MFA

1(A)-Day

Date: 01-10-2016

Course no.: Math 2213

Course name: Complex Variable, Differential Equations and Harmonic Analysis

Ref. Book:

Theory and Problems of Complex Analysis

— M.R. Spiegel

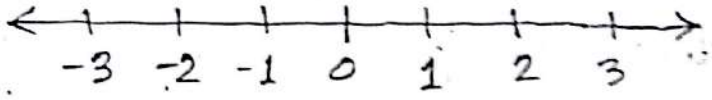
Complex Number

$2+3i = z$

$x^2+1=0$

$x = \pm\sqrt{-1}$

i
j
k
$= \sqrt{-1} \rightarrow$ imaginary number



$z = x+iy$

where x is the real part and y is the imaginary part

In polar form,

$z = x+iy$

$= r\cos\theta + ir\sin\theta$

$= r(\cos\theta + i\sin\theta)$

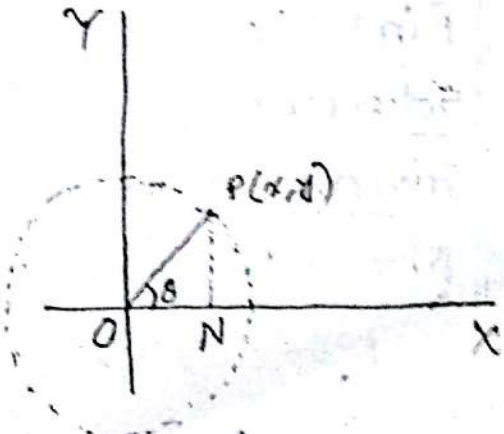
$= re^{i\theta}$

where, $r = |z| = \sqrt{x^2+y^2}$

$y = r\sin\theta$

$x = r\cos\theta$

$\theta = \tan^{-1}\frac{y}{x}$



θ is called the argument of amplitude of z.

Complex Conjugate:

if $z = x+iy$

* Modulus of z is always real number and represents an equation of circle.

Prob-1

Prove that

i) $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$

ii) $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

iii) $z\bar{z} = |z|^2$

Prob-2

Prove that

i) $|z_1+z_2| \leq |z_1| + |z_2|$

ii) $|z_1-z_2| \geqslant |z_1| - |z_2|$

iii) $|\bar{z}| = |z|$

Prob-3

Find $|e^z|$ and $|e^{iz}|$ if $z = x+iy$

Solution:

Given, $z = x+iy$

Now, $|e^z| = |e^{x+iy}|$

$= |e^x \cdot e^{iy}|$

$= |e^x(\cos y + i\sin y)|$

$= \sqrt{(e^x\cos y)^2 + (e^x\sin y)^2}$

$= e^x\sqrt{\cos^2 y + \sin^2 y}$

$= e^x$

Again, $|e^{iz}| = |e^{i(x+iy)}|$

$\Rightarrow |e^{iz}| = |e^{ix-y}|$

$= |e^{ix} \cdot e^{-y}|$

$= |e^{-y}(\cos x + i\sin x)|$

$\therefore |e^{iz}| = \sqrt{(e^{-y}\cos x)^2 + (e^{-y}\sin x)^2}$

$= e^{-y}\sqrt{\cos^2 x + \sin^2 x}$

$= e^{-y}$

Prob-4

Prove that i) $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

ii) $\arg \bar{z} = -\arg z$

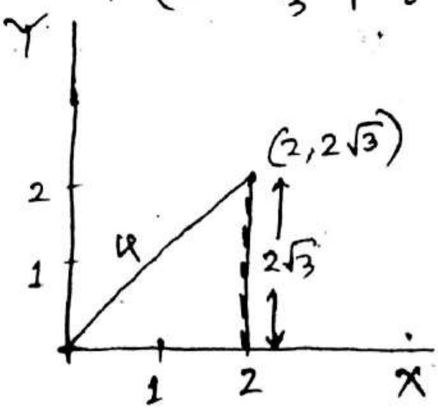
Prob-5 Express $2+2\sqrt{3}\, i$ in polar form.

Solution:

Modulus $= |z| = r = \sqrt{(2)^2 + (2\sqrt{3})^2} = 4$.

Amplitude, $\theta = \sin^{-1}\frac{2\sqrt{3}}{4} = \sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$

Then $2+2\sqrt{3}\, i = 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 4e^{\frac{\pi i}{3}}$



Prob-6: Express $-\sqrt{6} - \sqrt{2}\, i$ in polar form.

$r = |z| = \sqrt{(-\sqrt{6})^2 + (-\sqrt{2})^2} = \sqrt{8} = 2\sqrt{2}$

$\theta = \tan^{-1}\frac{-\sqrt{2}}{-\sqrt{6}} = \frac{7\pi}{6}$.

$\therefore -\sqrt{6} - \sqrt{2}\, i = 2\sqrt{2}\, e^{\frac{7\pi i}{6}}$

MRK

Mst. Rupali Khatun

1(B)-Day Date: 2/10/2016

Ref. Book:

Ordinary & Partial Differential Equation
— M.D. Raisingharia
S. Chand

Single independent varr. → Ordinary differentiation

$$\frac{d^2x}{dy^2} + \frac{dx}{dy} = 0$$

degree = 1

Order = 2

Partial Differential Equation (PDE):

A differential equation involving partial derivatives of one or more than one independent variable(s) is called PDE.

For example, i) $\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} + \frac{\delta^2 u}{\delta z^2} = 0$

ii) $\frac{\delta v}{\delta x} + x = v$

$$\frac{dy}{dx} = \left(\frac{dy}{dx}\right)^{\frac{1}{2}}$$

$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{dy}{dx}$ [Degree 2]

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$\frac{d^2y}{dx^2} + e^{y} = x$ → non-linear differential equation

y → dependent variable

Trancendental function → can be written as infinite series

$p = \frac{\partial z}{\partial x}$

$q = \frac{\partial z}{\partial y}$

$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial p}{\partial x}$

$s = \frac{\partial^2 z}{\partial x \cdot \partial y}$

$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial q}{\partial y}$

$\frac{1}{0}$ = undefined (অসংজ্ঞায়িত)

→ $\frac{a}{b} = z$

$\Rightarrow \frac{1}{0} = z$

There is no z.

$\frac{0}{0}$ = indeterminate (অনির্ণেয়)

→ $\frac{0}{0} = z$

$\therefore 0 = z \times 0$

For all z

1FA

1(E)-Day Date: 05/10/2016

De Moivre's Theorem:

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

where n is positive integer.

Proof:

Let, $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$\therefore z_1 z_2 = r_1 r_2 \{(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\}$$

$$= r_1 r_2 \{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\} \quad \ldots\ldots (i)$$

A generalization of (i)

$$z_1 z_2 \ldots z_n = r_1 r_2 \ldots\ldots r_n \{\cos(\theta_1+\theta_2+\cdots\cdots+\theta_n) + i\sin(\theta_1+\theta_2+\cdots+\theta_n)\} \quad \ldots\ldots (ii)$$

If $z_1 = z_2 = z_3 = \cdots\cdots = z_n$, (ii) becomes

$$z^n = r^n \{\cos n\theta + i\sin n\theta\}$$

We use principle of Mathematical induction.
Assume that the result is true for the particular positive integer k.

So, $(\cos\theta + i\sin\theta)^K = \cos K\theta + i\sin K\theta$

Then multiplying both sides by $(\cos\theta + i\sin\theta)$ we find

$$(\cos\theta + i\sin\theta)^{K+1} = \cos(K+1)\theta + i\sin(K+1)\theta$$

Thus if the result is true for $n=K$, then it is true for $n=K+1$. But since the result is clearly true for $n=1$, it must be true for $n=1+1=2$ a[nd] $n=2+1=3$ etc. and so must be true for all positive integer.

Roots of Complex Numbers:

From De Moivre's theorem we can show that if n is a positive integer

generalized

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left\{\cos\left(\frac{\theta+2K\pi}{n}\right) + i\sin\left(\frac{\theta+2K\pi}{n}\right)\right\} ;\ K=0,1,2,\ldots,(n-1)$$

provided $z \neq 0$

Prob: Prove $e^{i\theta} = e^{i(\theta+2K\pi)}$; $K=0,\pm1,\pm2,\ldots$

$$\begin{aligned} e^{i(\theta+2K\pi)} &= \cos(\theta+2K\pi) + i\sin(\theta+2K\pi) \\ &= \cos\theta + i\sin\theta \\ &= e^{i\theta} \end{aligned}$$

Prob:

a) Find the roots of $z^5+32=0$ and

b) Locate the values of z^5 in complex plane.

Solution:

(a) $z^5+32=0$

$\Rightarrow z^5=-32$

$\Rightarrow z^5 = 32(\cos\pi + i\sin\pi)$

$= 2^5\{\cos(\pi+2k\pi) + i\sin(\pi+2k\pi)\}$

$k=0,1,2,\ldots\ldots$

$\therefore z = 2\left\{\cos\left(\frac{\pi+2k\pi}{5}\right) + i\sin\left(\frac{\pi+2k\pi}{5}\right)\right\}$

Now, $z_1 = 2\left\{\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right\} = 2e^{\frac{\pi i}{5}}$ if $k=0$

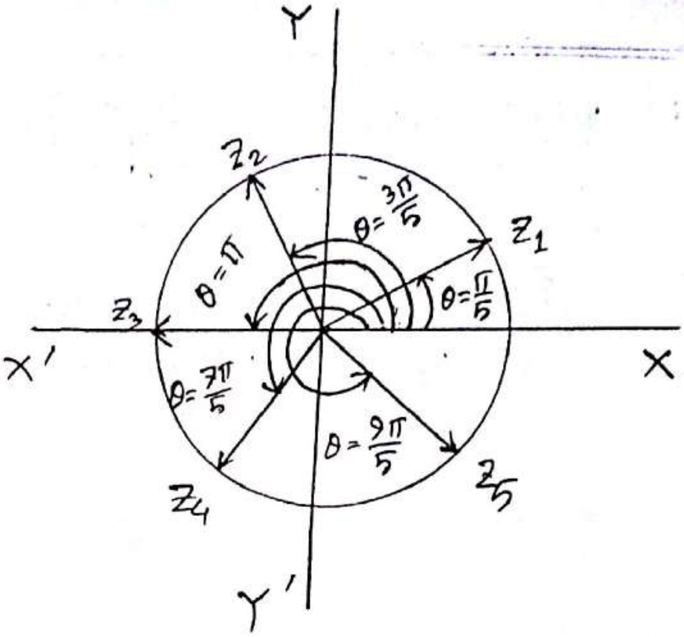
$z_2 = 2e^{\frac{3\pi i}{5}}$ if $k=1$

$z_3 = 2e^{\pi i}$ if $k=2$

$z_4 = 2e^{\frac{7\pi i}{5}}$ if $k=3$

$z_5 = 2e^{\frac{9\pi i}{5}}$ if $k=4$

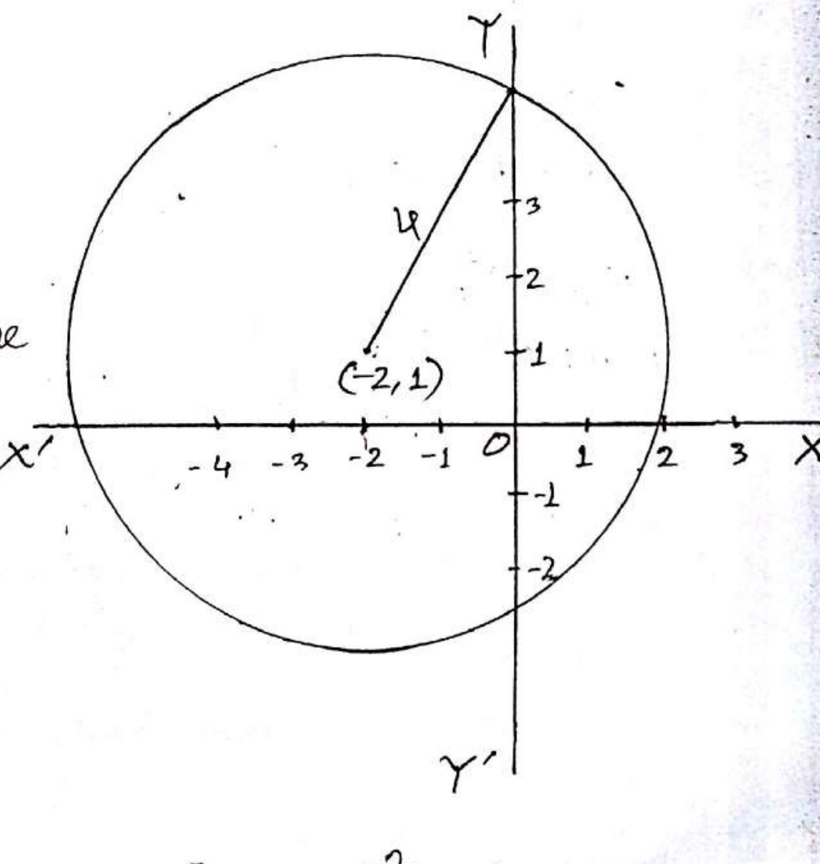
(b) $|z| = r = \sqrt{x^2+y^2}$



<u>Prob:</u> Find the equation for the circle of radius 4 with centre at $(-2,1)$ in complex plane.

<u>Solution:</u>

The ~~curve~~ centre can be represented by the complex number $-2+i$.
If z is any point on the circle, the distance from z to $(-2+i)$ is
$|z-(-2+i)| = 4$ or
$|z+2-i| = 4$ is the required equation



$|x+iy+2-i| = 4$

$\Rightarrow |(x+2)+i(y-1)| = 4$

$\Rightarrow \sqrt{(x+2)^2+(y-1)^2} = 4$

$\Rightarrow (x+2)^2 + (y-1)^2 = 16$

Date: 15/10/2016

<u>12(A)-Day</u>

Prob: Given a complex number z, interpret geometrically $ze^{i\alpha}$ where α is real.

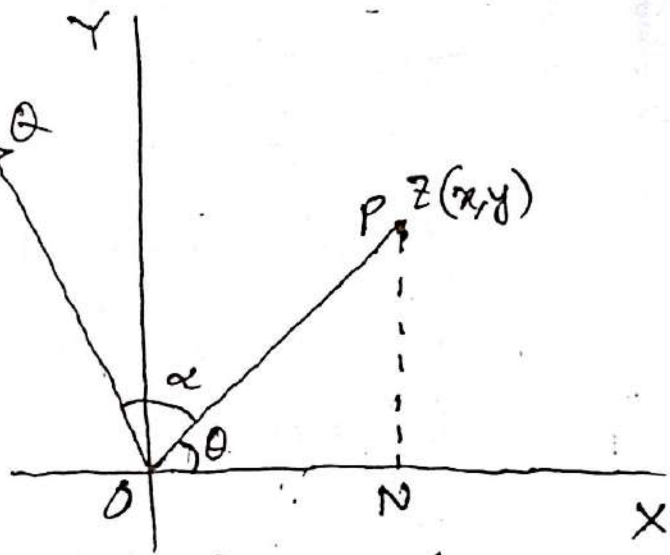


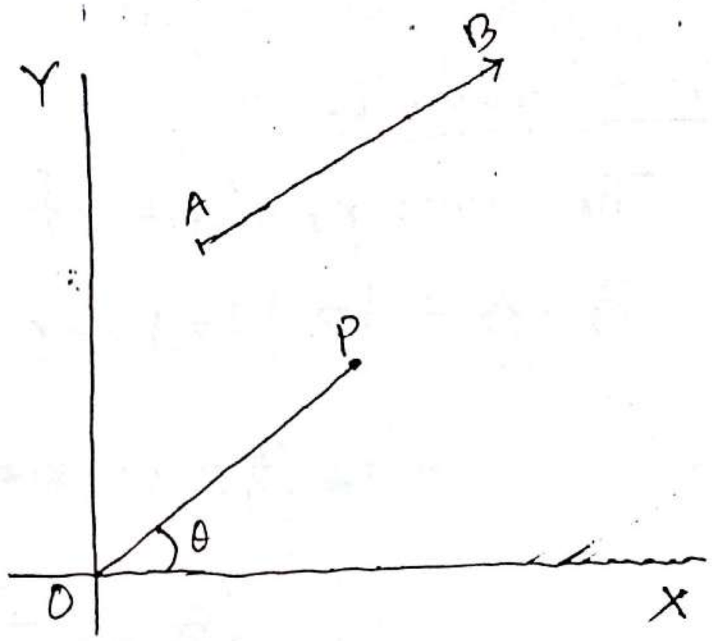
fig. 1

Solution:

Let, $z = re^{i\theta}$ be represented graphically by a vector OP in fig. 1.

Then $ze^{i\alpha} = re^{i\theta}e^{i\alpha} = re^{i(\theta+\alpha)}$ is the vector represented by OQ. Hence multiplication of a vector z by $e^{i\alpha}$ amount to relating z counterclockwise through an angle α!

Vector interpretation of a complex number:



$AB \parallel OP$

$|AB| = |OP|$

fig. 1

$$z = \vec{OP} = \vec{AB} = x + iy$$

Dot and Cross product of complex number

1. Dot product:

The dot (or scalar) product of z_1 & z_2 is defined by $z_1 \cdot z_2 = |z_1||z_2| \cos\theta = x_1 x_2 + y_1 y_2$

$$= Re\{\bar{z}_1 z_2\} = \frac{1}{2}\{\bar{z}_1 z_2 + z_1 \bar{z}_2\}$$

where θ is the angle between z_1 & z_2.

