Physics

Phy 1213

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Contents

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1. Structure of Matter

In terms of atomic structure:

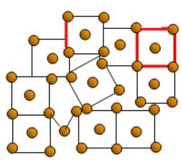
- Solid $\rightarrow$ long range order
- Liquid $\rightarrow$ short range order
- Gas $\rightarrow$ no order

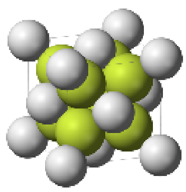
1.1 Solid

Crystalline If the atoms of molecules in solid are arranged in a highly regular ordered manner and the order continues to the whole solid, such a group of atoms are called crystals or crystalline solids. It is just a solid, having uniform chemical composition and and some recognizable features, which are same for all specimens. Though two crystals of same substance may look different from outside, the angle between the corresponding faces are always same.

Amorphous/Non-Crystalline Any substance that lacks long range order or geometrical shape, is known as amorphous substance. The atoms in such solids are distributed randomly.

1.1.1 Crystal Systems





Poly Crystal If the order does not continue over the whole solid, but interrupted by grain boundaries and within the grains, the molecules are arranged in an ordered manner; such a group of atoms is referred to as Poly Crystal.

Unit Cell The unit cell is usually the smallest area by repeated translation of which the whole crystal can be produced. It may be defined in general as the volume of the solid from which the entire solid crystal can be constructed, by translational repetition in three dimensions. In fact, an unit cell is the smallest part of a crystal, having all the structural properties of the given lattice.

The three sides of a unit cell are called the crystallographic axes. The intercepts define the dimensions of a unit cell, and are known as primitives, or characteristic intercepts of the axes.

The angles between the three axes are called interfacial angles. The primitives and the interfacial angles constitute the lattice parameters of the unit cell. These are also referred to as the geometrical constants of a given crystalline substance.

Bravais Lattice Any of 14 possible three-dimensional configurations of points used to describe the orderly arrangement of atoms in a crystal. Each point represents one or more atoms in the actual crystal, and if the points are connected by lines, a crystal lattice is formed; the lattice is divided into a number of identical blocks, or unit cells, characteristic of the Bravais lattices. The French scientist Auguste Bravais demonstrated in 1850 that only these 14 types of unit cells are compatible with the orderly arrangements of atoms found in crystals. The lattices listed by Bravais are divided into six or seven major crystal symmetry systems. All crystallographers recognize the isometric, orthorhombic, monoclinic, tetragonal, triclinic, and hexagonal systems; some, however, define the hexagonal system so as to include the trigonal or rhombohedral system, which is considered a seventh system by others.

Non-Bravais Lattice Some of the lattice points are at non-equivalent positions.

Co-ordinate Number Also called co-ordination number, it is defined as the number of nearest neighbors, which an atom has in the unit cell of any crystal structure.

Packing Fraction Also called relative density of packing, it is defined as the ratio of volume occupied by the spherical atoms to the available volume of the crystal structure, *i.e.* $P = v/V$.

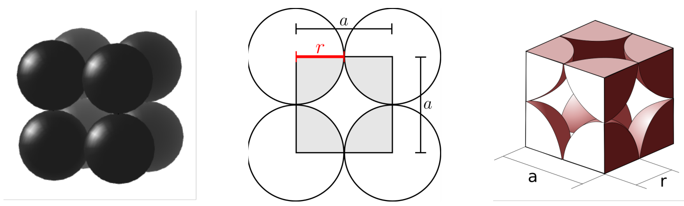
Lattice A lattice is a periodic structure of points in a space that looks net like.

Crystal Structure When a single or group of atoms are attached to every lattice point called **'basis'**, it forms a crystal structure.

$$Lattice + Basis \to \text{Crystal Structure}$$

1.1.2 Types of Space Lattices of Cubic System

A cubic structure is the simplest type of array, in which the atoms take positions at the corner of the cube. However, depending on the position of the lattice points in the unit cell, three different types of lattices are possible in this system.



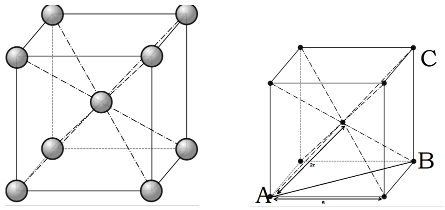
Simple Cubic

Simple Cubic: The crystal structure is cubic. Have atoms in each corners only. Each atom has 6 nearest neighbors. The co-ordinate no. for Simple Cubic is 6. Each atoms are shared by 8 atoms.
Share of each cube$= \frac{1}{8}$.
No. of corner atoms=8.
Atoms per unit cell$= 8 \times \frac{1}{8} = 1$.
Volume of one atom:

$$\begin{aligned} \text{volumne, } v &= \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{a}{2}\right)^3 \qquad ; r = \frac{a}{2} \\ &= \frac{\pi a^3}{6} \\ \text{packing fraction, } P &= \frac{\frac{\pi a^3}{6}}{a^3} = \frac{1}{6}\pi \end{aligned}$$

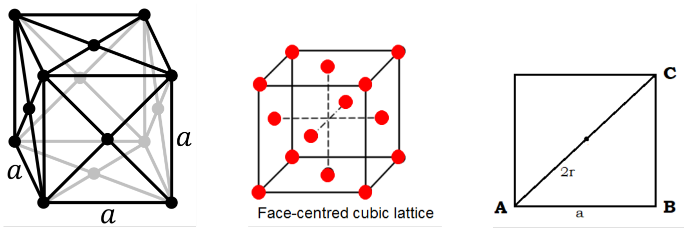
Body-Centric Cubic: The nearest atom to the corner is the body-centric atom. A corner atom is surrounded by 8 unit cell and 8 centric atoms. The co-ordinate no. for bcc is 8. There are 8 corner atoms and 1 center atoms.
Atoms per unit cell$= 8 \times \frac{1}{8} + 1 = 2$.

$$\begin{aligned} &AC^2 = AB^2 + BC^2 = (a^2 + a^2) + a^2 = 3a^2 \\ &\therefore AC = \sqrt{3}a = 4r \end{aligned}$$



Body-Centric Cubic

$$\therefore r = \frac{\sqrt{3}}{4}a$$
$$\therefore P = \frac{\frac{4}{3}\pi\left(\frac{\sqrt{3}}{4}a\right)^3}{a^3} = \frac{\sqrt{3}}{8}\pi$$



Face Centric Cubic

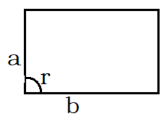
Face Centric Cubic: The nearest atom for a corner atom are the face centric atoms. The corner atom has 4 on same plane, 4 below and 4 above. The co-ordinate no. for fcc is 12. There are 8 corner atoms and 6 faced atoms. Each faced atom is shared by two atoms.
Atom per unit cell$= 8 \times \frac{1}{8} + 6 \times \frac{1}{2} = 1 + 3 = 4$.

$$AC^2 = AB^2 + BC^2 = a^2 + a^2 = 2a^2$$
$$\therefore AC = \sqrt{2}a = 4r$$
$$\therefore r = \frac{1}{2\sqrt{2}}a$$
$$\therefore P = \frac{\frac{4}{3}\pi\left(\frac{1}{2\sqrt{2}}a\right)^3}{a^3} \times 4 = \frac{1}{3\sqrt{2}}\pi$$

1.1.3 Crystal Structure

1.1.4 Crystal Structure in 2D

1. Square: $a = b,\ r = 90^\circ$
2. Rectangular: $a \neq b,\ r = 90^\circ$
3. Oblique: $a \neq b,\ r = arbitrary$



$$\left.\begin{matrix} a, b \\ r \end{matrix}\right\} \text{lattice parameter}$$

1.1.5 Crystal Structure in 3D

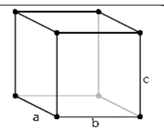
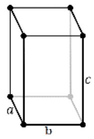
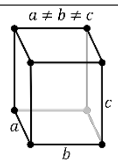
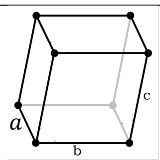
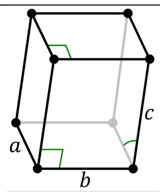
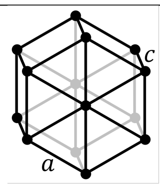
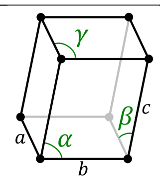
Table **??**Crystal Structure in 3D summarizes the crystal structures in 3D.

1.2 Different types of bonds in solids

1.2.1 Ionic Bond

Ionic bond, also called electrovalent bond, type of linkage formed from the electrostatic attraction between oppositely charged ions in a chemical compound. Such a bond forms when the valence (outermost) electrons of

3D Crystal Structure

Name	**Bravais Lattices**	**Specification**	**Figure**
Cubic	3	$a=b=c$ $\alpha=\beta=\gamma=90^\circ$	
Tetragonal	2	$a=b\neq c$ $\alpha=\beta=\gamma=90^\circ$	
Orthorhombic	4	$a\neq b\neq c$ $\alpha=\beta=\gamma=90^\circ$	
Rhombohedral	1	$a=b=c$ $\alpha=\beta\neq\gamma=90^\circ$	
Monoclinic	2	$a\neq b\neq c$ $\alpha=\beta=90^\circ\neq\gamma$	
Hexagonal	1	$a=b\neq c$ $\alpha=\beta=90^\circ;\gamma=120^\circ$	
Triclinic	1	$a\neq b\neq c$ $\alpha\neq\beta\neq\gamma=90^\circ$	

one atom are transferred permanently to another atom. The atom that loses the electrons becomes a positively charged ion (cation), while the one that gains them becomes a negatively charged ion (anion).

Ionic bonding results in compounds known as ionic, or electrovalent, compounds, which are best exemplified by the compounds formed between nonmetals and the alkali and alkaline-earth metals. In ionic crystalline solids of this kind, the electrostatic forces of attraction between opposite charges and repulsion between similar charges orient the ions in such a manner that every positive ion becomes surrounded by negative ions and vice versa. In short, the ions are so arranged that the positive and negative charges alternate and balance one another, the overall charge of the entire substance being zero. The magnitude of the electrostatic forces in ionic crystals is considerable. Accordingly, these substances tend to be hard and nonvolatile.

An ionic bond is actually the extreme case of a polar covalent bond, the latter resulting from unequal sharing of electrons rather than complete electron transfer. Ionic bonds typically form when the difference in the electronegativities of the two atoms is great, while covalent bonds form when the electronegativities are similar.

Examples: $NaCl, LiF, CaCl_2, Na_2SO_4$

1.2.2 Covalent Bond

Covalent bond, in chemistry, the interatomic linkage that results from the sharing of an electron pair between two atoms. The binding arises from the electrostatic attraction of their nuclei for the same electrons. A covalent bond forms when the bonded atoms have a lower total energy than that of widely separated atoms.
Molecules that have covalent linkages include the inorganic substances hydrogen, nitrogen, chlorine, water, and ammonia $(H_2, N_2, Cl_2, H_2O, NH_3)$ together with all organic compounds. In structural representations of molecules, covalent bonds are indicated by solid lines connecting pairs of atoms.
A single line indicates a bond between two atoms (*i.e.*, involving one electron pair), double lines (=) indicate a double bond between two atoms (*i.e.*, involving two electron pairs), and triple lines (≡) represent a triple bond, as found, for example, in carbon monoxide ($C \equiv O$). Single bonds consist of one sigma (σ) bond, double bonds have one σ and one pi (π) bond, and triple bonds have one σ and two π bonds.
Covalent bonds are directional, meaning that atoms so bonded prefer specific orientations relative to one another; this in turn gives molecules definite shapes, as in the angular (bent) structure of the H_2O molecule. Covalent bonds between identical atoms (as in H_2) are nonpolar— *i.e.*, electrically uniform—while those between unlike atoms are polar—*i.e.*, one atom is slightly negatively charged and the other is slightly positively charged. This partial ionic character of covalent bonds increases with the difference in the electronegativities of the two atoms. When none of the elements in a compound is a metal, no atoms in the compound have an ionization energy low enough for electron loss to be likely. In such a case, covalence prevails. As a general rule, covalent bonds are formed between elements lying toward the right in the periodic table (*i.e.*, the nonmetals). Molecules of identical atoms, such as H_2 and buckminsterfullerene (C_{60}), are also held together by covalent bonds.

1.2.3 Metallic

Metallic bond, force that holds atoms together in a metallic substance. Such a solid consists of closely packed atoms. In most cases, the outermost electron shell of each of the metal atoms overlaps with a large number of neighbouring atoms. As a consequence, the valence electrons continually move from one atom to another and are not associated with any specific pair of atoms. In short, the valence electrons in metals, unlike those in covalently bonded substances, are nonlocalized, capable of wandering relatively freely throughout the entire crystal. The atoms that the electrons leave behind become positive ions, and the interaction between such ions and valence electrons gives rise to the cohesive or binding force that holds the metallic crystal together.
Many of the characteristic properties of metals are attributable to the non-localized or free-electron character of the valence electrons. This condition, for example, is responsible for the high electrical conductivity of metals. The valence electrons are always free to move when an electrical field is applied. The presence of the mobile valence electrons, as well as the nondirectionality of the binding force between metal ions, account for the malleability and ductility of most metals. When a metal is shaped or drawn, it does not fracture, because the ions in its crystal structure are quite easily displaced with respect to one another. Moreover, the nonlocalized valence electrons act as a buffer between the ions of like charge and thereby prevent them from coming together and generating strong repulsive forces that can cause the crystal to fracture.

1.2.4 Van Der Waals'

Van der Waals forces, relatively weak electric forces that attract neutral molecules to one another in gases, in liquefied and solidified gases, and in almost all organic liquids and solids. The forces are named for the Dutch physicist Johannes Diderik van der Waals, who in 1873 first postulated these intermolecular forces in developing a theory to account for the properties of real gases. Solids that are held together by van der Waals forces characteristically have lower melting points and are softer than those held together by the stronger ionic, covalent, and metallic bonds.
Van der Waals forces may arise from three sources. First, the molecules of some materials, although electrically neutral, may be permanent electric dipoles. Because of fixed distortion in the distribution of electric charge in the very structure of some molecules, one side of a molecule is always somewhat positive and the opposite side somewhat negative. The tendency of such permanent dipoles to align with each other results in a net attractive force. Second, the presence of molecules that are permanent dipoles temporarily distorts the electron charge in other nearby polar or nonpolar molecules, thereby inducing further polarization. An additional attractive force results from the interaction of a permanent dipole with a neighbouring induced dipole. Third, even though no molecules of a material are permanent dipoles (*e.g.*, in the noble gas argon or the organic liquid benzene), a

force of attraction exists between the molecules, accounting for condensing to the liquid state at sufficiently low temperatures.
The nature of this attractive force in molecules, which requires quantum mechanics for its correct description, was first recognized (1930) by the Polish-born physicist Fritz London, who traced it to electron motion within molecules. London pointed out that at any instant the centre of negative charge of the electrons and the centre of positive charge of the atomic nuclei would not be likely to coincide. Thus, the fluctuation of electrons makes molecules time-varying dipoles, even though the average of this instantaneous polarization over a brief time interval may be zero. Such time-varying dipoles, or instantaneous dipoles, cannot orient themselves into alignment to account for the actual force of attraction, but they do induce properly aligned polarization in adjacent molecules, resulting in attractive forces. These specific interactions, or forces, arising from electron fluctuations in molecules (known as London forces, or dispersion forces) are present even between permanently polar molecules and produce, generally, the largest of the three contributions to intermolecular forces.

1.2.5 Hydrogen Bond

Hydrogen bonding, interaction involving a hydrogen atom located between a pair of other atoms having a high affinity for electrons; such a bond is weaker than an ionic bond or covalent bond but stronger than van der Waals forces. Hydrogen bonds can exist between atoms in different molecules or in parts of the same molecule. One atom of the pair (the donor), generally a fluorine, nitrogen, or oxygen atom, is covalently bonded to a hydrogen atom ($-FH$, $-NH$, or $-OH$), whose electrons it shares unequally; its high electron affinity causes the hydrogen to take on a slight positive charge. The other atom of the pair, also typically F, N, or O, has an unshared electron pair, which gives it a slight negative charge. Mainly through electrostatic attraction, the donor atom effectively shares its hydrogen with the acceptor atom, forming a bond. Because of its extensive hydrogen bonding, water (H_2O) is liquid over a far greater range of temperatures that would be expected for a molecule of its size. Water is also a good solvent for ionic compounds and many others because it readily forms hydrogen bonds with the solute. Hydrogen bonding between amino acids in a linear protein molecule determines the way it folds up into its functional configuration. Hydrogen bonds between nitrogenous bases in nucleotides on the two strands of DNA (guanine pairs with cytosine, adenine with thymine) give rise to the double-helix structure that is crucial to the transmission of genetic information.

1.3 Packing in Solids

1.3.1 Inter Atomic Distances and Forces of Equilibrium

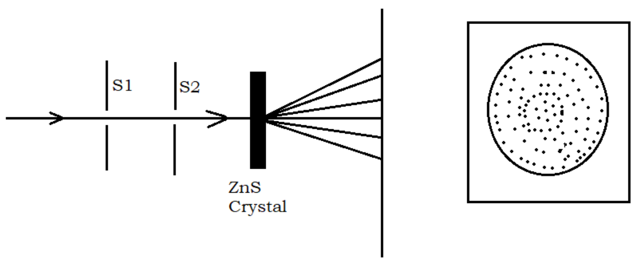
Valence Electron The outermost electron of an atom, *i.e.* those in the shell furthest from the nucleus.

Valence Band It is the band of energy occupied by the valence electron. The valence band may be completely filled or partially filled.

Conduction Band It is the band above the valence band that is of next higher permitted energies.

Fermi Level At the absolute zero temperature, all the energy levels below a certain level, will be filled with electrons, and all levels above this will be empty. The energy level that divides the filled and empty levels, are referred to as Fermi level.

1.3.2 X-Ray Diffraction

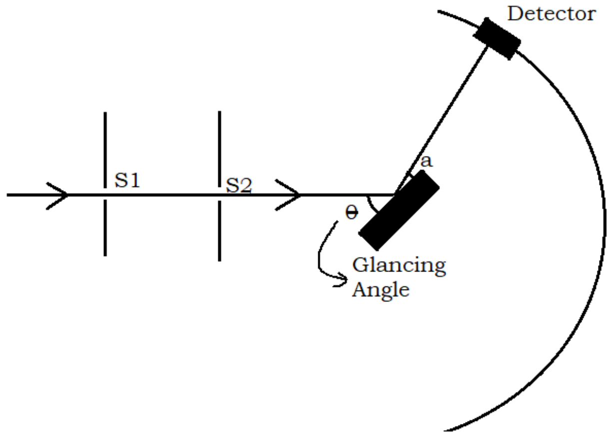


1.3.3 Braggs' Law

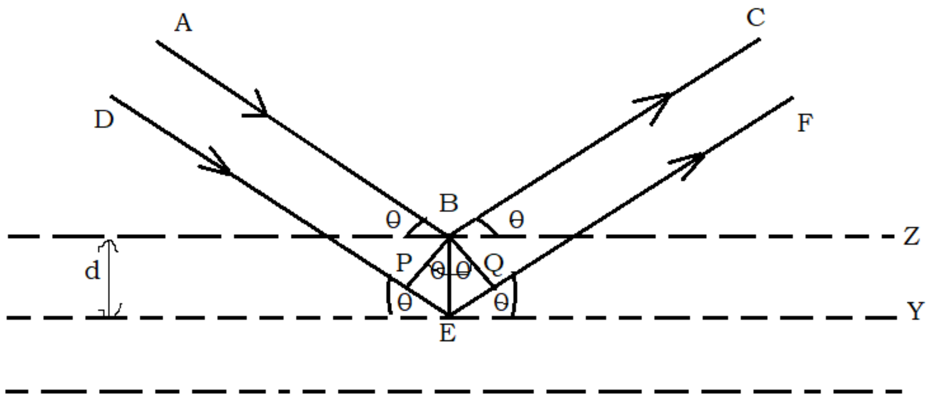
When monochromatic X-Rays are incident upon the atoms in a crystal lattice, each atom acts as a scattered radiation of the same wavelength. The crystal acts as a series of parallel reflecting planes. Thus, X-Rays would

be reflected according to the ordinary law of reflection and the planes would reflect X-Rays at all angles. X-Rays that enter more deeply into the crystal, are reflected from the lower planes.
The intensity of the reflected beams at a certain angle will be maximum, where the two reflected waves from two reflected planes have phase difference equal to an integral multiple of the wavelength of the X-Rays; while for some other angles, the intensity of the reflected beams would be minimum.

Consider a beam of monochromatic X-Rays of wavelength λ incident on a crystal and after reflection from the planes Z and Y, goes along BC and EF respectively.



Let, the crystal lattice spacing between two planes be d and θ be the Glancing angle. The path difference= $PE+EQ$.



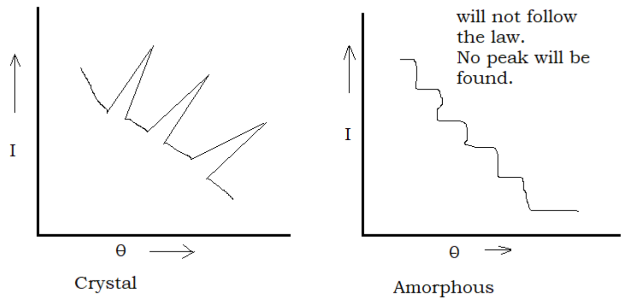
Now, from ΔBPE

$$\sin\theta = \frac{PE}{BE}$$
$$\Rightarrow PE = BE\sin\theta = d\sin\theta \quad ;where,\ d = BE$$
$$Similarly,\ EQ = d\sin\theta$$
$$\therefore \text{Path difference} = 2d\sin\theta$$

So, for constructive interference/the reflected beam to be maximum intensity,

$$2d\sin\theta = n\lambda \qquad (n = 1,2,3,\ldots)$$
$$\text{For anti-phase, } 2d\sin\theta = (2n+1)\frac{\lambda}{2} \qquad (n = 0,1,2,3,\ldots)$$

Condition: If $\sin\theta \le 1$, hence, $\lambda \le 2d$ is satisfied, and diffraction will occur.



λ from condition $= 2 \times 10^{-8} cm$. λ of X-Ray is $10^{-8} cm$. Only X-Ray can be diffracted by crystal.

1.4 Distinction Insulator

1.5 Semiconductor and Conductor

Solids can be classified into 3 classes according to their conductivity.

1.5.1 Conductor

A conductor facilitates the easy flow of an electron from one atom to another atom when proper voltage is applied. The reason is, there are no band gaps between the valence and conduction bands. In some materials, there is an overlapping of the conductor and valence bands, which means electrons can move between the two overlapping bands. As there is space for electrons to move into in the conduction band, valence band electron moves into the other band and conduction is allowed.
Silver is probably the best electrical conductor we encounter in everyday life. Other metals, such as gold, copper, steel, aluminium and brass also represent good conductors.
Solids are normally the best types of conductors, however, some liquids, including liquid metals (*e.g.* mercury), are also good at permitting the transmission of energy through them. Some materials are classed as superconductors. At extremely low temperatures these materials will conduct without resistance.

1.5.2 Insulator

An insulator prevents the flow of energy between two objects. For example, insulators may prevent the flow of electric, heat or sound. A substance that does not conduct electricity is called a dielectric material. These substances can be polarised by an applied electric field so electric charges do not flow through them as they would through a conductor. Therefore, the internal electrical field reduces the overall field within the dielectric. In insulators, there are larger gaps between the conduction and valence bands. The electrons cannot move into the conduction band and this means the material cannot conduct.

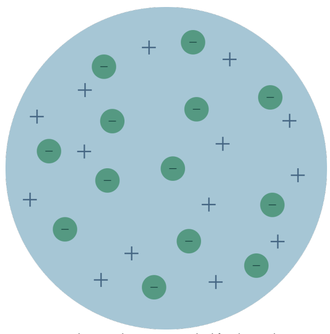
1.5.3 Semi-conductor

With moderate conductivity, a semiconductor has a conductivity value between that of a conductor such as silver and an insulator, such as the mica. The resistance of a semiconductor falls as its temperature rises. Elements like silicon (Si), germanium (Ge), selenium (Se); compounds like gallium arsenide ($GaAs$) and indium antimonide ($InSb$) are all examples of semiconductor elements. Silicon represents the most widely used semiconductor.
There is a gap between the valance and conduction bands in a semiconductor. However, it's small enough to facilitate the movement of electrons at room temperature, enabling some conduction. A rise in temperature increases the conductivity of a semiconductor because more electrons will have enough energy to move into the conduction band.
Ordinarily, gases are poor conductors due to the space between atoms. However, in some circumstances - such as when it contains a large number of ions – gasses can be fair conductors and act as semiconductors.

2. Atomic Physics

2.1 Atom Models

2.1.1 Thomson Atom Model



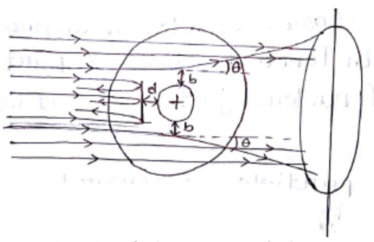
Also called Thomson's plum pudding model. In 1897, J.J. Thomson discovered the electron, a negatively charged particle thousand times lighter than a Hydrogen atom. Thomson originally believed that the Hydrogen atom must be made up of more than two thousands electrons, to account for its mass. An atom made of thousands of electrons would have a very high negative electric charge. In 1906, Thomson suggested that atoms contained far fewer electrons, a number roughly equal to the atomic number. These electrons must have been balanced by some sort of positive charge.

Thompson discovered the value of e/m, and as a result of this discovery two facts were clearly established:

1. Electrons enter into constitution of all atoms.
2. Since, as a whole, atom is electrically neutral, the quantity of positive and negative charges in it must be same.

Thompson suggested that the atom was spherical in shape. The whole mass of the atom was evenly distributed and positive charge was distributed all over the mass. The electrons with negative charge were embedded within the atom. The whole positive charge of the atom was equal to the total charge of the electrons. The atom was like a palm pudding. The electrons oscillate or vibrate about their mean position. Thompson model could successfully be applied in simple gas laws and in simple physical and chemical problems.
Failure: The scattering of particle by the heavy atoms studied by Rutherford led him to the idea that the positive charge could not be distributed evenly in the whole mass of the atom.

2.1.2 Rutherford Atom Model



In 1911, Rutherford experimented with α particle $({}_2^4He^{2+})$ and gold plate. He showed that, if α particle is thrown over a gold plate, on the core of the gold plate, α particle reflected 180° back. Form this incident, he described an atom model where he denoted the core as 'nucleus'.

In the experiment, a beam of α particles are aimed at a thin foil of gold. About 99% of the α particles pass straight through the foil. Some of the α particles are deflected through small angles. A very small amount of

α particles are bounced off the gold foil. The result of this experiment is the Rutherford Atomic model. The assumptions of the experiment:

1. Atom has a lot of space inside it which could explain why most of the particles went through the gold paper with no deviation from their path.
2. Atom has a dense center which gives the mass of the atom. This center, called nucleus, is positively charged. This could explain why the alpha particles suffered deviation from their path when they approach to a nucleus. Comparatively, the size of the nucleus is smaller than the atom.
3. Since the atom is an electrically neutral particle, the electrons must be around the nucleus, going round it in orbit like planets do around the sun. The number of electrons is equal (but with opposite sign) to the charge present in the nucleus.

Failure/Drawback

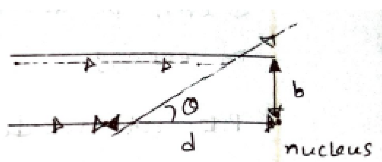
1. This model was made analogous to the solar system, where the nucleus may be compared to the sun and the electrons to the planets. In space, the planets are neutral and attract each other according to the law of gravitation. However, the electrons orbiting the nucleus are negatively charged and repel each other.
2. It was shown by Clark Maxwell that a charged body moving under the influence of attractive force, loses energy continuously in the form of electromagnetic radiation. Unlike a planet, the electron is a charged body, and it emits radiations, while revolving around nucleus. As a result, the electron should loose energy at every turn, and move closer to the nucleus following a spiral path. Consequently, the orbit would become smaller, and eventually, the electrons would fall into the nucleus. In other words, the atom should collapse. However, this never happens. This indicates that there is something wrong in the Rutherford's mass nuclear model of atom.
3. This model does not give any idea about the shape and size of the orbits of the electrons.
4. This model of atom does not say anything about the arrangement of electrons in an atom.
5. This model also failed to explain the existence of certain definite lines in the hydrogen spectrum.
6. This model failed to describe an atom with more than one electron.

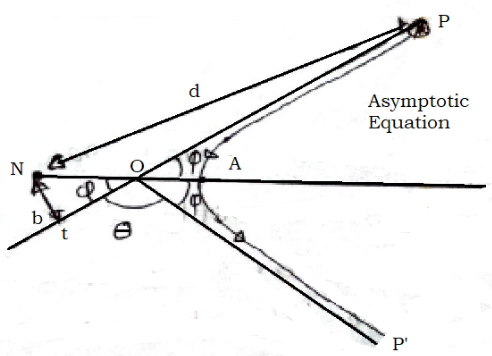
2.1.3 Rutherford Scattering Formula

2.1.3.1 Experiment of particle scattering

Basic assumptions

1. The particle and the nucleus with which it interacts are small enough to consider as point mass and charges.
2. The only force acting between the particle and the nucleus is the electrostatic repulsive force.
3. Compared to the particle, the nucleus is so massive that its motion during impact can be neglected.





Terms

1. **Impact Parameter** (b) is the minimum distance from nucleus, where no force will be felt. It is defined as the perpendicular distance between the path of a projectile and the center of a potential field, created by an object that the projectile approaching. This is the perpendicular distance of N from OP.
2. **Scattering angle** (θ) is the angle between asymptotic direction PO in which the alpha particle approaches the nucleus and the asymptotic direction in which it recedes/deflects.

3. As the particle gets closer and closer to the nucleus, its kinetic energy will get less and less due to the strong repulsive force between them until at a certain distance d from the nucleus, the particle will be forced to retrace its path, in which case, the angle of scattering will be 180°. This distance d is the distance of closest approach between the particle and nucleus.

