} + 4te^{2t}$

$\therefore\ y = -7e^{t} + 4e^{2t} + 4te^{2t}$ (Ans:)

Q: $ty'' + 2y' + ty = 0,\ y(0) = 1,\ y(\pi) = 0$

Solⁿ: $ty'' + 2y' + ty = 0$

$\Rightarrow -\frac{d}{ds}\{s^2 y - s\,y(0) - y'(0)\} + 2\{sy - y(0)\} + (-1)\frac{d}{ds}y$

$\Rightarrow \cancel{-2sy} - s^2 y' + 1 + \cancel{2sy} - 2 - y' = 0$

$\Rightarrow -y'(s^2+1) = 1$

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$\Rightarrow y' = \frac{-1}{s^2+1}$

integrating, $y = -\tan^{-1}s + A$

Since $y \to 0$ as $s \to \infty$ we must have $A = \pi/2$

$\Rightarrow y = \pi/2 - \tan^{-1}s = \tan^{-1}\frac{1}{s}$

$\Rightarrow L\{y\} = \tan^{-1}\frac{1}{s}$

$\Rightarrow y = L^{-1}\{\tan^{-1}\frac{1}{s}\}$

$\therefore y = \frac{\sin t}{t}$ (Ans:-)

Q. 11 :- $ty'' + (1-2t)y' - 2y = 0$, $y(0) = 1$, $y'(0) = 2$

$\Rightarrow L(ty'') + L(1-2t)y' - 2L(y) = 0$

$\Rightarrow L(ty'') + L(y') - 2L(ty') - 2L(y) = 0$

$\Rightarrow \frac{-d}{ds}\{s^2L(y) - sy(0) - y'(0)\} + sL(y) - y(0) + 2\cdot \frac{d}{ds}\{sL(y) - y(0)\} - 2L\{y\} = 0$

$\Rightarrow \frac{-d}{ds}(s^2y - s - 2) + sy - 1 + 2\,\frac{d}{ds}(sy - 1) - 2y = 0$

$\Rightarrow -(2sy + s^2\frac{dy}{ds} - 1) + sy - 1 + \cancel{2y} - \cancel{2y} + 0 + 2s\frac{dy}{ds} = 0$

$\Rightarrow -2sy - s^2\frac{dy}{ds} + sy + 0 + 2s\,\frac{dy}{ds} = 0$

$\Rightarrow sy + s^2\frac{dy}{ds} - 2s\,\frac{dy}{ds} = 0$

$\Rightarrow y + s\frac{dy}{ds} - 2\frac{dy}{ds} = 0$

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$\Rightarrow y + (s-2)\dfrac{dy}{ds} = 0$

$\Rightarrow \dfrac{dy}{y} + \dfrac{ds}{s-2}$ [variable seperation]

$\Rightarrow \log y + \log(s-2) = \log c$

$\Rightarrow y = \dfrac{c}{s-2}$

$\Rightarrow L(y) = \dfrac{c}{s-2}$

$\Rightarrow y = c\,e^{2t}$

$\therefore y(0) = 1$. so, $1 = ce^0$

$\therefore c = 1$

$\therefore y = e^{2t}$ (Ans:-)

$\boxed{Q}$ 12:- Solve by Laplace trasforms of the boundary value problem $\dfrac{\delta u}{\delta t} = 4\dfrac{\delta^2 u}{\delta x^2}$, $u(0,t) = 0$, $u(3,t) = 0$, $u(x,0) = 10\sin 2\pi x - 6\sin 4\pi x$

Solⁿ:- Given that, $\dfrac{\delta u}{\delta t} = 4\dfrac{\delta^2 u}{\delta x^2}$

$\Rightarrow L\left\{\dfrac{\delta u}{\delta t}\right\} = 4L\left\{\dfrac{\delta^2 u}{\delta x^2}\right\}$

