Heaven's light is our guide"

Rajshahi University of Engineering & Technology
Department of Computer Science & Engineering

Discrete Mathematics

Course No. : 305

Chapter 1: The Foundations: Logic and Proofs

Prepared By : Julia Rahman

1.1 Propositional Logic

- **proposition :**
  - ✓ Is a declarative sentence (a sentence that declares a fact) that is either true or false, but not both.
  - ✓ Example: Are the following sentences propositions?
    1) Dhaka is the capital of Bangladesh. (Yes)
    2) Read this carefully. (No)
    3) 1+2=3 (Yes)
    4) x+1=2 (No)
    5) What is your name? (No)
- **Propositional Logic** –
  - ✓ the area of logic that deals with propositions.
  - ✓ Also called propositional calculus.
- **Propositional Variables** –
  - ✓ variables that represent propositions.
  - ✓ Just as letters are used to denote numerical variables: *p*, *q*, *r*, *s*
  - ✓ E.g. Proposition *p* – "Today is Friday."
- **Truth values** – T, F
- **Compound proposition** – statements constructed by combining one or more proposition.

1.1 Propositional Logic(cont.)

- **Logical operators**
  - ✓ are used to form new propositions from two or more existing propositions.
  - ✓ The logical operators are also called connectives.
  - ✓ ¬, ⋀, V, →,↔ are logical operators
- **DEFINITION 1(Negation)**
  - ✓ Let *p* be a proposition.
  - ✓ The negation of *p*, denoted by ¬*p*,
  - ✓ is the statement "It is not the case that *p*."
  - ✓ The proposition ¬*p* is read "not *p*."
  - ✓ The truth value of the negation of *p*, ¬*p* is the opposite of the truth value of *p*.

Examples:

- ➢ Negation of the proposition "Today is Friday." and express this in simple English.

Solution: The negation is "It is not the case that *today is Friday*."

In simple English, "Today is not Friday." or "It is not Friday today."

1.1 Propositional Logic(cont.)

- Negation of the proposition "At least 10 inches of rain fell today in Miami." and express this in simple English

Solution: The negation is "It is not the case that *at least 10 inches of rain fell today in Miami*."

In simple English, "Less than 10 inches of rain fell today in Miami."

- **Note:** Always assume fixed times, fixed places, and particular people unless otherwise noted.
- **Truth table:**

The Truth Table for the Negation of a Proposition.	
p	$\neg p$
T	F
F	T

1.1 Propositional Logic(cont.)

- **DEFINITION 2(Conjunction):**
  - ✓ Let *p* and *q* be propositions.
  - ✓ The *conjunction* of *p* and *q*, denoted by $p \Lambda q$, is the proposition "*p* and *q*".
  - ✓ The conjunction $p \Lambda q$ is true when both *p* and *q* are true and is false otherwise.

Examples:

- ➢ Conjunction of the propositions *p* and *q* where *p* is the proposition "Today is Friday." and *q* is the proposition "It is raining today.", and the truth value of the conjunction.

Solution: The conjunction is the proposition "Today is Friday and it is raining today." The proposition is true on rainy Fridays.

Truth table:

p	q	$p \Lambda q$
T	T	T
T	F	F
F	T	F
F	F	F

1.1 Propositional Logic(cont.)

- **DEFINITION 3(Disjunction):**
  - ✓ Let *p* and *q* be propositions. The *disjunction* of *p* and *q*, denoted by *p* v *q*, is the proposition "*p* or *q*".
  - ✓ The conjunction *p* v *q* is false when both *p* and *q* are false and is true otherwise.

Note: ***inclusive or*** **:** The disjunction is true when at least one of the two propositions is true.

E.g. "Students who have taken calculus or computer science can take this class." – those who take one or both classes.

Definition 3 uses inclusive or.

Truth table:

p	q	*p* v *q*
T	T	T
T	F	T
F	T	T
F	F	F

1.1 Propositional Logic(cont.)

- **DEFINITION 4(*exclusive or*):**
  - ✓ Let p and q be propositions. The *exclusive or* of p and q, denoted by $p \oplus q$,
  - ✓ is the proposition that is true when exactly one of p and q is true and is false otherwise.
  - ✓ E.g. "Students who have taken calculus or computer science, but not both, can take this class." – only those who take one of them.

