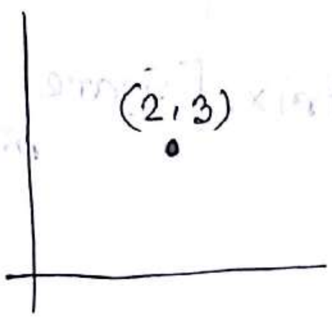
Suzan Ahamed (SA) sir. Date: 23.01.17

MATH 2113.

Book:

(i) Matrices - PN chattarjee.

(ii) Linear Algebra - S Lipschutz.



$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$$

$A_{1\times n} \longrightarrow$ row matrix.

$A_{m\times 1} \longrightarrow$ coloumn matrix.

$A = [a_{ij}]$

$a_{ij} = 0$; $i > j \longrightarrow$ Upper triangular matrix.

$a_{ij} = 0$; $i < j \longrightarrow$ Lower " "

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

→ Upper triangular matrix
→ Lower " "
→ Diagonal matrix

* Diagonal matrix হতে গেলে upper + lower triangular matrix হতে হবে।

$A^2 = A \rightarrow$ Idempotent matrix. [Sqare matrix হওয়া লাগবে।]

$A_{m\times n} \pm B_{m\times n}$

$$\left.\begin{aligned} A &= m \times n \\ B &= n \times m \end{aligned}\right\} AB = m \times m$$

$A^7 = A$

$A^k = \bar{0} \rightarrow$ Nilpotent matrix.

$A' \xrightarrow{\text{transpose}}$ Transpose matrix.

$A_{m\times n} = A'_{n\times m} \rightarrow$ সারিকে কলাম আর কলামকে সারি।

$A' = A \rightarrow$ Symmetric matrix.

$A' = -A \longrightarrow$ skeio-symmetric matrix.

$$A' = \begin{bmatrix} 0 & -3 & 4 \\ 3 & 0 & -10 \\ -4 & 10 & 0 \end{bmatrix}$$

operation

$$A^{-1} = \frac{\text{Adj } A}{|A|}$$

$A \pm B$

AB

A'

Post-multiplication

$AB = BA = I$

Pre-multiplication

$\therefore A^{-1} = B$

A, B পরস্পরের বিপরীত।

$\boxed{AA^{-1} = I}$, $\boxed{A^{-1}I = A^{-1}}$

$\boxed{AI = A}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Date: 20.02.17

Theorem-1: Every square matrix can be uniqely expressed as the sum of a symmetric and a skew-symmetric matrices.

Proof:

Let, A be any sqare matrix of order $n \times n$.

Then we can write,

$$A = \frac{1}{2}A + \frac{1}{2}A' + \frac{1}{2}A - \frac{1}{2}A'$$

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

$$\therefore A = B + C \text{ ; where } B = \frac{1}{2}(A + A')$$

$$C = \frac{1}{2}(A - A').$$

Now, we show that B is symmetric and C is skew-symmetric.

Now,

$$B' = \left\{ \frac{1}{2}(A + A') \right\}'$$

$$= \frac{1}{2}(A + A')'$$

$$= \frac{1}{2}(A' + (A')')$$

$$= \frac{1}{2}(A' + A)$$

$$\therefore B' = \frac{1}{2}(A + A')$$

$$\therefore B' = B$$

which implies that B is symmetric.

Again, $$C' = \left\{ \frac{1}{2}(A - A') \right\}'$$

$$= \frac{1}{2}(A - A')'$$

$$= \frac{1}{2}(A' + ((-1)A')')$$

$$= \frac{1}{2}(A' - A)$$

$$= -\frac{1}{2}(A - A')$$

$$\therefore C' = -C$$

which implies that C is skew symmetric.

suppose, A has another representation,

$$A = A_1 + A_2 \quad \text{--- (i)}$$; where A_1 is symmetric and A_2 is skew-symmetric.

Then, $A' = (A_1 + A_2)'$

$\Rightarrow A' = A_1' + A_2'$

$\therefore A' = A_1 - A_2$ --- (ii)

Addition of (i) & (ii)

$A + A' = 2A_1$

$\therefore A_1 = \frac{1}{2}(A + A')$

$\therefore A_1 = B$

Again, (i) $-$ (ii)

$A - A' = 2A_2$

$\therefore A_2 = \frac{1}{2}(A - A')$

$\therefore A_2 = C$

Which implies that the unique property of this expression/representation.

Theorem-2:

If A and B are two non-singular matrices of order $n \times n$ then AB also non-singular and $(AB)^{-1} = B^{-1}A^{-1}$

$A_{n\times n} \rightarrow$ matrix, $|A| \neq 0 \leftarrow$ non-singular.

$$\boxed{AB = BA = I}$$

Proof:

Given, A and B are non-singular matrices

i.e. $|A| \neq 0$ and $|B| \neq 0$

we know that, $|AB| = |A||B|$

$\therefore |AB| \neq 0$ since $|A| \neq 0$ and $|B| \neq 0$

which implies that AB is non-singular.

Now, $ABB^{-1}A^{-1} = A(BB^{-1})A^{-1}$

$= AIA^{-1}$

$= (AI)A^{-1}$

$= AA^{-1}$

$\therefore ABB^{-1}A^{-1} = I$ ——— (1)

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Again, $B^{-1}A^{-1}AB = B^{-1}(A^{-1}A)B$

$$= B^{-1}IB$$

$$= B^{-1}(IB)$$

$$= B^{-1}B$$

$\therefore B^{-1}A^{-1}AB = I$ —— (ii)

(i) & (ii) implies that, $(AB)^{-1} = B^{-1}A^{-1}$

<u>Theorem-3:</u> If a square matrix A has an inverse, then it is unique.

<u>Proof:</u> let, A has two inverse B and C

i.e. $AB = BA = I$

and $AC = CA = I$

Associative property:

$$ABC = (AB)C = A(BC)$$

Now,

$$B = BI = BAC = (BA)C = IC = C$$

which implies that A has unique inverse.

Date: 26.02.17

Rank of a matrix:

A non-zero matrix A of order $n \times n$ is said to be have rank r if at least one of its $r \times r$ square minor is different from zero while all the other (if any) minors of order $(r+1) \times (r+1)$ are zero.

$$\begin{bmatrix} 2 & 1 & 2 & 3 \\ 3 & 2 & 1 & 6 \\ 1 & 1 & 2 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

sub-matrix.

$$\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 2 - 1 = 1 \rightarrow \text{minor of the matrix.}$$

↳ Determine of the sub-matrix.

* If determine of the sub-matrix is non-zero then this order is called rank of a matrix.

Elementary Raw/Column operators:

(i) $R_{ij} \rightarrow$ Interchanging i-th and j-th rows.