Re $\rightarrow$ Real

2. Cross product:

The cross product of z_1 & z_2 is defined by $z_1 \times z_2 = |z_1||z_2| \sin\theta = x_1 y_2 - y_1 x_2$

$$= Im\{\bar{z}_1 z_2\}$$

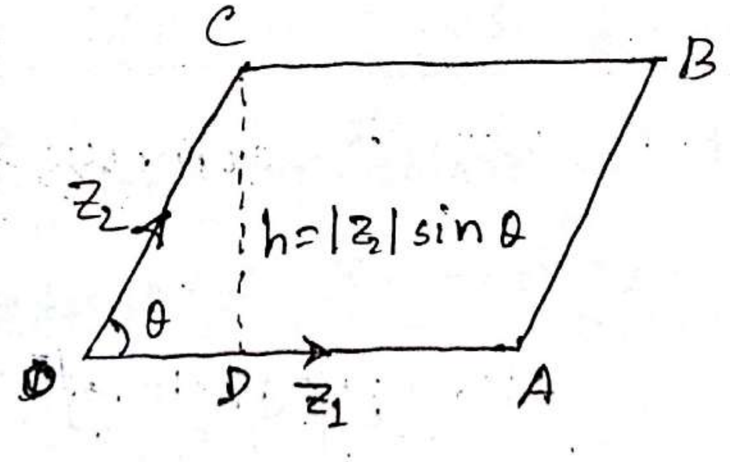
* Im → imaginary

$$= \frac{1}{2i}\{\bar{z}_1 z_2 - z_1 \bar{z}_2\}$$

Prob:

Prove that the area of a parallelogram having sides z_1 and z_2 is $|z_1 \times z_2|$

Soln:



Area of parallelogram = (base)(height)

$$= (OA)(h)$$

$$= |z_1||z_2| \sin\theta$$

$$= |z_1 \times z_2|$$

* Explain the fallacy:

$$-1 = \sqrt{-1}\cdot\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$$

Hence $-1 = 1$.

$$\downarrow$$

$$(i)(i) = i^2 = -1$$

H.W **Prob:**

Represent graphically the set of z for which

$$\left|\frac{z-3}{z+3}\right| = 2$$

Solution

$$\left|\frac{z-3}{z+3}\right| = 2$$

Or, $|z-3| = 2|z+3|$

Or, $|x+iy-3| = 2|x+iy+3|$

Or, $\sqrt{(x-3)^2+y^2} = 2\sqrt{(x+3)^2+y^2}$

Or, $(x-3)^2+y^2 = 4(x+3)^2+4y^2$ [squaring and simplifying]

Or, $x^2 - 6x+9+y^2 = 4x^2+24x+36+4y^2$

Or $3x^2+30x+3y^2+27 = 0$

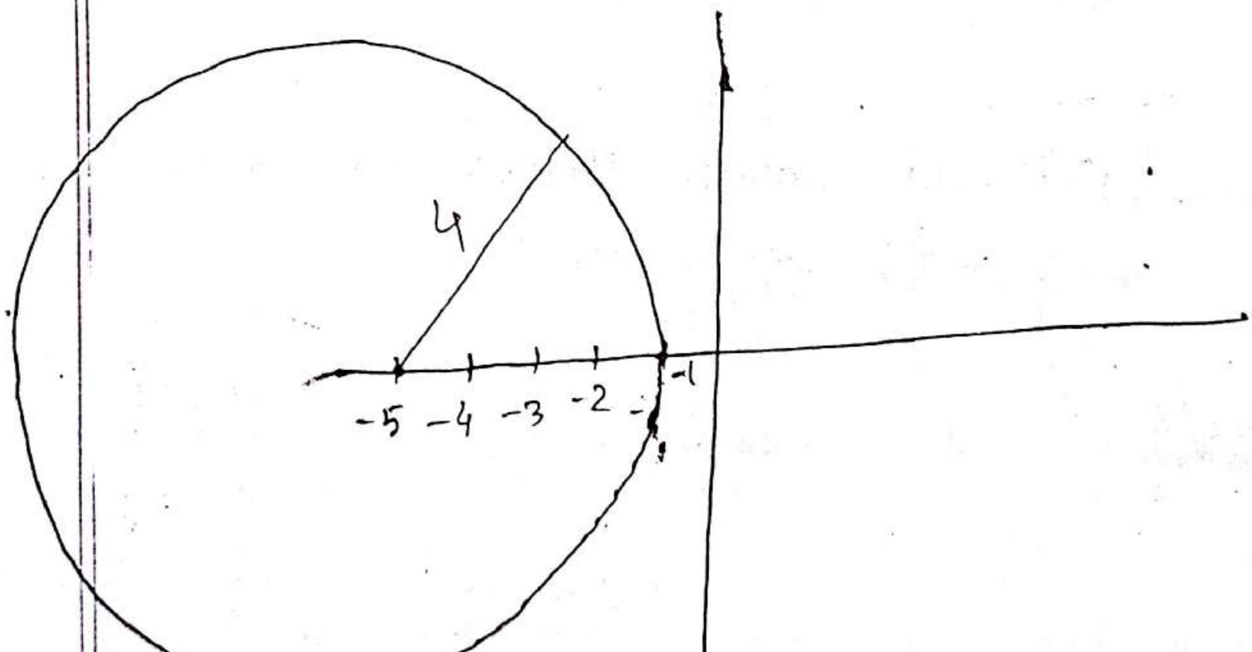
Or, $x^2 + 10x + y^2 + 9 = 0$

Or, $x^2 + 2\cdot5\cdot x + 5^2 + y^2 + 9 - 25 = 0$

Or, $(x+5)^2 + y^2 - 16 = 0$

Or, $(x+5)^2 + y^2 = 16$

Or, $|z+5| = 4$ is equation of circle of radius 4 with ~~circle~~ centre $(-5, 0)$



* Solution of 1st order DE by Lagrange's Method (Linear)

→ The Lagrange's equation of the form

$$Pp + Qq = R, \quad \cancel{whe}$$

where P, Q, R are functions of x, y, z and
$p = \frac{\delta z}{\delta x}, \; q = \frac{\delta z}{\delta y}$.

* The auxiliary equations of Lagrange's equ^n are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

* General solution of Lagrange's equ^n is

$f(u, v) = 0$ where, $u = c_1$ & $v = c_2$

or, $u = f(v)$

$\boxed{+}$ $Pp + Qq = R$

* $xp + yq = z$
* $x^2 p - y^2 q = -z^2$

} Lagrange's equation

Problem:

Solve the PDE $yq - xp = z$

Solution:

Given equation can be written as

$$-xp + yq = z \quad \ldots\ldots (i)$$

Equation (i) is Lagrange's equation, then the auxiliary equations are

$$\frac{dx}{-x} = \frac{dy}{y} = \frac{dz}{z} \quad \ldots\ldots (ii)$$

Taking 1st and 2nd fraction from (ii),

$$\frac{dx}{-x} = \frac{dy}{y}$$

$$\Rightarrow -\log x = \log y - \log c_1$$

$$\Rightarrow xy = c_1 \quad \ldots\ldots (iii)$$

Again taking 2nd and 3rd fraction from (ii),

$$\frac{dy}{y} = \frac{dz}{z}$$

$$\Rightarrow \log y = \log z + \log c_2$$

$$\Rightarrow \frac{y}{z} = c_2 \quad \ldots\ldots (iv)$$

Hence, the general solution of (i) is

$f(xy, y/z) = 0$

Or, $xy = f(y/z)$

> Any one should be written

H.W

* (i) Solve $y^2p - xyq = x(z-2y)$

(ii) " $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$.

Soln (i) Given,

$$y^2p - xyq = x(z-2y)$$

$$\Rightarrow y^2p - xyq = xz - 2xy \quad \text{—(i)}$$

Auxiliary equations are

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

Taking 1st and 2nd →

$$\frac{dx}{y^2} = \frac{dy}{-xy}$$

$$\Rightarrow \frac{dx}{y} = \frac{dy}{-x}$$

$$\Rightarrow -x\,dx = y\,dy$$

$$\Rightarrow -\frac{x^2}{2} = \frac{y^2}{2} - c_1 \quad \text{[Integrating]}$$

$$\therefore c_1 = \frac{x^2+y^2}{2}$$

see page 15

Taking ~~1st~~ 2nd & 3rd

$$\cancel{\frac{dy}{-xy} = \frac{dz}{x(z-2y)}}$$

$$\cancel{\Rightarrow \frac{dy}{-y} = \frac{dz}{z-2y}}$$

$$\cancel{\Rightarrow -\ln y = \ln(z-2y) - \ln c_2}$$

$$\cancel{c_2 = y(z-2y)}$$

~~Hence, the G.S of (i) is~~

$$\cancel{f\left(\frac{x^2+y^2}{2}, y(z-2y)\right) = 0}$$

(ii) Given,

$$(x^2-yz)p+(y^2-zx)q=z^2-xy \quad \text{--- (i)}$$

Auxiliary equations

$$\frac{dx}{x^2-yz}=\frac{dy}{y^2-zx}=\frac{dz}{z^2-xy}$$

~~Taking 1st and 2nd~~

$$\Rightarrow \frac{dx-dy}{x^2-yz-y^2+zx}=\frac{dy-dz}{y^2-zx-z^2+xy}=\frac{dz-dx}{z^2-xy-x^2+yz}$$

$$\Rightarrow \frac{dx-dy}{(x-y)(x+y+z)}=\frac{dy-dz}{(y-z)(x+y+z)}=\frac{dz-dx}{(z-x)(x+y+z)}$$

$$\Rightarrow \frac{dx-dy}{x-y}=\frac{dy-dz}{y-z}=\frac{dz-dx}{z-x}$$

Taking 1st 2 factions

$$\log(x-y)=\log(y-z)+\log c_1$$

$$c_1=\frac{x-y}{y-z}$$

Taking last 2 fractions

$$\log(y-z)=\log(z-x)+\log c_2$$

$$\therefore c_2=\frac{y-z}{z-x}$$

General Sol^n $\rightarrow$

$$f\left(\frac{x-y}{y-z},\frac{y-z}{z-x}\right)=0$$

Method of multipliers:

$$Pp + Qq = R$$

Auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{(l\,dx + m\,dy + n\,dz)}{(lP + mQ + nR)}$$

$= 0$

$= 0$

when P, Q and R are not simple

Problem:

Solve $x(z^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)$

Solution:

The Lagrange's auxiliary equations are

[Multiplying by x, y, z respecti

$$\frac{dx}{x(z^2 - y^2)} = \frac{dy}{y(x^2 - z^2)} = \frac{dz}{z(y^2 - x^2)} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

Now, from the 4th fraction, we get

$$x\,dx + y\,dy + z\,dz = 0 \Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_1/2$$

$x^2 + y^2 + z^2 = c_1$ — (i

and from the 5th fraction we get

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

$$\Rightarrow \log x + \log y + \log z = \log c_2 \Rightarrow xyz = c_2 \quad \text{— (ii)}$$

Hence the general solution of ~~(i)~~ is

$$f(x^2+y^2+z^2,\ xyz)=0$$

H.W

* **Solve:** i) $(x^2-y^2-z^2)p + 2xyq = 2xz$

ii) $x^2 p + y^2 q = (x+y)z$

Soln

(i) Given,

$$(x^2-y^2-z^2)\,p + 2xyq = 2xz$$

Auxiliary equations are

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$$

Using 2nd & 3rd fraction

$$\frac{dy}{2xy} = \frac{dz}{2xz}$$

$$\Rightarrow \frac{dy}{y} = \frac{dz}{z}$$

$$\Rightarrow \ln y = \ln z + \ln c_1$$

$$\therefore c_1 = \frac{y}{z} \quad \text{---(1)}$$

Using multipliers x, y, z we get

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} = \frac{x\,dx+y\,dy+z\,dz}{x(x^2+y^2+z^2)}$$

Taking last two fractions,

~~So,~~

$$\cancel{x\,dx+y\,dy+z\,dz=0}$$

$$\cancel{\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=\frac{c_2}{2}}$$

$$\cancel{\therefore c_2 = x^2+y^2+z^2 \quad \text{(ii)}}$$

$$\therefore \frac{dz}{2xz} = \frac{x\,dx+y\,dy+z\,dz}{x(x^2+y^2+z^2)}$$

$$\Rightarrow \frac{dz}{z} = \frac{2x\,dx+2y\,dy+2z\,dz}{x^2+y^2+z^2}$$

$$\Rightarrow \ln z = \ln(x^2+y^2+z^2) - \ln c_2$$

Hence the general solution is $c_2 = \frac{x^2+y^2+z^2}{z}$

$$f\left(\frac{y}{z}, \frac{x^2+y^2+z^2}{z}\right) = 0$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

(ii) Auxiliary equations are

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z} = \frac{yz\,dx+zx\,dy-xy\,dz}{0}$$

Using first two fractions

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

$$\Rightarrow x^{-2}\,dx = y^{-2}\,dy$$

$$\therefore -\frac{1}{x} = -\frac{1}{y} + c_2$$

$$\therefore c_2 = \frac{1}{y} - \frac{1}{x}$$

Using 4th fraction

$$yz\,dx + zx\,dy - xy\,dz = 0$$

Integrating,

$$\therefore xyz + xyz - xyz = c_1$$

$$\therefore c_1 = xyz$$

Hence the general solution is

$$f\left(xyz, \frac{1}{y} - \frac{1}{x}\right) = 0$$

2(E)-Day

Date: 19/10/2016

Problems

1. Find the real and imaginary parts of the followings-

   i) $3x + 2iy - ix + 5y = 7 + 5i$

   ii) $f(z) = 2e^{iz}$, iii) $f(z) = ize^{-z}$

   iv) $f(z) = \sqrt{z}$, v) $f(z) = \dfrac{-1 + i\sqrt{3}}{2}$

2. Find two complex numbers whose sum is 4 and whose product is 8.

3. If $z = 6e^{\pi i/3}$, evaluate $|e^{iz}|$

4. If $w = 3z - z^2$ and $z = x + iy$, find $|w|^2$ in terms of x and y

5. If $z = x + iy$, prove that $\sqrt{x} + \sqrt{y} \leq \sqrt{2}\,|x+iy|$

Complex function

Function, Limit, Continuity

Function: (def^n দেখতে হবে)

$w = f(z)$

i) **Single valued function:**

If only one value of w corresponds to each value z, we say it w is a single valued function.

Example:- $w = f(z) = z^2$ is a single valued function

ii) **Multiple valued function:**

If more than one value of w corresponds to each value z, we say that w is a multiple valued function.

Example: $w = f(z) = \sqrt{z}$ is a multiple valued function

Limit:

Let $f(z)$ be defined and single valued. We say that the number l is the limit of $f(z)$ as z approaches z_0 and write $\lim_{z \to z_0} f(z) = l$ if for any

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positive number ϵ (however small) we can find another positive number δ (depending on ϵ) such that $|f(z)-l|<\epsilon$ whenever $0<|z-z_0|<\delta$.

<u>Prob-1</u>. Evaluate $\lim_{z\to 0} \frac{\bar{z}}{z}$

<u>Solution:</u>

$$\lim_{z\to 0} \frac{\bar{z}}{z} = \lim_{(x,y)\to 0} \frac{x-iy}{x+iy}$$

$$= \lim_{(x,y=0)\to 0} \frac{x}{x}$$

$$= \lim_{x\to 0} \frac{x}{x}$$

$$=1 \quad , \{\text{if } y=0\}$$

Similarly, $\lim_{z\to 0} \frac{\bar{z}}{z} = \lim_{(x=0,y)\to 0} \frac{-iy}{iy}$

$$= \lim_{y\to 0} \frac{-iy}{iy}$$

$$= -1 \,, \quad \text{if } x=0$$

since two approaches do not give same result, the limit does not exist.

Prob-2

Evaluate the followings

i) $\lim_{z \to i} \left\{ (z-i) \frac{e^{imz}}{z^2+1} \right\}$

ii) $\lim_{z \to ae^{\pi i/4}} \left\{ (z - ae^{\pi i/4}) \frac{1}{z^4+a^4} \right\}$

iii) $\lim_{z \to \pi i} \frac{d}{dz} \left\{ (z-\pi i)^2 \frac{e^{imz}}{(z^2+1)^2} \right\}$

Solution:

(i) Given,

$$\lim_{z \to i} \left\{ (z-i) \frac{e^{imz}}{z^2+1} \right\}$$

$$= \lim_{z \to i} \left\{ (z-i) \frac{e^{imz}}{(z+i)(z-i)} \right\}$$

$$= \lim_{z \to i} \frac{e^{imz}}{z+i}$$

$$= \frac{e^{-m}}{2i}$$

MRK

3(A)-Day

Date: 22/10/2016

Problem: Solve $(y+z)p+(z+x)q=x+y$

Solution

The Lagrange's auxiliary equations are

$$\frac{dx}{y+z}=\frac{dy}{z+x}=\frac{dz}{x+y}=\frac{(y-z)dx+(z-x)dy+(x-y)dz}{y^2-z^2+z^2-x^2+x^2-y^2}$$

$$=\frac{(y-z)dx+(z-x)dy+(x-y)dz}{0}$$

$$=\frac{dx-dy}{y-x}=\frac{d(x-y)}{-(x-y)}$$

$$=\frac{dy-dz}{z-y}=\frac{d(y-z)}{-(y-z)}$$

Exact differential

$$=\frac{dx+dy+dz}{2(x+y+z)}=\frac{d(x+y+z)}{2(x+y+z)}$$

Taking 4th fraction, we get

$(y-z)dx+(z-x)dy+(x-y)dz=0$

$\Rightarrow ydx-zdx+zdy-xdy+xdz-ydz=0$

$\Rightarrow ydx-xdy+zdy-ydz+xdz-zdx=0$

$\Rightarrow$

Taking 5th and 6th ratios,

$$\frac{d(x-y)}{-(x-y)} = \frac{d(y-z)}{-(y-z)}$$

$$\Rightarrow \frac{d(x-y)}{(x-y)} = \frac{d(y-z)}{(y-z)}$$

$$\Rightarrow \ln(x-y) = \ln(y-z) + \ln c_1 \quad [\text{Integrating}]$$

$$\Rightarrow \frac{x-y}{y-z} = c_1 \quad \text{———} (*)$$

Again, taking 5th and 7th fractions

$$\frac{d(x-y)}{-(x-y)} = \frac{d(x+y+z)}{2(x+y+z)}$$

$$\Rightarrow 2\frac{d(x-y)}{(x-y)} + \frac{d(x+y+z)}{(x+y+z)} = 0$$

$$\Rightarrow 2\ln(x-y) + \ln(x+y+z) = \ln c_2$$

$$\Rightarrow (x-y)^2(x+y+z) = c_2 \quad \text{———} (**)$$

Hence, the general solution is

$$f\left(\frac{x-y}{y-z}, (x-y)^2(x+y+z)\right) = 0$$

Problem

Find the integral surface of the LPDEs
$x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$ —(i) which
contains the straight lines $x+y=0,\ z=1$ —(ii)

Solution

The Lagrange's auxiliary equations are

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)} = \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$$

$$= \frac{x\,dx + y\,dy - dz}{0}$$

Taking 4th ratio, $\ln x + \ln y + \ln z = \ln c_1$

$$\Rightarrow xyz = c_1 \quad \text{—(iii)}$$

Taking 5th ratio, $\frac{x^2}{2} + \frac{y^2}{2} - z = \frac{c_2}{2}$

$$\Rightarrow x^2+y^2-2z = c_2 \quad \text{—(iv)}$$

Taking t as a parameter, the given equation of the st. lines $x+y=0,\ z=1$ can be put in a parametric equ^n

$$x = t,\ y = -t,\ z = 1 \quad \text{—(v)}$$

Using (v) in (iii) and (iv), we get

$$-t^2 = c_1 \ , \ 2t^2 - 2 = c_2$$

These give,

$$2(-c_1) - 2 = c_2$$

$$\Rightarrow 2c_1 + c_2 + 2 = 0 \quad \text{——— (vi)}$$

Putting values of c_1 & c_2 in (vi), we get

$$2xyz + x^2 + y^2 - 2z + 2 = 0$$

(Ans)

(1) $2y(z-3)p + (2x-z)q = y(2x-3)$, $z=0$, $x^2+y^2=2x$

(2) $(x-y)p + (y-x-z)q = z$, $z=1$, $x^2+y^2=1$

Soln

(i) $$\frac{dx}{2y(z-3)} = \frac{dy}{2x-z} = \frac{dz}{y(2x-3)} = \frac{\frac{1}{2}dx + ydy - dz}{0}$$

Taking 1st and 3rd fractions,

$$\frac{dx}{2y(z-3)} = \frac{dz}{y(2x-3)}$$

$\Rightarrow (2x-3)\,dx = (2z-6)\,dz$

$\Rightarrow x^2 - 3x = z^2 - 6z + c_1$

$\therefore c_1 = x^2 - z^2 - 3x + 6z$ ——— (i)

Again, from 4th fraction

$\frac{1}{2}dx + ydy - dz = 0$

$\therefore \frac{1}{2}x + \frac{y^2}{2} - z = c_2/2$

$\Rightarrow c_2 = x + y^2 - 2z$ ——— (ii)