Truth table:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Conditional Statements

- **DEFINITION 5:**
  - ✓ Let p and q be propositions. The *conditional statement* $p \rightarrow q$, is the proposition "if p, then q."
  - ✓ The conditional statement is false when p is true and q is false, and true otherwise.
  - ✓ In the conditional statement $p \rightarrow q$, p is called the ***hypothesis*** (or *antecedent* or *premise*) and q is called the ***conclusion*** (or *consequence*).
  - ✓ A conditional statement is also called an implication.
  - ✓ Example: "If I am elected, then I will lower taxes." $p \rightarrow q$

Truth table:

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

Conditional Statements(cont.)

Example:

Let p be the statement "Maria learns discrete mathematics." and q the statement "Maria will find a good job." Express the statement $p \rightarrow q$ as a statement in English.

Solution: Any of the following -

"If Maria learns discrete mathematics, then she will find a good job.

"Maria will find a good job when she learns discrete mathematics."

"For Maria to get a good job, it is sufficient for her to learn discrete mathematics."

"Maria will find a good job unless she does not learn discrete mathematics."

- Other conditional statements:
  - Converse of $p \rightarrow q$: $q \rightarrow p$
  - Contrapositive of $p \rightarrow q$: $\neg q \rightarrow \neg p$
  - Inverse of $p \rightarrow q$: $\neg p \rightarrow \neg q$

Conditional Statements(cont.)

- **DEFINITION 6(biconditional):**
  - ✓ Let p and q be propositions. The *biconditional statement* $p \leftrightarrow q$ is the proposition "p if and only if q."
  - ✓ The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise.
  - ✓ Biconditional statements are also called *bi-implications*.
  - ✓ $p \leftrightarrow q$ has the same truth value as $(p \rightarrow q) \land (q \rightarrow p)$
  - ✓ "*if and only if*" can be expressed by "*iff*"

Example:

Let p be the statement "You can take the flight" and let q be the statement "You buy a ticket." Then $p \leftrightarrow q$ is the statement "You can take the flight if and only if you buy a ticket."

Implication:

If you buy a ticket you can take the flight.
If you don't buy a ticket you can't take the flight.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Precedence of Logical Operators

- We can use parentheses to specify the order in which logical operators in a compound proposition are to be applied.
- To reduce the number of parentheses, the precedence order is defined for logical operators.

Precedence of Logical Operators.	
Operator	Precedence
$\neg$	1
$\wedge$ $\vee$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

E.g. $\neg p \wedge q = (\neg p) \wedge q$

$p \wedge q \vee r = (p \wedge q) \vee r$

$p \vee q \wedge r = p \vee (q \wedge r)$

Truth Tables of Compound Propositions

- We can use connectives to build up complicated compound propositions involving any number of propositional variables,
- then use truth tables to determine the truth value of these compound propositions.
- Example: Construct the truth table of the compound proposition

$$(p \vee \neg q) \rightarrow (p \wedge q).$$

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Translating English Sentences

- English (and every other human language) is often ambiguous.
- Translating sentences into compound statements removes the ambiguity.

Example - 1: How can this English sentence be translated into a logical expression?

"You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old."

Solution: Let

q, r, and s represent "You can ride the roller coaster," "You are under 4 feet tall," and "You are older than 16 years old."

The sentence can be translated into: $(r \land \neg s) \rightarrow \neg q$.

Example - 2: How can this English sentence be translated into a logical expression?

"You can access the Internet from campus only if you are a computer science major or you are not a freshman."

Solution: Let

a, c, and f represent "You can access the Internet from campus," "You are a computer science major," and "You are a freshman."

The sentence can be translated into: $a \rightarrow (c \lor \neg f)$.

Logic and Bit Operations

- Computers represent information using bits.
- A **bit** is a symbol with two possible values, 0 and 1.
- By convention, 1 represents T (true) and 0 represents F (false).
- A variable is called a Boolean variable if its value is either true or false.
- Bit operation – replace true by 1 and false by 0 in logical operations.

