$$\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4....}}}}$$

$$1-1+1-1+1........=?$$

$$\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4....}}}}$$

Discrete mathematics

Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

$$\exists_{x\in\Re}\exists_{y\in\Re}(x=y)$$

$$\forall_x(\Re\,|\,x)$$

$$\sum_{x=1}^{\infty}\frac{1}{x}=?$$

$$\sum_{x=1}^{\infty}x=?$$

Chapter 2

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Functions

Section 2.3

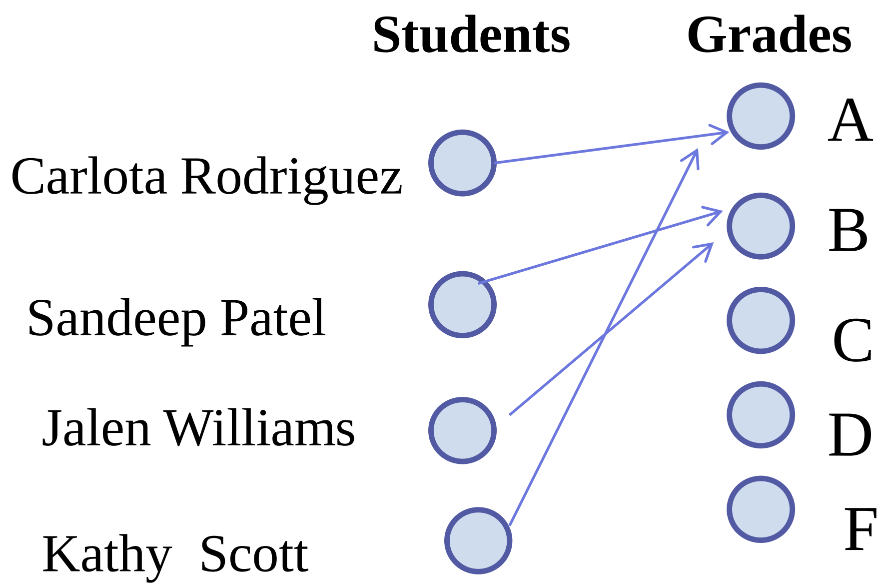
Section Summary

- Definition of a Function.
  - Domain, Codomain
  - Image, Preimage
- Injection, Surjection, Bijection
- Inverse Function
- Function Composition
- Graphing Functions
- Floor, Ceiling, Factorial
- Partial Functions (optional)

Functions

Definition: Let A and B be nonempty sets. A *function* f from A to B, denoted $f: A \to B$ is an assignment of each element of A to exactly one element of B. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.

- Functions are sometimes called *mappings* or *transformations*.



Functions

- A function f: $A \rightarrow B$ can also be defined as a subset of $A \times B$ (a relation). This subset is restricted to be a relation where **no two elements of the relation have the same first element**.
- Specifically, a function f from A to B contains one, and only one ordered pair (a, b) for every element $a \in A$.

and

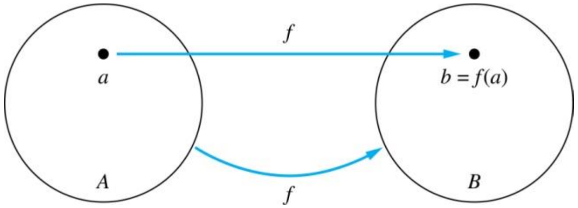
$$\forall x[x \in A \rightarrow \exists y[y \in B \wedge (x, y) \in f]]$$

$$\forall x, y_1, y_2[[(x, y_1) \in f \wedge (x, y_2) \in f] \rightarrow y_1 = y_2]$$

Functions

Given a function $f: A \rightarrow B$:

- We say f *maps* A to B *or*
  f is a *mapping* from A to B.
- A is called the ***domain*** of f.
- B is called the ***codomain*** of f.
- If f(a) = b,
  - then b is called the image of a under f.
  - a is called the preimage of b.
- The range of f is the set of all images of points in **A** under f. We denote it by $f(A)$.
- Two functions are *equal* when they have the same domain, the same codomain and map each element of the domain to the same element of the codomain.