Now, the parametric equation given,

$x = t,\ y = (2t - t^2)^{\frac{1}{2}},\ z = 0$

Substituting these values in (i) & (ii) we get

$c_1 = t^2 - 3t \quad , \quad c_2 = t + 2t - t^2 = 3t - t^2$

These give, $c_1 + c_2 = 0$

Substituting the values of c_1 & c_2 from (i) & (ii)

$x^2 - z^2 - 3x + 6z + x + y^2 - 2z = 0$

$\Rightarrow x^2 + y^2 - z^2 - 2x + 4z = 0$

(ii) Given, $(x-y)p + (y-x-z)q = z$ and $x^2+y^2=1$, $z=1$

$$\frac{dx}{x-y} = \frac{dy}{y-x-z} = \frac{dz}{z} = \frac{dx+dy+dz}{0}$$

Taking 4th fraction,

$$dx + dy + dz = 0$$

$$\therefore c_1 = x + y + z \quad \text{——— (i)}$$

Taking the 2nd and 3rd fraction.

$$\frac{dy}{y-x-z} = \frac{dz}{z}$$

$$\Rightarrow \frac{dy}{y-(x+y+z-y)} = \frac{dz}{z}$$

$$\Rightarrow \frac{dy}{y-(c_1-y)} = \frac{dz}{z}$$

$$\Rightarrow \frac{2dy}{2y-c_1} = \frac{2dz}{z}$$

$$\Rightarrow \ln(2y-c_1) - 2\ln z = \ln c_2$$

$$\therefore c_2 = \frac{2y-c_1}{z^2} \quad \text{——— (ii)}$$

putting $z=1$ in (i) and (ii)

$$x+y = c_1 - 1$$

and $y-x = c_2+1$

But,

$$2(x^2+y^2) = (x+y)^2 + (y-x)^2$$

$$2 = (c_1-1)^2 + (c_2+1)^2$$

$$\Rightarrow c_1^2 + c_2^2 - 2c_1 + 2c_2 = 0$$

Putting the values of c_1 & c_2 from (i) & (ii),

$$(x+y+z)^2 + (y-x-z)^2/z^4 - 2(x+y+z) + 2(y-x-z)/z^2 = 0$$

or, $z^4(x+y+z)^2 + (y-x-z)^2 - 2z^4(x+y+z) + 2z^2(y-x-z) = 0$

3(B)-Day

Date: 23/10/2016

* Solution of 1st order non-LPDE using Charpit's Method:

Linear partial diff. equ^n

* Let the equation be $f(x, y, z, p, q) = 0$ —— (*)
* The Charpit's auxiliary equations are

to page details

$$\frac{dp}{\frac{\delta f}{\delta x} + p\frac{\delta f}{\delta z}} = \frac{dq}{\frac{\delta f}{\delta y} + q\frac{\delta f}{\delta z}} = \frac{dz}{-p\frac{\delta f}{\delta p} - q\frac{\delta f}{\delta q}} = \frac{dx}{-\frac{\delta f}{\delta p}} = \frac{dy}{-\frac{\delta f}{\delta q}}$$

* The complete integral (solution) is

$$dz = p\,dx + q\,dy$$

Non-Linear → Trancendental function of dependent variable

Problem

Find a complete integral of $px + qy = pq$

Sol^n:

Given that, $px + qy = pq$ —— (i)

Let $f(x,y,z,p,q) = px + qy - pq = 0$ —— (1)

The Charpit's auxiliary equ^{n}s are

$$\frac{dp}{\frac{\partial f}{\delta x} + p\frac{\partial f}{\delta z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\delta q}} = \frac{dx}{-\frac{\delta f}{\delta p}} = \frac{dy}{-\frac{\delta f}{\delta q}}$$

$$\Rightarrow \frac{dp}{p + p\cdot 0} = \frac{dq}{q + q\cdot 0} = \frac{dz}{-p(x-q) - q(y-p)} = \frac{dx}{-(x-q)} = \frac{dy}{-(y-p)}$$

Taking 1st and 2nd fractions, $\frac{dp}{p} = \frac{dq}{q}$

$$\Rightarrow \log p = \log q + \log a \text{ [Int.]}$$

$$\Rightarrow p = qa \quad \text{—— (ii)}$$

Putting (ii) in (i),

$$q \cdot ax + qy = qa \cdot q$$

$$\Rightarrow ax + y = qa$$

$$\therefore q = \frac{1}{a}(ax+y)$$

From (ii)

$$p = (ax+y)$$

Hence, the complete integral is

$$dz = p\,dx + q\,dy$$

$$\Rightarrow dz = (ax+y)dx + \frac{1}{a}(ax+y)dy$$

$$\Rightarrow a\,dz = a(ax+y)dx + (ax+y)dy$$

$$\Rightarrow a\,dz = (ax+y)(a\,dx + dy)$$

$$\Rightarrow a\,dz = u\,du$$

Put $ax + y = u$
$\Rightarrow a\,dx + dy = du$

$$\Rightarrow az = \frac{u^2}{2} + b \text{ [Integrating]}$$

$$\therefore az = \frac{1}{2}(ax+y)^2 + b$$

which is the req. complete integral of the given equⁿ.

H.W.
Must

① Find a complete integral of $q = 3p^2$

② Find a complete integral of $p^2 - y^2 q = y^2 - x^2$

MFA

3(E)-Day

Date : 1/11/2016

Continuity :

$f(z)$ be continuous at $z = z_0$ if

i) $\lim_{z \to z_0} f(z) = l$ must exist

ii) $f(z_0)$ must exist i.e. $f(z)$ is defined at z_0.

iii) $l = f(z_0)$

Prob

If $f(z) = \begin{cases} z^2 ; & z \neq i \\ 0 ; & z = i \end{cases}$

Is the function continuous at $z = i$? If not, redefine the function to be continuous.

Soln

$f(z)$ be continuous at $z = i$ if

i) $\lim_{z \to z_0} f(z) = l$ i.e $\lim_{z \to i} f(z) = \lim_{z \to i} z^2$

$= i^2 = -1$ exist

ii) $f(i) = 0$ exist

iii) $l \neq f(z_0)$ i.e. $-1 \neq 0$

So the function $f(z)$ is not continuous at $z = i$.

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If we redefine the function $f(z)$ as $f(z) = z^2$; for all value of z | $z = 1, 2, 2i, 3i, \ldots$

Prob

If $f(z) = \begin{cases} \dfrac{z^2+4}{z-2i} & ; z \neq 2i \\ 3+4i ; & z = 2i \end{cases}$ Is the function continuous at $z = 2i$? If not, redefine the function $f(z)$ to be continuous.

Soln

$f(z)$ be continuous at $z = 2i$ if

i) $\lim\limits_{z \to z_0} f(z) = l$ i.e. $\lim\limits_{z \to 2i} f(z) = \lim\limits_{z \to 2i} \dfrac{z^2+4}{z-2i}$

$$= \lim_{z \to 2i} \frac{z^2 - (2i)^2}{z - 2i}$$

$$= \lim_{z \to 2i} z + 2i$$

ii) $f(z_0) = f(2i) = 3+4i$ exist $= 4i$ exist

iii) $l \neq f(z_0)$ i.e. $4i \neq 3+4i$

So, the function $f(z)$ is not continuous at $z = 2i$.

If we redefine the function $f(z) = \frac{z^2+4}{z-2i}$ for all value of z - -

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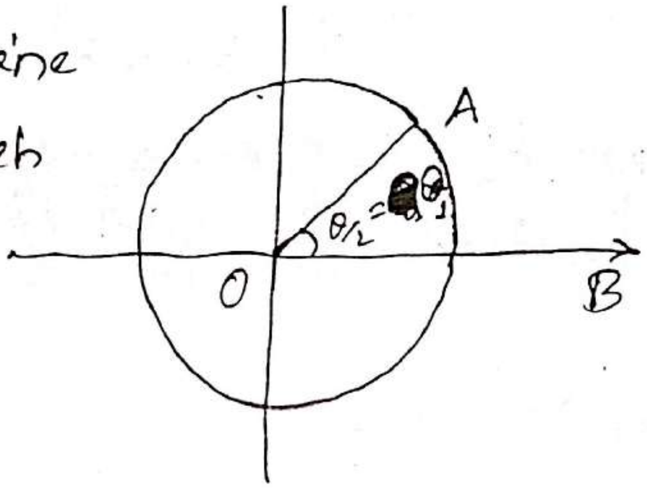
: Branch point and Branch line:

$$w = f(z) = \sqrt{z} = z^{\frac{1}{2}} = (re^{i\theta})^{\frac{1}{2}} = \sqrt{r}\, e^{i\theta/2}$$

OB is called Branch line and O is called branch point.



Prob

1. Prove that zeros of (i) $\sin z$ (ii) $\cos z$ are all real and find them.

2. Evaluate $\lim\limits_{z \to \frac{-a+\sqrt{a^2-b^2}}{b}} \left\{ \left(z - \frac{-a+\sqrt{a^2-b^2}}{b} \right) \frac{2}{bz^2+2aiz-b} \right\}$

4(A)-Day Date: 2/11/2016

Complex Differentiation:

Derivative:

If $f(z)$ is single valued in some region R of the z-plane, the derivative of $f(z)$ is defined as $f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$ provided that the limit exists independent of the manner in which $\Delta z \to 0$.

M.I

Analytic function:

If the derivative $f'(z)$ exists at all points z of a region R. then $f(z)$ is said to be analytic

Difference →

Derivative ⟶ some points

Analytic ⟶ all "

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Prob

Test the analyticity of $f(z) = \bar{z}$.

Solution

By definition:

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$ if the limit exist independent of the manner in which $\Delta z \to 0$.

$$\therefore f'(z) = \lim_{\Delta z \to 0} \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{x - iy + \Delta x - i\Delta y - (x - iy)}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

If $\Delta y = 0$, the required limit is $\lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$

If $\Delta x = 0$, " " " " $\lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1$

Since the limits depends in the manner in which $\Delta z \to 0$, the derivative does not exist. So, it is not analytic.

* Prove that i) necessary and ii) sufficient condition that $w = f(z) = u(x,y) + iv(x,y)$ be analytic in a region R is that the Cauchy-Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ are satisfied in R.

Proof

Necessity: In order for $f(z)$ to be analytic, the limit

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)\} - \{u(x,y) + iv(x,y)\}}{\Delta x + i\Delta y} \quad \ldots\ldots (1)$$

must exist independent of the manner in which Δz (Δx and Δy) approaches zero.

We consider two possible cases:

Case I: $\Delta y = 0$, $\Delta x \to 0$. In this case ① becomes:

$$\lim_{\Delta x \to 0} \left\{ \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i\left[\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}\right] \right\}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$ provided the partial derivative exists.

Case II: $\Delta x = 0$, $\Delta y \to 0$. In this case ① becomes

$$\lim_{\Delta y \to 0} \left\{ \frac{u(x,y+\Delta y) - u(x,y)}{i\Delta y} + \frac{v(x,y+\Delta y) - v(x,y)}{\Delta y} \right\}$$

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -i\frac{\partial u}{\delta y} + \frac{\partial v}{\partial y}$$

Now $f(z)$ can not possibly be analytic unless these two limits (two cases) are identical.

Thus a necessary condition that $f(z)$ be analytic is

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Sufficient : See book

Prob

Test the analyticity of the followings:

i) $f(z) = ze^{-z}$ ii) $f(z) = ize^{-z}$

Definition: 1. Laplacian: The operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is often called Laplacian.

2. Harmonic function: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ are Harmonic function.

Prob

Prove that real and imaginary points of an analytic function $f(z)$ of a complex variable ~~syste~~ satisfy Laplaces equation.

Proof

since the function $w = f(z) = u(x,y) + iv(x,y)$ be analytic function, so it satisfies the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{--- (2)}$$

Differentiation (i) w.r. to x, we get $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 v}{\partial y \partial x}$ --- (3)

(ii) w.r. to y, we get $\dfrac{\partial^2 u}{\partial x \partial y} = -\dfrac{\partial^2 u}{\partial y^2}$

or $\dfrac{\partial^2 u}{\partial y^2} = -\dfrac{\partial^2 v}{\partial x \partial y}$

--- (4)

Adding (3) & (4), we get Laplaces equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = - - - - = 0$$

Proof:

Similarly dif. (1) w.r. to y and

" (ii) " " x, we get

① Given, $q = 3p^2$

Let, $f(x,y,z,p,q) = 3p^2 - q = 0$ —— (i)

Charpit's auxiliary equations

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\therefore \frac{dp}{0 + p\cdot 0} = \frac{dq}{0 + q\cdot 0} = \frac{dz}{-p\cdot 6p - q(-1)} = \frac{dx}{-6p} = \frac{dy}{1}$$

Taking first fraction

$dp = 0$

$\therefore p = a$ [Integrating]

Putting $p = a$ in (i),

$q = 3a^2$

~~Sol~~ Hence the complete integral is

$$dz = p\,dx + q\,dy$$

$$\Rightarrow dz = a\,dx + 3a^2\,dy$$

$$\therefore z = ax + 3a^2y + b \quad \text{[Integrating]}$$

where a, b are arbitrary constants.

[Ans]

(2) Given,

$$p^2 - y^2 q = y^2 - x^2$$

Let, $f(x,y,z,p,q) = p^2 - y^2 q - y^2 + x^2 = 0$ —— (i)

Charpit's auxiliary equations

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + p\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-p\frac{\partial f}{\partial p}} = \frac{dy}{-q\frac{\partial f}{\partial q}}$$

$$\therefore \frac{dp}{2x} = \frac{dq}{-2qy - 2y} = \frac{dz}{-2p^2 - q(-y^2)} = \frac{dx}{-2p} = \frac{dy}{y^2}$$

—— (ii)

Taking 1st & 4th fraction

$$\frac{dp}{2x} = \frac{dx}{-2p}$$

$\Rightarrow \quad p\,dp = -x\,dx$

$\Rightarrow \quad \frac{p^2}{2} + \frac{x^2}{2} = \frac{a^2}{2}$ [integrating

$\Rightarrow \quad x^2 + p^2 = a^2$ —— (iii)

$\therefore p^2 = a^2 - x^2$

$\Rightarrow p = \sqrt{a^2 - x^2}$

Or, $x^2 = a^2 - p^2$

$p^2 = a^2 - x^2$, putting this in (i)

$$a^2 - x^2 - qy^2 - y^2 + x^2 = 0$$

$\Rightarrow \quad -qy^2 = y^2 - a^2$

$\Rightarrow qy^2 = a^2 - y^2$

$\Rightarrow q = a^2/y^2 - 1$

Hence the complete integral is

$dz = p\,dx + q\,dy$

$\Rightarrow dz = \sqrt{a^2 - x^2}\,dx + (a^2/y^2 - 1)\,dy$

$\therefore z = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} - \frac{a^2}{y} - y + b.$

where a,b are arbitrary constant. [Ans.]

Problem

Find a complete, singular and general integral of $(p^2+q^2)y = qz$

Soln Let. $f(x,y,z,p,q) = (p^2+q^2)y - qz = 0$ —— (i)
We know, Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\therefore \frac{dp}{0 + p(-q)} = \frac{dq}{p^2+q^2-q^2} = \frac{dz}{-p(2py) - q(2qy - z)} = \frac{dx}{-2py} = \frac{dy}{-(2qy-z)}$$

$$\Rightarrow \frac{dp}{-pq} = \frac{dq}{p^2} = \frac{dz}{-p(2py) - q(2qy - z)} = \frac{dx}{-2py} = \frac{dy}{z - 2qy}$$

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Taking first two ratios,

$$\frac{dp}{-pq} = \frac{dq}{p^2} \Rightarrow \frac{dp}{-q} = \frac{dq}{p} \Rightarrow p\,dp + q\,dq = 0$$

$$\Rightarrow p^2 + q^2 = a^2 \quad [\text{Integrating}]$$

$$\therefore p^2 = a^2 - q^2$$

From (i) we get,

$$\therefore a^2 y = qz \Rightarrow q = \frac{a^2 y}{z}$$

$$\therefore p^2 = a^2 - \frac{a^4 y^2}{z^2}$$

$$= \frac{a^2}{z^2}\left(z^2 - a^2 y^2\right)$$

$$\therefore p = \frac{a}{z}\sqrt{z^2 - a^2 y^2}$$

Now, complete integral is

$$dz = p\,dx + q\,dy$$

$$\Rightarrow dz = \frac{a}{z}\sqrt{z^2 - a^2 y^2}\,dx + \frac{a^2 y}{z}\,dy$$

$$\Rightarrow z\,dz = a\sqrt{z^2 - a^2 y^2}\,dx + a^2 y\,dy$$

$$\Rightarrow \frac{z\,dz - a^2 y\,dy}{\sqrt{z^2 - a^2 y^2}} = a\,dx$$

$$\Rightarrow \frac{t\,dt}{t} = a\,dx$$

$$\Rightarrow dt = a\,dx$$

$$\Rightarrow t = ax+b \quad \text{[integrating]}$$

$$\Rightarrow \sqrt{z^2-a^2y^2} = ax+b$$

$$\Rightarrow z^2-a^2y^2 = (ax+b)^2$$

$$\Rightarrow z^2 = a^2y^2+(ax+b)^2 \text{ ——— (ii)}$$

Put,

$$z^2-a^2y^2 = t^2$$

$$\Rightarrow 2z\,dz - 2a^2y\,dy = 2t\,dt$$

$$\Rightarrow z\,dz - a^2y\,dy = t\,dt$$

Which is the required complete integral.

Singular integral: (constant elimination)

Differentiating partially (ii) w.r. to a and b, we get

$$0 = 2ay^2+2(ax+b).x \text{ ——— (iii)}$$

$$0 = 0+2(ax+b) \text{ ——— (iv)}$$

Now, eliminating a and b from (ii), (iii) & (iv) we get,

$z=0$, which is the the required S.I. Since $z=0$ satisfies the given PDE (i).

একটা const. এ নিয়ে যেতে হবে

G.I :

Replacing b by $\phi(a)$ in (ii), we get

$$z^2 - a^2y^2 = [ax + \phi(a)]^2 \quad \text{—— (v)}$$

Differentiating (v) partially w.r. to a, we get.

$$-2ay^2 = 2\,[ax + \phi(a)] \cdot [x + \phi'(a)] \quad \text{—— (vi)}$$

General ~~equation~~ integral is obtained by eliminating a from (v) & (vi).

[করে দেখলে better]

H.W

must (i) Find a complete integral and S.I of $2xz - px^2 - 2qxy + pq = 0$

(ii) Find a C.I. of $pxy + pq + qy = yz$

(iii) Find a C.I of $p^2x + q^2y = z$

Raisinghania →

Page → 66 → example – 20, 22, 24, 27

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Prob. 2:

Show that (a) $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic.

(b) Find v such that $f(z) = u + iv$ is analytic.

(c) Also find $f(z)$ in terms of z.

Solution:

(a)

$$\frac{\partial u}{\partial x} = 3x^2 - \cancel{6xy}\ 3y^2 + 6x; \quad \frac{\partial^2 u}{\partial x^2} = 6x + 6 \quad \text{---(1)}$$

$$\frac{\partial u}{\partial y} = -6xy - 6y; \quad \frac{\partial^2 u}{\partial y^2} = -6x - 6 \quad \text{---(2)}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x + 6 - 6x - 6$$
$$= 0$$

So, u is harmonic.

