Number Theory and Cryptography

Chapter 4

Primes and Greatest Common Divisors

Section 4.3

Section Summary

- Prime Numbers and their Properties
- Conjectures and Open Problems About Primes
- Greatest Common Divisors and Least Common Multiples
- The Euclidian Algorithm
- gcd as Linear Combinations

Primes

Definition: A positive integer p greater than 1 is called ***prime*** if the only positive factors of p are 1 and p. A positive integer that is greater than 1 and is not prime is called ***composite***.

Example: The integer 7 is prime because its only positive factors are 1 and 7, but 9 is composite because it is divisible by 3.

The Fundamental Theorem of Arithmetic

Theorem: Every positive integer greater than 1 can be written uniquely as a prime or as the product of two or more primes where the prime factors are written in order of nondecreasing size.

Examples:

- $100 = 2 \cdot 2 \cdot 5 \cdot 5 = 2^2 \cdot 5^2$
- $641 = 641$
- $999 = 3 \cdot 3 \cdot 3 \cdot 37 = 3^3 \cdot 37$
- $1024 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^{10}$

Erastothenes
(276-194 B.C.)

The Sieve of Erastosthenes

- The ***Sieve of Erastosthenes*** can be used to find all primes not exceeding a specified positive integer. For example, begin with the list of integers between 1 and 100.
  - a. Delete all the integers, other than 2, divisible by 2.
  - b. Delete all the integers, other than 3, divisible by 3.
  - c. Next, delete all the integers, other than 5, divisible by 5.
  - d. Next, delete all the integers, other than 7, divisible by 7.
  - e. Since all the remaining integers are not divisible by any of the previous integers, other than 1, the primes are:

{2,3,5,7,11,15,1719,23,29,31,37,41,43,47,53,59,61,67,71,73,79,83,89, 97}

The Sieve of Erastosthenes

TABLE 1 The Sieve of Eratosthenes.

Integers divisible by 2 other than 2 receive an underline.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Integers divisible by 3 other than 3 receive an underline.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Integers divisible by 5 other than 5 receive an underline.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Integers divisible by 7 other than 7 receive an underline; integers in color are prime.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

If an integer n is a composite integer, then it has a prime divisor less than or equal to $\sqrt{n}$.

To see this, note that if $n = ab$, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Trial division, a very inefficient method of determining if a number n is prime, is to try every integer $i \leq \sqrt{n}$ and see if n is divisible by i.

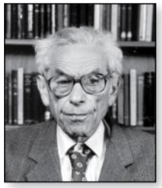
Infinitude of Primes

Theorem: There are infinitely many primes. (Euclid)

Proof: Assume finitely many primes: $p_1, p_2, \ldots.., p_n$

- Let $q = p_1 p_2 \cdots p_n + 1$
- Either q is prime or by the fundamental theorem of arithmetic it is a product of primes.
  - But none of the primes p_j divides q since if $p_j \mid q$, then p_j divides $q - p_1 p_2 \cdots p_n = 1$.
  - Hence, there is a prime not on the list $p_1, p_2, \ldots.., p_n$. It is either q, or if q is composite, it is a prime factor of q. This contradicts the assumption that $p_1, p_2, \ldots.., p_n$ are all the primes.
- Consequently, there are infinitely many primes.

Euclid
(325 B.C.E. – 265 B.C.E.)



Paul Erdős
(1913-1996)

Mersene Primes

Definition: Prime numbers of the form $2^p - 1$, where p is prime, are called *Mersene primes*.

– $2^2 - 1 = 3$, $2^3 - 1 = 7$, $2^5 - 1 = 37$, and $2^7 - 1 = 127$ are Mersene primes.

– $2^{11} - 1 = 2047$ is not a Mersene prime since $2047 = 23 \cdot 89$.

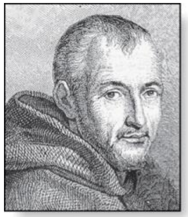
– There is an efficient test for determining if $2^p - 1$ is prime.

– The largest known prime numbers are Mersene primes.

– As of mid 2011, 47 Mersene primes were known, the largest is $2^{43,112,609} - 1$, which has nearly 13 million decimal digits.