(ii) $R_i(k) \rightarrow$ Multiplying each element of i-th row by non-zero number k.

(iii) $R_{ij}(k) \rightarrow$ Multiplying each element of j-th row by a non-zero number k and then adding with the corresponding elements of i-th row.

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 1 & 3 \\ 1 & 1 & 4 & 6 \end{bmatrix}$$

For column:

(i) $C_{ij} \rightarrow$ Interchanging i-th and j-th column.

(ii) $C_i(k) \rightarrow$ Multiplying each element of i-th column by non-zero number k.

(iii) $C_{ij}(k) \rightarrow$ Multiplying each element of j-th column by a non-zero number k and then adding with the corresponding elements of i-th column.

এগুলোও zero করা লাগবে।

1 রাখতে হবে।

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 1 & 3 \\ 1 & 1 & 4 & 6 \end{bmatrix}$$

এখন zero করা লাগবে।

$R_{21}(-2)$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ \vdots & & & \end{bmatrix}$$

equivalence to (equal sign দেওয়া যাবে না)

* Normal form of Matrix:

A non-zero matrix A of order $m \times n$ can be reduced by using elementary row or column operations to one of the four following forms:

(i) $[I_r]$ (ii) $\begin{bmatrix} I_r \\ \bar{0} \end{bmatrix}$ (iii) $\begin{bmatrix} I_r & \bar{0} \end{bmatrix}$

(iv) $\begin{bmatrix} I_r & \bar{0} \\ \bar{0} & \bar{0} \end{bmatrix}$

where I_r is the unit matrix of order $r \times r$ these four forms are called the normal forms of A.

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 4 \\ 4 & 5 & 6 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\sim \begin{bmatrix} I_2 & \bar{0} \\ \bar{0} & \bar{0} \end{bmatrix}$$

?

$[I_4]$

$$\sim \begin{bmatrix} I_3 & \bar{0} \end{bmatrix}$$

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28.02.17

Normal Forms:

Q-1: Reduce the following matrix to the normal form and hence find its rank.

$$A = \begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}_{4\times 4}$$

$$\overset{R_{12}}{\sim} \begin{bmatrix} 1 & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$\begin{array}{c} C_{21}(-2) \\ C_{31}(-3) \\ \sim \\ C_{41}(1) \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 3 & 3 & -3 \\ 1 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$\begin{array}{c} R_{21}(2) \\ \sim \\ R_{31}(-1) \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 3 & -3 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$\overset{R_2\left(\frac{1}{3}\right)}{\sim}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$\underset{C_{42}(1)}{\overset{C_{32}(-1)}{\sim}}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\underset{R_{42}(-1)}{\overset{R_{32}(2)}{\sim}}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} I_2 & \bar{O} \\ \bar{O} & \bar{O} \end{bmatrix}$$

which is the required normal form of the given matrix and hence rank of A, $\rho(A) = 2$.

$$* \quad A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$$\underset{R_{12}}{\sim} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$$\underset{C_{41}(-1)}{\overset{C_{31}(-1)}{\sim}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & -3 & -1 \\ 1 & 1 & -3 & -1 \end{bmatrix}$$

$$\underset{R_{41}(-1)}{\overset{R_{31}(-3)}{\sim}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

$$\underset{C_{42}(1)}{\overset{C_{32}(1)}{\sim}} \cancel{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 6 \\ 0 & 1 & 2 & 0 \end{bmatrix}} \qquad \underset{R_{32}}{\sim}$$

$$\underset{C_{42}(1)}{\overset{C_{32}(3)}{\sim}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\underset{R_{42}(-1)}{\overset{R_{32}(-1)}{\sim}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} I_2 & \bar{O} \\ \bar{O} & \bar{O} \end{bmatrix}$$

Which is the required normal form of the given matrix and hence rank of A, $\rho(A) = 2$ Ans.

$$* \quad A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$$

$$\underset{\substack{C_{31}(-3) \\ C_{41}(-4)}}{\overset{C_{21}(-2)}{\sim}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & -2 & -3 \\ 3 & -2 & -4 & -6 \\ 4 & -3 & -6 & -9 \end{bmatrix} \underset{\substack{R_{31}(-3) \\ R_{41}(-4)}}{\overset{R_{21}(-2)}{\sim}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{bmatrix}$$

$$\underset{\substack{R_3(-1) \\ R_4(-1)}}{\overset{R_2(-1)}{\sim}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & +2 & +4 & +6 \\ 0 & +3 & +6 & +9 \end{bmatrix} \underset{C_{42}(-3)}{\overset{C_{32}(-2)}{\sim}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{bmatrix}$$

$$\underset{R_{42}(-3)}{\overset{R_{32}(-2)}{\sim}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} I_2 & \bar{O} \\ \bar{O} & \bar{O} \end{bmatrix}$$

which is the required normal form of the given matrix and hence ~~matri~~ rank of A, $\rho(A) = 2$ (A)

05.03.17

Echelon form: (শুধুমাত্র Row operator use করা হয়।

$$\begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 4 & 6 & 9 \\ 3 & 4 & 5 & 6 \\ 1 & 2 & 1 & 9 \end{bmatrix}$$

$0 \ \boxed{1} \ 5 \ 0 \longrightarrow 1$

$0 \ 0 \ \boxed{3} \ 5 \longrightarrow 2$

Q-1: Reduce the following matrix to the echelon form and hence find its rank:

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

Given,

$$\Rightarrow A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$$\underset{\sim}{R_{12}} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$$\underset{R_{41}(-1)}{\overset{R_{31}(-3)}{\sim}} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

$$\underset{R_{42}(-1)}{\overset{R_{32}(-1)}{\sim}} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the required ecolon form of the given matrix and rank of $\rho(A) = 2$.

<u>Q-2:</u>

$$A = \begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$\overset{R_{12}}{\sim} \begin{bmatrix} 1 & 2 & 3 \\ -2 & -1 & -3 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\underset{R_{31}(-1)}{\overset{R_{21}(2)}{\sim}} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & -2 & -2 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$\overset{R_{24}}{\sim} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & -2 & -2 & 0 \\ 0 & 3 & 3 & -3 \end{bmatrix}$$

$$\underset{R_{42}(-3)}{\overset{R_{32}(2)}{\sim}} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Rank, $\rho(A) = 2$.

* (i) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$

(ii). $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$

Solution:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$$

$$\underset{\substack{R_{31}(-3) \\ R_{41}(-4)}}{\overset{R_{21}(-2)}{\sim}} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \\ 0 & -3 & -6 & -9 \end{bmatrix}$$

$$\underset{\substack{R_3(-1) \\ R_4(-1)}}{\overset{R_2(-1)}{\sim}} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 9 \end{bmatrix}$$

$$\underset{R_{42}(-3)}{\overset{R_{32}(-2)}{\sim}} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the required ecolon form of the given matrix and rank of A, $\rho(4) = 2$ An.