Table for the Bit Operators *OR*, *AND*, and *XOR*				
x	y	$x \vee y$	$x \wedge y$	$x \oplus y$
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	1	1	0

Logic and Bit Operations(cont.)

- **DEFINITION 7(String):**
  - ✓ A *bit string* is a sequence of zero or more bits.
  - ✓ The *length* of this string is the number of bits in the string.

Example: Find the bitwise *OR*, bitwise *AND*, and bitwise *XOR* of the bit string 01 1011 0110 and 11 0001 1101.

Solution:

01 1011 0110

11 0001 1101

----

11 1011 1111 bitwise *OR*
01 0001 0100 bitwise *AND*
10 1010 1011 bitwise *XOR*

1.2 Propositional Equivalence

Propositional Equivalence

- **DEFINITION 1:**
  - ✓ A compound proposition that is always true, no matter what the truth values of the propositions that occurs in it, is called a ***tautology***.
  - ✓ A compound proposition that is always false is called a ***contradiction***.
  - ✓ A compound proposition that is neither a tautology or a contradiction is called a ***contingency*.**

Examples of a Tautology and a Contradiction.			
p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Logical Equivalences

- **DEFINITION 2(Logical Equivalence):**
  - ✓ The compound propositions p and q are called *logically equivalent* if $p \leftrightarrow q$ is a tautology.
  - ✓ The notation $p \equiv q$ denotes that p and q are logically equivalent.
  - ✓ Compound propositions that have the same truth values in all possible cases are called logically equivalent.

Example: Show that $\neg p \vee q$ and $p \rightarrow q$ are logically equivalent.

Truth Tables for $\neg p \vee q$ and $p \rightarrow q$.				
p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

- ✓ In general, 2^n rows are required if a compound proposition involves n propositional variables in order to get the combination of all truth values.

Logical Equivalences(cont.)

- **Some example of logical equivalence:**

De Morgan Laws: $\neg(p \wedge q) \equiv \neg p \vee \neg q; \quad \neg(p \vee q) \equiv \neg p \wedge \neg q.$

Identity laws: $p \wedge T \equiv p$

$p \vee F \equiv p$

Domination laws: $p \vee T \equiv T$

$p \wedge F \equiv F$

Idempotent laws: $p \vee p \equiv p$

$p \wedge p \equiv p$

Double negation law: $\neg(\neg p) \equiv p$

Commutative laws: $p \vee q \equiv q \vee p$

$p \wedge q \equiv q \wedge p$

Logical Equivalences(cont.)

Associative laws:

$$p \vee (q \vee r) \equiv (p \vee q) \vee r$$

$$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$$

Distributive law of conjunction over disjunction:

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

Equivalences involving conditional statements and biconditionals:

$$p \rightarrow q \equiv \neg p \vee q$$

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$

Using the existing one can derive new logical equivalences.

Constructing New Logical Equivalences

Example - 1: Show that $\neg(p \rightarrow q)$ and $p \land \neg q$ are logically equivalent.

Solution:

$\neg(p \rightarrow q) \equiv \neg(\neg p \lor q)$ *by example on slide 3*

$\equiv \neg(\neg p) \land \neg q$ *by the second De Morgan law*

$\equiv p \land \neg q$ *by the double negation law*

Example - 2: *Show that* $(p \land q) \rightarrow (p \lor q)$ is a tautology.

Solution: To show that this statement is a tautology, we will use logical equivalences to demonstrate that it is logically equivalent to T.

$(p \land q) \rightarrow (p \lor q) \equiv \neg(p \land q) \lor (p \lor q)$ *by example on slide 3*

$\equiv (\neg p \lor \neg q) \lor (p \lor q)$ *by the first De Morgan law*

$\equiv (\neg p \lor p) \lor (\neg q \lor q)$ *by the associative and communicative law for disjunction*

$\equiv \mathrm{T} \lor \mathrm{T}$

$\equiv \mathrm{T}$

Note*: The above examples can also be done using truth tables*.

