7.3 FUZZY LOGIC

In crisp logic, the truth values acquired by propositions or predicates are 2-valued, namely *True*, *False* which may be treated numerically equivalent to (0, 1). However, in fuzzy logic, truth values are multivalued such as *absolutely true, partly true, absolutely false, very true*, and so on and are numerically equivalent to (0–1).

Fuzzy propositions

A *fuzzy proposition* is a statement which acquires a fuzzy truth value. Thus, given $\tilde{P}$ to be a fuzzy proposition, $T(\tilde{P})$ represents the truth value (0–1) attached to $\tilde{P}$. In its simplest form, fuzzy propositions are associated with fuzzy sets. The fuzzy membership value associated with the fuzzy set $\tilde{A}$ for $\tilde{P}$ is treated as the fuzzy truth value $T(\tilde{P})$.

i.e. $T(\tilde{P}) = \mu_{\tilde{A}}(x)$ where $0 \le \mu_{\tilde{A}}(x) \le 1$ (7.17)

Example

$\tilde{P}$: Ram is honest.

$T(\tilde{P}) = 0.8$, if $\tilde{P}$ is partly true.

$T(\tilde{P}) = 1$, if $\tilde{P}$ is absolutely true.

Fuzzy connectives

Fuzzy logic similar to crisp logic supports the following connectives:

(i) *Negation* : $-$

(ii) *Disjunction* : $\vee$

(iii) *Conjunction* : $\wedge$

(iv) *Implication* : $\Rightarrow$

Table 7.3 illustrates the definition of the connectives. Here $\tilde{P}$, $\tilde{Q}$ are fuzzy propositions and $T(\tilde{P})$, $T(\tilde{Q})$, are their truth values.

Table 7.3 Fuzzy connectives

Symbol	Connective	Usage	Definition
$-$	Negation	$\bar{\tilde{P}}$	$1 - T(\tilde{P})$
$\vee$	Disjunction	$\tilde{P} \vee \tilde{Q}$	$\max(T(\tilde{P}), T(\tilde{Q}))$
$\wedge$	Conjunction	$\tilde{P} \wedge \tilde{Q}$	$\min(T(\tilde{P}), T(\tilde{Q}))$
$\Rightarrow$	Implication	$\tilde{P} \Rightarrow \tilde{Q}$	$\sim\tilde{P} \vee \tilde{Q} = \max(1 - T(\tilde{P}), T(\tilde{Q}))$

$\tilde{P}$ and $\tilde{Q}$ related by the '$\Rightarrow$' operator are known as antecedent and consequent respectively. Also, just as in crisp logic, here too, '$\Rightarrow$' represents the IF-THEN statement as

IF x is $\tilde{A}$ THEN y is $\tilde{B}$, and is equivalent to

$$\tilde{R} = (\tilde{A} \times \tilde{B}) \cup (\bar{\tilde{A}} \times \tilde{Y}) \tag{7.18}$$

The membership function of $\tilde{R}$ is given by

$$\mu_{\tilde{R}}(x, y) = \max(\min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)), 1 - \mu_{\tilde{A}}(x)) \tag{7.19}$$

Also, for the compound implication IF x is $\tilde{A}$ THEN y is $\tilde{B}$ ELSE y is $\tilde{C}$ the relation R is equivalent to

$$\tilde{R} = (\tilde{A} \times \tilde{B}) \cup (\bar{\tilde{A}} \times C) \tag{7.20}$$

The membership function of $\tilde{R}$ is given by

$$\mu_{\tilde{R}}(x, y) = \max(\min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)), \min(1 - \mu_{\tilde{A}}(x), \mu_{\tilde{C}}(y))) \tag{7.21}$$

Example

$\tilde{P}$: Mary is efficient, $T(\tilde{P}) = 0.8$

$\tilde{Q}$: Ram is efficient, $T(\tilde{Q}) = 0.65$

(i) $\bar{\tilde{P}}$: Mary is not efficient.

$$T(\bar{\tilde{P}}) = 1 - T(\tilde{P}) = 1 - 0.8 = 0.2$$

(ii) $\tilde{P} \wedge \tilde{Q}$: Mary is efficient and so is Ram.

$$\begin{aligned} T(\tilde{P} \wedge \tilde{Q}) &= \min(T(\tilde{P}), T(\tilde{Q})) \\ &= \min(0.8, 0.65) \\ &= 0.65 \end{aligned}$$

(iii) $T(\tilde{P} \vee \tilde{Q})$: Either Mary or Ram is efficient.

$$\begin{aligned} T(\tilde{P} \vee \tilde{Q}) &= \max(T(\tilde{P}), T(\tilde{Q})) \\ &= \max(0.8, 0.65) \\ &= 0.8 \end{aligned}$$

(iv) $\tilde{P} \Rightarrow \tilde{Q}$: If Mary is efficient then so is Ram.

$$\begin{aligned} T(\tilde{P} \Rightarrow \tilde{Q}) &= \max(1 - T(\tilde{P}), T(\tilde{Q})) \\ &= \max(0.2, 0.65) \\ &= 0.65 \end{aligned}$$

Example 7.10

Let $X = \{a, b, c, d\}$ $Y = \{1, 2, 3, 4\}$

and $\tilde{A} = \{(a, 0)(b, 0.8)(c, 0.6)(d, 1)\}$

$\tilde{B} = \{(1, 0.2)(2, 1)(3, 0.8)(4, 0)\}$

$\tilde{C} = \{(1, 0)(2, 0.4)(3, 1)(4, 0.8)\}$

Determine the implication relations

(i) IF x is $\tilde{A}$ THEN y is $\tilde{B}$.

(ii) IF x is $\tilde{A}$ THEN y is $\tilde{B}$ ELSE y is $\tilde{C}$.

Solution

To determine (i) compute

$$\tilde{R} = (\tilde{A} \times \tilde{B}) \cup (\bar{\tilde{A}} \times Y) \quad \text{where}$$

$$\mu_{\tilde{R}}(x, y) = \max(\min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)), 1 - \mu_{\tilde{A}}(x))$$

$$\tilde{A} \times \tilde{B} = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.2 & 0.8 & 0.8 & 0 \\ 0.2 & 0.6 & 0.6 & 0 \\ 0.2 & 1 & 0.8 & 0 \end{bmatrix} \end{array}$$

$$\bar{\tilde{A}} \times Y = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.2 & 0.2 & 0.2 & 0.2 \\ 0.4 & 0.4 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Here, Y the universe of discourse could be viewed as $\{(1, 1)\ (2, 1)\ (3, 1)\ (4, 1)\}$ a fuzzy set all of whose elements x have $\mu(x) = 1$.

Therefore.

$$\tilde{R} = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.2 & 0.8 & 0.8 & 0.2 \\ 0.4 & 0.6 & 0.6 & 0.4 \\ 0.2 & 0.1 & 0.8 & 0 \end{bmatrix} \end{array}$$

which represents IF x is $\tilde{A}$ THEN y is $\tilde{B}$.

To determine (ii) compute

$$\tilde{R} = (\tilde{A} \times \tilde{B}) \cup (\bar{\tilde{A}} \times \tilde{C}) \text{ where}$$

$$\mu_{\tilde{R}}(x, y) = \max(\min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)), \min(1 - \mu_{\tilde{A}}(x), \mu_{\tilde{C}}(y)))$$

$$\tilde{A} \times \tilde{B} = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.2 & 0.8 & 0.8 & 0 \\ 0.2 & 0.6 & 0.6 & 0 \\ 0.2 & 1 & 0.8 & 0 \end{bmatrix} \end{array}$$

$$\bar{\tilde{A}} \times \tilde{C} = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 0.4 & 1 & 0.8 \\ 0 & 0.2 & 0.2 & 0.2 \\ 0 & 0.4 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Therefore,

$$\tilde{R} = \max((\tilde{A} \times \tilde{B}), (\bar{\tilde{A}} \times \tilde{C})) \text{ gives}$$

$$\tilde{R} = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 0.4 & 1 & 0.8 \\ 0.2 & 0.8 & 0.8 & 0.2 \\ 0.2 & 0.6 & 0.6 & 0.4 \\ 0.2 & 1 & 0.8 & 0 \end{bmatrix} \end{array}$$

The above $\tilde{R}$ represents IF x is $\tilde{A}$ THEN y is $\tilde{B}$ ELSE y is $\tilde{C}$.