(11) Given,

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$$

$$\underset{R_{31}(2)}{\overset{R_{21}(-3)}{\sim}} \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & -2 & 1 & 5 \\ 0 & 7 & 2 & 2 \end{bmatrix}$$

$$\overset{R_2(-\frac{1}{2})}{\sim} \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & -1/2 & -5/2 \\ 0 & 7 & 2 & 2 \end{bmatrix}$$

$$\overset{R_{32}(-7)}{\sim} \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & -1/2 & -5/2 \\ 0 & 0 & 11/2 & 39/2 \end{bmatrix}$$

$\frac{7}{2}+2$
$= \frac{7+4}{2} = \frac{11}{2}$

$\frac{35}{2}+2$
$= \frac{35+4}{2}$
$= \frac{39}{2}$

$$\overset{R_3(\frac{2}{11})}{\sim} \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & -1/2 & -5/2 \\ 0 & 0 & 1 & \frac{39}{11} \end{bmatrix}$$

$\therefore$ This is the required ecolon form of the given matrix and rank of A is, $\rho(A) = 3$ (Ans)

Date: 07.03.17

System of Linear Equation:

$$y^1 = mx^1 + cx^0 \rightarrow$$ variable গুলোর power

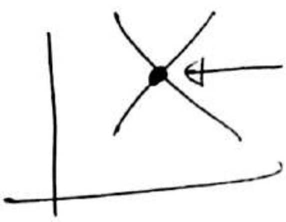
$\{0, 1\}$ হবে। length finite হতে হবে। তাহলেই সমীকরণটি linear হবে।

$x = 4 \rightarrow Y\text{-}Z$ plane এর parallel একটা সমীকরণ।

$$1 \cdot x + 0 \cdot y + 0 \cdot z = 4 \longrightarrow \text{plane}.$$

$(4, 0, 0)$.

$$\left.\begin{aligned} x + y &= 15 \\ 2x - 3y &= 20 \end{aligned}\right\} (x, y) \equiv (13, 2)$$



← একটা বিন্দুতেই intersect করছে।

* দুইটা সরলরেখা parallel হলে এরা কখনও ছেদ করবে না।

* যদি একটা সরলরেখা অন্য একটা সরলরেখাকে একাধিক বিন্দুতে ছেদ করাতে চাই তাহলে এরা একে

অপরের উপর overlap করবে।

* $Ax = B$, this formula is called <u>system of linear matrix.</u>

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$- - - - - - - - - - - - -$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

এগুলো equation কে matrix আকারে লিখা যায়।

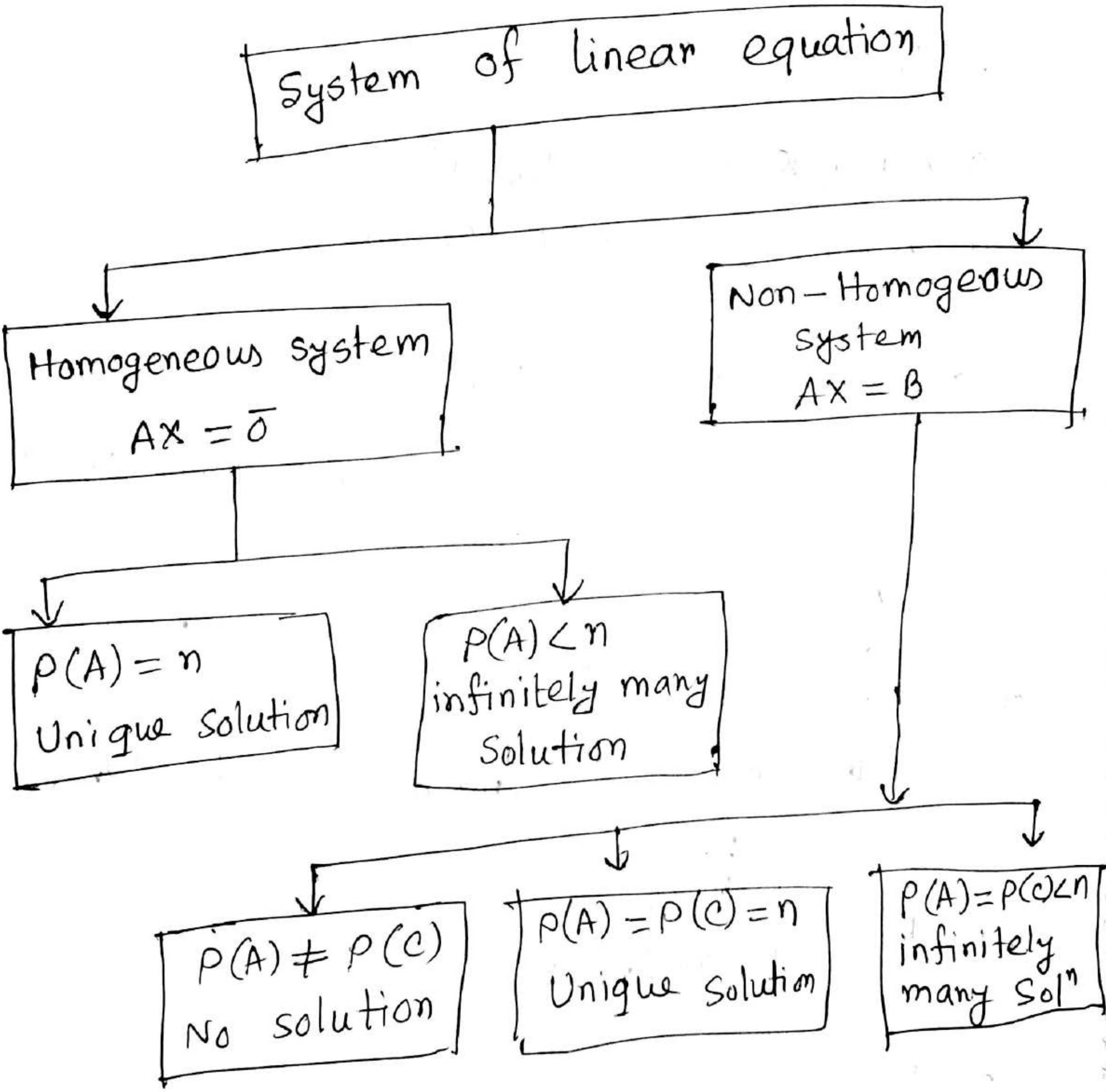
$$Ax = B$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m \times 1}$$

$$\begin{aligned} x - y &= 0 \\ 2x + 3y &= 0 \end{aligned} \longrightarrow \text{Homogeneous system.}$$

$$\left.\begin{aligned} x+y&=15 \\ 2x+3y&=21 \end{aligned}\right\} \longrightarrow \text{Non-homogeneous system.}$$



Where,

01. A is the co-efficient Matrix.
02. C is the augmented matrix.
03. $\rho(A)$ is the rank of A.
04. $\rho(C)$ " " " " C
05. n is the number of unknowns/variables.

