Simple Harmonic Free Vibrations

Waves on passing through a medium cause the particles comprising the medium to vibrate. A piece of wood on the surface of water in a pond, will bob up and down under the influence of water waves. The physical characteristics of the waves are determined by observing the manner in which a particle in the path of wave executes vibrations and thus a proper understanding of waves will require, as a prerequisite, the knowledge of the nature and theory of vibrations. We initiate the discussion with the simplest type of vibration called *simple harmonic* (also called *sinusoidal*) *motion*. It is abbreviated as SHM.

2.1 SIMPLE HARMONIC MOTION (MATHEMATICAL REPRESENTATION)

The simplest vibration of a single particle on one dimensional system is provided by a point mass m attached to a spring of negligible mass, Fig. 2.1. When the mass is displaced from its equilibrium position through a small

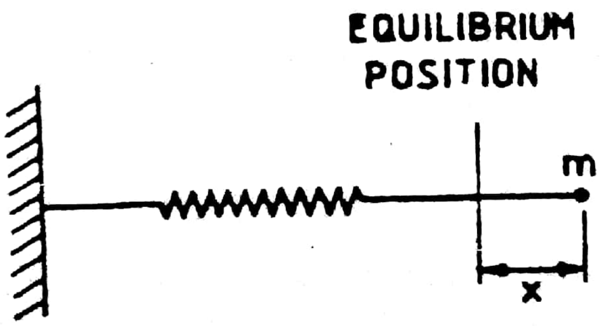

Fig. 2.1 A point mass attached to a spring of negligible mass

displacement x, the only force acting on it is an elastic restoring force proportional to x and acting in a direction towards the equilibrium position. Assuming Hooke's (empirical) law of elasticity to hold and ignoring the forces of air friction and internal elastic friction (which will cause the damping of the vibrations), the equation of motion is

$$m \frac{d^2x}{dt^2} = - Sx \tag{2.1}$$

where S is the restoring force per unit displacement for small displacements. It is called the ***stiffness constant***. The negative sign in Eq. (2.1) indicates that the direction of the restoring force is always opposite to the displacement x. Rewriting Eq. (2.1) as

$$\frac{d^2x}{dt^2} = - \frac{S}{m} x$$

$$= - \omega_0^2 x \tag{2.2}$$

where $\omega_0 = \sqrt{\frac{S}{m}}$ is the characteristic frequency of the simple harmonic motion.

Calling the differential operator, $D \equiv \frac{d}{dt}$, the Eq. (2.2) becomes

$$D^2 x = - \omega_0^2 x \tag{2.3}$$

which is an algebraic equation and gives the solution

$$D = \pm i\omega_0 \tag{2.4}$$

The general solution of Eq. (2.2) becomes

$$x = Ae^{i\omega_0 t} + Be^{-i\omega_0 t} \tag{2.5}$$

where A and B are unknown constants to be determined from the initial conditions. This solution of the harmonic force equation, is called the exponential form of solution. Are the constants A and B real, imaginary or complex? This is easily decided as follows. Making use of the relation between exponential and trigonometrical quantities,

$$e^{\pm i\omega_0 t} = \cos \omega_0 t \pm i \sin \omega_0 t$$

Eq. (2.5) becomes

$$x = (A + B) \cos \omega_0 t + i (A - B) \sin \omega_0 t \tag{2.6}$$

From physical considerations it is evident that the displacement of a moving body must be real which is possible if $A - B = 0$ or $A = B$. Under this condition, the solution becomes

$$x = 2 A \cos \omega_0 t \tag{2.7}$$

This solution is no longer general since it does not have two arbitrary constants, as it should, being a solution of a differential equation of the second order. If, however, A and B are taken as imaginary numbers, the solution again will be found to contain one constant. Thus A and B cannot be imaginary either.

Lastly, let us suppose that A and B are complex numbers and are represented as

$$A = C_1 + i D_1 \tag{2.8}$$

$$B = C_2 + i D_2$$

Equation (2.6) for displacement becomes

$$x = (C_1 + C_2) \cos \omega_0 t - (D_1 - D_2) \sin \omega_0 t$$
$$+ i (D_1 + D_2) \cos \omega_0 t + i (C_1 - C_2) \sin \omega_0 t \tag{2.9}$$

The displacement will be real for all values of t, provided

$$D_1 + D_2 = 0 \text{ or } \quad D_1 = -D_2$$
$$C_1 - C_2 = 0 \text{ or } \quad C_1 = C_2$$

Thus the expressions for A and B become, Eq. (2.8),

$$A = C_1 + i D_1$$
$$B = C_1 - i D_1 \tag{2.10}$$

which are obviously complex conjugate of each other. Thus the general solution Eq. (2.5) becomes

$$x = (C_1 + iD_1) e^{i\omega_0 t} + (C_1 - i D_1) e^{-i\omega_0 t} \tag{2.11}$$

which is called the complex exponential form of solution. The unknown constants C_1 and D_1 are to be determined from the initial conditions which are normally the displacement x and the velocity dx/dt at time $t = 0$.

Calling $x = x_0$ at $t = 0$, one gets by inserting $t = 0$ in Eq. (2.11), $C_1 = \frac{x_0}{2}$

and

$$\dot{x} = \frac{dx}{dt}$$
$$= i\omega_0 (C_1 + iD_1)e^{i\omega_0 t} - i\omega_0 (C_1 - iD_1) e^{-i\omega_0 t}$$

which at $t = 0$ becomes,

$$x_0 = i\omega_0 (C_1 + iD_1) - i\omega_0 (C_1 - iD_1)$$
$$= -2\omega_0 D_1$$

or
$$D_1 = -\frac{x_0}{2\omega_0} \tag{2.12}$$

Alternatively the Eq. (2.9) is put into an equivalent trigonometric form as follows. Since the displacement will be real, its imaginary part may be dropped and it becomes

$$x = (C_1 + C_2) \cos \omega_0 t - (D_1 - D_2) \sin \omega_0 t \tag{2.13}$$

Putting
$$C_1 + C_2 = C \sin \phi_0$$
and
$$-(D_1 - D_2) = C \cos \phi_0,$$
then

$$x = C \cos \omega t \sin \phi_0 + C \sin \omega t \cos \phi_0$$
$$= C \sin (\omega_0 t + \phi_0) \tag{2.14}$$

where

$$C = [(C_1 + C_2)^2 + (D_1 - D_2)^2]^{1/2}$$

and

$$\phi_0 = \tan^{-1}\left(\frac{C_1 + C_2}{D_2 - D_1}\right)$$

The maximum value of sin $(\omega_0 t + \phi_0)$ is unity, so the constant C is the maximum values of x, called the amplitude. The system will oscillate between the values $\pm C$ and it will be shown that the value of C is determined from the total energy of the vibrating system.

The angle ϕ_0, called phase constant defines the position in the cycle of oscillation at the time $t = 0$ and it is obtained from Eq. (2.14) by putting $t = 0$. It is given by

$$x = C \sin \phi_0 \tag{2.15}$$

The phase constant ϕ_0, when increased by an integral multiple of 2π, describes the same motion.

Further let us understand the physical significance of ω_0. When the time t is increased by $\frac{2\pi}{\omega_0}$, the equation (2.14) gives

$$\begin{aligned} x &= C \sin \left[\omega_0 \left(t + \frac{2\pi}{\omega_0}\right) + \phi_0\right] \\ &= C \sin (\omega_0 t + 2\pi + \phi_0) \\ &= C \sin (\omega_0 t + \phi_0) \end{aligned}$$

It is evident that the function repeats itself after a period of time $\frac{2\pi}{\omega_0}$. Calling it the periodic time T, one gets

$$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{\sqrt{\frac{S}{m}}} = 2\pi\sqrt{\frac{m}{S}} \tag{2.16}$$

Thus the time period T is the time to complete one vibration or the time in which the phase angle of the particle increases by 2π. The frequency of oscillation ν, is the number of complete vibrations per second. Thus

$$\nu = \frac{1}{T} = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{S}{m}} \tag{2.17}$$

The stiffness constant S has the units of newtons per metre and the dimensions of $\frac{S}{m} = \frac{MLT^{-2}}{LM} = T^{-2}$.

That is the reason for equating $\frac{S}{m}$ to the second power of ω_0 (which has the dimensions of T^{-1}).

The amplitude is decided by the initial speed of the particle. Thus if v is the velocity at $x = 0$, then from Eq. (2.14), one obtains

$$\begin{aligned} \frac{dx}{dt} &= C\,\omega_0 \cos (\omega_0 t + \phi_0) \\ &= C\,\omega_0 [1 - \sin^2 (\omega_0 t + \phi_0)]^{1/2} \end{aligned}$$

$$= \omega_0 [C^2 - x^2]^{1/2}$$

or

$$v = \omega_0 C$$

$$C = \frac{v}{\omega_0} \tag{2.18}$$

Therefore, if the velocity v is high, the amplitude C will also be large.

As is obvious from Eq. (2.14), simple harmonic motion can be expressed by a sine function. That is why such a motion is a sinusoidal function of time.

Putting $\phi_0 = \delta + \frac{\pi}{2}$ in Eq. (2.14), one gets

$$x = C \sin\left(\omega_0 t + \delta + \frac{\pi}{2}\right)$$

$$= C \cos (\omega_0 t + \delta) \tag{2.19}$$

Thus the displacement of a point executing simple harmonic motion can be described by a cosine function of time. As such this type of motion is also called cosinusoidal.

A system whose displacement can be described by either a sine or cosine function of time, is said to be linear. Herein the stiffness S is constant with displacement x. However, nonlinearity sets in if S does not remain constant with displacement.

2.2 ENERGY OF A SIMPLE HARMONIC OSCILLATOR

A particle executing simple harmonic motion is subject to the action of a restoring force, so its energy can be both kinetic and potential. If no energy is dissipated, then the amplitude of displacement remains constant and it will follow that the law of conservation of mechanical energy holds for harmonic oscillations. Rewriting Eq. (2.14), the displacement is

$$x = C \sin (\omega_0 t + \phi_0) \tag{2.14}$$

Now velocity

$$v = \dot{x} = C\omega_0 \cos (\omega_0 t + \phi_0) \tag{2.15}$$

and acceleration $$a = \ddot{x} = -C\omega_0^2 \sin (\omega_0 t + \phi_0) = -\omega_0^2 x \tag{2.16}$$

$$= -Cx$$

The displacement, velocity and acceleration curves are plotted in Fig. 2.2 taking the initial phase $\phi_0 = 0$. Thus when the displacement is maximum, the speed is zero as the velocity changes direction there. Further when the displacement is zero, the speed is maximum and acceleration is zero, implying thereby that the whole energy is kinetic.

The potential energy is obtained by summing all the increments of work $Sx\,dx$, done by the system against the restoring force over the range zero to x. The system has the minimum potential energy at the point where $x = 0$.

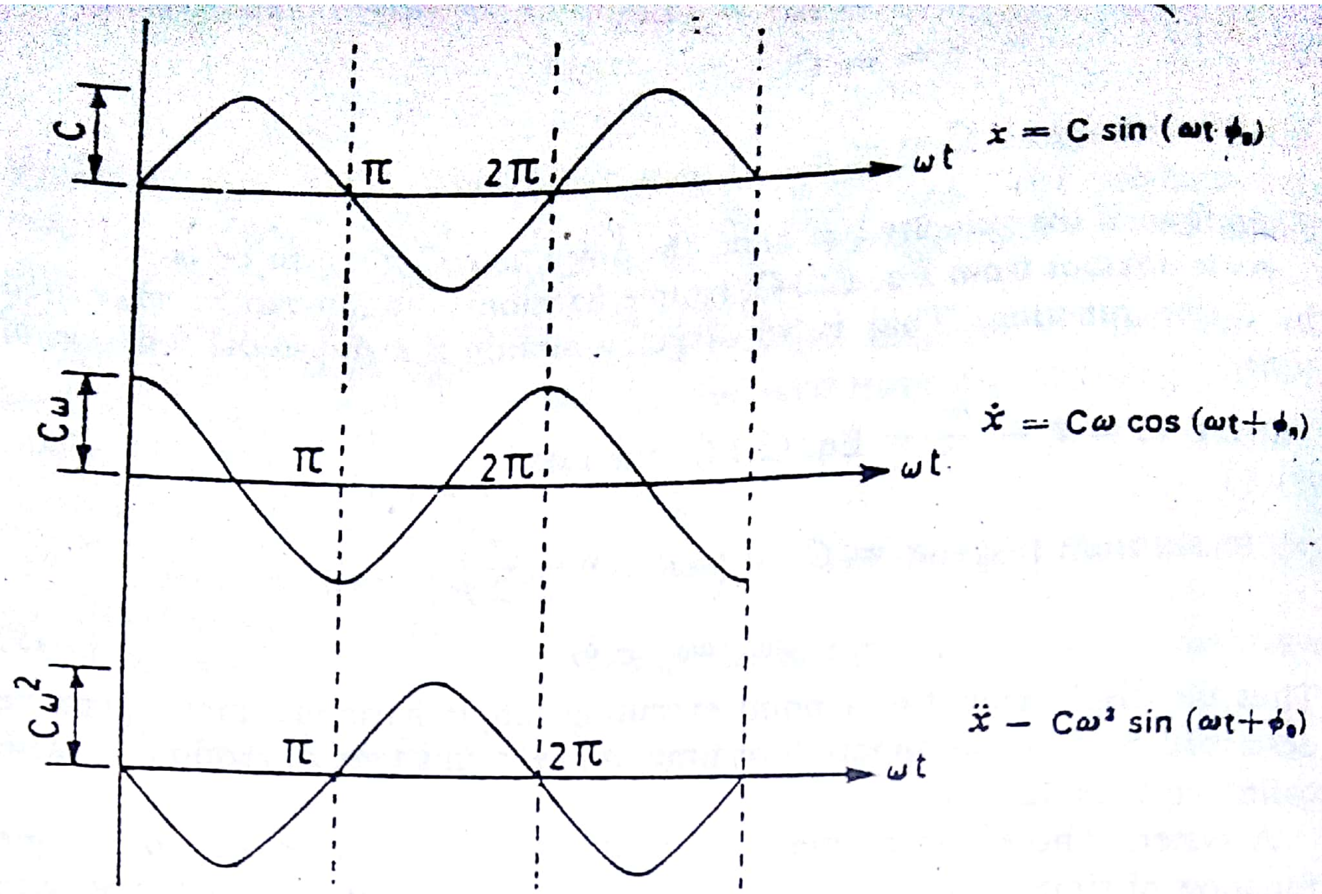

Fig. 2.2 Displacement, velocity and acceleration plotted as a function of ωt. The initial phase constant ϕ_0 has been taken equal to zero. The displacement and acceleration are out of phase, whereas displacement lags behind velocity by $\pi/2$ radians

Thus the potential energy at any instant

$$PE = \int_0^x Sx\,dx = \frac{1}{2}Sx^2$$

$$= \frac{1}{2}SC^2 \sin^2(\omega_0 t + \phi_0) \tag{2.17}$$

The maximum potential energy occurs at $x = \pm C$ and is

$$PE_{max} = \frac{1}{2}SC^2$$

$$= \frac{1}{2}m\omega_0^2C^2 \qquad \left(\omega_0 = \sqrt{\frac{S}{m}}\right)$$

The kinetic energy is

$$KE = \frac{1}{2}m\dot{x}^2$$

$$= \frac{1}{2}m\omega_0^2C^2 \cos^2(\omega_0 t + \phi_0) \tag{2.18}$$

and its maximum value is

$$KE_{max} = \frac{1}{2}m\omega_0^2C^2 \tag{2.19}$$

The maximum values of the potential and kinetic energies are equal which implies that the energy exchange is complete. The total energy E at any instant (from Eq. 2.17 and Eq. 2.18) is

$$E = \frac{1}{2}Sx^2 + \frac{1}{2}m\dot{x}^2$$

$$= \frac{1}{2}mC^2\omega_0^2\,[\sin^2(\omega_0 t + \phi_0) + \cos^2(\omega_0 t + \phi_0)]$$

$$= \frac{1}{2}mC^2\omega_0^2$$

$$= \frac{1}{2}SC^2 \tag{2.20}$$

Equation (2.20) shows that the total energy does not depend upon time and is constant. The potential and kinetic energies are plotted as a function of displacement in Fig. 2.3. The potential energy curve is parabolic wrt x and is symmetric about the position of equilibrium, $x = 0$. Therefore, energy is stored in the spring whether it is compressed or extended, as is the case when a gas is compressed or rarefied. The kinetic energy curve is parabolic both wrt x and $\dot{x}$. One curve is inverted wrt the other and exhibits the $\pi/2$ phase difference between the displacement and the velocity. For any arbitrary value of displacement x, the sum of the ordinates of the PE and KE curves, equals E, the total constant energy of the particle. In addition the total energy is equal to the maximum value of either of them.

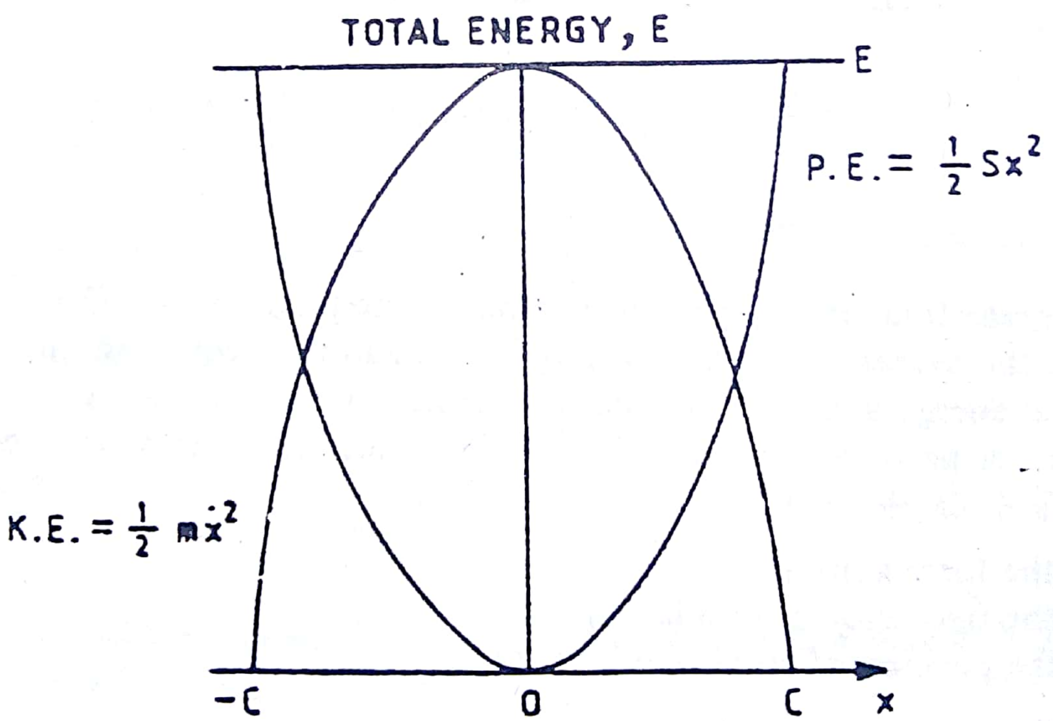

Fig. 2.3 Potential and kinetic energies plotted as a function of displacement for simple harmonic motion

Ex. 2.1 Prove that the average kinetic energy of a harmonic oscillator is equal to its average potential energy and each is equal to one half the total energy.