1.3 Predicates and Quantifiers

Predicates and Quantifiers

Predicates:

- ✓ A predicate is a statement that contains variables those are neither true nor false.
- ✓ E.g. "x > 3", "x = y + 3", "x + y = z"
  "x is greater than 3"
  "x": subject of the statement
  "is greater than 3": the ***predicate***

We can denote the statement "x is greater than 3" by *P(x)*, where *P* denotes the predicate and *x* is the variable.

Once a value is assigned to the variable *x*, the statement *P(x)* becomes a proposition and has a truth value.

- ✓ It is the verbal statement that describes the property of a variable. Usually represented by the letter P, the notation P(*x*) is used to represent some unspecified property or predicate that *x* may have
  e.g. P(*x*) = *x* has 30 days.
  P(April) = April has 30 days.
- ✓ Predicate logic builds on propositional logic by introducing quantifiers to handle special types of "variables" in a mathematical statement. With quantifiers and predicates, we can express a great deal more than we could using propositional logic.

Predicates and Quantifiers

- **Example 1:** Let *P(x)* denote the statement "x > 3." What are the truth values of *P*(4) and *P*(2)?

Solution: *P*(4) – "4 > 3", *true*

P(3) – "2 > 3", *false*

- **Example 3:** Let *Q*(x,y) denote the statement "x = y + 3." What are the truth values of the propositions *Q*(1,2) and *Q*(3,0)?

Solution: *Q*(1,2) – "1 = 2 + 3" , *false*

Q(3,0) – "3 = 0 + 3", *true*

- **Example 4:** Let *A(c,n)* denote the statement "Computer *c* is connected to network *n*", where *c* is a variable representing a computer and *n* is a variable representing a network. Suppose that the computer MATH1 is connected to network CAMPUS2, but not to network CAMPUS1. What are the values of A(MATH1, CAMPUS1) and A(MATH1, CAMPUS2)?

Solution: *A(MATH1, CAMPUS1) – "MATH1 is connect to CAMPUS1", false*

A(MATH1, CAMPUS2) – "MATH1 is connect to CAMPUS2", true

Predicates and Quantifiers

- A statement involving n variables $x_1, x_2, \ldots, x_n$ can be denoted by $P(x_1, x_2, \ldots, x_n)$.
- A statement of the form $P(x_1, x_2, \ldots, x_n)$ is the value of the propositional function P at the n-tuple $(x_1, x_2, \ldots, x_n)$, and P is also called a ***n*-place predicate** or a ***n*-array predicate**.
- Example:

  " if x > 0 then x:= x + 1"

  When the statement is encountered, the value if x is inserted into $P(x)$.

  If $P(x)$ is true, x is increased by 1.

  If $P(x)$ is false, x is not changed.

Predicates and Quantifiers

Quantifiers:

- **Quantification:**
  - ✓ express the extent to which a predicate is true over a range of elements.
  - ✓ Quantifiers are phrases that refer to given quantities, such as "for some" or "for all" or "for every," indicating how many objects have a certain property.

Two kinds of quantifiers: Universal and Existential

- **Universal quantification**:
  - ✓ a predicate is true for every element under consideration.
  - ✓ The *universal quantification* of $P(x)$ is the statement

"$P(x)$ for all values of x in the domain."

The notation $\forall x P(x)$ denotes the universal quantification of $P(x)$. Here $\forall$ is called the **Universal Quantifier**. We read $\forall x P(x)$ as "for all $xP(x)$" or "for every $xP(x)$." An element for which $P(x)$ is false is called a **counterexample** of $\forall x P(x)$.

Predicates and Quantifiers

Example 8: Let *P(x)* be the statement "$x + 1 > x$." What is the truth value of the quantification $\forall x P(x)$, where the domain consists of all real numbers?

Solution: *Because P(x) is true for all real numbers, the quantification is true.*

- A statement $\forall x P(x)$ is false, if and only if *P(x)* is not always true where x is in the domain. One way to show that is to find a counterexample to the statement $\forall x P(x)$.

Example 9: Let *Q(x)* be the statement "x < 2". What is the truth value of the quantification $\forall x Q(x)$, where the domain consists of all real numbers?