7.3.1 Fuzzy Quantifiers

Just as in crisp logic where predicates are quantified by quantifiers, fuzzy logic propositions are also quantified by *fuzzy quantifiers*. There are two classes of fuzzy quantifiers such as

(i) Absolute quantifiers and

(ii) Relative quantifiers

While absolute quantifiers are defined over $\Re$, relative quantifiers are defined over [0–1].

Example

Absolute quantifier	Relative quantifier
round about 250	almost
much greater than 6	about
some where around 20	most

7.3.2 Fuzzy Inference

Fuzzy inference also referred to as *approximate reasoning* refers to computational procedures used for evaluating linguistic descriptions. The two important inferring procedures are

(i) Generalized Modus Ponens (GMP)

(ii) Generalized Modus Tollens (GMT)

GMP is formally stated as

$$\begin{array}{ll} \text{IF} & x \text{ is } \tilde{A} \text{ THEN } y \text{ is } \tilde{B} \\ & \quad x \text{ is } \tilde{A}' \\ \hline & \quad y \text{ is } \tilde{B}' \end{array} \tag{7.22}$$

Here, $\tilde{A}$, $\tilde{B}$, $\tilde{A}'$ and $\tilde{B}'$ are fuzzy terms. Every fuzzy linguistic statement above the line is analytically known and what is below is analytically unknown.

To compute the membership function of $\tilde{B}'$, the max-min composition of fuzzy set $\tilde{A}'$ with $\tilde{R}\,(x, y)$ which is the known implication relation (IF-THEN relation) is used. That is,

$$\tilde{B}' = \tilde{A}' \circ \tilde{R}(x, y) \tag{7.23}$$

In terms of membership function,

$$\mu_{\tilde{B}'}(y) = \max\,(\min\,(\mu_{\tilde{A}'}(x), \mu_{\tilde{R}}(x, y))) \tag{7.24}$$

where $\mu_{\tilde{A}'}(x)$ is the membership function of $\tilde{A}'$, $\mu_{\tilde{R}}(x, y)$ is the membership function of the implication relation and $\mu_{\tilde{B}'}(y)$ is the membership function of $\tilde{B}'$.

On the other hand, GMT has the form

$$\begin{array}{ll} \text{IF} & x \text{ is } \tilde{A} \text{ THEN } y \text{ is } \tilde{B} \\ & \quad y \text{ is } \tilde{B}' \\ \hline & \quad x \text{ is } \tilde{A}' \end{array}$$

The membership of $\tilde{A}'$ is computed on similar lines as

$$\tilde{A}' = \tilde{B}' \circ \tilde{R}(x, y)$$

In terms of membership function,

$$\mu_{\tilde{A}'}(x) = \max\,(\min\,(\mu_{\tilde{B}'}(y), \mu_{\tilde{R}}(x, y))) \tag{7.25}$$

Example

Apply the fuzzy Modus Ponens rule to deduce Rotation is quite slow given

(i) If the temperature is high then the rotation is slow.

(ii) The temperature is very high.

Let H (High), VH (Very High), $\tilde{S}$ (Slow) and $\tilde{QS}$ (Quite Slow) indicate the associated fuzzy sets as follows:

For $X = \{30, 40, 50, 60, 70, 80, 90, 100\}$, the set of temperatures and $Y = \{10, 20, 30, 40, 50, 60\}$, the set of rotations per minute,

$$\tilde{H} = \{(70, 1)\ (80, 1)\ (90, 0.3)\}$$

$$\tilde{VH} = \{(90, 0.9)\ (100, 1)\}$$

$$\tilde{QS} = \{(10, 1)\ (20, 0.8)\}$$

$$\tilde{S} = \{(30, 0.8)\ (40, 1)\ (50, 0.6)\}$$

To derive $\tilde{R}(x, y)$ representing the implication relation (i), we need to compute

$$\tilde{R}(x, y) = \max(\tilde{H} \times \tilde{S}, \bar{\tilde{H}} \times Y)$$

$$\tilde{H} \times \tilde{S} = \begin{array}{c} \\ 30 \\ 40 \\ 50 \\ 60 \\ 70 \\ 80 \\ 90 \\ 100 \end{array} \begin{array}{c} \begin{array}{cccccc} 10 & 20 & 30 & 40 & 50 & 60 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0.8 & 1 & 0.6 & 0 \\ 0 & 0 & 0.8 & 1 & 0.6 & 0 \\ 0 & 0.3 & 0.3 & 0.3 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\bar{\tilde{H}} \times Y = \begin{array}{c} \\ 30 \\ 40 \\ 50 \\ 60 \\ 70 \\ 80 \\ 90 \\ 100 \end{array} \begin{array}{c} \begin{array}{cccccc} 10 & 20 & 30 & 40 & 50 & 60 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0.7 & 0.7 & 0.7 & 0.7 & 0.7 & 0.7 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$\tilde{R}(x,y) = \begin{array}{c|cccccc} & 10 & 20 & 30 & 40 & 50 & 60 \\ \hline 30 & 1 & 1 & 1 & 1 & 1 & 1 \\ 40 & 1 & 1 & 1 & 1 & 1 & 1 \\ 50 & 1 & 1 & 1 & 1 & 1 & 1 \\ 60 & 1 & 1 & 1 & 1 & 1 & 1 \\ 70 & 0 & 0.8 & 0.8 & 1 & 0.6 & 0 \\ 80 & 0 & 0 & 0.8 & 1 & 0.6 & 0 \\ 90 & 0.7 & 0.7 & 0.7 & 0.7 & 0.7 & 0.7 \\ 100 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}$$

To deduce Rotation is quite slow we make use of the composition rule

$$\tilde{QS} = \tilde{VH} \circ \tilde{R}(x,y)$$

$$= [0\ 0\ 0\ 0\ 0\ 0\ 0.9\ 1] \times \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0.8 & 0.8 & 1 & 0.6 & 0 \\ 0 & 0 & 0.8 & 1 & 0.6 & 0 \\ 0.7 & 0.7 & 0.7 & 0.7 & 0.7 & 0.7 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= [1\ 1\ 1\ 1\ 1\ 1]$$

7.4 FUZZY RULE BASED SYSTEM

Fuzzy linguistic descriptions are formal representations of systems made through fuzzy IF-THEN rules. They encode knowledge about a system in statements of the form—
IF (a set of conditions) are satisfied THEN (a set of consequents) can be inferred.
Fuzzy IF-THEN rules are coded in the form—

$$\text{IF } (x_1 \text{ is } \tilde{A}_1, x_2 \text{ is } \tilde{A}_2, \ldots, x_n \text{ is } \tilde{A}_n) \text{ THEN } (y_1 \text{ is } \tilde{B}_1, y_2 \text{ is } \tilde{B}_2, \ldots, y_n \text{ is } \tilde{B}_n).$$

where linguistic variables x_i, y_j take the values of fuzzy sets A_i and B_j respectively.

Example

If there is heavy rain and strong winds
then there must be severe flood warning.

Here, heavy, strong, and severe are fuzzy sets qualifying the variables rain, wind, and flood warning respectively.

A collection of rules referring to a particular system is known as a *fuzzy rule base*. If the conclusion C to be drawn from a rule base R is the conjunction of all the individual consequents C_i of each rule, then

$$C = C_1 \cap C_2 \cap \ldots \cap C_n \tag{7.26}$$

where

$$\mu_C(y) = \min(\mu_{C_1}(y), \mu_{C_2}(y), \ldots, \mu_{C_n}(y)), \forall y \in Y \tag{7.27}$$

where Y is the universe of discourse.

On the other hand, if the conclusion C to be drawn from a rule base R is the disjunction of the individual consequents of each rule, then

$$C = C_1 \cup C_2 \cup C_3 \ldots \cup C_n \tag{7.28}$$

where

$$\mu_C(y) = \max(\mu_{C_1}(y), \mu_{C_2}(y), \ldots, \mu_{C_n}(y)), \forall y \in Y \tag{7.29}$$

7.5 DEFUZZIFICATION

In many situations, for a system whose output is fuzzy, it is easier to take a crisp decision if the output is represented as a single scalar quantity. This conversion of a fuzzy set to single crisp value is called *defuzzification* and is the reverse process of *fuzzification*.

Several methods are available in the literature (Hellendoorn and Thomas, 1993) of which we illustrate a few of the widely used methods, namely *centroid method, centre of sums,* and *mean of maxima*.