– The *Great Internet Mersene Prime Search* (*GIMPS*) is a distributed computing project to search for new Mersene Primes.

http://www.mersenne.org/



Marin Mersenne
(1588-1648)

Distribution of Primes

- Mathematicians have been interested in the distribution of prime numbers among the positive integers. In the nineteenth century, the *prime number theorem* was proved which gives an asymptotic estimate for the number of primes not exceeding x.
- **Prime Number Theorem**: The ratio of the number of primes not exceeding x and $x/\ln x$ approaches 1 as x grows without bound. ($\ln x$ is the natural logarithm of x)
  - The theorem tells us that the number of primes not exceeding x, can be approximated by $x/\ln x$.
  - The odds that a randomly selected positive integer less than n is prime are approximately $(n/\ln n)/n = 1/\ln n$.

Generating Primes

- So far, no useful closed formula that always produces primes has been found. There is no simple function $f(n)$ such that $f(n)$ is prime for all positive integers n.
- But $f(n) = n^2 - n + 41$ is prime for all integers 1,2,…, 40. Because of this, we might conjecture that $f(n)$ is prime for all positive integers n. But $f(41) = 41^2$ is not prime.

Conjectures about Primes

- Even though primes have been studied extensively for centuries, many conjectures about them are unresolved, including:
- **Goldbach's Conjecture:** Every even integer n, $n > 2$, is the sum of two primes. It has been verified by computer for all positive even integers up to $1.6 \cdot 10^{18}$. The conjecture is believed to be true by most mathematicians.
- There are infinitely many primes of the form $n^2 + 1$, where n is a positive integer. But it has been shown that there are infinitely many primes of the form $n^2 + 1$, where n is a positive integer or the product of at most two primes.
- **The Twin Prime Conjecture:** The twin prime conjecture is that there are infinitely many pairs of twin primes. Twin primes are pairs of primes that differ by 2. Examples are 3 and 5, 5 and 7, 11 and 13, etc. The current world's record for twin primes (as of mid 2011) consists of numbers $65{,}516{,}468{,}355 \cdot 2^{333{,}333} \pm 1$, which have 100,355 decimal digits.

Greatest Common Divisor

Definition: Let a and b be integers, not both zero. The largest integer d such that $d \mid a$ and also $d \mid b$ is called the greatest common divisor of a and b.

The greatest common divisor of a and b is denoted by $\gcd(a,b)$.

One can find greatest common divisors of small numbers by inspection.

Example: What is the greatest common divisor of 24 and 36?

Solution: $\gcd(24, 36) = 12$

Example: What is the greatest common divisor of 17 and 22?

Solution: $\gcd(17,22) = 1$

Greatest Common Divisor

Definition: The integers *a* and *b* are *relatively prime* if their greatest common divisor is 1.

Example: 17 and 22

Definition: The integers a_1, a_2, …, a_n are *pairwise relatively prime* if $\gcd(a_i, a_j) = 1$ whenever $1 \leq i < j \leq n$.

Example: Determine whether the integers 10, 17 and 21 are pairwise relatively prime.

Solution: Because gcd(10,17) = 1, gcd(10,21) = 1, and gcd(17,21) = 1, 10, 17, and 21 are pairwise relatively prime.

Example: Determine whether the integers 10, 19, and 24 are pairwise relatively prime.

Solution: Because gcd(10,24) = 2, 10, 19, and 24 are not pairwise relatively prime.

Finding the GCD Using Prime Factorizations

- Suppose the prime factorizations of *a* and *b* are:

$$a = p_1^{a_1} p_2^{a_2} \ldots p_n^{a_n} \ , \qquad b = p_1^{b_1} p_2^{b_2} \ldots p_n^{b_n} \ ,$$

where each exponent is a nonnegative integer, and where all primes occurring in either prime factorization are included in both. Then:

$$\gcd(a,b) = p_1^{\min(a_1,b_1)} p_2^{\min(a_2,b_2)} \ldots p_n^{\min(a_n,b_n)} \ .$$

- This formula is valid since the integer on the right (of the equals sign) divides both *a* and *b*. No larger integer can divide both *a* and *b*.

Example: $120 = 2^3 \cdot 3 \cdot 5 \qquad 500 = 2^2 \cdot 5^3$

$\gcd(120,500) = 2^{\min(3,2)} \cdot 3^{\min(1,0)} \cdot 5^{\min(1,3)} = 2^2 \cdot 3^0 \cdot 5^1 = 20$

- Finding the gcd of two positive integers using their prime factorizations is not efficient because there is no efficient algorithm for finding the prime factorization of a positive integer.

Least Common Multiple

Definition: The least common multiple of the positive integers *a* and *b* is the smallest positive integer that is divisible by both *a* and *b*. It is denoted by lcm(a,b).