Date: 12.3.17

Ex-1: Solve the following system.

$$2x-2y+5z+3w=0$$
$$4x-y+z+w=0$$
$$3x-2y+3z+4w=0$$
$$x-3y+7z+6w=0$$

The co-efficient matrix for the given system is,

$$A = \begin{bmatrix} 2 & -2 & 5 & 3 \\ 4 & -1 & 1 & 1 \\ 3 & -2 & 3 & 4 \\ 1 & -3 & 7 & 6 \end{bmatrix}$$

$$\overset{R_{14}}{\sim} \begin{bmatrix} 1 & -3 & 7 & 6 \\ 4 & -1 & 1 & 1 \\ 3 & -2 & 3 & 4 \\ 2 & -2 & 5 & 3 \end{bmatrix}$$

$$\underset{R_{41}(-2)}{\overset{R_{21}(-4)}{\overset{R_{31}(-3)}{\sim}}} \begin{bmatrix} 1 & -3 & 7 & 6 \\ 0 & 11 & -27 & -23 \\ 0 & 7 & -18 & -14 \\ 0 & 4 & -9 & -9 \end{bmatrix}$$

$$\overset{R_{24}(-3)}{\sim} \begin{bmatrix} 1 & -3 & 7 & 6 \\ 0 & -1 & 0 & 4 \\ 0 & 7 & -18 & -14 \\ 0 & 4 & -9 & -9 \end{bmatrix}$$

$$\overset{R_{2}(-1)}{\sim} \begin{bmatrix} 1 & -3 & 7 & 6 \\ 0 & 1 & 0 & -4 \\ 0 & 7 & -18 & -14 \\ 0 & 4 & -9 & -9 \end{bmatrix}$$

$$R_{32}(-7) \atop {\sim \atop R_{42}(-4)} \begin{bmatrix} 1 & -3 & -7 & 6 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & -18 & 14 \\ 0 & 0 & -9 & 7 \end{bmatrix}$$

$$\underset{\sim}{R_3(-\tfrac{1}{18})} \begin{bmatrix} 1 & -3 & -7 & 6 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -7/9 \\ 0 & 0 & -9 & 7 \end{bmatrix}$$

$$\underset{\sim}{R_{43}(9)} \begin{bmatrix} 1 & -3 & -7 & 6 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -7/9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is the ecolon form of the co-efficient matrix A and since there are 3 non-zero raw.

$$\therefore \rho(A) = 3$$

Again,
since $\rho(A) < n$
Therefore there are infinitely many solution in the given system.

In this case, number of free variables $= n - \rho(A)$
$= 4 - 3$
$= 1$

let, w be the free variable, also let $w = k$.
where k is any real number.

Now, the equivalent system is,

$$x - 3y + 7z + 6w = 0$$
$$y - 4w = 0$$
$$z - \frac{7}{9}w = 0$$

which implies that,

$$z = \frac{7}{9}k$$
$$y = 4k$$
$$x = 12k - \frac{49}{7}k - 6k$$
$$= \frac{108k - 49k - 54k}{9}$$
$$= \frac{5}{9}k.$$

Therefore, the solution of the given system is,

$$\left.\begin{array}{l} x = \frac{5}{9}k \\ y = 4k \\ z = \frac{7}{9}k \\ w = k. \end{array}\right\} \text{(Ans)}$$

* if $n - \rho(A) = 0$

$\therefore 4 - 4 = 0$ $\qquad \rho(A) = 4$, $n = 4$

There is a unique solution.

$\therefore x = 0,\ y = 0,\ z = 0,\ w = 0.$

<u>H.W</u> Solve the following system,

$$x - 2y + z - w = 0$$
$$x + y - 2z + 3w = 0$$
$$4x + y - 5z + 8w = 0$$
$$5x - 7y + 2z - w = 0$$

The co-efficient matrix for the given system is -

$$A = \begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & 1 & -2 & 3 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix}$$

$$\underset{R_{41}(-5)}{\overset{R_{21}(-1),\ R_{31}(-4)}{\sim}} \begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 3 & -3 & 4 \\ 0 & 9 & -9 & 12 \\ 0 & 3 & -3 & 4 \end{bmatrix}$$

$$\overset{R_2\left(\frac{1}{3}\right)}{\sim} \begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 1 & -1 & 4/3 \\ 0 & 9 & -9 & 12 \\ 0 & 3 & -3 & 4 \end{bmatrix}$$

$$\underset{R_{42}(-3)}{\overset{R_{32}(-9)}{\sim}} \begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 1 & -1 & 4/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is the ecolon form of the given co-efficient matrix A and since there are 2 non-zero row.

$\therefore \quad \rho(A) = 2$

Again since $\rho(A) < n$.

Therefore, there are infinitely many solution in the given system.

In this case, number of free variables

$$= n - \rho(A)$$
$$= 4 - 2$$
$$= 2$$

let, w & z be the free variable, also let $w = k_1$, $z = k_2$ where k_1 & k_2 are any real number.

Now, the equivalent system is;

$$x - 2y + z - w = 0$$
$$y - z + \frac{4}{3} w = 0$$

which implies that,

$\therefore w = k_1$

$\therefore z = k_2$

$$\therefore y = k_2 - \frac{4}{3} k_1$$

$$\therefore x = 2k_2 - \frac{8}{3} k_1 - k_2 + k_1$$
$$= \frac{6k_2 - 8k_1 - 3k_2 + 3k_1}{3}$$
$$= \frac{3k_2 - 5k_1}{3}$$

Date: 14.03.17

Ex-1: Solve the following system of linear equations:

$$x + y + z = 6$$
$$x + 2y + 3z = 14$$
$$x + 4y + 7z = 30$$

The Augmented matrix for the given system is,

$$C = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{bmatrix}$$

$$\underset{R_{31}(-1)}{\overset{R_{21}(-1)}{\sim}} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 6 & 24 \end{bmatrix}$$

$$\begin{array}{cc} & \begin{array}{ccc|c} & A & & C \end{array} \\ \overset{R_{32}(-3)}{\sim} & \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

which implies that $\rho(A) = \rho(C) = 2$

Since, $\rho(A) = \rho(C) < n$, therefore the given system has infinitely many solution.

Now, the number of free variable $= n - \rho(A)$

$$= 3 - 2$$
$$= 1$$

let, z be the free variable..