Solution

The instantaneous value of the potential energy is

$$\text{PE} = \frac{1}{2} m\omega_0^2 C^2 \sin^2 (\omega t + \phi)$$

The average PE for one complete time period is

$$\text{PE} = \frac{1}{T}\int_0^T \frac{1}{2} m\omega_0^2 C^2 \sin^2 (\omega_0 t + \phi)\, dt$$

$$= \frac{m\omega_0^2 C^2}{4T}\int_0^T 2\sin^2 (\omega_0 t + \phi)\, dt$$

$$= \frac{m\omega_0^2 C^2}{4T}\int_0^T (1 - \cos 2(\omega_0 t + \phi))\, dt$$

$$= \frac{m\omega_0^2 C^2}{4T}\Big[t\Big]_0^T = \frac{1}{4} m\omega_0^2 C^2 \qquad \text{(i)}$$

Now instantaneous kinetic energy is

$$\text{KE} = \frac{1}{2} m\dot{x}^2$$

$$= \frac{1}{2} m\omega_0^2 C^2 \cos^2 (\omega_0 t + \phi)$$

The average KE for one complete time period is

$$\text{KE} = \frac{1}{T}\int_0^T \frac{1}{2} m\omega_0^2 C^2 \cos^2 (\omega_0 t + \phi)\, dt$$

$$= \frac{m\omega_0^2 C^2}{4T}\int_0^T [1 + \cos 2(\omega_0 t + \phi)]\, dt$$

$$= \frac{1}{4} m\omega_0^2 C^2 \qquad \text{(ii)}$$

The average total energy per one complete time period is $= 1/2 m\omega_0^2 C^2$

Thus the average KE of a harmonic oscillator is equal to the average potential energy and each is equal to one half of the total energy.

Ex. 2.2 A particle of mass 1 g moves in a potential energy well given by $U = U_0 + 6x + x^2$. Find

(i) the force constant,
(ii) the frequency of oscillation, and
(iii) the position of stable equilibrium.

Solution

The force is given by

$$F = -\frac{dU}{dx}$$

$$= -6 - 2x$$

Now at the position of equilibrium, the force is zero.

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Thus $x = -3$ is the point of equilibrium.
The equation of motion is

$$m\frac{d^2x}{dt^2} = -6 - 2x = -2(3 + x)$$

Changing the variable by putting

$$x + 3 = y$$

one gets

$$\frac{dx}{dt} = \frac{dy}{dt}$$

$$\frac{d^2x}{dt^2} = \frac{d^2y}{dt^2}$$

Thus the equation of motion becomes

$$m\frac{d^2y}{dt^2} = -2y$$

The force constant is 2 N/m.
The frequency of oscillation

$$\nu = \frac{\omega}{2\pi}$$

$$= \frac{1}{2\pi}\sqrt{\frac{2}{1}}$$

$$= 0.225 \text{ Hz}$$

Ex. 2.3 A particle of mass 2 g is making simple harmonic motion along the x-axis. At distances 6 cm and 10 cm from the equilibrium position, the velocities of the particle are 5 cm/s and 4 cm/s respectively. Find the time period of oscillation, the amplitude and the maximum kinetic energy.

Solution

The equation governing simple harmonic motion is

$$x = C \sin(\omega_0 t + \phi) \tag{i}$$

The velocity $v = \dfrac{dx}{dt} = C\omega_0 \cos(\omega_0 t + \phi)$

$$= \omega_0\sqrt{C^2 - x^2} \tag{ii}$$

Thus if we call the velocities v_1 and v_2 at the respective distances of x_1 and x_2 from the position of equilibrium, we get

$$v_1 = \omega_0\sqrt{C^2 - x_1^2} \tag{iii}$$

$$v_2 = \omega_0\sqrt{C^2 - x_2^2} \tag{iv}$$

nerefore $v_1^2 - v_2^2 = \omega_0^2(x_2^2 - x_1^2)$

or $$\omega_0 = \sqrt{\frac{(v_1^2 - v_2^2)}{(x_2^2 - x_1^2)}}$$

Putting

$$v_1 = 5 \text{ cm/s}$$
$$v_2 = 4 \text{ cm/s}$$
$$x_1 = 6 \text{ cm}$$
$$x_2 = 10 \text{ cm}$$

we get

$$\omega_0 = \frac{3}{8} \text{ radians/s}$$

or $$\frac{\omega_0}{2\pi} = \frac{3}{16\pi} \text{ radians/s}$$

or $$T = \frac{16\pi}{3} \text{ s}$$
$$= 16.76 \text{ s}$$

Rewriting Eqs (iii) and (iv),

$$v_1 = \omega_0 \sqrt{(C^2 - x_1^2)} \tag{v}$$
$$v_2 = \omega_0 \sqrt{(C^2 - x_2^2)} \tag{vi}$$

one gets by squaring

$$v_1^2 = \omega_0^2 (C^2 - x_1^2) \tag{v}$$
$$v_2^2 = \omega_0^2 (C^2 - x_2^2) \tag{vi}$$

From Eq. (v)

$$C^2 = \frac{v_1^2}{\omega_0^2} + x_1^2$$
$$= v_1^2 \frac{(x_2^2 - x_1^2)}{(v_1^2 - v_2^2)} + x_1^2$$
$$= \frac{v_1^2(x_2^2 - x_1^2) + x_1^2(v_1^2 - v_2^2)}{(v_1^2 - v_2^2)}$$

Therefore $$C = \left[\frac{v_1^2 x_2^2 - v_2^2 x_1^2}{(v_1^2 - v_2^2)}\right]^{1/2}$$

Putting the values, we get

$$C = \left[\frac{2500 - 576}{9}\right]^{1/2} = 14.62 \text{ cm}$$

Lastly the maximum kinetic energy $= \frac{1}{2} m v_{\max}^2$

$$= \frac{1}{2} m \omega^2 C^2$$
$$= \frac{1}{2} \times 2 \times 10^{-3} \left(\frac{3}{8}\right)^2 \times 213.78 \times 10^{-4}$$
$$= 2.99 \times 10^{-6} \text{ J}$$

Ex. 2.4 If the earth were a homogeneous sphere of radius R and a straight hole were drilled through the centre of it, show that a ball dropped into the hole will execute simple harmonic motion. Find the frequency of oscillation of the particle.

Solution

Let the ball be at a distance x from the centre of the earth and in this position, the force of attraction is exerted by the inner sphere of volume $4/3\pi x^3$. The mass of the sphere is

$$\frac{4}{3} m x^3 \times \rho$$

where ρ is the density of the earth (assumed uniform). The force acting on the ball

$$= \frac{4}{3}\pi x^3 \rho m \frac{G}{x^2}$$

$$= \frac{4}{3}\pi \rho m G x$$

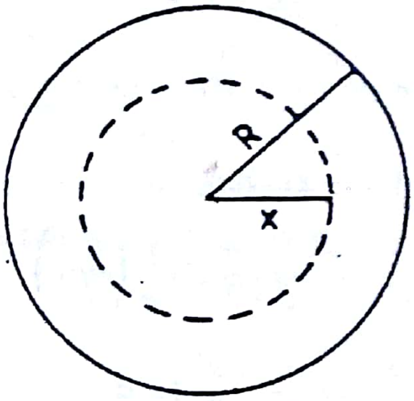


Fig. 2.4

where m is the mass of the ball. Furthermore, the force is directed towards the centre of the earth.

It constitutes the simple harmonic motion, because the force is proportional to x and it is directed towards the position of equilibrium.

Now the frequency, $\nu = \frac{\omega_0}{2\pi}$

$$= \frac{1}{2\pi}\sqrt{\frac{4}{3}G\pi\rho}$$

Now if R is the radius of earth, then

$$\frac{G4/3\pi R^3 \rho \times m}{R^2} = mg$$

or $$\frac{4}{3}G\pi\rho = \frac{g}{R}$$

Thus $$\nu = \frac{1}{2\pi}\sqrt{\frac{g}{R}}$$

Ex. 2.5 Show that the frequency of oscillations of a body of mass M suspended from a uniform spring of force constant S and mass m is given by

$$= \frac{1}{2\pi}\sqrt{\frac{S}{M + m/3}}$$

Solution

If the spring were weightless, the frequency of oscillation of mass M would be

$$= \frac{1}{2\pi}\sqrt{\frac{S}{M}}$$

Let l be the length of the spring and if m be its mass, then the mass per unit length is m/l. Consider a small element dx at a distance x from the upper end of the spring.

When the mass M undergoes a displacement vertically downward; the different parts of the spring elongate through different extensions; the upper end not moving at all whereas the mass M undergoing the maximum displacement. The displacement varies linearly with distance from the upper end. The displacement of the element dx at a distance x from the upper end is proportional to x. Its velocity is x/lv where v is the instantaneous velocity of the lower end of the spring.

Thus the kinetic energy of the element dx

$$= \frac{1}{2}\left(\frac{m}{l}dx\right)\left(\frac{x}{l}v\right)^2$$

$$= \frac{1}{2}\frac{mv^2}{l^3}x^2\,dx$$

Therefore the kinetic energy of the whole spring at that instant

$$= \int_0^l \frac{1}{2}\frac{mv^2}{l^3}x^2\,dx$$

$$= \frac{1}{2}\frac{mv^2}{l^3}\frac{l^3}{3} = \frac{mv^2}{6}$$

The total kinetic energy of the spring

$$= \frac{1}{2}Mv^2 + \frac{mv^2}{6}$$

$$= \frac{1}{2}\left(M + \frac{m}{3}\right)v^2$$

If the instantaneous displacement of the mass M is x at time t, then its potential energy

$$PE = \int_0^x Sx\,dx$$

$$= \frac{1}{2}Sx^2$$

Thus the total energy of the system is

$$E = \text{KE} + \text{PE}$$

$$= \frac{1}{2}\left(M + \frac{m}{3}\right)\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}Sx^2$$

Since the total energy of the system is conserved, therefore

$$\frac{dE}{dt} = 0$$

or

$$\left(M + \frac{m}{3}\right)\frac{d^2x}{dt^2} + Sx = 0$$

or

$$\frac{d^2x}{dt^2} = -\frac{S}{\left(M + \frac{m}{3}\right)}x$$

which represents SHM with time period

$$T = 2\pi\sqrt{\frac{M + \frac{m}{3}}{S}}$$

or

$$\nu = \frac{1}{2\pi}\sqrt{\frac{S}{M + \frac{m}{3}}}$$

2.3 EXAMPLES OF FREE VIBRATIONS

2.3.1 Angular Vibrations

As illustrations of mechanical systems executing simple harmonic vibrations, we consider the angular vibrations where the return torque is provided either by a spring (torsional vibrations) or by gravity (pendulum vibrations). Let us consider these cases one by one.

Torsional Pendulum It consists of a heavy mass suspended by a wire from a rigid support (Fig. 2.5). Normally either a disc or a spherical bob is used. The disc can be turned about the axis of the wire in a plane perpendicular to its length. If I is the moment of inertia of the disc about the wire, and θ is the rotational displacement, then the equation of motion is

$$I\frac{d^2\theta}{dt^2} = -C\theta \tag{2.21}$$

where C is the restoring couple per unit twist.
Rewriting it as

$$\frac{d^2\theta}{dt^2} + \frac{C}{I}\theta = 0,$$

it is obvious that it represents simple harmonic motion with angular frequency

$$\omega_0 = \sqrt{\frac{C}{I}} \tag{2.22}$$

and the time period

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{I}{C}}$$

Simple Pendulum It is constituted by a point mass suspended from a rigid support by a light, inextensible and flexible string.

Let the bob be displaced through an angular displacement θ. If m be the mass of the point mass and l the length of the string, then a clockwise torque $-mg \sin\theta \times l$, tries to bring the bob back to the equilibrium position, Fig. 2.6.

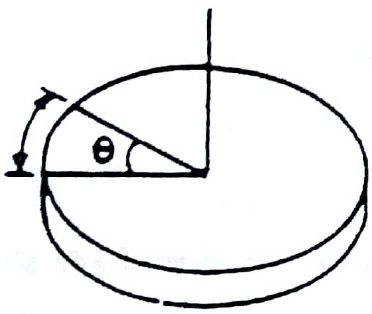


Fig. 2.5 Torsional pendulum

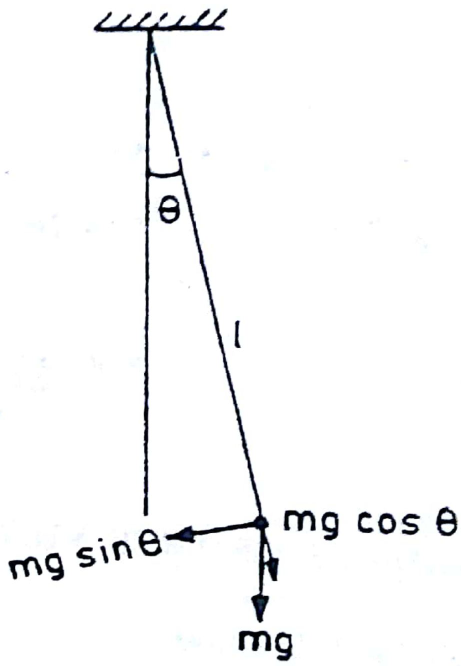


Fig. 2.6 A Simple pendulum

According to Newton's second law of motion

$$-mgl \sin\theta = ml^2 \times \frac{d^2\theta}{dt^2}$$

where ml^2 is the moment of inertia of the bob about a horizontal axis passing through centre of suspension. $d^2\theta/dt^2$ is the angular acceleration of the bob.

Thus

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$$

For a small angular displacement, $\sin\theta \approx \theta$, thus

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0 \tag{2.23}$$

which is the equation of simple harmonic motion with angular velocity

$$\omega_0 = \sqrt{\frac{g}{l}} \tag{2.24}$$

and the time period $T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{l}{g}}$ (2.25)

Ex. 2.6 A mass m is attached to a weightless spring and it has a time period T when oscillating in the horizontal position. Show that the time period will not change if the system is turned into a vertical direction.

Solution

Let S be the force constant of the spring and when the mass m is displaced through a small distance x horizontally, the restoring force is $-Sx$. The equation of motion is

$$m\ddot{x} = -Sx \tag{i}$$

or

$$\ddot{x} = -\frac{S}{m}x$$

which is SHM with time period $T = 2\pi\sqrt{\frac{m}{S}}$. When the system is made vertical, the force of gravity mg, extends the spring in the downward direction, say, through a distance x_0. The equilibrium is reached when $mg = Sx_0$. If the spring is displaced further through a distance x, the restoring force due to the spring acting upwards is

$$= -S(x + x_0)$$

The force due to gravity mg is still acting downwards and the net restoring force

$$= -[S(x + x_0) - mg]$$
$$= -S(x + x_0) + mg$$
$$= -Sx$$

which is identical with (i). It represents SHM with the same period

$$T = 2\pi\sqrt{\frac{m}{S}}$$

Thus the time period of oscillation of the mass m, is unaltered when the system is made vertical from the horizontal position.

Ex. 2.7 Using the law of conservation of energy, show that the angular speed of a simple pendulum is given by

$$\frac{d\theta}{dt} = \left[\frac{2}{ml^2}\{E - mgl(1 - \cos\theta)\}\right]^{1/2}$$

where E is the total energy of oscillations, l and m are length and mass of the pendulum and θ is the angular displacement of the pendulum from the vertical.

Solution

Let us get the total energy in term of the constants of motion. Thus the total energy

$$E = \text{KE} + \text{PE}$$

$$= \frac{1}{2}mv^2 + mgh$$

$$= \frac{1}{2}ml^2\omega^2 + mgl(1 - \cos\theta) \quad \text{(i)}$$

Thus $E - mgl(1 - \cos\theta) = \frac{1}{2}ml^2\omega^2$

or
$$\omega^2 = \frac{2}{ml^2}\{E - mgl(1 - \cos\theta)\}$$

Therefore the angular velocity

$$\frac{d\theta}{dt} = \left[\frac{2}{ml^2}\{E - mgl(1 - \cos\theta)\}\right]^{1/2}$$

Compound Pendulum A compound or physical pendulum is constituted by any rigid body which oscillates freely around any horizontal axis under the action of gravity. Let O be the point of suspension and XOX' the horizontal axis, around which the body is oscillating, Fig. 2.7. Let C, the centre of mass of the body, be situated at a distance b from the axis XOX'.

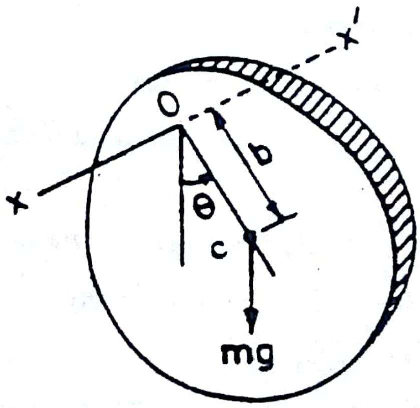


Fig. 2.7 A compound pendulum

If θ is the angle of swing at any time t, then the torque acting on the body is

$$-mgb\sin\theta$$

and the equation of motion is

$$I\frac{d^2\theta}{dt^2} = -mgb\sin\theta$$

where I is the moment of inertia, around the axis XOX'. Assuming that the amplitude of oscillation is small, we can approximate $\sin\theta \approx \theta$ and the equation of motion becomes

$$I\frac{d^2\theta}{dt^2} = -mgb\,\theta$$

or

$$\frac{d^2\theta}{dt^2} = -\frac{mgb}{I}\theta$$

This is the equation of SHM with the angular frequency

$$\omega_0 = \sqrt{\frac{mgb}{I}}$$

Putting $I = mk^2$ where k is the radius of gyration, one gets

$$\omega_0 = \sqrt{\frac{gb}{k^2}}$$

The time period is

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{k^2}{gb}}$$

k^2/b is called the length of the equivalent simple pendulum, since a simple pendulum with this length will have the same time period as the compound pendulum.

The time period of a compound pendulum is independent of the mass of the body and is the same so far as the ratio k^2/b is the same.