- The least common multiple can also be computed from the prime factorizations.

$$\mathrm{lcm}(a,b) = p_1^{\max(a_1,b_1)} p_2^{\max(a_2,b_2)} \cdots p_n^{\max(a_n,b_n)}$$

This number is divided by both *a* and *b* and no smaller number is divided by *a* and *b*.

Example: $\mathrm{lcm}(2^3 3^5 7^2, 2^4 3^3) = 2^{\max(3,4)}\, 3^{\max(5,3)}\, 7^{\max(2,0)} = 2^4\, 3^5\, 7^2$

Least Common Multiple

Theorem 5: Let a and b be positive integers. Then

$$ab = \gcd(a,b) \cdot \mathrm{lcm}(a,b)$$

Home Task: (*proof is Exercise* 31)

Euclid
(325 B.C.E. – 265 B.C.E.)

Euclidean Algorithm

- The Euclidian algorithm is an efficient method for computing the greatest common divisor of two integers. It is based on the idea that gcd(a,b) is equal to gcd(a,c) when $a > b$ and c is the remainder when a is divided by b.

Example: Find gcd(91, 287):

- $287 = 91 \cdot 3 + 14$ Divide 287 by 91
- $91 = 14 \cdot 6 + 7$ Divide 91 by 14
- $14 = 7 \cdot 2 + 0$ Divide 14 by 7

Stopping condition

gcd(287, 91) = gcd(91, 14) = gcd(14, 7) = 7

Euclidean Algorithm

- The Euclidean algorithm expressed in pseudocode is:

procedure *gcd*(a, b: positive integers)
$x := a$
$y := b$
while $y \neq 0$
    $r := x$ **mod** y
    $x := y$
    $y := r$
return x {gcd(a,b) is x}

- In Section 5.3, we'll see that the time complexity of the algorithm is $O(\log b)$, where $a > b$.

Correctness of Euclidean Algorithm

Lemma 1: Let $a = bq + r$, where a, b, q, and r are integers. Then $\gcd(a,b) = \gcd(b,r)$.

Proof:

- Suppose that d divides both a and b. Then d also divides $a - bq = r$ (by Theorem 1 of Section 4.1). Hence, any common divisor of a and b must also be any common divisor of b and r.
- Suppose that d divides both b and r. Then d also divides $bq + r = a$. Hence, any common divisor of a and b must also be a common divisor of b and r.
- Therefore, $\gcd(a,b) = \gcd(b,r)$.

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Correctness of Euclidean Algorithm

- Suppose that a and b are positive integers with $a \geq b$.
  Let $r_0 = a$ and $r_1 = b$.
  Successive applications of the division algorithm yields:

$$
\begin{array}{lll}
r_0 & = r_1 q_1 + r_2 & 0 \leq r_2 < r_1, \\
r_1 & = r_2 q_2 + r_3 & 0 \leq r_3 < r_2, \\
 & \ \ \cdot & \\
 & \ \ \cdot & \\
 & \ \ \cdot & \\
r_{n-2} & = r_{n-1} q_{n-1} + r_2 & 0 \leq r_n < r_{n-1}, \\
r_{n-1} & = r_n q_n \ . &
\end{array}
$$

- Eventually, a remainder of zero occurs in the sequence of terms: $a = r_0 > r_1 > r_2 > \cdots \geq 0$. The sequence can't contain more than a terms.
- By Lemma 1
  $\gcd(a,b) = \gcd(r_0,r_1) = \cdots = \gcd(r_{n-1},r_n) = \gcd(r_n, 0) = r_n$.
- Hence the greatest common divisor is the last nonzero remainder in the sequence of divisions.

gcd as Linear Combinations

Bézout's Theorem: If a and b are positive integers, then there exist integers s and t such that $\gcd(a,b) = sa + tb$.

(*proof in exercises of Section* 5.2)

Definition: If a and b are positive integers, then integers s and t such that $\gcd(a,b) = sa + tb$ are called *Bézout coefficients* of a and b. The equation $\gcd(a,b) = sa + tb$ is called *Bézout's identity.*

- By Bézout's Theorem, the gcd of integers a and b can be expressed in the form $sa + tb$ where s and t are integers. This is a *linear combination* with integer coefficients of a and b.
  - $\gcd(6,14) = (-2)\cdot 6 + 1\cdot 14$

Étienne Bézout
(1730-1783)

Finding gcd as Linear Combinations

Example: Express $\gcd(252,198) = 18$ as a linear combination of 252 and 198.