Centroid method

Also known as the *centre of gravity* or the *centre of area* method, it obtains the centre of area (x^*) occupied by the fuzzy set. It is given by the expression

$$x^* = \frac{\int \mu(x)\, x\, dx}{\int \mu(x)\, dx} \tag{7.30}$$

for a continuous membership function, and

$$x^* = \frac{\sum_{i=1}^{n} x_i \cdot \mu(x_i)}{\sum_{i=1}^{n} \mu(x_i)} \tag{7.31}$$

for a discrete membership function.

Here, n represents the number of elements in the sample, x_i's are the elements, and $\mu(x_i)$ is its membership function.

Centre of sums (COS) method

In the centroid method, the overlapping area is counted once whereas in *centre of sums*, the overlapping area is counted twice. COS builds the resultant membership function by taking the algebraic sum of outputs from each of the contributing fuzzy sets $\tilde{A}_1$, $\tilde{A}_2, \ldots$, etc. The defuzzified value x^* is given by

$$x^* = \frac{\sum_{i=1}^{N} x_i \cdot \sum_{k=1}^{n} \mu_{\tilde{A}_k}(x_i)}{\sum_{i=1}^{N} \sum_{k=1}^{n} \mu_{\tilde{A}_k}(x_i)} \tag{7.32}$$

COS is actually the most commonly used defuzzification method. It can be implemented easily and leads to rather fast inference cycles.

Mean of maxima (MOM) defuzzification

One simple way of defuzzifying the output is to take the crisp value with the highest degree of membership. In cases with more than one element having the maximum value, the mean value of the maxima is taken. The equation of the defuzzified value x^* is given by

$$x^* = \frac{\sum_{x_i \in M} x_i}{|M|} \tag{7.33}$$

where $M = \{x_i \mid \mu(x_i)$ is equal to the *height* of fuzzy set$\}$
$|M|$ is the cardinality of the set M. In the continuous case, M could be defined as

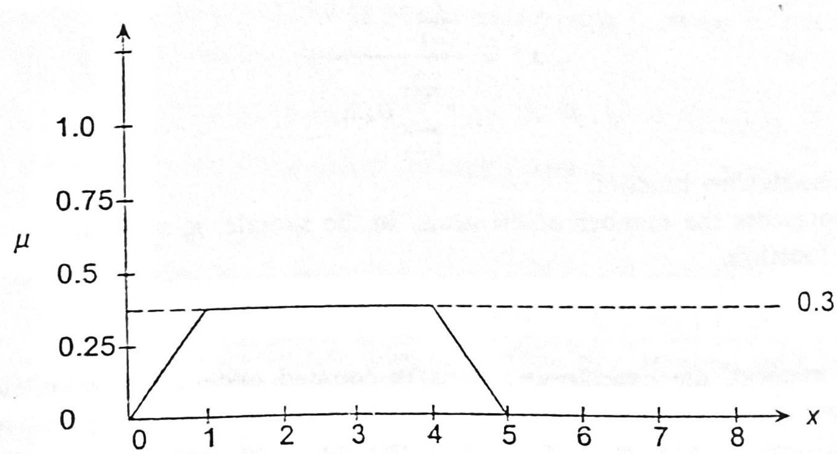
$$M = \{x \in [-c, c] \mid \mu(x) \text{ is equal to the height of the fuzzy set}\} \tag{7.34}$$

In such a case, the *mean of maxima* is the arithmetic average of mean values of all intervals contained in M including zero length intervals.

The *height* of a fuzzy set A, i.e. $h(A)$ is the largest membership grade obtained by any element in that set.

Example

$\tilde{A}_1$, $\tilde{A}_2$, and $\tilde{A}_3$ are three fuzzy sets as shown in Fig. 7.1(a), (b), and (c). Figure 7.2 illustrates the aggregate of the fuzzy sets.



(a) $\tilde{A}_1$

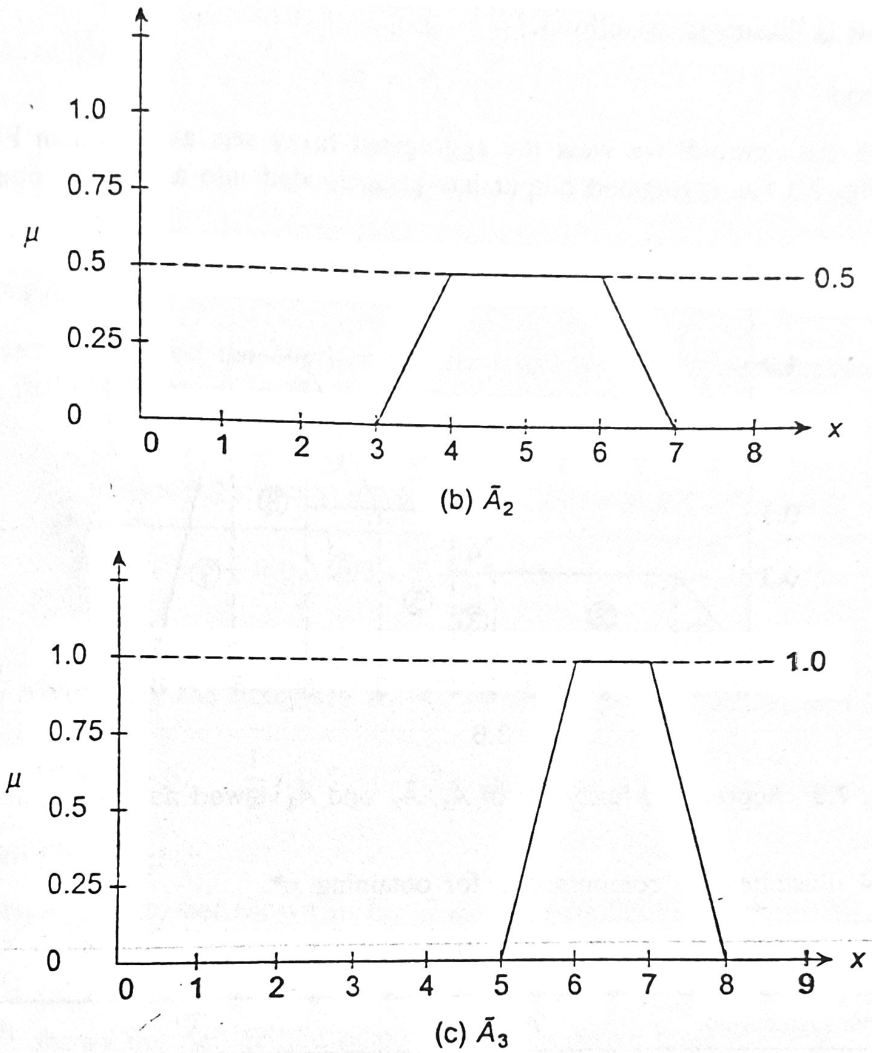


Fig. 7.1 Fuzzy sets $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{A}_3$.

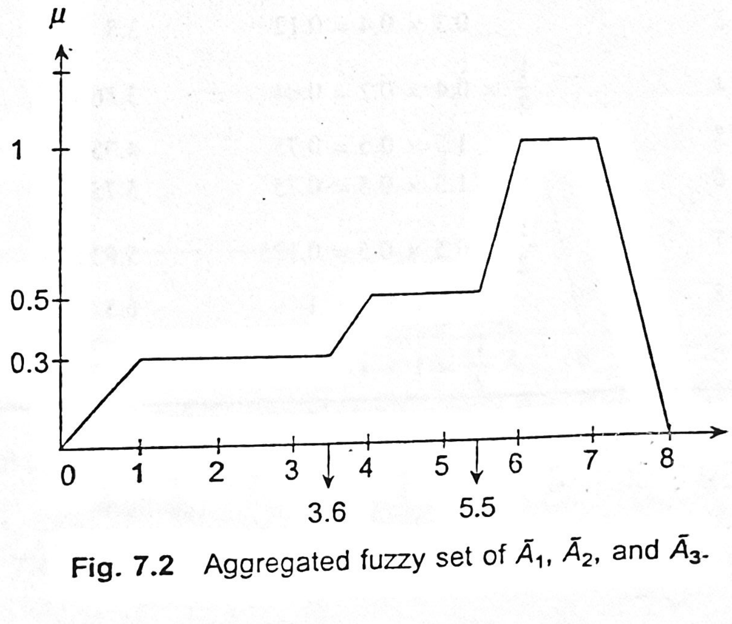


Fig. 7.2 Aggregated fuzzy set of $\tilde{A}_1$, $\tilde{A}_2$, and $\tilde{A}_3$.