We treat the case of a sphere of radius R suspended by a wire and the distance between the centre of the sphere and the point of suspension measures l. Let us calculate the time period in this case and examine whether it depends on the finite size of the sphere.

Since the sphere has a finite size, one cannot treat the system as a simple pendulum.

The moment of inertia of sphere of radius R and mass m, around an axis passing through its centre is $2/5mR^2$.

The moment of inertia around an axis passing through the point of suspension is

$$\frac{2}{5}mR^2 + ml^2$$

Thus

$$K^2 = \left(\frac{2}{5}R^2 + l^2\right)$$

$$= l^2\left(1 + 0.4\frac{R^2}{l^2}\right)$$

The time period

$$T = 2\pi\sqrt{\frac{l^2(1 + 0.4R^2/l^2)}{gl}}$$

$$= 2\pi\sqrt{\frac{l}{g}}\left(1 + 0.4\frac{R^2}{l^2}\right)^{1/2}$$

$$= 2\pi\sqrt{\frac{l}{g}}(1 + 0.2R^2/l^2)$$

The second term gives the correction which is necessitated by the finite size of the bob. However, this is generally negligibly small.

2.3.2 Longitudinal Vibrations in a (Gas The Helmholtz Resonator)

A gas column vibrating with its natural frequency is called resonator. The Helmholtz resonator consists of a spherical cavity with two necks, the wider one to receive the incoming sound and the smaller one to be inserted into the ear to hear the sound, Fig. 2.8.

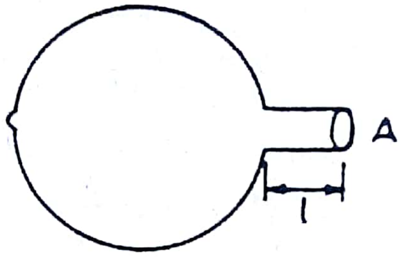


Fig. 2.8 The Helmholtz resonator

The only inertia we have to consider is that of the gas in the neck which moves to and fro like a piston of mass $\rho A l$ where l is the length of the neck and ρ, the density of the gas.

There is a change in pressure when a change of volume Ax is caused by the movement of the volume of air through a displacement x from its equilibrium position. This pressure change is calculated from the equation of state for adiabatic change, i.e.,

$$PV^{\gamma} = \text{constant}$$

where γ is the ratio of the two specific heats of the gas, that of specific heat at constant pressure to the one at constant volume. Taking logarithms and differentiating

$$dP + \frac{\gamma}{V} P\, dV = 0$$

$$\frac{dP}{P} = -\frac{\gamma}{V} dV \tag{2.26}$$

The equation of motion for the plug of air is

$$\rho A l \frac{d^2x}{dt^2} = -\gamma P \frac{Ax}{V} A$$

or

$$\frac{d^2x}{dt^2} + \frac{\gamma P A}{l \rho V} x = 0 \tag{2.27}$$

This is the equation of simple harmonic vibration with angular frequency

$$\omega_0 = \sqrt{\frac{\gamma P A}{l \rho V}}$$

and time period

$$T = \frac{2\pi}{\omega_0} = 2\pi \sqrt{\frac{l \rho V}{\gamma P A}}$$

The velocity of sound in air is given by the relation

$$v = \sqrt{\frac{\gamma P}{\rho}}$$

so that

$$T = \frac{2\pi}{v}\sqrt{\frac{Vl}{A}}$$

Therefore the frequency $(= 1/T)$ of the resonator depends upon the volume of the vessel and the length and area of cross-section of its neck.

2.3.3 Electrical Oscillations

Consider an electrical circuit in which an inductance L is connected across a capacitance $C \cdot q$, the charge on the condenser is the only source of emf. If the instantaneous value of the current is I then the voltage equation becomes

$$L\frac{dI}{dt} + \frac{q}{C} = 0$$

$$\frac{d^2q}{dt^2} + \frac{1}{LC}q = 0 \tag{2.29}$$

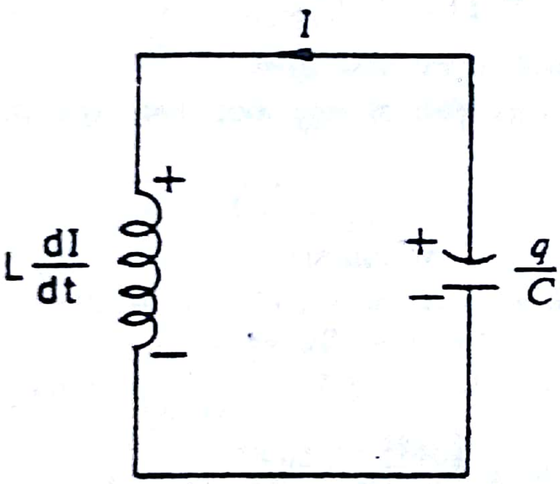


Fig. 2.9 An electrical circuit

where $I = dq/dt$. This is the equation of the simple harmonic motion with the angular frequency.

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

and time period

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{LC} \tag{2.30}$$

The charge on the capacitor varies harmonically and is given, out of analogy with Eq. (2.14), by

$$q = q_0 \sin(\omega_0 t \div \phi_0) \tag{2.31}$$

where q_0 is the maximum value (or the amplitude) of the charge.

The current

$$I = \frac{dq}{dt}$$

$$= \omega_0 q_0 \cos(\omega_0 t + \varphi_0) \tag{2.32}$$

and the voltage

$$V = \frac{q_0}{C} \sin(\omega_0 t + \varphi_0)$$

vary harmonically with the frequency ω_0.

The energy at a particular instant can be calculated when the condenser is charged to charge q.

Thus $$E = \tfrac{1}{2}CV^2 = \tfrac{1}{2}C\left(\frac{q}{C}\right)^2 = \frac{q^2}{2C} \tag{2.33}$$

This is electrostatic in nature.

Alternatively the inductive energy can be calculated when the current I is flowing through the inductance. Thus

$$E = \int VIdt = \int L\frac{dI}{dt}Idt = \int LIdI$$

$$= \tfrac{1}{2}LI^2 = \tfrac{1}{2}L\dot{q}^2 \tag{2.34}$$

The energy is magnetic in nature now.

There is an obvious semblance between the mechanical and electrical oscillators.

Thus

	Mechanical	*Electrical*
Eq. of motion	$m\ddot{x} + Sx = 0$	$L\ddot{q} + \frac{q}{C} = 0$
Total energy	$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}Sx^2$	$\frac{1}{2}L\dot{q}^2 + \frac{1}{2}\frac{q^2}{C}$

Depending upon the value of L and C, a very wide range of electrical frequencies can be generated. Thus a combination of a coil with $L = 100$ mH and a capacitor $C = 100\ \mu$F, will generate vibrations of approximately 50 Hz. This lies very low in the audio range. However with $L = 1\ \mu$H and $C = 10$ pF, the frequency is approximately 50 MHz and lies in the very high frequency region.

2.3.4 Plasma Vibrations

A plasma, which is defined as the fourth state of matter, is a gas which is comprised either wholly or partially of charged particles. It is electrically neutral as the positive and negative charges present are always equal.

A plasma slab is created whenever any ionizing collimated radiation like ultraviolet light strikes a gas. When it passes through the gas, energy is transferred to some of the molecules and as a consequence, the gas is ionized, removing the negative electrons which dart around leaving behind the positive ions. When the light is switched off, it leaves behind N electrons and N positive ions per unit volume of the plasma. However, in addition, the unionized neutral molecules will also be present but their presence does not affect the discussion below and as such these can be ignored.

For a brief while an external electric field, say, pointing downwards, is switched on. The electrons move in the upward direction and due to their higher mobility, are able to move faster. The electrons continue to move upwards even after the field is switched off. The positive ion flux, due to heavier mass, move very slowly and as such their motion can be ignored.

At some later time, the electrons have all covered a distance x, thus producing a sheet of unbalanced negative charge $-Nex$ per unit area at the top of the slab. The positive ions produce a sheet of unbalanced charge $+Nex$ per unit area at the bottom of the slab. These sheets produce an electric field E in the vertically upward direction between them given by

$$E = \frac{Ne}{\epsilon_0} x$$

where ϵ_0 is the permittivity of vacuum. The field exerts a return force $-(Ne^2/\epsilon_0)x$ on each electron in the plasma.

The equation of motion for the electron is

$$m_e \ddot{x} + \left(\frac{Ne^2}{\epsilon_0}\right) x = 0$$

or

$$\ddot{x} + \left(\frac{Ne^2}{m_e \epsilon_0}\right) x = 0 \tag{2.35}$$

This is the equation of simple harmonic motion with angular frequency

$$\omega_0 = \left(\frac{e^2}{m_e \epsilon_0}\right)^{1/2} \cdot N^{1/2} \tag{2.36}$$

Thus all the electrons vibrate verticaly with the plasma frequency ω_0. Putting the values of the following constants,

$$m_e = 9.11 \times 10^{-31} \text{ kg}, \; e = 1.60 \times 10^{-19} \text{ C}; \; \epsilon_0 = 8.85 \times 10^{-12} \text{ F m}^{-1},$$

one gets

$$\frac{e^2}{m_e \epsilon_0} = 3.18 \times 10^3 \text{ m}^3 \text{ s}^{-2} \tag{2.37}$$

Thus the value of plasma frequency serves as a measure of the electron density in the particular plasma.

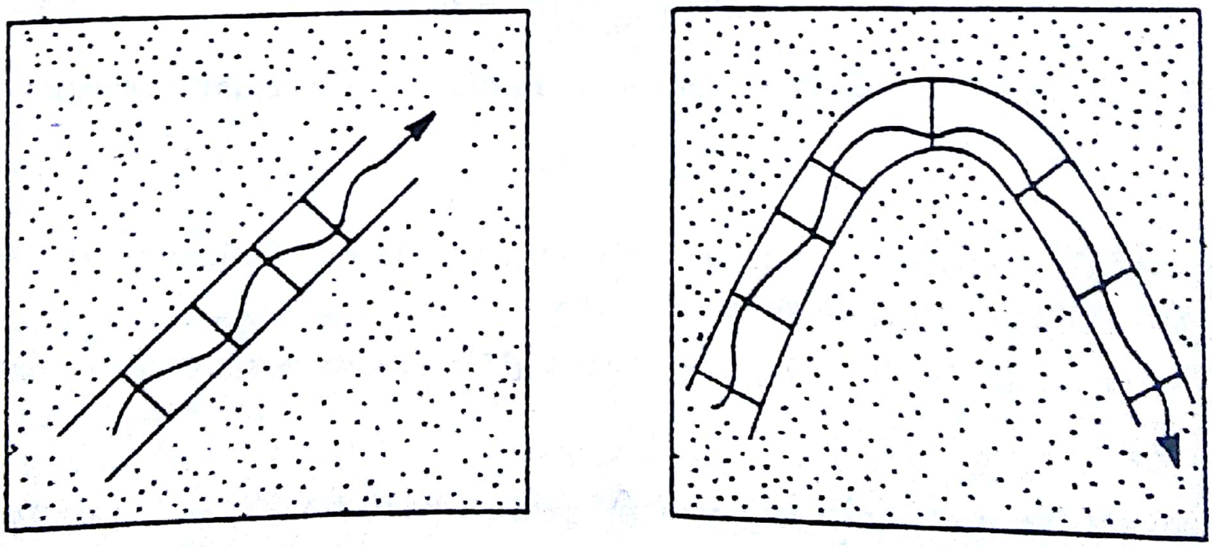
Let us use the result (2.36) for finding the plasma frequencies in the case of the ionosphere and free electrons in metals.

The Ionosphere The flux of ultraviolet light and X-rays emanating from the sun in the day is able to ionize the gas in the upper atmosphere. The highest electron densities are found in a region called ionosphere which has a layered structure and extends from 60 km up above the sea level. The density of the atmosphere falls off with height, but the electron density is not a simple function of height. However, the amount of ultraviolet radiation reaching a particular level of the atmosphere depends how much of it has been absorbed. The lowest layer called, D layer has electron densities in the range around $N \approx 10^9\ m^{-3}$ at mid-day. The highest layer, called F_2 layer, has electron densities around $N \approx 10^{12}\ m^{-3}$.

The corresponding plasma frequencies from Eqs. (2.36) and (2.37) turn out to be $\omega_0 \approx 2 \times 10^6\ s^{-1}$ for the D layer and $\omega_0 \approx 6 \times 10^7\ s^{-1}$ for the F_2 layer. These lie in the radio frequency range.

The presence of plasma has a direct bearing on the radio communication in the atmosphere. The F_2 layer reflects short wave radio signals back to the ground and thereby enables the signals to be received at great distances despite the curvature of the earth. It will be clear from the following considerations.

It can be shown that the index of refraction in the case of ionized gases is $n = \sqrt{(1 - (\omega_p^2/\omega^2))}$ where $\omega_p = Ne^2/m\epsilon_0$. It is obvious that for frequencies $\omega \gg \omega_p$, i.e., in regions of very little ionization and or high frequency range, the index of refraction n is real and so the waves propagate as in a dielectric medium, Fig. 2.10(a). However in the regions of the ionosphere like F_2 layer where ω_p increases with distance due to higher ionisation, the values of n will decrease with altitude. The electromagnetic radiation will bend in a direction away from the normal as the radiation is moving into a region of lower index of refraction from a region of higher n (Fig. 2.10(b)). Thus the bending of high frequency electromagnetic waves by F_2 layer is used in long distance transmission.



(a) (b)

Fig. 2.10 **Propagation of electromagnetic waves in (a) ionised medium of constant ω_p. (b) medium where $\omega_p \propto$ height (F_2 layer)**

The metals The metals have free conduction electrons which being detached from their parent atoms, are free to move. The positive ions form a lattice through which the electrons can travel and constitute an electric current, when under an applied field.

Let us take the case of Cu and assume that each Cu atom contributes exactly one conduction electron to the metal. The number of Cu atoms per unit volume is

$$N = 1000 \frac{N_A \rho}{A}$$

where N_A is the Avogadro No. (6.02×10^{23} mol^{-1}), A is the atomic weight and ρ, the density of Cu.
Putting the values for Cu

$$A = 63.5$$

$$\rho = 8900 \text{ kg m}^{-3}$$

one gets

$$N = 8.4 \times 10^{28} \text{ m}^{-3}$$

The plasma frequency for Cu is

$$\omega_0 \approx 2 \times 10^{16} \text{ s}^{-1}$$

This frequency lies in the infra-red region.

2.3.5 Lattice Vibrations

The force that binds the atoms together in a crystal is attractive with a repulsive core. It is the repulsive core that prevents the molecule or the solid from collapsing. The equilibrium distance between two atoms bound together in the solid, is that distance at which the attractive and repulsive forces exactly balance each other. Any small change in the position of the atom, brings into play the return force which makes the vibration possible. Although qualitatively this force is known but not its detailed dependence on distance. From a study of the vibration which takes place under the influence of this interatomic force, it is possible to have the detailed dependence on distance.

One can easily show that for a complex potential energy function of an arbitrary shape, a particle oscillating about an equilibrium position, will always perform simple harmonic motion provided the oscillations are sufficiently small. Figure 2.11 shows a potential energy curve which has two minima and two maxima but the stable equilibrium is represented by the minima alone.' At the point $x = x_0$, the particle is in stable equilibrium. When the point mass is at x, its displacement is $(x - x_0)$. The potential energy in the region of the minimum $(x = x_0)$ can be expressed by a Taylor series in x,

$$V = V_{x_0} + \left(\frac{d^2V}{dx^2}\right)_{x=x_0} \frac{(x - x_0)^2}{2!} + \left(\frac{d^3V}{dx^3}\right)_{x=x_0} \frac{(x - x_0)^3}{3!} + \ldots$$

Defining the displacement $X = (x - x_0)$ one gets

$$V = V_{x_0} + \frac{1}{2}X^2\left(\frac{d^2V}{dx^2}\right)_{x=x_0} + \frac{1}{6}X^3\left(\frac{d^3V}{dx^3}\right)_{x=x_0} + \ldots \qquad (2.38)$$

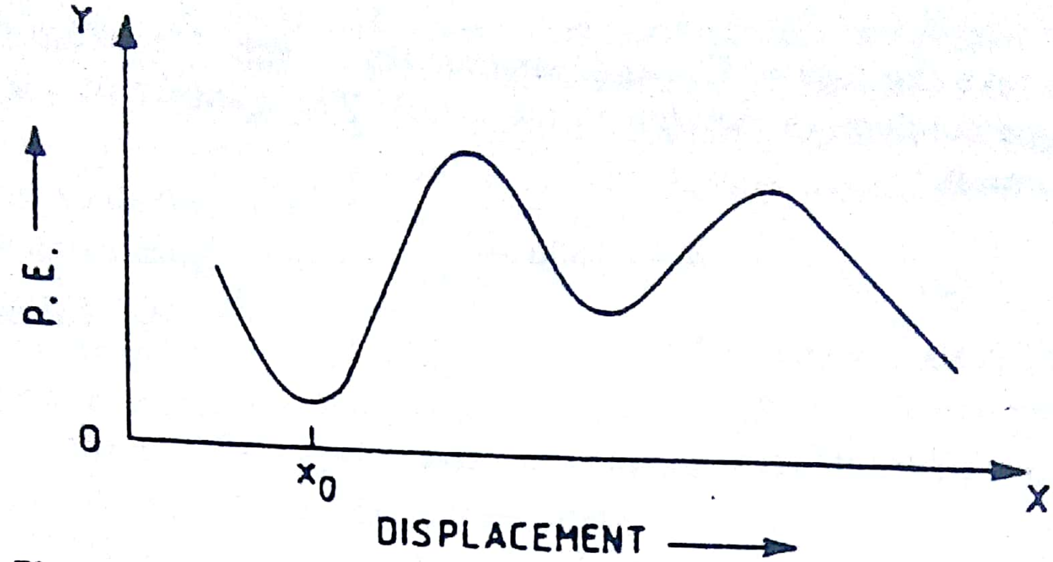


Fig. 2.11 A potential energy curve. At $x = x_0$, the particle is in stable equilibrium

There is no term proportional to X, since the potential energy V is at a minimum when $X = 0$ or $x = x_0$, i.e., $(dV/dx)_{x=x_0} = 0$

The restoring force F for a displacement X, is

$$F = -\frac{dV}{dx}$$

Thus

$$F = -X\left(\frac{d^2V}{dx^2}\right)_{x=x_0} - \frac{1}{2}X^2\left(\frac{d^3V}{dx^3}\right)_{x=x_0} - \ldots$$

For a small displacement X from the equilibrium position, the terms X^2, X^3 can be neglected. Thus

$$F = -X\left(\frac{d^2V}{dx^2}\right)_{x=x_0} \tag{2.39}$$

Comparing it with Eq. (2.1), one gets the approximate expression for stiffness

$$S \approx \left(\frac{d^2V}{dx^2}\right)_{x=x_0} \tag{2.40}$$

and the angular frequency

$$\omega_0 \approx \left[\frac{1}{m}\left(\frac{d^2V}{dx^2}\right)_{x=x_0}\right]^{1/2} \tag{2.41}$$

This implies that even in a complex potential energy function a particle vibrates in simple harmonic motion when displaced slightly from the equilibrium position. The frequency of oscillation is a measure of the d^2V/dx^2 term at the minimum of the potential energy curve.