2.1.3.2 Measuring Distance of the closest approach (d)

It is the minimum distance between α particle and center of nucleus, just before it reflects back by 180°. When α particle go towards nucleus, then it has kinetic energy only. If the velocity and mass of the α-particles are m and v_0, then total energy = kinetic energy, *i.e.*

$$T.E. = \frac{1}{2}mv_0^2$$

When α-particle is brought momentarily to rest, the work done in bringing it to rest, is equal to its initial kinetic energy. When the velocity, and hence, the kinetic energy is 0, all the energy is electrostatic potential energy. When the α particle is directed straight to N, $b = 0$. On account of the repulsive force, the α particle will be stopped at a certain distance d from the nucleus and made to retrace its path. In this case, $\theta = 180°$. This event is called **momentarily stopped**. d can be determined from the conservation of energy. Potential Energy at distance d is

$$\frac{Ze}{4\pi\varepsilon_0 d} \times 2e = \frac{Ze^2}{2\pi\varepsilon_0 d}$$

Kinetic Energy of particle will be changed into potential energy.

$$T.E. = \frac{Ze^2}{2\pi\varepsilon_0 d}$$
$$\Rightarrow \frac{1}{2}mv_0^2 = \frac{Ze^2}{2\pi\varepsilon_0 d}$$
$$\therefore d = \frac{Ze^2}{\pi\varepsilon_0 mv_0^2} \tag{2.1}$$

2.1.3.3 Relation between b and θ

As it is not possible, in practice, to direct the α particle exactly towards the nucleus, we must consider the case $b \neq 0$. In such case, the α particle will be deflected through an angle θ, less than 180°, and will travel through the hyperbolic path PAP'. Let v be the velocity of the α particle at A. Using the principles of conservation of energy,

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + \frac{Ze^2}{2\pi\varepsilon_0 l} \qquad (l = NA)$$
$$\Rightarrow mv_0^2 = mv^2 + \frac{\pi\varepsilon_0 mv_0^2 d}{\pi\varepsilon_0 l} \qquad \text{(From Equation 2.1)}$$
$$\Rightarrow v_0^2 = v^2 + \frac{v_0^2 d}{l}$$
$$\Rightarrow v^2 = v_0^2\left(1 - \frac{d}{l}\right)$$
$$\Rightarrow \frac{v^2}{v_0^2} = 1 - \frac{d}{l} \tag{2.2}$$

From the law of conservation of angular momentum,

$$mv_0 b = mvl$$
$$\Rightarrow \frac{v}{v_0} = \frac{b}{l} \Rightarrow \frac{v^2}{v_0^2} = \frac{b^2}{l^2}$$
$$\Rightarrow b^2 = l^2\left(1 - \frac{d}{l}\right) \qquad \text{(From Equation 2.2)}$$

$$\Rightarrow b^2 = l^2 - dl = l(l-d) \tag{2.3}$$

Using the property of hyperbola:

$$\varepsilon = \frac{1}{\cos\phi} \qquad \pi = \theta + 2\phi \Rightarrow \phi = \frac{1}{2}(\pi - \theta)$$

$$From\ \Delta NOZ,\ \sin\phi = \frac{b}{NO} \Rightarrow NO = \frac{b}{\sin\phi}.$$

$$Again,\ NO = \varepsilon OA$$

$$\Rightarrow OA = \frac{NO}{\varepsilon} = \frac{\frac{b}{\sin\phi}}{\frac{1}{\cos\phi}} = b\frac{\cos\phi}{\sin\phi}$$

$$\begin{aligned} l &= NO + OA \\ &= \frac{b}{\sin\phi} + b\frac{\cos\phi}{\sin\phi} \\ &= b\left(\frac{1+\cos\phi}{\sin\phi}\right) \\ &= b\left(\frac{2\cos^2\frac{\phi}{2}}{2\sin\frac{\phi}{2}\cos\frac{\phi}{2}}\right) \end{aligned}$$

$$\therefore l = b\cot\frac{\phi}{2}$$

Putting the value of l in Equation 2.3,

$$b^2 = b\cot\frac{\phi}{2}\left(b\cot\frac{\phi}{2} - d\right)$$

$$\Rightarrow b = b\cot^2\frac{\phi}{2} - d\cot\frac{\phi}{2} \qquad \text{(Since, } b \neq 0\text{)}$$

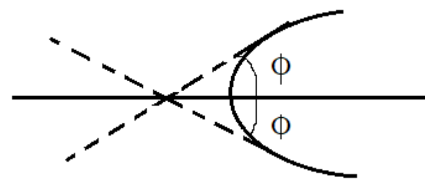
$$\Rightarrow d\cot\frac{\phi}{2} = b\left(\cot^2\frac{\phi}{2} - 1\right)$$

$$\begin{aligned} \Rightarrow d &= b\frac{\cot^2\frac{\phi}{2} - 1}{\cot\frac{\phi}{2}} \\ &= b\frac{2\cos^2\frac{\phi}{2} - 1}{\sin\frac{\phi}{2}\cos\frac{\phi}{2}} \\ &= 2b\frac{\cos\phi}{2\sin\frac{\phi}{2}\cos\frac{\phi}{2}} \\ &= 2b\frac{\cos\phi}{\sin\phi} = 2b\cot\phi \end{aligned}$$

$$\Rightarrow \cot\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \frac{d}{2b} = \tan\frac{\theta}{2}$$

$$\therefore \tan\frac{\theta}{2} = \frac{Ze^2}{2\pi\varepsilon_0 m v_0^2 b}$$

$$\Rightarrow b = \frac{Ze^2}{2\pi\varepsilon_0 m v_0^2}\cot\frac{\theta}{2}$$



For asymptotic ϕ,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

$$\Rightarrow y = \pm\frac{b}{a}x$$

$$\therefore \tan\phi = \pm\frac{b}{a}$$

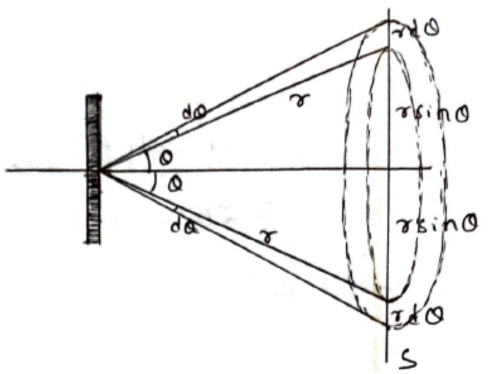
Essentially,

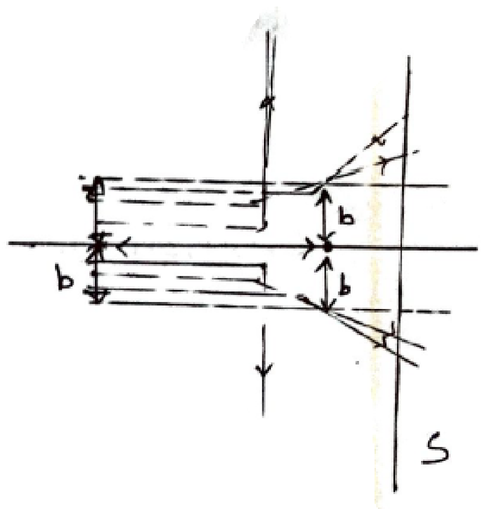
$$\begin{aligned} \varepsilon &= \sqrt{1 + \frac{b^2}{a^2}} \\ &= \sqrt{1 + \tan^2\phi} = \sqrt{\sec^2\phi} \\ &= \frac{1}{\cos\phi} \end{aligned}$$

This is the relation between Scattering angle and Impact parameter.

2.1.3.4 Number of scattered α particle

All the α particles approaching a target nucleus with impact parameter from 0 to b will be scattered through an angle θ or more. It means, the α particles should be directed in the area πb^2 around the nucleus. It is called the





cross section of the interaction, $\sigma(=\pi b^2)$.
Let, the thickness of foil be t, atoms per unit volume be n, and α particles incident upon area A.

$$\begin{aligned}
\therefore\ & \text{Nuclei per unit area} = nt \\
& \alpha \text{ particles will encounter nuclei} = ntA \\
& \text{area of } \alpha \text{ particles scatter through } \theta \text{ or more} = ntA\sigma = ntA\pi b^2 \\
f &= \text{Number of scattered } \alpha \text{ particles through } \theta \text{ or more angle for incident of } \alpha \\
& \quad \text{particles per unit area} \\
&= \frac{\text{Area of } \alpha \text{ particles scattered through } \theta \text{ or more angles}}{\text{Area of incidence}} \\
&= \frac{ntA\pi b^2}{A} = nt\pi b^2 \\
&= \frac{ntZ^2e^4}{4\pi\varepsilon_0^2 m^2 v_0^4}\cot^2\frac{\theta}{2} \\
\therefore \frac{\mathrm{d}f}{\mathrm{d}\theta} &= \text{change of fraction } f \text{ for change of small angle } d\theta \\
&= -\frac{ntZ^2e^4}{4\pi\varepsilon_0^2 m^2 v_0^4}\cot\frac{\theta}{2}\operatorname{co sec}^2\frac{\theta}{2} \\
\therefore \mathrm{d}f &= -\frac{ntZ^2e^4}{4\pi\varepsilon_0^2 m^2 v_0^4}\cot\frac{\theta}{2}\operatorname{co sec}^2\frac{\theta}{2}\,\mathrm{d}\theta
\end{aligned}$$

In this experiment, a fluorescent screen is placed behind the foil and area struck by these particles is given by,

$$\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}\theta} &= \text{change in the annular area for change of small angle } d\theta \\
&= \text{circumference} \times \text{width} \\
&= 2\pi r\sin\theta \times r = 4\pi r^2 \sin\frac{\theta}{2}\cos\frac{\theta}{2} \\
\therefore \mathrm{d}S &= 4\pi r^2 \sin\frac{\theta}{2}\cos\frac{\theta}{2}\,\mathrm{d}\theta
\end{aligned}$$

$\therefore$ Number of α particle that strike the foil is, $\mathrm{d}N = N_i\,\mathrm{d}f$. The number of α particle scattered into angle $d\theta$ and θ would be $N_i\,\mathrm{d}f$.

$$\begin{aligned}
\therefore \text{Per unit stroken area, } N(\theta) &= \frac{N_i\,\mathrm{d}f}{\mathrm{d}S} \\
&= -N_i\frac{\frac{ntZ^2e^4}{4\pi\varepsilon_0^2 m^2 v_0^4}\cot\frac{\theta}{2}\operatorname{co sec}^2\frac{\theta}{2}\,\mathrm{d}\theta}{4\pi r^2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,\mathrm{d}\theta}
\end{aligned}$$

$$= \frac{-ntN_iZ^2e^4}{16\pi^2r^2m^2\varepsilon_0^2v_0^4\sin^4\frac{\theta}{2}}$$

This equation represents the Rutherford scattering formula and indicates that the number of α-particles per unit area reaching the screen from distance r from the foil is

1. directly proportional to the thickness of the foil (t).
2. directly proportional to the squares of the atomic number of the foil (2^2).
3. inversely proportional to $\sin^4\frac{\theta}{2}$.
4. inversely proportional to the square of the initial kinetic energy (mv_0^2).

2.1.4 Bohr Atom Model

Main statements for the atom model given by Bohr are:

1. An atom consists of positively charged nucleus at the center. The negatively charged electrons move round the nucleus in various orbits known as stationary energy levels. The electrons do not emit radiations when they are moving in their own stationary levels.
2. Angular momentum of an electron is given by,

$$mvr = \frac{nh}{2\pi}$$
$$\begin{aligned} where,\ n &= \text{orbit no.}\\ m &= \text{mass of electron}\\ v &= \text{velocity of electron}\\ r &= \text{radius of orbit}\end{aligned}$$

3. Energy of an electron in one of its allowed orbits is fixed. As long as an electron remains in one of its allowed orbit, it neither absorbs nor radiates energy.
4. When an electron jumps from a higher level to lower level, it gives out radiation and vice versa.
5. Energy released, or absorbed by an electron, is equal to the difference of energy of two energy levels. Let, an electron jumps from higher energy level E_2 to a lower energy level E_1. The energy is emitted in the form of light. The amount of energy emitted, is given by-

$$\Delta E = E_1 - E_2 = h\upsilon$$
$$\begin{aligned} where,\ h &= \text{Plank's constant} = 6.6256\times 10^{-34}Js\\ \upsilon &= \text{Frequency of radiant light}\end{aligned}$$

6. Spectrum of light emitted from an electron is a **Line spectrum**.

Limitations:

1. This model violates the Heisenberg uncertainty principle. This theory considers electrons to have both a known radius and orbit, *i.e.* known position and momentum at the same time, which is impossible, according to Heisenberg.
2. This model made correct predictions for smaller sized atoms, *e.g.* Hydrogen, but poor spectral predictions are obtained when larger atoms are considered.
3. It failed to explain the Zeeman effect when the spectral line is split into several components in the presence of a magnetic field.
4. It also failed to explain the Stark effect when the spectral line gets split up into fine lines in the presence of electric field.

2.1.4.1 Energy Levels and Spectra

Radii of permitted orbit: According to Bohr model, F_1 is the electrostatic force of attraction between the nucleus and the electron, and F_2 the centrifugal force of repulsion between the nucleus and the electron from the circular motion of electron.

$$F_1 = k\frac{Ze^2}{r^2} \qquad \text{(Coulomb's constant, } k = \tfrac{1}{4\pi\varepsilon_0} = 9\times 10^9 Js,\ Z = \text{atomic number)}$$

$$F_2 = \frac{mv^2}{r}$$

$$Now,\ F_1 = F_2 \Rightarrow \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r^2} = \frac{mv^2}{r} \qquad (\varepsilon_0 = \text{permitivity of space} = 8.854\times10^{-12}Fm^{-1})$$

Again, angular momentum,

$$\begin{aligned}
mvr &= \frac{nh}{2\pi} \\
\Rightarrow v &= \frac{nh}{2\pi mr} && (2.4)\\
\Rightarrow v^2 &= \frac{n^2h^2}{4\pi^2m^2r^2} \\
\therefore \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r^2} &= \frac{m}{r}\frac{n^2h^2}{4\pi^2m^2r^2} \\
\Rightarrow r &= \frac{n^2h^2\varepsilon_0}{Z\pi me^2} = n^2r_1 \\
\Rightarrow r &\propto n^2
\end{aligned}$$

For Hydrogen atom,

$$r_1 = \frac{1\times(6.626\times10^{-34})^2\times8.854\times10^{-12}}{1\times3.1416\times9.11\times10^{-31}\times(1.6\times10^{-19})^2}m = 5.29\times10^{-11}m = 0.529\dot{A}$$

Orbit of the Hydrogen atom is called **Bohr radius**.
Velocity of Electron: From Equation 2.4

$$\begin{aligned}
v &= \frac{nh}{2\pi mr} \\
and,\ r &= \frac{n^2h^2\varepsilon_0}{Z\pi me^2} \\
\therefore v &= \frac{nh}{2\pi m}\times\frac{Z\pi me^2}{n^2h^2\varepsilon_0} \\
&= \frac{Ze^2}{2nh\varepsilon_0}
\end{aligned}$$

Energy of permitted orbit

$$\begin{aligned}
\text{Kinetic Energy, } K.E. &= \frac{1}{2}mv^2 = \frac{1}{2}m\times\frac{Z^2e^4}{4n^2h^2\varepsilon_0^2} \\
&= \frac{mZ^2e^4}{8n^2h^2\varepsilon_0^2}
\end{aligned}$$

As electron lies in the electron field of the nucleus, potential energy of an electron in an orbit is given by work done bringing the electron from infinite to the orbit.

$$\begin{aligned}
\therefore P.E. &= \frac{Ze}{4\pi\varepsilon_0 r}(-e) = -\frac{Ze^2}{4\pi\varepsilon_0 r} \\
&= -\frac{mZ^2e^4}{4n^2h^2\varepsilon_0^2} && \text{[negative sign represents energy spent]} \\
\therefore \text{Total Energy, } E_n &= P.E. + K.E. \\
&= -\frac{mZ^2e^4}{4n^2h^2\varepsilon_0^2} + \frac{mZ^2e^4}{8n^2h^2\varepsilon_0^2} \\
&= -\frac{mZ^2e^4}{8n^2h^2\varepsilon_0^2}
\end{aligned}$$

$$\therefore E_n \propto \frac{1}{n^2}$$

Here, energy is negative, which implies that the electron is bound to nucleus by attractive forces so that energy must be supplied to the electron in order to separate it completely from the nucleus.

Atomic Spectra: The spectrum frequencies of electromagnetic radiation, emitted or absorbed, during transmissions of electrons between energy levels within an atom, is called atomic spectrum. Atomic spectra can be of two types:

1. **Absorption spectra:** When an electron jumps from low energy level to high energy level, it absorbs a certain amount of energy, which is equal to the energy difference of those orbits. The resulting absorption line spectrum consists of a bright background crossed by dark lines that correspond to the missing wavelengths.
2. **Emission spectra:** When an electron jumps from high energy level to low energy level, it emits a certain amount of energy equal to the difference of the energy levels. The resulting emission spectral consists of bright lines on a dark background.

The radiation from atoms, or spectral lines can be classified into two types:

1. Continuous Spectra (visible light): can be found by heating solid or liquid.
2. Discrete Spectra (absorption spectra): can be found by heating gas.

Color can be divided into two types:

1. Monochromatic line ($\lambda = 1$): one wavelength
2. Non-monochromatic line ($\lambda > 1$): multiple wavelengths

Origin of spectral lines/spectral series: We know

$$\begin{aligned}
&E_{n_2} - E_{n_1} = h\nu \\
&\quad \Rightarrow \nu = \frac{1}{h}\left[-\frac{mZ^2e^4}{8n_2^2h^2\varepsilon_0^2} - \left(-\frac{mZ^2e^4}{8n_1^2h^2\varepsilon_0^2}\right)\right] \\
&\qquad = \frac{c}{\lambda} = \frac{mZ^2e^4}{8h^3\varepsilon_0^2}\left(\frac{1}{n_2^2} - \frac{1}{n_1^2}\right) && (c = \text{speed of light}) \\
&\quad \Rightarrow \frac{1}{\lambda} = \overline{\nu} = \frac{mZ^2e^4}{8h^3\varepsilon_0^2c}\left(\frac{1}{n_2^2} - \frac{1}{n_1^2}\right) = R_H\left(\frac{1}{n_2^2} - \frac{1}{n_1^2}\right) && (2.5)
\end{aligned}$$

$$\begin{bmatrix} R_H = \text{Rhydberg Constant} = \frac{mZ^2e^4}{8h^3\varepsilon_0^2c} = 1.097 \times 10^7 m^{-1} \\ \overline{\nu} = \text{number of waves per meter} \end{bmatrix}$$

Equation 2.5 is called the origin of spectral line. When $n_2 = \infty$, λ is then called series limit.

Lymann series: $\overline{\nu_n} = R_H\left(\frac{1}{1^2} - \frac{1}{n_2^2}\right)$ $\quad [n_2 = 2, 3, 4, \ldots]$

Balmer series: $\overline{\nu_n} = R_H\left(\frac{1}{2^2} - \frac{1}{n_2^2}\right)$ $\quad [n_2 = 3, 4, 5, \ldots]$

Paschen series: $\overline{\nu_n} = R_H\left(\frac{1}{3^2} - \frac{1}{n_2^2}\right)$ $\quad [n_2 = 4, 5, 6, \ldots]$

Bracket series: $\overline{\nu_n} = R_H\left(\frac{1}{4^2} - \frac{1}{n_2^2}\right)$ $\quad [n_2 = 5, 6, 7, \ldots]$

Pfund series: $\overline{\nu_n} = R_H\left(\frac{1}{5^2} - \frac{1}{n_2^2}\right)$ $\quad [n_2 = 6, 7, \ldots]$

How can a Hydrogen atom create so many spectral lines? Because even a simple system with two particles (proton & electron) has an infinite number of energy levels available to it. And transition from one level to another creates a spectral line.

Resonance potential: The minimum potential required to provide energy to the electron in the ground state to the first excited state (i.e. from $n = 1$ to $n = 2$) is called resonance potential. For Hydrogen, $E_1 = -13.6eV$, $E_2 =$

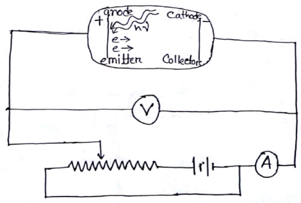
$-3.4eV$.
$\therefore E = E_2 - E_1 = (-3.6 + 13.6)eV = 10.2eV$ is the resonance potential.
Excitation potential: The potential required to provide energy to raise the electron from ground state to the state $n > 1$.
Ionization potential: It is the minimum potential required to bring the electron from the ground state to out of the atom. For Hydrogen atom, **IP** is $13.6eV$.
Binding energy: The minimum energy required to escape from the attraction of the nucleus.

2.2 Particle Properties of Waves

2.2.1 Photoelectric Effect

When a metal surface is illuminated with light or any other radiation of suitable wavelength, electron can be emitted from the surface. This phenomenon is called the photoelectric effect. The electrons emitted from the surface are called photo electrons. The whole experiment was done by Frank and Hazard.

- When UV light falls on neutral Zn plate, the plate becomes positively charged.
- When UV light falls on negatively charged Zn plate, the plate becomes neutral.
- When UV light falls on positively charged Zn plate, the plate gains more positive charge.



$$eV = \frac{1}{2}mv_{max}^2$$

From the figure

- There will be no photo electricity for stopping potential (V_s).
- Threshold frequency(υ_0) is the minimum frequency required for ejecting one electron. It is defined as the minimum frequency of light, for which electron just emit from the metal surface, if the light is exposed to metal surface.

2.2.1.1 Work Function ω_0

The energy needed for electron to reach the collector, is called work function.

$$\omega_0 = h\upsilon_0$$

2.2.1.2 Laws of Photoelectric Emission

The experimental observations on photoelectric effect may be summarized as follows, which are known as the fundamental laws of photoelectric emission.

1. The strength of photoelectric current is directly proportional to the intensity of the light, when the frequency is constant, $i.e.\ current,\ i \propto intensity,\ I$.
2. For every substance, there is a limiting frequency below which no photo electrons are emitted. This limiting frequency is called threshold frequency (v_0). The corresponding wavelength is called threshold wavelength (λ_0). No photo electrons are emitted for more than λ_0.
3. Velocity of electron is independent of intensity, but varies with the frequency of the incident light. Hence, the maximum kinetic energy of electron is independent of intensity of incident light.
4. Photoelectric emission is an instantaneous process. The estimated delay time is $3ns$.

2.2.1.3 Drawbacks/Limitations of classical/electromagnetic theory to explain photo electric effect

1. The electric and magnetic fields which are attributed to light waves in the electromagnetic theory, may well be able to exert sufficient force on the electrons in the metal and release them from the surface. But, $I \propto A^2$. On that basis, photoelectrons ejected from the metal should then have energies which are dependent on the intensity. But the experimental result does not show that.
2. It was expected that light of low frequency but sufficient intensity would be as effective as high frequency. But the experiment results show that minimum frequency is required.
3. It was also difficult to explain that there is no time lag between the arrival of radiation on the metal surface and the emission of photoelectrons. For example, it almost takes a year for Na light to accumulate and remove the electron. Why is that?

2.2.2 Einstein's Photoelectric Equation

1. Part of the energy is used for freeing the electrons from metal surface. This energy is known as the photoelectric work function of the metal and is represented by ω_0.
2. The balance of the photon energy is used up in imparting to the freed electron a kinetic energy of $\frac{1}{2}mv^2$.

Einstein expressed these assumptions in the form, Total energy = energy to free the electron + energy for the elctron to reach the collector

$$h\upsilon = h\upsilon_0 + \frac{1}{2}mv^2 = \omega_0 + \frac{1}{2}mv^2,$$

$$\begin{aligned} where,\ h\upsilon &= \text{energy content of each quantum of the incident light} \\ \upsilon_0 &= \text{threshold frequency} \\ \omega_0 &= \text{photoelectric work function of the metal surface being irradiated} \\ \frac{1}{2}mv^2 &= \text{kinetic energy of the ejected photoelectron} \end{aligned}$$

2.2.2.1 Characteristics of Einstein's photo electric equation

1. The maximum velocity of the ejected photoelectron is directly proportional to the frequency of the incident radiation.

$$\begin{aligned} h\upsilon &= \omega_0 + \frac{1}{2}mv^2 \\ \frac{1}{2}mv^2 &= h\upsilon - \omega_0 \\ \omega_0 \text{ is constant, } &\therefore v \propto \sqrt{\upsilon} \end{aligned}$$

2. The velocity of the photoelectron is independent of the intensity of the radiation. Increasing intensity only increases the amount of photons striking the metal. But each photoelectron will get same kinetic energy. However, the increased number of photoelectrons increase the strength of photoelectric current. This is why i varies directly with I.
3. Existence of a threshold frequency that varies with the nature of emitter.

$$\omega_0 = h\upsilon_0 = \frac{hc}{\lambda_0}$$

4. There is no time lag in the process of photoelectric emission. As the phenomenon is considered as a collision process between two particles, the photon and the atom, there is no time lag between the incidence of the photon on the metal surface and therefore emission of electron.
5. Emission of photoelectron with all possible velocity up to certain maxima.

$$\begin{aligned} h\upsilon &= \omega_0 + \frac{1}{2}mv_{max}^2 \\ &= \omega_0 + (K.E.)_{max} \\ \therefore\ eV \text{ (Stopping potential)} &= \frac{1}{2}mv_{max}^2 \end{aligned}$$

If the stopping potential is equal to kinetic energy, electron won't reach the collector. Electron will reach the collector if kinetic energy is higher than stopping potential

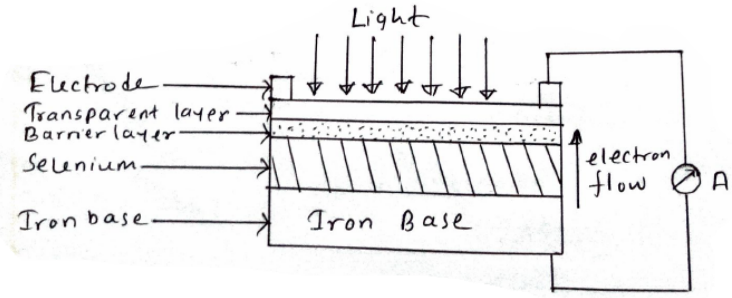
2.2.3 Photovoltaic Cells

Photoelectric cell is an arrangement to transform light energy into electricity. The three types are worth considering.

1. Photo emissive cell: It depends on the emission of electrons from a metal cathode when it is exposed to light or other radiation.
2. Photo voltaic cell: The sensitive element here is semiconductor which generates voltage proportional to the light incident on it.
3. Photo conductive cell: Semiconductor materials whose resistance changes in accordance with the radiant energy.

Photo voltaic cells are famous among them.

2.2.3.1 Photo voltaic cell structure and how it works

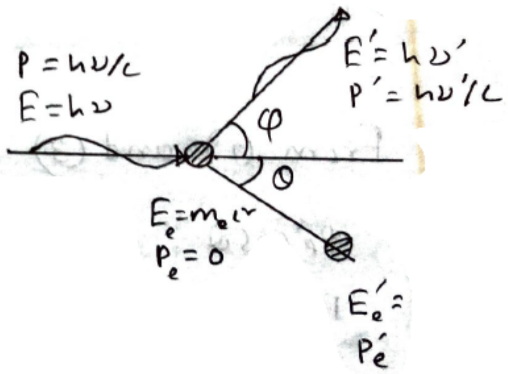


It is self-generating cell which employs semiconductor contacts against metals when light incident on such a combination and interval voltage generates which causes the current to flow to internal circuit. When light or radiation falls on the semiconductor, *i.e.* Selenium, it ejects electron and travels from selenium to the front silver electrode as shown in figure. As the barrier layer acts as a rectifier, it does not permit the flow of electron in the opposite direction. The **emf** generated internally between silver electrode and selenium is almost directly proportional to the incident flux.
The main advantage of photovoltaic cell is that no external battery is required for its operation, *i.e.* it is self-generating[1]. Moreover internal emf, and hence, current generated by it are large enough to be measured on a pointer galvanometer.

2.2.4 Compton Effect

In 1923, A.H. Compton, while making a spectroscopic study of X-rays scattered by matter, discovered that, when a beam of X-rays of well-defined wavelength λ is scattered through an angle θ by sending the radiation through a metallic foil, the scattered radiations consisted of two components; one of longer wavelength than the incident radiation and another one of the same wavelength as incident radiation. This incident is called the **Compton Effect**.
Radiation scattered from loosely bound, nearly free electron partly the energy of radiation is given to the electron which is released from the atom. The remainder of the energy is released as electromagnetic radiation.

The compton effect is the result of high energy photon colliding with a target, which releases loosely bound electrons from the outer shell of the atom.



Before collision:

Energy of photon, $E = h\upsilon$

Momentum of photon, $P = \dfrac{h\upsilon}{c}$

Energy of electron, $E_e = m_e c^2$

Momentum of electron, $P_e = 0$

After collision,

Energy of photon, $E' = h\upsilon'$

Momentum of photon, $P' = \dfrac{h\upsilon'}{c}$

Energy of electron, $E_e{}'$

Momentum of electron, $P_e{}'$

From the conservation of energy-

[1]In case of other photoelectric cells, a positive potential is to be applied to the collector to attract the electrons and therefore an external battery is necessary.

$$E_{initial} = E_{final}$$
$$\Rightarrow E + E_e = E' + E_e{}'$$
$$\Rightarrow E + m_e c^2 = E' + E_e{}' \tag{2.6}$$
$$\Rightarrow E'_e = E - E' + m_e c^2 \tag{2.7}$$

From the conservation of momentum,

$$(P_x)_{initial} = (P_x)_{final}$$
$$\Rightarrow P + 0 = P' \cos\phi + P_e{}' \cos\theta$$
$$\Rightarrow P_e{}' \cos\theta = P - P' \cos\phi \tag{2.8}$$
$$(P_y)_{initial} = (P_y)_{final}$$
$$\Rightarrow 0 + 0 = P' \sin\phi - P_e{}' \sin\theta \qquad \text{(negative since direction is opposite)}$$
$$\Rightarrow P_e{}' \sin\theta = P' \sin\phi \tag{2.9}$$

From $(\text{Equation 2.8})^2 + (\text{Equation 2.9})^2$

$$P'^2_e = P^2 + P'^2 - 2PP' \cos\phi \tag{2.10}$$

$$\begin{aligned} Now,\ P &= mv \\ &= \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}} \\ \Rightarrow P^2 &= \frac{m_0^2 v^2}{1 - \frac{v^2}{c^2}} \\ \Rightarrow P^2 c^2 &= \frac{m_0^2 v^2 c^4}{c^2 - v^2} \end{aligned} \qquad \begin{aligned} E &= mc^2 \\ &= \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \\ \Rightarrow E^2 &= \frac{m_0^2 c^4}{1 - \frac{v^2}{c^2}} \\ &= \frac{m_0^2 c^6}{c^2 - v^2} \end{aligned}$$

$$\therefore E^2 - P^2 c^2 = \frac{m_0^2 c^6}{c^2 - v^2} - \frac{m_0^2 v^2 c^4}{c^2 - v^2} = \frac{m_0^2 c^4}{c^2 - v^2}(c^2 - v^2) = m_0^2 c^4$$
$$\therefore E'^2_e - P'^2_e c^2 = m_e^2 c^4 \tag{2.11}$$
$$\Rightarrow (E - E' + m_e c^2)^2 = m_e^2 c^4 + (P^2 + P'^2 - 2PP' \cos\phi) c^2 \qquad \text{(From Equation 2.7 and 2.10)}$$
$$\Rightarrow E^2 + E'^2 + m_e^2 c^4 - 2EE' - 2E' m_e c^2 + 2E m_e c^2 = m_e^2 c^4 + P^2 c^2 + P'^2 c^2 - 2PP' c^2 \cos\phi$$
$$\Rightarrow E^2 + E'^2 - 2EE' - 2m_e c^2 (E - E') = E^2 + E'^2 - 2EE' \cos\phi \qquad \text{(Since, } E = Pc\text{)}$$
$$\Rightarrow EE' + m_e c^2 (E - E') = EE' \cos\phi$$
$$\Rightarrow (1 - \cos\phi) EE' = m_e c^2 (E - E')$$
$$\Rightarrow \frac{h^2 c^2}{\lambda \lambda'}(1 - \cos\phi) = m_e c^2 \left[\frac{hc}{\lambda} - \frac{hc}{\lambda'}\right] = h m_e c^3 \left[\frac{1}{\lambda} - \frac{1}{\lambda'}\right] = h m_e c^3 \left[\frac{\lambda' - \lambda}{\lambda \lambda'}\right] \qquad \text{(Since } E = h\upsilon = \tfrac{hc}{\lambda}\text{)}$$
$$\Rightarrow \lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\phi)$$

where, λ = wavelength of incident photon

λ' = wavelength of incident electron

Compton Shift, $\Delta\lambda = \lambda' - \lambda$ Compton Wavelength $= \dfrac{h}{m_e c}$

For visible light, ϕ is very small, hence, $\Delta\lambda$ is very small.