Aso let, $z = k$; where k is any real number.

Now, the equivalent system is

$$x + y + z = 6$$
$$y + 2z = 8$$

which implies that

$$y = 8 - 2k$$
$$x = 6 - y - z$$
$$= 6 - 8 + 2k - k$$
$$\therefore x = k - 2$$

Therefore the sol^n of the given system is—

$$\left.\begin{aligned} x &= k - 2 \\ y &= 8 - 2k \\ z &= k \end{aligned}\right\} \text{(Ans)}$$

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Ex-2: Solve the following system of linear equations—

$$x+y+4z=6$$
$$3x+2y-2z=9$$
$$5x+y+2z=13$$

The Augmented matrix for the given system is,

$$C = \begin{bmatrix} 1 & 1 & 4 & 6 \\ 3 & 2 & -2 & 9 \\ 5 & 1 & 2 & 13 \end{bmatrix}$$

$$\underset{R_{31}(-5)}{\overset{R_{21}(-3)}{\sim}} \begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & -1 & -14 & -9 \\ 0 & -4 & -18 & -17 \end{bmatrix}$$

$$\overset{R_2(-1)}{\sim} \begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & 1 & 14 & 9 \\ 0 & -4 & -18 & -17 \end{bmatrix}$$

$$\overset{R_{32}(4)}{\sim} \begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & 1 & 14 & 9 \\ 0 & 0 & 38 & 19 \end{bmatrix}$$

$$R_3\left(\frac{1}{38}\right) \sim \left[\begin{array}{ccc|c} 1 & 1 & 4 & 6 \\ 0 & 1 & 14 & 9 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right]$$

$\rho(A) = 3$ $\quad \rho(C) = 3$

which implies that $\rho(A) = \rho(C) = 3$

since, $\rho(A) = \rho(C) = n$

Therefore the given system has unique solution.

Now, the number of free variable is zero.

Now, The equivalent system is,

$$x + y + 4z = 6$$
$$y + 14z = 9$$
$$z = \frac{1}{2}$$

which implies that, $y = 2$
$x = 2$

Therefore, the sol^n of the given system is

$$\left.\begin{array}{l} x = 2 \\ y = 2 \\ z = \frac{1}{2} \end{array}\right\} \text{(Ans)}$$

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$\underline{Ex-3}$ Solve the following system of linear equation,

$$2x+6y+11=0$$
$$6x+20y-6z=-3$$
$$6y-18z=-1$$

The augmented matrix for the given system is,

$$C=\begin{bmatrix} 2 & 6 & 0 & -11 \\ 6 & 20 & -6 & -3 \\ 0 & \cancel{18}\,6 & -18 & -1 \end{bmatrix}$$

$$\underset{\sim}{R_1(\tfrac{1}{2})}\begin{bmatrix} 1 & 3 & 0 & -11/2 \\ 6 & 20 & -6 & -3 \\ 0 & 6 & -18 & -1 \end{bmatrix}$$

$-\frac{11}{\cancel{2}}\cdot\cancel{6}^{3}$

33-3

$$\underset{\sim}{R_{21}(-6)}\begin{bmatrix} 1 & 3 & 0 & -11/2 \\ 0 & 2 & -6 & 30 \\ 0 & 6 & -18 & -1 \end{bmatrix}$$

$$\underset{\sim}{R_2(\tfrac{1}{2})}\begin{bmatrix} 1 & 3 & 0 & -11/2 \\ 0 & 1 & -3 & 15 \\ 0 & 6 & -18 & -1 \end{bmatrix}$$

$$\underset{\sim}{R_{32}(-6)}\begin{bmatrix} 1 & 3 & 0 & -11/2 \\ 0 & 1 & -3 & 15 \\ 0 & 0 & 0 & -91 \end{bmatrix}$$

which implies that, $\rho(A) \neq \rho(C)$

$\rho(A) = 2$ $\therefore \rho(C) > \rho(A)$

and $\rho(C) = 3$

Therefore, the given system has no solution.

Ex-4 :

For what values of λ and μ, the following system—

$$x+y+z=6$$
$$x+2y+3z=10$$
$$x+2y+\lambda z=\mu$$

has (i) No solution (ii) Unique solution (iii) Infinitely many solutions.

Sol^n :

The Augmented matrix for the given system is,

$$C = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}$$

$$\underset{R_{31}(-1)}{\overset{R_{21}(-1)}{\sim}} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{bmatrix} \overset{R_{32}(-1)}{\sim} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{bmatrix}$$

case-(I) when $\lambda = 3$ and $\mu \neq 10$ then $\rho(A) = 2$ and $\rho(C) = 3$

In this case the given system has no solution.

case-(II) when $\lambda \neq 3$ then $\rho(A) = \rho(C) = 3$ this case the given system has unique solⁿ.

case-(III) when $\lambda = 3$ and $\mu = 10$ than $\rho(A) = \rho(C) < n$ this case the given system has infinitely many solutions.

Ex-5: Given,

$$\lambda^2 - 3\lambda + 3 - 1$$
$$\underline{\lambda^2 - 3\lambda + 2}$$

$$x+y+z=1$$
$$x+2y+4z=\lambda$$
$$x+4y+10z=\lambda^2$$

Sol^n:

The Augmented matrix for the given system is,

$$C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & \lambda \\ 1 & 4 & 10 & \lambda^2 \end{bmatrix}$$

$$\underset{R_{31}(-1)}{\overset{R_{21}(-1)}{\sim}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda-1 \\ 0 & 3 & 9 & \lambda^2-1 \end{bmatrix} \overset{R_{32}(-3)}{\sim} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda-1 \\ 0 & 0 & 0 & \lambda^2-3\lambda+2 \end{bmatrix}$$

Case-1, when $\lambda^2-3\lambda+2 \neq 0$, $\lambda - 1 \neq 0$,

$\rho(A) = 2$ and $\rho(C) = 3$

$\therefore$ In this case the given system has no unique sol^n.

Case-2: when $\lambda^2 - 3\lambda + 2 = 0$

$$\therefore \lambda = 1, 2$$

than $\rho(A) = \rho(C) = 3$. this case the given system has unique sol^n.

Case-III: when $\lambda = 2$ then $\rho(A) = \rho(c) < n$. This case the given system has infinitely many solution.