$\Rightarrow su - u(x,0) = 4\dfrac{d^2 u}{dx^2}$

$\Rightarrow su - 10\sin 2\pi x + 6\sin 4\pi x = 4\dfrac{d^2 u}{dx^2}$

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$$\Rightarrow 4\frac{d^2u}{dx^2} - su = 6\sin 4\pi x - 10\sin 2\pi x$$

$$\Rightarrow \frac{d^2u}{dx^2} - \frac{s}{4}u = 3/2\sin 4\pi x - 5/2\sin 2\pi x$$

The general sol^n is,

$$u(x,s) = u = c_1 e^{\sqrt{s/2}\,x} + c_2 e^{-\sqrt{s/2}\,x} - \frac{3/2\sin 4\pi x}{s/4 + 16\pi^2} + \frac{5/2\sin 2\pi x}{s/4 + 4\pi^2}$$

$$= c_1 e^{\sqrt{s/2}\,x} + c_2 e^{-\sqrt{s/2}\,x} - \frac{6\sin 4\pi x}{s + 64\pi^2} + \frac{10\sin 2\pi x}{s + 16\pi^2}$$

$\left[\because \frac{\sqrt{s}}{2}\right]$

$u(0,t) = 0$

$\therefore L\{u(0,t)\} = u(0,s) = c_1 + c_2 = 0$

$\therefore L\{u(3,t)\} = u(3,s) = c_1 e^{\sqrt{s/2}\cdot 3} + c_2 e^{-\sqrt{s/2}\cdot 3} = 0$

$\therefore c_1 = c_2 = 0$

$$\therefore L\{u(x,t)\} = u(x,s) = \frac{10\sin 2\pi x}{s + 16\pi^2} - \frac{6\sin 4\pi x}{s + 64\pi^2}$$

$$\Rightarrow u(x,t) = L^{-1}\left(\frac{10\sin 2\pi x}{s + 16\pi^2}\right) - L^{-1}\left(\frac{6\sin 4\pi x}{s + 64\pi^2}\right)$$

$$= 10\sin 2\pi x\, e^{-16\pi^2} - 6\sin 4\pi x\, e^{-64\pi^2} \quad \text{(Ans:-)}$$

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Q. 13 Solve by L.T. $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$, $u(0,t) = u(5,t) = 0$, $u(x,0) = 10\sin 4\pi x$

Sol^n:- Given that, $L\left(\frac{\partial u}{\partial t}\right) = 2L\left(\frac{\partial^2 u}{\partial x^2}\right)$

$\Rightarrow s(u) - u(x,0) = 2L\left(\frac{\partial^2 u}{\partial x^2}\right)$

$\Rightarrow 2\frac{d^2u}{dx^2} = su - 10\sin 4\pi x$

$\Rightarrow \frac{d^2u}{dx^2} - \frac{s}{2}u = -5\sin 4\pi x$ ---- (i)

The general sol^n is (i) $u = u(x,s) = c_1 e^{\sqrt{s/2}\,x} + c_2 e^{-\sqrt{s/2}\,x} + \frac{5\sin 4\pi x}{s/2 + 16\pi^2}$

$\Rightarrow u = c_1 e^{\sqrt{s/2}\,x} + c_2 e^{-\sqrt{s/2}\,x} + \frac{10\sin 4\pi x}{s + 32\pi^2}$ ---- (ii) $\left[\sqrt{\frac{s}{2}}\right.$

$\therefore u(0,t) = 0$

$L(u(0,t)) = u(0,s) = 0 \quad \therefore c_1 + c_2 = 0$

$L\{u(5,t)\} = u(5,s) = 0 \quad \therefore c_1 e^{\sqrt{s/2}\cdot 5} + c_2 e^{-\sqrt{s/2}\cdot 5} = 0$

$\therefore c_1 = c_2 = 0$

$u(x,s) = \frac{10\sin 4\pi x}{s + 32\pi^2}$

$\therefore L\{u(x,t)\} = \frac{10\sin 4\pi x}{s + 32\pi^2}$

$\therefore u(x,t) = L^{-1}\left(\frac{10\sin 4\pi x}{s + 32\pi^2}\right)$

$= 10e^{-32\pi^2 t}\sin 4\pi x$ (Ans.)