2.3 Wave Properties of Particle

2.3.1 De Broglie Waves

A moving body behaves in certain ways as though it has wave nature. This type of moving body is called De Broglie wave. The wavelength associated with a particle is referred to as **De Broglie wavelength**. Let, the equation of a particle

$$\psi = \psi_0 \sin(2\pi \upsilon_0 t_0) \tag{2.12}$$

$\psi =$ wave function $\qquad \psi_0 =$ amplitude

$\upsilon_0 =$ frequency observed by the observer

$$t_0 = \text{fixed time} \tag{2.13}$$

Let, the particle be given a velocity v along x-axis. From Lorentz transformation, we get,

$$t_0 = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\therefore \psi = \psi_0 \sin\left(2\pi \upsilon_0 \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}\right) \tag{2.14}$$

General wave equation,

$$y = a \sin \frac{2\pi}{\lambda}(ut - x) \qquad (u = \text{velocity of wave} = \upsilon\lambda)$$

$$= a \sin \frac{2\pi}{\frac{u}{\upsilon}}(ut - x)$$

$$= a \sin 2\pi\upsilon(t - \frac{x}{u})$$

$$= a \sin \frac{2\pi}{T}(t - \frac{x}{u}) \tag{2.15}$$

Comparing equation Equation 2.14 and Equation 2.15

$$\upsilon = \frac{1}{T} = \frac{\upsilon_0}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{2.16}$$

$$u = \frac{c^2}{v} \tag{2.17}$$

From Einstein's mass-energy relation,

$$h\upsilon_0 = m_0 c^2$$

$$\Rightarrow \upsilon_0 = \frac{m_0 c^2}{h}$$

Putting the value in Equation 2.16

$$\upsilon = \frac{1}{T} = \frac{m_0 c^2}{h\sqrt{1 - \frac{v^2}{c^2}}}$$

$$= \frac{mc^2 \sqrt{1 - \frac{v^2}{c^2}}}{h\sqrt{1 - \frac{v^2}{c^2}}} \qquad \left[m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}\right]$$

$$\therefore \upsilon = \frac{mc^2}{h} \tag{2.18}$$

$$\lambda = \frac{\text{velocity}}{\text{frequency}} = \frac{u}{\upsilon} = \frac{c^2}{v} \times \frac{h}{mc^2} \qquad \text{[From Equation 2.17 and 2.18]}$$

$$= \frac{h}{mv} = \frac{h}{p} \tag{2.19}$$

Equation 2.19 is De Broglie equation and λ is De Broglie wavelength.

2.3.1.1 De Broglie wavelength of electron

Let, an electron be accelerated through a potential difference of V volts. Then the kinetic energy of e^-,

$$\frac{1}{2}mv^2 = eV$$
$$\Rightarrow m^2v^2 = 2meV$$
$$\Rightarrow mv = \sqrt{2meV}$$

Putting the value in Equation 2.19,

$$\lambda = \frac{h}{\sqrt{2meV}}$$

Ignoring relativistic corrections,

$$m = m_0$$
$$\lambda = \frac{h}{\sqrt{2m_0eV}}$$

2.3.1.2 Characteristics of De Broglie wave

1. Since $\lambda = \frac{h}{2m_0eV}$, hence, larger the mass of the particle/matter, smaller the associated wave.
2. These waves are not electromagnetic waves but new kind of wave. They may be regarded as **pilot waves** as their only function is to pilot or guide the matter particle.
3. Two different velocities are associated with the material particle in motion, one is the mechanical motion of the particle (υ), another, propagation of associated wave (u). The phase or wave velocity of the matter wave is, $u = \frac{c^2}{v}$. Since, v cannot exceed c, hence, u is greater than c.

2.3.1.3 Pilot Wave Theory

According to this theory, the point particle and the matter wave, are both real and distinct physical entities.

2.3.1.4 Why the rest mass of photon is zero?

2.3.2 Group Velocity and Phase Velocity

Phase velocity of a wave is the rate at which the phase of the wave propagates in space.
Group velocity of a wave is the velocity with which the overall shape of the wave's amplitudes, known as modulation or envelope of the wave, propagates through space.
According to De Broglie's concept of matter waves, each particle of matter, while in motion, may be regarded as consisting of a group of waves or a wave packet as it is called. The wave packet, formed by the superposition of a number of waves and travelling with the velocity of the particle. Behaves very much like a corpuscle. Each component wave propagates with a definite velocity, called the wave velocity or phase velocity. But when a disturbance consists of a number of component waves, each traveling with slightly different velocity, the resultant velocity will be that of a periodicity. This periodicity, with which the velocity advances, is called the group velocity.

$$y_1 = a\cos(\omega_1 t - k_1 x) \qquad (k = \tfrac{2\pi}{\lambda} \text{ and } \tfrac{\omega}{k} = v)$$
$$y_2 = a\cos(\omega_2 t - k_2 x)$$
$$\begin{aligned} y &= y_1 + y_2 \\ &= a\cos(\omega_1 t - k_1 x) + a\cos(\omega_2 t - k_2 x) \\ &= 2a\cos\left(\frac{\omega_1+\omega_2}{2}t - \frac{k_1+k_2}{2}x\right)\cos\left(\frac{\omega_1-\omega_2}{2}t - \frac{k_1-k_2}{2}x\right) \\ &= 2a\cos(\omega t - kx)\cos\left(\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x\right) \end{aligned}$$

Here,

$$\underbrace{\omega = \frac{\omega_1+\omega_2}{2} \qquad k = \frac{k_1+k_2}{2}}_{particle} \qquad\qquad \underbrace{\Delta\omega = \omega_1 - \omega_2 \qquad \Delta k = k_1 - k_2}_{group}$$

General equation of wave,

$$y = a\cos(\omega t - kx) = a\cos k\left(\frac{\omega}{k}t - x\right) = a\cos\frac{2\pi}{\lambda}\left(\frac{\omega}{k}t - x\right)$$

The resultant wave has two parts:

1. A wave of frequency ω and propagation constant k. If velocity of phase is v_p then,

$$v_p = \frac{\omega}{k} = \frac{2\pi v}{\frac{2\pi}{\lambda}} = v\lambda$$

This is the phase velocity or wave velocity.

2. A second wave of frequency $\frac{\Delta\omega}{2}$ and propagation constant $\frac{\Delta k}{2}$ named group velocity (v_g).

$$\begin{aligned} v_g &= \frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk} = \frac{d}{dk}(v_p k) \\ &= v_p + k\frac{dv_p}{dk} \\ &= v_p + \frac{2\pi}{\lambda}\frac{1}{-\frac{2\pi}{\lambda^2}}\frac{dv_p}{d\lambda} && \left[k = \frac{2\pi}{\lambda} \Rightarrow dk = -\frac{2\pi}{\lambda^2}d\lambda\right] \\ \therefore v_g &= v_p - \lambda\frac{dv_p}{d\lambda} \end{aligned}$$

2.3.2.1 Particle velocity

It is the velocity of a particle (real or imagined) in a medium, as it transmits a wave.

2.3.2.2 Relationship between Group velocity and Particle velocity

From the definition of particle velocity

$$v_{p_a} = \frac{P}{m} \tag{2.20}$$

The group velocity of a matter wave is given by

$$v_g = \frac{d\omega}{dk} \qquad where,\ \omega = 2\pi v \qquad k = \frac{2\pi}{\lambda}$$

From Plank's equation, $E = h\upsilon$. From De Broglie wavelength, we can write, $\lambda = \frac{h}{P}$. Using these, we can rewrite the expressions for ω and k

$$\omega = \frac{2\pi E}{h} \qquad k = \frac{2\pi P}{h}$$

Differentiating these expressions,

$$d\omega = \frac{2\pi}{h}dE \qquad dk = \frac{2\pi}{h}dP$$

Hence,

$$v_g = \frac{\frac{2\pi}{h}dE}{\frac{2\pi}{h}dP} = \frac{dE}{dP}$$

Since, we are dealing with matter-waves, E can be the kinetic energy of particle in wave motion. Using the relation

$$E = \frac{P^2}{2m}$$

and differentiating it w.r.t. p, we get,

$$v_g = \frac{dE}{dP} = \frac{d}{dP}\left(\frac{P^2}{2m}\right) = \frac{2P}{2m} = \frac{P}{m} \tag{2.21}$$

From Equation 2.20 and 2.21

$$v_{p_a} = v_g$$

Hence, group velocity = particle velocity

3. Waves and Oscillations

3.1 Oscillations

3.1.1 Simple Harmonic Motion

Whenever a force acting on a particle and hence the acceleration of the particle is proportional to its displacement from the equilibrium position or any other fixed point on its path, but is always directed in a direction opposite to the direction of displacement and if the maximum displacement of the particle is the same on either side of the mean position, the particle is said to execute simple harmonic motion.

Frequency: The number of complete oscillations or cycles in unite time is called frequency of vibration.

Time period: It is the time taken by a particle to complete one vibration.

Amplitude: It is the maximum displacement of the particle from its mean position of rest.

Wave length: It is the distance travelled by the wave in the time in which the particle completes one vibration.

Angular frequency: The rate of changes of phase with time. It is expressed by ω.

Angular wave number: The rate of changes of phase with distance.

Wave motion: The motion which transports energy and momentum from one point in space to another without transport of matter.

Restoring force is a force which always try to bring back its oscillation in mean position.

Harmonic Oscillator is a system, that when displaced from its equilibrium position, experiences a restoring force, which is proportional to displacement. The motion of harmonic oscillator is called harmonic oscillation. Harmonic oscillation can be of two types.

- Quantum harmonic oscillation: for small particles, *e.g.* electron
- Classical harmonic oscillation: for heavier matter

Characteristics of SHM

1. It is periodic, oscillatory, linear motion.
2. Acceleration is proportional to its displacement from equilibrium point reacted toward its equilibrium point.
3. Acceleration is directed opposite to displacement and towards mean position.
4. The body moves up and down/to and fro/back and forth in its path with respect to a mean position.
5. The body/particle always returns to its mean position after a regular interval of time.
6. The force acting on the body will always try to bring the object back to its original/mean position. The force is called restoring force.
7. The maximum displacements on either side of the mean position/equilibrium position/position of grace will be the same.

Applications of SHM: Mechanics, Light, Atomic physics, Music, Electric oscillations, Alternating current etc.

3.1.1.1 Differential equation of SHM

If F be the restoring force acting on a particle executing simple harmonic motion and y be displacement from its mean position, then,

$$\begin{aligned} &F \propto -y \\ \Rightarrow &F = -ky && \text{(k is proportionary constant)} \end{aligned}$$

If the mass and acceleration of the particle be m and a respectively, then according to Newton's Law of Motion,

$$\begin{aligned} &F = ma \\ \Rightarrow &-ky = ma \\ \Rightarrow &-ky = m\frac{\mathrm{d}^2y}{\mathrm{d}t^2} \\ \Rightarrow &\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{k}{m}y = 0 \\ \Rightarrow &\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \omega^2 y = 0 && \left[\omega = \text{angular velocity of the particle} = \sqrt{\tfrac{k}{m}}\right] \end{aligned}$$

It is the differential equation of SHM. Solution of this equation:

$$2\frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 2\frac{\mathrm{d}y}{\mathrm{d}t} \times \omega^2 y = 0 \qquad \text{(multiplying both sides by } 2\tfrac{\mathrm{d}y}{\mathrm{d}t}\text{)}$$

Integrating,

$$\begin{aligned} &\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = -\omega^2 y^2 + c && \text{(c = integrating constant)} \\ &\frac{\mathrm{d}y}{\mathrm{d}t} = 0,\ when\ y = \pm a && \text{($i.e.$ velocity will be 0 at maximum displacement)} \\ &\therefore c = \omega^2 a^2 \\ &\therefore \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = -\omega^2 y^2 + \omega^2 a^2 = \omega^2(a^2 - y^2) \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = \pm\omega\sqrt{a^2 - y^2} \\ &\Rightarrow \frac{dy}{\sqrt{a^2 - y^2}} = \omega dt \end{aligned}$$

Integrating,

$$\begin{aligned} \sin^{-1}\frac{y}{a} &= \omega t + \varphi && \text{(φ = integrating constant/initial phase)} \\ \Rightarrow\ y &= a\sin(\omega t + \varphi) \\ \varphi &= \text{Initial phase} \\ \omega t + \varphi &= \text{Total phase} \\ \omega t &= \text{phase angle} \end{aligned}$$

This is the solution to the differential equation of simple harmonic motion, which gives the displacement of the particle at an instant time (t).
Expanding the equation, we have,

$$\begin{aligned} y &= a\sin\omega t\cos\varphi + a\cos\omega t\sin\varphi \\ &= A\sin\omega t + B\cos\omega t \\ where, A &= a\cos\varphi,\ and,\ B = a\sin\varphi \end{aligned}$$

This is the general form of the differential equation.

Special case:

$$When,\ A = 0,\ y = B\cos\omega t$$
$$When,\ B = 0,\ y = A\sin\omega t$$

Velocity

$$y = a\sin(\omega t+\varphi)$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = a\omega\cos(\omega t+\varphi)$$
$$= a\omega\sqrt{1-\sin^2(\omega t+\varphi)}$$
$$= a\omega\sqrt{1-\frac{y^2}{a^2}}$$
$$= a\omega\frac{\sqrt{a^2-y^2}}{a}$$
$$\therefore v = \omega\sqrt{a^2-y^2}$$

Acceleration

$$\frac{\mathrm{d}y}{\mathrm{d}t} = a\omega\cos(\omega t+\varphi)$$
$$\therefore \frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -a\omega^2\sin(\omega t+\varphi)$$
$$= -a\omega^2\frac{y}{a}$$
$$\therefore a = -\omega^2 y = -\frac{k}{m}y$$
$$\Rightarrow \omega = \sqrt{\frac{a}{y}}$$

(considering magnitude only)

Time Period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{y}{a}} = 2\pi\sqrt{\frac{m}{k}}$$
$$y = a\sin[\omega(t+T)+\varphi]$$
$$= a\sin(\omega t+2\pi+\varphi) = a\sin(\omega t+\varphi)$$

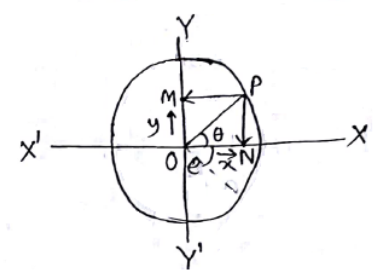
Frequency

$$\omega = 2\pi n = \frac{2\pi}{T}$$
$$\Rightarrow n = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{\sqrt{\frac{k}{m}}}{2\pi}$$

3.1.1.2 Phase and Phase Constant

Phase is the position and direction of motion at any particular time. **Initial phase/Phase constant** is the position and direction of motion when the counting of time started. Epoch is an instant in time chosen as the origin of a particular wave.

Angular displacement $= \theta = \omega t$



$$OP = a$$
$$ON = PM = x = OP\cos\theta = a\cos\theta = a\cos\omega t$$
$$OM = PN = y = OP\sin\theta = a\sin\theta = a\sin\omega t$$

M oscillates about y-axis, and N oscillates about x-axis.

Phase constant

1. If counting started at standard position $y = 0$ at $t = 0$

$$\therefore 0 = a\sin(\omega\times 0+\varphi)$$
$$\Rightarrow \sin\varphi = 0 \Rightarrow \varphi = 0$$
$$\therefore y = a\sin\omega t$$

2. If counting started at standard position $y = a$ at $t = 0$

$$\therefore a = a\sin(\omega\times 0+\varphi)$$
$$\Rightarrow \sin\varphi = 1 \Rightarrow \varphi = \frac{\pi}{2}$$
$$\therefore y = a\sin\left(\omega t+\frac{\pi}{2}\right) = a\cos\omega t$$

3. If counting started before particle had passed through its mean position $y = 0$ at $t = t'$

$$\therefore 0 = a\sin\left(\omega t'+\varphi\right)$$
$$\Rightarrow \sin\left(\omega t'+\varphi\right) = 0$$

$$\Rightarrow \omega t' + \varphi = 0$$
$$\Rightarrow \varphi = -\omega t' = -e \qquad \text{(Epoch)}$$
$$\therefore y = a\sin(\omega t - e)$$

4. If counting started after particle had passed through its mean position $y = 0$ at $t = -t'$

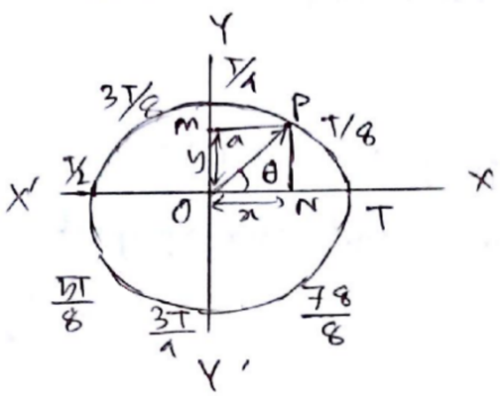
$$\therefore 0 = a\sin(-\omega t' + \varphi)$$
$$\Rightarrow \sin(-\omega t' + \varphi) = 0$$
$$\Rightarrow -\omega t' + \varphi = 0$$
$$\Rightarrow \varphi = \omega t' = e \qquad \text{(Epoch)}$$
$$\therefore y = a\sin(\omega t + e)$$

5. If particle complete one round

$$y = y \text{ at } t = t + \frac{2\pi}{\omega}$$
$$\therefore y = a\sin\left(\omega\left(t + \frac{2\pi}{\omega}\right) + \varphi\right)$$
$$\Rightarrow y = a\sin(\omega t + 2\pi + \varphi)$$
$$\therefore y = a\sin(\omega t + \varphi)$$

3.1.1.3 SHM & Circular Motion

At any instant,
The displacement along X axis, $ON = OP\cos\theta = a\cos\omega t$
The displacement along Y axis, $OM = OP\sin\theta = a\sin\omega t$



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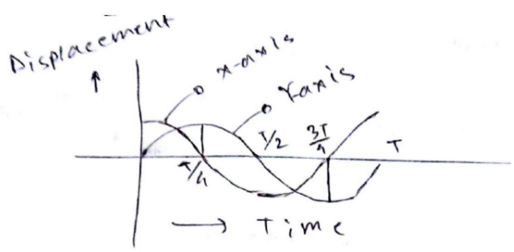


1. An SHM may be regarded as the projection of a rotating vector or a circular motion of the particle of the diameter of the circle or any other line of the circle.
2. The difference between the phases is always $\frac{\pi}{2}$ (or 90°).

3.1.1.4 Mechanical energy of SHM

In the absence of non-conservative forces, total mechanical energy of SHM = Kinetic Energy + Potential Energy = $\frac{1}{2}ka^2$. Prove that, total energy of a simple harmonic motion is constant.
Proof: Let, $y = a\sin(\omega t + \varphi)$ be the equation of the wave.

If dW is the work done to displace the particle by a distance dy against the restoring force F, then

$$\mathrm{d}W = -F\,\mathrm{d}y = -(-ky)\,\mathrm{d}y = ky\,\mathrm{d}y$$

Total work done to displace the particle from 0 to y is,

$$\int \mathrm{d}W = k\int_0^y y\,\mathrm{d}y$$

$$\Rightarrow W = \frac{1}{2}ky^2$$
$$= \frac{1}{2}ka^2\sin^2(\omega t + \varphi)$$

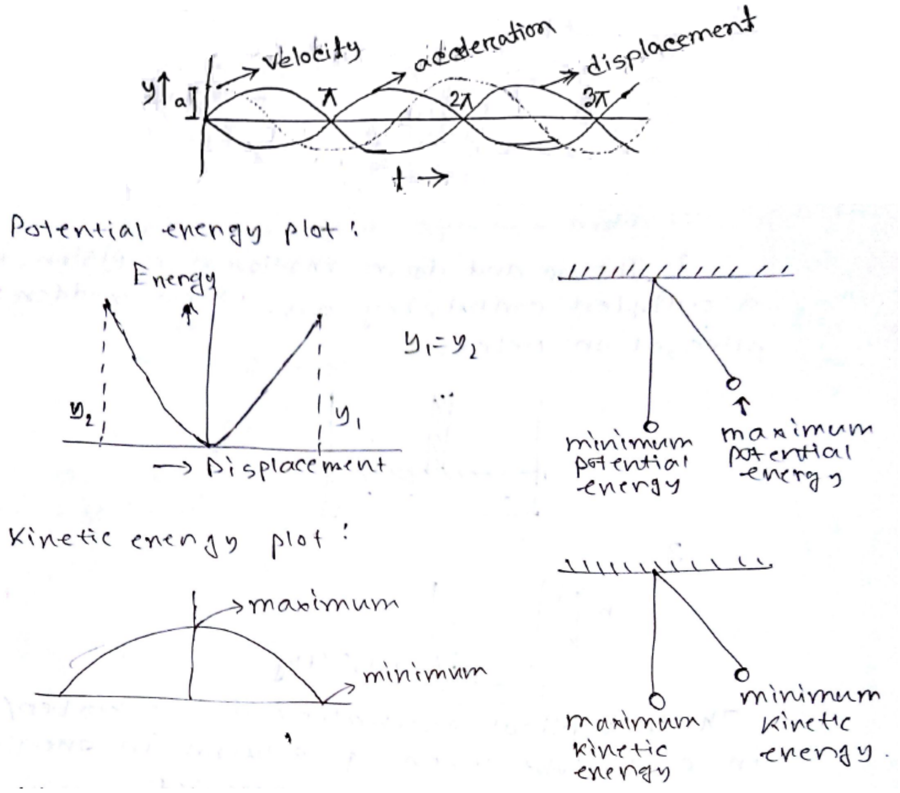
The total work done here is stored in the form of potential energy. Therefore, Potential energy is $\frac{1}{2}ka^2\sin^2(\omega t + \varphi)$

Kinetic Energy

$$
\begin{aligned}
K.E. &= \frac{1}{2}m\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 \\
&= \frac{1}{2}ma^2\omega^2\cos^2(\omega t+\varphi) \\
&= \frac{1}{2}ka^2\cos^2(\omega t+\varphi)
\end{aligned}
$$

Total Energy $= P.E. + K.E.$

$$
\begin{aligned}
&= \frac{1}{2}ka^2\sin^2(\omega t+\varphi)+\frac{1}{2}ka^2\cos^2(\omega t+\varphi) \\
&= \frac{1}{2}ka^2[\sin^2(\omega t+\varphi)+\cos^2(\omega t+\varphi)] \\
&= \frac{1}{2}ka^2
\end{aligned}
$$



At displacement = amplitude, $P.E. = \frac{1}{2}ka^2$ and $K.E. = 0$. At mean position, $P.E. = 0$ and $K.E. = \frac{1}{2}ka^2$. At any position, total energy $= \frac{1}{2}ka^2$.

3.1.1.5 The average value of Kinetic Energy over a complete cycle

$$
\begin{aligned}
&\frac{1}{T}\int_0^T K.E.\,\mathrm{d}t \\
=&\frac{1}{T}\int_0^T \frac{1}{2}m\omega^2a^2\cos^2(\omega t+\varphi)\,\mathrm{d}t \\
=&\frac{m\omega^2a^2}{4T}\int_0^T 2\cos^2(\omega t+\varphi)\,\mathrm{d}t \\
=&\frac{m\omega^2a^2}{4T}\int_0^T [1+\cos 2(\omega t+\varphi)]\,\mathrm{d}t \\
=&\frac{m\omega^2a^2}{4T}\int_0^T \mathrm{d}t+\frac{m\omega^2a^2}{4T}\int_0^T \cos 2(\omega t+\varphi)\,\mathrm{d}t \\
=&\frac{m\omega^2a^2T}{4T}+\frac{m\omega^2a^2}{4T}\left[\frac{\sin 2(\omega t+\varphi)}{2\omega}\right]_0^T \\
=&\frac{1}{4}ka^2+\frac{m\omega^2a^2}{4T}\left[\frac{\sin 2(\omega T+\varphi)}{2\omega}-\frac{\sin 2\varphi}{2\omega}\right] \\
=&\frac{1}{4}ka^2+\frac{m\omega^2a^2}{4T}\left[\frac{\sin 2(\frac{2\pi}{T}\times T+\varphi)}{2\omega}-\frac{\sin 2\varphi}{2\omega}\right]
\end{aligned}
$$

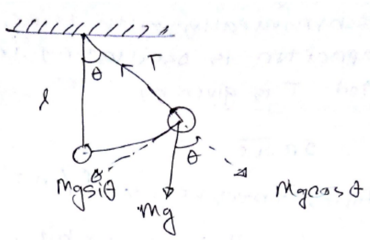
$$
\begin{aligned}
&= \frac{1}{4}ka^2 + \frac{m\omega^2 a^2}{4T}\left[\frac{\sin(4\pi+2\varphi)}{2\omega} - \frac{\sin 2\varphi}{2\omega}\right] \\
&= \frac{1}{4}ka^2 + \frac{m\omega^2 a^2}{4T}\left[\frac{\sin 2\varphi}{2\omega} - \frac{\sin 2\varphi}{2\omega}\right] \\
&= \frac{1}{4}ka^2 + 0 \\
\therefore K.E._{avg} &= \frac{1}{4}ka^2
\end{aligned}
$$

3.1.1.6 The average value of Potential Energy over a complete cycle

$$
\begin{aligned}
&\frac{1}{T}\int_0^T P.E.\,\mathrm{d}t \\
&= \frac{1}{T}\int_0^T \frac{1}{2}m\omega^2 a^2 \sin^2(\omega t+\varphi)\,\mathrm{d}t \\
&= \frac{m\omega^2 a^2}{4T}\int_0^T 2\sin^2(\omega t+\varphi)\,\mathrm{d}t \\
&= \frac{m\omega^2 a^2}{4T}\int_0^T [1-\cos 2(\omega t+\varphi)]\,\mathrm{d}t \\
&= \frac{m\omega^2 a^2}{4T}\int_0^T \mathrm{d}t - \frac{m\omega^2 a^2}{4T}\int_0^T \cos 2(\omega t+\varphi)\,\mathrm{d}t \\
&= \frac{m\omega^2 a^2 T}{4T} - \frac{m\omega^2 a^2}{4T}\left[\frac{\sin 2(\omega t+\varphi)}{2\omega}\right]_0^T \\
&= \frac{1}{4}ka^2 - \frac{m\omega^2 a^2}{4T}\left[\frac{\sin 2(\omega T+\varphi)}{2\omega} - \frac{\sin 2\varphi}{2\omega}\right] \\
&= \frac{1}{4}ka^2 - \frac{m\omega^2 a^2}{4T}\left[\frac{\sin 2\left(\frac{2\pi}{T}\times T+\varphi\right)}{2\omega} - \frac{\sin 2\varphi}{2\omega}\right] \\
&= \frac{1}{4}ka^2 - \frac{m\omega^2 a^2}{4T}\left[\frac{\sin(4\pi+2\varphi)}{2\omega} - \frac{\sin 2\varphi}{2\omega}\right] \\
&= \frac{1}{4}ka^2 - \frac{m\omega^2 a^2}{4T}\left[\frac{\sin 2\varphi}{2\omega} - \frac{\sin 2\varphi}{2\omega}\right] \\
&= \frac{1}{4}ka^2 - 0 \\
\therefore P.E._{avg} &= \frac{1}{4}ka^2
\end{aligned}
$$

3.1.1.7 Examples of SHM

- **Motion of pendulum:**
  The force is acting upon vertically and will be resolved into two components.
  1. $Mg\cos\theta \rightarrow$ acting towards the length of the pendulum
  2. $Mg\sin\theta \rightarrow$ acting perpendicular to the length of the pendulum
  $Mg\cos\theta$ will be balanced by the tension, T. So, the only force acting on the pendulum is, $F = Mg\sin\theta$



$$If\ \theta \leq 4^\circ,\ then,\ \sin\theta = \theta \qquad \left[\sin\theta = \theta - \tfrac{\theta^3}{3!} + \tfrac{\theta^5}{5!} - \ldots\right]$$

$\therefore$ the restoring force, $F = Mg\theta$

Linear Displacement, $x = l\theta$

$$\therefore \text{Acceleration, } a = l\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$$

$$\therefore F = Ml\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$$

$$\therefore Ml\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -Mg\theta \qquad \text{(since, they are acting in opposite direction)}$$

$$\Rightarrow l\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + g\theta = 0$$

$$\Rightarrow \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{l}\theta = 0$$

$$\Rightarrow \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \omega^2\theta = 0 \qquad \text{where, } \omega = \sqrt{\tfrac{g}{l}}$$

This is similar to the differential equation of SHM. Now,

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}} \qquad n = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$$

- **Motion of a body suspended from a coil spring:**
As $F = -k\Delta l$ and $F = mg$. These forces are equal and working on opposite direction.

$$\therefore\ k\Delta l = mg \rightarrow k = \frac{mg}{\Delta l}$$

After expansion, $F = k(\Delta l - x) - mg = -kx$
According to Newton's second law of motion,

$$F = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx$$

$$\Rightarrow \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{k}{m}x = 0$$

$$\Rightarrow \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \omega^2 x = 0 \qquad \text{where, } \omega = \sqrt{\tfrac{k}{m}}$$

Hence, the motion of a mass suspended from a coil spring, is SHM.

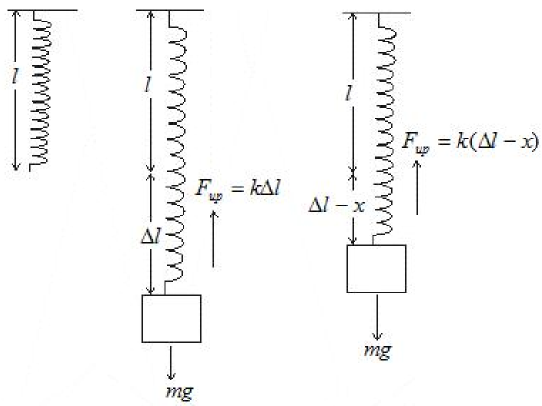
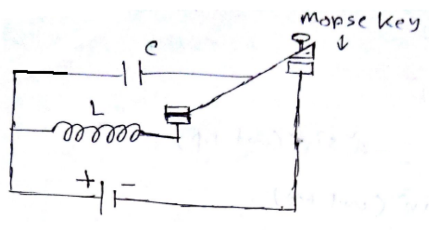


Figure 6(a) Figure 6(b) Figure 6(c)

$$\therefore\ T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

- **LC Circuit:**
When the key is pressed, the capacitor gets directly connected to the battery and thus gets changed. On being released, the key gets disconnected to the inductance coil. The capacitor, thus, discharges itself through the inductance coil. If Q be the charge on the capacitor, then the voltage across capacitor $V_C = \frac{Q}{C}$ will drive current through the inductor. *EMF* induced in the inductor will be $V_L = L\frac{\mathrm{d}i}{\mathrm{d}t}$.