Representing Functions

- Functions may be specified in different ways:
  - An explicit statement of the assignment.

    Students and grades example.
  - A formula.

    $f(x) = x + 1$
  - A computer program.
    - A Java program that when given an integer n, produces the nth Fibonacci Number (covered in the next section and also in Chapter 5).

Questions

f(a) = ?   z

The image of d is ?   z

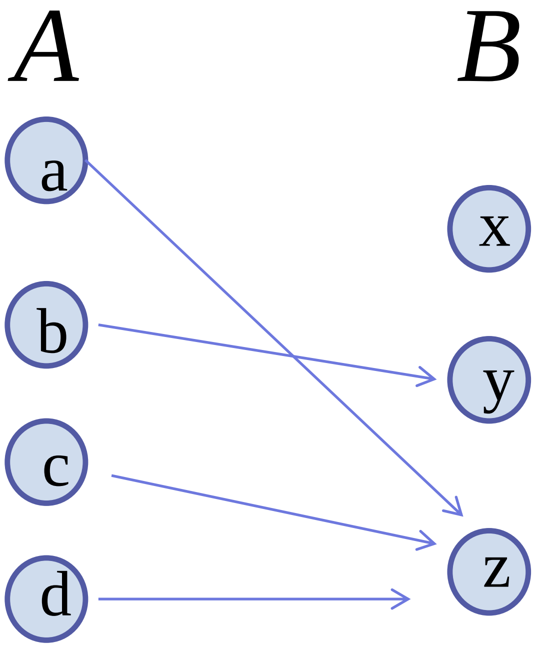
The domain of f is ? A

The codomain of f is ? B

The preimage of y is ?   b

$f(A)$ = ?   {y,z}

The preimage(s) of z is (are) ?   {a,c,d}



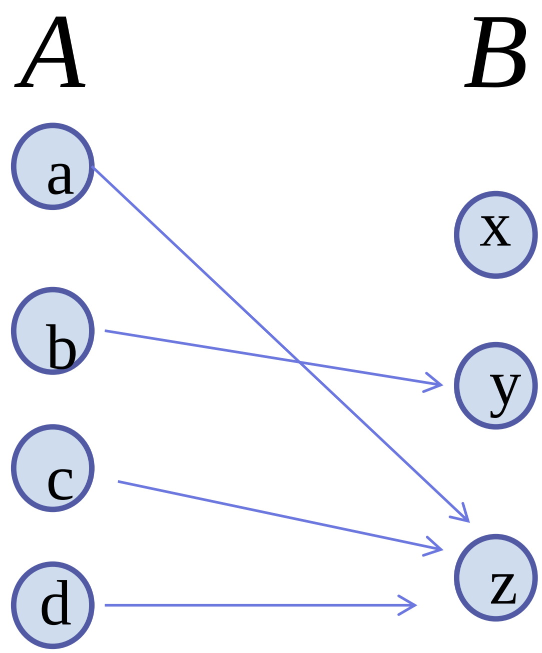
Question on Functions and Sets

- If $f : A \rightarrow B$ and S is a subset of A, then

$$f(S) = \{f(s) | s \in S\}$$

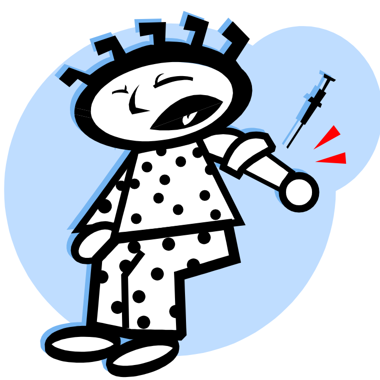
f {a,b,c,} is ? {y,z}

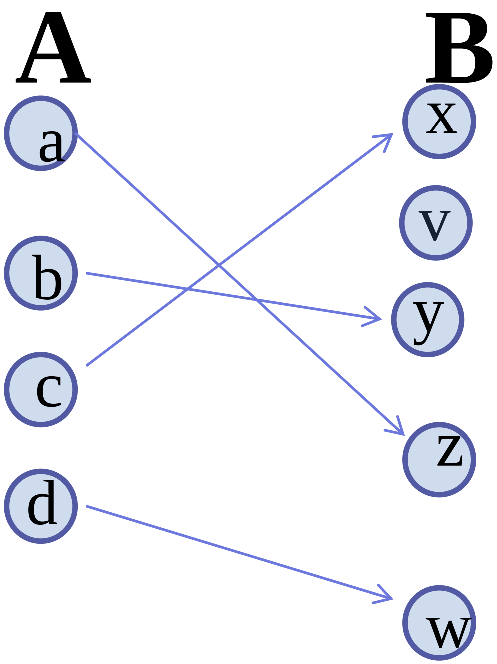
f {c,d} is ? {z}



Injections