Let us consider a molecule of an ionic solid like NaCl; where the cation has lost one electron and the anion has gained one electron, so as to form closed shell spherical structure individually. The interaction energy between the ions can be conveniently represented by the relation

$$V(r) = -\frac{e^2}{4\pi\epsilon_0 r} + \frac{B}{r^9} \tag{2.42}$$

The first term is the well-known electrostatic interaction energy between the charges e and $-e$, separated by a distance r. The second term is the repulsive part of the interaction. The choice of the exponent of r is only dictated by convenience so as to ensure that it should increase steeply with the decrease of r. The results we will discuss, are insensitive to the numerical value of this index.

The equilibrium separation (the bond length) R is the distance where $(-dV/dr)_{r=R} = 0$. Thus from Eq. (2.42),

$$\left.\frac{dV}{dr}\right|_{r=R} = \frac{e^2}{4\pi\epsilon_0 R^2} - \frac{9B}{R^{10}} = 0$$

or
$$B = \frac{e^2 R^8}{36\pi\epsilon_0} \tag{2.43}$$

The stiffness S can be found from the result (2.40)

Thus
$$S = \left(\frac{d^2V}{dr^2}\right)_{r=R}$$

$$= -\frac{2e^2}{4\pi\epsilon_0 R^3} + \frac{90B}{R^{11}}$$

$$= -\frac{e^2}{2\pi\epsilon_0 R^3} + \frac{90e^2}{36\pi\epsilon_0 R^3}$$

$$= +\frac{2e^2}{\pi\epsilon_0 R^3}$$

$$= 18.4\times 10^{-28}\ \text{Nm}^2/R^3 \tag{2.44}$$

The bond length of a diatomic molecule is found from the rotational spectrum. For an ionic molecule like HCl, $R = 0.13$ Nm and the stiffness turns out to be 836 Nm^{-1}. The Cl^- ion is 35 times heavier than H^+ ion, one may asssume that the negative ion is stationary and the vibrations are performed by H^+. Thus the vibration frequency may be estimated to be

$$\omega_0 = \left(\frac{S}{m_H}\right)^{1/2}$$

$$= 7.06\times 10^{14}\ \text{s}^{-1} \tag{2.45}$$

where $m_H = 1.67\times 10^{-27}$ kg. The angular frequency above corresponds to $\nu = 1.12\times 10^{14}$ Hz. The HCl molecule has an electric dipole moment and when it vibrates, the electric dipole moment will vary with the displacement and thus the molecule will emit light at about 10^{14} Hz. This frequency lies in the infrared region of the spectrum.

The energy states of a harmonic oscillator are given by*

$$E = (n + \tfrac{1}{2})\hbar\omega_0 \tag{2.46}$$

where the quantum number n can be any positive integer or zero and ω_0 is the natural frequency of oscillation. The oscillator will emit or absorb energy,

*A result quoted without proof. Any elementary book on Quantum Mechanics may be consulted for its derivation.

according to the selection rule $\Delta n = \pm 1$, only when n changes by ± 1. According to Quantum Mechanics the vibrational transition must be accompanied by a unit change in a second quantum number which describes the rotational energy of the molecule. Thus energy will be absorbed or emitted in quanta $\hbar\omega_0$. The ν_0 deduced experimentally for HCl is 8.9×10^{13} Hz and this value compares favourably with our estimate. This analysis tantamounts to the conclusion that the type of potential energy assumed for HCl (Eq. 2.42) describes the actual binding force at least in the region near the equilibrium distance R.

2.3.6 Transverse Vibrations of a Mass on a String

A point mass m attached to the middle of a light elastic string of length l, when displaced slightly from the equilibrium position executes simple harmonic motion.

It is assumed that T, the tension in the string remains constant when the mass is displaced through a small displacement y, Fig. 2.12. Resolving the tension in the longitudinal and transverse components, it is obvious that the longitudinal forces balance out and the transverse forces are the only remaining ones.

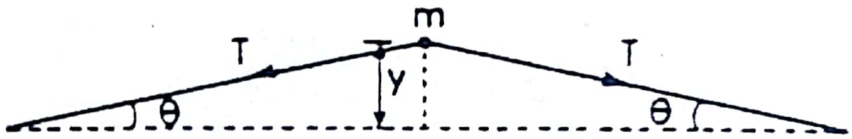


Fig. 2.12 A point mass on a stretched string

The equation of motion is

$$m \frac{d^2y}{dt^2} = -2T \sin \theta \tag{2.47}$$

The negative sign has been included because the displacement y and the tension T are in the opposite directions.

Now for small displacements

$$\sin \theta = \frac{2y}{l}$$

and the equation of motion becomes

$$m \frac{d^2y}{dt^2} = -\frac{4T}{l} y \tag{2.48}$$

This represents the SHM with angular frequency

$$\omega_0 = 2\sqrt{\frac{T}{ml}}$$

and time period

$$T = \frac{2\pi}{\omega_0} = \pi \sqrt{\frac{ml}{T}} \tag{2.49}$$

2.4 COMPOSITION OF SIMPLE HARMONIC MOTIONS

When a particle is simultaneously under the influence of two or more simple harmonic motions, it will describe a motion which is the resultant of all these motions. The constituent vibrations may be in a straight line or at right angles to each other and in the latter case the resultant curves are called Lissajous figures. Let us treat the following cases.

2.4.1 Two Simple Harmonic Vibrations of the Same Angular Frequency but Different Amplitudes and Phases

Let the displacements of the component vibrations be represented by

$$y_1 = a_1 \sin(\omega t + \phi_1) \tag{2.51}$$

and

$$y_2 = a_2 \sin(\omega t + \phi_2) \tag{2.52}$$

where ϕ_1 and ϕ_2 are their phases

The resultant displacement

$$\begin{aligned} y &= y_1 + y_2, \\ &= a_1 \sin(\omega t + \phi_1) + a_2 \sin(\omega t + \phi_2) \\ &= a_1 (\sin \omega t \cos \phi_1 + \cos \omega t \sin \phi_1) \\ &\quad + a_2 (\sin \omega t \cos \phi_2 + \cos \omega t \sin \phi_2) \\ &= \sin \omega t \, (a_1 \cos \phi_1 + a_2 \cos \phi_2) \\ &\quad + \cos \omega t \, (a_1 \sin \phi_1 + a_2 \sin \phi_2) \end{aligned}$$

Putting

$$R \cos \theta = a_1 \cos \phi_1 + a_2 \cos \phi_2 \tag{2.53}$$

and

$$R \sin \theta = a_1 \sin \phi_1 + a_2 \sin \phi_2$$

we get

$$\begin{aligned} y &= R \cos \theta \sin \omega t + R \cos \omega t \sin \theta \\ &= R \sin(\omega t + \theta) \end{aligned} \tag{2.54}$$

The resultant R is obtained from Eq. 2.53 by squaring and adding as

$$R^2 = a_1^2 + a_2^2 + 2a_1 a_2 \cos(\phi_1 - \phi_2) \tag{2.55}$$

Also by dividing from Eq. 2.53 we get the phase of the resultant motion as

$$\tan \theta = \frac{a_1 \sin \phi_1 + a_2 \sin \phi_2}{a_1 \cos \phi_1 + a_2 \cos \phi_2} \tag{2.56}$$

Equation 2.54 shows that the resultant motion has the same period as the constituent vibrations and in the same straight line with amplitude R (Eq. 2.55) and phase θ. (Eq 2.56).

Let us consider the following special cases

(i) If $\phi_1 - \phi_2 = 2n\pi$ where $n = 0, 1, 2,$ etc. Then the component vibrations are in the same phase and $R^2 = (a_1 + a_2)^2$ or $R = a_1 + a_2$. As the amplitude is essential positive; the negative sign of the square root is omitted.

(ii) If $\phi_1 - \phi_2 = (2n + 1)\pi$ where $n = 0, 1, 2,$ etc. Then the component vibrations are in the opposite phase and $R^2 = (a_1 - a_2)^2$ or

$R = a_1 - a_2$. Obviously $R = 0$, i.e., the particle will remain at rest if $a_1 = a_2$.

(iii) If $a_1 = a_2$ and φ_1 and φ_2 are different, then

$$R^2 = 2a_1^2 (1 + \cos (\varphi_1 - \varphi_2)) = 4a_1^2 \cos^2 \tfrac{1}{2}(\varphi_1 - \varphi_2)$$

or
$$R = 2a_1 \cos \frac{1}{2} (\varphi_1 - \varphi_2) \tag{2.57}$$

From Eq. (2.56) one gets

$$\tan \theta = \frac{\sin \varphi_1 + \sin \varphi_2}{\cos \varphi_1 + \cos \varphi_2}$$

$$= \tan \frac{1}{2} (\varphi_1 + \varphi_2)$$

or
$$\theta = \frac{1}{2} (\varphi_1 + \varphi_2) \tag{2.58}$$

The amplitude will be maximum, i.e. $R = 2a_1$ when $\cos \tfrac{1}{2}(\varphi_1 - \varphi_2) = 1$ or $(\varphi_1 - \varphi_2) = 2n\pi$ where n is an integer. This will be the case when the component vibrations are in phase.

The amplitude will be zero when $\cos \tfrac{1}{2}(\varphi_1 - \varphi_2) = 0$ or $\varphi_1 - \varphi_2 = (2n + 1)\pi$. This is the case when the component vibrations are in opposition.

2.4.2 A Large Number of Vibrations of the Same Angular Frequency but with Different Amplitudes

Let the component vibrations be

$$\begin{aligned} y_1 &= a_1 \sin (\omega t + \varphi_1) \\ y_2 &= a_2 \sin (\omega t + \varphi_2) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ y_n &= a_n \sin (\omega t + \varphi_n) \end{aligned} \tag{2.59}$$

The resultant displacement is given by

$$y = \sum_{n=1}^{n} a_n \sin (\omega t + \varphi_n)$$

$$= \sin \omega t \left(\sum_{n=1}^{n} a_n \cos \varphi_n\right) + \cos \omega t \left(\sum_{n=1}^{n} a_n \sin \varphi_n\right) \tag{2.60}$$

This may be represented by the motion

$$y = R \sin (\omega t + \theta)$$

where
$$R = \left[\left(\sum_{n=1}^{n} a_n \cos \varphi_n\right)^2 + \left(\sum_{n=1}^{n} a_n \sin \varphi_n\right)^2\right]^{1/2} \tag{2.61}$$

and
$$\tan \theta = \frac{\sum_{n=1}^{n} (a_n \sin \varphi_n)}{\sum_{n=1}^{n} (a_n \cos \varphi_n)}$$

Any simple harmonic motion may by represented by a vector diagram. The phase angle $(\omega t + \varphi)$ in the simple harmonic motion

$$y = a \sin (\omega t + \varphi)$$

increases uniformly with time as the vibration is taking place. The displacement at any time t is proportional to the sine of this angle. There is a convenient geometrical method of generating the displacement in simple harmonic motion both in magnitude and phase by a radius vector of length a rotating in the positive (anticlockwise) direction with constant angular velocity ω (Fig. 2.13).

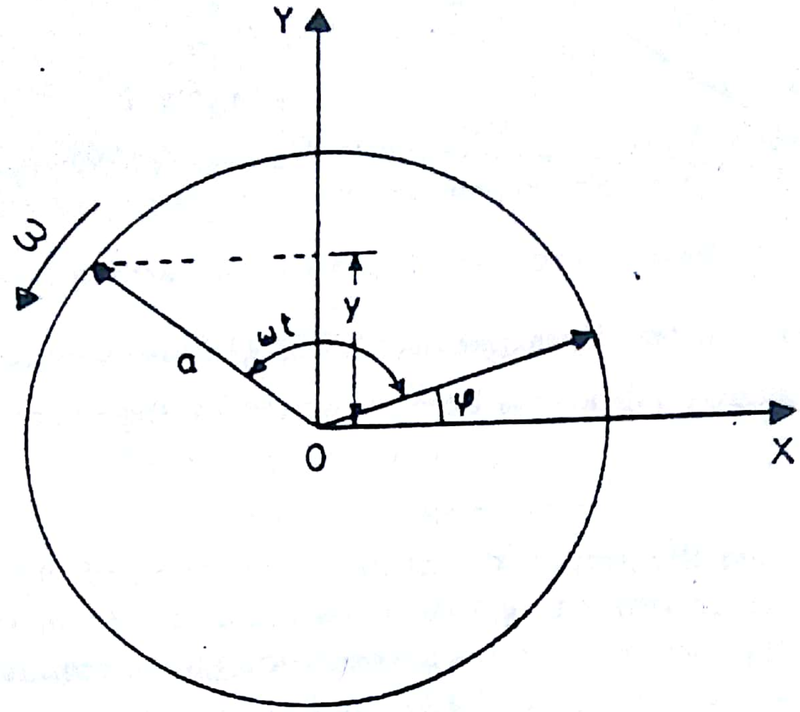


Fig. 2.13 The rotating vector generates the displacement as a function of time

The projection of the vector a on a fixed axis, say the y-axis, through the origin, gives the displacement. The fixed angle φ that this vector makes with the x-axis at $t = 0$, is the initial phase (or epoch).

This method is handy in getting the resultant motion when there are two or more simultaneous simple harmonic motions of equal angular frequencies but of different amplitudes and phases. Representing the displacements Eqs (2.51) and (2.52) by the sides of a parallelogram Fig. (2.14), the amplitude R of the resultant motion

$$y = R \sin (\omega t + \theta) \tag{2.62}$$

$$R^2 = (a_1 + a_2 \cos (\varphi_2 - \varphi_1)^2 + (a_2 \sin (\varphi_2 - \varphi_1))^2$$

$$= a_1^2 + a_2^2 + 2a_1 a_2 \cos (\varphi_2 - \varphi_1) \tag{2.63}$$

where the phase difference $= \varphi_2 - \varphi_1$

The phase constant θ of the resultant R is

$$\tan \theta = \frac{a_1 \sin \varphi_1 + a_2 \sin \varphi_2}{a_1 \cos \varphi_1 + a_2 \cos \varphi_2} \tag{2.64}$$

These are the same results as obtained by the analytical method before.

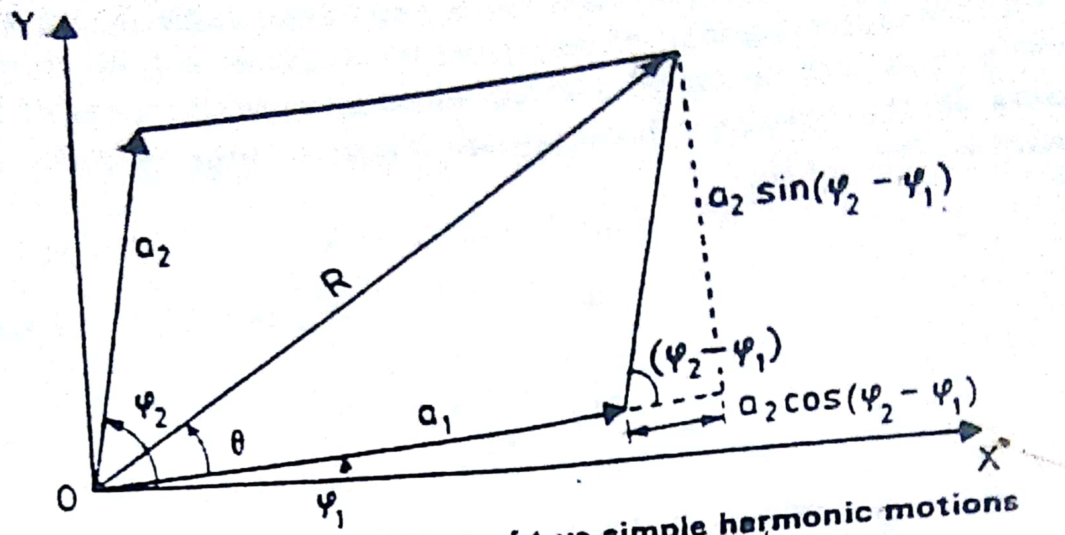


Fig. 2.14 Vector addition of two simple harmonic motions

2.4.3 Composition of two Perpendicular SHM of Same Period

Let the perpendicular vibrations be represented by the equations

$$x = a_1 \sin(\omega t + \varphi_1) \tag{2.65}$$

and

$$y = a_2 \sin(\omega t + \varphi_2) \tag{2.66}$$

where a_1 and a_2 are the amplitudes of the vibrations and their phase difference is $(\varphi_2 - \varphi_1)$. In order to get the equation of the locus (the path of the resultant vibration), we eliminate t between the above equations.

Rewriting the Eqs (2.65) and (2.66), we have

$$\frac{x}{a_1} = \sin \omega t \cos \varphi_1 + \cos \omega t \sin \varphi_1 \tag{2.67}$$

and

$$\frac{y}{a_2} = \sin \omega t \cos \varphi_2 + \cos \omega t \sin \varphi_2 \tag{2.68}$$

Multiplying Eq. (2.67) by $\sin \varphi_2$ and Eq. (2.68) by $\sin \varphi_1$ and subtracting, we obtain

$$\left(\frac{x}{a_1} \sin \varphi_2 - \frac{y}{a_2} \sin \varphi_1\right)^2 = \sin^2 \omega t \sin^2 (\varphi_2 - \varphi_1) \tag{2.68}$$

Analogously we obtain

$$\left(-\frac{x}{a_1} \cos \varphi_2 + \frac{y}{a_2} \cos \varphi_1\right)^2 = \cos^2 \omega t \sin^2 (\varphi_2 - \varphi_1) \tag{2.69}$$

Adding Eqs (2.68) and (2.69) we get

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - \frac{2xy}{a_1 a_2} \cos(\varphi_2 - \varphi_1) = \sin^2 (\varphi_2 - \varphi_1) \tag{2.70}$$

which is the equation of an ellipse inclined to the axes of coordinates and the ellipse may be inscribed in a rectangle of sides $2a$ and $2b$.

Let us examine the following particular cases:

(i) When $\varphi_2 - \varphi_1 = 0$, i.e., when the component vibrations are in phase. Equation (2.70) becomes

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - \frac{2xy}{a_1 a_2} = 0$$

$$\pm\left(\frac{x}{a_1} - \frac{y}{a_2}\right) = 0$$

or

These represent a pair of coincident straight lines lying in the first and third quadrants, shown in the curve of Fig. 2.15(i).