Since, there is no external *EMF* in the circuit (the battery being cut-off), then according to *KVL*, total/net *EMF* in the circuit is,

$$V_C + V_L = 0$$

$$\Rightarrow \frac{Q}{C} + L\frac{\mathrm{d}i}{\mathrm{d}t} = 0 \tag{3.1}$$

But current is the rate of flow of charge, hence, $i = \frac{\mathrm{d}Q}{\mathrm{d}t}$.

$$\therefore \frac{Q}{C} + L\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} = 0$$

$$\Rightarrow \frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + \frac{Q}{LC} = 0$$

$$\Rightarrow \frac{d^2Q}{dt^2} + \omega^2 Q = 0 \tag{3.2}$$

$$where,\ \omega = \sqrt{\frac{1}{LC}} \text{ is constant}$$

Thus, the charge of the capacitor oscillates simple harmonically with time, *i.e.* the discharge of the capacitor is oscillatory in nature. Its time period T is given by,

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{LC}$$

And therefore, the frequency, $f = \frac{1}{T} = \frac{1}{2\pi\sqrt{LC}}$. The solution of Equation 3.2 is given by,

$$Q = Q_0 \sin(\omega t + \varphi)$$
$$\therefore\ i = \frac{dQ}{dt} = \omega Q_0 \cos(\omega t + \varphi)$$

i will be maximum where $cos(\omega t + \varphi) = 1$. Considering this maximum value by i_0, we get,

$$i = i_0 \cos(\omega t + \varphi) \qquad where,\ i_0 = \omega Q_0$$

Thus, this is an example of SHM.

Why does capacitor blocks DC current?

A capacitor can store charge as it has two electrodes with di-electric media in between. It does not allow any current to pass through it. It stores energy as a function of voltage and changes in voltage by drawing current from, or by supplying current to the source of the voltage change, in opposition to change.

DC has a constant voltage. If constant voltage is applied to a capacitor, it charges upto some value decided by the voltage and capacitance. The current that is allowed to pass through by the capacitor is zero.

3.1.2 Composition of Simple Harmonic Motions and Lissajous Figures

3.1.2.1 Lissajous Figures

When a particle is influenced simultaneously by two simple harmonic motions at right angles to each other, the resultant motion of the particle traces a curve. These curves are called Lissajous figures.

Applications:

- To determine the ratio of time period of two vibrations.
- To determine/compare the frequencies of two tuning forks.

3.1.2.2 Composition of two SHMs of same frequency but different phases and amplitudes

Let, the individual displacements are,

$$y_1 = a_1 \sin(\omega t + \alpha_1) \tag{3.3}$$
$$y_2 = a_2 \sin(\omega t + \alpha_2) \tag{3.4}$$

The resultant displacement,

$$\begin{aligned} y &= y_1 + y_2 \\ &= a_1 \sin(\omega t + \alpha_1) + a_2 \sin(\omega t + \alpha_2) \\ &= a_1(\sin\omega t \cos\alpha_1 + \cos\omega t \sin\alpha_1) + a_2(\sin\omega t \cos\alpha_2 + \cos\omega t \sin\alpha_2) \\ &= \sin\omega t(a_1\cos\alpha_1 + a_2\cos\alpha_2) + \cos\omega t(a_1\sin\alpha_1 + a_2\sin\alpha_2) \end{aligned}$$

$$\text{Putting, } a_1\cos\alpha_1 + a_2\cos\alpha_2 = A\cos\varphi \tag{3.5}$$
$$a_1\sin\alpha_1 + a_2\sin\alpha_2 = A\sin\varphi \tag{3.6}$$
$$\therefore\ y = A\sin\omega t\cos\varphi + A\cos\omega t\sin\varphi = A\sin(\omega t + \varphi) \tag{3.7}$$

So, the resultant equation is simple harmonic and very similar to Equation 3.5 and Equation 3.6. Adding the square of these equations,

$$A^2\cos^2\varphi + A^2\sin^2\varphi = a_1^2\cos^2\alpha_1 + a_2^2\cos^2\alpha_2 + 2a_1a_2\cos\alpha_1\cos\alpha_2 +$$

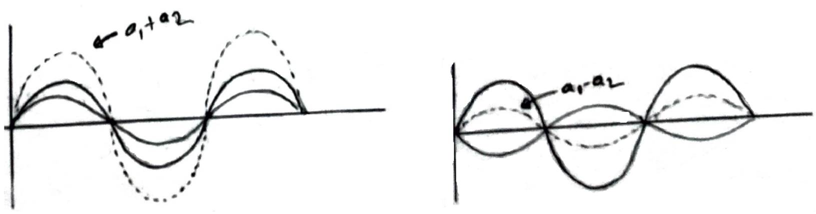
$$a_1^2 \sin^2 \alpha_1 + a_2^2 \sin^2 \alpha_2 + 2a_1 a_2 \sin \alpha_1 \sin \alpha_2$$
$$\Rightarrow A^2 = a_1^2 + a_2^2 + 2a_1 a_2 \cos(\alpha_1 - \alpha_2)$$
$$\Rightarrow A = \sqrt{a_1^2 + a_2^2 + 2a_1 a_2 \cos(\alpha_1 - \alpha_2)}$$

Dividing (3.6) by (3.5)

$$\tanh \varphi = \frac{a_1 \sin \alpha_1 + a_2 \sin \alpha_2}{a_1 \cos \alpha_1 + a_2 \cos \alpha_2}$$
$$\therefore \varphi = \tan^{-1} \frac{a_1 \sin \alpha_1 + a_2 \sin \alpha_2}{a_1 \cos \alpha_1 + a_2 \cos \alpha_2}$$

Special cases:

1. If the two vibrations are in same phase, *i.e.* $\alpha_1 - \alpha_2 = 0, 2\pi, 4\pi, ..., 2n\pi$, then, $A = a_1 + a_2$
2. If the two vibrations are in opposite phases, *i.e.* $\alpha_1 - \alpha_2 = \pi, 3\pi, 5\pi, ..., (2n+1)\pi$, then, $A = a_1 - a_2$



3.1.2.3 Composition of two SH vibrations at 90° angle to each other having equal frequencies but different phases and amplitudes

Let the two equation be

$$x = a \sin(\omega t + \varphi) \qquad (\varphi \text{ is the difference between two vibrations})$$
$$\Rightarrow \frac{x}{a} = \sin \omega t \cos \varphi + \cos \omega t \sin \varphi$$
$$and,\ y = b \sin \omega t \Rightarrow \frac{y}{b} = \sin \omega t$$
$$\therefore \frac{x}{a} = \frac{y}{b} \cos \varphi + \sqrt{1 - \frac{y^2}{b^2}} \sin \varphi$$
$$\Rightarrow \frac{x}{a} - \frac{y}{b} \cos \varphi = \sqrt{1 - \frac{y^2}{b^2}} \sin \varphi$$
$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} \cos^2 \varphi - 2\frac{xy}{ab} \cos \varphi = \sin \varphi - \frac{y^2}{b^2} \sin^2 \varphi$$
$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} - 2\frac{xy}{ab} \cos \varphi = \sin \varphi \tag{3.8}$$

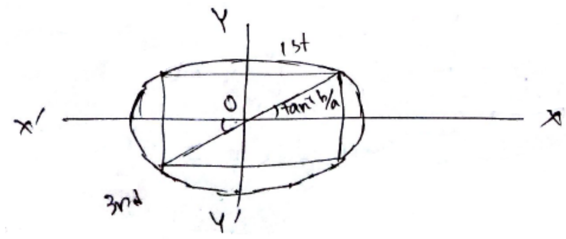
This is the general equation of an eclipse whose shape will depend upon the value of the phase difference of the two vibrations.

Special cases:

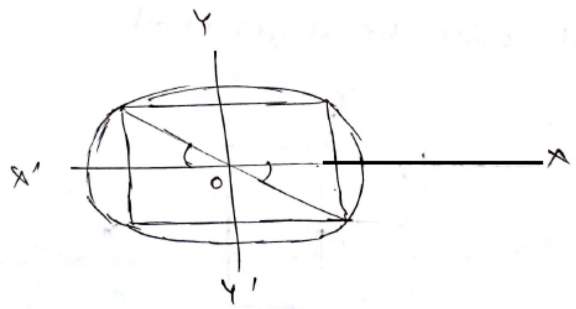
1. If the vibrations are in same phase, *i.e.*, $\varphi = 0, 2\pi, 4\pi, ..., 2n\pi$, then, $\sin \varphi = 0$ and $\cos \varphi = 1$, and Equation 3.8 becomes,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 2\frac{xy}{ab} = 0$$
$$\Rightarrow \left(\frac{x}{a} - \frac{y}{b}\right)^2 = 0$$
$$\Rightarrow \frac{x}{a} - \frac{y}{b} = 0$$
$$\Rightarrow y = \frac{b}{a} x$$

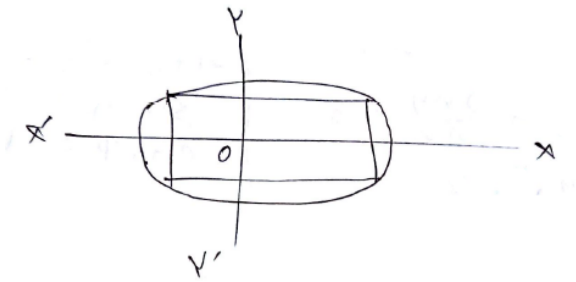
This is the equation of a straight line. Here, amplitude $= \sqrt{a^2 + b^2}$ and angle $= \tan^{-1} \frac{b}{a}$.



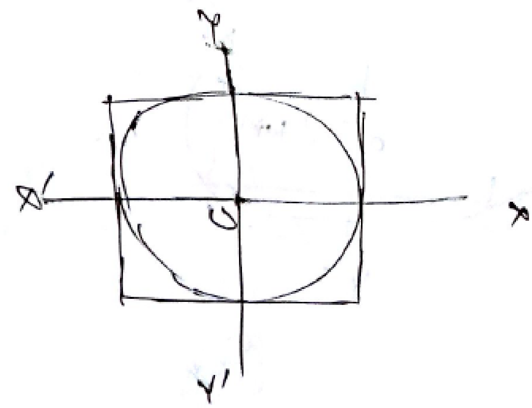
(a) Special case-1



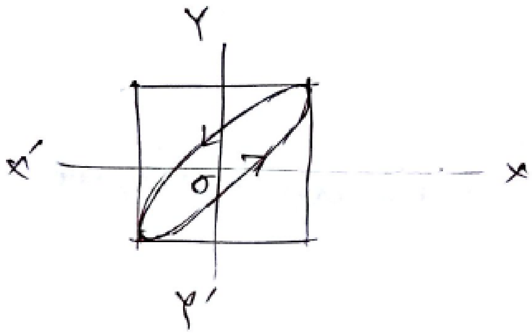
(b) Special case-2



(c) Special case-3



(d) Special case-3 (when $a = b$)



(e) Special case-4

Figure 3.2: Lissajous' figures for 3.1.2.3

2. If $\varphi = \pi, 3\pi, 5\pi, ..., (2n+1)\pi$, then, $\sin\varphi = 0$ and $\cos\varphi = -1$, and Equation 3.8 becomes,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + 2\frac{xy}{ab} = 0$$
$$\Rightarrow \left(\frac{x}{a} + \frac{y}{b}\right)^2 = 0$$
$$\Rightarrow \frac{x}{a} + \frac{y}{b} = 0$$
$$\Rightarrow y = -\frac{b}{a}x$$

This also represents the equation of a straight line. Here, amplitude $= \sqrt{a^2+b^2}$ and angle $= \tan^{-1}\left(-\frac{b}{a}\right)$.

3. When, $\varphi = \frac{\pi}{2}$, then, $\sin\varphi = 1$ and $\cos\phi = 0$. Equation 3.8 becomes-

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

This is the equation of a symmetrical eclipse. If $a = b$, then, $x^2 + y^2 = a^2$, which is the general equation of circle.

4. When, $\varphi = \frac{\pi}{4}$, then, $\sin\varphi = \cos\phi = \frac{1}{\sqrt{2}}$. Equation 3.8 becomes-

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\frac{1}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}\right)^2$$
$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{\sqrt{2}xy}{ab} = \frac{1}{2}$$

This is the equation of an oblique eclipse.

3.1.2.4 Composition of two SHMs at right angles to each other and having time periods in the ratio 1:2

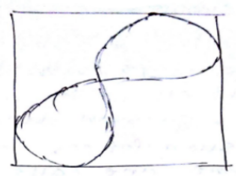
Watch this tutorial if necessary.

$$y = b\sin\omega t \Rightarrow \frac{y}{b} = \sin\omega t$$
$$x = a\sin(2\omega t + \varphi)$$
$$\Rightarrow \frac{x}{a} = \sin(2\omega t + \varphi) = \sin(2\omega t)\cos\varphi + \cos(2\omega t)\sin\varphi$$
$$= 2\sin\omega t\cos\omega t\cos\varphi + (1 - 2\sin^2\omega t)\sin\varphi$$
$$= \frac{2y}{b}\sqrt{1 - \frac{y^2}{b^2}}\cos\varphi + \left(1 - \frac{2y^2}{b^2}\right)\sin\varphi$$
$$= \frac{2y\sqrt{b^2 - y^2}}{b^2}\cos\varphi + \left(1 - \frac{2y^2}{b^2}\right)\sin\varphi$$
$$\Rightarrow \frac{x}{a} - \sin\varphi + \frac{2y^2}{b^2}\sin\varphi = \frac{2y\sqrt{b^2 - y^2}}{b^2}\cos\varphi$$
$$\Rightarrow \left(\frac{x}{a} - \sin\varphi\right)^2 + \frac{4y^2}{b^2}\left(\frac{x}{a} - \sin\varphi\right)\sin\varphi + \frac{4y^4}{b^4}\sin^2\varphi = \frac{4y^2(b^2 - y^2)}{b^4}\cos^2\varphi$$
$$\Rightarrow \left(\frac{x}{a} - \sin\varphi\right)^2 + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2} + \frac{x}{a}\sin\varphi - 1\right) = 0 \tag{3.9}$$

Special cases:

1. When $\varphi = 0, 2\pi, 4\pi, \ldots, 2n\pi$, then $\sin\varphi = 0$ and Equation 3.9 becomes,

$$\frac{x^2}{a^2} + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2} - 1\right) = 0$$



The resultant motion gives the figure of eight.

2. When $\varphi = \frac{\pi}{2}$, then $\sin\varphi = 1$ and Equation 3.9 becomes,

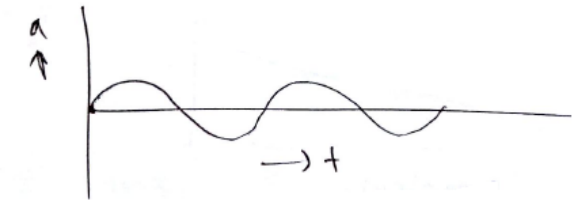
$$\left(\frac{x}{a} - 1\right)^2 + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2} + \frac{x}{a} - 1\right) = 0$$

This is the equation of parabola with vertex $(a, 0)$.

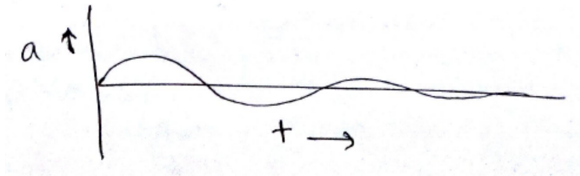
3.1.3 Damped Oscillations

If vibration of a body vibrates without any external force impressed upon it, then it is called **free vibration**. In actual practice, a simple harmonic oscillation almost always vibrates in a resisting medium, like air, oil etc. The amplitude of vibration, therefore, goes on decreasing progressively with time. Such forces, which are non-conservative in nature, are called free damped vibrations.

In the absence of any dissipative forces, the amplitude of oscillations depends upon time and gradually decreases to zero with time and finally, the oscillation die out. Such type of vibrations are called damped free vibration.



No damping vibration



Damped Vibration

3.1.3.1 Differential Equation of a damped vibration

A harmonic oscillation oscillating in a damping medium, will be simultaneously subjected to the restoring force and the Damping/resistive force. If there is nothing but restoring force, then there will be no damping vibration. But if there is air/oil/friction/resistance/resistive force, then the oscillation eventually dies out.
Let the equation of SHM,

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+\omega^2x=0 \qquad F_R \text{ (restoring force)} \propto x \text{ (displacement)}$$

$$F_D \propto \frac{\mathrm{d}y}{\mathrm{d}t} \qquad \textit{i.e.} \text{ damping force} \propto \text{velocity}$$

If there is no loss of energy of a pendulum by friction or otherwise, the amplitude of oscillation will be independent of time and the body will go on oscillating with the same time period for any length of time. This type of oscillations will be undamped free vibration.
On the other hand, in the presence of resistive forces, the amplitude of oscillation depends upon time and gradually decreases with time and finally the oscillation dies out. Such vibrations are called free damped vibration. The equation of motion of a particle executing SHM in a damping medium will be,

$$m\frac{\mathrm{d}^2y}{\mathrm{d}t^2}+b\frac{\mathrm{d}y}{\mathrm{d}t}+ay=0$$

Rearranging, we get,

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2}+\frac{b}{m}\frac{\mathrm{d}y}{\mathrm{d}t}+\frac{a}{m}y=0$$

$$\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}t^2}+2\lambda\frac{\mathrm{d}y}{\mathrm{d}t}+\omega^2y=0$$

Notes:

$$\frac{b}{m}=2\lambda \quad \omega=\sqrt{\frac{a}{m}}$$

λ = damping factor/constant

a = stiffness constant

$$F_R \propto y \quad F_D \propto \frac{\mathrm{d}y}{\mathrm{d}t}$$

This is the differential equation of damped oscillation. It is a 2nd order linear homogeneous equation.

Trial solution,

$$y=Ae^{kt}$$

$$\Rightarrow y'=kAe^{kt} \qquad \text{differentiating w.r.t. } t$$

$$\Rightarrow y''=k^2Ae^{kt} \qquad \text{differentiating w.r.t. } t$$

Auxiliary equation,

$$k^2Ae^{kt}+2\lambda kAe^{kt}+\omega^2Ae^{kt}=0$$

$$\Rightarrow (k^2+2\lambda k+\omega^2)Ae^{kt}=0$$

$$\Rightarrow k^2+2\lambda k+\omega^2=0 \qquad \text{since } Ae^{kt}\neq 0$$

$$\Rightarrow k=\frac{-2\lambda\pm\sqrt{4\lambda^2-4\omega^2}}{2}=-\lambda\pm\sqrt{\lambda^2-\omega^2}$$

Hence, there are two solutions.

$$y_1=A_1e^{\left(-\lambda+\sqrt{\lambda^2-\omega^2}\right)t} \qquad y_2=A_2e^{\left(-\lambda-\sqrt{\lambda^2-\omega^2}\right)t}$$

The constants A_1 and A_2 are determined from the initial conditions. Therefore, the general solution,

$$y=y_1+y_2=A_1e^{\left(-\lambda+\sqrt{\lambda^2-\omega^2}\right)t}+A_2e^{\left(-\lambda-\sqrt{\lambda^2-\omega^2}\right)t} \tag{3.10}$$

$$\Rightarrow y'=A_1\left(-\lambda+\sqrt{\lambda^2-\omega^2}\right)e^{\left(-\lambda+\sqrt{\lambda^2-\omega^2}\right)t}+A_2\left(-\lambda-\sqrt{\lambda^2-\omega^2}\right)e^{\left(-\lambda-\sqrt{\lambda^2-\omega^2}\right)t} \tag{3.11}$$

Let, the maximum value of the displacement be, $y_{max}=a_0$ at $t=0$. From Equation 3.10

$$y_{max}=A_1e^0+A_2e^0=A_1+A_2=a_0 \tag{3.12}$$

Therefore, the velocity is zero at maximum displacement. From Equation 3.11

$$
\begin{aligned}
0 &= A_1\left[-\lambda+\sqrt{\lambda^2-\omega^2}\right]+A_2\left[-\lambda-\sqrt{\lambda^2-\omega^2}\right]\\
\Rightarrow &-(A_1+A_2)\lambda+(A_1-A_2)\sqrt{\lambda^2-\omega^2}\\
\Rightarrow A_1-A_2 &= \frac{\lambda(A_1+A_2)}{\sqrt{\lambda^2-\omega^2}}=\frac{\lambda a_0}{\sqrt{\lambda^2-\omega^2}} \qquad (3.13)
\end{aligned}
$$

Solving Equation 3.12 and 3.13

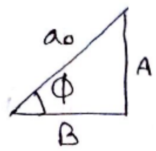
$$
\begin{aligned}
A_1 &= \frac{1}{2}a_0\left[1+\frac{\lambda}{\sqrt{\lambda^2-\omega^2}}\right]\\
A_2 &= \frac{1}{2}a_0\left[1-\frac{\lambda}{\sqrt{\lambda^2-\omega^2}}\right]\\
\therefore y_1 &= \frac{1}{2}a_0\left[1+\frac{\lambda}{\sqrt{\lambda^2-\omega^2}}\right]e^{\left(-1+\sqrt{\lambda^2-\omega^2}\right)t}\\
y_2 &= \frac{1}{2}a_0\left[1-\frac{\lambda}{\sqrt{\lambda^2-\omega^2}}\right]e^{\left(-1-\sqrt{\lambda^2-\omega^2}\right)t}
\end{aligned}
$$

These are main solutions.

Cases:

1. When $\lambda < \omega$, then the term $\sqrt{\lambda^2-\omega^2}$ will be imaginary.

$$
\begin{aligned}
\therefore \sqrt{\lambda^2-\omega^2} &= i\sqrt{\omega^2-\lambda^2}\\
&= ig \qquad\qquad g=\sqrt{\omega^2-\lambda^2}\\
\therefore y &= e^{-\lambda t}\left[A_1e^{igt}+A_2e^{-igt}\right]\\
&= e^{-\lambda t}\left[A_1(\cos gt+i\sin gt)+A_2(\cos gt-i\sin gt)\right]\\
&= e^{-\lambda t}\left[(A_1+A_2)\cos gt+i(A_1-A_2)\sin gt\right]\\
&= e^{-\lambda t}[A\cos gt+B\cos gt]\\
&= e^{-\lambda t}\left[a_0\frac{A}{a_0}\cos gt+a_0\frac{B}{a_0}\sin gt\right]\\
&= a_0e^{-\lambda t}[\cos gt\sin\varphi+\sin gt\cos\varphi]\\
&= a_0e^{-\lambda t}\sin(gt+\varphi)
\end{aligned}
$$

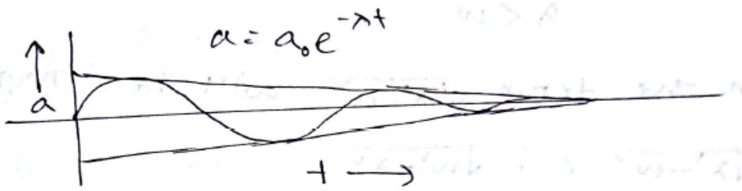


$\therefore$ Amplitude, $a=a_0e^{-\lambda t}$ and frequency, $f=\dfrac{g}{2\pi}=\dfrac{\sqrt{\omega^2-\lambda^2}}{2\pi}$

Here, damping has two effects:

- The amplitude will decay exponentially with time to zero.
- Damping decreases frequency of the oscillator.

Solution is oscillatory but frequency and amplitude will decay exponentially.



2. When $\lambda=\omega, \sqrt{\lambda^2-\omega^2}$, each of the two terms of the R.H.S. of the solution become infinite and the solution breaks down. If λ^2 is nearly equal to ω^2, *i.e.* $\sqrt{\lambda^2-\omega^2}=h$, where h is very small,

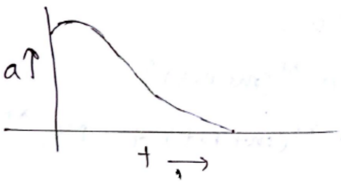
$$y=A_1e^{(-\lambda+h)t}+A_2e^{(-\lambda-h)t}$$

$$
\begin{aligned}
&= e^{-\lambda t}\left(A_1 e^{ht} + A_2 e^{-ht}\right) \\
&= e^{-\lambda t}\left[A_1\left(1 + ht + \frac{h^2t^2}{2!} + \ldots\right) + A_2\left(1 - ht + \frac{h^2t^2}{2!} - \ldots\right)\right] && \left(e^x = 1 + x + \tfrac{x^2}{2!} + \tfrac{x^3}{3!} + \ldots\right) \\
&= e^{-\lambda t}[A_1(1+ht) + A_2(1-ht)] && (h \text{ is very small, so we can ignore higher order terms}) \\
&= e^{-\lambda t}[A_1 + A_2 + (A_1 + A_2)ht] \\
&= e^{-\lambda t}(M + Nt) && (3.14)
\end{aligned}
$$

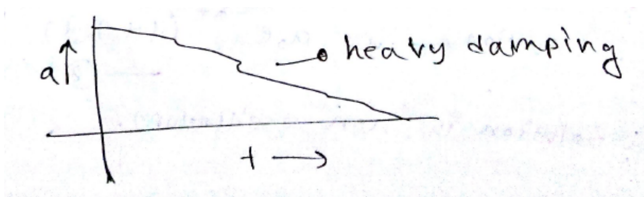
At $t = 0, y = y_{max} = a_0 = M$. Now,

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\lambda t}(M + Nt)\right) \\
&= -\lambda e^{-\lambda t}(M + Nt) + Ne^{-\lambda t} \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \; at \; t = 0 \\
\therefore\; &-\lambda M + N = 0 \Rightarrow N = \lambda a_0 && (M = a_0) \\
\therefore\; y &= a_0 e^{-\lambda t}(1 + \lambda t)
\end{aligned}
$$

In Equation 3.14, the second term decays less rapidly than the first term. As t increases the exponential term becomes more dominant and the displacement reaches the value 0 for a finite value of t. This is called the cause of critical damping or dead bit. Application: pointer type instruments.

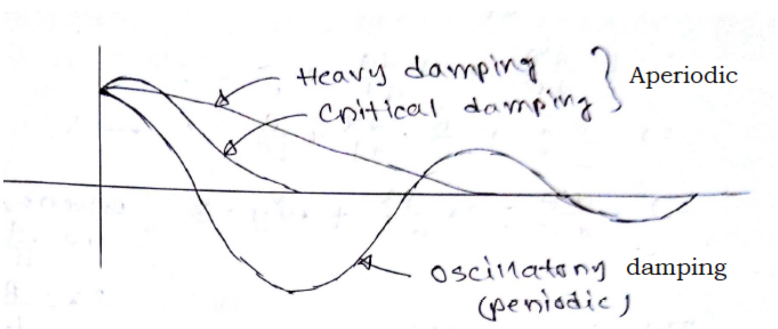


Case 2



Case 3

3. When $\lambda > \omega, \sqrt{\lambda^2 - \omega^2}$ is a real quantity. Then there will occur over damping. Then the motion is not periodic but aperiodic. The term $e^{-\lambda t}$ dominate and displacement goes to zero as t becomes infinite.



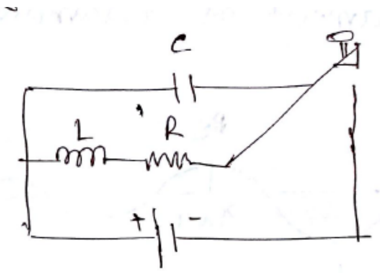
For all three cases

LCR circuit is an oscillatory damper.

3.1.3.2 LC Circuit in Damped Medium

When $R = 0$, for LC Circuit

$$\frac{Q}{C} + L\frac{\mathrm{d}i}{\mathrm{d}t} = 0$$



That is simple harmonic. But when, $R \neq 0$ then,

$$L\frac{\mathrm{d}i}{\mathrm{d}t} + Ri + \frac{Q}{C} = 0$$

This is the differential equation for LCR circuit in damped medium.
But $i = \frac{\mathrm{d}Q}{\mathrm{d}t}$

$$L\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + R\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{Q}{c} = 0$$

$$\Rightarrow \frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{Q}{LC} = 0$$

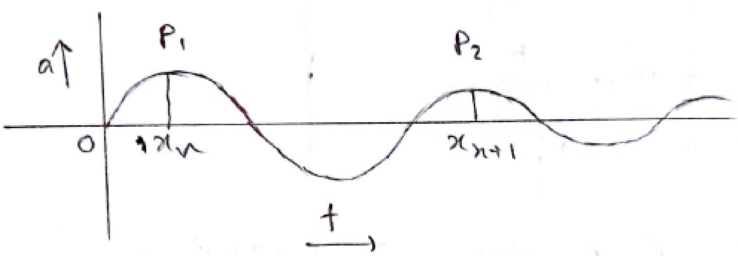
$$\Rightarrow \frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + 2\lambda\frac{\mathrm{d}Q}{\mathrm{d}t} + \omega^2 Q = 0 \qquad \text{(where, } 2\lambda = \tfrac{R}{L} \text{ and } \omega = \sqrt{\tfrac{1}{LC}}\text{)}$$

3.1.3.3 Effect of Damping

Damping clearly produces two effects.

1. The frequency of the damped harmonic oscillator $(\frac{g}{2\pi})$ is smaller than its natural, or undamped frequency, *i.e.* damping decreases frequency, or increases time period. In actual practice, the damping is small in majority of cases and its effect on frequency is negligible, *e.g.* musical instrument.
2. The amplitude of the oscillation does not remain constant at a_0, which represents amplitude in the absence of any damping, but decays exponentially with time to zero, in accordance with the term $e^{-\lambda t}$, which is called the **damping factor**.

3.1.3.4 Logarithmic decrement of a damped oscillator



The logarithmic decrement gives a measure of the rate at which the amplitude of vibration decreases with time. If P_1 and P_2 are successive maxima at displacements x_n and x_{n+1} separated by a time period $\frac{2\pi}{\omega}$. The maxima occurs at P_1 at time t, and minima occurs at P_2 at time $t + \frac{2\pi}{\omega}$.

$$\therefore x_n = a_0 e^{-\lambda t} \qquad x_{n+1} = a_0 e^{-\lambda\left(t + \frac{2\pi}{\omega}\right)}$$

Therefore,

$$\frac{x_{n+1}}{x_n} = \frac{a_0 e^{-\lambda\left(t+\frac{2\pi}{\omega}\right)}}{a_0 e^{-\lambda t}} = e^{-\lambda\frac{2\pi}{\omega}} = e^{-\lambda T}$$

$$\therefore -\ln\left(\frac{x_{n+1}}{x_n}\right) = \lambda T = \delta$$

This defines the logarithmic decrement (δ) of a damp oscillation as the natural logarithm of the ratio of amplitudes at time t and $t + T$.