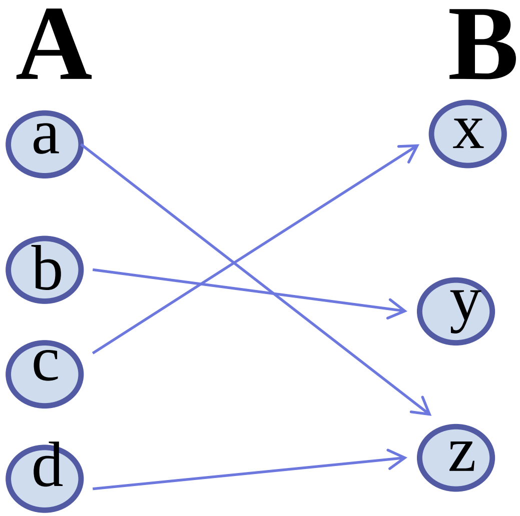
Definition: A function f is said to be *one-to-one* , or *injective*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be an *injection* if it is one-to-one.





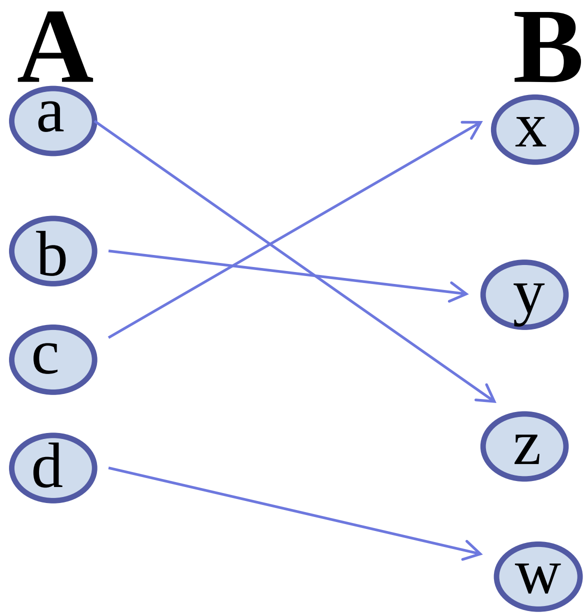
Surjections

Definition: A function f from A to B is called *onto* or *surjective*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function *f* is called a *surjection* if it is *onto*.



Bijections

Definition: A function f is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto (surjective and injective).



Showing that *f* is one-to-one or onto

Example 1: Let f be the function from $\{a,b,c,d\}$ to $\{1,2,3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

Solution: Yes, f is onto since all three elements of the codomain are images of elements in the domain. If the codomain were changed to $\{1,2,3,4\}$, f would not be onto.