The defuzzification using (i) centroid method, (ii) centre of sums method, and (iii) mean of maxima method is illustrated as follows.

Centroid method

To compute x^*, the centroid, we view the aggregated fuzzy sets as shown in Figs. 7.2 and 7.3. Note that in Fig. 7.3 the aggregated output has been divided into areas for better understanding.

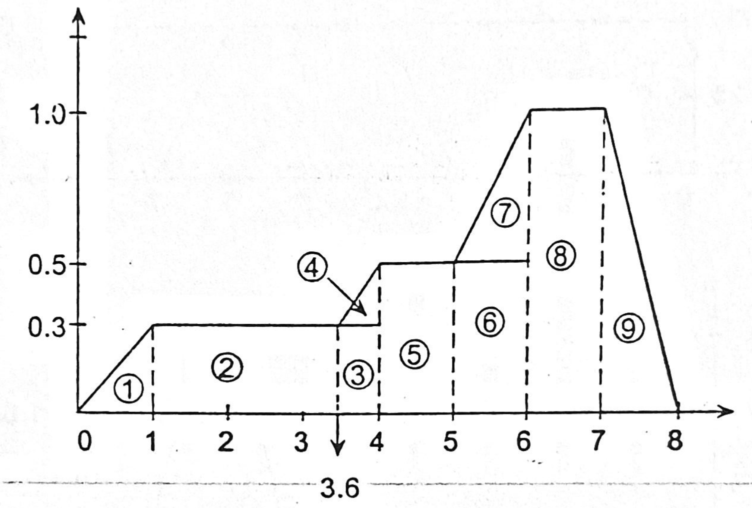


Fig. 7.3 Aggregated fuzzy set of $\tilde{A}_1$, $\tilde{A}_2$, and $\tilde{A}_3$ viewed as area segments.

Table 7.4 illustrates the computations for obtaining x^*.

Table 7.4 Computation of x^*

Area segment no.	Area (A)	$\bar{x}$	$A\bar{x}$
1	$\frac{1}{2} \times 0.3 \times 1 = 0.15$	0.67	0.1005
2	$2.6 \times 0.3 = 0.78$	2.3	1.748
3	$0.3 \times 0.4 = 0.12$	3.8	0.456
4	$\frac{1}{2} \times 0.4 \times 0.2 = 0.04$	3.8667	0.1546
5	$1.5 \times 0.5 = 0.75$	4.75	3.5625
6	$1.5 \times 0.5 = 0.75$	5.75	1.4375
7	$\frac{1}{2} \times 0.5 \times 0.5 = 0.125$	5.833	0.729
8	$1 \times 1 = 1$	6.5	6.5
9	$\frac{1}{2} \times 1 \times 1 = 0.5$	7.33	3.665

In Table 7.4, Area (A) shows the area of the segments of the aggregated fuzzy set and $\bar{x}$ shows the corresponding centroid. Now,

$$x^* = \frac{\sum A\bar{x}}{\sum \bar{x}}$$

i.e.

$$x^* = 18.353/3.695$$
$$= 4.9$$

Centre of sums method

Here, unlike centroid method the overlapping area is counted not once but twice. Making use of the aggregated fuzzy set shown in Fig.7.2, the centre of sums, x^* is given by

$$x^* = \frac{\frac{1}{2}\times 0.3\times(3+5)\times 2.5+\frac{1}{2}\times 0.5\times(4+2)\times 5+\frac{1}{2}\times 1\times(3+1)\times 6.5}{\frac{1}{2}\times 0.3\times(3+5)+\frac{1}{2}\times 0.5\times(4+2)+\frac{1}{2}\times 1\times(3+5)}$$
$$= 2.3$$

Here, the areas covered by the fuzzy sets $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3$ (Refer Figs. 7.1(a), (b), and (c)) are given by

$\frac{1}{2}\times 0.3\times(3+5)$, $\frac{1}{2}\times 0.5\times(4\div 2)$, and $\frac{1}{2}\times 1\times(3+1)$ respectively.

Mean of maxima method

Since the aggregated fuzzy set shown in Fig. 7.2 is a continuous set, x^* the mean of maxima is computed as $x^* = 6.5$.

Here, $M = \{X \in [6, 7] | \mu(x) = 1\}$ and the height of the aggregated fuzzy set is 1.

Figure 7.4 shows the defuzzified outputs using the above three methods.

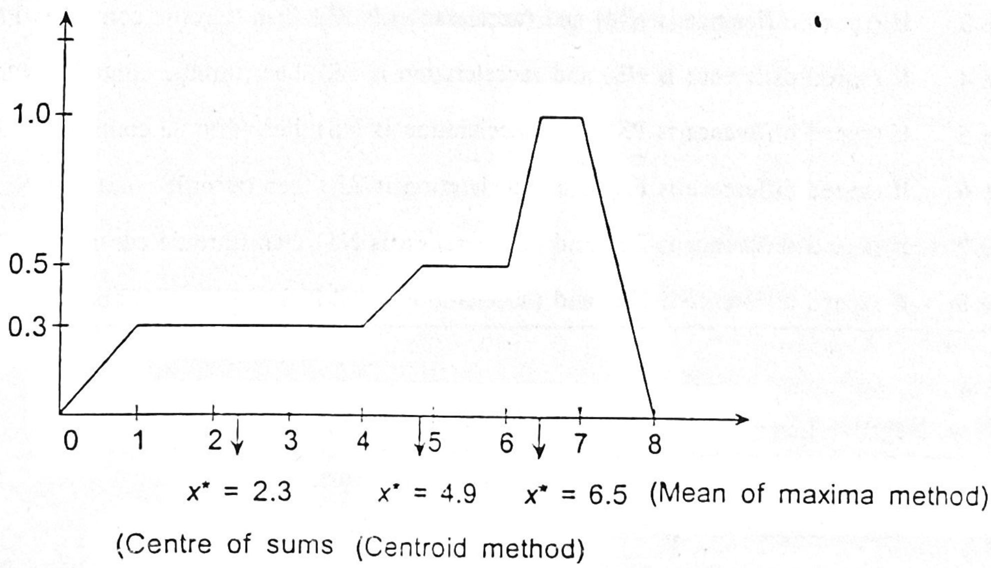


Fig. 7.4 Defuzzified outputs of the aggregate of $\tilde{A}_1$, $\tilde{A}_2$, and $\tilde{A}_3$.

7.6 APPLICATIONS

In this section we illustrate two examples of Fuzzy systems, namely

(i) *Greg Viot's* (Greg Viot, 1993) *Fuzzy Cruise Control System*

(ii) *Yamakawa's* (Yamakawa, 1993) *Air Conditioner Controller*

7.6.1 Greg Viot's Fuzzy Cruise Controller

This controller is used to maintain a vehicle at a desired speed. The system consists of two fuzzy inputs, namely speed difference and acceleration, and one fuzzy output, namely throttle control as illustrated in Fig. 7.5.

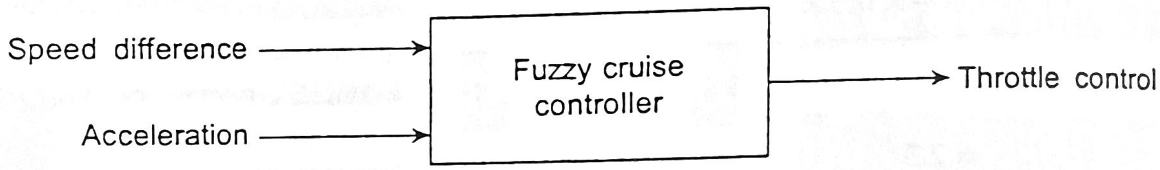


Fig. 7.5 Fuzzy cruise controller.

Fuzzy rule base

A sample fuzzy rule base R governing the cruise control is as given in Table 7.5.