3.1.3.5 Relaxation time

The time required to elapse for the amplitude of a damp oscillator to decay to $\frac{1}{e}th$ of its original amplitude a_0. According to definition,

$$\begin{aligned}
&a = a_0 e^{-\lambda t} \\
\Rightarrow &\frac{a_0}{e} = a_0 e^{-\lambda t} \\
\Rightarrow &e^{-1} = e^{-\lambda t} \\
\Rightarrow &t = \frac{1}{\lambda} = \tau && \text{(where } \tau \text{ is relaxation time)}
\end{aligned}$$

3.1.3.6 Power dissipation of a damped oscillator

We know, the displacement of an oscillator (considering under damping), $y = a_0 e^{-\lambda t} \sin(gt+\varphi)$

$$\begin{aligned}
\therefore velocity,\ v = \frac{\mathrm{d}y}{\mathrm{d}t} &= -\lambda a_0 e^{-\lambda t} \sin(gt+\varphi) + a_0 g e^{-\lambda t} \cos(gt+\varphi) \\
&= -a_0 e^{-\lambda t}[g\cos(gt+\varphi) - \lambda \sin(gt+\varphi)] \\
\therefore K.E. &= \frac{1}{2} m v^2 \\
&= \frac{1}{2} m a_0^2 e^{-2\lambda t}\left[g^2\cos^2(gt+\varphi) + \lambda^2 \sin^2(gt+\varphi) - 2g\lambda\cos(gt+\varphi)\sin(gt+\varphi)\right]
\end{aligned}$$

The average value of $\sin(gt+\varphi)$ and $\cos(gt+\varphi)$ over a complete cycle is 0. If λ is small, we can neglect the last term in the right hand side. Then the average value of $\sin^2(gt+\varphi)$ over a complete cycle

$$\int_0^T \frac{\sin^2(gt+\varphi)}{T} dt = \int_0^T \frac{\cos^2(gt+\varphi)}{T} dt = \frac{1}{2}$$

Therefore, the average K.E. of a damped oscillator,

$$\begin{aligned}
&= \frac{1}{2} m a_0^2 e^{-2\lambda t}\left[\frac{1}{2}g^2 + \frac{1}{2}\lambda^2 - 0\right] \\
&= \frac{1}{4} m a_0^2 g^2 e^{-2\lambda t} && \text{(If damping is very small, } i.e.\ \lambda \approx 0) \\
\therefore P.E. &= \int F\,\mathrm{d}y = \frac{1}{2} m\omega^2 y^2 \\
&= \frac{1}{2} m\omega^2 a_0^2 e^{-2\lambda t} \sin^2(gt+\varphi) \\
&= \frac{1}{4} m\omega^2 a_0^2 e^{-2\lambda t} \\
&= \frac{1}{4} m g^2 a_0^2 e^{-2\lambda t} && \text{(Since damping is small, } g = \omega) \\
\therefore \text{Average total energy } &= P.E. + K.E. \\
&= \frac{1}{4} m g^2 a_0^2 e^{-2\lambda t} + \frac{1}{4} m g^2 a_0^2 e^{-2\lambda t} \\
&= \frac{1}{2} m g^2 a_0^2 e^{-2\lambda t} \\
E &= E_0 e^{-2\lambda t} && \text{(where, } E_0 = \tfrac{1}{2} m g^2 a_0^2)
\end{aligned}$$

where, E_0 is the total average energy of the undamped oscillator, which is similar to $a = a_0 e^{-\lambda t}$. This equation shows that the energy of a damped oscillator decreases at a faster rate $(e^{-2\lambda t})$ than its amplitude, which decay at a slower rate $(e^{-\lambda t})$.

Problem: Show that the decay of energy is faster than its amplitude.

Solution: Prove the two equations above. Therefore, the energy decreases exponentially at a faster rate than the amplitude which decays at the rate.

Average power dissipation-

$$\begin{aligned}P &= \text{rate of loss of energy}\\ &= -\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}\left(E_0 e^{-2\lambda t}\right)\\ &= 2\lambda E_0 e^{-2\lambda t} = 2\lambda E\end{aligned}$$

Hence, as λ increases, loss increases. The loss of energy is due to the alter against the dissipative force and appears in the form of heat.

3.1.3.7 Quality Factor

$$Q = \frac{2\pi \times \text{Energy Stored}}{\text{Energy loss per period}} = \frac{2\pi E}{\frac{P}{T}} = \frac{2\pi E}{2\lambda E \frac{2\pi}{g}} = \frac{g}{2\lambda}$$

If λ is small, $g \approx w$, then, $Q = \frac{\omega}{2\lambda} = \frac{\omega\tau}{2}$
The quality factor, as the name suggests, measures the quality of a harmonic oscillator in so far damping is concerned. The smaller the damping, the greater/the better the quality of the oscillator.

$$\begin{aligned}Since,\ \omega &= \sqrt{\frac{a}{m}}\ and\ 2\lambda = \frac{b}{m}\\ Hence,\ Q &= \frac{\omega}{2\lambda} = \sqrt{\frac{a}{m}} \times \frac{b}{m}\\ &= \frac{\sqrt{am}}{b}\end{aligned}$$

When $\lambda = 0$ (no damping, which does not happen), $Q = \infty$ which is not possible in reality.

- $Q < \frac{1}{2} \rightarrow$ overdamped
- $Q = \frac{1}{2} \rightarrow$ critical damping
- $Q > \frac{1}{2} \rightarrow$ better quality (can be upto 10^{14})

3.1.4 Forced Vibration

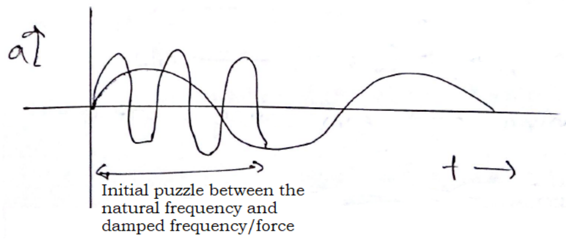
When vibrating system is subjected to an external periodic force, the system is said to be in a state of forced vibration. Initially, the body will vibrate with its own natural frequency, but in a very short time, the normal frequency dies out due to damping force and the body will oscillate with the frequency of the applied force. Such vibrations are called forced vibrations.
Let, the periodic force applied to a damped oscillator be,

$$F = F_0 \sin pt \qquad \frac{p}{2\pi}\text{is the frequency of the applied force}$$

Hence, the equation of motion will be,

$$\begin{aligned}&m\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -b\frac{\mathrm{d}y}{\mathrm{d}t} - ay + F\\ \Rightarrow &\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{b}{m}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{a}{m}y = \frac{F_0}{m}\sin pt\\ \Rightarrow &\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 2\lambda\frac{\mathrm{d}y}{\mathrm{d}t} + \omega^2 y = f_0 \sin pt \qquad (3.15)\end{aligned}$$

This is the differential equation of forced vibration which is a non-homogeneous differential equation.



Let, a particular solution of Equation 3.15 after steady state has been reached,

$$y = A\sin(pt-\theta) \tag{3.16}$$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = pA\cos(pt-\theta) \qquad \text{(differentiating w.r.t } t\text{)}$$

$$\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -p^2\sin(pt-\theta) \qquad \text{(differentiating w.r.t } t\text{)}$$

Putting these in Equation 3.15,

$$-p^2A\sin(pt-\theta)+2\lambda pA\cos(pt-\theta)+\omega^2A\sin(pt-\theta) = f_0\sin(pt-\theta+\theta)$$
$$\Rightarrow A(\omega^2-p^2)\sin(pt-\theta)+2\lambda pA\cos(pt-\theta) = f_0\sin(pt-\theta)\cos\theta+f_0\sin\theta\cos(pt-\theta)$$

Equating the coefficients of $\sin(pt-\theta)$ and $\cos(pt-\theta)$ from both sides,

$$A(\omega^2-p^2) = f_0\cos\theta \tag{3.17}$$
$$2\lambda pA = f_0\sin\theta \tag{3.18}$$

Adding the squares of Equation 3.17 and Equation 3.18,

$$A = \frac{f_0}{\sqrt{(\omega^2-p^2)^2+4\lambda^2p^2}}$$

Dividing Equation 3.18 by Equation 3.17,

$$\tan\theta = \frac{2\lambda p}{\omega^2-p^2}$$
$$\Rightarrow \theta = \tan^{-1}\left(\frac{2\lambda p}{\omega^2-p^2}\right)$$
$$\therefore y = A\sin(pt-\theta) = \frac{f_0}{\sqrt{(\omega^2-p^2)^2+4\lambda^2p^2}}\sin\left(pt-\tan^{-1}\left(\frac{2\lambda p}{\omega^2-p^2}\right)\right)$$

To get the complete solution of Equation 3.15, the complementary function of the corresponding homogeneous equation, *i.e.* $y = a_0e^{-\lambda t}\sin(gt-\varphi)$, must be added to the particular solution. Therefore, the complete solution of Equation 3.15 be,

$$y = a_0e^{-\lambda t}\sin(gt-\varphi)+A\sin(pt-\theta)$$

The first term on the RHS, called **tangient term**, represents an initial damped oscillation of frequency $\frac{g}{2\pi}$ with its amplitude decaying exponentially to zero. And second term represents forced vibration of frequency $\frac{p}{2\pi}$ at constant amplitude A. The former oscillation dies out quickly due to damping, while the later one only remains active.

3.1.5 Resonance

As we know,

$$A = \frac{f_0}{\sqrt{(\omega^2-p^2)^2+4\lambda^2p^2}} \tag{3.19}$$

For a finite value of $'\lambda'$, the amplitude $'a'$ will be maximum when the denominator of Equation 3.19 has its minimum value, *i.e.*

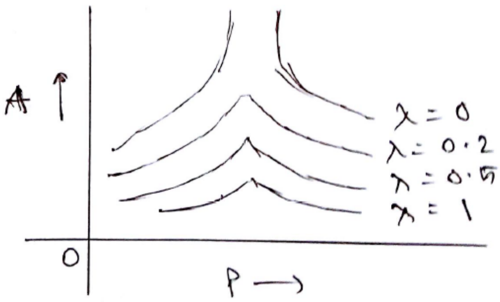
$$\frac{\mathrm{d}}{\mathrm{d}p}[(\omega^2-p^2)^2+4\lambda^2p^2] = 0$$
$$\Rightarrow -4p(\omega^2-p^2)+8\lambda^2p = 0$$

$$\Rightarrow 2p\left[4\lambda^2 - 2(\omega^2 - p^2)\right] = 0 \qquad (p \neq 0)$$
$$\Rightarrow 4\lambda^2 - 2(\omega^2 - p^2) = 0$$
$$\Rightarrow p = \sqrt{\omega^2 - 2\lambda^2}$$
$$\therefore A_{max} = \frac{f_0}{\sqrt{(\omega^2 + 2\lambda^2 - \omega^2)^2 + 4\lambda^2(\omega^2 - 2\lambda^2)}} = \frac{f_0}{2\lambda\sqrt{\omega^2 - \lambda^2}}$$

This maximum value of amplitude is known as resonance.
Comments: If there is no damping, then $\lambda = 0$. The amplitude A_{max} will tend to be ∞, which is ascintotic behavior. The curve being ascintotic to Y-axis. This, however, never happens, since damping is actually never zero in real life. When λ is small, λ^2 is smaller.

$$\therefore A_{max} = \frac{f_0}{2\omega\lambda}$$

Hence, the maximum value of A is dependent on different degree of damping.

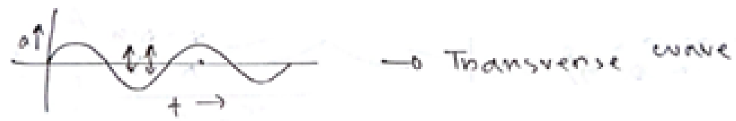


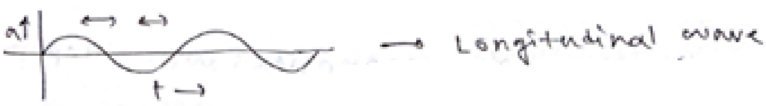
Sharpness of resonance depends on damping. The more the damping, the less the sharpness will be.

3.2 Waves

Wave motion can be thought as the transport of energy and momentum from one point in medium to another without the transport of matter. Types of wave motion:

1. Mechanical wave (material medium is necessary)
   (a) Progressive
   (b) Stationary
2. Non-mechanical wave (electromagnetic/radioactive)





The wave motion is the disturbance produced in the medium by the repeated periodic motions of the particles of the medium only the wave travels forward whereas the particles of the medium vibrate about their mean position. There is regular phase difference between only two successive particles of the medium. The velocity of wave is different from the velocity with which the particles of the medium vibrates about their mean position.
Characteristics:

- Wave motion is a disturbance, produced in the medium by the repeated periodic motion of the particle of the medium.
- Only the wave travels onward, while the particles of the medium oscillates about the mean position.
- While the velocity of the wave is constant, the velocity of the particles is different at different positions.
- There is a phase difference between the particles of the medium. The particles ahead starts vibrating a little later than the particle just preceding it.

3.2.1 Travelling/progressive waves

A plane progressive wave is defined as the onward transmission of the vibratory motion of a body in an elastic medium from one particle to the successive particle. Progressive waves are of two types:

1. Transverse wave
2. Longitudinal wave

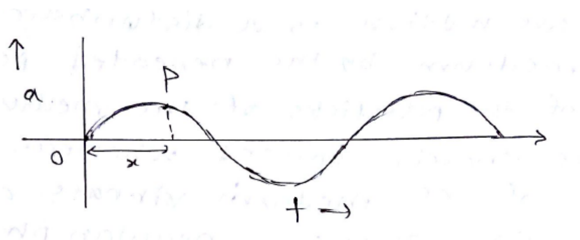
Distinction between Transverse and Longitudinal Wave:

Transverse wave	Longitudinal wave
It is the wave motion where the particles of the medium oscillate up and down about their mean position in a direction at their right angles to the direction of propagation of the wave motion itself.	It is the wave motion where the particles of the medium oscillate to and fro about their mean position along the direction of propagation of the wave motion itself.
This type of wave travels in the form of crests and troughs with one crest and one adjoining trough making up one wave.	This type of wave travels in the form of compressions and rarefactions.
This wave motion is possible in media which possess elasticity of shape or rigidity, resistivity. It is possible in solids and liquids.	This wave motion is possible in media possessing elasticity of volume, i.e. solids, liquids and gasses.
This type of wave motion can be polarized.	Cannot be polarized.

3.2.1.1 Equation of plane progressive wave

The equation of motion of particle at o passing through the mean position in the positive x-direction is,

$$y = a\sin\omega t \tag{3.20}$$



Considering another particle at p which is at a distance x away from o. Hence, the equation of motion of the particle at p,

$$y = a\sin(\omega t - \varphi) \tag{3.21}$$
$$where,\ \varphi \text{ is the phase difference}$$

For the path difference x, the phase difference $\varphi = \frac{2\pi}{\lambda}x$. From Equation 3.21

$$y = a\sin\left(\omega t - \frac{2\pi}{\lambda}x\right) \tag{3.22}$$

$$y = a\sin(\omega t - kx) \qquad where,\ \vec{\mathbf{k}} \text{ is a vector} \tag{3.23}$$

$$y = a\sin\frac{2\pi}{\lambda}(vt - x) \tag{3.24}$$

$$y = a\sin k(vt - x) \tag{3.25}$$

Equation 3.22 and 3.25 are the expressions for the equation of plane progressive wave in different forms.

3.2.1.2 Phase velocity/Wave velocity

The compressions and rarefactions of longitudinal wave or crest and through of transverse wave advances through a medium with a constant velocity. In other words, advance of phase through a medium takes place with same velocity. This velocity of advance is known as phase velocity.

$$y = a\sin(\omega t - kx) \qquad (\omega t - kx \text{ is the constant phase})$$

$$\text{Phase velocity, } \frac{\mathrm{d}}{\mathrm{d}t}(\omega t - kx) = 0 \qquad \text{(since, phase is constant)}$$

$$\Rightarrow \omega - k\frac{\mathrm{d}x}{\mathrm{d}t} = 0$$

$$\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\omega}{k} = v = n\lambda$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \text{phase velocity} \qquad v = \text{wave velocity}$$

3.2.1.3 Differential Equation of Wave Motion

$$y = a\sin\frac{2\pi}{\lambda}(vt - x) \tag{3.26}$$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{2\pi a v}{\lambda}\cos\frac{2\pi}{\lambda}(vt - x)$$

$$\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -\frac{4\pi^2v^2}{\lambda^2}a\sin\frac{2\pi}{\lambda}(vt - x) = -\frac{4\pi^2v^2}{\lambda^2}y \tag{3.27}$$

To find the value of compression, differentiating Equation 3.26 with respect to x,

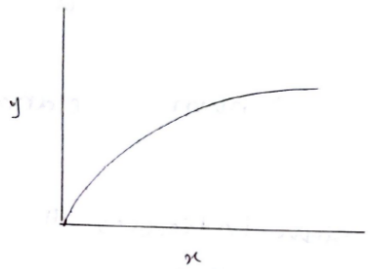
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2\pi a}{\lambda}\cos\frac{2\pi}{\lambda}(vt - x)$$

$$\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{4\pi^2}{\lambda^2}a\sin\frac{2\pi}{\lambda}(vt - x) = -\frac{4\pi^2}{\lambda^2}y \tag{3.28}$$

Comparing Equation 3.27 and 3.28, we get,

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = v^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = k\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \qquad \text{(where, } k = v^2\text{)}$$

This is the differential equation of the wave motion. Here, $\frac{\mathrm{d}^2y}{\mathrm{d}t^2}$ is particle acceleration and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is compression/expansion. Now, curvature of the displacement curve at that point can be described by the following curve-



3.2.1.4 Relation between Particle velocity and Wave velocity

The equation of a plane progressive wave is given by,

$$y = a\sin\frac{2\pi}{\lambda}(vt - x) \tag{3.29}$$

$$\text{Particle velocity, } U = \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{2\pi a v}{\lambda}\cos\frac{2\pi}{\lambda}(vt - x)$$

$$\therefore U_{max} = \frac{2\pi a}{\lambda}v$$

Maximum particle velocity = $\frac{2\pi a}{\lambda}\times$wave velocity.
The acceleration of particle is given by,

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -\frac{4a\pi^2v^2}{\lambda^2}\sin\frac{2\pi}{\lambda}(vt - x) = -\frac{4\pi^2v^2}{\lambda^2}y$$

The acceleration is maximum when $y = a$. Hence, the maximum acceleration $= -\frac{4\pi^2 v^2}{\lambda^2} a$. The negative sign that the acceleration is directed towards the mean position. Now, differentiating Equation 3.29 w.r.t. x,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2\pi a}{\lambda} \cos \frac{2\pi}{\lambda}(vt - x)$$

Comparing two equations,

$$U = \frac{\mathrm{d}y}{\mathrm{d}t} = -v\frac{\mathrm{d}y}{\mathrm{d}x}$$

Particle velocity at a point = − wave velocity × slope of the displacement curve at that point.

3.2.2 Energy Calculation

The equation of a progressive wave

$$\begin{aligned} y &= a \sin \frac{2\pi}{\lambda}(vt - x) \\ \text{Particle Velocity} = \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{2\pi a v}{\lambda} \cos \frac{2\pi}{\lambda}(vt - x) \\ \therefore \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} &= -\frac{4a\pi^2 v^2}{\lambda^2} \sin \frac{2\pi}{\lambda}(vt - x) = -\frac{4\pi^2 v^2}{\lambda^2} y \end{aligned}$$

3.2.2.1 Kinetic energy per unit volume

Let, the mass of element $= P$. Then, kinetic energy per unit volume

$$\begin{aligned} &= \frac{1}{2} P \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 \\ &= \frac{1}{2} P \times \frac{4a^2\pi^2 v^2}{\lambda^2} \cos^2 \frac{2\pi}{\lambda}(vt - x) \\ &= \frac{2a^2\pi^2 v^2 P}{\lambda^2} \cos^2 \frac{2\pi}{\lambda}(vt - x) \end{aligned}$$

3.2.2.2 Potential energy per unit volume

The work done per unit volume for a small displacement $\mathrm{d}y$ of a layer,

$$\begin{aligned} \mathrm{d}W &= force \times displacement \\ &= P \times \frac{4\pi^2 v^2}{\lambda^2} y \times \mathrm{d}y && \text{(considering magnitude only)} \\ \text{Total work done} &= \int_0^y \mathrm{d}W \\ &= \int_0^y \frac{4\pi^2 v^2 P}{\lambda^2} y \,\mathrm{d}y \\ &= \frac{4\pi^2 v^2 P}{\lambda^2} \left[\frac{y^2}{2}\right]_0^y \\ &= \frac{2\pi^2 v^2 P}{\lambda^2} y^2 \\ &= \frac{2\pi^2 v^2 P}{\lambda^2} a^2 \sin^2 \frac{2\pi}{\lambda}(vt - x) \end{aligned}$$

3.2.2.3 Energy density

$$\begin{aligned} E &= \frac{2a^2\pi^2 v^2 P}{\lambda^2} \left[\cos^2 \frac{2\pi}{\lambda}(vt - x) + \sin^2 \frac{2\pi}{\lambda}(vt - x)\right] \\ &= \frac{2a^2\pi^2 v^2 P}{\lambda^2} \end{aligned}$$

$$= 2a^2\pi^2 P\left(\frac{v}{\lambda}\right)^2 = 2a^2\pi^2 n^2 P$$

$$where,\ n = \frac{v}{\lambda} = \text{frequency of the wave}$$

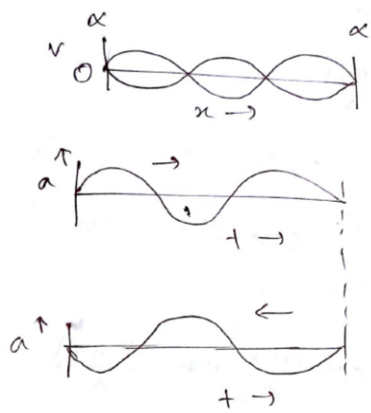
Total energy is independent of time and position, whereas potential energy and kinetic energy are dependent upon both of them. Hence, total energy always remains constant.

3.2.3 Stationary Wave

When two simple harmonic waves of the same amplitude, frequency and time period travel in opposite direction in straight line, the resultant wave is called the stationary of standard wave.

3.2.3.1 When reflection occurs at a fixed boundary

In tuning fork experiment, if two ends are fixed, then potential in two ends are infinitive.
The equation of motion due to incident wave,



$$y_1 = a\sin\frac{2\pi}{\lambda}(vt - x)$$

The equation of motion due to reflected wave,

$$y_2 = -a\sin\frac{2\pi}{\lambda}(vt + x)$$

$$\therefore \text{The resultant wave, } y = y_1 + y_2$$

$$= a\sin\frac{2\pi}{\lambda}(vt - x) - a\sin\frac{2\pi}{\lambda}(vt + x)$$

$$= -2a\sin\frac{2\pi x}{\lambda}\cos\frac{2\pi vt}{\lambda}$$

$$\Rightarrow y = -A\cos\frac{2\pi vt}{\lambda} \quad where,\ A = 2a\sin\frac{2\pi x}{\lambda}$$

$$\therefore \frac{dy}{dt} = \frac{4\pi av}{\lambda}\sin\frac{2\pi x}{\lambda}\sin\frac{2\pi vt}{\lambda}$$

$$\Rightarrow \frac{d^2y}{dt^2} = \frac{8a\pi^2v^2}{\lambda^2}\sin\frac{2\pi x}{\lambda}\cos\frac{2\pi vt}{\lambda}$$

$$Again, \frac{dy}{dx} = -\frac{4a\pi}{\lambda}\cos\frac{2\pi x}{\lambda}\cos\frac{2\pi vt}{\lambda}$$

Changes with respect to position:

1. Considering the points where $\sin\frac{2\pi x}{\lambda} = 0$ and $\cos\frac{2\pi x}{\lambda} = \pm 1$

$$\therefore displacement,\ y = 0$$

$$Amplitude,\ A = 0$$

$$Velocity,\ \frac{dy}{dt} = 0$$

$$Acceleration,\ \frac{d^2y}{dt^2} = 0$$

$$Strain,\ \frac{dy}{dx} = \mp\frac{4a\pi}{\lambda}\cos\frac{2\pi vt}{\lambda}$$

It is, hence, clear that strain at these points will always be maximum whereas other terms will always be zero. Such points are called **nodes**. Now,

$$\sin\frac{2\pi x}{\lambda} = 0$$

$$\Rightarrow \frac{2\pi x}{\lambda} = m\pi \qquad (m = 0, 1, 2, ...)$$

$$\Rightarrow x = \frac{m\lambda}{2} = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, \ldots$$

Thus, the nodes on nodal points are obviously equidistant and separated by $\frac{\lambda}{2}$. At $x = 0$, the position of interface is a node.

2. Considering the points where $\sin \frac{2\pi x}{\lambda} = \pm 1$ and $\cos \frac{2\pi x}{\lambda} = 0$

$$\therefore displacement,\ y = \mp 2a \cos \frac{2\pi vt}{\lambda}$$

$$Amplitude,\ A = 2a$$

$$Velocity,\ \frac{\mathrm{d}y}{\mathrm{d}t} = \pm \frac{4a\pi v}{\lambda} \sin \frac{2\pi vt}{\lambda}$$

$$Acceleration,\ \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \pm \frac{8a\pi^2 v^2}{\lambda^2} \cos \frac{2\pi vt}{\lambda}$$

$$Strain,\ \frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

Therefore, it is clear that, at these points, displacement will always be maximum, and such points are called **anti-nodes**, or **anti-nodal points**.

$$\cos \frac{2\pi x}{\lambda} = 0$$

$$\Rightarrow \frac{2\pi x}{\lambda} = (2m+1)\frac{\pi}{2} \qquad (m = 0, 1, 2, \ldots)$$

$$\Rightarrow x = (2m+1)\frac{\lambda}{4} = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \ldots$$

These points are anti-nodes separated by $\frac{\lambda}{2}$.

3.2.3.2 When reflection occurs at a free boundary

$$y = 2a \cos \frac{2\pi x}{\lambda} \sin \frac{2\pi vt}{\lambda}$$

$$= A \sin \frac{2\pi vt}{\lambda}$$

$$U = \frac{\mathrm{d}y}{\mathrm{d}t} = A \frac{2\pi v}{\lambda} \cos \frac{2\pi vt}{\lambda}$$

$$= \frac{4\pi av}{\lambda} \cos \frac{2\pi x}{\lambda} \cos \frac{2\pi vt}{\lambda}$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -\frac{8\pi^2 v^2}{\lambda^2} a \cos \frac{2\pi x}{\lambda} \sin \frac{2\pi vt}{\lambda}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4\pi}{\lambda} a \sin \frac{2\pi x}{\lambda} \sin \frac{2\pi vt}{\lambda}$$

3.2.3.3 Energy Transfer in a Stationary Wave

When a wave propagate through a medium, the bulk modulus,

$$k = -\frac{P}{\frac{\mathrm{d}y}{\mathrm{d}x}} \qquad \left[k = \frac{Stress}{Strain}; \quad \text{Stress}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) \text{and Pressure are same here} \right]$$

$$\Rightarrow P = -k\frac{\mathrm{d}y}{\mathrm{d}x}$$

In case of a stationary wave formed by reflection at a free boundary, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4\pi}{\lambda} a \sin \frac{2\pi x}{\lambda} \sin \frac{2\pi vt}{\lambda}$$

$$\text{Also, } v = \sqrt{\frac{k}{\rho}} \Rightarrow k = v^2 \rho \qquad \text{(jadu!)}$$

$$\therefore P = \frac{4\pi v^2 \rho}{\lambda} a \sin\frac{2\pi x}{\lambda} \sin\frac{2\pi vt}{\lambda}$$

P is maximum when

$$\begin{aligned}
\sin\frac{2\pi x}{\lambda} &= \sin\frac{2\pi vt}{\lambda} = 1 \\
\therefore P_{max} = P_0 &= \frac{4\pi v^2 \rho}{\lambda} a \\
\therefore P &= P_0 \sin\frac{2\pi x}{\lambda} \sin\frac{2\pi vt}{\lambda} \\
\text{Putting } P_0 \sin\frac{2\pi x}{\lambda} &= P_x, \text{ we get} \\
P &= P_x \sin\frac{2\pi vt}{\lambda}
\end{aligned}$$

The particle velocity at the point

$$U = \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{4\pi v}{\lambda} a \cos\frac{2\pi x}{\lambda} \cos\frac{2\pi vt}{\lambda}$$

Putting, $\frac{4\pi v}{\lambda} a \cos\frac{2\pi x}{\lambda} = U_x$, we have

$$U = U_x \cos\frac{2\pi vt}{\lambda}$$

Now, work done/energy transferred per unit area in small interval $\mathrm{d}t$,

$$\mathrm{d}I = PU\,\mathrm{d}t$$

Energy transferred at total time period T is

$$\int_0^T \mathrm{d}I = I = \int_0^T PU\,\mathrm{d}t$$

The rate of energy transfer

$$\begin{aligned}
&= \frac{1}{T}\int_0^T PU\,\mathrm{d}t \\
&= \frac{1}{T}\int_0^T P_x \sin\frac{2\pi vt}{\lambda} U_x \cos\frac{2\pi vt}{\lambda}\,\mathrm{d}t \\
&= \frac{P_x U_x}{2T}\int_0^T \sin\frac{4\pi vt}{\lambda}\,\mathrm{d}t \\
&= \frac{P_x U_x}{2T} \times 0 = 0
\end{aligned}$$

Energy will be distributed, but the net energy transfer is zero.

3.2.3.4 Distinction between Progressive and Stationary wave

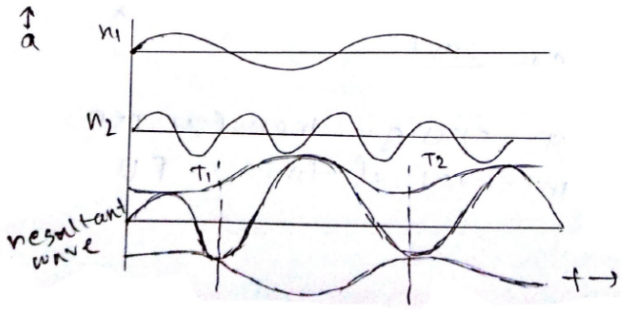
Progrssive wave	Stationary wave
The vibration characteristics in each particle in the path of a progressive wave is the same.	The vibration characteristic of each particle of the medium is its own.
The maximum displacement of all particles of the medium is some maximum velocity but one after another.	The maximum displacement of the particles are not some but they attain their respective maximum displacement simultaneously.
All particles pass through their mean positions with the same maximum velocity but one after another.	All particles pass through their mean positions simultaneously but with different maximum velocities.
No particles of the medium permanently at rest.	Certain particles of the medium are permanently st rest.
All particles of the medium undergo the same changes of pressure but one after another.	The change of pressure is maximum at the nodal points and zero at the anti-nodal points, but occur simultaneously at all points.
A regular transfer of energy takes place across every section of the medium.	There is no transfer of energy across any section of the medium.

3.2.4 Intensity of Waves

3.2.5 Beats

If frequency and amplitude of two SHMs are nearly same ($f_1 \sim f_2 \approx 10Hz$), then the interference of sound wave causes beat.

When two sound waves traveling in the same direction with nearly the same frequency and amplitude, the resultant sound heard alternates between loud and soft. This throbbing effect in the intensity of sound is called beat.