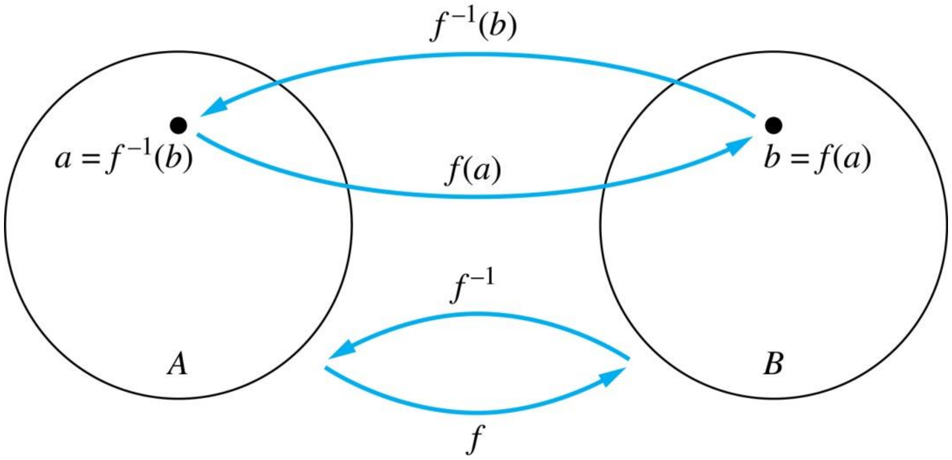
Example 2: Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

Solution: No, f is not onto because there is no integer x with $x^2 = -1$, for example.