Q. 14:- Solve by L.T. $\frac{\partial u}{\partial t} = k^2\frac{\partial^2 u}{\partial x^2}$, $u(0,t) = 0$, $u(\pi,t) = 0$

$u(x,0) = \sin x$.

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$\underset{\sim\sim}{\text{SOL}^n}$:- Given that, $\frac{\delta u}{\delta t} = k^2 \frac{\delta^2 u}{\delta x^2}$

$\Rightarrow L(\delta u/\delta t) = k^2 L(\delta^2 u/\delta x^2)$

$\Rightarrow su - u(x,0) = h^2 \frac{d^2 u}{dx^2}$

$\Rightarrow su - \sin x = h^2 \frac{d^2 u}{dx^2}$

$\Rightarrow \frac{d^2 u}{dx^2} - \frac{s}{h^2} u = -\frac{\sin x}{h^2}$ --- (i)

The general sol^n is, $u(x,s) = c_1 e^{\sqrt{s}/h\, x} + c_2 e^{-\sqrt{s}/h\, x} + \frac{\frac{1}{h^2}\sin x}{s/h^2 + 1}$

$u(0,t) = 0$

$L\{u(0,t)\} = u(0,s) = 0 \quad \therefore c_1 + c_2 = 0$

$L\{u(\pi,t)\} = u(\pi,s) = c_1 e^{\sqrt{s}/h \cdot 0} + c_2 e^{-\frac{\sqrt{s}}{h}\cdot 0} = 0$

$\therefore c_1 = c_2 = 0$

$U(x,s) = \frac{\sin x}{s + h^2}$

$\therefore L\{U(x,t)\} = \frac{\sin x}{s+h^2}$

$\therefore U(x,t) = L^{-1}\left\{\frac{\sin x}{s+h^2}\right\}$

$= \sin x \cdot e^{-h^2 t}$ (Ans:)

**Q.15:- $y'' + a^2 y = F(t)$, $y(0) = 1$, $y'(0) = -2$

$\underset{\sim\sim}{\text{SOL}^n}$:- $L(y'') + L(a^2 y) = L\{F(t)\}$

$\Rightarrow (s^2 + a^2)y - s + 2 = f(s)$

$\Rightarrow (s^2 + a^2)y = s - 2 + f(s)$

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$$\Rightarrow y = \frac{s}{s^2+a^2} - \frac{2}{s^2+a^2} + \frac{f(s)}{s^2+a^2}$$

$$\Rightarrow y = L^{-1}\left(\frac{s}{s^2+a^2}\right) - L^{-1}\left(\frac{2}{s^2+a^2}\right) + L^{-1}\left(\frac{f(s)}{s^2+a^2}\right)$$

$$= \cos at - \frac{2}{a}\sin at + F(t)\,\frac{1}{a}\sin at$$

$$\therefore y = \cos at - 2/a \sin at + \int_0^t \frac{1}{a} F(u) \sin(at - au)\, du \text{ (Ans)}$$

Ex. 16:- $\int_0^\alpha \frac{\sin t}{t} dt$

Let $F(t) = \sin t$, $L\{F(t)\} = f(s) = \frac{1}{s^2+1}$

Then, $\int_0^\alpha \frac{\sin t}{t} dt = \int_0^\alpha \frac{du}{u^2+1} = [\tan^{-1} u]_0^\alpha = \cancel{-\tan^{-1}(0)} = \pi/2$

$= [\tan^{-1}\alpha - \tan^{-1}(0)]$

$= \tan\{\tan^{-1}(\pi/2)\}$

$= \pi/2$ (Ans.)