(b) Since $f(z)$ is analytic function. so from Cauchy-Riemann equation, we get --

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x \quad \text{---(3)}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 6xy + 6y \quad \text{---(4)}$$

Integrating (3) w.r. to y, keeping x constant.

$$\int dv = \int (3x^2 - 3y^2 + 6x)\, dy = 3x^2y - y^3 + 6xy + F(x) \quad (5)$$

where $F(x)$ is an arbitrary real function of x.

Now, substitute (5) into (4) and obtain

$$6xy + 6y + F'(x) = 6xy + 6y$$

$$\text{or,} \quad F'(x) = 0$$

$$\therefore \quad F(x) = C$$

So (5) becomes $v = 3x^2y - y^3 + 6xy + C$

(c)

$$f(z) = f(x+iy) = u(x,y) + iv(x,y)$$

$$= x^3 - 3xy^2 + 3x^2 - 3y^2 + 1 + i(3x^2y - y^3 + 6xy + \ldots$$

Ans

Putting $y=0$. $f(x) = u(x,0) + iv(x,0) = x^3+3x^2+1+ic$

Replacing x by z

$$f(z) = u(z,0) + iv(z,0) = z^3+3z^2+1+ic$$

Prob 3

Show that (a) $u = 2x(1-y)$ is harmonic

(b) Find v such that $f(z) = u+iv$ is analytic

Tama: $v = 2y - y^2 + x^2$ (ভুল হলে জানি না :p)

$\cancel{f(z) = 2x + ix^2}$

(c) Also find $f(z)$ in terms of z.

$f(z) = \cancel{2x + ix^2}\ 2z + iz^2$

Prob 4

If $v = x^2 - y^2 + 2xy - 3x - 2y$, find u such that $f(z) = u+iv$ is analytic

$u = x^2 - y^2 - 2x + 3y - 2xy$ (ভুল থাকলে জানিয়েন :( )

Prob

If $u_1(x,y) = \frac{\partial u}{\partial x}$ and $u_2(x,y) = \frac{\partial u}{\partial y}$, prove that

$$f'(z) = u_1(z,0) - iu_2(z,0) = \left[\frac{\partial u}{\partial x}\right]_{\substack{x=z\\y=0}} - i\left[\frac{\partial u}{\partial y}\right]_{\substack{x=z\\y=0}}$$

Proof

Let $f(z) = u + iv$ be analytic. So, from Cauchy Riemann equation,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{——— (1)}$$

Now. $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$

$= \dfrac{\partial u}{\partial x} - i\dfrac{\partial u}{\partial y}$ [using (1)]

Or, $f'(x+iy) = \dfrac{\partial u}{\partial x} - i\dfrac{\partial u}{\partial y} = u_1(x,y) - i u_2(x,y)$ ——— (2)

Putting $x = z$ & $y = 0$ in (2),

$$f'(z) = u_1(z,0) - i u_2(z,0)$$

Prob

If $v_1(x,y) = \frac{\partial v}{\partial y}$ and $v_2(x,y) = \frac{\partial v}{\partial x}$, prove that

$$f'(z) = v_1(z,0) + i v_2(z,0) = \left[\frac{\partial v}{\partial y}\right]_{\substack{x=z \\ y=0}} + i\left[\frac{\partial v}{\partial x}\right]_{\substack{x=z \\ y=0}}$$

Proof

Let, $f(z) = u + iv$ be analytic. So, from Cauchy - Riemann equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{---} (1)$$

Now, $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$ [Using (i)]

$$\therefore f'(x+iy) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} = v_1(x,y) + i v_2(x,y); \quad \text{---} (2)$$

Putting $x = z$ & $y = 0$ in (2)

So, $f'(z) = v_1(z,0) + i v(z,0)$

Prob. 5.

Vi Must

If $Im\{f'(z)\} = 6x(2y-1)$ and $f(0) = 3-2i$, $f(1) = 6-5i$. Find $f(z)$ and $f(1+i)$

Ans.

$f(z) = 2z^3 - 3iz^2 + z + 3 - 2i$

Tama: The solution is in Tuhin's khata.

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Singular points:

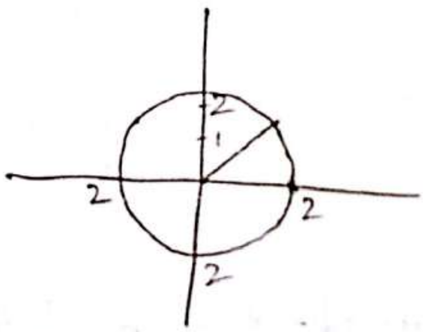
A point at which $f(z)$ fails to be analytic is called a singular points.

Ex. 1 : $f(z) = \frac{2z}{z^2+4} = \frac{2z}{(z+2i)(z-2i)}$. the singular points at $z = \pm 2i$

Various types of singular points:

1. Isolated singularities: If we can find $\delta > 0$ such that the circle $|z - z_0| = \delta$ enclose no singular point other than z_0.

$|z - 2i|$



2. Poles: If we can find a positive integer n such that $\lim_{z \to z_0} (z - z_0)^n f(z) = A \neq 0$, then $z = z_0$ is called a pole of order n. If $n = 1$, is called simple pole.

Ex. 2 :

Ex-2: $f(z) = \dfrac{2z}{(z^2+4)^2} = \dfrac{2z}{(z+2i)^2(z-2i)^2}$ order

has a pole at $z = \pm 2i$ of order 2.

3. Branch point: Ex. 3 $f(z) = (z-3)^{\frac{1}{2}}$ has a branch point at $z = 3$.

4. Removable singularities:

z_0 is called removable singularities of $f(z)$ if $\lim\limits_{z \to z_0} f(z)$ exist.

Ex-4: $f(z) = \dfrac{\sin z}{z}$, since $\lim\limits_{z \to 0} \dfrac{\sin z}{z} = 1$

5. Essential singularity: 4 এর একটাও না হলে

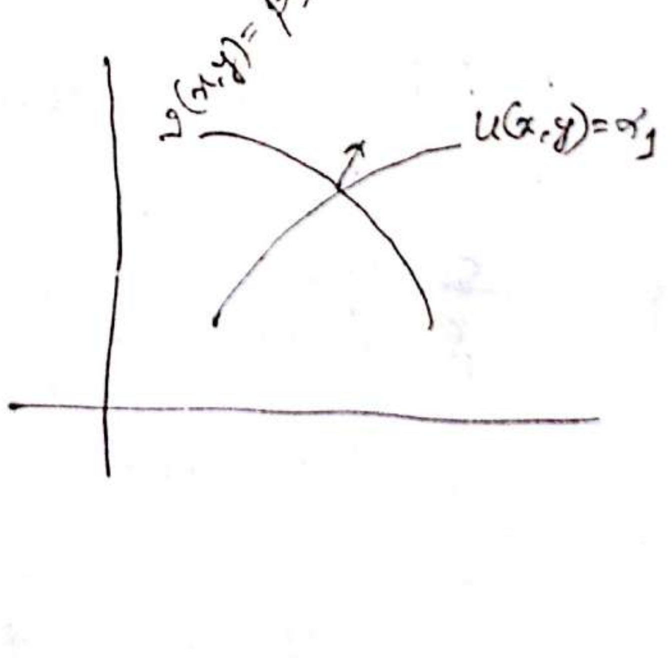
Ex.5: $f(z) = e^{\frac{1}{z-2}}$ has essential singularities at $z = 2$.

Orthogonal families of curves:

Let $u(x,y)=\alpha$ and $v(x,y)=\beta$. where u & v are the real and imaginary points of an analytic function $f(z)$ and α and β are any constants, represent two families of curves. Prove that the families are orthogonal.

Proof

Consider any two members of the respective families, say $u(x,y)=\alpha_1$ and $v(x,y)=\beta_1$, where α_1 & β_1 are particular constant in fig. 1.



Differentiate $u(x,y)=\alpha_1$, w. r. to x yields

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dx}=0$$

Then the slope of $u(x,y)=\alpha_1$ is $\frac{dy}{dx}=-\frac{\partial u}{\partial x}\Big/\frac{\partial u}{\partial y}=$

Similarily the slop of $v(x,y)=\beta_1$ is $m_2=\frac{dy}{dx}=-\frac{\partial v}{\partial x}\Big/\frac{\partial v}{\partial y}$

The product of the slopes is, using Cauchy-Riemann equations

$$\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}\Big/\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}=-\frac{\partial v}{\partial y}\cdot\frac{\partial u}{\partial y}\Big/\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}$$

Thus the curves are orthogonal.

Prob:

Find the orthogonal set of curves $x^2-y^2=a^2$

Solution: Let $u(x,y)=x^2-y^2-a^2=0$. find $v(x,y)=?$

The slope of $u(x,y)=x^2-y^2-a^2$ is $\cdots=\frac{x}{y}=m_1$

" " $v(x,y)=-2xy$ is $=-\frac{y}{x}=m_2$

Prob. Find the orthogonal tragetories of the families of curves in the xy plane defined by $e^{-x}(x \sin y - y \cos y) = \alpha$, where α is constant.

Problems

1. Test the analyticity of the followings:
   a) $f(z) = \bar{z}$, (b) $f(z) = |z|^4$. (c) $f(z) = \frac{1+z}{1-z}$

2. Show that (a) $v = 2y + x^2 - y^2$ is harmonic.
   (b) Find u such that $f(z) = u + iv$ is ~~ala~~ analytic. (c) also find $f(z)$ in terms of z.

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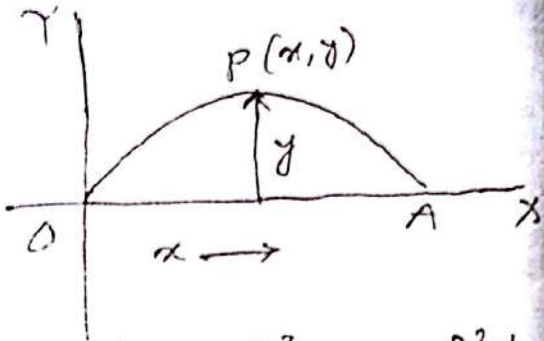
Solution of Wave Equation:

* $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$



Problem: Solve the wave equation $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ $\cancel{\partial^2}$ using seperation of variables.

Soln:

Here, given equn is $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ ——— (i)

(ii)

Let $y = XT$, where X is a function of x only and T is a function of t only.

$$\therefore \frac{\partial y}{\partial t} = X \frac{dT}{dt} \quad \text{and} \quad \frac{\partial y}{\partial x} = T \frac{dX}{dx}$$

$$\Rightarrow \frac{\partial^2 y}{\partial t^2} = X \frac{d^2 T}{dt^2} \qquad \Rightarrow \frac{\partial^2 y}{\partial x^2} = T \frac{d^2 X}{dx^2}$$

Since, X and T are functions of single variable.

From (i), we get

$$X \frac{d^2 T}{dt^2} = c^2 T \frac{d^2 X}{dx^2}$$

$$y = e^{mx}$$

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$$

$$\boxed{m^2 + 5m + 6 = 0}$$

$$\Rightarrow \frac{d^2T}{dt^2} \Big/ c^2 T = \frac{d^2X}{dx^2} \Big/ X = K \text{ (say)}$$

$$\therefore \frac{d^2T}{dt^2} = c^2 K T \text{ — (iii) and } \frac{d^2X}{dx^2} = K X \text{ — (iv)}$$

Auxiliary equation of (iii)

$$m^2 - c^2 K = 0$$

$$\Rightarrow m = \pm c\sqrt{K}$$

Also A.E. of (iv) is

$$m^2 - K = 0$$

$$\Rightarrow m = \pm \sqrt{K}$$

Sol^n of (iii) & (iv) are when $K > 0$. $m = \pm c_1$

$$T = c_1 e^{c\sqrt{K}t} + c_2 e^{-c\sqrt{K}t}$$

$$X = c_3 e^{\sqrt{K}x} + c_4 e^{-\sqrt{K}x}$$

$$m = a \pm ib$$

$$y = e^{ax}(c_1 \cos bx + c_2 \sin bx)$$

When $K < 0$ then

$$T = (c_5 \cos c\sqrt{K}t + c_6 \sin c\sqrt{K}t)$$

$$X = (c_7 \cos \sqrt{K}x + c_8 \sin \sqrt{K}x)$$

$$m = 1, 1$$

$$y = c_1 + c_2 x$$

when $K = 0$ then

$$T = c_9 + c_{10} t$$

$$X = c_{11} + c_{12} x$$

$T(n) = \sqrt{n}\,.T(\sqrt{n}) + n$

There are the three cases depending upon the particulars problems. Here we are dealing with wave eqn (k<0)

$$y = XT$$
$$= (C_5 \cos c\sqrt{K}\,t + C_6 \sin c\sqrt{K}\,t)(C_7 \cos\sqrt{K}\,x + C_8 \sin\sqrt{K}\,x)$$

This is the required soln.

wave equn
এর জন্য sin-cos
নেয়া হয়

Problem

Find the soln of wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ such that $y = P_0 \cos pt$ when $x = l$ and $y = 0$ when $x = 0$.

Soln Given that, $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ —— (i)

with $y = P_0 \cos pt$ when $x = l$

and $y = 0$ when $x = 0$

H·W page -X10 — 5,6

From above, we know that the soln of (1) is

$$y = (c_1 \cos c\sqrt{K}t + c_2 \sin c\sqrt{K}t)$$
$$(c_3 \cos \sqrt{K}x + c_4 \sin \sqrt{K}x) \quad \text{—— (ii)}$$

Put $y = 0$ when $x = 0$ in (ii)

$$0 = (c_1 \cos c\sqrt{K}t + c_2 \sin c\sqrt{K}t)\, c_3$$
$$\Rightarrow c_3 = 0$$

Now, from (ii)

$$y = (c_1 \cos c\sqrt{K}t + c_2 \sin c\sqrt{K}t)\, c_4 \sin \sqrt{K}x \quad \text{—— (iii)}$$
$$= c_1 c_4 \cos c\sqrt{K}t \sin \sqrt{K}x + c_2 c_4 \sin c\sqrt{K}t \sin \sqrt{K}x$$

Put $y = P_0 \cos pt$ when $x = l$, then from (iii)

$$P_0 \cos pt = c_1 c_4 \cos c\sqrt{K}t \cdot \sin \sqrt{K}l + c_2 c_4 \sin c\sqrt{K}t \cdot \sin \sqrt{K}l$$
$$y = \quad \text{—— (iv)}$$

Equating the co-efficients of sin & cos on both sides

$$P_0 = c_1 c_4 \sin \sqrt{K}l \Rightarrow c_1 c_4 = \frac{P_0}{\sin \sqrt{K}l}$$

$$0 = c_2 c_4 \sin \sqrt{K}l \Rightarrow c_2 = 0$$

And $p = c\sqrt{K} \Rightarrow \frac{p}{c} = \sqrt{K}$

~~Putting~~ values of $c_1 c_4$, c_2 and $\sqrt{K}$ at (iv)

MFA

6(A)-Day Date: 23/11

Complex Integration:

If $f(z) = u(x,y) + iv(x,y) = u + iv$ then complex line integral $\int f(z)\,dz$ can be expressed interms of real line integral as $\int f(z)dz = \int_c (u+iv)(dx + idy) = \int_c udx - vdy + i\int_c vdx + udy$ [Open

$\oint_c f(z)dz$ closed curve integration

Prob-1: Evaluate $\int_c \bar{z}\,dz$ from $z=0$ to $z=4+2i$ along the curve c given by $z=t^2+it$.

Solution:

The limit point if $z=0$, then $t=0$ and when $z=4+2i$ then $t=2$

So the line integral $\int_c \bar{z}\,dz = \int_{t=0}^{2} (\overline{t^2+it})\,d(t^2+it)$

$= \dots\dots$

$= \int_{t=0}^{2} (2t^3 - it^2 + t)\,dt$

$= 10 - \frac{8i}{3}$

Simply connected region and multiply connected region.

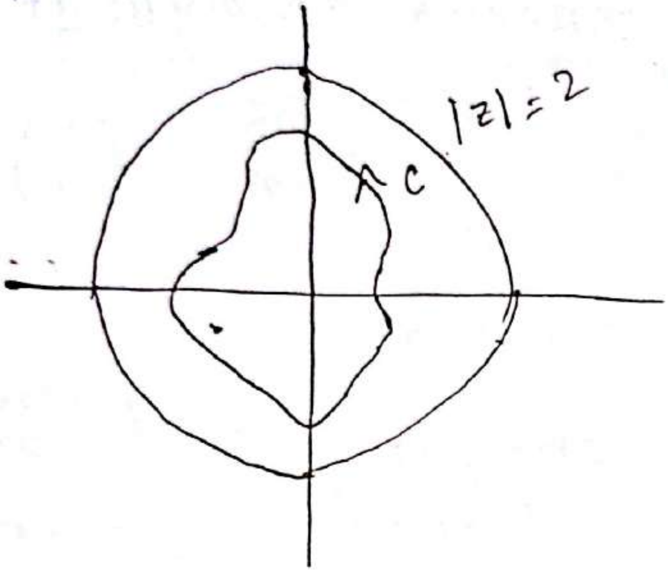


Fig. 1

Simply connected region

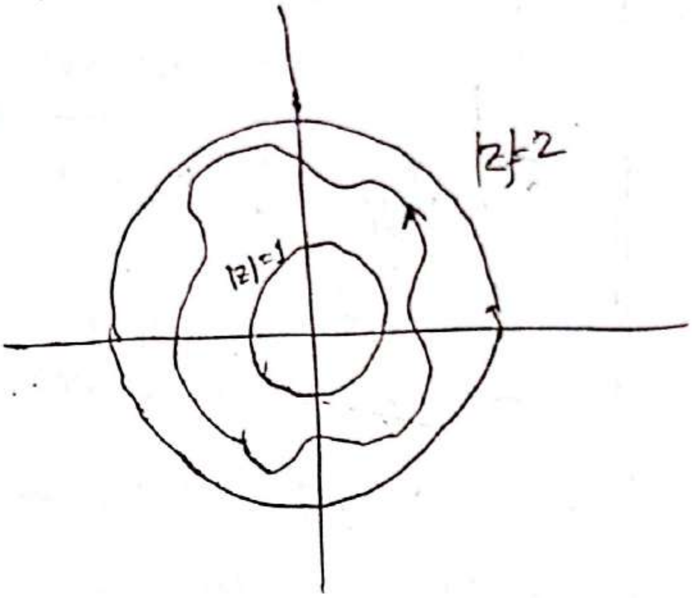


Fig. 2

Multiply connected

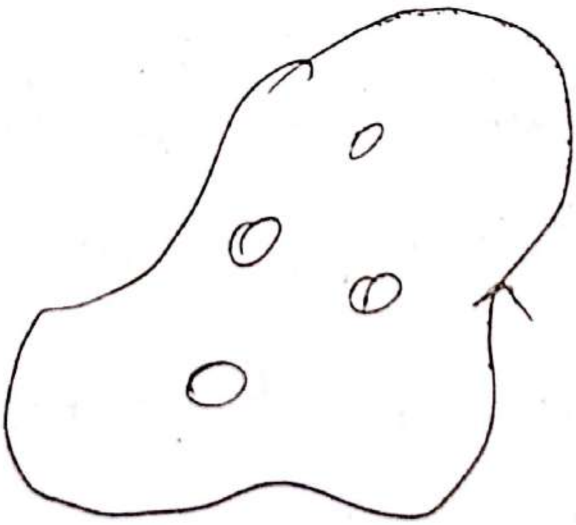


Fig. 3

Multiply connected

State and prove Green's theorem in the plane $\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

see book

Cauchy's integral theorem:

If $f(z)$ be analytic inside and on a simple closed curve C, then

$$\oint_C f(z)\,dz = 0$$

Proof

Since $f(z)$ be analytic, so it is satisfied Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{--- (1)}$$

By difinition of complex integrations, we have

$$\oint_C f(z)\,dz = \oint_C (u+iv)(dx+idy) = \oint_C u\,dx - v\,dy$$
$$+ i\oint_C v\,dx + u\,dy \quad \text{--- (2)}$$

Applying Green's theorem in (2), we get

$$\oint_C f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$$= \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$$= 0 + 0$$

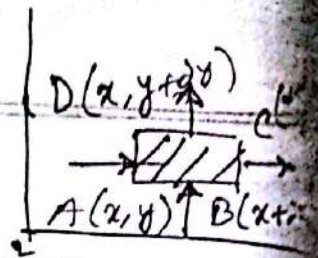
$$= 0$$

Prob. 2: Prove that (i) $\oint_C dz = 0$ (ii) $\oint_C z\,dz = 0$, where C is any simple closed curve.