Table 7.5 Sample cruise control rule base

Rule 1	If (speed difference is NL) and (acceleration is ZE) then (throttle control is PL).
Rule 2	If (speed difference is ZE) and (acceleration is NL) then (throttle control is PL).
Rule 3	If (speed difference is NM) and (acceleration is ZE) then (throttle control is PM).
Rule 4	If (speed difference is NS) and (acceleration is PS) then (throttle control is PS).
Rule 5	If (speed difference is PS) and (acceleration is NS) then (throttle control is NS).
Rule 6	If (speed difference is PL) and (acceleration is ZE) then (throttle control is NL).
Rule 7	If (speed difference is ZE) and (acceleration is NS) then (throttle control is PS).
Rule 8	If (speed difference is ZE) and (acceleration is NM) then (throttle control is PM).

Key

NL – Negative Large

ZE – Zero

PL – Positive Large

NM – Negative Medium

PM – Positive Medium

NS – Negative Small

PS – Positive Small

Fuzzy sets

The fuzzy sets which characterize the inputs and output are as given in Fig. 7.6.

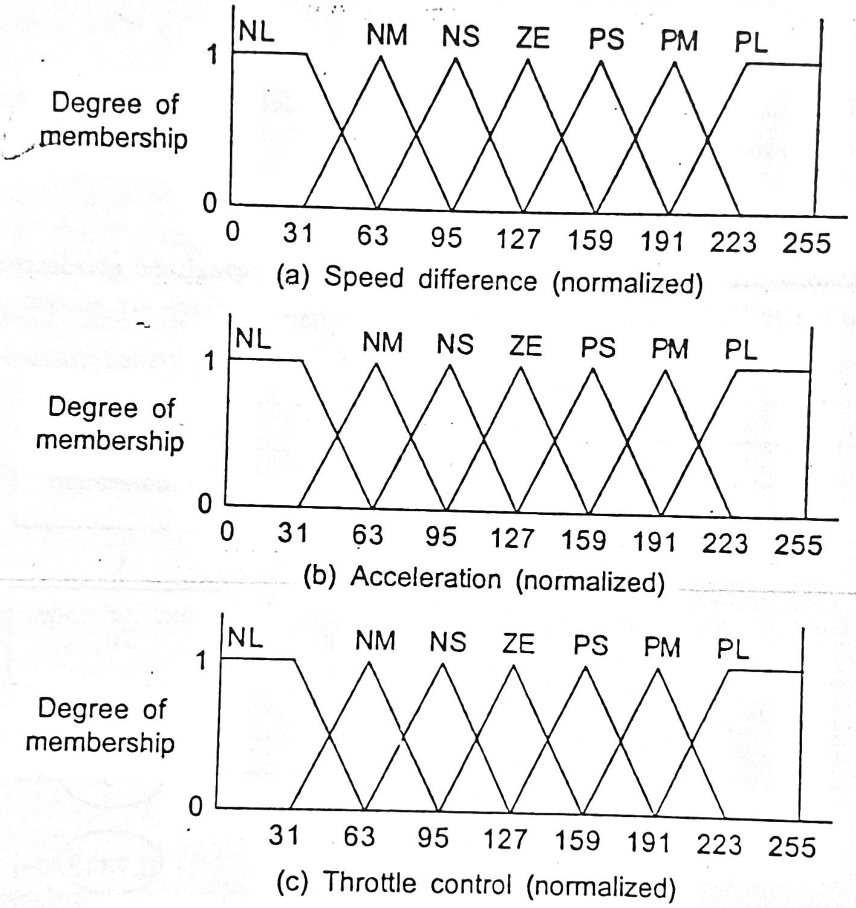


Fig. 7.6 Fuzzy sets characterising fuzzy cruise control.

Fuzzification of inputs

For the *fuzzification* of inputs, that is, to compute the membership for the antecedents, the formula illustrated in Fig. 7.7 is used.

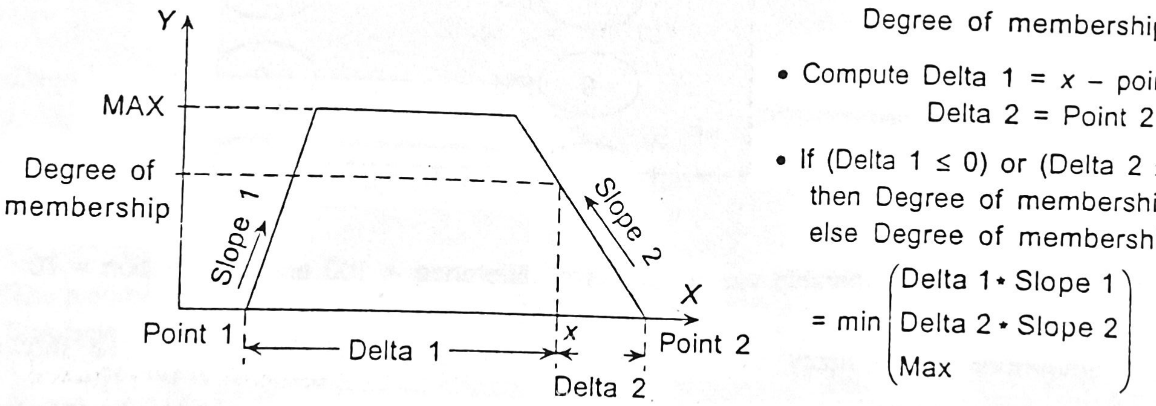


Fig. 7.7 Computation of fuzzy membership value.

Here, x which is the system input has its membership function values computed for all fuzzy sets. For example, the system input speed difference deals with 7 fuzzy sets, namely NL, NM, NS, ZE, PS, PM, and PL. For a measured value of the speed difference x', the membership function of x' in each of the seven sets is computed using the formula shown in Fig. 7.7. Let μ_1', μ_2',..., μ_7' be the seven membership values. Then, all these values are recorded for the input x' in an appropriate data structure.

Similarly, for each of the other system inputs (acceleration in this case), the fuzzy membership function values are recorded.

Example

Let the measured normalized speed difference be 100 and the normalized acceleration be 70, then the fuzzified inputs after computation of the fuzzy membership values are shown in Fig. 7.8.

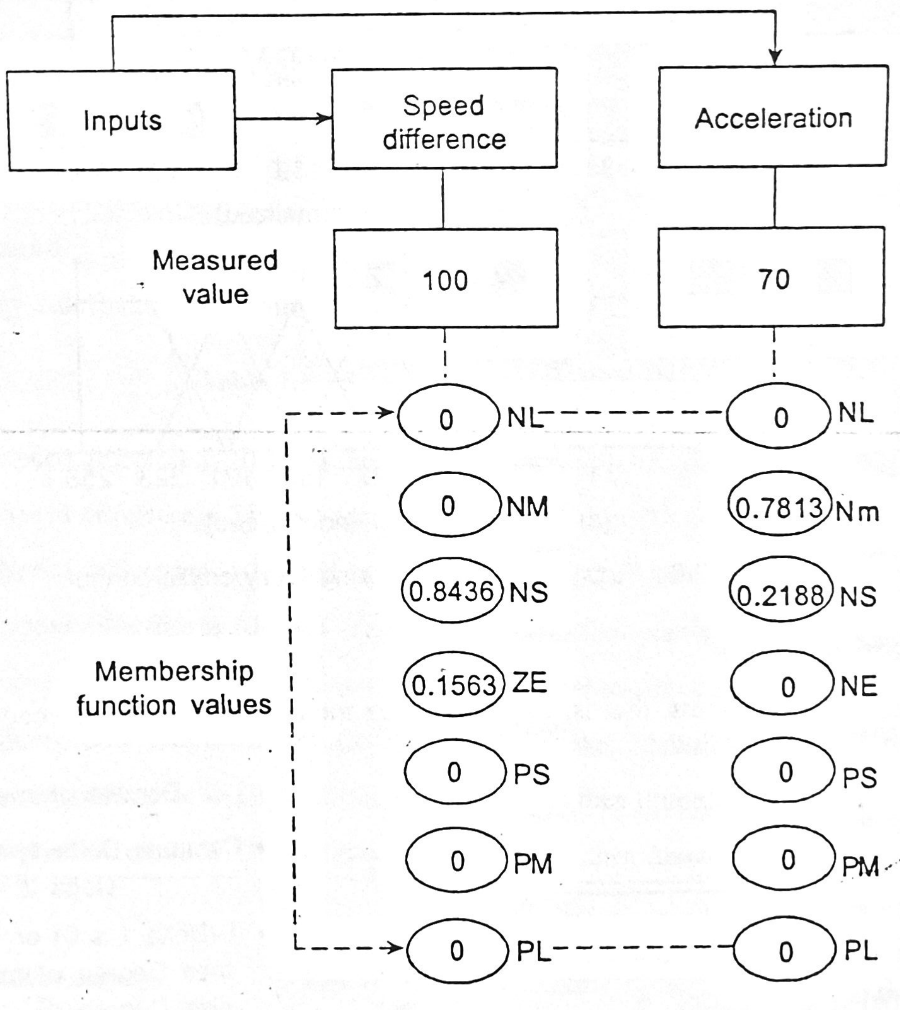


Fig. 7.8 Fuzzy membership values for speed difference = 100 and acceleration = 70.