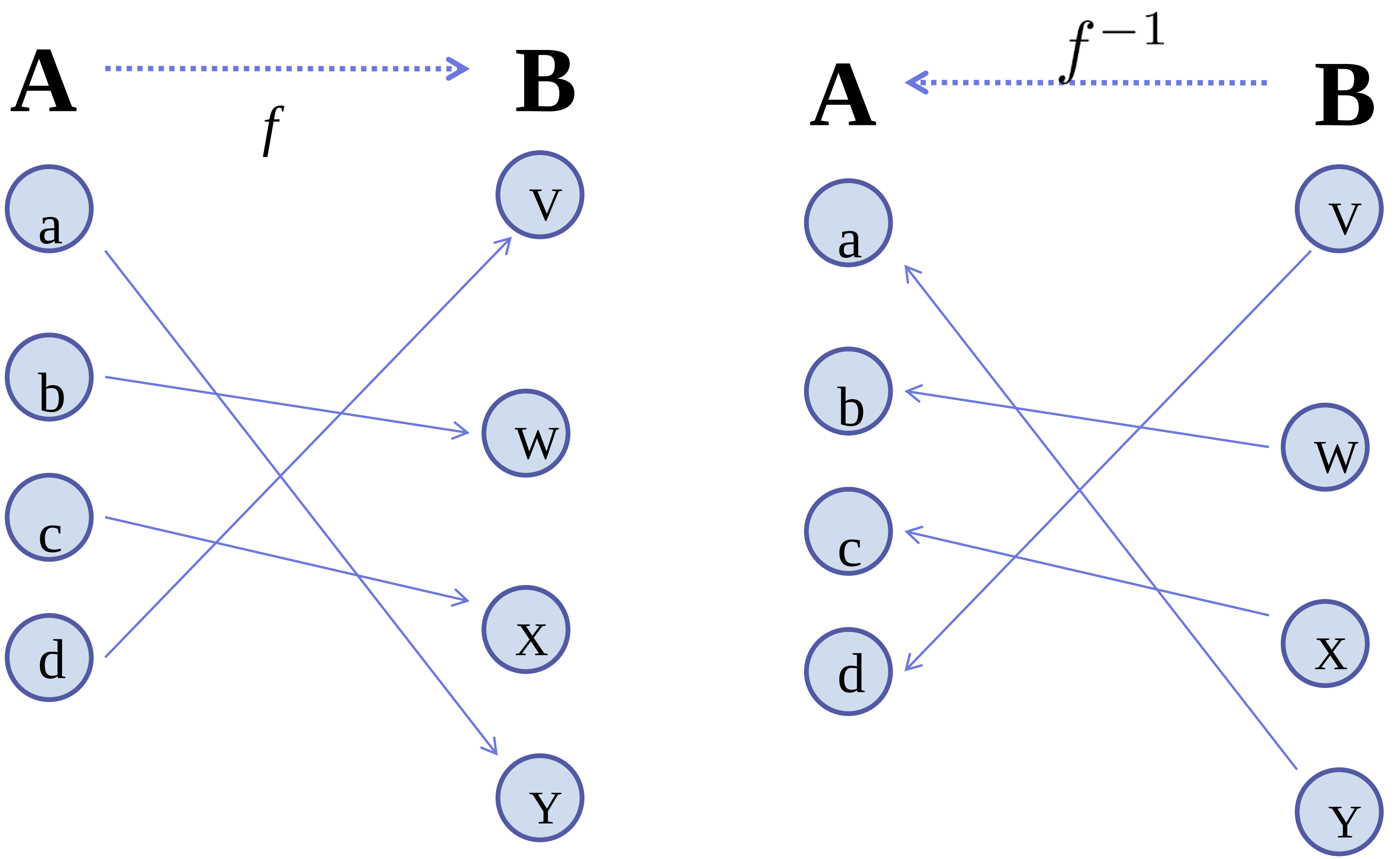
Inverse Functions

Definition: Let f be a bijection from A to B. Then the *inverse* of f, denoted f^{-1}, is the function from B to A defined as $f^{-1}(y) = x$ iff $f(x) = y$

No inverse exists unless f is a bijection. Why?



Inverse Functions



Questions

Example 1: Let f be the function from $\{a,b,c\}$ to $\{1,2,3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible and if so what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

Questions

Example 2: Let $f: \mathbf{Z} \to \mathbf{Z}$ be such that $f(x) = x + 1$. Is f invertible, and if so, what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence so $f^{-1}(y) = y - 1$.

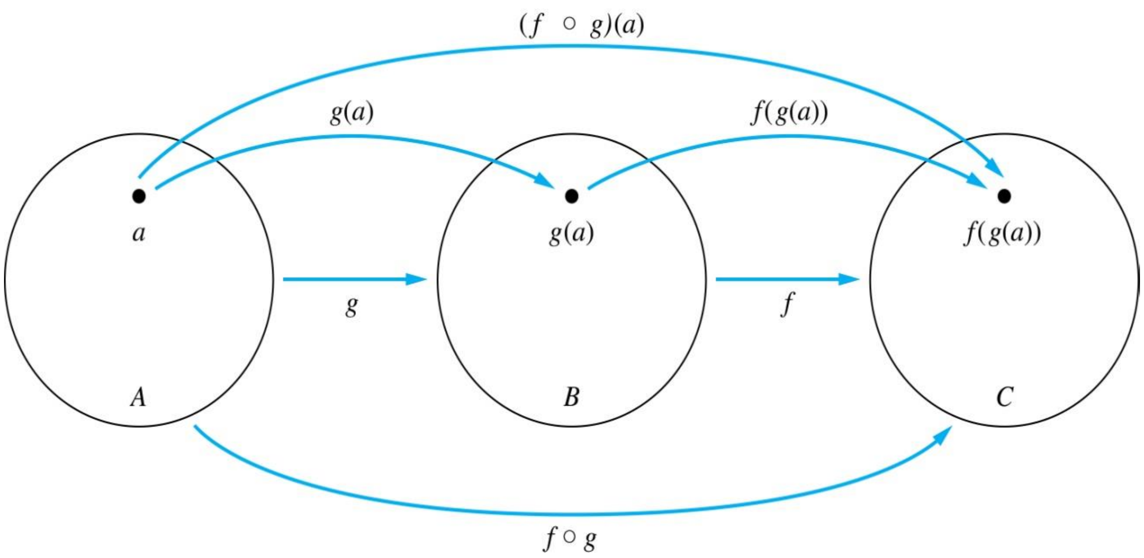
Questions

Example 3: Let f: $\mathbf{R} \rightarrow \mathbf{R}$ be such that $f(x) = x^2$. Is f invertible, and if so, what is its inverse?