Let, two sound waves be,

$$
\begin{aligned}
y_1 &= a\sin\omega_1 t = a\sin 2\pi f_1 t \\
y_2 &= b\sin\omega_2 t = b\sin 2\pi f_2 t \\
\therefore \text{Resultant wave, } y &= y_1 + y_2 \\
&= a\sin 2\pi f_1 t + b\sin 2\pi f_2 t \\
&= a\sin 2\pi f_1 t + b\sin 2\pi (f_1 - f_1 + f_2)t \\
&= a\sin 2\pi f_1 t + b\sin 2\pi f_1 t\cos 2\pi (f_1 - f_2)t - b\cos 2\pi f_1 t\sin 2\pi (f_1 - f_2)t \\
&= \sin 2\pi f_1 t[a + b\cos 2\pi (f_1 - f_2)t] - b\cos 2\pi f_1 t\sin 2\pi (f_1 - f_2)t \\
&= \sin 2\pi f_1 t.A\cos\theta - A\sin\theta.\cos 2\pi f_1 t \\
&= A\sin(2\pi f_1 t - \theta) \\
where,\ A\cos\theta &= a + b\cos 2\pi (f_1 - f_2)t \\
A\sin\theta &= b\sin 2\pi (f_1 - f_2)t \\
\therefore A &= \sqrt{a^2 + b^2 + 2ab\cos 2\pi (f_1 - f_2)t}
\end{aligned}
$$

The amplitude is maximum when

$$\cos 2\pi(f_1 - f_2)t = 1$$
$$\Rightarrow 2\pi(f_1 - f_2)t = 0, 2\pi, 4\pi, ..., 2k\pi \qquad (k = 0, 1, 2, ...)$$
$$\therefore t = 0, \frac{1}{f_1 - f_2}, \frac{2}{f_1 - f_2}, ..., \frac{k}{f_1 - f_2}$$

The max value is

$$A = \sqrt{a^2 + b^2 + 2ab} = a + b$$

The amplitude is minimum when

$$\cos 2\pi(f_1 - f_2)t = -1$$
$$\Rightarrow 2\pi(f_1 - f_2)t = \pi, 3\pi, 5\pi, ..., (2k+1)\pi \qquad (k = 0, 1, 2, ...)$$
$$\therefore t = \frac{1}{2(f_1 - f_2)}, \frac{3}{2(f_1 - f_2)}, ..., \frac{2k+1}{2(f_1 - f_2)}$$

The min value is

$$A = \sqrt{a^2 + b^2 - 2ab} = a - b$$

The time interval between two successive maxima and minima $\frac{1}{f_1 - f_2}$. Therefore, the number of beats produced per second,

$$\frac{1}{t} = f_1 - f_2 = \Delta f = \text{frequency difference}$$

Applications:

- To detect the speed of auto-mobiles.
- To track the earth satellite.
- To discover the double stars.
- Red shift
- Discovery of Saturn's rings

3.2.6 Doppler Effect

The apparent change in the frequency due to the relative motion between the source and the observer, is known as the doppler effect in sound.

3.2.6.1 Calculation of Apparent frequency

Case-1: Both source and observer are at rest

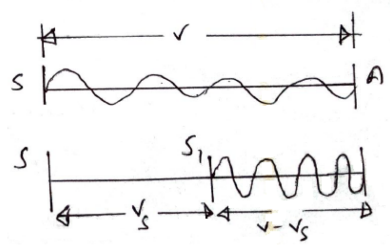
Let a source, which is at rest, sends out f waves per second. These waves travel through a distance with velocity v in air. The observer will hear a sound of f frequency.

$$\therefore \text{Apparent frequency} = f$$

Case-2: Source is in motion, Observer at rest

S be the position of source, and A be the observer at rest. The observer is at v distance from the source. The source is emitting f waves per second and it is traveling towards the observer with a velocity v and the source follows the waves with velocity v_s.

At the end of $1sec$, the sound wave reaches point A, where $SA = v$. At the same time, the source reaches S_1, where $SS_1 = v_s$.



The f waves emitted in $1sec$ would have occupied the length SA. But since the source itself has moved towards the observer SS_1 distance, these f waves occupy the length $S_1A = v - v_s$. Thus, the motion of the source has shortened the wavelength to λ_1

$$\lambda_1 = \frac{v - v_s}{f}$$

Since the velocity is independent of wavelength, the observer now receives an increased number of waves per second.

$$\therefore f' = \frac{v}{\lambda_1'} = \frac{v}{\frac{v-v_s}{f}} = \frac{fv}{v-v_s}$$

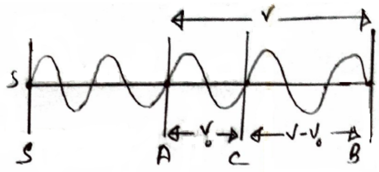
$\therefore$ apparent pitch/frequency, $f' = \left(\frac{v}{v-v_s}\right)f$

$f' > f$

When the source moves away from the observer,

$$f' = \left(\frac{v}{v+v_s}\right)f \qquad f' < f$$

Case-3: Observer is in motion and Source is at rest
The observer moves away from A at velocity v_0 and wave reach A to B in $1sec$, and $AB = v$. At the same time, the observer reaches C, and $AC = v_0$. The observer will hear fewer waves between B and C, and $BC = v - v_0$.



v length occupy f waves

$\therefore v - v_0$ length occupy $\frac{f(v-v_0)}{v}$ waves

$$\therefore f' = \left(\frac{v-v_0}{v}\right)f$$

If the observer moves towards the source, then

$$f' = \left(\frac{v+v_0}{v}\right)f$$

Case-4: Both the Source and Observer are in motion
When the source is moving and the observer is still, $f_1 = \frac{fv}{v-v_s}$, and when the source is still and the observer is moving, $f' = \left(\frac{v-v_0}{v}\right)f_1$

$$\therefore f' = \left(\frac{v-v_0}{v}\right)\frac{fv}{v-v_s} = \left(\frac{v-v_0}{v-v_s}\right)f$$

f' will be greater or less than f, depending on the value of v_s is greater than v_0 or not.

1. When the Source and Observer move towards each other

$$f' = \left(\frac{v-(-v_0)}{v-v_s}\right)f = \left(\frac{v+v_0}{v-v_s}\right)f$$

2. When the Source and observer move away from each other

$$f' = \left(\frac{v-v_0}{v-(-v_s)}\right)f = \left(\frac{v+v_0}{v+v_s}\right)f$$

3. Source moving away from observer and observer moving towards the source

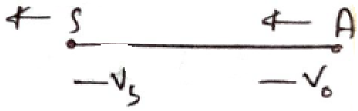
$$f' = \left(\frac{v-(-v_0)}{v-(-v_s)}\right)f = \left(\frac{v+v_0}{v+v_s}\right)f$$

4. Source moving towards the observer and Observer moving away from the Source

$$f' = \left(\frac{v+v_0}{v-v_s}\right)f$$

1

2



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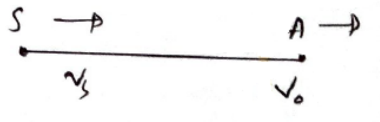

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3.2.6.2 Applications of Doppler Effect

- tracking of earth satellite
- detect the speed of automobiles
- discovery of double stars
- red shift or blue shift
- RADAR
- Saturn's rings
- velocity and rotation of Sun

Some Tutorials

- Wave and Oscillation playlist

4. Theories of Light

Physical optics studies the nature of light, whether it is wave or particle, propagation through medium and its phenomena.

4.1 Wave Theory

4.1.1 Huygens Wave Theory

According to this theory, light is transferred through a medium called Ether. A vibrating particle placed at a point in a homogeneous medium (ether) extending in all directions, will communicate its motion to all its neighboring particles. As a result, from their mean position of rest, the particle execute a periodic motion. Due to periodic motion of the particle, a wave motion is produced.
Huygens suggested that light has a wave character and that it travels in ether as a longitudinal wave. His theory could explain reflection and refraction, as well as the principles that govern them. This hypothesis, however, was unable to account for light propagation in a rectilinear path.

4.1.2 Huygens Principle and Construction

To explain the propagation of light through ether, Hygen proposed the following principles:

- Every point on a primary wave front may be considered as a secondary source of disturbance.
- Secondary waves on wavelets spread out from each one of these secondary source into the medium with same velocity as the original wave.
- The envelope of all the secondary wave fronts in the wavelets (in the forward direction) after any given interval of time gives rise to the secondary wave front.

4.1.3 Superposition of Light Waves

Principles of Superposition of wave motion, first enunciated by Thomus Young in 1801, states that

- When a medium is disturbed simultaneously by more than one wave, the instantaneous resultant displacement of the medium at every point at every instant is the algebraic sum of the displacement of the medium that would be produced at the point by the individual wave trains if each were present alone.

$$y = A\sin\frac{2\pi}{\lambda}(vt - x)$$

If there are $y_1, y_2, y_3, \ldots$

$$y = y_1 + y_2 + y_3 + \ldots \Rightarrow y = \sum_{i=1}^{\infty} y_i$$

- After the superposition at the region of crossover, the wave trains emerge unimpeded as if they have not met each other at all. Each wave retains its individual characteristics and behave as if other wave is absent and they never met.

4.1.4 Electromagnetic Theory

Also called Maxwell's theory. According to this theory, light consists of electromagnetic oscillations perpendicular to the direction of travel of the wave motion. If calculated,

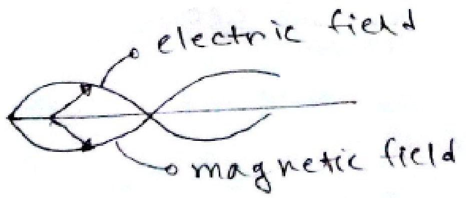
from experiment, $c = 2.99 \times 10^8 ms^{-1}$ theoretically, $c = 2.93 \times 10^8 ms^{-1}$

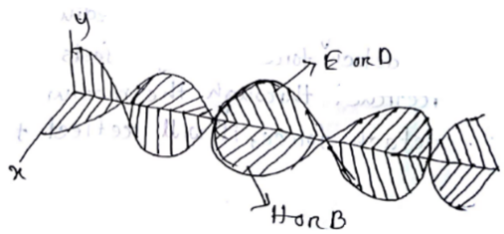
$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

$$\varepsilon_0 = \text{permitivity constant or electric field constant}$$

$$\mu_0 = \text{magnetic field constant}$$

The directions of E and H in a uniform plane wave,

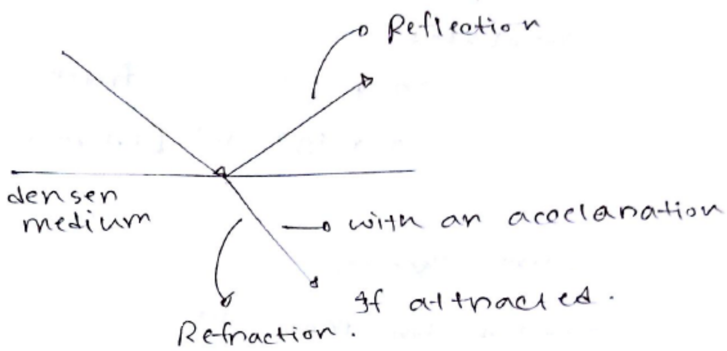




4.2 Particle Theory

4.2.1 Newton's Corpuscular Theory

The theory/concept of reflection/refraction is called corpuscular theory. It says, light is elastic particles which are called corpuscle. According to this theory, these particles are tiny, light, elastic and invisible, and travels in straight line with tremendous velocity. When light comes to a medium, it will feel attach or detach force. If light feels attach force, it will refract through the medium. If light feels, detach force, it will reflect from the medium. Newton theorized that different colored particles must have varying diameters.



4.2.2 Quantum Theory of Light

Invented by Max Planck in 1890 and elaborated by Einstein. It states that both light and matter consists of tiny particles which have wave like properties associated with them. Energy of a photon, $E = h\upsilon$. According to this theory, $c = 3 \times 10^8 ms^{-1}$.

comes by a packet

chargeless /
non-charged /
charge- neutral.

mass less ,

5. Interference

5.1 Introduction

Let, two beams of light cross each other at a certain point. According to principle of superposition, the two lights will continue unimpeded, without being influenced by each other in any way. The resultant intensity will either be greater or less than that which would be given by one beam alone. This modification of intensity due to superposition of two or more beams of light is known as interference of light.

Interference of light is an incident in which two or more waves of light of the same frequency either reinforce or cancel each other; the amplitude of resulting wave being equal to the sum of the amplitudes of the combining waves.

There are two types of interference.

1. Constructive interference
2. Destructive interference

Interference can be created in three ways:

1. Young's double slit
2. Plano-convex lens and glass-plate
3. Virtual coherent source

5.1.1 Coherent sources

For stationary interference pattern to be observed, the two sources must start either exactly in phase or with a constant phase difference. This can be possible only if the two sources are derived from a single parent source, so the source from which a wave will come and by some process it converts into two such that it seems they are coming from different sources, then the source is called coherent source. Example: a real source and its virtual image.

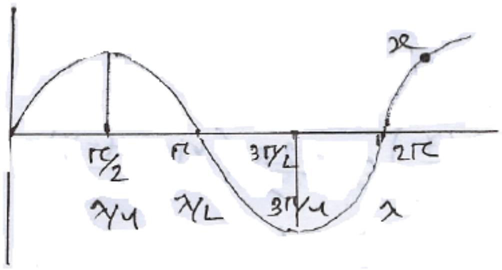
Two sources are said to be coherent if they emit lightwaves of same frequency, nearly the same amplitude, and are always in phase, or have a constant phase difference with each other; in other words, the sources have point-to-point relationship

In actual practice, it is not possible to have two independent sources to be coherent, since they do not have any fixed phase relationship. But, if the two interfering waves are derived from the same wavefront, then they both will change phases simultaneously, and will always have a constant phase relationship.

5.1.2 Conditions of Interference

1. Minimum two beams of light, which interfere, must be coherent.
2. The interfering waves must have the same amplitude.
3. The original source must be monochromatic.
4. Phase difference must be constant.
   - In phase $\rightarrow$ constructive/bright light
   - Out of phase $\rightarrow$ destructive/dark light
5. The fringe width should reasonably be as large as possible so that each fringe can be recognized distinctly. The separation between sources should be as small as possible. While the distance of the screen from the sources should be as large as possible.
6. The interfering waves must be propagated in almost the same direction, or the interfering wavefronts must intersect at a very small angle.

5.1.3 Phase difference and Path difference



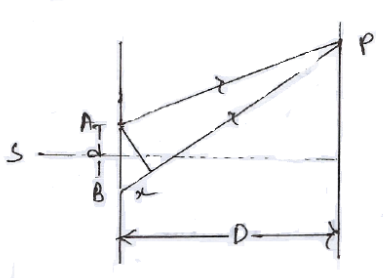
For a path difference of λ, the phase difference is 2π
For a path difference of x, the phase difference is $\frac{2\pi}{\lambda}x$

$$\delta = \frac{2\pi}{\lambda} \times x$$

δ = Phase difference
x = Path difference

5.1.4 Analytical Treatment of Interference

Consider a monochromatic light source S emitting waves of wavelength λ. Two narrow slits A and B are equidistant from S and act as two virtual coherent sources. Let, a be the amplitude of the waves. The phase difference two waves, at reaching point P, at any instance is δ, and the path difference is x. If y_1 and y_2 are the displacements,



$$y_1 = a\sin\frac{2\pi}{\lambda}vt \tag{5.1}$$

$$y_2 = a\sin\left(\frac{2\pi vt}{\lambda} + \delta\right)$$
$$= a\sin\frac{2\pi}{\lambda}(vt + x) \tag{5.2}$$

Now, $y = y_1 + y_2$

$$= a\sin\frac{2\pi}{\lambda}vt + a\sin\frac{2\pi}{\lambda}(vt+x)$$
$$= 2a\sin\frac{2\pi}{\lambda}\left(vt + \frac{x}{2}\right)\cos\frac{\pi x}{\lambda}$$
$$= A\sin\frac{2\pi}{\lambda}\left(vt + \frac{x}{2}\right)$$
(where $A = 2a\cos\frac{\pi x}{\lambda}$)

For resultant amplitude to be maximum, *i.e.* $A_{max} = 2a$

$$\cos\frac{\pi x}{\lambda} = 1 = \cos n\pi \quad (\text{where, } n = 0,1,2,\ldots)$$
$$\Rightarrow \frac{\pi x}{\lambda} = n\pi \Rightarrow x = n\lambda$$

At these points, the particle is violently disturbed producing what is known as constructive interference which results in bright light.
For resultant amplitude to be minimum, *i.e.* $A_{min} = 0$

$$\cos\frac{\pi x}{\lambda} = 0 = \cos(2n+1)\frac{\pi}{2}$$
$$(\text{where, } n = 0,1,2,\ldots)$$
$$\Rightarrow \frac{\pi x}{\lambda} = (2n+1)\frac{\pi}{2} \Rightarrow x = (2n+1)\frac{\lambda}{2}$$

At these points, the particle remains stationary and the intensity is minimum. This is known as destructive interference which results in dark light.

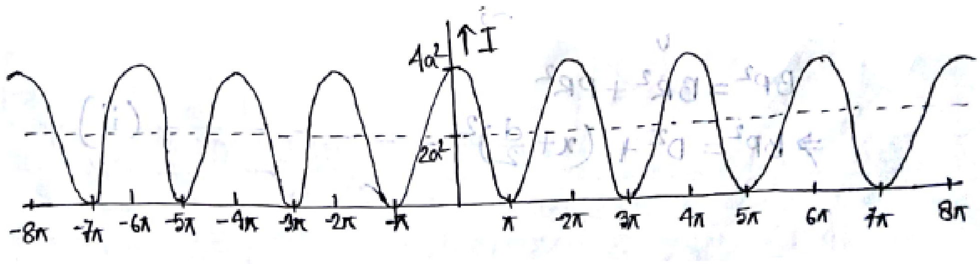
5.1.5 Energy distribution

Intensity of light is directly proportional to the square of the amplitude, *i.e.* $I \propto A^2$. Let's consider two light waves,

$$y_1 = a\sin\omega t \tag{5.3}$$
$$y_2 = a\sin(\omega t+\delta) \tag{5.4}$$
$$\delta \rightarrow \text{initial phase/phase difference}$$
$$\begin{aligned}Now,\ y &= y_1 + y_2\\ &= a\sin\omega t + a\sin(\omega t+\delta)\\ &= a\sin\omega t + a\sin\omega t\cos\delta + a\cos\omega t\sin\delta\\ &= a(1+\cos\delta)\sin\omega t + a\cos\omega t\sin\delta\end{aligned}$$
$$Let,\ a(1+\cos\delta) = A\cos\theta \tag{5.5}$$
$$and\ a\sin\delta = A\sin\theta \tag{5.6}$$
$$\therefore y = A\sin\omega t\cos\theta + A\cos\omega t\sin\theta = A\sin(\omega t+\theta)$$

It is general equation of S.H.M. Adding the squares of Equation 5.5 and 5.6,

$$\begin{aligned}A^2\sin^2\theta + A^2\cos^2\theta &= a^2(1+2\cos\delta+\cos^2\delta) + a^2\sin^2\delta\\ \Rightarrow A^2 &= a^2 + 2a^2\cos\delta + a^2\cos^2\delta + a^2\sin^2\delta\\ &= 2a^2 + 2a^2\cos\delta\\ &= 2a^2(1+\cos\delta) = 4a^2\cos^2\frac{\delta}{2}\\ \therefore I &= 4a^2\cos^2\frac{\delta}{2}\end{aligned}$$
$$where,\ a \rightarrow \text{amplitude of each of the individual waves}$$
$$\delta \rightarrow \text{phase difference between the waves}$$



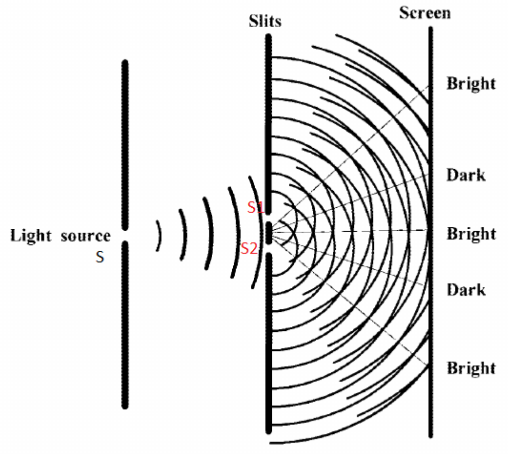
Special Cases:

1. When $\delta = 0, 2\pi, 4\pi, \ldots, 2n\pi$,
$$\cos^2\frac{\delta}{2} = 1$$
Hence, $I = 4a^2$

2. When $\delta = \pi, 3\pi, 5\pi, \ldots, (2n+1)\pi$,
$$\cos^2\frac{\delta}{2} = 0$$
Hence, $I = 0$

5.2 Young's double slit Experiment

Historically, the phenomenon of interference of light was first demonstrated by Thomas Young in about 1801 by a simple experiment. He allowed sunlight to pass through a pinhole S and then at same distance through two sufficiently close pinholes S_1 and S_2 in an opaque screen. Finally the light was received on a screen on which he observed uneven distribution of light intensity consisting of many alternate bright and dark spots.

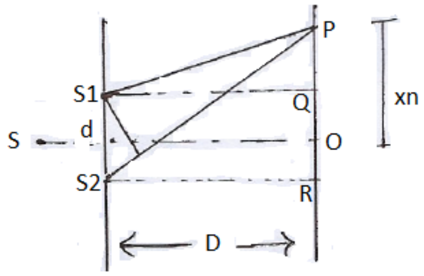


5.2.1 Theory of interference fringes: expression for the width of a fringe

S is a narrow monochromatic source of light. S_1 and S_2 are two narrow parallel slits separated by a distance d and are equal distance from S. Let the point P, located on the screen at distance x_n from the point O. In the right angled triangle S_1PQ,

$$S_1P^2 = S_1Q^2 + PQ^2 = D^2 + \left(x_n - \frac{d}{2}\right)^2$$

Similarly, from the right angled triangle S_2PR,



$$S_2P^2 = D^2 + \left(x_n + \frac{d}{2}\right)^2$$

$$\therefore S_2P^2 - S_1P^2 = \left[D^2 + \left(x_n + \frac{d}{2}\right)^2\right] - \left[D^2 + \left(x_n - \frac{d}{2}\right)^2\right]$$

$$= 2x_nd$$

$$\Rightarrow (S_1P + S_2P)(S_2P - S_1P) = 2x_nd$$

$$\Rightarrow S_2P - S_1P = \frac{2x_nd}{S_1P + S_2P}$$

Both x_n and d are small compared to D, which is usually several thousand times longer than x_n or d. One can, therefore, write,

$$S_1P = S_2P \cong D \qquad or,\ S_1P + S_2P \cong 2D$$

So, the equation can be written as,

$$S_2P - S_1P = \text{path difference} = \frac{2x_nd}{2D} = \frac{x_nd}{D}$$

For bright fringe

$$\frac{x_nd}{D} = n\lambda$$

$$\therefore x_n = \frac{n\lambda D}{d}$$

$$x_1 = \frac{\lambda D}{d}$$

$$x_2 = \frac{2\lambda D}{d}$$

$$\therefore x_2 - x_1 = \frac{\lambda D}{d}$$

For dark fringe

$$\frac{x_nd}{D} = (2n+1)\frac{\lambda}{2}$$

$$\therefore x_n = (2n+1)\frac{\lambda D}{2d}$$

$$x_1 = \frac{3\lambda D}{2d}$$

$$x_2 = \frac{5\lambda D}{2d}$$

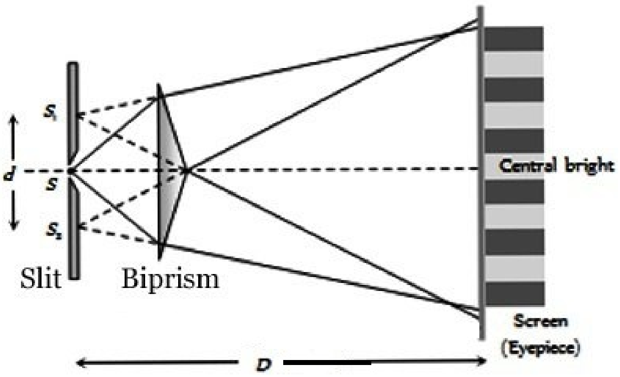
$$\therefore x_2 - x_1 = \frac{\lambda D}{d}$$

Therefore, distance between any two consecutive bright or dark fringe = width = $\frac{\lambda D}{d}$, and distance between any dark or bright fringe = bridth = $\frac{\lambda D}{2d}$.
Width of fringe: Distance between any two consecutive dark or bright fringe is called width of the fringe. The width increases with increase in wavelength, distance D, and bringing S_1 and S_2 closer.
Wavelength

$$\Delta x = \frac{\lambda D}{d} \Rightarrow \lambda = \frac{\Delta x d}{D}$$

5.3 Fresnel's Bi-prism

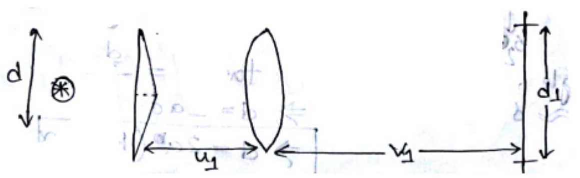
A bi-prism is essentially two prisms, each of very small refracting angle, placed base to base. In reality, the bi-prism is constructed from a single plate of glass by suitably grinding and polishing it. The obtuse angle of the prism is only slightly less than 180° and the other angles are equal. The obtuse angle may be $179°20'$ and the other angles be $20'$ each.
In the experimental arrangement, the bi-prism is placed with its refracting edge accurately parallel to the slit S, which is illuminated by a source of monochromatic light of wavelength λ.

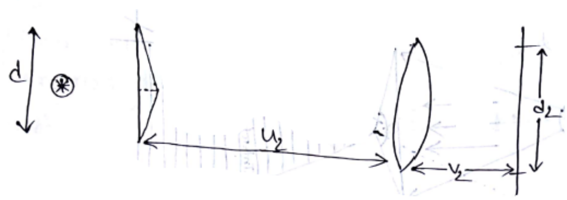


S_1 and S_2 being the image of the slit S, obviously function as coherent sources in the experiment. Moreover, the two emergent wave fronts intersect at small angles and hence the fundamental condition of interference is satisfied. In reality, diffraction bands are present over the whole region but equally spaced interference fringes are observed superposed on them in portion between E and F, and $d = S_1S_2$, which is very small.
From the experiment, λ can be determined by the law, $\lambda = \frac{\Delta x d}{D}$.

5.3.1 Measurements

1. **Fringes width (x):** It can be determined directly by setting eye-piece at a suitable distance from the bi-prism.
2. **The distance of eye-piece from the slit (D):** It can be directly read out from the positions of the respective uprights on the bench.
3. **Measurements of d:**
   (a) **Magnification method**





$$M = \frac{v}{u}$$

$v \rightarrow$ distance of shadow
$u \rightarrow$ distance of object

$$M = \frac{v_1}{u_1} = \frac{d_1}{d} \quad and \quad M = \frac{v_2}{u_2} = \frac{d_2}{d} \qquad (u_1 = v_2,\ u_2 = v_1)$$

$$\therefore \frac{u_1}{v_1} = \frac{d_2}{d}$$

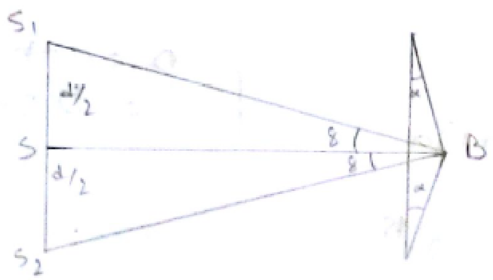
$$\Rightarrow \frac{d}{d_1} = \frac{d_2}{d} \Rightarrow d = \sqrt{d_1 d_2}$$

(b) **Method of deviation:** Since the refracting angle of the bi-prism is very small, the deviation (δ) suffered by the light beam in passing through the bi-prism is given by

$$\delta = (\mu - 1)\alpha$$

Where μ is the refractive index of the material of the bi-prism and α is its refracting angle. From the triangle SBS_1 in the figure

$$\tan\delta = \frac{\frac{d}{2}}{a}$$



Where d is the distance of separation between the sources S_1 and S_2 and a is the distance between the slit and the refracting edge of the bi-prism. Since, δ is very small,

$$\frac{d}{2} = a\delta$$

$$\Rightarrow d = 2a\alpha(\mu - 1)$$

5.4 Thin Film Interference

5.4.1 Interference due to reflected light

Everyone is familiar with the magnificent colors produced by a thin film of oil on the surface of water, a thin film of a soap bubble, or coatings of oxides on heated materials. This phenomenon can be shown through Figure 5.2.

$$x = \mu(BD + DE) - BP \qquad \text{(total path difference within } BDEF\text{)}$$

$$\Rightarrow x = 2\mu BD - BP \qquad (BD = DE)$$

In the triangle BDJ,

$$\cos r = \frac{BJ}{BD} \Rightarrow \cos r = \frac{t}{BD} \Rightarrow BD = \frac{t}{\cos r}$$

Again, in the triangle BDQ,

$$\tan r = \frac{BQ}{DQ} = \frac{BQ}{t} = \frac{EQ}{t} \qquad (\because BQ = EQ)$$

$$\Rightarrow BQ = EQ = t \cdot \tan r$$

Now, in the triangle BPE

$$\cos(90^\circ - i) = \sin i = \frac{BP}{BE} = \frac{BP}{BQ + EQ} = \frac{BP}{2BQ} \qquad (\because BQ = EQ)$$

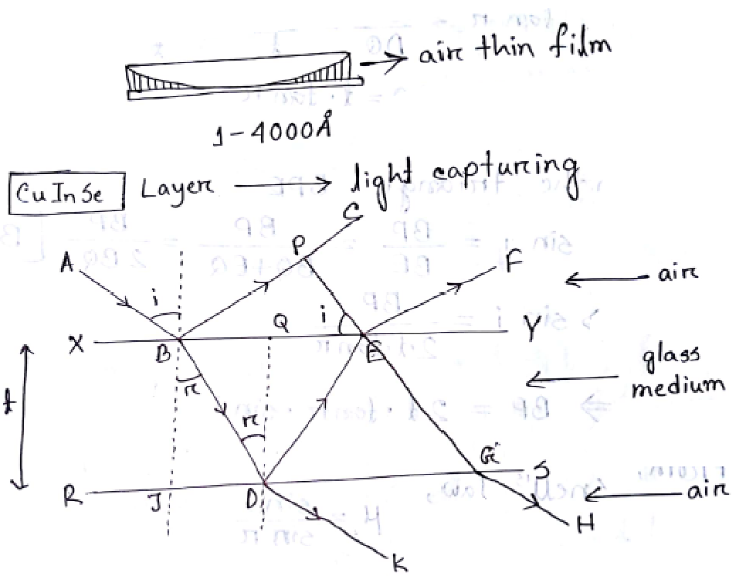


Figure 5.2: Interference due to reflected light from a plane parallel thin film.
$Transmission + Reflection + Absorption = 100\%$

$$\Rightarrow \sin i = \frac{BP}{2t \cdot \tan r}$$

From Snell's law

$$\mu = \frac{\sin i}{\sin r} \Rightarrow \sin i = \mu \cdot \sin r$$
$$\Rightarrow \frac{BP}{2t \cdot \tan r} = \mu \cdot \sin r$$
$$\Rightarrow BP = 2\mu t \cdot \frac{\sin^2 r}{\cos r}$$
$$\therefore x = \frac{2\mu t}{\cos r} - 2\mu t \cdot \frac{\sin^2 r}{\cos r}$$
$$= \frac{2\mu t(1 - \sin^2 r)}{\cos r} = 2\mu t \cos r$$

But, because of reflection, a phase change π occurs. If phase difference is π, then the path difference is $\frac{\lambda}{2}$.
Hence, actual path difference $= 2\mu t \cos r \pm \frac{\lambda}{2}$.
When t is very small, *i.e.* $t \approx 0$, then path difference $= \lambda/2$. Therefore, the film will appear dark.