Solution: *Q(x) is not true for every real numbers, e.g. Q(3) is false. x = 3 is a counterexample for the statement* $\forall x Q(x)$*. Thus the quantification is false.*

$\forall x P(x)$ is the same as the conjunction

$$P(x_1) \land P(x_2) \land \ldots . \land P(x_n).$$

Example 12: What does the statement $\forall x N(x)$ mean if *N(x)* is "Computer *x* is connected to the network" and the domain consists of all computers on campus?

Solution: *"Every computer on campus is connected to the network."*

Predicates and Quantifiers

- **Existential quantification**:
  - ✓ a predicate is true for one or more element under consideration.
  - ✓ The *existential quantification* of *P(x)* is the statement

    "There exists an element x in the domain such that *P(x)*."

    We use the notation $\exists x P(x)$ for the existential quantification of *P(x)*. Here is called the **Existential Quantifier**.
  - ✓ The existential quantification $\exists x P(x)$ is read as

    "There is an *x* such that *P(x)*," or

    "There is at least one *x* such that *P(x)*," or

    "For some *x*, *P(x)*."

Example 14: Let *P(x)* denote the statement "$x > 3$". What is the truth value of the quantification $\exists x P(x)$, where the domain consists of all real numbers?

Solution: *"x > 3" is sometimes true – for instance when x = 4. The existential quantification is true.*

- $\exists x P(x)$ is false if and only if *P*(x) is false for every element of the domain.

Predicates and Quantifiers

Example 15: Let *Q*(x) denote the statement "x = x + 1". What is the true value of the quantification ∃xQ(x), where the domain consists for all real numbers?

Solution: *Q(x) is false for every real number. The existential quantification is false*.

- If the domain is empty, ∃x*Q*(x) is false because there can be no element in the domain for which *Q*(x) is true.

The existential quantification ∃x*P*(x) is the same as the disjunction $P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$

Quantifiers		
Statement	***When True?***	***When False?***
∀*xP(x)*	*xP(x) is true for every x.*	*There is an x for which xP(x) is false.*
∃*xP(x)*	*There is an x for which P(x) is true.*	*P(x) is false for every x.*

Predicates and Quantifiers

- **Precedence of Quantifiers**

  $\forall$ and $\exists$ have higher precedence than all logical operators.

  E.g. $\forall x P(x) \vee Q(x)$ is the same as $(\forall x P(x)) \vee Q(x)$

- **Translating from English into Logical Expressions**

Example 23: Express the statement "Every student in this class has studied calculus" using predicates and quantifiers.

Solution: If the domain consists of students in the class –

$$\forall x C(x)$$

where $C(x)$ is the statement "x has studied calculus.

If the domain consists of all people –

$$\forall x(S(x) \rightarrow C(x))$$

where $S(x)$ represents that person x is in this class.

If we are interested in the backgrounds of people in subjects besides calculus, we can use the two-variable quantifier $Q(x,y)$ for the statement "student x has studies subject y." Then we would replace $C(x)$ by $Q(x, \text{calculus})$ to obtain $\forall x Q(x, \text{calculus})$ or $\forall x(S(x) \rightarrow Q(x, \text{calculus}))$

Predicates and Quantifiers

Example 26: Consider these statements. The first two are called *premises* and the third is called the *conclusion*. The entire set is called an *argument*.

"All lions are fierce."

"Some lions do not drink coffee."

"Some fierce creatures do not drink coffee."

Solution:

Let $P(x)$ be "x is a lion."

$Q(x)$ be "x is fierce."

$R(x)$ be "x drinks coffee."

$\forall x(P(x) \rightarrow Q(x))$

$\exists x(P(x) \land \neg R(x))$

$\exists x(Q(x) \land \neg R(x))$

1.4 Nested Quantifiers

Nested Quantifiers

- **Nested Quantifier:** *Quantifier that appears within the scope of another quantifier.*

Examples: $\forall x \exists y (x + y = 0)$ x,y real numbers.

$\forall x \, \exists y(x+y=0) \Leftrightarrow \forall x Q(x)$, where $Q(x)= \exists y P(x,y)$, where $P(x,y)$ is $(x+y=0)$.