The computations of the fuzzy membership values for the given inputs have been shown in Fig. 7.9.

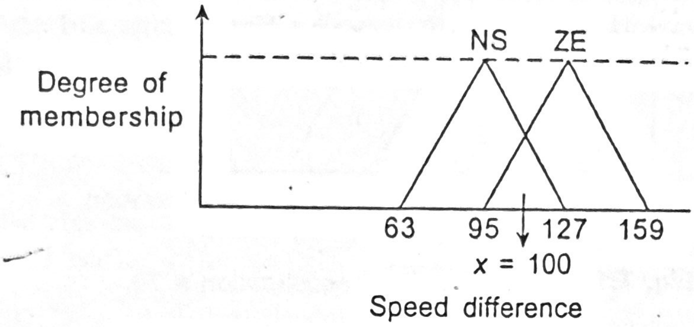


Fig. 7.9 Fuzzification of speed difference = 100.

For speed difference ($x = 100$), the qualifying fuzzy sets are as shown in Fig. 7.9.
Fuzzy membership function of x for NS where

$$\text{Delta 1} = 100 - 63 = 37$$
$$\text{Delta 2} = 127 - 100 = 27$$
$$\text{Slope 1} = 1/32 = 0.03125$$
$$\text{Slope 2} = 1/32 = 0.03125$$

Degree of membership function

$$\mu_{NS}(x) = \min\begin{pmatrix} 37 \times 0.03125 \\ 27 \times 0.03125 \\ 1 \end{pmatrix}$$

$$= 0.8438$$

Fuzzy membership function of x for ZE where

$$\text{Delta 1} = 100 - 95 = 5$$
$$\text{Delta 2} = 159 - 100 = 59$$
$$\text{Slope 1} = \frac{1}{32} = 0.03125$$
$$\text{Slope 2} = 0.03125$$

Degree of membership function

$$\mu_{ZE}(x) = \min\begin{pmatrix} 5 \times 0.03125 \\ 59 \times 0.03125 \\ 1 \end{pmatrix} = 0.1563$$

The membership function of x with the remaining fuzzy sets, namely NL, NM, PS, PM, PL is zero. Similarly for acceleration ($x = 70$), the qualifying fuzzy sets are as shown in Fig. 7.10.

The fuzzy membership function of $x = 70$ for NM is $\mu_{NM}(x) = 0.7813$ and for NS is $\mu_{NS}(x) = 0.2188$.

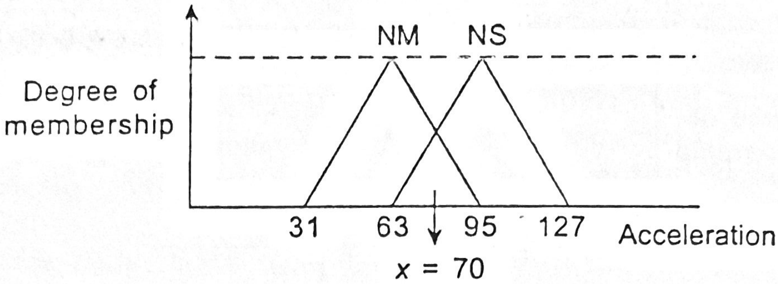


Fig. 7.10 Fuzzification of acceleration = 70.

Rule strength computation

The *rule strengths* are obtained by computing the minimum of the membership functions of the antecedents.

Example

For the sample rule base R given in Table 7.5, the rule strengths using the fuzzy membership values illustrated in Fig. 7.8 are

Rule 1: min (0, 0) = 0

Rule 2: min (0.1563, 0) = 0

Rule 3: min (0, 0) = 0

Rule 4: min (0.8438, 0) = 0

Rule 5: min (0, 0.2188) = 0

Rule 6: min (0,0) = 0

Rule 7: min (0.1563, 0.2188) = 0.1563

Rule 8: min (0.1563, 0.7813) = 0.1563

Fuzzy output

The *fuzzy output* of the system is the 'fuzzy OR' of all the fuzzy outputs of the rules with non-zero rule strengths. In the event of more than one rule qualifying for the same fuzzy output, the stronger among them is chosen.

Example

In the given rule base R, the competing fuzzy outputs are those of Rules 7 and 8 with strengths of 0.1563 each.

However, the fuzzy outputs computed here do not aid a clear-cut decision on the throttle control. Hence, the need for defuzzification arises.

Defuzzification

The centre of gravity method is applied to defuzzify the output. Initially, the centroids are computed for each of the competing output membership functions. Then, the new output membership areas are determined by shortening the height of the membership value on the Y axis as dictated by the rule strength value. Finally, the Centre of Gravity (CG) is computed using the weighted average of the X-axis centroid points with the newly computed output areas, the latter serving as weights.

Example

Figure 7.11 illustrates the computation of CG for the two competing outputs of rules 7 and 8 with strength of 0.1563 each.

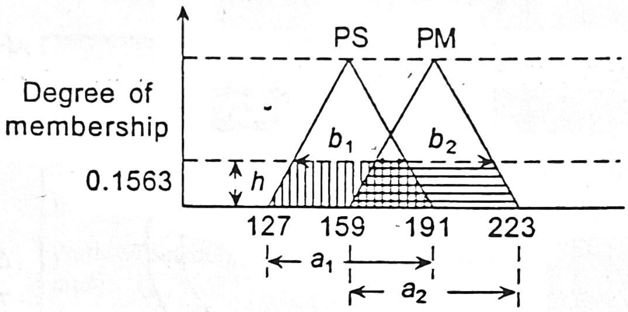


Fig. 7.11 Computation of CG for fuzzy cruise control system.

For the fuzzy set PS,

X-axis centroid point	=	159
Rule strength applied to determine output area	=	0.1563
Shaded area	=	$\frac{1}{2} h \cdot (a_1 + b_1)$
	=	$\frac{1}{2}(0.1563)(64 + 63.82)$
	=	9.99

For the fuzzy set PM,

X-axis centroid point	=	191
Rule strength applied to determine output area	=	0.1563
Shaded area	=	$\frac{1}{2} h \cdot (a_1 + b_1)$
	=	$\frac{1}{2}(0.1563)(64 + 63.82)$
	=	9.99

Therefore,

$$\text{Weighted average,} \quad (CG) = \frac{9.99 \times 159 + 9.99 \times 191}{19.98} = 175$$

In crisp terms, the throttle control (normalized) is to be set as 175.

7.6.2 Air Conditioner Controller

The system as illustrated in Fig. 7.12 comprises a dial to control the flow of warm/hot or cool/cold air and a thermometer to measure the room temperature (T°C). When the dial is turned positive, warm/hot air is supplied from the air conditioner and if it is turned negative, cool/cold air is supplied. If set to zero, no air is supplied.