MRK

6(B)-Day

Solution of Laplace Equation



Heat flow equ^n: $\dfrac{\partial u}{\partial t} = c^2\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right)$

When temperature u doesn't depend on time then

$$\frac{\partial u}{\partial t} = 0$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$ This is the two dimensional L.E.

Problem

Solve $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ which satisfies the conditions $u(0, y) = u(l, y) = u(x, 0) = 0$ and $u(x, a) = \sin\dfrac{n\pi x}{l}$

Sol^n

Given that,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{--- (i)}$$

Let, $u = X_{(x)}\,Y_{(y)}$

$$\Rightarrow \frac{\partial u}{\partial x} = Y\frac{dX}{dx}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = Y\frac{d^2X}{dx^2} \text{ and } \frac{\partial^2 u}{\partial y^2} = X\frac{d^2Y}{dy^2}$$

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from (i) we get

$$Y \frac{d^2X}{dx^2} + X \frac{d^2Y}{dy^2} = 0$$

$$\Rightarrow \frac{d^2X}{dx^2}\Big/X = - \frac{d^2Y}{dy^2}\Big/Y = -p^2 \quad \text{(say)}$$

$$\therefore \frac{d^2X}{dx^2} = -Xp^2 \quad \text{and} \quad \frac{d^2Y}{dy^2} = Yp^2$$

$$\therefore \frac{d^2X}{dx^2} + Xp^2 = 0 \quad \text{——(ii)} \quad \text{and} \quad \frac{d^2Y}{dy^2} - Yp^2 = 0 \quad \text{——(iii)}$$

A.E of (ii) is

$m^2 + p^2 = 0$

$\therefore m = \pm ip$

A.E. of (iii) is

$m^2 - p^2 = 0$

$\therefore m = \pm p$

$\therefore X = c_1 \cos px + c_2 \sin px$

$\therefore Y = c_3 e^{py} + c_4 e^{-py}$

The general sol^n of (i) is

$$u = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \quad \text{——(iv)}$$

Now putting $x=0$, $u=0$ in (iv)

$$0 = c_1 (c_3 e^{py} + c_4 e^{-py}) \Rightarrow c_1 = 0$$

From (iv)

$$u = c_2 \sin px (c_3 e^{py} + c_4 e^{-py}) \quad \text{——— (v)}$$

Second condition,

$x = l$, $u = 0$ in (v)

$$0 = c_2 \sin pl (c_3 e^{py} + c_4 e^{-py})$$

$$\therefore \sin pl = 0 = \sin n\pi \quad (\because c_2 \neq 0)$$

$$pl = n\pi$$

$$p = \frac{n\pi}{l}$$

From (v) we have

$$u = c_2 \sin \frac{n\pi x}{l} \left(c_3 e^{\frac{n\pi y}{l}} + c_4 e^{-\frac{n\pi y}{l}}\right) \quad \text{——— (vi)}$$

~~Now~~

Now putting $u=0$, $y=0$ at (vi)

$$0 = c_2 \sin\frac{n\pi x}{l}(c_3 + c_4)$$

$$c_3 + c_4 = 0$$

$$\therefore c_3 = -c_4$$

From (vi)

$$u = c_2 c_4 \sin\frac{n\pi x}{l}\left(-e^{\frac{n\pi y}{l}} + e^{\frac{-n\pi y}{l}}\right) \quad \text{—— (vii)}$$

On putting $y=a$ and $u = \sin\frac{n\pi x}{l}$ at (vii)

$$\sin\frac{n\pi x}{l} = c_2 c_4 \sin\frac{n\pi x}{l}\left(-e^{\frac{n\pi a}{l}} + e^{\frac{-n\pi a}{l}}\right)$$

$$c_2 c_4 = \frac{1}{e^{\frac{-n\pi a}{l}} - e^{\frac{n\pi a}{l}}}$$

From (vii) we get

$$u = \sin\frac{n\pi x}{l}\,\frac{e^{\frac{-n\pi y}{l}} - e^{\frac{n\pi y}{l}}}{e^{-\frac{n\pi a}{l}} - e^{\frac{n\pi a}{l}}}$$

$$= \sin\frac{n\pi x}{l}\,\frac{\sinh\frac{n\pi y}{l}}{\sinh\frac{n\pi a}{l}}.$$

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MRK

Solution of DE by Frobenius Method:

* Analytic function:

If a function $f(x)$ is differentiable every points in its domain, then the function $f(x)$ is said to be analytic.

For example, e^x, $\sin x$, $\cos x$, x, x^2 etc.

$f(x) = \frac{1}{x-1}$, $\mathbb{R}$

$x = 2 \rightarrow$ ordinary point

$x = 1 \rightarrow$ singular "

↳ define হবেনা

not analytic

but if $x \neq 1$ is defined, then analytic.

Singular Point

Regular | Irregular

(i) Regular Singular Point:

$$y'' + P(x)\, y' + Q(x)\, y = 0$$

$P(x) = \frac{1}{x}$, $Q = \frac{1}{x^2}$

$$y'' + \frac{1}{x} y' + \frac{1}{x^2} y = 0$$

$x=0$, $P(x)$ and $Q(x)$ undefined.

$x\,p(x) = 1 \qquad x^2 Q(x) = 1$

Removable $\longrightarrow$ so regular

* Frobenius Method:

If $x=0$ is a regular singularity of the equ^n $y'' + P(x)y' + Q(x)y = 0$ —— (i)

The solution of (i) is

$$y = x^m(a_0 + a_1 x + a_2 x^2 + \cdots) = \sum_{K=0}^{\alpha} a_K x^{m+K}$$

(i) Case I :

When $m_1 \neq m_2$, and $|m_1 - m_2|$ not integer, i.e. $m_1 = \frac{1}{2}$, $m_2 = 2$

then, $y = c_1(y)_{m_1} + c_2(y)_{m_2}$

(ii) Case II :

When $m_1 = m_2$, then $y = c_1(y)_{m_1} + c_2\left(\frac{\partial y}{\partial m}\right)_{m_1}$

(iii) Case III :

When $m_1 \neq m_2$ and differ by an integer, then

$y = c_1(y)_{m_1} + c_2\left(\frac{\partial y}{\partial m}\right)_{m_2}$ [replacing a_0 by $b_0(m-m_1)$]

(iv) Case IV :

When $m_1 \neq m_2$, differ by an integer, making some coefficient indeterminate, then

$$y = C_1 (y)_{m_1} + C_2 (y)_{m_1}$$

Problem Solve $3xy'' + 2y' + y = 0$

Soln:

Given that $3xy'' + 2y' + y = 0$ ——— (i) $\quad y'' + \frac{2}{3x}y' + \frac{1}{3x}y$, $x=0$ undefin

Since, $x=0$ is a regular singular point of (i), then

$$y = \sum_{K=0}^{\alpha} a_K x^{m+K} \text{ ——— (ii)}$$

$$y' = \sum_{K=0}^{\alpha} a_K (m+K) x^{m+K-1} \text{ ——— (iii)}$$

$$y'' = \sum_{K=0}^{\alpha} a_K (m+K)(m+K-1) x^{m+K-2} \text{ ——— (iv)}$$

Substituting (ii), (iii) and (iv) in (i)

$$3x \sum_{K=0}^{\alpha} a_K (m+K)(m+K-1) x^{m+K-2}$$

$$+ 2 \sum_{K=0}^{\alpha} a_K (m+K) x^{m+K-1} + \sum_{K=0}^{\alpha} a_K x^{m+K} = 0$$

$\cancel{3a+3b+3c}$

$$\Rightarrow \sum_{K=0}^{\alpha} 3(m+K)(m+K-1)\, a_K x^{m+K-1} + 2\sum_{K=0}^{\alpha} a_K (m+K)\, x^{m+K-1} + \sum_{K=0}^{\alpha} a_K x^{m+K} = 0$$

$$\Rightarrow \sum_{K=0}^{\alpha} a_K \left[3(m+K)(m+K-1) + 2(m+K)\right] x^{m+K-1} + \sum_{K=0}^{\alpha} a_K x^{m+K} = 0$$

$$\Rightarrow \sum_{K=0}^{\alpha} a_K (m+K)(3m+3K-1)\, x^{m+K-1} + \sum_{K=0}^{\alpha} a_K x^{m+K} = 0 \quad \text{——— (v)}$$

Equating the lowest degree to zero, by putting $K=0$ in (v), we get

$$a_0\, m(3m-1) = 0$$

$$\Rightarrow m(3m-1) = 0 \quad \because a_0 \neq 0$$

$$\Rightarrow m = 0,\ m = \frac{1}{3}$$

Equating the next lowest degree term x^m from (v)

$$a_1 (m+1)(3m+2) + a_0 = 0$$

$$\Rightarrow a_1 = -\frac{a_0}{(m+1)(3m+2)}$$

To obtain the recurrance relation, equate to zero the term x^{m+k},

$$a_{k+1}(m+k+1)(3m+3k+2) + a_k = 0$$

$$\Rightarrow \quad a_{k+1} = -\frac{a_k}{(m+k+1)(3m+3k+2)} \quad \text{—— (vi)}$$

Put $k = 1, 2, 3, \ldots$ in (vi).

$$a_2 = -\frac{a_1}{(m+2)(3m+5)} = \frac{a_0}{(m+1)(3m+2)(3m+2)(3m+5)}$$

$$a_3 = -\frac{a_2}{(m+3)(3m+8)} = -\frac{a_0}{(m+1)(m+2)(m+3)(3m+2)(3m+5)(3m+8)}$$

Now,
$$y = \sum_{k=0}^{\infty} a_k x^{m+k} = x^m \sum_{k=0}^{\infty} a_k x^k$$

$$= x^m(a_0 + a_1 x + a_2 x^2 + \cdots)$$

$$= x^m\left(a_0 - \frac{a_0}{(m+1)(3m+2)}x + \frac{a_0}{(m+1)(m+2)(3m+2)(3m+5)}x^2\right.$$

$$= a_0 x^m \left[1 - \frac{1}{(m+1)(3m+2)} x + \frac{1}{(m+1)(m+2)(3m+2)(3m+5)} x^2 - \cdots \cdots\right] \cdots \cdots \quad (vii)$$

Putting $m=0$ in (vii)

$$y = a_0 \left[1 - \frac{1}{2}x + \frac{1}{20}x^2 - \frac{1}{480}x^3 + \cdots\right) = c_1 y_1$$

Putting $m = \frac{1}{3}$ in (vii),

$$y = a_0 x^{\frac{1}{3}} \left[1 - \frac{1}{4}x + \frac{1}{56}x^2 - \frac{1}{1680}x^3 + \cdots\right] = c_2 y_2$$

Hence the g.s of (i) is

$$y = c_1 y_1 + c_2 y_2 = a_0\left(1 - \frac{1}{2}x + \frac{1}{20}x^2 - \frac{1}{480}x^3 + \cdots\right)$$

$$+ a_0 x^{\frac{1}{3}}\left(1 - \frac{1}{4}x + \frac{1}{56}x^2 - \frac{1}{1680}x^3 + \cdots\right)$$

$\boxed{\text{Ans.}}$

*Problem $x(x-1)y'' + (3x-1)y' + y = 0$

Soln

* $m = 0, 0$ | $m_1 = m_2$

* Recurrance relation $\rightarrow a_{k+1} = a_k$

i.e, $a_0 = a_1 = a_2 = a_3 = \cdots$

$$y = x^m(a_0 + a_1 x + a_2 x^2 + \cdots)$$
$$= a_0 x^m(1 + x + x^2 + x^3 + \cdots) \quad [a_0 = a_1 = a_2 = \cdots$$

$m = 0 \longrightarrow$

$$C_1(y) = a_0(1 + x + x^2 + x^3 + \cdots)$$

$$\frac{\partial y}{\partial m} = a_0 x^m \log x\,(1 + x + x^2 + x^3 + \cdots)$$

$$\Rightarrow \left(\frac{\partial y}{\partial m}\right)_{m=0} = a_0 \log x\,(1 + x + x^2 + x^3 + \cdots)$$

Page $\rightarrow$ 625 $\longrightarrow$ 6, 7, 8, 9, 10 (Example)

Bessel's Equations

* The DE $x^2y'' + xy' + (x^2 - n^2)y = 0$ is called

Bessel's

* The

Bessel's Equations:

* The DE $x^2y'' + xy' + (x^2 - n^2)y = 0$ is called the Bessel's DE and the particular solns of it are called Bessel's functions of order n.

$$m^2 = n^2 \Rightarrow m = n, -n$$

$$J_{\{n\}}(x) + J_{\{-n\}}(x)$$

MRK

7(B)-Day Date: 4/12

On

$$J_n(x) = \sum_{r=0}^{\alpha} \frac{(-1)^r}{r!\,(n+r)!}\left(\frac{x}{2}\right)^{n+2r} = \sum_{r=0}^{\alpha} \frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

$$J_{-n}(x) = \sum_{r=0}^{\alpha} \frac{(-1)^r}{r!\,(-n+r)!}\left(\frac{x}{2}\right)^{-n+2r}$$

* Prove that, $J_{-n}(x) = (-1)^n J_n(x)$, where n is +ve integer.

Proof

We know,

$$J_{-n}(x) = \sum_{r=0}^{\alpha} \frac{(-1)^r}{r!\,(-n+r)!}\left(\frac{x}{2}\right)^{-n+2r}$$

$$= \sum_{r=0}^{n-1} \frac{(-1)^r}{r!\,(-n+r)!}\left(\frac{x}{2}\right)^{-n+2r} + \sum_{r=n}^{\alpha} \frac{(-1)^r}{r!\,(-n+r)!}\left(\frac{x}{2}\right)^{-n+2r}$$

$$= 0 + \sum_{r=n}^{\alpha} \frac{(-1)^r}{r!\,(-n+r)!}\left(\frac{x}{2}\right)^{-n+2r} \quad [\because (-\text{ve integer})! = \alpha]$$

$$= \sum_{r=n}^{\alpha} \frac{(-1)^r}{r!\,(-n+r)!}\left(\frac{x}{2}\right)^{-n+2r} \qquad \text{——— (i)}$$

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On putting $r = n+k$ in (i)

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^{n+k}}{(n+k)!\, k!} \left(\frac{x}{2}\right)^{n+2k}$$

$$= (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(n+k)!} \left(\frac{x}{2}\right)^{n+2k}$$

$$\therefore J_{-n}(x) = (-1)^n J_n(x)$$

(\#) Prove that. (i) $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

(ii) $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

(iii) $[J_{-\frac{1}{2}}(x)]^2 + [J_{\frac{1}{2}}(x)]^2 = \frac{2}{\pi x}$

(iv) Show that $\int_0^{\frac{\pi}{2}} \sqrt{\pi x}\, J_{\frac{1}{2}}(2x)\, dx = 1$

Proof:

(i) From the definition of Bessel's function we know,

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(n+r)!} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \frac{x^n}{2^n n!} - \frac{x^{n+2}}{2^{n+2}(n+1)!} + \frac{x^{n+4}}{2! \cdot 2^{n+4}(n+2)!} - \cdots$$

$$= \frac{x^n}{2^n n!}\left[1 - \frac{x^2}{2^2(n+1)} + \frac{x^4}{2\cdot 2^4(n+1)(n+2)} - \cdots\right]$$

$$J_n(x) = \frac{x^n}{2^n \cancel{n!}\,\Gamma(n+1)}\left[1 - \frac{x^2}{4(n+1)} + \frac{x^4}{32(n+1)(n+2)} - \cdots\right]$$

$[\because n! = \Gamma(n+1)]$

$$\text{——— (i)}$$

$$J_{-\frac{1}{2}}(x) = \frac{x^{-\frac{1}{2}}}{2^{-\frac{1}{2}}\,\Gamma\left(-\frac{1}{2}+1\right)}\left[1 - \frac{x^2}{4\left(-\frac{1}{2}+1\right)} + \frac{x^4}{32\left(-\frac{1}{2}+1\right)\left(-\frac{1}{2}+2\right)} - \cdots\right]$$

$$= \frac{\sqrt{2}}{\sqrt{x}\,\Gamma\frac{1}{2}}\left[1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right]$$

$$= \frac{\sqrt{2}}{\sqrt{x}\,\sqrt{\pi}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right]$$

$\left[\Gamma\frac{1}{2} = \sqrt{\pi}\right]$

$$= \sqrt{\frac{2}{\pi x}}\cos x \quad \text{[Proved]}$$

We know,

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(-n+r)!} \left(\frac{x}{2}\right)^{-n+2r}$$

$$= \frac{x^{-n}}{2^{-n}\cdot(-n)!} - \frac{x^{-n+2}}{2^{-n+2}\cdot(-n+1)!} + \frac{x^{-n+4}}{2!\,2^{-n+4}(-n+2)!} - \frac{x^{-n+6}}{3!\,2^{-n+6}(-n} + \cdots$$

$$= -\frac{x^{-n}}{2^{-n}(-n+1)!} \left[\frac{x^2}{2^2} \right.$$

Putting $n = \frac{1}{2}$ in (i)

$$J_{\frac{1}{2}}(x) = \frac{\sqrt{x}}{\sqrt{2}\,\Gamma\left(\frac{1}{2}+1\right)} \left[1 - \frac{x^2}{4\left(\frac{1}{2}+1\right)} + \frac{x^4}{32\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)} - \cdots \right]$$

$$= \frac{\sqrt{x}}{\sqrt{2}\,\Gamma\left(\frac{3}{2}\right)} \left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots \right]$$

$$= \frac{\sqrt{x}}{\sqrt{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} \frac{1}{x} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right]$$

$$= \frac{1\times\sqrt{2}}{\sqrt{x}\,\sqrt{\pi}} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right]$$

$$= \sqrt{\frac{2}{\pi x}}\, \sin x$$

(iv)

$$J_{1/2}(2x) = \sqrt{\frac{2}{\pi \cdot 2x}}\ \sin 2x$$

$$= \sqrt{\frac{1}{\pi x}}\ \sin 2x$$

$$\int_0^{\pi/2} \sqrt{\pi x}\ J_{1/2}(2x)\, dx$$

$$= \int_0^{\pi/2} \sqrt{\pi x}\ \sqrt{\frac{1}{\pi x}}\ \sin 2x\, dx$$

$$= \int_0^{\pi/2} \sin 2x\, dx$$

$$= \left[-\frac{\cos 2x}{2}\right]_0^{\pi/2} = -\frac{1}{2}\left[\cos 2x\right]_0^{\pi/2}$$

$$= -\frac{1}{2}(-1-1)$$

$$= 1$$

* Recurrence formula for Bessel's functions

Formula 1: $x J_n'(x) = n J_n(x) - x J_{n+1}(x)$

Page-65, 66. (2-6 formula)

Proof:

We know, $$J_n(x) = \sum_{r=0}^{\alpha} \frac{(-1)^r}{r!(n+r)!} \left(\frac{x}{2}\right)^{n+2r} \quad \text{--- (i)}$$

Differentiating (i) w.r.to x, we get

$$J_n'(x) = \sum_{r=0}^{\alpha} \frac{(-1)^r (n+2r)}{r!(n+r)!} \left(\frac{x}{2}\right)^{n+2r-1} \cdot \frac{1}{2}$$

$$\Rightarrow x J_n'(x) = n \cancel{x} \sum_{r=0}^{\alpha} \frac{(-1)^r}{r!(n+r)!} \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \left(\frac{x}{2}\right)^{-1} \cdot \frac{1}{2}$$

$$+ x \sum_{r=0}^{\alpha} \frac{(-1)^r \cdot 2r}{2 \cdot r!(n+r)!} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= n J_n(x) + x \sum_{r=1}^{\alpha} \frac{(-1)^r}{(r-1)!(n+r)!} \left(\frac{x}{2}\right)^{n+2r-1}$$

[$\because$ $r=0$ ... $(-1)!$ ...