\# Given,

$$-2x + y + z = a$$
$$x - 2y + z = b$$
$$x + y - 2z = c$$

The augmented matrix for the given system is,

$$C = \begin{bmatrix} -1 & 1 & 1 & a \\ 1 & -2 & 1 & b \\ 1 & 1 & -2 & c \end{bmatrix}$$

$$\underset{R_{21}}{\sim} \begin{bmatrix} 1 & -2 & 1 & b \\ -1 & 1 & 1 & a \\ 1 & 1 & -2 & c \end{bmatrix} \underset{R_{31}(-1)}{\overset{R_{21}(1)}{\sim}} \begin{bmatrix} 1 & -2 & 1 & b \\ 0 & -1 & 2 & a+b \\ 0 & 3 & -3 & c-b \end{bmatrix}$$

$\sim$

$3a+3b+c-b$

$\Rightarrow 3a+2b+c$

$$R_2(-1) \sim \begin{bmatrix} 1 & -2 & 1 & b \\ 0 & 1 & -2 & -(a+b) \\ 0 & 3 & -3 & c-b \end{bmatrix}$$

$$R_{32}(-3) \sim \begin{bmatrix} 1 & -2 & 1 & b \\ 0 & 1 & -2 & -(a+b) \\ 0 & 0 & 3 & 3a+2b+c \end{bmatrix}$$

$$R_3\left(\tfrac{1}{3}\right) \sim \begin{bmatrix} 1 & -2 & 1 & b \\ 0 & 1 & -2 & -(a+b) \\ 0 & 0 & 1 & \frac{1}{3}(3a+2b+c) \end{bmatrix}$$

For no solution:

$$-(a+b)=0 \quad \text{or} \quad \frac{1}{3}(3a+2b+c)=0$$

$\therefore \rho(A)=3$ and $\rho(C)=2$.

$\therefore \rho(A) \neq \rho(C)$

For unique solution:

$$-(a+b)\neq 0 \quad \text{and} \quad \frac{1}{3}(3a+2b+c)\neq 0$$

$\therefore \rho(A)=\rho(C)=n$

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Date: 19.03.17

Characteristic roots and characteristic vectors
or
Eigen values and Eigen vectors:

* let, A be any non-zero square matrix of order $n \times n$, then.

(i) The matrix $A - \lambda I$ is called the characteristic matrix of A.

(ii) The determinent $|A - \lambda I|$ is called the characteristic polynomial of A.

(iii) The equation $|A - \lambda I| = 0$ is called the characteristic equation of A.

(iv) The values of λ for which the equation $|A - \lambda I| = 0$ is satisfied are called the eigen values of characteristic roots of A.

(v) Corresponding to each ch. root these is a non-zero vector X which satisfied the equation $(A - \lambda I) X = \bar{O}$. These non-zero vectors are called the ch. vectors or eigen vectors of A.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

(i) $A - \lambda I = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1-\lambda & 2 \\ 2 & 3-\lambda \end{bmatrix}$$

(ii) $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix}$

$$= (1-\lambda)(3-\lambda) - 4$$

$$= 3 - \lambda - 3\lambda + \lambda^2 - 4$$

$$= \lambda^2 - 4\lambda - 1$$

(iii) $|A - \lambda I| = 0$

$\Rightarrow \lambda^2 - 4\lambda - 1 = 0$

Q-1: Find all the eigen values and eigen vectors of the following matrix.

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

The characteristic equan for the given matrix, A is,

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\{(3-\lambda)(2-\lambda)-2\} - 2\{(2-\lambda)-1\} + 1\{2-(3-\lambda)\} = 0$$

$$\Rightarrow (2-\lambda)(4-5\lambda+\lambda^2) - 2(2-\lambda-1) + (2-3+\lambda) = 0$$

$$\Rightarrow 8-10\lambda+2\lambda^2-4\lambda+5\lambda^2-\lambda^3-4+2\lambda+2+2-3+\lambda=0$$

$$\Rightarrow -\lambda^3+7\lambda^2-11\lambda+5=0$$

$$\Rightarrow \lambda^3-7\lambda^2+11\lambda-5=0$$

$$\Rightarrow \lambda^2(\lambda-1)-6\lambda(\lambda-1)+5(\lambda-1)=0$$

$$\Rightarrow (\lambda-1)(\lambda^2-6\lambda+5)=0$$

$$\therefore \lambda = 1, 1, 5.$$

which are the required eigen vectors.

For $\lambda = 5$, the equation $[A - \lambda I]x = \bar{0}$ becomes,

$$\left\{ \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -3x_1 + 2x_2 + x_3 \\ x_1 - 2x_2 + x_3 \\ x_1 + 2x_2 - 3x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which implies that,

$$-3x_1 + 2x_2 + x_3 = 0$$
$$x_1 - 2x_2 + x_3 = 0$$
$$x_1 + 2x_2 - 3x_3 = 0$$

The co-efficient matrix of the above system is,

$$A_1 = \begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}$$

$$\overset{R_{12}}{\sim} \begin{bmatrix} 1 & -2 & 1 \\ -3 & 2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \underset{R_{31}(-1)}{\overset{R_{21}(3)}{\sim}} \begin{bmatrix} 1 & -2 & 1 \\ 0 & -4 & 4 \\ 0 & 4 & -4 \end{bmatrix}$$

$$\overset{R_2(-\frac{1}{4})}{\sim} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & -4 & 4 \end{bmatrix} \overset{R_{32}(4)}{\sim} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

which implies that $\rho(A_1) = 2$. since $\rho(A_1) < n$ therefore the above system has infinitely many solution.

In this case number of free variable $= n - \rho(A_1)$

$$= 3 - 2$$
$$= 1$$

let, x_3 be the free variable. Also let, $x_3 = k$, where k is any real number.

Now, the equivalent system is,

$$x_1 - 2x_2 + x_3 = 0$$
$$x_2 - x_3 = 0$$

$$\therefore x_2 = x_3 = k$$

Therefore,

$$x_1 = k, \; x_2 = k, \; x_3 = k.$$

Therefore, the eigen vector correspondency to the eigen value.

$$X_{\lambda=5} = \begin{bmatrix} k \\ k \\ k \end{bmatrix};$$ where k is any real number.

For, $\lambda = 1$, the equation $(A - \lambda I)X = \bar{0}$ becomes,

$$\left(\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 + 2x_2 + x_3 \\ x_1 + 2x_2 + x_3 \\ x_1 + 2x_2 + x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which implies that,

$$x_1 + 2x_2 + x_3 = 0$$
$$x_1 + 2x_2 + x_3 = 0$$
$$x_1 + 2x_2 + x_3 = 0$$

The co-efficient matrix of the above system is,

$$A_2 = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\underset{R_{31}(-1)}{\overset{R_{21}(-1)}{\sim}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which implies that $\rho(A_2) = 1$. Since $\rho(A_2) < n$ therefore the above system has infinitely many solution.

In this case number of free variabe $= n - \rho(A)$
$= 3 - 1$
$= 2$

let, x_2, x_3 be free variable.