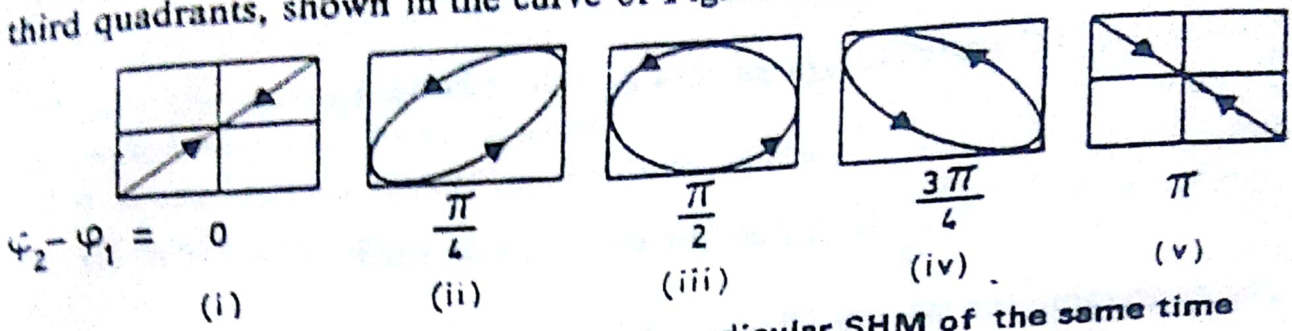


Fig. 2.15 Lissajous figures for two perpendicular SHM of the same time period and different phase differences

(ii) When $\varphi_2 - \varphi_1 = \pi/4$, the Eq. (2.70) becomes

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - \frac{\sqrt{2}\,xy}{a_1 a_2} = \frac{1}{2}$$

which represents an oblique ellipse shown in Fig. 2.15 (ii)

(iii) When $\varphi_2 - \varphi_1 = \pi/2$, the general equation becomes

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} = 1$$

which is the equation of a symmetrical ellipse with its major a_1 and minor a_2 axes coinciding with the x-and y-axes respectively, Fig. 2.15 (iii).

If in addition to $\varphi_2 - \varphi_1 = \pi/2$, $a_1 = a_2$, the general equation reduces to the equation of a circle

$$x^2 + y^2 = a_1^2$$

A uniform circular motion may be considered to be constituted by two similar SHM at right angles to each other and having a phase difference of $\pi/2$.

(iv) When $\varphi_2 - \varphi_1 = \pi$, the general equation becomes

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} + \frac{2xy}{a_1 a_2} = 0$$

or

$$\pm\left(\frac{x}{a_1} + \frac{y}{a_2}\right) = 0$$

This represents a pair of straight lines lying in the second and fourth quadrants, Fig. 2.15 (v).

2.4.4 Composition of Two Perpendicular Vibrations of Time Periods in the Ratio 1 : 2

Let the component vibrations be represented by

$$\frac{x}{a_1} = \sin(2\omega t + \varphi_1) \qquad (2.71)$$

and $$\frac{y}{a_2} = \sin(\omega t + \varphi_2) \tag{2.72}$$

Rewriting these equations as follows:

$$\frac{x}{a_1} = \sin 2\omega t \cos \varphi_1 + \cos 2\omega t \sin \varphi_1$$

and

$$\frac{y}{a_2} = \sin \omega t \cos \varphi_2 + \cos \omega t \sin \varphi_2$$

and expressing the $\sin 2\omega t$ and $\cos 2\omega t$ terms in terms of $\sin \omega t$, we get

$$\frac{x}{a_1} = 2 \sin \omega t \cos \omega t \cos \varphi_1 + (1 - 2\sin^2 \omega t) \sin \varphi_1 \tag{2.73}$$

$$\frac{y}{a_2} = \sin \omega t \cos \varphi_2 + \cos \omega t \sin \varphi_2 \tag{2.74}$$

In order to simplify the mathematics involved let us assume for the present that $\varphi = \varphi_1 - \varphi_2$ and $\varphi_2 = 0$. Then the above equations become

$$\frac{x}{a_1} = 2 \sin \omega t\,(1 - \sin^2 \omega t)^{1/2} \cos \varphi + (1 - 2\sin^2 \omega t) \sin \varphi \tag{2.75}$$

and $$\frac{y}{a_2} = \sin \omega t \tag{2.76}$$

Eliminating t between them, we get

$$\frac{x}{a_1} = 2\frac{y}{a_2}\sqrt{\left(1 - \frac{y^2}{a_2^2}\right)} \cos \varphi + \left(1 - 2\frac{y^2}{a_2^2}\right) \sin \varphi$$

or $$\left[\frac{x}{a_1} - \left(1 - \frac{2y^2}{a_2^2}\right) \sin \varphi\right]^2 = \frac{4y^2}{a_2^2}\left(1 - \frac{y^2}{a_2^2}\right) \cos^2 \varphi$$

or $$\left[\left(\frac{x}{a_1} - \sin \varphi\right) + \frac{2y^2}{a_2^2} \sin \varphi\right]^2 = \frac{4y^2}{a_2^2} \cos^2 \varphi - \frac{4y^4}{a_2^4} \cos^2 \varphi$$

Rearranging the terms we get

$$\left(\frac{x}{a_1} - \sin \varphi\right)^2 + \frac{4y^4}{a_2^4} \sin^2 \varphi + \frac{4y^2}{a_2^2}\frac{x}{a_1} \sin \varphi - \frac{4y^2}{a_2^2} \sin^2 \varphi$$

$$= \frac{4y^2}{a_2^2} \cos^2 \varphi - \frac{4y^4}{a_2^4} \cos^2 \varphi$$

or $$\left(\frac{x}{a_1} - \sin \varphi\right)^2 + \frac{4y^2}{a_2^2}\frac{x}{a_1} \sin \varphi$$

$$= \frac{4y^2}{a_2^2} - \frac{4y^2}{a_2^4}$$

or

$$\left(\frac{x}{a_1} - \sin \varphi\right)^2 + \frac{4y^2}{a_2^2}\left(\frac{y^2}{a_2^2} + \frac{x}{a_1} \sin \varphi - 1\right) = 0$$

Reverting to our original notation, we get

$$\left[\frac{x}{a_1} - \sin(\varphi_1 - \varphi_2)\right]^2 + \frac{4y^2}{a_2^2}\left[\frac{y^2}{a_2^2} + \frac{x}{a_1}\sin(\varphi_1 - \varphi_2) - 1\right] = 0 \tag{2.77}$$

This is the equation of a curve with two loops in general for any value of the phase difference and amplitudes.

Let us examine the particular cases one by one.

(i) When $\varphi_1 - \varphi_2 = 0$ or π, the general equation (Eq. 2.77) becomes

$$\frac{x^2}{a_1^2} + \frac{4y^2}{a_2^2}\left(\frac{y^2}{a_2^2} - 1\right) = 0$$

This equation represents a figure of 8.

(ii) When $\varphi_1 - \varphi_2 = \pi/2$, one gets

$$\left(\frac{x}{a_1} - 1\right)^2 + \frac{4y^2}{a_2^2}\left(\frac{y^2}{a_2^2} + \frac{x}{a_1} - 1\right) = 0$$

or

$$\left(\frac{x}{a_1} - 1\right)^2 + \left(\frac{2y^2}{a_2^2}\right)^2 + \frac{4y^2}{a_2^2}\left(\frac{x}{a_1} - 1\right) = 0$$

$$\left[\left(\frac{x}{a_1} - 1\right) + \frac{2y^2}{a_2^2}\right]^2 = 0$$

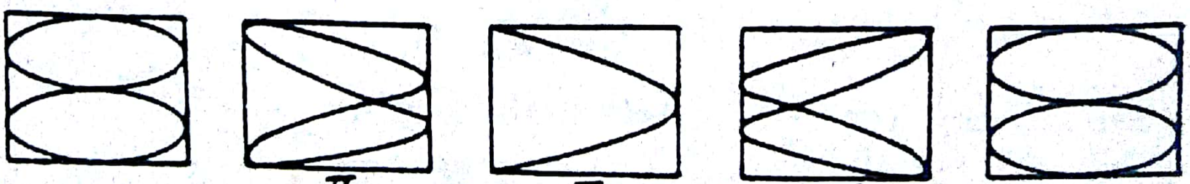
which represents two coincident parabolas and each of which has the equation

$$\left(\frac{x}{a_1} - 1\right) + \frac{2y^2}{a_2^2} = 0$$

or

$$y^2 = -\frac{a_2^2}{2a_1}(x - a_1)$$

The curves giving the resultant motion depending on the value of the phase difference are given in Fig. 2.16.



$\varphi_2 - \varphi_1 = 0$ (i); $\frac{\pi}{4}$ (ii); $\frac{\pi}{2}$ (iii); $\frac{3\pi}{4}$ (iv); π (v)

Fig. 2.16 The curves giving the resultant motion for two perpendicular SHM with time periods in the ratio 1 : 2 and different phase differences

2.5 SUPERPOSITION OF A LARGE NUMBER OF SIMPLE HARMONIC VIBRATIONS OF THE SAME AMPLITUDE AND FREQUENCY

2.5.1 Successive Phase Difference

Such a case has special importance when we deal with the phenomenon of diffraction of waves. Let there be component vibrations of the same amplitude a, and frequency ω but with a progressive phase difference φ. Then the successive waves may be represented by the terms

$$\begin{array}{l} a\, e^{i(\omega t+\varphi)} \\ a\, e^{i(\omega t+2\varphi)} \\ a\, e^{i(\omega t+3\varphi)} \\ \dots \quad \dots \\ a\, e^{i(\omega t+n\varphi)} \end{array} \tag{2.78}$$

The resultant is given by their sum

$$a\, e^{i(\omega t+\varphi)} + a\, e^{i(\omega t+2\varphi)} + a\, e^{i(\omega t+3\varphi)} + \dots + a\, e^{i(\omega t+n\varphi)}$$

$$= a\, e^{i(\omega t+\varphi)}[1 + e^{i\varphi} + e^{i2\varphi} + \dots + e^{i(n-1)\varphi}]$$

The expression within the brackets is a geometric series and can be summed as

$$= a e^{i(\omega t+\varphi)} \frac{1 - e^{in\varphi}}{1 - e^{i\varphi}}$$

$$= a\, e^{i(\omega t+\varphi)} \frac{e^{in\varphi/2}(e^{-in\varphi/2} - e^{in\varphi/2})}{e^{i\varphi/2}(e^{-i\varphi/2} - e^{i\varphi/2})}$$

$$= a\, e^{i(\omega t+(n+1)\varphi/2)} \frac{\sin n\,\varphi/2}{\sin \varphi/2}$$

Recovering the sine terms from the complex function, one gets the resultant motion

$$a \sin\left[\omega t + (n+1)\frac{\varphi}{2}\right] \frac{\sin n\,\varphi/2}{\sin \varphi/2} \tag{2.79}$$

The amplitude of motion is

$$R = a\frac{\sin n\varphi/2}{\sin \varphi/2}$$

When n is very large and φ is very small

$$(n-1)\frac{\varphi}{2} \approx \frac{n\varphi}{2}$$

and

$$\sin\frac{\varphi}{2} \rightarrow \frac{\varphi}{2}$$

and in the limit, one gets

$$R = a\frac{\sin n\varphi/2}{\sin \varphi/2}$$

$$= na \frac{\sin \frac{n\varphi}{2}}{n \frac{\varphi}{2}} \tag{2.80}$$

The intensity (which is proportional to the square of the amplitude) is proportional to n^2.

The resultant phase θ is given by

$$\tan \theta = \frac{\sum_{n=1}^{n} a \sin n\varphi}{\sum_{n=1}^{n} a \cos n\varphi} \tag{2.81}$$

It is shown below that

$$\sum_{n=1}^{n} a \sin n\varphi = a \sin (n+1) \frac{\varphi}{2} \frac{\sin \left(\frac{n\varphi}{2}\right)}{\sin \left(\frac{\varphi}{2}\right)}$$

Now

$$\sum_{n=1}^{n} a \sin n\varphi = a [\sin \varphi + \sin 2\varphi + \ldots + \sin n\varphi]$$

Multiplying each term within the bracket in turn by $2 \sin \left(\frac{\varphi}{2}\right)$ one gets

$$2 \sin \frac{\varphi}{2} \sin \varphi = \cos \frac{\varphi}{2} - \cos \frac{3\varphi}{2}$$

$$2 \sin \frac{\varphi}{2} \sin 2\varphi = \cos \frac{3\varphi}{2} - \cos \frac{5\varphi}{2}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$2 \sin \frac{\varphi}{2} \sin n\varphi = \cos (2n-1) \frac{\varphi}{2} - \cos (2n+1) \frac{\varphi}{2}$$

Adding the terms on both sides leads to the result

$$2 \sin \frac{\varphi}{2} \sum_{n=1}^{n} \sin n\varphi = \cos \frac{\varphi}{2} - \cos (2n+1) \frac{\varphi}{2}$$

$$= 2 \sin (n+1) \frac{\varphi}{2} \sin \frac{n\varphi}{2}$$

or

$$\sum_{n=1}^{n} \sin n\varphi = \sin (n+1) \frac{\varphi}{2} \frac{\sin \frac{n\varphi}{2}}{\sin \frac{\varphi}{2}}$$

Therefore

$$\sum_{n=1}^{n} a \sin n\varphi = a \sin (n+1) \frac{\varphi}{2} \frac{\sin \left(\frac{n\varphi}{2}\right)}{\sin \left(\frac{\varphi}{2}\right)}$$

In an identical manner it can be shown that

$$\sum_{n=1}^{n} a \cos n\varphi = a \cos (n+1)\frac{\varphi}{2} \frac{\sin\left(\frac{n\varphi}{2}\right)}{\sin\left(\frac{\varphi}{2}\right)}$$

The resultant phase thus becomes

$$\tan \theta = \tan (n+1)\frac{\varphi}{2} \tag{2.82}$$

The same result can be obtained vectorially by applying the graphical method. The lines separating the displacements to be combined become the sides of an incomplete regular polygon and the line closing the polygon is the resultant displacement R and having a phase θ, Fig. 2.17(a)

When the amplitude a and the successive phase difference φ is infinitely small, i.e., both na and $n\varphi$ are finite, then the vibration polygon coincides with its circumscribing circle. The resultant displacement R is represented by the chord OA of the circular arc OBA and the resultant phase by the $\angle AOD$, Fig. 2.17(b)

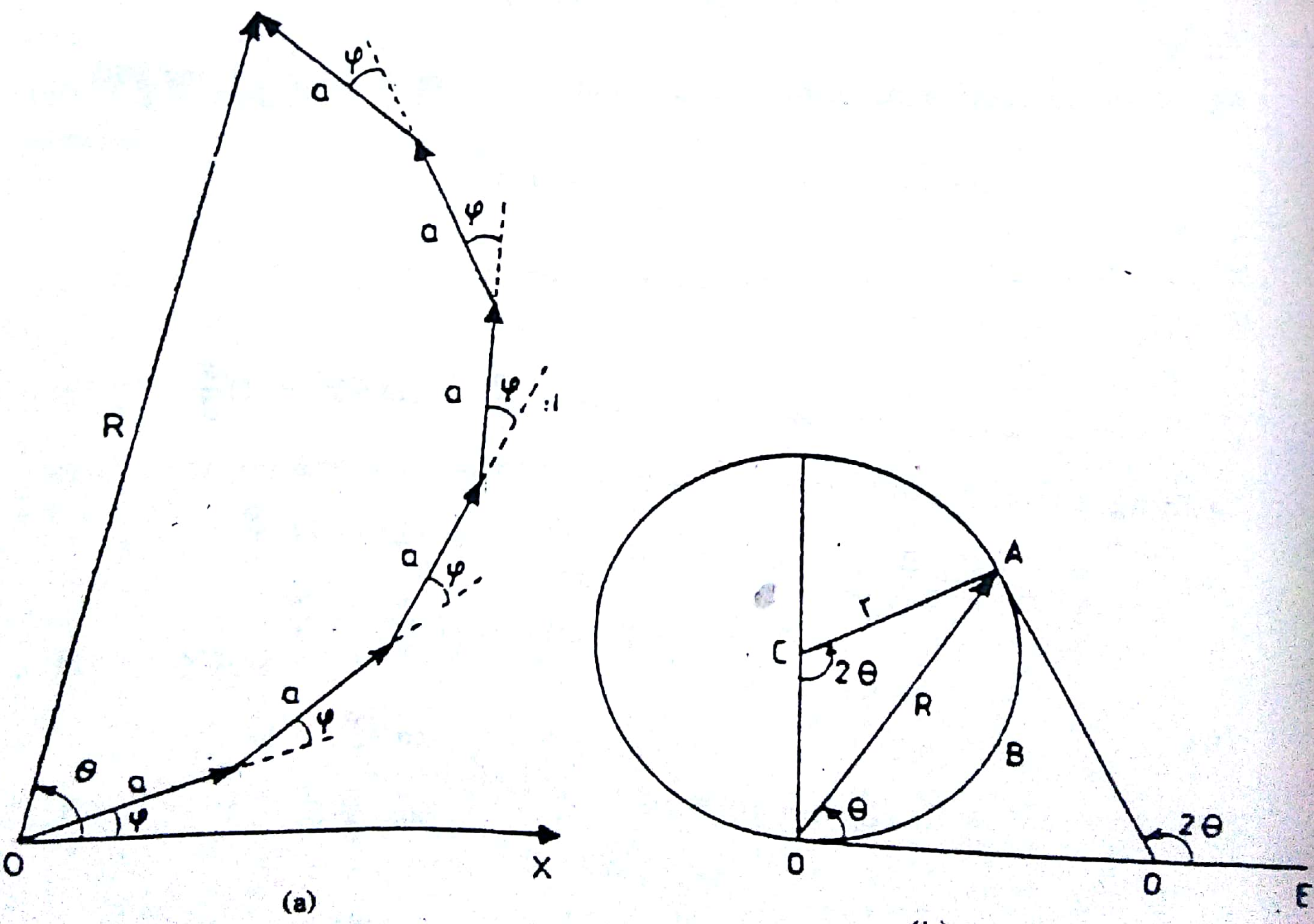


Fig. 2.17 (a) Vibration polygon of simple harmonic motions of equal amplitude a and successive phases φ. The resultant displacement is represented by R. (b) The resultant displacement of a large number of waves of equal amplitude and successive phase difference ϕ is represented by OA, the chord of the circular arc OBA.

Now geometrically from Fig. 2.12(b), we have

$$\angle ADE = np = 2\theta$$

and

$$\angle AOD = \theta$$

$$R = OA = 2r \sin\theta = 2r \sin\left(\frac{np}{2}\right)$$

where r is the radius of the circumscribing circle.