Putting $r = s+1$ in (ii)

$$\therefore \; x J_n'(x) = n J_n(x) + x \sum_{s=0}^{\infty} \frac{(-1)^{s+1}}{s!\,(n+s+1)!} \left(\frac{x}{2}\right)^{n+2s+1}$$

$$= n J_n(x) - x \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,[(n+1)+s]!} \left(\frac{x}{2}\right)^{(n+1)+2s}$$

$$\therefore \; x J_n'(x) = n J_n(x) - x J_{n+1}(x)$$ [Proved]

7th (E)-Day স্যারের থেকে

$$(1-x^2) y'' - 2xy' + n(n+1) y = 0$$

$$P_n(x) = \sum_{r=0}^{n/2} \frac{(-1)^r \,(2n-2r)!}{2^n r!\,(n-r)!\,(n-2r)!} x^{n-2r}$$

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{3x^2}{2} - \frac{1}{2}$

$P_3(x) = \frac{5x^3}{2} - \frac{3x}{2}$

$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

ARK

8(E)-Day Date: 26/12/20

Prob: Express $f(x) = 4x^3 + 6x^2 + 7x + 2$ interms of Legendre's polynomials.

Soln:

Let, $4x^3 + 6x^2 + 7x + 2 = aP_3(x) + bP_2(x) + cP_1(x) + d$

$$= a\left(\frac{5x^3}{2} - \frac{3x}{2}\right) + b\left(\frac{3x^2}{2} - \frac{1}{2}\right) + cx + d \cdot 1$$

$$= \frac{5ax^3}{2} - \frac{3ax}{2} + \frac{3bx^2}{2} - \frac{b}{2} + cx + d$$

$$= \frac{5ax^3}{2} + \frac{3bx^2}{2} + \left(c - \frac{3a}{2}\right)x + d - \frac{b}{2}$$

Equating the coefficients of like powers of x, we get,

$$4 = \frac{5a}{2}, \quad 6 = \frac{3b}{2}, \quad 7 = c - \frac{3a}{2} = c - \frac{3}{2} \cdot \frac{8}{5}$$

$$\Rightarrow a = \frac{8}{5}, \quad b = 4, \quad \Rightarrow c = \frac{47}{5}$$

$$2 = d - \frac{b}{2} = d - 4 \cdot \frac{1}{2}$$

$$\Rightarrow d = 4$$

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$$f(x) = 4x^3 + 6x^2 + 7x + 2$$

$$= \frac{8}{5} P_3(x) + 4 \cdot P_2(x) + \frac{47}{5} P_1(x) + 4P_0$$

(Ans).

* How Legendre's polynomial is expressed?

* Generative function of Legendre's polynomial is $(1-2xz+z^2)^{-\frac{1}{2}}$ i.e. $(1-2xz+z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\alpha} P_n(x) z^n$

Problem

Show that (i) $P_{2n}(0) = (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n}$

(ii) $P_{2n+1}(0) = 0$

Soln:

We know, the generative function of Legendre's polynomial is

$$(1-2xz+z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\alpha} P_n(x) z^n$$

$$\Rightarrow \sum_{n=0}^{\alpha} P_{2n}(x) z^{2n} = (1-2xz+z^2)^{-\frac{1}{2}}$$

$$\Rightarrow \sum_{n=0}^{\infty} P_{2n}(0) z^{2n} = (1+z^2)^{-\frac{1}{2}}$$

$$= 1 + \left(-\frac{1}{2}\right) z^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!} (z^2)^2$$

$$+ \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!} (z^2)^3 + \cdots$$

$$\cdots + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!} (z^2$$

Equating the coefficient of z^{2n}, we get

$$P_{2n}(0) = \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!}$$

$$= (-1)^n \frac{1\cdot 3\cdot 5\cdots\cdots(2n-1)}{2^n\cdot n!}$$

$$= (-1)^n \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}$$

[showed]

*Show that the recurrence formula for legendre's polynomials, $nP_n = (2n-1)xP_{n-1} - (n-1)P_{n-2}$

Proof:

We know, $(1-2xz+z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\alpha} P_n(x) z^n$

Differentiating w.r.to z, we get

$$-\frac{1}{2}(1-2xz+z^2)^{-\frac{3}{2}}(-2x+2z) = \sum_{n=0}^{\alpha} P_n(x)\cdot nz^{n-1}$$

$$\Rightarrow (1-2xz+z^2)^{-\frac{1}{2}}(x-z) = (1-2xz+z^2)\sum_{n=0}^{\alpha} P_n(x)\cdot nz^{n-1}$$

$$\Rightarrow (x-z)\sum_{n=0}^{\alpha} P_n(x) z^n = (1-2xz+z^2)\sum_{n=0}^{\alpha} P_n(x)\, nz^{n-1} \quad \text{--- (1)}$$

Equating the coefficients z^{n-1} from both sides, we get,

$$x P_{n-1}(x) - P_{n-2}(x) = \cancel{n P_n(x) - 2x P_{n-1}(x)}$$
$$nP_n(x) - 2x(n-1)P_{n-1}(x) + (n-2)P_{n-2}(x)$$

$$\Rightarrow nP_n = (2n-1)x P_{n-1} - (n-1)P_{n-2}$$

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9(B) - Day Date: 29/12

Orthogonality of Legendre's Polynomial:

Problem Show that $\int_{-1}^{+1} P_m(x) P_n(x)\, dx = 0$ if $m \neq n$ ← orthogonality property

$= \frac{2}{2n+1}$ if $m = n$

Soln:

$P_n(x)$ is ~~the~~ a solution of

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \quad \text{---(i)}$$

and $P_m(x)$ is a solution of

$$(1-x^2)\frac{d^2z}{dx^2} - 2x\frac{dz}{dx} + m(m+1)z = 0 \quad \text{---(ii)}$$

Now, multiplying (i) by z and (ii) by y and subtracting them, we get.

$$(1-x^2)\left\{z\frac{d^2y}{dx^2} - y\frac{d^2z}{dx^2}\right\} - 2x\left\{z\frac{dy}{dx} - y\frac{dz}{dx}\right\}$$
$$+ \{n(n+1) - m(m+1)\}yz = 0$$

$$\Rightarrow (1-x^2)\left\{\left(z\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot\frac{dz}{dx}\right) - \left(\frac{dy}{dx}\cdot\frac{dz}{dx} + y\frac{d^2z}{dx^2}\right)\right\}$$
$$- 2x\left\{z\frac{dy}{dx} - y\frac{dz}{dx}\right\} + \{n(n+1) - m(m+1)\}yz$$

$$\Rightarrow \frac{d}{dx}\left\{(1-x^2)\left(z\frac{dy}{dx} - y\frac{dz}{dx}\right)\right\} + \{n(n+1) - m(m+1)\}yz = 0$$

Integrating above equ^n between the limits -1 to $+1$, we get,

$$\int_{-1}^{+1} \frac{d}{dx}\left[(1-x^2)\left(z\frac{dy}{dx} - y\frac{dz}{dx}\right)\right]dx + (n-m)(n+m+1)\int_{-1}^{+1} yz\,dx = 0$$

$$\Rightarrow \left[(1-x^2)\left(z\frac{dy}{dx} - y\frac{dz}{dx}\right)\right]_{-1}^{+1} + (n-m)(n+m+1)\int_{-1}^{+1} yz\,dx = 0$$

$$\Rightarrow 0 + (n-m)(n+m+1)\int_{-1}^{+1} P_n(x)\,P_m(x)\,dx = 0$$

(i) dep-var $\to y$
(ii) $\to z$

$$\therefore \int_{-1}^{+1} P_n(x)\,P_m(x)\,dx = 0 \quad \text{if } m \neq n$$

Page - 660
Example - 31

Rodrick formula
Page $\to$ 654
Exercise $\to$ 8.21

* Prove that $(1-2xz+z^2)^{-\frac{1}{2}}$ is a solⁿ of the

$$z\frac{\partial^2(zv)}{\partial z^2}+\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial v}{\partial x}\right]=0$$

<u>Sol^n:</u>

Let, $v=(1-2xz+z^2)^{-\frac{1}{2}}=\sum_{n=0}^{\infty}P_n(x)\,z^n$

$$\Rightarrow zv=\sum_{n=0}^{\infty}P_n(x)\,z^{n+1}$$

$$\Rightarrow \frac{\partial}{\partial z}(zv)=\sum_{n=0}^{\infty}P_n(x)\,(n+1)\,z^n$$

$$\Rightarrow \frac{\partial^2}{\partial z^2}(zv)=\sum_{n=0}^{\infty}P_n(x)\cdot n(n+1)\,z^{n-1}$$

Let, $v=(1-2xz+z^2)^{-\frac{1}{2}}=\sum_{n=0}^{\infty}P_n(x)\,z^n$

$$\Rightarrow \frac{\partial v}{\partial x}=\sum_{n=0}^{\infty}P_n'(x)\,z^n$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2}=\sum_{n=0}^{\infty}P_n''(x)\,z^n$$

$$\Rightarrow (1-x^2)\frac{\partial^2 v}{\partial x^2}-2x\frac{\partial v}{\partial x}=(1-x^2)\sum_{n=0}^{\infty}P_n''(x)z^n-2x\sum_{n=0}^{\infty}P_n'(x)\,z^n$$

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$P_0(x)$
$= 0$

$$\Rightarrow z\frac{\partial^2}{\partial z^2}(zv) = \sum_{n=0}^{\alpha} P_n(x)\cdot n(n+1)\, z^n$$

$$\Rightarrow \frac{\partial v}{\partial x} = \sum_{n=0}^{\alpha} P_n'(x)\, z^n$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} = \sum_{n=0}^{\alpha} P_n''(x) z^n$$

$$L.H.S. = z\frac{\partial^2(zv)}{\partial z^2} + \frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial v}{\partial x}\right]$$

$$= z\frac{\partial^2(zv)}{\partial z^2} + (1-x^2)\frac{\partial^2 v}{\partial x^2} - 2x\frac{\partial v}{\partial x}$$

$$= \sum_{n=0}^{\alpha} P_n(x)\cdot n(n+1) z^n + (1-x^2)\sum_{n=0}^{\alpha} P_n''(x) z^n$$
$$- 2x \sum_{n=0}^{\alpha} P_n'(x)\cdot z^n$$

$$= \sum_{n=0}^{\alpha}\left[(1-x^2)P_n''(x) - 2x P_n'(x) + n(n+1)P_n\right] z^n$$

$= 0$ as P_n is a Legendre's Polynomial.

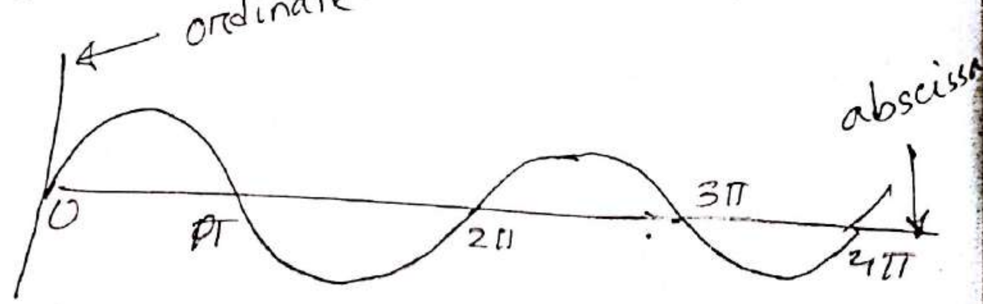
10 (A)—Day Date: 03/01/2

Fourier Series:

* Periodic function:

If the value of each ordinate $f(t)$ repeats itself at equal interval in the abscissa, then $f(t)$ is said to be periodic function.

If $f(t) = f(t+T) = f(t+2T) = \dots$, then T is called the period of $f(t)$



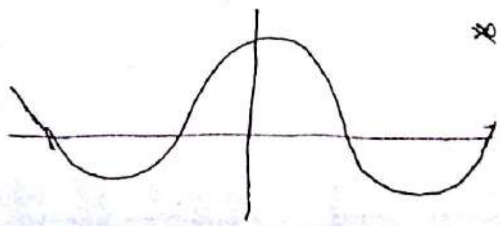
$\sin x = \sin(x+2\pi) = \sin(x+4\pi) = \dots$

Even & Odd Function:

A function $f(x)$ is said to be even function if $f(-x) = f(x)$.

For example $\cos x, x^2, x^4, \dots$ etc.

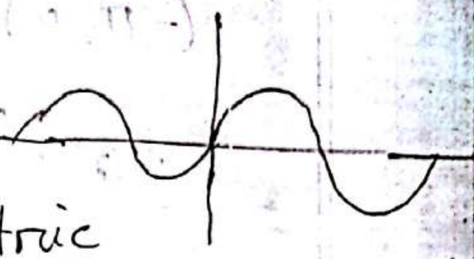
$\cos(-x) = \cos x$



* Even function Symmetric

Odd function $\longrightarrow$ $f(-x) = -f(x)$

For example $\longrightarrow$ $\sin x, x^3, x, x^5$



Odd function $\longrightarrow$ Anti Symmetric

Page $\rightarrow$ 850

Fourier Series:

If a function $f(x)$ is defined in the interval $(-L, L)$ and can be expanded in a trigonometric series. i.e,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right)$$

then $f(x)$ is said to be Fourier Series where,

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x \, dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x \, dx$$

$(-\pi, \pi) \leftarrow$ Interval

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\alpha} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\alpha} a_n \cos nx + \sum_{n=1}^{\alpha} b_n \sin nx$$

Half range cosine Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\alpha} a_n \cos nx \quad , \quad b_n = 0$$

sine, $f(x) = \sum_{n=1}^{\alpha} b_n \sin nx$, $a_0 = 0$, $a_n = 0$

$(0, \pi) \longrightarrow$

$\frac{2}{\text{period}} \leftarrow \frac{2}{\pi} \int_0^{\pi}$

$\frac{1}{\frac{p}{2}}$

Advantage of Fourier Series:

→

Application of Fourier Series in Engineering:

→

Problem:

Determine the fourier coefficients within the interval $(-\pi, \pi)$.

Soln

We know,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{(i)}$$

Useful Integrals:

(i) $\int_0^{2\pi} \sin nx \, dx = \int_0^{2\pi} \cos nx \, dx = 0$

(ii) $\int_0^{2\pi} \sin^2 nx \, dx = \int_0^{2\pi} \cos^2 nx \, dx = \pi$

(iii) $\int_0^{2\pi} \sin nx \cdot \cos nx \, dx = 0$

Integrating (i) ~~w.r. to~~ x between $(-\pi, \pi)$

$$\cancel{\int_{-\pi}^{\pi} f(x)\,dx = a_0 \pi}$$

Multiplying both sides of (i) by $\cos nx$ and integrating (i) w.r. to x between $(-\pi, \pi)$,

$$\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi}\cos nx\,dx +$$

$$+ a_1 \int_{-\pi}^{\pi} \cos x \cos nx\,dx + \cdots + a_n \int_{-\pi}^{\pi} \cos^2 nx$$

$$+ b_1 \int_{-\pi}^{\pi} \sin x \cos nx\,dx + b_2 \int_{-\pi}^{\pi} \sin 2x \cos nx\,dx$$

$$+ \cdots$$

$$= a_n \int_{-\pi}^{\pi} \cos^2 nx\,dx$$

$$= \frac{1}{2} a_n \int_{-\pi}^{\pi} 2\cos^2 nx\,dx$$

$$= \frac{1}{2} a_n \int_{-\pi}^{\pi} (1 + \cos 2nx)\, dx$$

$$= \frac{1}{2} a_n \left\{ \int_{-\pi}^{\pi} dx + \int_{-\pi}^{\pi} \cos 2nx \, dx \right\}$$

$$= \frac{1}{2} a_n \left\{ 2\pi + \frac{1}{2n} \left[\sin 2nx \right]_{-\pi}^{\pi} \right\}$$

$$= a_n \pi$$

$$\therefore a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

Integrating (i) w.r.to x

$$\int_{-\pi}^{\pi} f(x)\, dx = \frac{a_0}{2} \int_{-\pi}^{\pi} dx + a_1 \int_{-\pi}^{\pi} \cos x \, dx + a_2 \int \cos 2x \, dx$$

$$+ \cdots + a_n \int_{-\pi}^{\pi} \cos nx \, dx + b_1 \int_{-\pi}^{\pi} \sin x \, dx$$

$$+ b_2 \int_{-\pi}^{\pi} \sin 2x \, dx + \cdots$$

$$= \frac{a_0}{2} \int_{-\pi}^{\pi} dx$$

$$= \frac{a_0}{2} \cdot 2\pi = a_0 \pi$$

$$\therefore a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

Multiplying (i) by sin nx and inte. w.r.to x

$$\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi}\sin nx\,dx + a_1\int_{-\pi}^{\pi}\cos x\sin nx$$

$$+\cdots + a_n\int_{-\pi}^{\pi}\cos nx\sin nx\,dx$$

$$+b_1\int_{-\pi}^{\pi}\sin x\sin nx\,dx + \cdots + b_n\int_{-\pi}^{\pi}\sin^2 n$$

$$= b_n\int_{-\pi}^{\pi}\sin^2 nx\,dx$$

$$= \frac{1}{2}b_n\int_{-\pi}^{\pi} 2\sin^2 nx\,dx$$

$$= \frac{1}{2}b_n\int_{-\pi}^{\pi} 2\sin^2 nx\,dx$$

$$= \frac{1}{2}b_n\int_{-\pi}^{\pi}(1-\cos 2nx)\,dx$$

$$= \frac{1}{2}b_n\left[2\pi - \int_{-\pi}^{\pi}\cos 2nx\,dx\right]$$

$$= \pi b_n - \frac{1}{2\cdot 2n}b_n\left[\sin 2nx\right]_{-\pi}^{\pi}$$

$$= \pi b_n$$

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Problem

Find the fourier series representing $f(x) = x$, $0 < x < 2\pi$.