Examples 1: Assume that the domain for the variables x and y consists of real number. The statement

$$\forall x \forall y(x+y=y+x)$$

says that $x+y=y+x$ for all real number x and y. This is the commutative law for addition of real numbers. The statement

$$\forall x \exists y \, (x+y=0)$$

says that for every real number x there is a real number y such that $x+y=0$. This states that every real number has an additive inverse.

Examples 2: Translate the statement into English $\forall x \forall y((x>0) \land (y<0) \rightarrow (xy<0))$.

Solution: This statement says that for every real number x and every real number y, if $x>0$ and $y<0$, then $xy<0$; “The product (multiplication) of a positive real number and a negative real number is (always) a negative real number.”

Nested Quantifiers

- **The Order of Quantifiers:**
  - ✓ The orders of mixed quantifiers are important.
  - ✓ If the quantifiers are of the same type, then order does not matter.
  - ✓ If the quantifiers are of different types, then order is important.

Example 4: Let $Q(x, y)$ be $x + y=0$. Then what are the truth values of the quantifiers $\exists y \forall x\ Q(x, y)$ and $\forall x \exists y\ Q(x, y)$,where the domains for all variables consists of real numbers?

Solution: $\exists y \forall x\ Q(x, y)$=False and $\forall x \exists y\ Q(x, y)$=True.

Example 5: Let $Q(x, y, z)$ be $x + y=z$. Then what are the truth values of the quantifiers $\forall x \forall y \exists z\ Q(x, y, z)$ and $\exists z \forall x \forall y Q(x, y, z)$, where the domains for all variables consists of real numbers?

Solution: $\forall x \forall y \exists z\ Q(x, y, z)$=T means

"For all x and for all y, there is a z such that $x + y=z$." is true, and

$\exists z \forall x \forall y\ Q(x, y, z)$=F means

"There is a z such that for all x and for all y it is true that $x + y=z$." is false.

Nested Quantifiers

Translating Mathematical Statements into Statements of Nested Quantifiers

Example 6: Translate the statement "The sum of two positive integers is always positive" into a logical expression.

Solution: $\forall x \forall y(x+y>0)$, where D=N – natural numbers. If D=Z – integers, then $\forall x \forall y((x>0) \wedge (y>0) \rightarrow (x+y>0))$.

Example 7: Translate the statement "Every real number except zero has a multiple inverse". (A *multiplicative inverse* of a real number x is a real number y such that xy=1).

Solution: $\forall x((x \neq 0) \rightarrow \exists y\ (xy=1))$, where D=R.

If $D=R\backslash\{0\}$, then $\forall x\ \exists y\ (xy=1)$.

Example 8: Express the definition of a limit using quantifiers.

Solution:

For every real number $\varepsilon>0$ there exists a real number $\delta>0$ such that $|f(x)-L| <\varepsilon$ whenever $0<|x-a|<\delta$.

If $D=[0, +\infty)$ for ε and δ, D=R for x, then $\forall \varepsilon \exists \delta \forall x(0<|x-a|<\delta \rightarrow |f(x)-L| <\varepsilon)$.

If D=R for all ε, δ and x, then $\forall \varepsilon>0\ \exists \delta>0\ \forall x((0<|x-a|<\delta) \rightarrow (|f(x)-L| <\varepsilon))$.

Nested Quantifiers

Translating from Nested Quantifiers into English

- ✓ Expressions with nested quantifiers can be complicated.
- ✓ The first step in translating such an expression is to write out what the quantifiers and predicates in the expression mean.
- ✓ The next step is to express this meaning in a simpler sentence.

Example 9: Translate the statement $\forall x\,(C(x) \lor \exists y(C(y) \land F(x,y)))$ into English, where $C(x)$ is "x has a computer", $F(x,y)$ is "x and y are friends", and the domain for both x and y consists of all students in the school.

Solution: Every student in the school has a computer or has friend who has a computer.

Example 10: Translate the statement $\exists x\, \forall y \forall z\, ((F(x,y) \land F(x,z) \land (y \neq z)) \rightarrow \neg F(y,z))$ into English, where $F(a, b)$ means a and b are friends and the domain for x and y and z consists of all students in school.