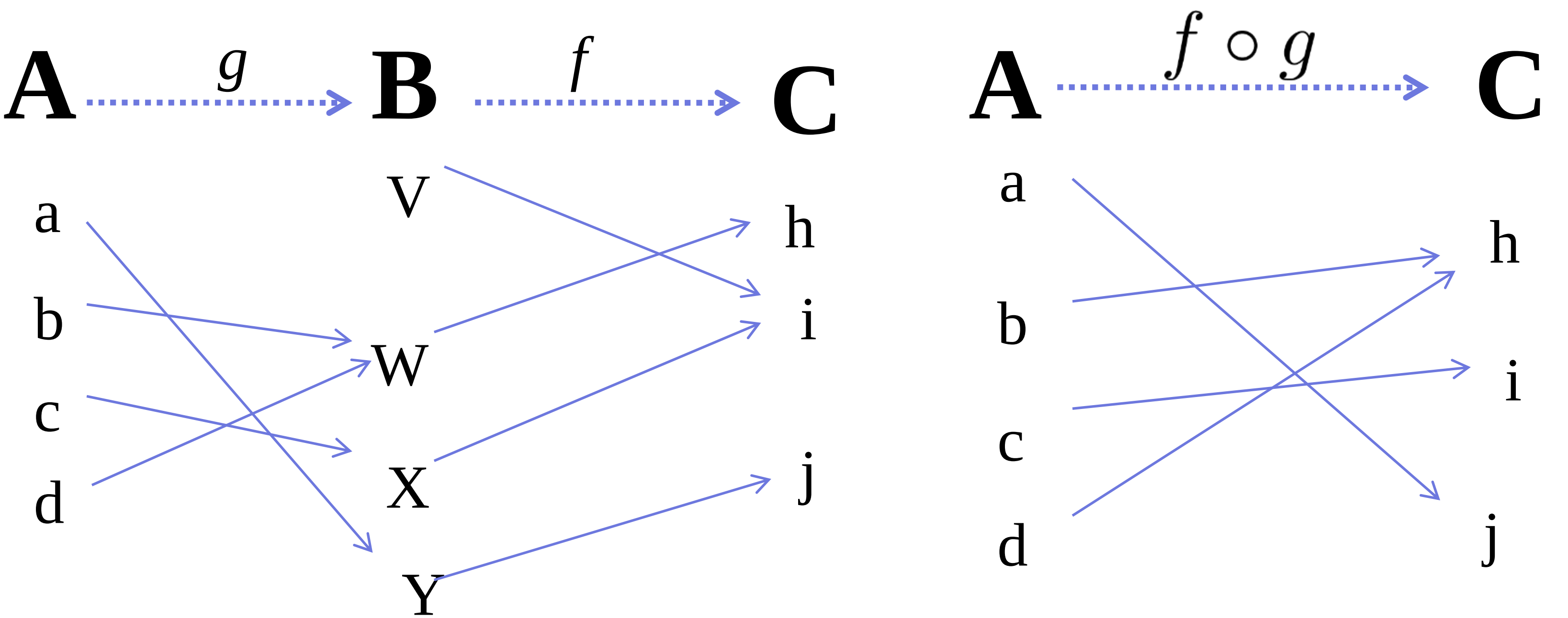
Solution: The function f is not invertible because it is not one-to-one .

Composition

- **Definition**: Let f: $B \to C$, g: $A \to B$. The *composition of f with g*, denoted $f \circ g$ is the function from A to C defined by $f \circ g(x) = f(g(x))$



Composition



Composition

Example 1: If $f(x) = x^2$ and $g(x) = 2x + 1$,

then
$$f(g(x)) = (2x+1)^2$$

and
$$g(f(x)) = 2x^2 + 1$$

Composition Questions

Example 2: Let g be the function from the set $\{a,b,c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a,b,c\}$ to the set $\{1,2,3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$.

What is the composition of f and g, and what is the composition of g and f.

Solution: The composition $f \circ g$ is defined by

$f \circ g\ (a) = f(g(a)) = f(b) = 2.$

$f \circ g\ (b) = f(g(b)) = f(c) = 1.$

$f \circ g\ (c) = f(g(c)) = f(a) = 3.$

Note that $g \circ f$ is not defined, because the range of f is not a subset of the domain of g.

Composition Questions

Example 2: Let f and g be functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$.

What is the composition of f and g, and also the composition of g and f?