Soln.

We know the fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{--- (i)}$$

Here,

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx = \frac{1}{\pi} \int_0^{2\pi} x\, dx = \frac{1}{\pi} \left[\frac{x^2}{2} \right]_0^{2\pi} = 2\pi$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx = \frac{1}{\pi} \int_0^{2\pi} x \cos nx\, dx$$

$$= \frac{1}{\pi} \left[x \frac{\sin nx}{n} - 1 \left(- \frac{\cos nx}{n^2} \right) \right]_0^{2\pi}$$

$$= 0$$

From (i),

$$f(x) = \pi - \sum_{n=1}^{\infty} \frac{2}{n} \sin nx$$

$$= \pi - \left(2 \sin x + \sin 2x + \frac{2}{3} \sin 3x + \cdots\right)$$

12.7

f.s. in complex var. ৭.০ বার পড়ে

$$f(x) = \begin{cases} -1, & -\pi < x < -\frac{\pi}{2} \\ 0, & -\pi/2 < x < \frac{\pi}{2} \\ 1, & \pi/2 < x < \pi \end{cases}$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi} \int_{-\pi}^{-\pi/2} f(x)\, dx + \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} f(x)\, dx$$

$$+ \frac{1}{\pi} \int_{\pi/2}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{-\pi/2} (-dx) + \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} 0\, dx + \frac{1}{\pi} \int_{\pi/2}^{\pi} dx$$

Problem:

show that for an even function there is no b_n terms appear.

Soln

We know, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\alpha} a_n \cos \frac{n\pi}{L} x + \sum_{n=1}^{\alpha} b_n \sin \frac{n\pi}{L} x$ ——— (i)

Also we know, $b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x \, dx$

$$= \frac{1}{L} \int_{-L}^{0} f(x) \sin \frac{n\pi}{L} x \, dx + \frac{1}{L} \int_{0}^{L} f(x) \sin \frac{n\pi}{L} x \, dx$$ ——— (ii)

Let $x = -u$ in the 1st term on the right side of (ii), we get,

$$\frac{1}{L} \int_{-L}^{0} f(x) \sin \frac{n\pi}{L} x \, dx = \frac{1}{L} \int_{L}^{0} f(-u) \sin \left(-\frac{n\pi}{L} u\right) (-du)$$

$$= -\frac{1}{L} \int_{0}^{L} f(-u) \sin \left(\frac{n\pi}{L} u\right) du$$

$$= -\frac{1}{L}\int_0^L f(u)\sin\left(\frac{n\pi}{L}u\right)du$$

$$= -\frac{1}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$$

V.V.I

867 — Ex-12

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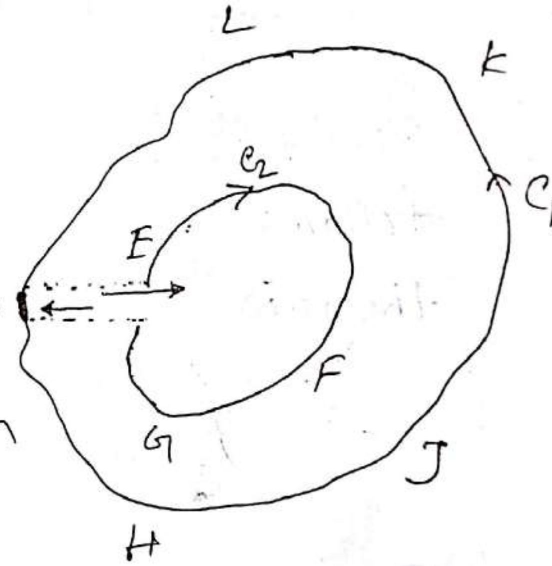
10(E)-Day

Date: 9/1/2017

Prob:

Let $f(z)$ be analytic in a region R bounded by two simple closed curve C_1 & C_2 and also on C_1 & C_2, then prove that $\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$ where C_1 & C_2 are both traversed in the positive sense to their interiors i.e. anticlockwise.



Proof:

Constant Gess cut DE since $f(z)$ is analytic in the region R, so by Cauchi's integral theorem

$$\oint_{DEFGEDHJKLD} f(z)\,dz = 0$$

Or, $$\oint_{DE} f(z)\,dz + \oint_{EFGE} f(z)\,dz + \oint_{ED} f(z)\,dz + \oint_{DHJKLD} f(z)\,dz = 0$$

Or, $$\oint_{DHJKLD} f(z)\,dz = -\oint_{EFGE} f(z)\,dz = \oint_{EGFE} f(z)\,dz$$ $\Rightarrow$ Or, $\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$

Prob

Evaluate $\oint_C \frac{dz}{z-a}$, where C is the simple closed curve and $z=a$ (i) outside C (ii) inside C.

Solution:

(i) Since $z=a$ is outside C,

So $f(z) = \frac{1}{z-a}$ is analytic everywhere & follows Cauchy's integral theorem.

$$\therefore \oint_C \frac{dz}{z-a} = 0.$$

(ii) Since $z=a$ is inside C. So. $f(z) = \frac{1}{z-a}$ non-analytic at $z=a$. Now we construct a circle Γ of radius ϵ at the point $z=a$. So, we can write $|z-a| = \epsilon$.

$\Gamma \to$ Gamma

$re^{i\theta} \leftarrow$ Polar form

$z - a = \epsilon e^{i\theta}$, $z = a + \epsilon e^{i\theta}$, $0 \le \theta \le 2\pi$ and

$dz = i\epsilon e^{i\theta} d\theta$.

a [illegible] value [illegible]

$$\text{So, } \oint_C \frac{dz}{z-a} = \int_{\theta=0}^{2\pi} \frac{i\epsilon e^{i\theta} d\theta}{\epsilon e^{i\theta}} = i\int_{\theta=0}^{2\pi} d\theta = 2\pi i$$

Cauchy's integral formula:

If $f(z)$ is analytic inside and on the boundary c of simply connected region R except at the point 'a' inside c, then

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz$$

$$f'(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^2} dz \quad \text{[for first derivative]}$$

$$f^n(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz \quad \text{[for nth "]}$$

Proof

Hints: $\frac{f(z)}{z-a}$ is analytic in a simple closed curve C except one point at $z=a$ inside C. Now we construct ... radius of ϵ. So we can write

$$\oint_C \frac{f(z)}{z-a}\,dz = \oint_{\Gamma} \frac{f(z)}{z-a}\,dz \quad \text{--- (1)}$$

Now, $|z-a| = \epsilon$, $z-a = \epsilon e^{i\theta}$, $0 \le \theta \le 2\pi$,

$dz = i\epsilon e^{i\theta}\,d\theta$

From (i) $\rightarrow$

$$\oint_C \frac{f(z)}{z-a}\,dz = \oint_{\Gamma} \frac{f(z)}{z-a}\,dz$$

$$= \oint_{\theta=0}^{2\pi} \frac{f_1(a+\epsilon e^{i\theta})}{\epsilon e^{i\theta}}\, i\epsilon e^{i\theta}\,d\theta$$

Or, $$\oint_C \frac{f(z)}{z-a}\,dz = i\int^{2\pi} f(a+\epsilon e^{i\theta})\,d\theta \quad \text{--- (2)}$$

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Taking limit at both sides of (2)

$$\lim_{\epsilon \to 0} \oint_C \frac{f(z)}{z-a}\, dz = \lim_{\epsilon \to 0} i \int_0^{2\pi} f(a+\epsilon e^{i\theta})\, d\theta$$

$$= i \int_0^{2\pi} \lim_{\epsilon \to 0} f(a+\epsilon e^{i\theta})\, d\theta$$

$$= i \int_0^{2\pi} f(a)\, d\theta$$

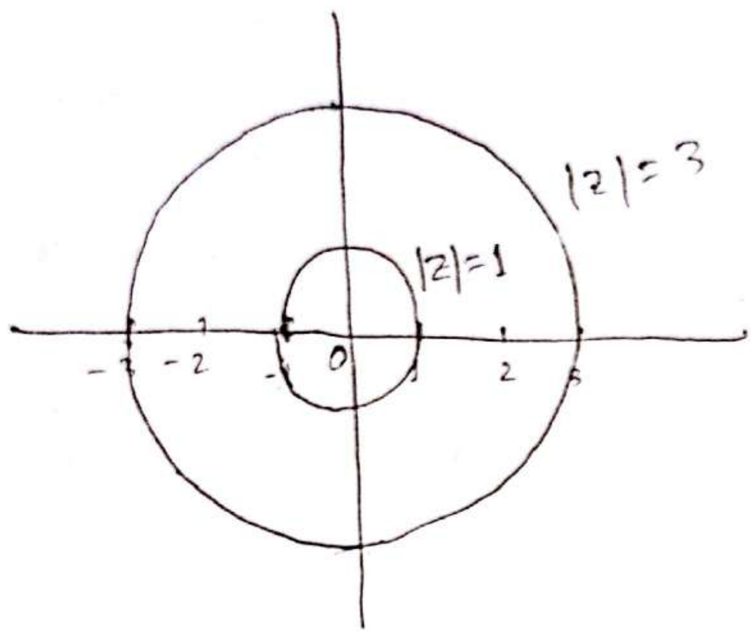
$$= f(a)\, 2\pi i$$

$$\Rightarrow f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\, dz$$

<u>Prob</u>

Evaluate $\frac{1}{2\pi i} \oint_C \frac{e^z}{(z-2)}\, dz$, where C is the circle (i) $|z|=3$, (ii) $|z|=1$

Soln



(ii) a=2 . so z is outside c, so it foll

Cauchy's integral theorem. then

$$\frac{1}{2\pi i}\oint_c \frac{e^z}{(z-2)}\,dz = 0$$

(i) a=2, and c is |z|=3, so a is [ans

inside c. Then it follows Cauchy's integral theorem formula.

11(A) - Day Date: 11/1/2017

Problem: Find the fourier series expansion of the periodic function of period 2π,

$$f(x) = x^2, \quad -\pi \leq x \leq \pi$$. Also obtain

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\cdots$$

Soln:

Given that.

$$f(x) = x^2, \quad -\pi \leq x \leq \pi .$$

Since, $f(x)$ is an even function, hence, $b_n = 0$

Now, $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{2}{\pi}\int_{0}^{\pi} x^2\, dx = \frac{2\pi^2}{3}$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{2}{\pi} \int_{0}^{\pi} x^2 \cos nx \, dx$$

$$= \cdots$$

$$= \frac{2}{\pi} \left[\frac{\pi^2 \sin n\pi}{n} + \frac{2\pi \cos n\pi}{n^2} - \frac{2 \sin n\pi}{n^3} \right]$$

$$= \frac{4}{n^2} \cos n\pi = \frac{4}{n^2} (-1)^n$$

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We know, the fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \Big[\text{since, } f(x) \text{ is eve}$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx \quad \text{Ans}$$

Now,
$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$$

$$= \cancel{\frac{\pi^2}{3} + 4\left(\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2}\right)}$$

$$= \frac{\pi^2}{3} + 4\left(-\frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \cdots\right)$$

$$= \frac{\pi^2}{3} - 4\left(\frac{1}{1^2}\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \cdots\right)$$

When $x=0$, we get,

$$0 = \frac{\pi^2}{3} - 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\right)$$

$$\Rightarrow \frac{\pi^2}{3} = 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\right)$$

$$\therefore \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots$$

F.T → Fourier Transform

Integral Transform:

The integral transform of $f(x)$ with the Kernel $K(s,)$ is defined as

$$I\{f(x)\} = F(s) = \int_a^b f(x)\, K(s,x)\, dx$$

I → Integral transform complex field

* Fourier Transform:

$$F\{f(x)\} = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx$$

and $$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, dx$$ [Inverse F.T

* F. cosine T.

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx\, dx$$

F. sin T. :

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx\, dx$$

Fourier Integral Theorem:

Statement:

It states that $f(x) = \frac{1}{\pi}\int_{0}^{\alpha}\int_{-\alpha}^{\alpha} f(t)\cos u(t-x)\,dt\,du$

Proof:

We know, fourier series in $(-c, c)$ is given

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\alpha} a_n \cos\frac{n\pi x}{c} + \sum_{n=1}^{\alpha} b_n \sin\frac{n\pi x}{c}$$

where,

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(t)\,dt$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(t)\cos\frac{n\pi t}{c}\,dt$$

$$b_n = \frac{1}{c}\int_{-c}^{c} f(t)\sin\frac{n\pi t}{c}\,dt$$

From (i) we get,

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)\,dt + \sum_{n=1}^{\infty}\frac{1}{c}\int_{-c}^{c} f(t)\cos\frac{n\pi t}{c}\cos\frac{n\pi x}{c}\,dt$$

$$+ \sum_{n=1}^{\infty}\frac{1}{c}\int_{-c}^{c} f(t)\sin\frac{n\pi t}{c}\sin\frac{n\pi x}{c}\,dt$$

$$= \frac{1}{2c}\int_{-c}^{c} f(t)\,dt + \sum_{n=1}^{\infty}\frac{1}{c}\int_{-c}^{c} f(t)\left[\cos\frac{n\pi t}{c}\cos\frac{n\pi x}{c} + \sin\frac{n\pi t}{c}\sin\frac{n\pi x}{c}\right]dt$$

$$\Rightarrow f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)\,dt + \sum_{n=1}^{\infty}\frac{1}{c}\int_{-c}^{c} f(t)\cos\frac{n\pi}{c}(t-x)\,dt$$

$$= \frac{1}{2c}\int_{-c}^{c} f(t)\left\{1 + 2\sum_{n=1}^{\infty}\cos\frac{n\pi}{c}(t-x)\right\}dt \quad \text{—— (ii)}$$

Since, cosine functions are even functions, i.e, $\cos(-\theta) = \cos\theta$, then the expression,

$$1 + 2\sum_{n=1}^{\infty}\cos\frac{n\pi}{c}(t-x) = \sum_{n=-\infty}^{\infty}\cos\frac{n\pi}{c}(t-x).$$

$-2, 2 \rightarrow$ same $\rightarrow$ এইজন্য-2

$0 \rightarrow 1+$

$$\therefore f(x) = \frac{1}{2c} \int_{-c}^{c} f(t) \left\{ \sum_{n=-\alpha}^{\alpha} \cos \frac{n\pi}{c} (t-x) \right\} dt \qquad \text{(ii)}$$

$$= \frac{1}{2\pi} \int_{-c}^{c} f(t) \left\{ \frac{\pi}{c} \sum_{n=-\alpha}^{\alpha} \cos \frac{n\pi}{c} (t-x) \right\} dt \qquad \text{(iii)}$$

Let us now assume that c increases indefinitely, so that, we may write

$$\frac{n\pi}{c} = u \text{ and } \frac{\pi}{c} = du,$$

$$\lim_{c \to \alpha} \left\{ \frac{\pi}{c} \sum_{n=-\alpha}^{\alpha} \cos \frac{n\pi}{c} (t-x) = \int_{-\alpha}^{\alpha} \cos u (t-x) \right.$$

Then (iii) gives,

$$f(x) = \frac{1}{2\pi} \int_{-\alpha}^{\alpha} f(t) \left\{ \int_{-\alpha}^{\alpha} \cos u (t-x) \, du \right\} dt$$

$\rightarrow \because$ even function

$$= \frac{1}{2\pi} \int_{-\alpha}^{\alpha} f(t) \left[2 \int_{0}^{\alpha} \cos u (t-x) \, du \right] dt$$

$$= \frac{1}{\pi} \int_{-\alpha}^{\alpha} f(t) \left[\int_{0}^{\alpha} \cos (t-x) \, du \right] dt$$

$$\therefore f(x) = \frac{1}{\pi} \int_{0}^{\alpha} \int_{-\alpha}^{\alpha} f(t) \cos u (t-x) \, dt \, du$$

Fourier transform:

$$F\{f(x)\} = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

* Find the fourier transform of $f(x) = \begin{cases} 1, & \text{for } |x| < a \\ 0, & \text{for } |x| > a \end{cases}$

Soln

We know, the F.T. of $f(x)$ is given by

$$F\{f(x)\} = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-a} f(x)\, e^{isx}\, dx + \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} f(x)\, e^{isx}\, dx$$

$$+ \frac{1}{\sqrt{2\pi}} \int_{a}^{\infty} f(x)\, e^{isx}\, dx$$

$$= 0 + \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} 1 \cdot e^{isx}\, dx + 0$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isx}}{is} \right]_{-a}^{a} = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{is} \left[e^{isa} - e^{-isa} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{is}\, 2i \sin sa = \frac{1}{\sqrt{2\pi}} \frac{2 \sin sa}{s}$$ (Ans).

$$\cancel{\int uv\,dx = u\int v\,dx - \int\left[\frac{du}{dx}\int v\,dx\right]dx}$$

$$\cancel{\int uv\,dx = u\int v\,dx + \int\left[\frac{du}{dx}\int v\,dx\right]dx}$$

Formula: $\int e^{ax}\sin bx\,dx = \dfrac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$

* Find the fourier sine & cosine transform of $f(x) = e^{-ax}$

<u>Soln</u>

We know the fourier sine transform is

$$F\{f(x)\} = F(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$$

$$\cancel{= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}} = \sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}(a\sin sx - s\cos sx)}{a^2+s^2}\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left(0 + \frac{s}{a^2+s^2}\right) \qquad [\because e^{-\infty} = $$

$$= \sqrt{\frac{2}{\pi}}\cdot\frac{s}{a^2+s^2} \quad \text{(Ans.)}$$

11(E)-Day

Date: 21/1/17

* Evaluate $\oint_C \frac{z\,dz}{(9-z^2)(z+i)}$, where C is the circle $|z|=2$

*** Cauchy's Residual Theorem:

(i) If $f(z)$ is analytic inside and on a simple closed curve C except for a pole of order m at $z=a$ inside C, then prove that

$$\frac{1}{2\pi i}\oint_C F(z)\,dz = \lim_{z\to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}\left\{(z-a)^m F(z)\right\}$$

(ii) If there are two poles at $z=a_1$ and $z=a_2$ inside C of order m_1 & m_2 respectively, then prove that

$$\frac{1}{2\pi i}\oint_C F(z)\,dz = \lim_{z\to a_1} \frac{1}{(m_1-1)!} \frac{d^{m_1-1}}{dz^{m_1-1}}\left\{(z-a_1)^{m_1} F(z)\right\}$$

$$+ \lim_{z\to a_2} \frac{1}{(m_2-1)!} \frac{d^{m_2-1}}{dz^{m_2-1}}\left\{(z-a_2)^{m_2} F(z)\right\}$$

(iii) In general if $F(z)$ has a number of poles inside C with ~~resident~~ residues $R_1, R_2, R_3, \dots,$ then

$$\oint_C F(z)\,dz = 2\pi i \{R_1 + R_2 + R_3 + \cdots\} = 2\pi i \{\text{sum of Residue}\}$$

<u>Proof</u>

If $F(z)$ has pole of order m at $z = a$ then $F(z) = \frac{f(z)}{(z-a)^m}$ where $f(z)$ is analytic inside and on C and $f(a) \neq 0$. Then by Cauchy's integral formula

$$f^{m-1}(a) = \frac{(m-1)!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^m}\,dz \quad \cdots\cdots (1)$$

Now $\frac{1}{2\pi i} \oint_C F(z)\,dz = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^m}\,dz = \frac{f^{m-1}(a)}{(m-1)!}$

[by using (1)]

$$= \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}} \{ f(z) \}$$

$$= \lim_{z\to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \{ (z-a)^m F(z) \}, \text{ since}$$

$$F(z) = \frac{f(z)}{(z-a)^m}.$$

* Evaluate: $\oint_C \frac{e^z}{(z^2+\pi^2)^2} dz$, where C is the circle $|z|=4$.