Also

$$na = \text{Arc } OBA = 2\theta r$$

or

$$\gamma = \frac{na}{2\theta},$$

therefore

$$R = \frac{2na}{2\theta} \sin\theta = na \frac{\sin\theta}{\theta} = na \frac{\sin\left(\frac{np}{2}\right)}{\left(\frac{np}{2}\right)} \tag{2.83}$$

This is the same result as obtained analytically.

2.5.2 Random Phase Difference

If the waves are randomly phased (the phase difference taking any value between 0 and 2π) and equal amplitude, the resultant may be found as follows. Calling the phases δ_1, δ_2 and amplitudes along the x and y-axes, the components of R

$$R_x = a\cos\delta_1 + a\cos\delta_2 + a\cos\delta_3 \ldots + a\cos\delta_n$$

$$= a \sum_{k=1}^{n} \cos\delta_k$$

and

$$R_y = a \sum_{k=1}^{n} \sin\delta_k$$

Thus

$$R_x^2 = a^2\left(\sum_{k=1}^{n} \cos\delta_k\right)^2$$

$$= a^2\left[\sum_{k=1}^{n} \cos^2\delta_k + \sum_{\substack{k=1\\k\neq l}} \cos\delta_k \sum_{l=1} \cos\delta_l\right]$$

Now

$$\sum_{k=1}^{n} \cos^2\delta_k = n\,\overline{\cos^2\delta}$$

and

$$\sum_{\substack{k=1\\k\neq l}}^{n} \cos\delta_k \sum_{l=1}^{n} \cos\delta_l = 0$$

Since $\cos\delta_k$ and $\cos\delta_l$ have random values between ± 1 and the sum of these product terms is zero. $\overline{\cos^2\delta}$ is the average value of $\cos^2\delta$ over the interval zero to 2π.

Therefore

$$\overline{\cos^2 \delta} = \frac{1}{2\pi}\int_0^{2\pi} \cos^2 \delta \, d\delta$$

$$= \frac{1}{2}$$

$$= \overline{\sin^2 \delta}$$

and the value of $\overline{R_x^2}$ becomes

$$\overline{R_x^2} = a^2 \sum_{k=1}^{n} \overline{\cos^2 \delta_k}$$

$$= na^2\, \overline{\cos^2 \delta} = \tfrac{1}{2}na^2$$

Furthermore $\qquad \overline{R_y^2} = \tfrac{1}{2}na^2 \qquad$ (2.84)

giving $\qquad R^2 = \overline{R_x^2} + \overline{R_y^2} = na^2$

or $\qquad R = \sqrt{n}\, a$

Thus the resultant intensity is proportional to n in this case.

The above discussion illustrates a very important case of a laser. The word *laser* is an acronym derived from light amplification from stimulated emission of radiation and laser is a coherent source of radiation, since the emitted radiations are in phase. If there are n excited atoms involved in a laser action, the resultant intensity is proportional to n^2. Incoherent sources have random phases. This is the case when an assembly of excited atoms lose their excitation energy through spontaneous emission, by emitting the radiation at random moments in time essentially independent, randomly phased wave packets of finite length. In this case the resultant intensity is proportional to n.

There is another feature of random processes which is important in transport phenomena such as viscosity, conductivity and diffusion which are characterised by the transport of momentum, energy and mass respectively. These phenomena are a result of random collision processes. If a represents the average mean free-path, then after n collisions, the distance travelled by the atom will be $\sqrt{n}\,a$. This obviously will be proportional to the square root of time elapsed, and not directly proportional to t. It may be remarked that $\sqrt{n}\,a$ is the statistical average of the distances travelled by the individual atoms.

2.6 ANHARMONIC OSCILLATIONS

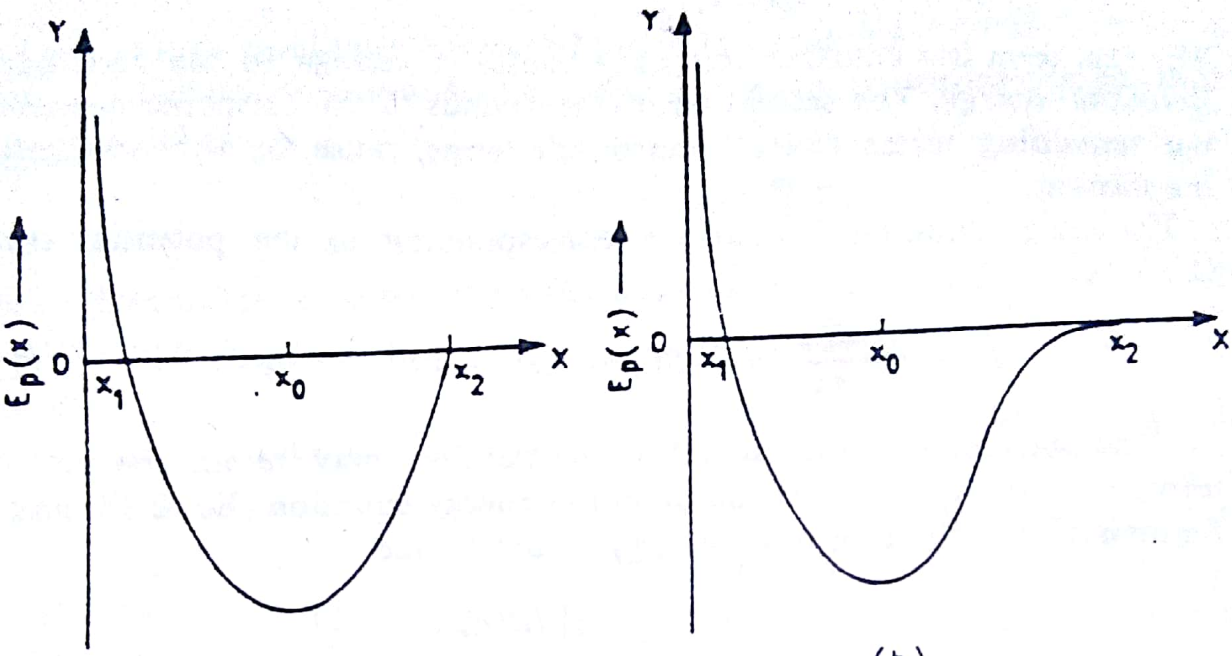
Simple harmonic motion is characterised by the force law according to which it is given by

$$F = -Sx$$

where S is the force constant, x is the displacement from the equilibrium position, which in this case is taken as origin. The potential energy, $E_p = \frac{1}{2}Sx^2$.

However, when the equilibrium position is taken at x_0 instead of the origin as in the adjoining Fig. 2.18, then the potential energy is

$$E_p = \tfrac{1}{2}S(x - x_0)^2$$



(b)

(a)

Fig. 2.18 (a) Harmonic oscillator with equilibrium position at x_0
(b) Anharmonic oscillator with equilibrium position at x_0

The plot of E_p as a function of x is shown in Fig. 2.18(a). It is a parabola and its points of intersection with the x-axis, x_1 and x_2 are symmetrically located wrt x_0, the position of equilibrium.

Now

$$\frac{dE_p}{dx} = S(x - x_0) \quad \text{and} \quad \frac{d^2E_p}{dx^2} = S$$

and the frequency of oscillation in the harmonic case is

$$\omega_0 = \sqrt{\frac{S}{m}} = \sqrt{\left(\frac{d^2E_p}{dx^2}\right)\Big/m} \tag{2.85}$$

Now consider the case where the potential energy is not a parabola but has a well-defined minimum at x_0. The extreme positions x_1 and x_2 between which the particle oscillates are not symmetrical wrt x_0, Fig. 2.18(b) and the motion is called anharmonic oscillatory motion. Let us see how to determine the frequency in this case. Expanding $E_p(x)$ around $x = x_0$ by the use of Taylor's theorem and noting that $\left(\frac{dE_p}{dx}\right)_{x=x_0} = 0$, we obtain

$$E_p(x) = E_p(x_0) + \tfrac{1}{2}\left(\frac{d^2E_p}{dx^2}\right)_{x=x_0}(x - x_0)^2$$

$$+ \frac{1}{6}\left(\frac{d^3E_p}{dx^3}\right)_{x=x_0}(x - x_0)^3 + \ldots$$

$$= E_p(x_0) + \tfrac{1}{2}S(x - x_0)^2 + \frac{1}{6}S'(x - x_0)^3 + \ldots \tag{2.86}$$

where $$S = \left(\frac{d^2E_p}{dx^2}\right)_{x=x_0}$$

and $$S' = \left(\frac{d^3E_p}{dx^3}\right)_{x=x_0}$$

The first term is a constant and only causes a change in the zero of the potential energy. The second term corresponds to the harmonic motion and the remaining terms called anharmonic terms, cause the anharmonicity of the motion.

The force acting on the particle corresponding to the potential energy (2.86) is

$$F = -\frac{dE_p}{dx} = -S(x - x_0) - \tfrac{1}{2}S'(x - x_0)^2 \tag{2.87}$$

If the amplitude of the oscillation is small, we may retain the first two terms in the expansion of the potential energy equation (Eq. 2.86) and the frequency of oscillation has the approximate value

$$\omega_0 = \sqrt{S/m} = \sqrt{\left[\left(\frac{d^2E_p}{dx^2}\right)_{x=x_0}\right]/m}$$

However, if the potential energy is large, the effect of anharmonic terms becomes perceptible and the above frequency ω_0 becomes in error from the actual frequency.

Ex. 2.8 The intermolecular potential between two gas atoms is approximated by the expression [J·Lenard-Jones potential]

$$E_p = E_{p,0}\left[2\left(\frac{r_0}{r}\right)^6 - \left(\frac{r_0}{r}\right)^{12}\right]$$

where $E_{p,0}$ and r_0 are positive constants and r is the distance of separation between the molecules. Calculate the frequency of oscillation corresponding to this potential.

Solution

The interatomic potential is

$$E_p = E_{p,0}\left[2\left(\frac{r_0}{r}\right)^6 - \left(\frac{r_0}{r}\right)^{12}\right]$$

where r_0 is the equilibrium separation, since $F = -\frac{dE_p}{dr} = 0$ at $r = r_0$.

Thus

$$\frac{d^2E_p}{dr^2} = -E_{p,0}\left[84\frac{r_0^6}{r^8} - 156\frac{r_0^{12}}{r^{14}}\right]$$

Putting $r = r_0$

$$\frac{d^2E_p}{dr^2} = \frac{72}{r_0^2}E_{p,0}$$

Thus the approximate frequency of oscillation is

$$\omega = \sqrt{\frac{72 E_{p,0}}{m r_0^2}}$$

Ex. 2.9 Large amplitude motion of a simple pendulum.

Show that the time period of a simple pendulum performing large amplitude motion is

$$T = T_0 \left(1 + \frac{1}{4} \sin^2 \frac{\theta_0}{2}\right)$$

where T_0 is the time period when the amplitude is small and θ_0 is the maximum angular displacement.

Solution

The equation of motion of a simple pendulum in terms of its angular displacement is

$$\frac{d^2\theta}{dt^2} + \omega_0^2 \sin\theta = 0$$

where $$\omega_0 = \sqrt{\frac{g}{l}}$$

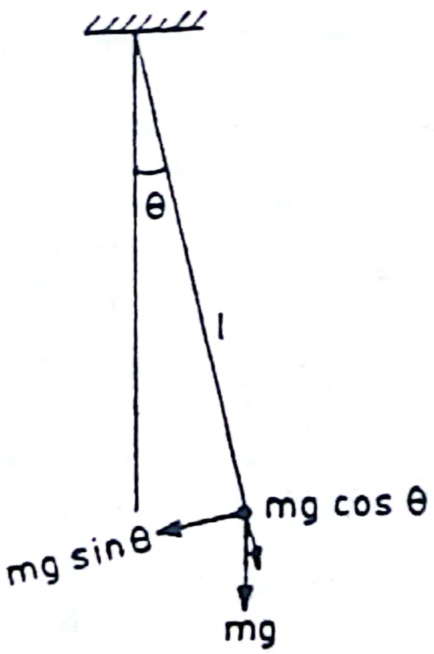


Fig. 2.19 A simple pendulum

Multiplying this equation by $2(d\theta/dt)$ and integrating with respect to t, one gets

$$\left(\frac{d\theta}{dt}\right)^2 = 2\omega_0^2 \cos\theta + C$$

where C is the constant of integration.

Now the angular velocity $d\theta/dt = 0$ when $\theta = \theta_0$, the maximum angular displacement. Thus

$$C = -2\omega_0^2 \cos\theta_0 \qquad \text{(i)}$$

so that

$$\frac{d\theta}{dt} = \omega_0[2(\cos\theta - \cos\theta_0)]^{1/2} \tag{ii}$$

Integrating it wrt t

$$\omega_0 t = \int \frac{d\theta}{[2(\cos\theta - \cos\theta_0)]^{1/2}}$$

Now $\theta = \theta_0$ when $t = 0$, and if T is the new period of oscillation, then at $\theta = \theta_0$, $t = T/4$.

Thus

$$\omega_0 \frac{T}{4} = \int_0^{\theta} \frac{d\theta}{2\left(\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right)^{1/2}} \tag{iii}$$

Expressing θ as a function of θ_0 by assuming

$$\sin\frac{\theta}{2} = \sin\left(\frac{\theta_0}{2}\right)\sin\varphi$$

where $-1 < \sin\varphi < 1$

Thus $$\tfrac{1}{2}\cos\left(\frac{\theta}{2}\right)\delta\theta = \sin\left(\frac{\theta_0}{2}\right)\cos\varphi\,\delta\varphi$$

and rewriting (iii) by putting $T_0 = 2\pi/\omega_0$, one gets

$$\frac{\pi T}{2T_0} = \int_0^{\pi/2} \frac{d\varphi}{\left[1 - \sin^2\frac{\theta_0}{2}\sin^2\varphi\right]^{1/2}}$$

$$= \int_0^{\pi/2}\left[1 - \sin^2\frac{\theta_0}{2}\sin^2\varphi\right]^{-1/2} d\varphi$$

$$= \int_0^{\pi/2}\left[1 + \tfrac{1}{2}\sin^2\frac{\theta_0}{2}\sin^2\varphi + \frac{3}{8}\sin^4\frac{\theta_0}{2}\sin^4\varphi + \ldots\right] d\varphi$$

$$= \frac{\pi}{2}\left(1 + \frac{1}{4}\sin^2\frac{\theta_0}{2} + \frac{9}{64}\sin^4\frac{\theta_0}{2}\right)$$

Hence $$T = T_0\left(1 + \frac{1}{4}\sin^2\frac{\theta_0}{2} + \frac{9}{64}\sin^4\frac{\theta_0}{2}\right)$$

or $$T = T_0\left(1 + \frac{1}{4}\sin^2\frac{\theta_0}{2}\right) \tag{iv}$$

when $\theta_0 = 60°$, we get

$$T = T_0\left(1 + \frac{1}{4}\left(\frac{1}{4}\right)\right)$$

$$= T_0\left(\frac{17}{16}\right)$$

The percentage change in the time period

$$\frac{T - T_0}{T_0} \times 100 = \frac{100}{16} = 6.25\%.$$

PROBLEMS

2.1 How long after the start of the motion, a harmonically oscillating point will be out of the equilibrium position by half the amplitude? The oscillation period is 24 s and the initial phase is zero. [2s]

2.2 The equation of a point is given as

$$x = 2 \sin \left(\frac{\pi}{2} t + \frac{\pi}{4}\right)$$

Find

(a) the period of oscillations
(b) the maximum velocity of the point
(c) its maximum acceleration [4 s; 3.14×10^{-2} m/s; 4.93×10^{-2} m/s]

2.3 A 100 g mass, attached to a spring, is set into oscillation from its equilibrium position with initial velocity of 5 cm/s. The time period of the oscillation is measured to be 2 s. Find the maximum displacement and the spring constant. [1.5 cm; 0.99 N/m]

2.4 Show that vertical vibrations of a mass m suspended on a spring of stiffness S, whose other end is tied to a rigid support, have angular frequency $(S/m)^{1/2}$.

2.5 A mass moves under a potential $V(x) = V_0 \cosh (x/x_0)$ where V_0 and x_0 are constants. (a) Find the position of stable equilibrium. (b) Show that the frequency of small vibrations about the equilibrium position is the same as it would be if the same mass was vibrating on a spring of stiffness V_0/x_0^2. [$x = 0$]

2.6 A spherical vessel of volume 1 litre has an attached tube of 10 cm length and 1 cm radius. Calculate the wavelength of sound for which this vessel could serve as a resonator. [3.54 m]

2.7 Calculate the frequency of oscillation of an L-C circuit in which the inductance and capacitance have the respective values 10 millihenry and 1 $\mu\mu$F. Calculate the energy of the system if the maximum voltage across the capacitor is 1 V.

$$\left[\frac{10^7}{2\pi}/\text{s};\ 0.5 \times 10^{-12}\ \text{J}\right]$$

2.8 A particle oscillates in a potential field

$$U = \frac{1}{2} kx - \frac{1}{2} k_1 x^3$$

where $k_1 \ll k$. Will the displacement be symmetrical around the equilibrium position? Calculate the mean position of the anharmonic oscillator and the potential energy at the mean position. $\left[\text{unsymmetrical; } x = \frac{k}{2k_1}, \frac{k^2}{8k_1}\right]$

Hint: At the equilibrium position $-\frac{\partial U}{\partial x} = 0$.

2.9 The amplitude of SHM is 2.5 cm and frequency 60 Hz. If the mass of the body executing the motion is 2 g, find the kinetic energy at the middle of the oscillation. [8.88×10^{-2} J]

2.10 A mass of 500 g is suspended from a fixed point by a light spiral spring and the spring is extended by 10 cm in the position of equilibrium. Further the mass is released after pulling it down by additional 5 cm. Find the time period, amplitude and energy of motion of the suspended mass.

$$\left[T = \frac{\sqrt{2}\pi}{7} \text{ s; } C = 5 \text{ cm; } E = 6.125 \times 10^{-2} \text{ J}\right]$$

2.11 The force of interaction between two atoms in a certain diatomic molecule is given by $F = -(a/r^2) + (b/r^3)$ where r is the distance between atoms, a and b are positive constants. Find

(a) the equilibrium separation,
(b) the effective spring constant for small displacements from equilibrium,
(c) the period of small oscillations about the equilibrium position. μ is the effective mass of the molecule. $\left[\frac{b}{a}; \frac{a^4}{b^3}; 2\pi\left(\frac{a^4}{b^3\mu}\right)^{\frac{1}{2}}\right]$

2.12 A particle is performing SHM along x-axis with amplitude a. Show that the probability of a particle to be between x and $x + dx$ is

$$\frac{dx}{\pi\sqrt{(a^2 - x^2)}}$$

Hint: If dt is the time for the particle to go from x to $x + dx$, then the probability of finding the particle between x and $x + dx = 2dt/T$ where T is the time period of motion. Assume the displacement, $x = a \cos(\omega t + \phi)$ and deduce the result.