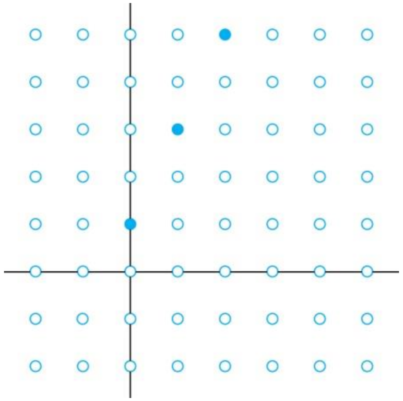
Solution:

$f \circ g\ (x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$

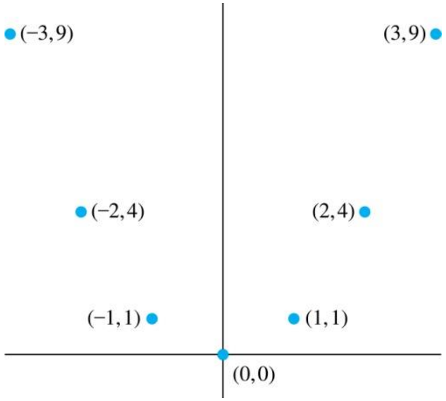
$g \circ f\ (x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$

Graphs of Functions

- Let f be a function from the set A to the set B. The *graph* of the function f is the set of ordered pairs $\{(a,b) \mid a \in A$ and $f(a) = b\}$.



Graph of $f(n) = 2n + 1$ from Z to Z



Graph of $f(x) = x^2$ from Z to Z

Some Important Functions

- The *floor* function, denoted

$$f(x) = \lfloor x \rfloor$$

is the largest integer less than or equal to *x*.

- The *ceiling* function, denoted

$$f(x) = \lceil x \rceil$$

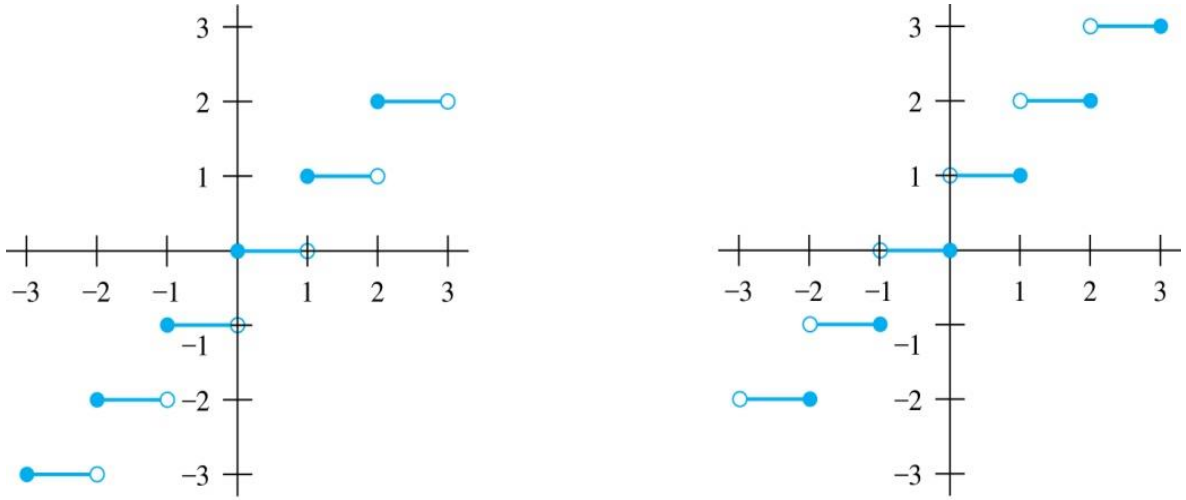
is the smallest integer greater than or equal to *x*