1. Bright film:

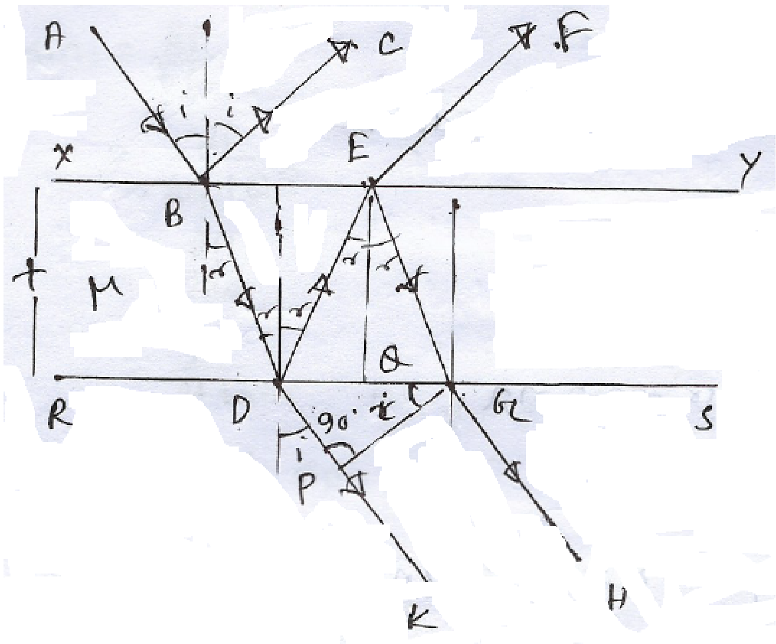
$$2\mu t \cos r \pm \frac{\lambda}{2} = n\lambda$$

2. Dark film:

$$2\mu t \cos r \pm \frac{\lambda}{2} = (2n+1)\frac{\lambda}{2}$$

5.4.2 Interference due to transmitted wave

$$\text{Optical path difference} = \mu(DE + EG) - DP$$
$$For\ \triangle DEQ,\ \cos r = \frac{EQ}{DE}$$
$$\therefore,\ DE = \frac{t}{\cos r} = EG$$
$$Again,\ \tan r = \frac{DQ}{EQ}$$



$$\therefore\ DQ = t\tan r = QG$$

$$From\ \triangle DEQ,\ \tan r = \frac{DQ}{EQ} \Rightarrow DQ = EQ\tan r = t\tan r = EQ$$

$$For\ \triangle GDP,\ \sin i = \frac{DP}{DG} = \frac{DP}{GQ+DQ} = \frac{DP}{2t\tan r} \qquad \left(\mu = \tfrac{\sin i}{\sin r} \Rightarrow \sin i = \mu \sin r\right)$$

$$\therefore\ DP = 2t\tan r\sin i = 2\mu t\tan r\sin r$$

$$\text{Path difference} = \mu \times \frac{2t}{\cos r} - 2\mu t\tan r\sin r$$

$$= \frac{2\mu t}{\cos r} - \frac{2\mu t\sin^2 r}{\cos r}$$

$$= 2\mu t\cos r$$

When t is very small, *i.e.* $t \approx 0$, then path difference is 0, and film will appear bright.

1. For bright:

$$2\mu t\cos r = n\lambda$$

2. For dark:

$$2\mu t\cos r = (2n+1)\frac{\lambda}{2}$$

5.4.3 Color of light wave

The wavelength that is equal to $2\mu t\cos r$ will show color.

5.5 Interference due to Multiple Reflection

From A Textbook of Optics - Brij Lal
Chapter 15 (15.11)

5.6 Newton's Ring

Newton's ring is the noteworthy illustration of the interference of light wave reflected from the opposite surfaces of a thin film of variable thickness. When a Plano-convex lens of large radius of curvature is placed on a glass plate so that its convex surface faces the plate. A thin air film of progressively increasing thickness in all

directions from the point of contact between the lens and the glass plate is very easily formed. The air film thus possess a radial symmetry about the point of contact.
When it is illuminated normally, preferably with monochromatic light, an interference pattern consisting of a series of alternate dark and bright circular rings, concentric with the point of contact is observed. Since, the phenomenon was first examined in detail by Newton, the rings are called Newton's ring.
Radius of curvature: The distance from the vertex to the center of the curvature is the radius of curvature of the surface.

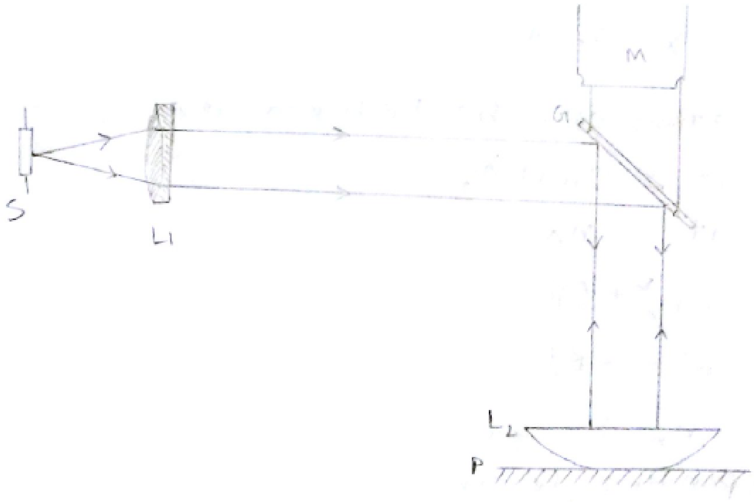
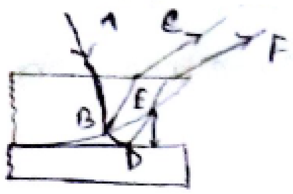


Figure 5.3: Experimental arrangement of Newton's ring

5.6.1 Theory of Newton's ring



The two rays will interfere constructively if,

$$2\mu t\cos(\theta+\pi)\pm\frac{\lambda}{2}=n\lambda$$

(λ is the wavelength of light in air)

$$\Rightarrow 2\mu t\cos(\theta+\pi)=(2n-1)\frac{\lambda}{2}$$

n cannot have a value of zero for bright fringes seen in reflected light. Again, the two rays will interfere destructively if,

$$2\mu t\cos(\theta+\pi)\pm\frac{\lambda}{2}=(2n\pm1)\frac{\lambda}{2}$$

$$\Rightarrow 2\mu t\cos(\theta+\pi)=n\lambda$$

The equations can be reduced to,

$$2\mu t=(2n-1)\frac{\lambda}{2}\quad and\quad 2\mu t=n\lambda$$

From the figure below,

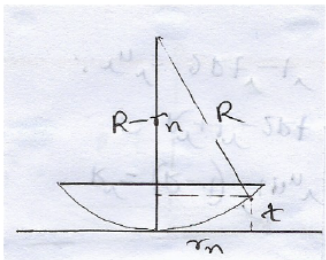
$$R^2=r_n^2+(R-t)^2\Rightarrow r_n^2=2Rt-t^2$$

Where r_n is the radius of the circular ring corresponding to the constant film thickness t, and R is the radius of curvature of lens. Since, $R>>t$

$$\therefore t=\frac{r_n^2}{2R}$$

$$\therefore r_n^2=(2n-1)\frac{\lambda R}{2\mu}\qquad\text{(bright ring)}$$

$$and\ r_n^2=\frac{n\lambda R}{\mu}\qquad\text{(dark ring)}$$



The square of the diameters of the bright and dark rings are, therefore, given by the expression,

$$D_n^2=2(2n-1)\frac{\lambda R}{\mu}\qquad\text{(bright ring)}$$

$$\Rightarrow D_n^2\propto 2n-1$$

$$and\ D_n^2=\frac{4n\lambda R}{\mu}\qquad\text{(dark ring)}$$

$$\Rightarrow D_n^2\propto n$$

Difference between the 4th and 5th dark rings is $0.46\sqrt{\frac{\lambda}{\mu}R}$. Difference between the 16th and 17th dark rings is $0.26\sqrt{\frac{\lambda}{\mu}R}$. Thus it is clear that the alternate bright and dark rings surrounding the central dark spot in Newton's rings gradually become narrower as their radii increase.

5.6.1.1 Determination of wavelength

For an air film, since $\mu = 1$, we have,

$$
\begin{aligned}
D_n^2 &= 2(2n-1)\lambda R && \text{(bright ring)}\\
\Rightarrow D_{n+p}^2 &= 2(2n+2p-1)\lambda R\\
\textit{and } D_n^2 &= 4n\lambda R && \text{(dark ring)}\\
\Rightarrow D_{n+p}^2 &= 4(n+p)\lambda R\\
\therefore D_{n+p}^2 - D_n^2 &= 4p\lambda R\\
\Rightarrow \lambda &= \frac{D_{n+p}^2 - D_n^2}{4pR}
\end{aligned}
$$

5.6.1.2 Determination of the radius of curvature of the lens

$$
\begin{aligned}
D_{n+p}^2 - D_n^2 &= 4p\lambda R\\
\therefore R &= \frac{D_{n+p}^2 - D_n^2}{4p\lambda}
\end{aligned}
$$

5.6.1.3 Miscellaneous

When is the central spot dark?

$$\text{Phase difference} = \frac{2\pi}{\lambda} \times 2\mu t + r$$

For $t = 0$, path difference $= r$. Hence, central spot is dark when the rings are viewed for reflected light.

When is the central spot white?

The central spot is white for transmitted light.

5.6.2 Determination of refractive index of a liquid with Newton's rings

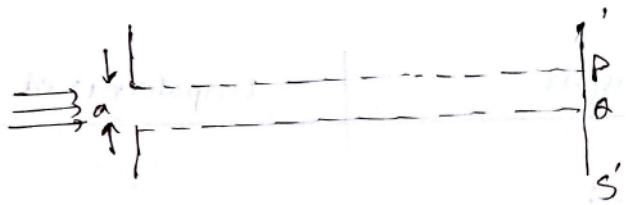
$$
\begin{aligned}
\text{For air film, } (D_{n+p}^2 - D_n^2)_{air} &= 4p\lambda R\\
\text{For liquid film, } (D_{n+p}^2 - D_n^2)_{liquid} &= \frac{4p\lambda R}{\mu}\\
\therefore \mu &= \frac{(D_{n+p}^2 - D_n^2)_{air}}{(D_{n+p}^2 - D_n^2)_{liquid}}
\end{aligned}
$$

Since, $\mu > 1$, the rings contract in the ratio $\sqrt{\frac{1}{\mu}}$ when the air film is replaced by the liquid film.

6. Diffraction

If light is passed through a long narrow slit of width a, it shows rhythmic variations in intensity. These rhythmic variations in intensity and the bending of light around the concerns of an obstacle or the encroachment of light into the region of geometrical shadow constitute a class of phenomena known as the diffraction of light.

Diffraction is a phenomena which is defined as the bending of light around the edges of an obstacle.



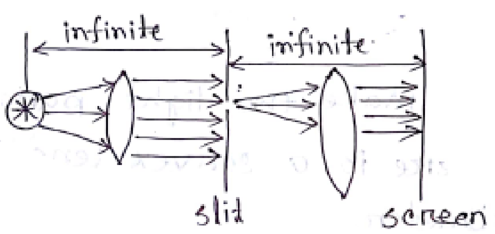
Interference	Diffraction
It is the result of interaction of light coming from different wave-fronts originating from a source.	It is the result of interaction of light coming from different parts of the save wave-front.
Interference fringes may or may not be be of same width.	Diffraction fringes are not of same width.
Regions of minimum intesity are perfectly dark.	Regions of minimum intensity are not perfectly dark.
All bright bands are of same intensity.	Different maxima of varying intensities, with maximum intensity for central maxima.

Diffraction can be of two types.

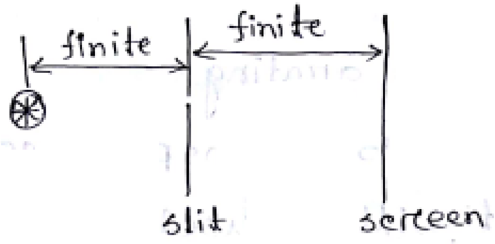
1. Fraunhoffer diffraction
2. Fresnel diffraction

6.1 Fraunhoffer diffraction

The diffraction in which the source of light and screen are effectively at infinite distance from the obstacle. The conditions required for Fraunhoffer diffraction are achieved using two convex lenses, one to make the light from the source parallel and other to focus the light after diffraction on the screen. The incident wave-front is plane and the secondary wavelets which originate from the unblocked portions of the wavefront, are in the same phase at every point in the plane of the obstacle.
This problem is simple to handle mathematically because the rays are parallel.

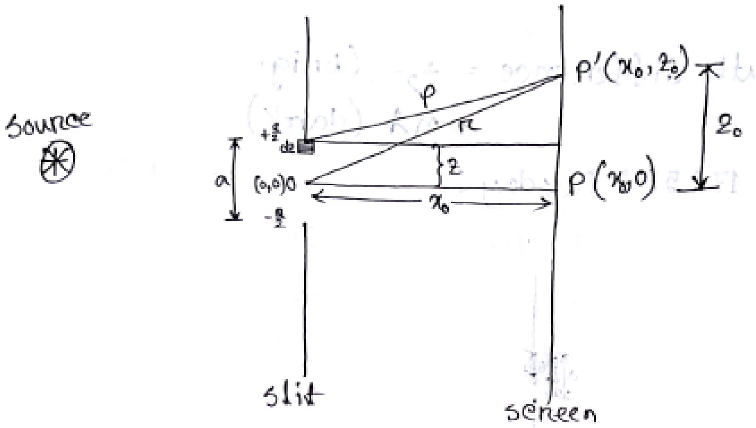


Fraunhoffer diffraction



Fresnel diffraction

6.1.1 Fraunhoffer diffraction by a single slit



The displacement of P' due the element dz,

$$\mathrm{d}y = k\sin 2\pi\left(\frac{t}{T}-\frac{\rho}{\lambda}\right)\mathrm{d}z \qquad (\rho \to \text{distance between d}z\text{ and }P')$$

$$\Rightarrow \int \mathrm{d}y = y = k\int_{-\frac{a}{2}}^{\frac{a}{2}} \sin 2\pi\left(\frac{t}{T}-\frac{\rho}{\lambda}\right)\mathrm{d}z \tag{6.1}$$

$$r^2 = x_0^2 + z_0^2$$

$$\rho^2 = x_0^2 + (z_0 - z)^2$$

$$= x_0^2 + z_0^2 - 2z_0z + z^2 = r^2 - 2z_0z + z^2$$

$$= r^2\left(1 - \frac{2z_0z}{r^2} + \frac{z^2}{r^2}\right)$$

$$But,\ r >> z,\ hence,\ r^2 >>>> z^2$$

$$\therefore \rho^2 = r^2\left(1 - \frac{2z_0z}{r^2}\right)$$

$$\Rightarrow \rho = r\sqrt{\left(1 - \frac{2z_0z}{r^2}\right)} = r - \frac{zz_0}{r} \qquad \text{(for calculation accuracy)}$$

In the triangle OPP'

$$\sin\theta = \frac{z_0}{r} \Rightarrow z_0 = r\sin\theta$$

$$\therefore p = r - z\sin\theta$$

Putting the value of p in Equation 6.1

$$y = k\int_{-\frac{a}{2}}^{\frac{a}{2}} \sin 2\pi\left(\frac{t}{T} - \frac{r - z\sin\theta}{\lambda}\right)\mathrm{d}z$$

$$= -\frac{k\lambda}{2\pi\sin\theta}\left[\cos 2\pi\left(\frac{t}{T} - \frac{r}{\lambda} + \frac{a\sin\theta}{2\lambda}\right) - \cos 2\pi\left(\frac{t}{T} - \frac{r}{\lambda} - \frac{a\sin\theta}{2\lambda}\right)\right]$$

$$
\begin{aligned}
&= \frac{k\lambda}{\pi \sin\theta} \sin 2\pi\left(\frac{t}{T}-\frac{r}{\lambda}\right) \sin\frac{\pi a \sin\theta}{\lambda} \\
&= ka\frac{\sin\beta}{\beta}\sin 2\pi\left(\frac{t}{T}-\frac{r}{\lambda}\right) \qquad \left(\text{let, } \frac{\pi a \sin\theta}{\lambda}=\beta\right)
\end{aligned}
$$

$$Here,\ Amplitude,\ A = ka\frac{\sin\beta}{\beta}$$

$$\therefore I = k^2a^2\frac{\sin^2\beta}{\beta^2}$$

$$If,\ \beta=0,\ then,\ \lim_{\beta\to 0}\frac{\sin\beta}{\beta}=1,\ hence,\ I=k^2a^2$$

6.1.2 Position of principle maxima

$$
\begin{aligned}
R &= \frac{R_0}{\beta}\left(\beta-\frac{\beta^3}{3!}+\frac{\beta^5}{5!}-\frac{\beta^7}{7!}+\ldots\right) \\
&= R_0\left(1-\frac{\beta^2}{3!}+\frac{\beta^4}{5!}-\frac{\beta^6}{7!}+\ldots\right)
\end{aligned}
$$

It is obvious that R will be maximum if the negative term vanish. This is only possible when,

$$\beta = \frac{\pi a \sin\theta}{\lambda} = 0 \quad or,\ \theta = 0$$

6.1.3 Position of minimum intensity

In case of minimum intensity,

$$\frac{\pi a \sin\theta}{\lambda} = \pm m\pi \qquad (m = 1,2,3,\ldots)$$

$$\Rightarrow a\sin\theta = \pm m\lambda$$

The value $m=0$ is not admissible as the value $\theta=0$ corresponds to the principle maximum. The first minimum occurs at,

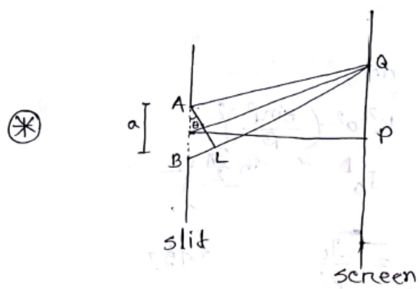
$$\theta = \pm\sin^{-1}\left(\frac{\lambda}{a}\right)$$

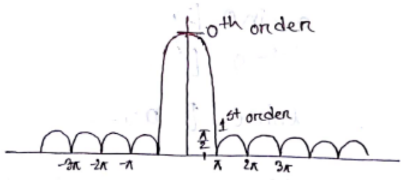
The second minimum at

$$\theta = \pm\sin^{-1}\left(\frac{2\lambda}{a}\right)$$

Since, $\sin\theta$ cannot exceed unity, the maximum value of m is the integer which is less than (and closest to) $\frac{a}{\lambda}$.

6.1.4 Secondary maxima





$$\begin{aligned} &From,\ \triangle ABL,\ \sin\theta = \frac{BL}{AB} \\ &\Rightarrow BL = a\sin\theta && (\because AB = a) \\ &\Rightarrow \lambda = a\sin\theta && (\because BL = \lambda) \\ &\therefore n\lambda = a\sin\theta && (\text{for } nth \text{ fringe}) \end{aligned}$$

If the path difference is odd multiple of $\frac{\lambda}{2}$, the direction of secondary maxima,

$$a\sin\theta = (2n+1)\frac{\lambda}{2}$$

Let's imagine a phase difference 2β.

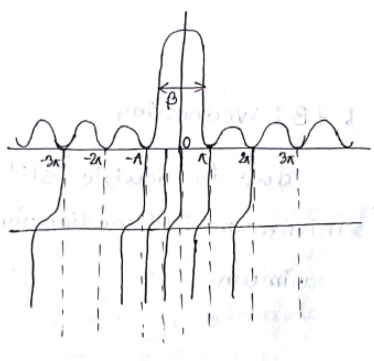
$$\begin{aligned} &\therefore 2\beta = \frac{2\pi}{\lambda} \times a\sin\theta && (\text{phase difference} = \tfrac{2\pi}{\lambda} \times \text{path difference}) \\ &\Rightarrow \beta = \frac{\pi}{\lambda} \times a\sin\theta \\ &\Rightarrow \beta = \frac{\pi}{\lambda} \times (2n+1)\frac{\lambda}{2} = (2n+1)\frac{\pi}{2} && (n = 1,2,3,...) \end{aligned}$$

For 1st order

$$\begin{aligned} I &= k^2a^2\left(\frac{\sin\beta}{\beta}\right)^2 \\ \Rightarrow I_1 &= I_0\left(\frac{\sin\frac{3\pi}{2}}{\frac{3\pi}{2}}\right)^2 = \frac{I_0}{22} \\ I_2 &= \frac{I_0}{66} \end{aligned}$$

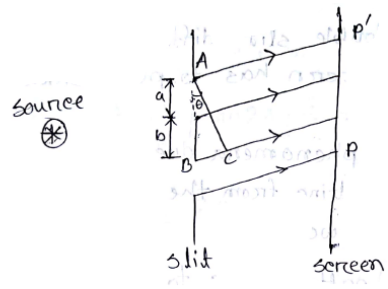
6.1.5 Secondary minima

$$\begin{aligned} &\frac{a}{2}\sin\theta = \frac{n\lambda}{2} \\ &\Rightarrow a\sin\theta = n\lambda \\ &\Rightarrow \frac{\beta\lambda}{\pi} = n\lambda \\ &\Rightarrow \beta = n\pi && (n = 1,2,3,...) \\ &I = I_0\left(\frac{\sin\beta}{\beta}\right)^2 \\ &\Rightarrow \frac{dI}{d\beta} = I_0\frac{d}{d\beta}\left(\frac{\sin\beta}{\beta}\right)^2 \\ &\Rightarrow 0 = I_0\left[\frac{2\beta^2\sin\beta\cos\beta - 2\beta\sin^2\beta}{\beta^2}\right] && (\text{for minimum } I,\ \tfrac{dI}{d\beta} = 0) \\ &\Rightarrow 2\beta^2\sin\beta\cos\beta - 2\beta\sin^2\beta && (I_0 \neq 0) \\ &\Rightarrow 2\beta\sin\beta(\beta\cos\beta - \sin\beta) = 0 \\ &\therefore \sin\beta = 0 \\ &or,\ \beta\cos\beta - \sin\beta = 0 \Rightarrow \tan\beta = \beta \end{aligned}$$



6.1.6 Fraunhoffer diffraction by a double slit

Here, $a = b$. From $\triangle ABC$,



$$\sin\theta = \frac{BC}{a+b} \Rightarrow BC = (a+b)\sin\theta$$

$$Maximum : (a+b)\sin\theta_n = n\lambda$$

$$Minimum : (a+b)\sin\theta_n = (2n+1)\frac{\lambda}{2}$$

$$\Rightarrow \sin\theta_n = \left(\frac{2n+1}{a+b}\right)\frac{\lambda}{2}$$

$$\sin\theta_1 - \sin\theta_0 = \frac{\lambda}{a+b} = \text{Angular Separation}$$

- The angular separation between any two consecutive minima or maxima is equal to $\frac{\lambda}{a+b}$.
- In the case of double slit diffraction, the diffraction pattern has to be considered in two parts.
  1. The interference phenomena due to secondary wave emitting from the corresponding points of the two slit.
  2. The diffraction pattern due to secondary wave from the two slit individually.

$$\begin{aligned}
y =& k\int_{-\frac{a}{2}}^{\frac{a}{2}} \sin 2\pi\left(\frac{t}{T}-\frac{r}{\lambda}+\frac{z\sin\theta}{\lambda}\right)\mathrm{d}z + k\int_{d-\frac{a}{2}}^{d+\frac{a}{2}} \sin 2\pi\left(\frac{t}{T}-\frac{r}{\lambda}+\frac{z\sin\theta}{\lambda}\right)\mathrm{d}z\\
=& ka\left(\frac{\sin\beta}{\beta}\right)\sin 2\pi\left(\frac{t}{T}-\frac{r}{\lambda}\right) - \frac{k\lambda}{2\pi\sin\theta}\left[\cos 2\pi\left(\frac{t}{T}-\frac{r}{\lambda}+\frac{z\sin\theta}{\lambda}\right)\right]_{d-\frac{a}{2}}^{d+\frac{a}{2}}\\
=& ka\left(\frac{\sin\beta}{\beta}\right)\sin 2\pi\left(\frac{t}{T}-\frac{r}{\lambda}\right) - \frac{k\lambda}{2\pi\sin\theta}\\
&\left[\cos 2\pi\left(\frac{t}{T}-\frac{r}{\lambda}+\frac{d\sin\theta}{\lambda}+\frac{a\sin\theta}{2\lambda}\right) - \cos 2\pi\left(\frac{t}{T}-\frac{r}{\lambda}+\frac{d\sin\theta}{\lambda}-\frac{a\sin\theta}{2\lambda}\right)\right]\\
=& ka\left(\frac{\sin\beta}{\beta}\right)\sin 2\pi\left(\frac{t}{T}-\frac{r}{\lambda}\right) - \frac{k\lambda}{2\pi\sin\theta}\left[2\sin 2\pi\left(\frac{t}{T}-\frac{r}{\lambda}+\frac{d\sin\theta}{\lambda}\right)\sin 2\pi\left(-\frac{a\sin\theta}{2\lambda}\right)\right]\\
=& ka\left(\frac{\sin\beta}{\beta}\right)\sin 2\pi\left(\frac{t}{T}-\frac{r}{\lambda}\right) + \frac{k\lambda}{\pi\sin\theta}\sin 2\pi\left(\frac{t}{T}-\frac{r}{\lambda}+\frac{d\sin\theta}{\lambda}\right)\sin\left(\frac{\pi a\sin\theta}{\lambda}\right)\\
=& ka\left(\frac{\sin\beta}{\beta}\right)\sin 2\pi\left(\frac{t}{T}-\frac{r}{\lambda}\right) + ka\left(\frac{\sin\beta}{\beta}\right)\sin 2\pi\left(\frac{t}{T}-\frac{r}{\lambda}+\frac{d\sin\theta}{\lambda}\right) \qquad \left(\text{Since, } \beta = \frac{\pi a\sin\theta}{\lambda}\right)\\
=& ka\left(\frac{\sin\beta}{\beta}\right)\left[\sin 2\pi\left(\frac{t}{T}-\frac{r}{\lambda}\right) + \sin 2\pi\left(\frac{t}{T}-\frac{r}{\lambda}+\frac{d\sin\theta}{\lambda}\right)\right]\\
=& 2ka\left(\frac{\sin\beta}{\beta}\right)\sin 2\pi\left(\frac{t}{T}-\frac{r}{\lambda}+\frac{d\sin\theta}{2\lambda}\right)\cos 2\pi\left(\frac{d\sin\theta}{2\lambda}\right)\\
=& \frac{2ka\sin\beta}{\beta}\sin 2\pi\left(\frac{t}{T}-\frac{r}{\lambda}+\frac{d\sin\theta}{2\lambda}\right)\cos\left(\frac{\pi d\sin\theta}{\lambda}\right)
\end{aligned}$$

$$Let,\ \frac{\pi d\sin\theta}{\lambda} = \gamma$$

$$\therefore y = 2ka\frac{\sin\beta}{\beta}\cos\gamma\sin 2\pi\left(\frac{t}{T}-\frac{r}{\lambda}+\frac{d\sin\theta}{2\lambda}\right)$$

$$\therefore I = 4k^2a^2\left(\frac{\sin^2\beta}{\beta^2}\right)\cos^2\gamma$$

$$But,\ I_0 = k^2a^2 \qquad \therefore I = 4I_0\left(\frac{\sin^2\beta}{\beta^2}\right)\cos^2\gamma$$

The first term $\frac{\sin^2\beta}{\beta}$ represents the diffraction pattern produced by a single slit of width a and the second term $\cos^2\gamma$ represents the interference pattern produced by two point sources of equal intensity and phase difference $\gamma = \frac{\delta}{2}$ and separated by a distance d.

6.1.7 Position of maxima and minima

The equation tells us that the resultant intensity will be zero when either of the two terms is zero. For the first term, this will occur when, $\beta = \pi, 2\pi, 3\pi, ...$ And for the second term when, $\gamma = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, ...$
Since, by definition,

$$\delta = \frac{\pi}{\lambda} d \sin\theta \quad and \quad \beta = \frac{\pi}{\lambda} a \sin\theta$$

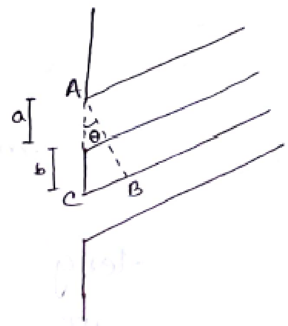
The corresponding angles of diffraction are given by the following relations,

$$a \sin\theta = \lambda, 2\lambda, 3\lambda, ... = m\lambda \quad where,\ m = 1, 2, 3, ... \qquad \text{(minima)}$$

$$and,\ d \sin\theta = \frac{\lambda}{2}, \frac{3\lambda}{2}, \frac{5\lambda}{2}, ... = (2n+1)\frac{\lambda}{2} \quad where,\ n = 0, 1, 2, ... \qquad \text{(minima)}$$

Points separated by a distance (a+b) in the two slits are known as corresponding points.

6.1.8 Missing order in a double slit diffraction pattern



The direction of interference maxima, $(a+b)\sin\theta = n\lambda$. From $\triangle ABC$,

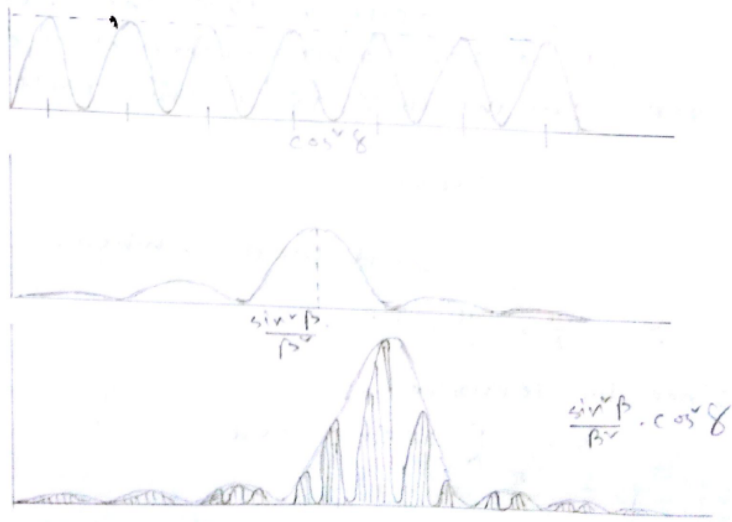
$$\sin\theta = \frac{BC}{a+b} \Rightarrow BC = (a+b)\sin\theta$$

The direction of interference minima, $a \sin\theta = p\lambda$. Here, n and p are natural numbers.
Note: interference maxima = diffraction minima and interference minima = diffraction maxima

$$if\ b = 0,\ then,\ \frac{n}{p} = 1 \Rightarrow n = p = 1, 2, 3, ...$$
$$if\ a = b,\ then,\ \frac{n}{p} = 2 \Rightarrow n = 2p = 2, 4, 6, ...$$
$$if\ 2a = b,\ then,\ \frac{n}{p} = 3 \Rightarrow n = 3p = 3, 6, 9, ...$$

These orders are missing in a double slit diffraction pattern.