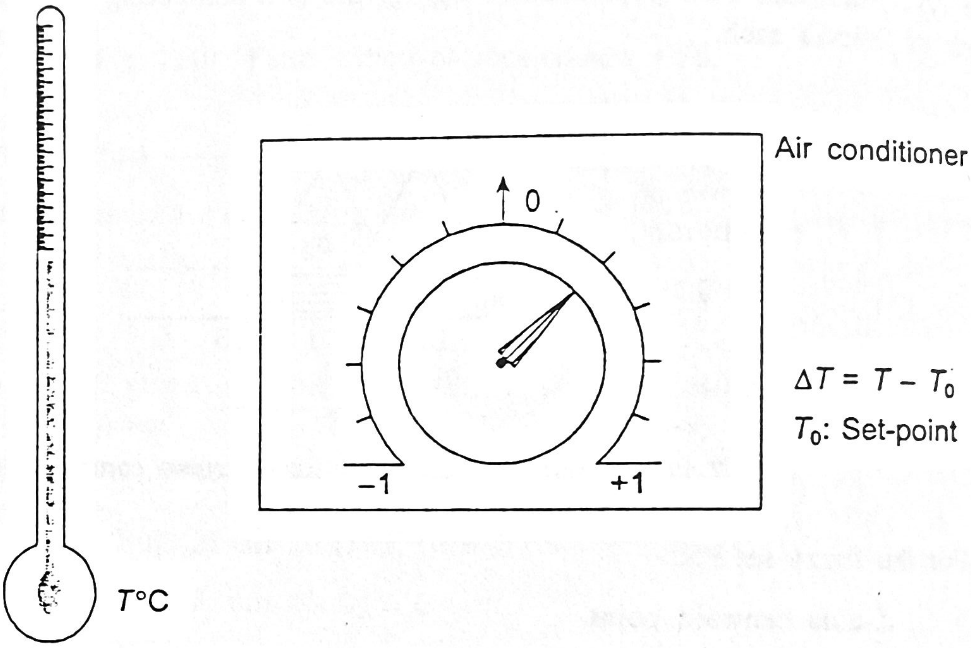


Fig. 7.12 Air conditioner control system.

A person now notices the difference in temperature (ΔT °C) between the room temperature (T °C) as measured by the thermometer and the desired temperature (T_0°C) at which the room is desired to be kept (set-point). The problem now is to determine to what extent the dial should be turned so that the appropriate supply of air (hot/warm/cool/cold) will nullify the change in temperature.

For the above problem the rule base is as shown in Table 7.6.

Table 7.6 Fuzzy rule base for the air conditioner control

S.no.		Fuzzy rule (Descriptive)		Fuzzy rule (Notational)
1	*If*	the room temperature is approximately equal to the set point T_0°C, ΔT is approximately Zero (ZE) *and* the temperature is rapidly changing higher, i.e. $\frac{d\Delta T}{dt}$ is positively large (PL)	*If*	ΔT is ZE *and* $\frac{d\Delta T}{dt}$ is PL
	then	blow cold air rapidly, i.e. turn the dial negative large (NL).	*then*	dial should be NL

(*Cont.*)

Table 7.6 Fuzzy rule base for the air conditioner control *(Cont)*

S.no.		Fuzzy rule (Descriptive)	Fuzzy rule (Notational)
2	*If*	the room temperature is high *and* there is no change in temperature, i.e. ΔT is positive large (PL) and $\frac{d\Delta T}{dt}$ is approximately zero (ZE)	*If* ΔT is PL *and* $\frac{d\Delta T}{dt}$ is ZE *then* dial should be NM.
	then	blow cold air at an intermediate level. i.e. turn the dial negative medium (NM).	
3	*If*	the room temperature is a little bit higher than the set-point *and* the temperature is gradually decreasing. i.e. ΔT is positively small (PS) and $\frac{d\Delta T}{dt}$ is negatively small (NS)	*If* ΔT is PS *and* $\frac{d\Delta T}{dt}$ is NS *then* dial should be ZE.
	then	there is no need to blow hot or cold air. i.e. turn the dial to approximately zero (ZE).	

The fuzzy sets for the system inputs, namely ΔT and $\frac{d\Delta T}{dt}$, and the system output, namely turn of the dial are as shown in Fig. 7.13.

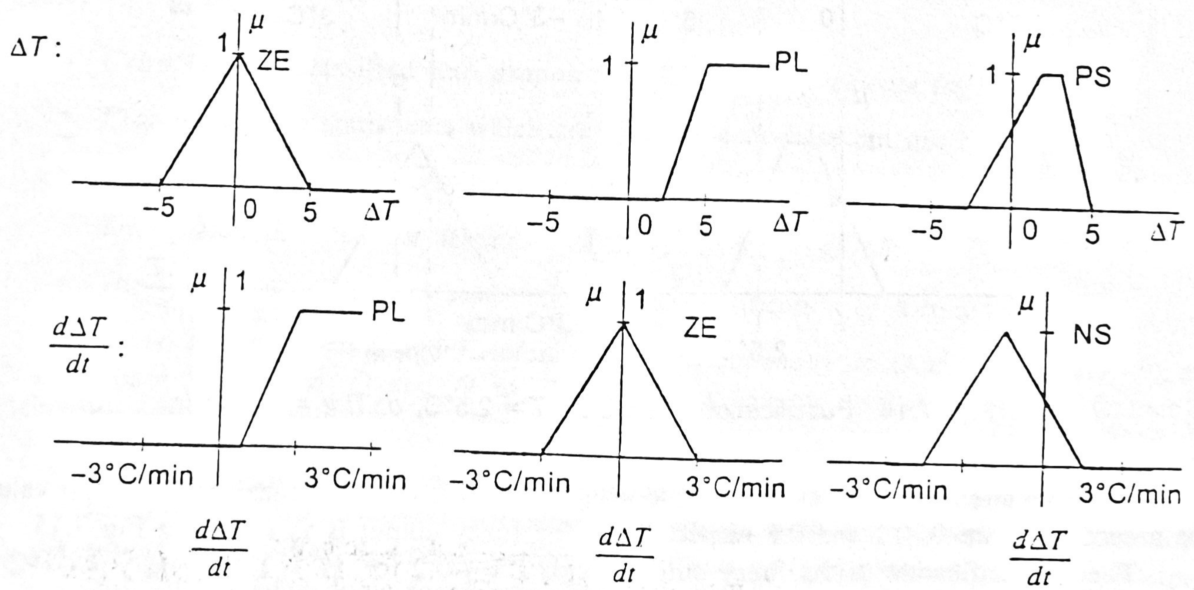


Fig. 7.13 Cont.

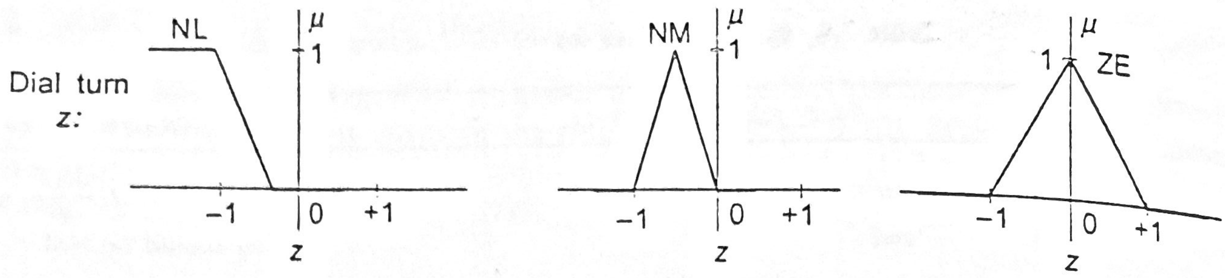


Fig. 7.13 Fuzzy sets for the air conditioner control system.

Consider the system inputs, ΔT = 2.5°C and $\frac{d\Delta T}{dt}$ = −1°C/min. Here the fuzzification of system inputs has been directly done by noting the membership value corresponding to the system inputs as shown in Fig. 7.14.