2.13 At one time the fundamental unit of time (second) was defined as the time required for a simple pendulum with a length of exactly 1 m to swing from one side to the other. If $g = 9.8$ m/s^2 find the error in this definition. Further find the value of g for which it is exactly correct. [0.37% too large; π^2 m/s^2]

3

Damped Simple Harmonic Vibrations

3.1 THE DECAY OF FREE VIBRATIONS DUE TO DAMPING (NATURE OF DAMPING FORCE)

The free vibrations of any real physical system always die away with time, since some energy is inevitably lost by a resistive or a viscous element. In actual practice an oscillating simple pendulum, a vibrating string or sounding tuning fork, have to inevitably experience resistance offered both at the supports and by the surrounding medium like air. As a consequence there is a gradual fall in the amplitude of the vibrating body. The resistance offered by the dissipative forces is called damping. Under the circumstances when the damping force is small so as not to cause any significant modification of the undamped motion of the body, it is easy to prove that the damping force is proportional to the velocity of the vibrating body.

Let a body executing simple harmonic motion

$$x = a \sin \omega t \tag{3.1}$$

be subjected to a small damping force f, which in its most general form may be assumed to be a function of displacement, velocity and acceleration. Thus

$$\mathbf{f} = A + Bx + C\frac{dx}{dt} + D\frac{d^2x}{dt^2} \tag{3.2}$$

where A, B, C and D are constants. The work done by the force $\mathbf{f}$, in displacing the vibrating body through a displacement dx,

$$\begin{aligned} dw &= \mathbf{f}\cdot dx \\ &= \mathbf{f}\cdot \mathbf{v}\, dt \end{aligned} \tag{3.3}$$

Therefore the work done per cycle is

$$= \int_0^{2\pi/\omega} dw$$

$$= \int_0^{2\pi/\omega} \left(A + Bx + C\frac{dx}{dt} + D\frac{d^2x}{dt^2}\right)\cdot\left(\frac{dx}{dt}\right) dt$$

$$= \int_0^{2\pi/\omega} (A + Ba \sin \omega t + Ca\omega \cos \omega t - Da\omega^2 \sin \omega t)\,(a\omega \cos \omega t)\, dt$$

$$= \int_0^{2\pi/\omega} Aa\omega \cos \omega t\, dt + \int_0^{2\pi/\omega} (B - D\omega^2)a^2\omega \sin \omega t \cos \omega t\, dt$$

$$+ \int_0^{2\pi/\omega} Ca^2\omega^2 \cos^2 \omega t\, dt$$

$$= \int_0^{2\pi/\omega} Aa\omega \cos \omega t\, dt + \int_0^{2\pi/\omega} (B - D\omega^2)\tfrac{1}{2}a^2\omega \sin 2\omega t\, dt$$

$$+ \int_0^{2\pi/\omega} Ca^2\omega^2 \frac{(1 + \cos 2\omega t)}{2}\, dt$$

$$= \tfrac{1}{2}\int_0^{2\pi/\omega} Ca^2\omega^2\, dt = Ca^2\,\omega\pi \tag{3.4}$$

Thus, the work done by the damping force is non-zero only for that term from Eq. (3.2) which is proportional to the velocity of the vibrating body. The presence of this damping term always implies the dissipation of energy.

The equation of motion for a mass m, executing simple harmonic oscillations then may be written as

$$m\frac{d^2x}{dt^2} = -Sx - C\frac{dx}{dt}$$

or

$$\frac{d^2x}{dt^2} = -\frac{Sx}{m} - \frac{C}{m}\frac{dx}{dt}$$

$$= -\omega_0^2 x - 2r\frac{dx}{dt} \tag{3.5}$$

where $\omega_0 = \sqrt{\dfrac{S}{m}}$ is the angular frequency of the undamped SHM oscillations. $\dfrac{C}{m}$ has been put equal to $2r$ for the sake of mathematical convenience. $2r$ is the damping force per unit mass at an instant when the vibrating body is moving with unit velocity. *r is called the damping factor.*

The differential Eq. (3.5) can be solved easily through the use of the differential operator, $D \equiv \dfrac{d}{dt}$. As a consequence, Eq. (3.5) is reduced to the following algebraic equation

$$D^2x = -\omega_0^2 x - 2r\,Dx \tag{3.6}$$

or

$$(D^2 + 2rD + \omega_0^2)x = 0$$

$$D^2 + 2rD + \omega_0^2 = 0$$

Hence

$$D = \frac{-2r \pm \sqrt{4r^2 - 4\omega_0^2}}{2}$$

$$= -r \pm \sqrt{r^2 - \omega_0^2} \tag{3.7}$$

Denoting the roots by α and β as

$$\alpha = -r + \sqrt{r^2 - \omega_0^2}$$

$$\beta = -r - \sqrt{r^2 - \omega_0^2} \tag{3.8}$$

Working with one root at a time one gets $dx/dt = \alpha x$ which on integrating gives $x = C_1 e^{\alpha t}$. Analogously the other root β will lead to the solution $x = C_2 e^{\beta t}$. The general expression for displacement x is given by the linear combination of both the terms.

$$x = C_1 e^{\alpha t} + C_2 e^{\beta t}$$

$$= C_1 \exp[(-r + \sqrt{r^2 - \omega_0^2})t] + C_2 \exp[(-r - \sqrt{r^2 - \omega_0^2})t]$$

$$= e^{-rt}[C_1 \exp[\sqrt{r^2 - \omega_0^2}\,t + C_2 \exp[-\sqrt{r^2 - \omega_0^2}\,t]] \tag{3.9}$$

The nature of the motion represented by Eq. (3.9) will depend upon the relative magnitudes of r and ω_0. The following three cases arise

(i) When $r > \omega_0$, aperiodic damping occurs. This case results when the damping resistance term r is greater than the stiffness term. Then the displacement is given by

$$x = C_1 \exp[(-r + \sqrt{r^2 - \omega_0^2})t] + C_2 \exp[-(r - \sqrt{r^2 - \omega_0^2})t]$$

Obviously it decreases monotonically with the increase in t. A system disturbed from the equilibrium will return to the state asymptotically, i.e., as $t \to \infty$. Thus heavy damping causes the motion to be non-oscillatory.

(ii) When $r = \omega_0$, critical damping occurs. The amplitude at any instant becomes

$$x = (C_1 + C_2)e^{-rt}$$

Depending on the value of r, the damping factor, the system comes to rest in a very short time. The characteristic of critical damping is employed in the construction of many pointer type instruments where the pointer moves and comes to stationary position in a very short time.

(iii) When $r < \omega_0$, the system makes damped vibrations.

In this case both the roots α and β (Eq. 3.8) become imaginary and Eq. (3.9) becomes

$$x = e^{-rt}[C_1 \exp(i\sqrt{\omega_0^2 - r^2}\,t) + C_2 \exp(-i\sqrt{\omega_0^2 - r^2}\,t)] \tag{3.10}$$

Putting $\sqrt{\omega_0^2 - r^2} = q$, it becomes

$$x = e^{-rt}[C_1 e^{iqt} + C_2 e^{-iqt}] \tag{3.11}$$

Strictly speaking the damped vibrations are not periodic since the amplitude decays with time according to the factor e^{-rt}, each successive swing having a lesser amplitude than its predecessor. However, if r is very small as compared with ω_0, the motion can be classified as nearly periodic.

The common concepts of period and frequency do not have the well defined meanings as in the case of SH oscillators, since in the case of non-periodic motion, there is no cycle, no repeat. However, when damping is small, i.e., $r \ll \omega_0$, the motion is almost periodic and the word frequency will have the conditioned meaning. Thus the frequency of an almost periodic motion is given by

$$q = \sqrt{\omega_0^2 - r^2}$$

Obviously its value is smaller than the natural frequency, however when $r \ll \omega_0$, then $q \approx \omega_0$.

The time period of damped vibrations is defined as the interval of time between two successive states of the system during which the displacement x passes through its equilibrium value, varying in a single direction, say increasing. Thus

$$T = \frac{2\pi}{q} = \frac{2\pi}{(\omega_0^2 - r^2)^{1/2}}$$

The above Eq. (3.11) can be put into the following equivalent trigonometric forms:

$$x = e^{-rt}[(C_1 + C_2)\cos qt + i(C_1 - C_2)\sin qt] \tag{3.12}$$

or alternately

$$x = C_0 e^{-rt}\sin(qt + \varphi_0) \tag{3.13}$$

where C_0 and φ_0 are constants determined from the initial conditions; q is the natural cyclic frequency of vibrations of a dissipative system.

The value $C(t) = C_0 e^{-rt}$ is called the amplitude of the damped vibrations. The values of the amplitude for the instants of time t, $t + \Delta t$, $t + 2\Delta t$, etc. constitute a geometric progression with a common ratio $e^{-r\Delta t}$. The dependence of the displacement x on t is shown in Fig. 3.1.

Ex. 3.1 Determine the nature of the constants C_1 and C_2 in the equation

$$x = e^{-rt}[C_1 e^{iqt} + C_2 e^{-iqt}]$$

and show explicitly that it may be put into the equivalent trigonometric forms given by Eqs. (3.12) and (3.13).

Solution

The equation (3.11) expressed in terms of trigonometrical functions is

$$\begin{aligned} x &= e^{-rt}[C_1(\cos qt + i\sin qt) + C_2(\cos qt - i\sin qt)] \\ &= e^{-rt}[(C_1 + C_2)\cos qt + i(C_1 - C_2)\sin qt] \end{aligned}$$

From physical considerations, it is clear that the displacement x must be a real quantity. This is possible if $C_1 - C_2 = 0$ or $C_1 = C_2$. Then the equation becomes

$$x = e^{-rt}\, 2C_1 \cos qt \tag{3.14}$$

However, this is not a general solution since the solution of a second order differential equation must have two constants instead of one, as is the case. However, if C_1 and C_2 are imagined to be imaginary, again the solution will not be general. Thus C_1 and C_2 cannot be imaginary as well.

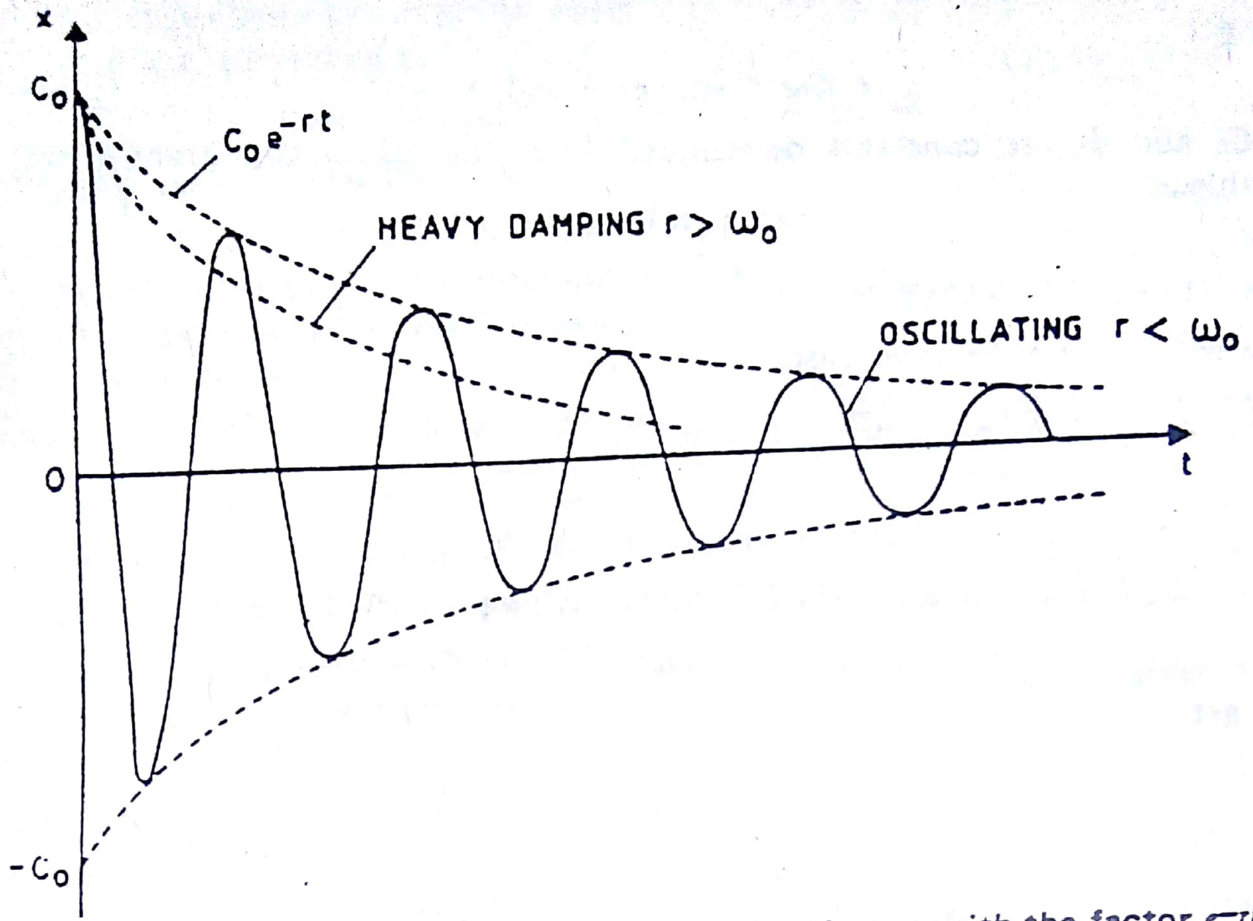


Fig. 3.1 The amplitude of the damped motion decays with the factor e^{-rt}

Assuming that C_1 and C_2 are complex numbers, we get

$$C_1 = c_1 + id_1$$
$$C_2 = c_2 + id_2 \tag{3.15}$$

where c_1 and c_2, d_1 and d_2 are real numbers.

Equation (3.11) in view of Eq. (3.15), becomes

$$x = e^{-rt}[(c_1 + c_2)\cos qt - (d_1 - d_2)\sin qt$$
$$+ i(d_1 + d_2)\cos qt + i(c_1 - c_2)\sin qt] \tag{3.16}$$

Again making use of the physical requirement that the displacement has to be real, we get

$$d_1 + d_2 = 0 \quad \text{or} \quad d_2 = -d_1$$
$$c_1 - c_2 = 0 \quad \text{or} \quad c_1 = c_2$$

Therefore $\quad C_1 = c_1 + id_1$

and $\quad C_2 = c_2 - id_1 \qquad (3.16a)$

Obviously C_1 and C_2 are complex conjugate to each other. Hence Eq. (3.16) becomes

$$x = e^{-rt}[(c_1 + c_2)\cos qt - (d_1 - d_2)\sin qt] \tag{3.17}$$

Putting

$$c_1 + c_2 = C_0 \sin\phi_0$$
$$-(d_1 - d_2) = C_0 \cos\phi_0$$

we get

$$x = C_0 e^{-rt} \sin (qt + \phi_0) \tag{3.13}$$

C_0 and ϕ_0 are constants determined from the initial conditions of the problem.

3.2 TYPES OF DAMPING

We will take up the three cases one by one

(i) *Over damping* When $r > \omega_0$, we can write Eq. (3.9) by putting $\sqrt{r^2 - \omega_0^2} = p$, as

$$x = e^{-rt}[C_1 e^{pt} + C_2 e^{-pt}]$$

where C_1 and C_2 are arbitrary constants. Calling

$$F = C_1 + C_2 \quad \text{and} \quad G = C_1 - C_2$$

we get

$$C_1 = \frac{F + G}{2}; \quad C_2 = \frac{F - G}{2}$$

Therefore

$$x = e^{-rt}\left[\frac{F}{2}(e^{pt} + e^{-pt}) + \frac{G}{2}(e^{pt} - e^{-pt})\right]$$

$$= e^{-rt}[F \cosh pt + G \sinh pt]$$

where F and G are arbitrary constants whose values will depend on the initial conditions. Assuming that $x = 0$ at $t = 0$, then $F = 0$. Hence

$$x = Ge^{-rt} \sinh \sqrt{r^2 - \omega_0^2}\,t \tag{3.18}$$

Thus, under heavy damping, the system does not perform oscillatory behaviour (because the displacement x does not depend on any periodic function of time like $\cos \omega t$ or $\sin \omega t$ but when once disturbed will return to equilibrium very slowly (Fig. 3.2).

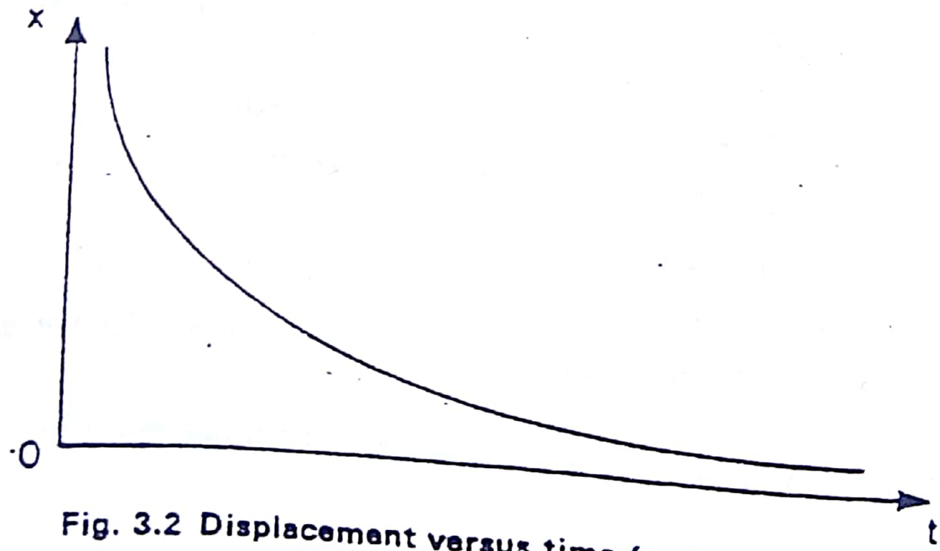


Fig. 3.2 Displacement versus time for heavy damping

The motion is aperiodic, neither oscillatory nor dead-beat, because it takes a long time to return to equilibrium.

(ii) *Light damping* $r < \omega_0$. Then $\sqrt{r^2 - \omega_0^2}$ is an imaginary quantity which may be written as

$$\sqrt{r^2 - \omega_0^2} = i\sqrt{\omega_0^2 - r^2} \equiv iq, \text{ say}$$

so that the displacement x, Eq. (3.13) is

$$x = C_0 e^{-rt} \sin (qt + \phi_0) \tag{3.13}$$

The displacement varies sinusoidally with time with a new reduced angular frequency, $q = (\omega_0^2 - r^2)^{1/2}$, which is less than ω_0. Moreover, its amplitude decays with time.