Solution: First use the Euclidean algorithm to show $\gcd(252,198) = 18$

i. $252 = 1\cdot 198 + 54$

ii. $198 = 3\cdot 54 + 36$

iii. $54 = 1\cdot 36 + 18$

iv. $36 = 2\cdot 18$

- Now working backwards, from iii and i above
  - $18 = 54 - 1\cdot 36$
  - $36 = 198 - 3\cdot 54$
- Substituting the 2nd equation into the 1st yields:
  - $18 = 54 - 1\cdot(198 - 3\cdot 54) = 4\cdot 54 - 1\cdot 198$
- Substituting $54 = 252 - 1\cdot 198$ (from i)) yields:
  - $18 = 4\cdot(252 - 1\cdot 198) - 1\cdot 198 = 4\cdot 252 - 5\cdot 198$

- This method illustrated above is a two pass method. It first uses the Euclidian algorithm to find the gcd and then works backwards to express the gcd as a linear combination of the original two integers. A one pass method, called the *extended Euclidean algorithm*, is developed in the exercises.

Consequences of Bézout's Theorem

Lemma 2: If a, b, and c are positive integers such that $\gcd(a, b) = 1$ and $a \mid bc$, then $a \mid c$.

Proof: Assume $\gcd(a, b) = 1$ and $a \mid bc$

- Since $\gcd(a, b) = 1$, by Bézout's Theorem there are integers s and t such that
  $$sa + tb = 1.$$
- Multiplying both sides of the equation by c, yields $sac + tbc = c.$
- From Theorem 1 of Section 4.1:
  $a \mid tbc$ (part ii) and a divides $sac + tbc$ since $a \mid sac$ and $a \mid tbc$ (part i)
- We conclude $a \mid c,$ since $sac + tbc = c.$

Lemma 3: If p is prime and $p \mid a_1a_2\cdots a_n$, then $p \mid a_i$ for some i.
(*proof uses mathematical induction; see Exercise* 64 *of Section* 5.1)

- Lemma 3 is crucial in the proof of the uniqueness of prime factorizations.

Uniqueness of Prime Factorization

- We will prove that a prime factorization of a positive integer where the primes are in nondecreasing order is unique. (This part of the fundamental theorem of arithmetic. The other part, which asserts that every positive integer has a prime factorization into primes, will be proved in Section 5.2.)

  Proof: (*by contradiction*) Suppose that the positive integer n can be written as a product of primes in two distinct ways:

  $$n = p_1 p_2 \cdots p_s \text{ and } n = q_1 q_2 \cdots p_t.$$

  - Remove all common primes from the factorizations to get

    $$p_{i_1} p_{i_2} \cdots p_{i_u} = q_{j_1} q_{j_2} \cdots q_{j_v}.$$

  - By Lemma 3, it follows that p_{i_1} divides q_{j_k}, for some k, contradicting the assumption that p_{i_1} and q_{j_k} are distinct primes.

  - Hence, there can be at most one factorization of n into primes in nondecreasing order.

◀

Dividing Congruence by an Integer

- Dividing by an integer relatively prime to the modulus does produce a valid congruence:

Theorem 7: Let m be a positive integer and let a, b, and c be integers. If $ac \equiv bc \pmod{m}$ and $\mathbf{gcd(}c,m\mathbf{) = 1}$, then $a \equiv b \pmod{m}$.

Proof: Since $ac \equiv bc \pmod{m}$, $m \mid ac - bc = c(a - b)$ by Lemma 2 and the fact that $\gcd(c,m) = 1$, it follows that $m \mid a - b$. Hence, $a \equiv b \pmod{m}$.

Query???

$$\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4\ldots.}}}}$$

$$\exists_{x\in\Re}\exists_{y\in\Re}(\,x = y\,) = ?$$

$$\sum_{x=1}^{\infty} x = ?$$

$$\sum_{x=1}^{\infty}\frac{1}{x} = ?$$

$$\forall_{x}(\,\Re \mid x\,) = ?$$

$$\exists_{x\in\Re}\exists_{y\in\Re}(\,x = y\,) = ?$$

$$\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4\ldots.}}}} = ?$$

$$1-1+1-1+1\ldots\ldots\ldots = ?$$

$$\sum_{x=1}^{\infty}\frac{1}{x} = ?$$