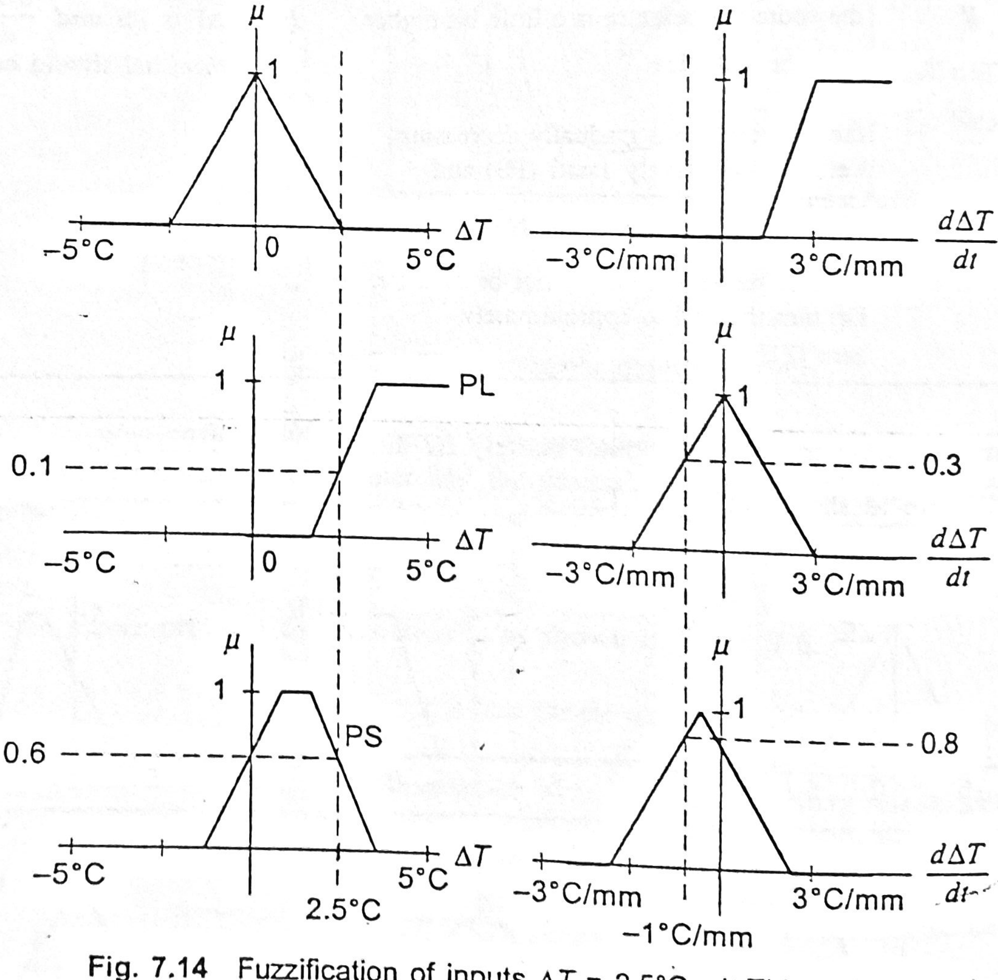


Fig. 7.14 Fuzzification of inputs ΔT = 2.5°C, $d\Delta T/dt$ = −1°C/min.

The rule strengths of rules 1, 2, 3 choosing the minimum of the fuzzy membership value of the antecedents are 0, 0.1 and 0.6 respectively. The fuzzy output is as shown in Fig 7.15.

The defuzzification of the fuzzy output yields Z = −0.2 for ΔT = 2.5°C and y = −1°C/min. Hence, the dial needs to be turned in the negative direction, i.e. −0.2 to achieve the desired temperature effect in the room.

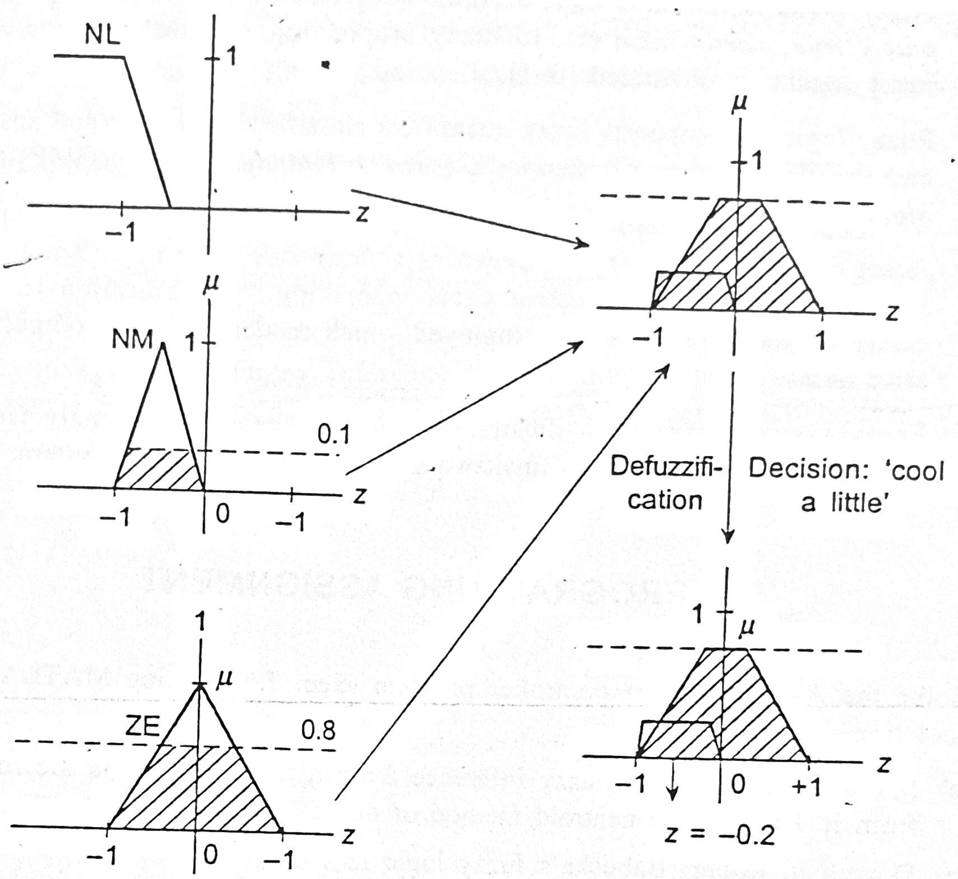


Fig. 7.15 Defuzzification of fuzzy outputs for z (turn of the dial).

SUMMARY

- *Crisp logic* is classified into *propositional logic* and *predicate logic.*
- *Propositions* are statements which are either true or false but not both.
- Propositional logic supports the five major *connectives* $\wedge$, $\vee$, $\sim$, $\Rightarrow$, $=$.

  Truth tables describe the semantics of these connectives.
- The laws of propositional logic help in the simplification of formulae.
- *Modus Ponens* ($P \Rightarrow Q$ and P, infers Q), *Modus Tollens* ($P \Rightarrow Q$ and $\sim Q$, infers $\sim P$), and *Chain rule* ($P \Rightarrow Q$ and $Q \Rightarrow R$ infers $P \Rightarrow R$) are useful rules of inference in propositional logic.
- Propositional logic is handicapped owing to its inability to quantify. Hence, the need for *predicate. logic arises*. Besides propositions and connectives, predicate logic supports *predicates, functions, variables, constants and quantifiers* ($\forall$, $\exists$). The interpretation of predicate logic formula is done over a domain D. The three rules of inference of propositional logic are applicable here as well.

- Fuzzy logic on the other hand accords multivalued truth values such as *absolutely true*, *partly true*, *partly false* etc. to fuzzy propositions. While crisp logic is two valued, fuzzy logic is multivalued [0–1].
- Fuzzy logic also supports fuzzy quantifiers classified as relative and absolute quantifiers and the Fuzzy rules of inference *Generalized Modus Ponens* (GMP) and *Generalized Modus Tollens* (GMT).
- A set of fuzzy *if-then* rules known as a *fuzzy rule base* describes a *fuzzy rule based system*. However, for effective decision making, defuzzification techniques such as center of gravity method are employed which render the fuzzy outputs of a system in crisp terms.
- Fuzzy systems have been illustrated using two examples, namely Greg Viot's fuzzy cruise control system and Yamakawa's air conditioner control system.

PROGRAMMING ASSIGNMENT

P7.1 Solve the Air conditioner controller problem (Sec. 7.6.2) using MATLAB®'s fuzzy logic tool box.

(a) Make use of the FIS (Fuzzy Inference System) editor to frame the rule base and infer from it. Employ the centroid method of defuzzification.

(b) Download Robert Babuska's fuzzy logic tool box. (http://lcewww.et.tudelft.nl/~babuska/) and implement the same problem.

SUGGESTED FURTHER READING

Fuzzy logic concepts are discussed in *A First Course in Fuzzy Logic* (Nguyen and Walker, 1999). The design and properties of fuzzy systems and fuzzy control systems could be found in *A Course in Fuzzy Systems and Control* (Wang, 1997). Several fuzzy system case studies have been discussed in *The Fuzzy Systems Handbook* (Earl Cox, 1998). The book is also supplemented by a CD-ROM containing Windows 95 fuzzy logic library with code to generate 32 bit DLLs for Visual BASIC and Visual C++. The applications of fuzzy systems for neural networks, knowledge engineering and chaos are discussed in *Foundations of Neural Networks, Fuzzy Systems and Knowledge Engineering* (Kasabov, 1996).

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