Solution: There is a student none of whose friends are also friends with each other.

Nested Quantifiers

Translating English Sentences into Logical Expressions

Example 11: Express the statement "If a person is female and is a parent, then this person is someone's mother" as a logical expression involving logical expression involving predicates, quantifiers with a domain consisting of all people and logical connectives.

Solution: equivalently "For every person x, if x is female and a parent, then there is a person y such that x is the mother for this person y." So, let F(x): x is female; P(x): x is parent; M(x, y): x is mother of y. Then the statement is:

$$\forall x\ ((F(x) \wedge P(x)) \rightarrow \exists y\ M(x, y))\ .$$

Example 12: Express the statement "Everyone has exactly one best friend" as a logical expression involving logical expression involving predicates, quantifiers with a domain consisting of all people and logical connectives.

Solution: equivalently "For every person x, x has exactly one best friend." Attention "exactly one"! . Let B(x, y): y is the best friend of x. So if a person z is not the person y, then z is not the best friend of x. Thus

$$\forall x \exists y (B(x, y) \wedge\ \forall z (z \neq y) \rightarrow \neg B(x, z)).$$

With the uniqueness quantifier $\exists!$ The statement is $\forall x \exists ! y B(x, y)$.

1.5 Rules of Inference

Rules of Inference

- A *rule of inference* is a pre-proved relation: any time the left hand side (LHS) is true, the right hand side (RHS) is also true.

- Therefore, if we can match a premise to the LHS (by substituting propositions), we can assert the (substituted) RHS
- Example:

  Modus Ponens

$$(p \wedge (p \rightarrow q)) \rightarrow q$$

$$\begin{array}{l} p \\ \underline{p \rightarrow q} \\ \therefore \quad q \end{array}$$

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	$(p \wedge (p \rightarrow q)) \rightarrow q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

Rules of Inference

Rule of inference	Tautology	Name
$p \to q$ $\underline{p \qquad}$ $\therefore q$	$[p \wedge (p \to q)] \to q$	Modus ponens
$\neg q$ $\underline{p \to q}$ $\therefore \neg p$	$[\neg q \wedge (p \to q)] \to \neg p$	Modus tollen
$p \to q$ $\underline{q \to r}$ $\therefore p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Hypothetical syllogism
$p \vee q$ $\underline{\neg p \quad}$ $\therefore q$	$((p \vee q) \wedge \neg p) \to q$	Disjunctive syllogism
$\underline{p \qquad}$ $\therefore p \vee q$	$p \to (p \vee q)$	Addition
$\underline{p \wedge q}$ $\therefore p$	$(p \wedge q) \to p$	Simplification
p $\underline{q \qquad}$ $\therefore p \wedge q$	$((p) \wedge (q)) \to (p \wedge q)$	Conjunction
$p \vee q$ $\underline{\neg p \vee r}$ $\therefore q \vee r$	$[(p \vee q) \wedge (\neg p \vee r)] \to (p \vee r)$	Resolution

Rules of Inference

Using the rules of inference to build arguments

Example 6: Show that the hypothesis "It is not sunny this afternoon and it is colder than yesterday," "We will go swimming only if it is sunny", "If we do not go swimming, then we will take a canoe trip" and "If we take a canoe trip, then we will be home by sunset" lead to the conclusion "we will be home by sunset."

Solution:

p	It is sunny this afternoon
q	It is colder than yesterday
r	We go swimming
s	We will take a canoe trip
t	We will be home by sunset (the conclusion)

1. $\neg p \wedge q$
2. $r \rightarrow p$
3. $\neg r \rightarrow s$
4. $s \rightarrow t$
5. t

Rules of Inference

	Step	Reason
1.	$\emptyset p \cup q$	Premise
2.	$\emptyset p$	Simplification using (1)
3.	$r \rightarrow p$	Premise
4.	$\emptyset r$	Modus Tollens (2) & (3)
5.	$\neg r \rightarrow s$	Premise
6.	s	Modus Ponens (4) & (5)
7.	$s \rightarrow t$	Premise
8.	t	Modus Ponens (6) & (7)