6.1.9 Intensity curve



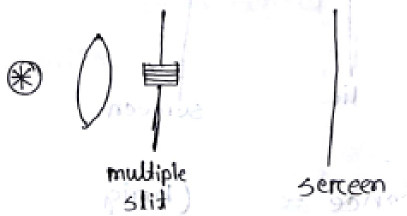
6.1.10 Difference between single slit diffraction and double slit diffraction

Single slit diffraction	Double slit diffraction
It is produced due to different part of same wavefront.	It is produced due to two different wavefronts.
Its central fringe is twice as wide as other fringes.	Its fringe width is of same size.
Intensity of fringes decreases as we go to successive maxima away from the center.	Fringes have same intensity.
At an angle $\frac{\lambda}{a}$, first minima is obtained.	At an angle $\frac{\lambda}{a}$, first maxima is obtained.

6.2 Diffraction Gratings

Any arrangement which is equivalent in its action to a large number of parallel equidistant slits of the same width, is known as the diffraction grating. The corresponding diffraction pattern is known as the grating spectrum.

A diffraction grating is an optical component with a periodic structure which splits and diffracts light into several beams traveling in different directions.



The diffraction gratings are of two kinds.

1. **Transmission grating:** When a wavefront is incident on a grating surface, light is transmitted through the slits and obstructed by the opaque portions, such a grating is called transmission grating.
2. **Reflection grating**

Gratings are prepared by ruling equidistant and parallel lines on a glass surface. The lines are drawn with a fine diamond point. The space in between any two lines is transparent to light and the lined portion is opaque to light. Such surface acts as transmission gratings. If on the other hand, the lines are drawn on a silver surface (plane or concave), then light is reflected from the position of the mirror between any two lines and such surfaces act as reflection gratings.

6.2.1 How to make a grating

1. Matter should be transparent.
2. Which light should be used, we should pull the number of line of light's wavelength.

6.2.2 The determination of wavelength

The wavelength λ can be measured by the law,

$$\lambda = \frac{(a+b)\sin\theta}{n}; \quad n = 1$$

Where θ is the angle of diffraction for 1st order. Thus the general formula

$$\lambda = \frac{(a+b)\sin\theta}{n}$$

6.2.3 Dispersive poser of grating

Dispersive power of a grating is defined as the ratio of the difference in the angle of diffraction of any two neighboring spectral lines to the difference in wavelength between two spectral lines. It is also defined as the difference in the angle of diffraction per unit change in wavelength. The diffraction of the $n-th$ order principle order can be written as

$$(a+b)\sin\theta = n\lambda$$

$$\Rightarrow (a+b)\cos\theta\, d\theta = n\, d\lambda$$

$$\Rightarrow \frac{d\theta}{d\lambda} = \frac{n}{(a+b)\cos\theta} = \frac{nN}{\cos\theta}$$

From this equation, it is clear that, the dispersive power of grating is

- directly proportional to the number of lines per *cm*
- inversely proportional to $cos\theta$

6.2.4 Angular dispersive power of the grating

If we differentiate $(a+b)\sin\theta = n\lambda$ with respect to λ keeping n constant, we get,

$$\frac{\Delta\theta}{\Delta\lambda} = \frac{n}{(a+b)\cos\theta} = \text{angular dispersive power of the grating}$$

$\therefore \frac{\Delta\theta}{\Delta\lambda}$, the ratio of the rate of change of the angle of diffraction with the change in wavelength is called the angular dispersive power of the grating.

6.2.5 Determination of wavelength of Sodium light by a diffraction grating

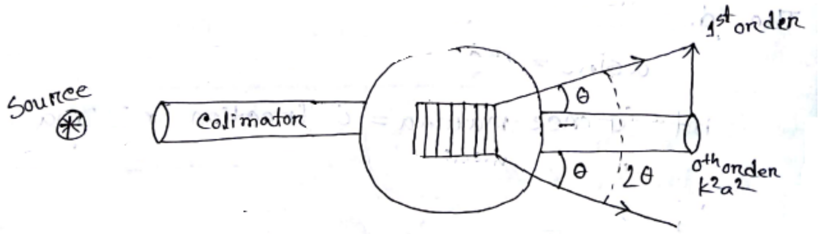
$$\frac{1}{a+b} = N$$

$$Here,\ N = \text{grating constant}$$

$$= 530nm^{-1}$$

$$a\sin\theta = n\lambda$$

$$\lambda = \frac{\sin 2\theta}{nN}$$

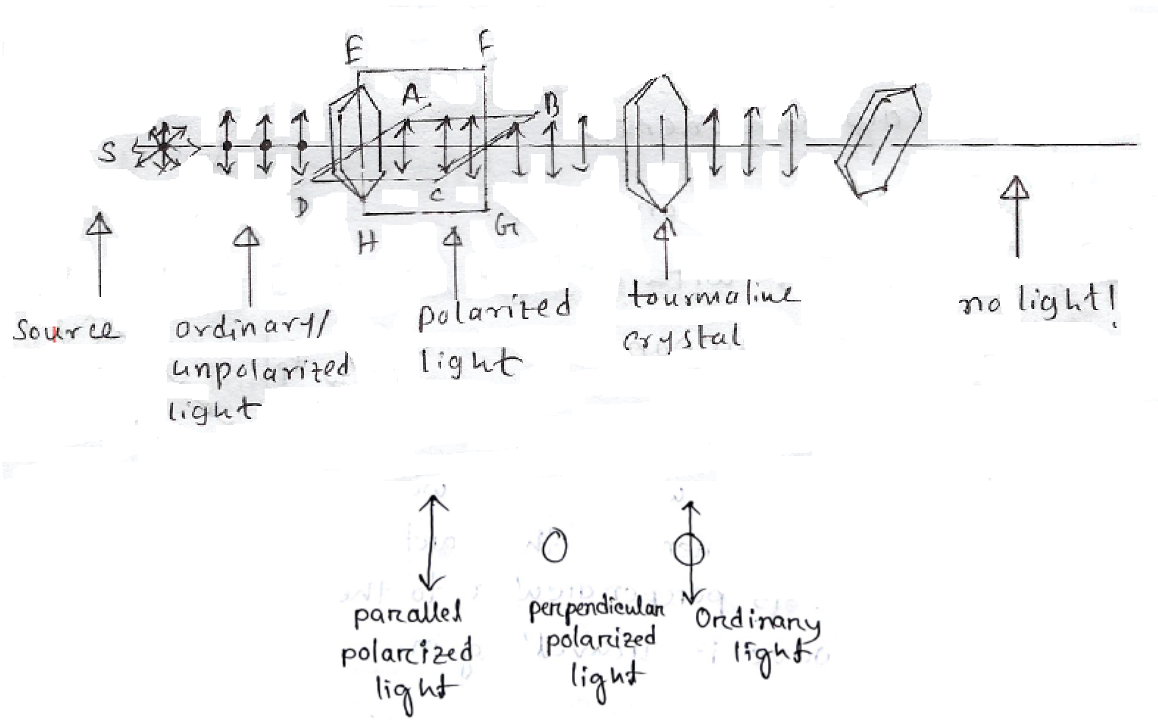


6.3 Fresnel's Diffraction

In this type of diffraction, the source of light and the screen are effectively at finite distances from the obstacle. Observation of Fresnel diffraction phenomenon does not require any lenses. The incident wave-front is not planar. As a result, the phase of secondary wavelets is not same at all points in the plane of the obstacle. The resultant amplitude at any point of screen is obtained by mutual interference of secondary wavelets from different wavelets of unblocked points of wavefront. It is experimentally simple but the analysis proves to be very complex.

7. Polarization

7.1 Introduction

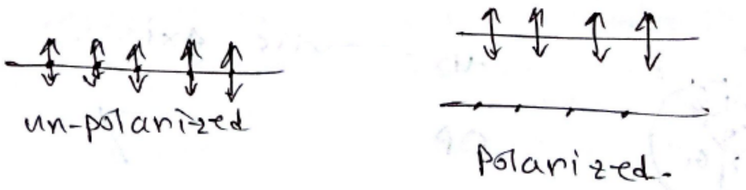


Polarization is the process by which lightwaves vibrating in different planes, can be made to vibrate in a particular plane.

Plane of Vibration The plane in which vibration takes place is known as the plane of vibration.
Plane of Polarization The plane in which no vibrations occur is the plane of polarization and is at right angle to the plane of vibration.

7.1.1 Polarized light and Unpolarized light

Light is an electromagnetic wave. It consists of an electric and magnetic field, oscillating at right angles to each other. In polarized light, oscillation takes place in single direction. Typically, when we consider the direction of oscillations, we consider the direction of oscillation of the electric field. Sunlight or light emitted by a filament lamp are unpolarized. This means that oscillations of light waves are not all in a single direction.



Plane Polarized Light The simplest type of electromagnetic wave, in which the direction of vibrations of electric vector E is strictly confined to a single direction in the plane perpendicular to the direction of propagation of the wave. Such a light is said to be plane polarized light. A plane polarized lightwave is a wave in which the electric vector is confined to a single plane.

Linearly Polarized Light If the wave is coming towards the eye, the electric vector appears executing a linear vibration normal to the ray direction. Such kind of wave is known as a linearly polarized light. A linearly polarized lightwave is a wave in which the electric vector oscillates in a given constant orientation.

Natural Light light in which plane of vibration are symmetrically distributed about the propagation direction of the wave, is known as the unpolarized light. An ordinary light source consists of a very large number of atomic emitters. Each atom radiates a plain polarized wave train for about $10^{-8} sec$. Natural light is unpolarized.

Polarized light waves are parallel to the plane. Ordinary light waves are perpendicular to the plane.

7.1.2 Specific rotation

Specific rotation is defined as the change in orientation of monochromatic plane-polarized light, per unit distance coneenfration product, as the light passes through a sample of a compound solution.

7.1.3 Distinction between polarized and unpolarized light

Polarized light	Unpolarized light
The oscillation is confined to only one plane.	The oscillation occurs in many planes.
It is absolutely coherent in nature.	It is incoherent in nature.
Its intensity depends on the nature of Polaroid used.	Its intensity depends on the nature of source.
In polarized light, electric vector is confined to a lane and magnetic vector H is normal to the plane.	In unpolarized light, plane of vibration of electric vector continuously and C random change.

7.2 Methods of producing Polarized Light

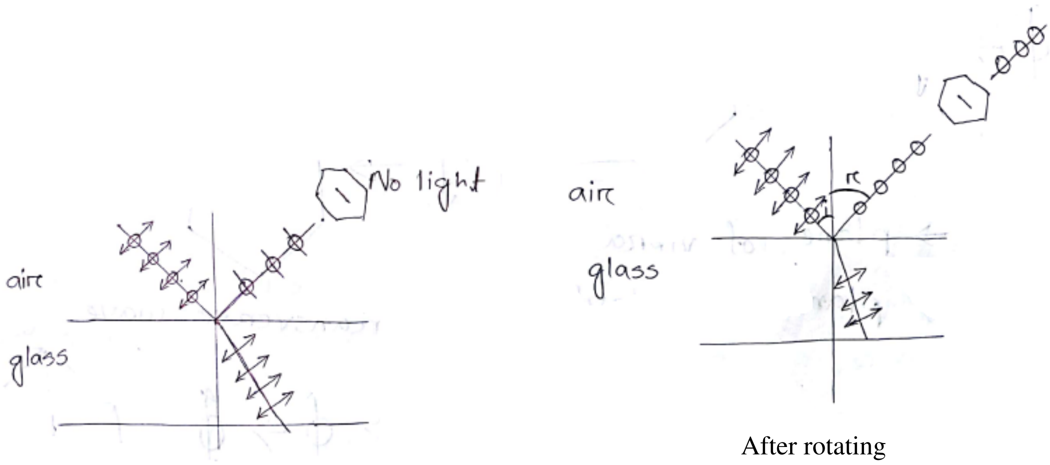
There are three important methods for producing plane-polarized light.

1. By reflection: polarization of light by reflection was first noted by Males in 1808.
2. By refraction: Brewster discussed thoroughly the phenomenon of polarization by refraction in 1812.
3. By double refraction: the phenomenon of polarization by double refraction was first observed by Bartholinus in 1669. However, comprehensive investigation were carried out by Hygens in 1690.

Polarization may also be produced by selective absorption and scattering of light.

7.2.1 Polarization by Reflection

In 1808, French engineer and scientist Males discovered polarization of natural light by reflection from the surface of glass. He noticed that, when natural light is incident on a plane sheet of glass at a certain angle, the reflected beam is plane-polarized. The reflected light consists mostly of dot components with a few arrow components. The refracted light consists of arrow components with a good number of dot components.This is true for all angle of incident, except for one angle. For this particular angle, none of the arrow components are reflected, but transmitted. The reflected light, although weak, is completely polarized. On the other hand, the



After rotating

Polarizing angle= 57.5°
$i \rightarrow$ angle of incident, $r \rightarrow$ angle of reflection. Here, $i = r$

refracted light is strong, but partially polarized. This particular angle of incidence, for which the reflected light becomes completely polarized, is known as the polarizing angle, which is 57.5° for glass.

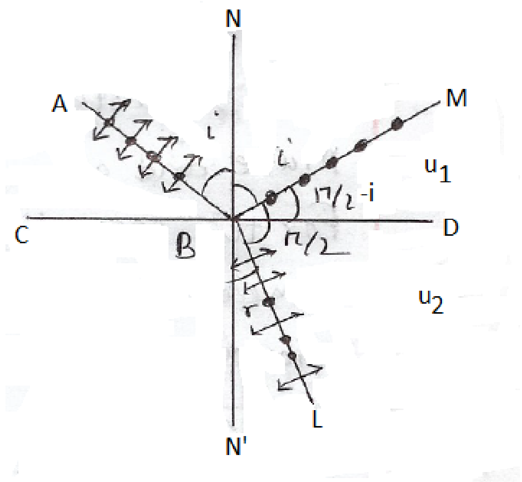
7.2.1.1 Brewster's law

Sir David Brewster, a Scottish physicist, found experimentally in 1812 that the polarizing angle depends upon the refractive indices of the reflective material and the surrounding medium in which it is placed. According to him,

"The tangent of the polarizing angle is equal to the refractive index of the reflective material with respect to its surroundings."

This is known as Brewster's law. It can be expressed for the figure as,

$$\tan i = \frac{\mu_2}{\mu_1} = {}_1\mu_2 \tag{7.1}$$



Where ${}_1\mu_2$ is the refractive index of the reflecting material with respect to its surrounding medium. Snell's law gives,

$$\frac{\sin i}{\sin r} = {}_1\mu_2 \tag{7.2}$$

From Equation 7.1 and 7.2,

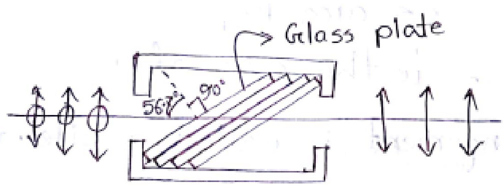
$$\sin r = \cos i = \sin\left(\frac{\pi}{2} - i\right)$$

$$\Rightarrow r = \frac{\pi}{2} - i$$
$$\Rightarrow r + i = \frac{\pi}{2}$$

Now, $\angle MBN = \angle NBA = i$ and $\angle MBN + \angle N'BL = 90°$. Hence, $\angle MBL = \angle MBD + \angle DBL = 90°$. This proves that, the reflected and refracted rays are 90° apart.

Brewster's angle: The direction of polarization (the way the electric field vectors point) is parallel to the plane of the interface. The special angle of incidence that produces a 90° angle between the reflected and refracted ray, is called the Brewster angle, θ_p.

7.2.2 Polarization by Refraction



When unpolarized light is incident at Brewster's angle on a smooth glass surface, the reflected light is totally polarized while the refracted light is partially polarized. If natural light is transmitted through a single plate, then it is partially polarized. If a stack of glass plates is used instead of single plate, reflections from successive surfaces to each glass plate filter the perpendicular component from the transmitted ray. The transmitted ray consists of only parallel components. About 15 glass plates are required for this purpose. The glass plates are kept inclined at an angle of 32.5° to the axis of the tube (dependant on μ). This arrangement is called **a pile of glass plate**. When unpolarized light is incident on the plates at Brewster angle, the transmitted light will be polarized and parallel to the plane of incidence.

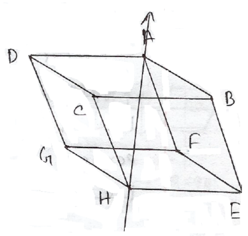
7.2.3 Polarization by Scattering

If a narrow beam of natural light is incident on a transparent medium containing a suspension of ultra-microscopic particles, the light scattered is partially polarized. The degree of polarization depends on the angle of scattering. The beam scattered at 90° with respect to the incident direction is linearly polarized. The maximum effect is observed on a clear day when the sun is near the horizon. The light reaching an observer on the ground from directly overhead, is polarized to the extent of 70% to 80%.

7.2.4 Polarization by double refraction

Erasmus Bartholinus, a Dutch philosopher, observed in 1669 that when a ray or ordinary light is incident on a calcite crystal ($CaCO_3$), it splits into two refracted rays. This phenomenon is known as double refraction. It was described later by Huygens.

7.2.4.1 Optical Axis of Crystal



The optical axis of a crystal is a direction within the crystal, parallel to straight line through either of the blunt corners and making equal angles with the three edges meeting there. The corners, where three obtuse angles meet, are known as blunt corners. A line drawn through A, making equal angles with each of the three edges, gives the direction of the optical axis. Any line parallel to this line, is also an optical axis. Therefore, optical axis is not a line but is a direction. Moreover, it is not defined by joining two blunt corners.

In a special case, when three edges of the crystal are equal, the line joining the blunt corners A and H coincide with crystallographic axis of the crystal, and it gives the direction of the optical axis. If a ray of light incident along the optic axis or parallel to it, then it will not split into two rays. In this case, double refraction would be absent.

7.2.4.2 Principle Section of Crystal

A plane which contains to optical axis and is perpendicular to the opposite faces of the crystal, is called the principle section of the crystal. It always cuts the surface of a calcite crystal in a parallelogram with angles 109° and 71°.

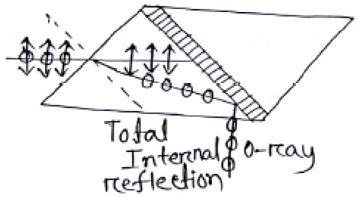
7.2.4.3 Ordinary Ray and Extraordinary Ray

The two rays produced in double refraction are linearly polarized in mutually perpendicular directions. The ray which obeys Snell's law of refraction is known as **Ordinary ray (O-ray)**. And that does not is known as **Extraordinary ray (E-ray)**.

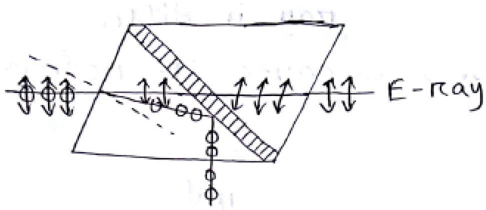
$$\mu_0 = \frac{\sin i}{\sin r_o} \qquad \mu_e = \frac{\sin i}{\sin r_e}$$

$$As\ r_e > r_o \quad \therefore \sin r_e > \sin r_o \quad \therefore \mu_e < \mu_o$$

Standard $\mu_o = 1.65$ and $\mu_e = 1.48$. Refractive index of Canada Balsam, $\mu_b = 1.55$.



O-Ray



E-Ray

7.2.4.4 Difference between e-ray and o-ray:

E-ray	**O-ray**
Extra-ordinary ray has refractive index of $$\mu_e = \frac{\sin i}{\sin r_e}$$.	Ordinary ray has refractive index of $$\mu_o = \frac{\sin i}{\sin r_o}$$.
It does not obey the law of refraction so its refractive index varies with angle of incidence and it is not fixed.	It obeys the law of refraction and its refractive index is constant.
Velocity of light inside the crystal is greater than o-ray.	Velocity of light inside the crystal is less than e-ray.
The velocity in the crystal is different in different directions as its refractive index varies.	It travels in the crystal with the same velocity in all directions.
Its vibration is parallel to the principle section of the crystal.	Its vibration is perpendicular to the principle section of the crystal.
It consists of dot components.	It consists of arrow components.

7.2.4.5 Superposition of O-ray and E-ray

When O-ray and E-ray overlap on each other after emerging from an anisotropic crystal plate, it is obvious that they cannot produce interference fringes as in a double slit experiment. Instead, they combine to produce different states of polarization, depending upon their optical path difference.

1. When the optical path difference is 0, or a multiple of $\lambda/2$, the resultant lightwave is linearly polarized.
2. When the optical path difference is $\lambda/4$, the resultant lightwave is elliptically polarized.
3. In a particular instance, when the wave amplitudes are equal and the optical path difference is $\lambda/4$, the resultant lightwave is circularly polarized.

7.2.4.6 Principle Plane

A plane in the crystal drawn through the principle axis and the O-ray, is called the principle plane of O-ray. Similarly, the plane through the optical axis and E-ray is called the principle plane of E-ray.

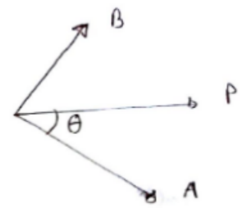
In general, the two planes do not coincide. In a particular case, when the plane of incidence is principle section, the principle plane of O-ray and E-ray coincide.

7.2.5 Intensity of polarized light: Malus' law

"The intensity of polarized light transmitted through the analyzer varies as the square of the cosine angle between the plane of transmission of the analyzer and the plane of polarizer, i.e. $I \propto \cos^2\theta$"

Here, $\theta =$ the angle between the transmission axes of the analyzer and the polarizer

$$
\begin{aligned}
\text{Intensity of } OA &= A\cos\theta \\
&\Rightarrow I = A^2\cos^2\theta \\
&\Rightarrow I = I_0\cos^2\theta \\
&\Rightarrow \frac{I}{I_0} = \cos^2\theta \\
\textit{when},\ \theta &= 90^\circ,\ I = 0 \\
\theta &= 0^\circ,\ I = I_0
\end{aligned}
$$

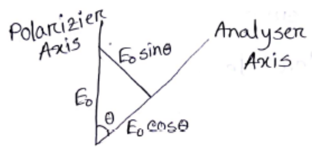


Alternate Question: describe the effect of analyzer on plane polarized light.

Proof: Here, θ is the angle between the transmission axes of the analyzer and the polarizer. The completely plane polarized light form the polarizer is incident on the analyzer. If E_0 is the amplitude of the electric vector transmitted by the polarizer, then intensity I_0 of the light incident on the analyzer is $I_0 \propto E_0^2$.

The electric field vector E_0 can be resolved into two components *i.e.* $E_0\cos\theta$ and $E_0\sin\theta$. The analyzer will transmit only the component $E_0\cos\theta$; $E_0\sin\theta$ will be absorbed by the analyzer. Therefore, the intensity I of the light transmitted by the analyzer is,

$$
\begin{aligned}
I &\propto (E_0\cos\theta)^2 \\
\therefore \frac{I}{I_0} &= \frac{(E_0\cos\theta)^2}{E_0^2} \\
\Rightarrow I &= I_0\cos^2\theta \\
\therefore I &\propto \cos^2\theta
\end{aligned}
$$



1. When the optical axes of both polarimeter and analyzer are parallel, *i.e.* $\theta = 0^\circ = 180^\circ$

$$I = I_0\cos^2 0^\circ = I_0$$

Intensity will be maximum.

2. When the optical axes of both polarimeter and analyzer are at right angle, *i.e.* $\theta = 90^\circ = 270^\circ$

$$I = I_0\cos^2 90^\circ = 0$$

Intensity will be zero.

7.2.6 Effect of Polarizer on Natural Light

Alternate question: Show that, if unpolarized light of intensity I_0 is incident on a polarizer, the intensity of light transmitted through the polarizer is $I_0/2$.

In unpolarized light, all values of θ, starting from 0 to 2π, are equally probable.

$$
\begin{aligned}
I &= \frac{I_0}{2\pi}\int_0^{2\pi}\cos^2\theta\, d\theta \\
&= \frac{I_0}{4\pi}\int_0^{2\pi}2\cos^2\theta\, d\theta \\
&= \frac{I_0}{4\pi}\int_0^{2\pi}(1+\cos 2\theta)\, d\theta \\
&= \frac{I_0}{4\pi}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}
\end{aligned}
$$

$$= \frac{I_0}{4\pi} \times 2\pi$$
$$= \frac{I_0}{2}$$

7.2.7 Effect of Polarizer on Transmission of Polarized light

If unpolarized light is incident on a polarizer, the intensity of transmitted polarized light will be half the intensity of unpolarized light incident on the polarizer. If partially polarized light is incident on a polarizer, the intensity of the transmitted light will be dependant on the direction of the transmission axis of the polarizer. Two positions of both I_{max} and I_{min} occur in one complete rotation.
If plane polarized light is incident on a polarizer, the intensity of transmitted light varies from zero to a maximum value. Two positions of zero intensity and two positions of maximum intensity occur in one complete rotation of the polarizer.
When circularly polarized light is incident on a polarizer, the intensity of the transmitted light stays constant in any position of the polarizer. In case of elliptically polarized light, intensity of light transmitted through the polarizer, varies with the rotation of the polarizer, from I_{max} to I_{min}.

7.2.8 Analysis of Polarized Light

The following steps are used in the analysis of the type of polarization-

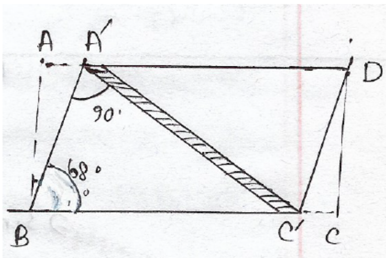
- The light of unknown polarization is allowed to fall normally on a polarimeter. The polarizer is slowly rotated through a full circle and the intensity of the transmitted light is observed. If the intensity of the light transmitted is extinguished twice in one full rotation of the polarizer, then the incident light is plane-polarized.
- If the intensity of transmitted light varies between a maximum and a minimum value, but does not become extinguished in any position of the polarizer, then the incident light is either ellyptically or partially polarized.
- If the intensity of transmitted light remains constant on rotation of polarizer, then the incident light is either circularly polarized, or completely unpolarized.

7.3 Nicole prism

Nicole prism is an optical device for producing and analyzing polarized light and is based on the phenomenon of double refraction. It is made from calcite crystal. It polarizes unpolarized light and analyzes plane polarized light in the practice.

7.3.1 Construction of Nicole Prism

A calcite crystal with length three time than its breadth is taken. Its principle section is a parallelogram with 71° and 109° angles. The two faces are ground in such a way that the angles become 68° and 90°. The crystal is then cut along AC' and joined together with Canada Balsam.

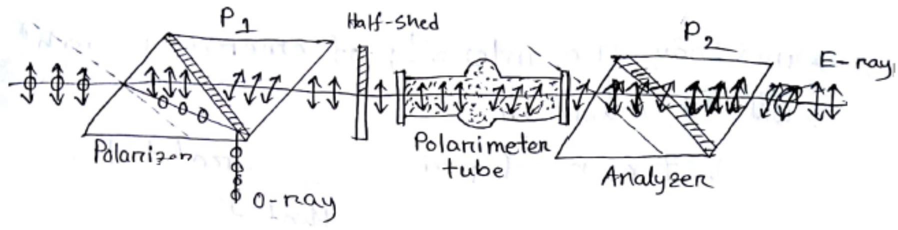


7.3.2 Optical axis and Principle axis

A plane which contains an optic axis and is perpendicular to the opposite phase of the crystal, is called principle section /plane of the crystal. For n phases principle plane will be $\frac{n}{2}$. When the three edges of the crystal are equal, the line joining two blunt corners coincides with the crystallo graphics axis of the crystal and it gives the direction of the optic axis. If there is one optical axis, then it is called uniaxial crystal, *e.g.* calcite crystal. If there are more than one optical axis, then it is called biaxial crystal, *e.g.* mica crystal.

7.3.3 Nicole prism as a polarizer and an analyzer

In order to produce and analyze plane polarized light, two Nicole prism needs to be arranged. When a beam of unpolarized light is incident on the Nicole prism, emergent beam from the prism is obtained as plain polarized



and which has vibrations parallel to the principle section. The prism is therefore known as polarizer. If this polarized beam falls on another parallel Nicole prism P_2 after passing through half-shed and polarimeter tube, whose principle section is parallel to that of P_1, the incident beam will behave as e-ray inside the Nicole prism P_2 and completely transmitted through it. This way the intensity of emergent light will be maximum.
Now, the Nicole prism P_2 is rotated about the axis, then we note that the intensity of emerging light decreases and becomes zero at 90° rotation of the second prism. In this position, the vibrations of e-ray becomes perpendicular to the principle section of the analyzer (Nicole prism P_2). Hence, this ray behaves as o-ray for prism P_2 and it is totally internally reflected by Canada balsam layer. This fact can be used for detecting the plane polarized light and the Nicole prism P_2 acts as an analyzer.
If the Nicole prism P_2 is further rotated about its axis, the intensity of light emerging from it increases and becomes maximum for the position when principle section of P_2 is again parallel to that of P_1.
Hence, the Nicole prisms P_1 and P_2 acts as polarizer and analyzer, respectively.

7.4 Optical Activity

The ability to rotate the plane of polarization of plane polarized light by certain substance, is called optical activity. Substances, which have the ability to rotate the plane of polarized light passing through them, are called optically active substances. For example: quartz and cinnabar.

7.4.1 Classification of Optically Active Substance

Optically active substances are classified into two types.

1. **Dextrorotatory Substance:** substances which rotate the plane of polarization of the light towards the right, are known as right-handed, or dextrorotatory.
2. **Laevorotatory Substance:** substances which rotate the plane of polarization of the light towards the left, are known as left-handed, or laevorotatory.

7.4.2 Specific Rotation

It is the rotation produced by one decimeter long column of the solution containing 1gm of optically active material per $c.c$ of solution.

7.5 Optics of Crystals

An optically isotropic material is one in which the index of refraction is the same in all directions. Glass, water, air are examples of isotropic materials. The atoms in a crystal are arranged in a regular periodic manner.
If the arrangement of atoms differs in different directions within a crystal, then the physical properties vary with the direction. The thermal conductivity, electric conductivity, velocity of light and hence refractive index, etc properties depends on the crystallographic direction along which the property is measured. Then we say that the crystal is anisotropic. In such crystal, the force of interaction between the electron cloud and the lattice, is different in different crystallographic directions.
There are two types of anisotropic crystals.

1. **Bi-axial crystal:** In this type of crystal, both refracted rays are e-rays *i.e.* mica, topaz, aragonite.
2. **Uniaxial crystal:** In this type of crystal, one of the refracted rays is o-ray and the other is an e-ray *i.e.* calcite, tourmaline, quartz.

7.6 Polarimeter

It is an instrument used for determining the optical rotation of solution. When used for determining the quantity of sugar in a solution, it is called Soceharimeter. Polarimeter makes the light parallel because there is a convex lens in it.

Resources

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