Example:

$$\lceil 3.5 \rceil = 4 \qquad \lfloor 3.5 \rfloor = 3$$

$$\lceil -1.5 \rceil = -1 \qquad \lfloor -1.5 \rfloor = -2$$

Floor and Ceiling Functions



(a) $y = \lfloor x \rfloor$ (b) $y = \lceil x \rceil$

Graph of (a) Floor and (b) Ceiling Functions

Floor and Ceiling Functions

TABLE 1 Useful Properties of the Floor and Ceiling Functions.
(n is an integer, x is a real number)

(1a)	$\lfloor x \rfloor = n$ if and only if $n \leq x < n + 1$
(1b)	$\lceil x \rceil = n$ if and only if $n - 1 < x \leq n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x - 1 < n \leq x$
(1d)	$\lceil x \rceil = n$ if and only if $x \leq n < x + 1$
(2)	$x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x + n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x + n \rceil = \lceil x \rceil + n$

Proving Properties of Functions

Example: Prove that x is a real number, then

$$\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$$

Solution: Let $x = n + \varepsilon$, where n is an integer and $0 \leq \varepsilon < 1$.

Case 1: $\varepsilon < ½$

- $2x = 2n + 2\varepsilon$ and $\lfloor 2x \rfloor = 2n$, since $0 \leq 2\varepsilon < 1$.
- $\lfloor x + 1/2 \rfloor = n$, since $x + ½ = n + (1/2 + \varepsilon)$ and $0 \leq ½ + \varepsilon < 1$.
- Hence, $\lfloor 2x \rfloor = 2n$ and $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + n = 2n$.

Case 2: $\varepsilon \geq ½$

- $2x = 2n + 2\varepsilon = (2n + 1) + (2\varepsilon - 1)$ and $\lfloor 2x \rfloor = 2n + 1$, since $0 \leq 2\varepsilon - 1 < 1$.
- $\lfloor x + 1/2 \rfloor = \lfloor n + (1/2 + \varepsilon) \rfloor = \lfloor n + 1 + (\varepsilon - 1/2) \rfloor = n + 1$ since $0 \leq \varepsilon - 1/2 < 1$.
- Hence, $\lfloor 2x \rfloor = 2n + 1$ and $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + (n + 1) = 2n + 1$.

Factorial Function

Definition: f: $\mathbf{N} \rightarrow \mathbf{Z}^+$, denoted by $f(n) = n!$ is the product of the first n positive integers when n is a nonnegative integer.

$$f(n) = 1 \cdot 2 \cdots (n-1) \cdot n, \qquad f(0) = 0! = 1$$

Examples:

$$f(1) = 1! = 1$$

$$f(2) = 2! = 1 \cdot 2 = 2$$

$$f(6) = 6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$$

$$f(20) = 2{,}432{,}902{,}008{,}176{,}640{,}000.$$

Stirling's Formula:

$$n! \sim \sqrt{2\pi n}(n/e)^n$$

$$f(n) \sim g(n) \doteq lim_{n\to\infty} f(n)/g(n) = 1$$

Partial Functions (*optional*)

Definition: A *partial function f* from a set A to a set B is an assignment to each element a in a subset of A, called the *domain of definition* of f, of a unique element b in B.

- The sets A and B are called the *domain* and *codomain* of f, respectively.
- We day that f is *undefined* for elements in A that are not in the domain of definition of f.
- When the domain of definition of f equals A, we say that f is a *total function*.

Example: f: $\mathbf{N} \rightarrow \mathbf{R}$ where $f(n) = \sqrt{n}$ is a partial function from $\mathbf{Z}$ to $\mathbf{R}$ where the domain of definition is the set of nonnegative integers. Note that f is undefined for negative integers.

Query???

$$\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4....}}}}$$

$$\exists_{x\in\Re}\exists_{y\in\Re}(x=y)=?$$

$$\sum_{x=1}^{\infty} x = ?$$

$$\sum_{x=1}^{\infty}\frac{1}{x}=?$$

$$\forall_{x}(\Re\,|\,x)=?$$

$$\exists_{x\in\Re}\exists_{y\in\Re}(x=y)=?$$

$$\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4....}}}}=?$$

$$1-1+1-1+1........=?$$

$$\sum_{x=1}^{\infty}\frac{1}{x}=?$$