The poles of $f(z) = \frac{e^z}{(z^2+\pi^2)^2} = \frac{e^z}{(z+\pi i)^2 (z-\pi i)^2}$ are at $z = \pm \pi i$ inside C. and both of order 2.

$\therefore$ Residue at $z = +\pi i$ is ~~lim~~

$$\lim_{z\to \pi i} \frac{1}{(2-1)!} \frac{d^{2-1}}{dz^{2-1}}$$

Problem:

Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{(z^2+1)^2}\,dz$ if $t>0$ and C is the circle $|z|=2$.

Ans: $\frac{1}{2}(\sin t - t\cos t)$

Theorem 1:

Consider the evaluation of integrals of t type $I = \int_{-\alpha}^{\alpha} F(x)\,dx$ where $F(z)$ is a com function that satisfies the following conditions –

1. It is ~~any~~ analytic in the upper hal plane except at finite number of poles.

2. It has no poles on the real ~~pole~~ axis

3. $z F(z) \to 0$ (Converge) as $|z| \to \alpha$ for $0 \leq \arg z \leq \pi$.

4. When x is real, $x F(x) \to 0$ as $x \to \pm\alpha$ in such way that $\int_{-\alpha}^{\alpha} F(x)\,dx$ converges. Then $I = \int_{-\alpha}^{\alpha} F(x)\,dx = \oint_C F(z)\,dz$

$= 2\pi i$ {sum of Resid

Theorem 1:

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$= 2\pi i$ {sum of Residu

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Proof:

$$\oint_C F(z)\,dz = \int_{-R}^{R} F(x)\,dx + \int_{\Gamma} F(z)\,dz$$

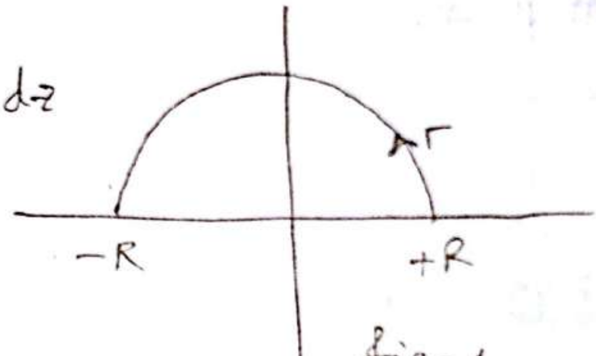

fig. 1

From fig. 1 .

~~From fig. 1b / Taking~~

Taking limit as $R \longrightarrow \alpha$

$$\oint_C F(z)\,dz = \int_{-\alpha}^{\alpha} F(x)\,dx + 0 = 2\pi i \{ \text{sum of Resid}$$

12(B)—Day Date: 21/1/2017

* State & Prove properties of Fourier Transform

i) Linear property: If $F_1(s)$ & $F_2(s)$ be the fourier transform of the functions $f_1(x)$ & $f_2(x)$ repectively then

$$F\left[a f_1(x) + b f_2(x)\right] = aF_1(s) + b F_2(s)$$, where a & b are constants.

Proof:

We know,

$$F\{f(x)\} = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} e^{isx} f(x)\, dx$$

Then, $$F\{f_1(x)\} = F_1(s) = \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\infty} e^{isx} f_1(x)\, dx$$

and $$F\{f_2(x)\} = F_2(s) = \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} e^{isx} f_2(x)\, dx$$

Now, $$F\{a f_1(x) + b f_2(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} e^{isx} \left[a f_1(x) + b f_2(x)\right] dx$$

$$= a \cdot \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} e^{isx} f_1(x)\, dx + b \cdot \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} e^{isx} f_2(x)\, dx$$

$$= a F_1(s) + b F_2(s)$$

[Proved]

ii) Change of scale property: If $F(s)$ be a fourier transform of $f(x)$ then

$$F\{f(ax)\} = \frac{1}{a} F\left(\frac{s}{a}\right),$$ where a is a constant

Proof:

We know,

$$F\{f(x)\} = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$$

$$\therefore F\{f(ax)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(ax)\, dx$$

Put $ax = t$
$\Rightarrow x = \frac{t}{a}$
$\therefore dx = \frac{1}{a} dt$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{is\left(\frac{t}{a}\right)} f(t) \cdot \frac{1}{a} dt$$

$$= \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\left(\frac{s}{a}\right)t} f(t) \cdot dt$$

$$= \frac{1}{a} F\left(\frac{s}{a}\right)$$ [Proved]

Page → 951 → theorem দেখতে হবে

(iii) shipting property: If $F(s)$ be the fourier transform of $f(x)$, then

$$F\{f(x-a)\} = e^{isa} F(s).$$

Page-949

<u>Proof</u>

$s \rightarrow$ complex varia[ble]

* Laplace Transform:

Let $f(t)$ be a function defined for all +ve values of t, i.e. $t \geq 0$, then

$$L\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t)\,dt$$

provided the integral exists, is called Laplace Transform of $f(t)$.

* $L\{1\} = \int_0^{\infty} e^{-st} \cdot 1\, dt$

t^0

$$= \left[\frac{e^{-st}}{-s}\right]_0^{\infty}$$

$$= -\frac{1}{s}[0-1]$$

$$= \frac{1}{s}$$

* $L\{t^n\} = \dfrac{n!}{s^{n+1}}$, where n & s are +ve.

$$= \frac{\Gamma(n+1)}{s^{n+1}}$$

* Show that, $L\{e^{at}\} = \frac{1}{s-a}$. where $s > a$

$$L\{e^{at}\} = \int_0^{\infty} e^{-st} \cdot e^{at}\, dt$$

$$= \int_0^{\infty} e^{-st+at}\, dt$$

$$= \int_0^{\infty} e^{t(a-s)}\, dt$$

$$= \left[\frac{e^{t(a-s)}}{a-s}\right]_0^{\infty}$$

$$= \frac{1}{a-s}\left[0-1\right]$$

$$= \frac{1}{s-a}$$

partial DE $\xrightarrow{\text{Lap. Trans}}$ Ordinary DE

* Show that,

$L\{\sin at\}$, $L\{\cos at\}$

L·T এর Property

RK

13(A) - Day

Date: 20/2/...

$L\{1\} = \frac{1}{s}$

$L^{-1}\{\frac{1}{s}\} = 1$

$L\{t^n\} = \frac{\Gamma n+1}{s^{n+1}}$ $\therefore L^{-1}\{\frac{1}{s^{n+1}}\} = \frac{t^n}{\Gamma n+1}$

Problem:

Find the inverse L.T of

(i) $\frac{s^2+s+2}{s^{3/2}}$, (ii) $\frac{2s-5}{9s^2-25}$ (iii) $\frac{s-2}{6s^2+20}$. (iv)

Soln

Taking the inverse L.T of expression (i)

$$L^{-1}\left\{\frac{s^2+s+2}{s^{3/2}}\right\} = L^{-1}\left\{s^{\frac{1}{2}} + s^{-\frac{1}{2}} + \frac{2}{s^{3/2}}\right\}$$

$$= L^{-1}\{s^{\frac{1}{2}}\} + L^{-1}\{s^{-\frac{1}{2}}\} + 2L^{-1}\{s^{-3/2}\}$$

$$= L^{-1}\left\{\frac{1}{s^{-\frac{1}{2}}}\right\} + L^{-1}\left\{\frac{1}{s^{1/2}}\right\} + 2L^{-1}\left\{\frac{1}{s^{3/2}}\right\}$$

$n+1 = -\frac{1}{2}$

$$= \frac{t^{-3/2}}{\Gamma -\frac{1}{2}} + \frac{t^{-\frac{1}{2}}}{\Gamma \frac{1}{2}} + 2 \cdot \frac{t^{\frac{1}{2}}}{\Gamma \frac{3}{2}}$$

$$= \frac{t^{-3/2}}{\Gamma\left(-\frac{1}{2}\right)} + \frac{t^{-1/2}}{\sqrt{\pi}} + \frac{2t^{1/2}}{\frac{1}{2}\sqrt{\pi}}$$

Taking the

Problem

Find the inverse L.T of $\frac{1}{s^2-5s+6}$

Taking the inverse L.T of the expression

$$\cancel{L^{-1}\left\{\frac{1}{s^2-5s+6}\right\} = L^{-1}\left\{\frac{1}{s^2}\right\} - 5L^{-1}\left\{\frac{1}{s}\right\}}$$

$$\cancel{= L^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{5}L^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{6}L^{-1}\{1\}}$$

$$\cancel{= \frac{t}{\Gamma 2} - \frac{1}{5}\cdot 1 + \frac{1}{s}}$$

$$L^{-1}\left\{\frac{1}{s^2-5s+6}\right\} = L^{-1}\left\{\frac{1}{(s-3)(s-2)}\right\}$$

$$= \cancel{L^{-1}\{s-3\}} = L^{-1}\left\{\frac{1}{s-3} - \frac{1}{s-2}\right\}$$

$$= L^{-1}\left\{(s-3)^{-1}\right\} - L^{-1}\left\{(s-2)^{-1}\right\}$$

$$= e^{3t} - e^{2t} \quad \text{(Ans.)}$$

$$\boxed{L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}}$$

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 0$$

Let, $L\{y(t)\} = Y(s) = \int_0^{\infty} e^{-st} y(t)\, dt$

$$L\{\dot{y}(t)\} = \int_0^{\infty} e^{-st} \dot{y}(t)\, dt$$

$$= e^{-st} y(t)\Big|_0^{\infty} - \int_0^{\infty} (-s) e^{-st} \cdot y(t)\, dt$$

$$= -y(0) + s\int_0^{\infty} e^{-st} y(t)\, dt$$

$$= -y(0) + sY(s)$$

$$L\{\ddot{y}(t)\} = \int_0^{\infty} e^{-st} \ddot{y}(t)\, dt$$

$$= e^{-st} \dot{y}(t)\Big|_0^{\infty} - \int_0^{\infty} (-s) e^{-st} \cdot \dot{y}(t)\, dt$$

$$= -\dot{y}(0) + s\int_0^{\infty} e^{-st} \dot{y}(t)\, dt$$

$$= s^2 Y(s) - s y(0) - \dot{y}(0)$$

* Solve the DE $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 1$ subject to the condition $y(0) = \dot{y}(0) =$

Taking L.T on both sides of given eq^n, we get

$$L\{\ddot{y}(t) + 3\dot{y}(t) + 2y(t)\} = L(1)$$

$$\Rightarrow L\{\ddot{y}(t)\} + 3L\{\dot{y}(t)\} + 2L\{y(t)\} = L(1)$$

$$\Rightarrow [s^2 Y(s) - s y(0) - \dot{y}(0)] + 3[-y(0) + sY(s)] + 2Y(s) = \frac{1}{s}$$

$$\Rightarrow s^2 Y(s) + 3s\,Y(s) + 2Y(s) = \frac{1}{s}$$

$$\Rightarrow (s^2 + 3s + 2)\,Y(s) = \frac{1}{s}$$

$$\Rightarrow Y(s) = \frac{1}{s(s^2+3s+2)}$$

$$= \frac{1}{s(s+2)(s+1)}$$

$$= \left[\frac{1/2}{s} + \frac{1/2}{s+2} - \frac{1}{s-1}\right]. \qquad \text{—— (i)}$$

Taking inverse L.T. on (ii)

$$y(t) = \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t}$$

Page → 916 ⟶ See all problems

47, 48, 49, ---

Replacing x by $-x$ in the first integral and combining with 3rd integral, using properties of definite integration, we find

$$\int_{HJA} \frac{e^{iz}}{z} dz + \int_{\epsilon}^{R} \frac{e^{ix} - e^{-ix}}{x} dx + \int_{BDEFG} \frac{e^{iz}}{z} dz = 0$$

Or, $$2i \int_{\epsilon}^{R} \frac{\sin x}{x} dx = -\int_{HJA} \frac{e^{iz}}{z} dz - \int_{BDEFG} \frac{e^{iz}}{z} dz$$

Let $\epsilon \to 0$ and $R \to \infty$, the 2nd integral on the right sides approaches zero. Letting $z = \epsilon e^{i\theta}$ in the 1st integral on the right sides we see that it approaches

$$-\int_{HJA} \frac{e^{iz}}{z} dz = -\lim_{\epsilon \to 0} \int_{\pi}^{0} \frac{e^{i\epsilon e^{i\theta}}}{\epsilon e^{i\theta}} i\epsilon e^{i\theta} d\theta$$

$$= \lim_{\epsilon \to 0} \int_{0}^{\pi} i e^{i\epsilon e^{i\theta}} d\theta = i \int_{0}^{\pi} d\theta = \pi i,$$

since the limit can be taken under the integral sign. Then we ham,

$$\lim_{\substack{\epsilon \to 0 \\ R \to \infty}} 2i \int_{\epsilon}^{R} \frac{\sin x}{x}\, dx = \pi i$$

Or,

$$\int_0^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi i}{2i} = \frac{\pi}{2}$$

(Ans).

Definite integral of the type $\int_0^{2\pi} F(\sin\theta, \cos\theta)$

Evaluate

i) $\displaystyle\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta}$.

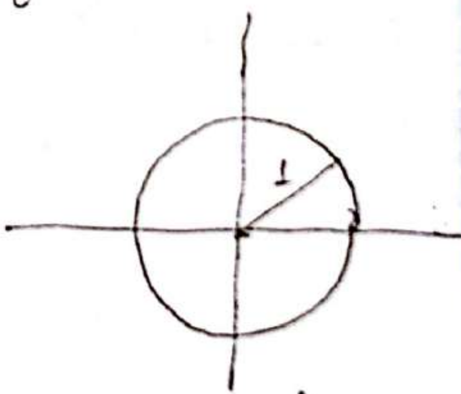

fig. 1.

(ii) $\displaystyle\int_0^{2\pi} \frac{d\theta}{5 - 3\sin\theta}$

iii) $\displaystyle\int_0^{2\pi} \frac{\cos^3\theta}{5 - 4\cos\theta}\, d\theta$

Solution:

Let $z = e^{i\theta}$. then $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i}$

and $dz = ie^{i\theta}\, d\theta = iz\, d\theta$ so that

$$\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \oint_C \frac{dz/iz}{a + b\left(\frac{z - z^{-1}}{2i}\right)} = \cdots = \oint_C \frac{2dz}{bz^2 + 2aiz - b},$$

where c is the circle of unit radius with centre at the origin as shown in fig. 1

The poles of $F(z) = \frac{2}{bz^2 + 2aiz - b}$ are ~~obtain~~ obtained by solving $bz^2 + 2aiz - b = 0$ and are given by

$$z = \frac{-2ai \pm \sqrt{(-2ai)^2 - 4b(-b)}}{2b}$$

$$=$$

$$= \frac{-ai \pm i\sqrt{a^2 - b^2}}{b}$$

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So, the poles are $z_1 = \frac{-a+\sqrt{a^2-b^2}}{b}\, i$ and $z_2 = \frac{-a-\sqrt{a^2-b^2}}{b}\, i$, but only the z_1 ~~is~~ lies inside C in fig. 1, since

$$|z_1| = \left| \frac{-a+\sqrt{a^2-b^2}}{b}\, i \right| = \left| \frac{-a+\sqrt{a^2-b^2}}{b} \times \frac{\sqrt{a^2-b^2}+a}{\sqrt{a^2-b^2}+a} \right|$$

$$= \dots$$

$$= \left| \frac{-b}{\sqrt{a^2-b^2}+a} \right|$$

$$= \frac{b}{\sqrt{a^2-b^2}+a} < 1$$

since $a > |$

Thus the residue at z_1 is

$$\lim_{z \to z_1} \left\{ (z - z_1) \frac{2}{bz^2 + 2aiz - b} \right\}$$

$$= \lim_{z \to \frac{-a+\sqrt{a^2-b^2}}{b} i} \left\{ \left(z - \frac{-a+\sqrt{a^2-b^2}}{b} i \right) \frac{2}{bz^2 + 2aiz - b} \right\}$$

$$=$$

$$= \frac{1}{\sqrt{a^2-b^2}\, i}$$

Hence $\displaystyle\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \oint \frac{2dz}{bz^2 + 2aiz - b} = 2\pi i \left\{ \text{sum of residue} \right\}$

$$= 2\pi i \times \frac{1}{\sqrt{a^2-b^2}\, i} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

(Ans).

Conformal Mapping

Definition:

A transformation which preserves the sense as well as the magnitude of angles is said to be conformable.

If $w = f(z)$ is analytic function, it follows that in the neighbourhood of any point where $f'(z) \neq 0$, the transform defined by $w = f(z)$ is conformable.

Conversely, it can be shown that if the mapping $u = u(x,y)$ & $v = v(x,y)$ is conformal and if the first partial derivative of u and v are continuous then $w = f(z) = u + iv$ is an analytic function.

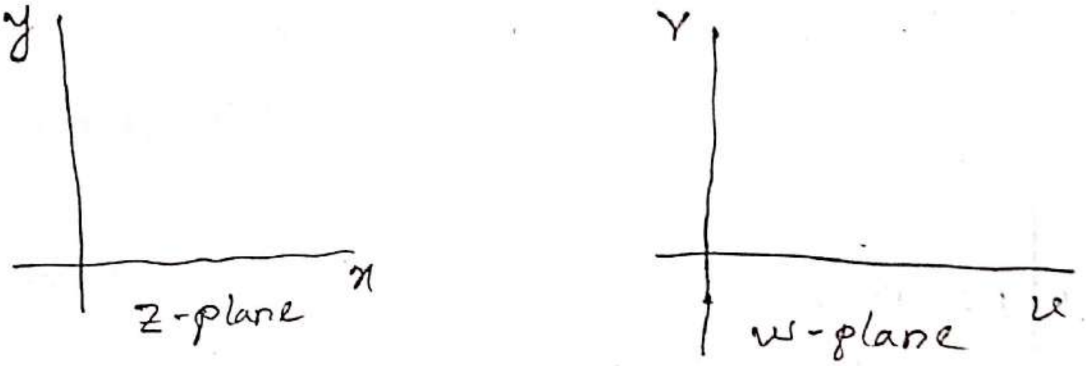
Example:

z-plane is mapped onto the w-plane by the function $w = f(z) = z^2$

Solution:

We have, $w = f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$

and thus $u = u(x,y) = x^2 - y^2$ and $v = v(x,y) = 2xy$ —(i)



Various Types of Transformations:

1. Translation: $w = z + B$

2. Rotation: $w = ze^{i\theta}$

3. Stretching $w = az$

4. Inversion: $w = \frac{1}{z}$

*5. Bilinear Transformation: $w = \frac{az+b}{cz+d}$

$ad - bc \neq 0,\ c \neq 0$

* 1. Prove that every bilinear transformation is the combination of translation, rotation, stretching and inversion.

2. Prove that every bilinear tranformation transform a circle (a line) of the z-plane into a circle (a line) of w-plane.

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3. page-216, chapter-8. # 09-15

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1. Chapter 1+2 (1 টা)
2. Control int. , conformable mapping (1 টা)
3. প্রাবস্থান (1 টা)