Also let, $x_2 = k_1$, $x_3 = k_2$, where k_1 & k_2 are any real numbers

Now, the equivalent system is,

$$x_1 + 2x_2 + x_3 = 0$$

$$\therefore x_1 = -2k_{\cancel{1}} - k_2$$

Therefore, $x_1 = -(2k_1 + k_2)$

$x_2 = k_1$

$x_3 = k_2$

Therefore the eigen vector corresponding to the eigen value.

$$\therefore X_{\lambda=1} = \begin{bmatrix} -(2k_1+k_2) \\ k_1 \\ k_2 \end{bmatrix}$$; where k_1 & k_2 are any real number.

Date : 27.03.17

Caley - Hamilton Theorem:

Statement: Every square matrix satisfies its characteristic equation.

or,

let, A be any non-zero square matrix of order $n \times n$. Aso let $|A - \lambda I| = (-1)^n (\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \cdots + a_n)$ be its ch polynomial. Then according to cayley Hamilton theorem, we have to show that,

$$A^n + a_1 A^{n-1} + \cdots + a_n I = \bar{O}$$

$$\boxed{|A - \lambda I| = 0}$$

$$A^2 - 3A - 4I = \bar{O}$$

$$\Rightarrow A^2 - 3A - 4 \cdot A A^{-1} = \bar{O}$$

$$\Rightarrow A - 3I - 4A^{-1} = \bar{O}$$

$$\therefore A^{-1} = \frac{1}{4}(A - 3I)$$

$A_{n \times n}$

$$|A - \lambda I| = (-1)^n \{ \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n \} = 0$$

$$|A - \lambda I| = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix}_{2\times 2} = 0$$

↪ λ এর power (Max) 1,

এর যেমন λ এর Maximum Power $= 2$.

3×3 " " " " " " $= n-1$.

$n \times n$ " " " "

$$A - \lambda I = \begin{bmatrix} 1+\lambda^2 & 2\lambda & 3 \\ 3 & 4\lambda^3 & 4 \\ 5 & 6 & \lambda^4 \end{bmatrix}$$

$$= \lambda^4 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \lambda^3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \lambda^2 \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

$$+ \lambda \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} + \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

$$\mathrm{Adj}(A-\lambda I) = \lambda^{n-1}\begin{bmatrix} & \\ & \end{bmatrix} + \lambda^{n-2}\begin{bmatrix} & \\ & \end{bmatrix} + \lambda^{n-3}\begin{bmatrix} & \\ & \end{bmatrix}$$

$$+ \cdots\cdots + \begin{bmatrix} & \\ & \end{bmatrix}$$

$$A(\mathrm{adj}A) = (\mathrm{adj}A)A = |A|I.$$

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$$

$$|A| = 7, \quad A(\mathrm{adj}A) = 7I = 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

Proof:

Since the elements of $(a-\lambda I)$ are at the most of 1st degree in λ, therefore the elements of $\mathrm{adj}(A-\lambda I)$ are at the most of degree $(n-1)$ in λ.

Thus the matrix $\mathrm{adj}(A-\lambda I)$ can be written as,

$$\mathrm{adj}(A-\lambda I) = B_0\lambda^{n-1} + B_1\lambda^{n-2} + \cdots\cdots + B_{n-1} \quad (1)$$

where, $B_0, B_1, \ldots\ldots, B_{n-1}$ are some square matrices of order $n \times n$ for any square matrix

A of order $n \times n$, we know that

$$A(\text{adj} A) = (\text{adj} A)A = |A| I.$$

Similarly for the matrix $(A - \lambda I)$, we can write,

$$(A - \lambda I)\, \text{adj}(A - \lambda I) = |A - \lambda I| I.$$

Then, $(A - \lambda I)\, \text{adj}\, (A - \lambda I) = |A - \lambda I| I$

$$\Rightarrow (A - \lambda I)\left(\lambda^{n-1} B_0 + \lambda^{n-2} B_1 + \cdots\cdots + B_{n-1}\right) =$$

$$(-1)^n \left(\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + a_3 \lambda^{n-3} + \cdots + a_n\right) I$$

Now, equating the co-efficients of equal powers of λ on both sides, we get,

λ^n : $\quad -I B_0 = (-1)^n I$

λ^{n-1} : $\quad A B_0 - I B_1 = (-1)^n a_1 I$

λ^{n-2} : $AB_1 - IB_2 = (-1)^n a_2 I$

- - - - - - -

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λ^0 : $AB_{n-1} = (-1)^n a_n I$

Now, premultiplying the above equation by $A^n, A^{n-1}, \ldots A, I$ respectively,

$$-A^n B_0 = (-1)^n A^n$$

$$A^n B_0 - A^{n-1} B_1 = (-1)^n A^{n-1}$$

- - - - - -

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Now, adding the above equations

$$(-1)^n (A^n + a_1 A^{n-1} + \ldots + a_n) I = \bar{O}$$

$$\therefore A^n + a_1 A^{n-1} + \ldots + a_n I = \bar{O}$$

(Proved)

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Date: 29.03.17

Ex-1: Verify Cayley-Hamilton theorem for the following matrix and hence find its inverse:

(i) $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Sol^n: (1)

The characteristic equation for the given matrix is,

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\{(2-\lambda)^2 - 1\} + 1\{-(2-\lambda)+1\} + 1\{1-(2-\lambda)\} = 0$$

$$\Rightarrow (2-\lambda)(4-4\lambda+\lambda^2-1)+\lambda-2+1+1-2+\lambda=0$$

$$\Rightarrow (2-\lambda)(\lambda^2-4\lambda+3)+2\lambda-2=0$$

$$\Rightarrow 2\lambda^2-8\lambda+6-\lambda^3+4\lambda^2-3\lambda+2\lambda-2=0$$

$$\Rightarrow -\lambda^3+6\lambda^2-9\lambda+4=0$$

$$\Rightarrow \lambda^3-6\lambda^2+9\lambda-4=0$$

To verify cayley-Hamilton theorem, we have to show that,

$$A^3-6A^2+9A-4I=\bar{O} \quad \text{—— (1)}$$

$$\text{L.H.S} = A^3-6A^2+9A-4I$$

$$= \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}^3 - 6\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}^2 + 9\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

$$-4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the cayley-Hamilton theorem verified.

Now premultiplying eq-1 by A^{-1} we get,

$$A^3A^{-1} - 6A^2A^{-1} + 9AA^{-1} - 4IA^{-1} = \bar{O}$$

$$\Rightarrow A^2 - 6A + 9I - 4A^{-1} = \bar{O}$$

$$\Rightarrow 4A^{-1} = A^2 - 6A + 9I$$

$$\therefore A^{-1} = \frac{1}{4}(A^2 - 6A + 9I)$$

$$= \frac{1}{4}\left(\begin{bmatrix} & & \end{bmatrix}^2 - 6\begin{bmatrix} & & \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$\therefore A^{-1} =$$

Ex-(2): Find the eigen values and eigen vectors of the following matrix:

(i) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$.

(ii) $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{bmatrix}$

Sol^n:

(i) The ch. equation for the given matrix is:

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 2-\lambda & 3 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda)^2 - 2\cdot 0 + 3\cdot 0 = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda)(2-\lambda) = 0$$

$$\therefore \lambda = 1, 2, 2$$

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For the eigen value $\lambda = 1$, the equation, $(A - \lambda I) X = \bar{0}$ becomes,

$$\left(\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This implies,

$$2x_2 + 3x_3 = 0$$
$$x_2 + x_3 = 0$$
$$x_3 = 0$$

The co-efficent matrix of the above system is,

$$A_1 = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \overset{R_{12}}{\sim} \begin{bmatrix} 0 & 1 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{R_{21}(-2)} \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & -3 \\ 0 & 0 & 1 \end{bmatrix} \underset{\sim}{R_{32}} \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\underset{\sim}{R_{32}(3)} \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

which implies that,

$$\rho(A_1) = 2$$

Since $\rho(A_1) < n$, therefore the above system has infinitely many solutions.

In this case number of free variable

$$= n - \rho(A_1)$$
$$= 3 - 2$$
$$= 1$$

let, x_1 be the free variable.

~~Al~~so let, $x_1 = k$. ; where k is any real number.

Now, the equivalent system is,

$$x_2 + 3x_3 = 0$$
$$x_3 = 0$$

Which implies, $x_2 = 0$
$x_3 = 0$
$x_1 = k$

Now the eigen vector corresponding to $\lambda = 1$ is,

$$X_{\lambda=1} = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}$$

For the eigen value $\lambda = 2$, the equation $(A - \lambda I)x = \bar{0}$ becomes

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Linear Combination of vectors:

Let V be a vector space over the scalar field K. Then a vector $v \in V$ is called the linear combination of vectors $u_1, u_2, \ldots, u_n$ in V if there exist some scalars $k_1, k_2, \ldots, k_n \in K$ such that,

$$v = k_1 u_1 + k_2 u_2 + \cdots + k_n u_n.$$

$(x, y) \in \mathbb{R}^2$, $(2, 3) \in \mathbb{R}$

$$(2,3) = \boxed{2}(1,0) + \boxed{3}(0,1)$$
$$= (2,0) + (0,3)$$
$$= (2,3)$$

$$V = k_1 u_1 + k_2 u_2$$

Ex-1:

Express $v = (3, 7, -4)$ as a linear combination of the vectors $u_1 = (1, 2, 3)$, $u_2 = (2, 3, 7)$, $u_3 = (3, 5, 6)$.

Soln: let, $v = x u_1 + y u_2 + z u_3$

$$\Rightarrow (3,7,-4) = x(1,2,3) + y(2,3,7) + z(3,5,6)$$

$$\Rightarrow (3,7,-4) = (x, 2x, 3x) + (2y, 3y, 7y) + (3z, 5z, 6z)$$

$$\Rightarrow (3,7,-4) = (x+2y+3z,\ 2x+3y+5z,\ 3x+7y+6z)$$

This implies,

$$x + 2y + 3z = 3$$
$$2x + 3y + 5z = 7$$
$$3x + 7y + 6z = -4$$

The augmentent matrix for the above system is-

$$C = \begin{bmatrix} 1 & 2 & 3 & 3 \\ 2 & 3 & 5 & 7 \\ 3 & 7 & 6 & -4 \end{bmatrix}$$

$$\underset{R_{31}(-3)}{\overset{R_{21}(-2)}{\sim}} \begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & -3 & -13 \end{bmatrix}$$

$$\overset{R_2(-1)}{\sim} \begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & -3 & -13 \end{bmatrix}$$

$$\xrightarrow{R_{32}(-1)} \begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -4 & -12 \end{bmatrix}$$

$$\xrightarrow{R_3(-\frac{1}{4})} \begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

which implies that,

$$\rho(A) = \rho(C) = 3$$

Since, $\rho(A) = \rho(C) = n$, therefore the above system has unique solution.

Now, the equivalent system is

$$x + 2y + 3z = 3$$
$$y + z = -1$$
$$z = 3$$

Thus, $z = 3$, $y = -4$

$\therefore x = 2$

Therefore the linear combination is,

$$(3, 7, -4) = 2(1, 2, 3) - 4(2, 3, 7) + 3(3, 5, 6)$$

(Answer)

Spanning Set:

let, V be a vector space over the scalar field K. Then a set $\{u_1, u_2, \ldots, u_n\}$ in V is said to form a spanning set of V if every $v \in V$ is a linear combination of that vectors. That is if there exist some scalars $k_1, k_2, \ldots, k_n$ in K such that

$$v = k_1 u_1 + k_2 u_2 + \cdots + k_n u_n.$$

$$(5, 6, 7) \in \mathbb{R}^3, \quad \{(1,0,0), (0,1,0), (0,0,1)\}$$

$$(5, 6, 7) = 5(1,0,0) + 6(0,1,0) + 7(0,0,1)$$

Ex-1:

Test whether the following vectors in $\mathbb{R}^3$ form a spanning set or not?

① $\{(1,1,1), (1,1,0), (1,0,0)\}$

② $\{(1,2,3), (1,3,5), (1,5,9)\}$

Solution:

0 0

① let, $a, b, c \in \mathbb{R}^3$ be any vector in $\mathbb{R}^3$.

Also let, $(a, b, c) = x(1,1,1) + y(1,1,0) + z(1,0,0)$

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Basis of a vector Space:

Let, V be a vector space over the scalar field K. The set $S = \{u_1, u_2 \ldots u_n\}$ of vectors in V is said to form a basis of V if it has the following two properties

(1) S is linearly independent.

(11) S span V or S is a spanning set of V.

Dimension of vector space:

A vector space V is said to be finite dimensional or said to be of dimension n if V has a basis of exactly n elements. In this case, we write $\dim V = n$.

$\mathbb{R}^2$

$S = \{\overset{u_1}{(1,0)}\ \overset{u_2}{(0,1)}\}$ → lin independent.

→ Spanning set.

$(3,4) = 3(1,0) + 4(0,1)$

$\mathbb{R}^3$ Basis of $\mathbb{R}^3$

$$S = \{(1,0,0), (0,1,0), (0,0,1)\}$$

$$(3,4,7) = 3(1,0,0) + 4(0,1,0) + 7(0,0,1)$$ ← spanning set.

$$\Rightarrow x(1,0,0) + y(0,1,0) + z(0,0,1) = (0,0,0)$$

$\therefore x = 0, \ y = 0, \ z = 0$

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