The variation of the amplitude of the damped oscillatory motion is depicted in Fig. 3.1. The constant C_0 obviously represents the maximum amplitude if no damping were present.

The decrease of amplitude therefore is caused by the presence of the damping term: $2r(dx/dt)$ in the equation of motion of the SHM; Eq. (3.5).

(iii) *Critical damping* $r = \omega_0$

Under this condition $\alpha = \beta = -r$ and the displacement x is given by

$$\begin{aligned} x &= (C_1 + C_2)e^{-rt} \\ &= Ce^{-rt} \end{aligned} \tag{3.19}$$

This is not a general solution it has only one arbitrary constant. For getting a general solution, assume that

$$x = Ze^{-rt} \tag{3.20}$$

where the expression for Z is yet to be determined.

Now $\qquad Dx = -rZe^{-rt} + e^{-rt}DZ$

and $\qquad D^2x = r^2Ze^{-rt} - re^{-rt}DZ - re^{-rt}DZ + e^{-rt}D^2Z$

$$= r^2e^{-rt}Z - 2re^{-rt}DZ + e^{-rt}D^2Z \tag{3.21}$$

Substituting Eq. (3.21) into Eq. (3.6) and dropping the factor e^{-rt} one obtains

$$(\omega_0^2 - r^2)Z + D^2Z = 0 \tag{3.22}$$

Since $\omega_0 = r$, it reduces to

$$D^2Z = 0 \tag{3.23}$$

Integrating it twice wrt time, one obtains

$$Z = C_3t + C_4 \tag{3.24}$$

Hence from Eq. (3.20), one gets

$$x = (C_3t + C_4)e^{-rt}$$

Thus the motion is damped simple harmonic motion for the case of critical damping.

The condition $r = \omega_0$ is called the condition for critical damping.

The vibrating system under this condition comes to rest comparatively quickly than when either $r > \omega_0$ or $r < \omega_0$. This characteristic is exploited in the construction of many pointer-type instruments. The pointer moves and comes to nearly rest position in a very short time. Such instruments are called dead-beat.

Such behaviour is desirable in the moving parts of electrical meters or the like, where one may like to have a steady reading as soon as the meter is connected in the circuit.

Ex. 3.2 Show that the energy of damped vibration decreases exponentially with time and rate of loss of its energy is the rate of doing work against the frictional force.

Solution

In the case of a simple harmonic motion, the energy is partly kinetic and partly potential. The instantaneous total energy from Eq. (2.20) is

$$= \tfrac{1}{2}SC^2$$

However, in damped motion,

$$C = C_0 e^{-rt}$$

Thus the total energy for damped motion

$$E = \tfrac{1}{2}SC_0^2 e^{-2rt}$$

$$= \tfrac{1}{2}m\omega^2 C_0^2 e^{-2rt} \qquad (\because\ S = m\omega^2)$$

$$= E_0 e^{-2rt}$$

where

$$E_0 = \tfrac{1}{2}m\omega^2 C_0^2$$

Now

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}Sx^2\right)$$

$$= \dot{x}(m\ddot{x} + S\dot{x}) \qquad (\because\ m\ddot{x} + 2r\dot{x} + Sx = 0$$

$$= \dot{x}(-2r\dot{x}) \qquad \therefore\ m\ddot{x} + Sx = -2r\dot{x})$$

$$= -2r\dot{x}^2$$

which gives the rate of doing work against the frictional force, since

$$2r\dot{x}^2 = (2r\dot{x})(\dot{x}) = \text{force} \times \text{velocity}$$

3.3 METHODS OF FINDING THE DAMPING COEFFICIENTS OF A DAMPED VIBRATING SYSTEM

The energy of an oscillator is proportional to the square of its amplitude. In the case of a damped oscillator, the amplitude decays exponentially with time as e^{-rt}, Eq. (3.13), implying thereby that the energy will decay as $(e^{-rt})^2$ or e^{-2rt}. Thus the decay rate of the energy depends upon $2r$ which is the damping force per unit mass at an instant when the vibrating body is moving with a unit velocity. There are three alternate methods of characterising a damped vibration.

3.3.1 Logarithmic Decrement

This is a measure of the rate at which the amplitude of vibration decays. Refer to the adjoining Fig. 3.3 and let P_1 and P_2 be the successive maxima (corresponding to displacements x_n and x_{n+1} and separated by a time period $(= 2\pi/q)$. Thus if the maxima at P_1 occurs at time t_1, the one at P_2 will

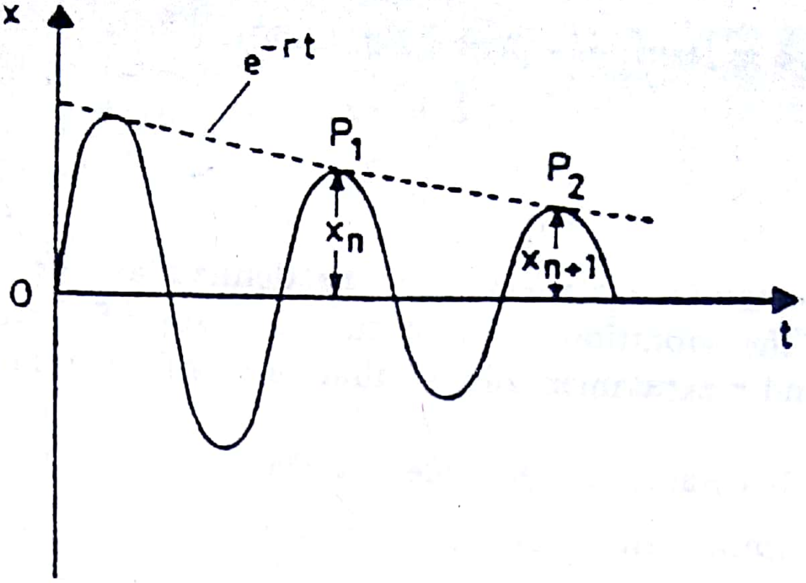


Fig. 3.3 Graphical representation of oscillatory motion. The amplitude decays with e^{-rt}

occur at $t = t_1 + 2\pi/q$. Hence

$$x_n = a_0 e^{-rt_1} \tag{3.26}$$

$$x_{n+1} = a_0 e^{-r\left(t_1 + \frac{2\pi}{q}\right)} \tag{3.27}$$

From the above equations, we get

$$\frac{x_{n+1}}{x_n} = \exp\left(-r\frac{2\pi}{q}\right) = e^{-rT} \tag{3.28}$$

Taking log on both sides, one gets

$$-\log \frac{x_{n+1}}{x_n} = rT \tag{3.29}$$

Defining the logarithmic decrement (δ) of the damped motion as the natural logarithm of the ratio of the amplitudes of vibration at the instants of time t and $t + T$, we get

$$\delta = rT = \frac{2\pi r}{q} = \frac{\pi C}{mq} \qquad \left(\because\ 2r = \frac{C}{m}\right) \tag{3.30}$$

It is clear that δ gives a method of evaluating C, the damping coefficient, since all other quantities can be determined either experimentally or by a displacement curve of the damped vibrations.

3.3.2 Relaxation Time or Modulus of Decay

An alternative method of describing the decay of amplitude of the damped vibration is by means of the time required to elapse for the amplitude to decay to $1/e$ of its original value, a_0.

The amplitude at a time t is

$$a_t = a_0 e^{-rt} \tag{3.31}$$

$$= a_0 e^{-1} \qquad \text{for} \qquad t = \frac{1}{r} = \tau$$

$\tau (= 1/r)$ is called the relaxation time or modulus of decay.

Let us express δ in terms of τ. From Eq. (3.30)

$$\delta = rT$$

$$= \frac{T}{\tau} \tag{3.32}$$

Thus the logarithmic decrement may be defined as the ratio of the time period of the vibration (T) and the modulus of decay (τ). One can determine T and τ experimentally, so that value of δ can be calculated.

3.3.3 The Quality Factor or Q-Value of a Damped Oscillator

Q-value of a damped oscillator is a factor which defines the quality of the oscillator so far as damping is concerned. The less the damping, the higher the quality factor Q. It is also called the figure of merit of a harmonic oscillator and is defined as 2π times the ratio of the energy stored in the system to the energy lost per cycle. Thus

$$Q = 2\pi \frac{\text{Energy stored}}{\text{Energy lost per cycle}} = \frac{2\pi E}{-dE} \tag{3.33}$$

where $-dE$ is the energy lost over a period.

Since $\frac{2\pi}{T} = 2\pi\nu = \omega$, we get

$$Q = \frac{E\omega}{-\frac{dE}{T}} = \frac{E\omega}{-\frac{dE}{dt}} = \frac{E\omega}{-\frac{d}{dt}(E_0 e^{-2rt})} = \frac{\omega\tau}{2} \tag{3.34}$$

where τ is the relaxation time of the oscillator.

When the damping is low, $\omega = \omega_0$ and therefore $Q = \frac{\omega_0 \tau}{2}$, which is a constant of the damped motion.

Since $$\omega_0 = \sqrt{\frac{S}{m}}$$

and $$\tau = \frac{1}{r},$$

we get $$Q = \frac{1}{2r}\sqrt{\frac{S}{m}}$$

showing thereby that the lower the damping, the higher the quality factor.

There is an alternative way of defining the quality factor. The time for the energy E to decay to $E_0 e^{-1}$ is given by $\tau = 1/2r$.

In this interval of time the oscillator executes $\omega_0/2\pi$ or $Q/2\pi$ oscillations, so that the phase of the oscillator shifts by Q.

Thus the quality factor Q of a damped oscillator is the phase change brought about in the oscillator during the time interval when the energy of the oscillator decays to $1/e$ of its initial value. Therefore, the quality factor can serve to form an idea of the speed with which the vibrations reduce in amplitude.

Ex. 3.3 A simple pendulum has a period of 1 second and an amplitude of 10°. After 10 complete oscillations, its amplitude has been reduced to 5°. What is the relaxation time of the pendulum. Calculate the quality factor.

Solution

The amplitude at a time t is

$$a_t = a_0 e^{-rt}$$
$$= a_0 e^{-t/\tau}$$

where a_0 is the initial amplitude

r, the damping coefficient, and

$\tau = 1/r$, the relaxation time

Thus $$5° = 10° e^{-10/\tau}$$

$$\frac{10}{\tau} = \log_e 2$$

or $$\tau = \frac{10}{\log_e 2} = \frac{10}{\log_{10} 2 \times \log_e 10}$$

$$= \frac{10}{0.3 \times 2.3} = 14.495$$

Thus $$Q = \frac{\omega_0 \tau}{2} = \frac{2\pi t}{T\,2} = \frac{2\pi}{1} \times 7.25 = 45.55$$

Ex. 3.4 If the quality factor of an undamped tuning fork of frequency 256 is 10^3, calculate the time in which its energy is reduced to $1/e$th of its energy in the absence of damping. How many oscillations the tuning fork will make in this time.

Solution

The energy of a damped harmonic oscillator at any time t is given by

$$E = E_0 e^{-2rt} = E_0 e^{-2t}$$

where E_0 is the initial energy in the absence of damping and τ is the relaxation time

When $$t = \frac{\tau}{2}, E = \frac{E_0}{e}$$

Also $$Q = \omega_0 \tau / 2$$

Hence $$\tau = \frac{2Q}{\omega_0} = \frac{2 \times 10^3}{2\pi \times 256} = \frac{2 \times 10^3}{512\pi} = 0.2.45$$

Thus the energy will reduce to the $1/e$th value in 0.62 s. The number of oscillations made by the tuning fork in time t

$$= \left(\frac{\omega_0}{2\pi}\right) \tau / 2$$

$$= \frac{Q}{2\pi} = \frac{10^3}{2\pi} = 159$$

Ex. 3.5 Show that the fractional change in the resonant frequency ω_0 of a damped SH mechanical oscillator is $(8Q^2)^{-1}$ where Q is quality factor.

Solution

Let ω_0 be the frequency of the undamped oscillator.

The only case when a damped oscillator will oscillate is when it is underdamped. Its frequency ω is given by

$$\omega = \sqrt{\omega_0^2 - r^2}$$

or
$$\omega^2 = \omega_0^2 - \frac{1}{\tau^2}$$

$$\frac{\omega^2}{\omega_0^2} = 1 - \frac{1}{\omega_0^2\tau^2}$$

$$= 1 - \frac{1}{4Q^2}$$

Thus
$$\frac{\omega}{\omega_0} = \left(1 - \frac{1}{4Q^2}\right)^{1/2} = 1 - \frac{1}{8Q^2}$$

or
$$\frac{2\pi\nu}{2\pi\nu_0} = 1 - \frac{1}{8Q^2}$$

The fractional change in the resonant frequency $= 1/8Q^2$

Ex. 3.6 A capacitance C with a charge q_0 at $t = 0$ discharges through a resistance R. Show that the relaxation time of this process is RC seconds, i.e.,

$$q = q_0 e^{-t/RC}$$

Solution

Let there be charge q on the condenser and a current $I = dq/dt$ be flowing through the circuit when the key k is flipped on to the terminal B for discharging the condenser. The voltage equation is

$$R\frac{dq}{dt} + \frac{q}{C} = 0$$

or
$$\frac{dq}{q} = -\frac{dt}{RC}$$

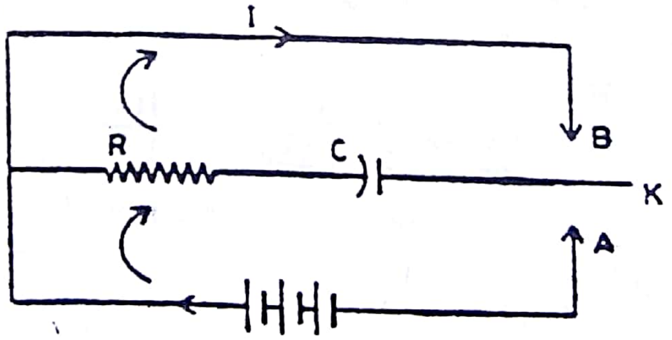


Fig. 3.4 A circuit containing the capacitance C and resistance R

Integrating

$$\log q = -\frac{t}{RC} + A$$

when $t = 0$, $q = q_0$, hence $A = \log q_0$

Therefore $$\log q = -\frac{t}{RC} + \log q_0$$

or $$q = q_0 e^{-t/RC} \quad \text{(ii)}$$

Thus the relaxation time of the discharge process is RC.

3.4 EXAMPLES OF DAMPING IN PHYSICAL SYSTEMS

In the earlier part of this section, the case of an electrical circuit containing inductance, capacitance and resistance capable of damped simple harmonic electrical oscillations, will be analysed and the points of semblance between its equation of motion and that of a damped simple harmonic oscillator brought out. This is the realistic case of resistance damping which invariably comes into play due to the presence of resistance in our oscillating electrical circuit. We then discuss below the cases of electromagnetic damping (due to the eddy currents) and collision damping in relation to the inosphere and conduction electrons in metals.

3.4.1 Resistance Damping. Oscillatory Discharge of a Condenser through a Circuit Containing Resistance and Inductance

The presence of resistance in a series LCR circuit provides the damping force for the oscillating discharge of the condenser because electrical energy is dissipated through Joule heating in the resistive element. The condenser C is charged by connecting the key K to A, Fig. 3.5. On connecting the key to B, the condenser gets discharged through the inductance L and resistance R.

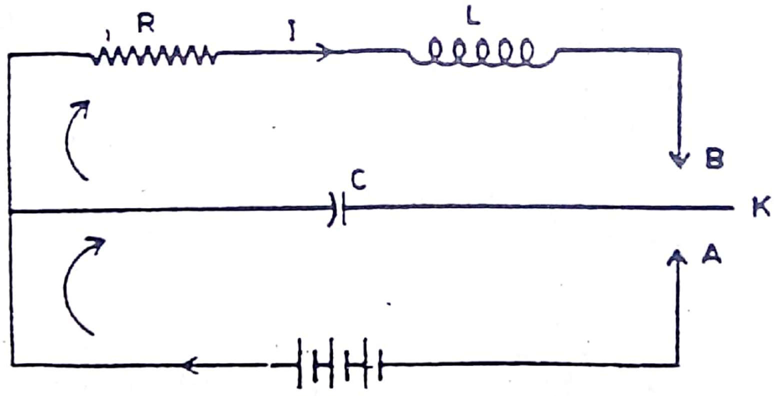


Fig. 3.5 A circuit containing resistance R, inductance L and capacity C

If I is the instantaneous current in the circuit at a time t, then the sum of the voltages around the circuit is

$$L\frac{dI}{dt} + RI + \frac{q}{C} = 0 \quad (3.36)$$

where q is the charge on the plates of the condenser at time t.

But $I = dq/dt$, therefore the voltage equation becomes

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0 \quad (3.37)$$

This equation is similar to the equation of damped simple harmonic oscillator, Eq. (3.5), i.e.

$$m\frac{d^2x}{dt^2} + C\frac{dx}{dt} + Sx = 0 \tag{3.5}$$

where the charge q is replaced by the displacement x.

Out of mathematical analogy between them, the solution for the electrical case is

$$q = \exp\left(-\frac{Rt}{2L}\right)\left[C_1 \exp\left(\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}t\right) + C_2 \exp\left(-\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}t\right)\right] \tag{3.38}$$

where C_1 and C_2 are constants to be determined from the initial conditions.

At $t = 0$, let $q = q_0$ and $I = dq/dt = 0$ then

$$q_0 = C_1 + C_2$$

$$0 = -\frac{R}{2L}(C_1 + C_2) + C_1\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} - C_2\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$
$$= 0$$

Putting $C_1 = q_0 - C_2$ in the above equation, we get

$$\frac{R}{2L}q_0 - (q_0 - C_2)\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} + C_2\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = 0$$

$$2\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\,C_2 = q_0\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} - \frac{R}{2L}q_0$$

Therefore
$$C_2 = \frac{q_0}{2}\left[1 - \frac{1}{2\frac{L}{R}\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}}\right]$$

and
$$C_1 = \frac{q_0}{2}\left[1 + \frac{1}{2\frac{L}{R}\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}}\right]$$

Three cases arise depending upon the relative values of $R^2/4L^2$ and $1/LC$. These are

Case i $R^2/4L^2 > 1/LC$. When the resistance is high so that this condition is fulfilled, then from Eq. (3.38), it is obvious that the discharge is non-oscillatory and decays with time, Fig. 3.6(i)

Case ii Critically damped motion. When the values of the circuit elements are such that

$$\frac{R^2}{4L^2} = \frac{1}{LC}$$

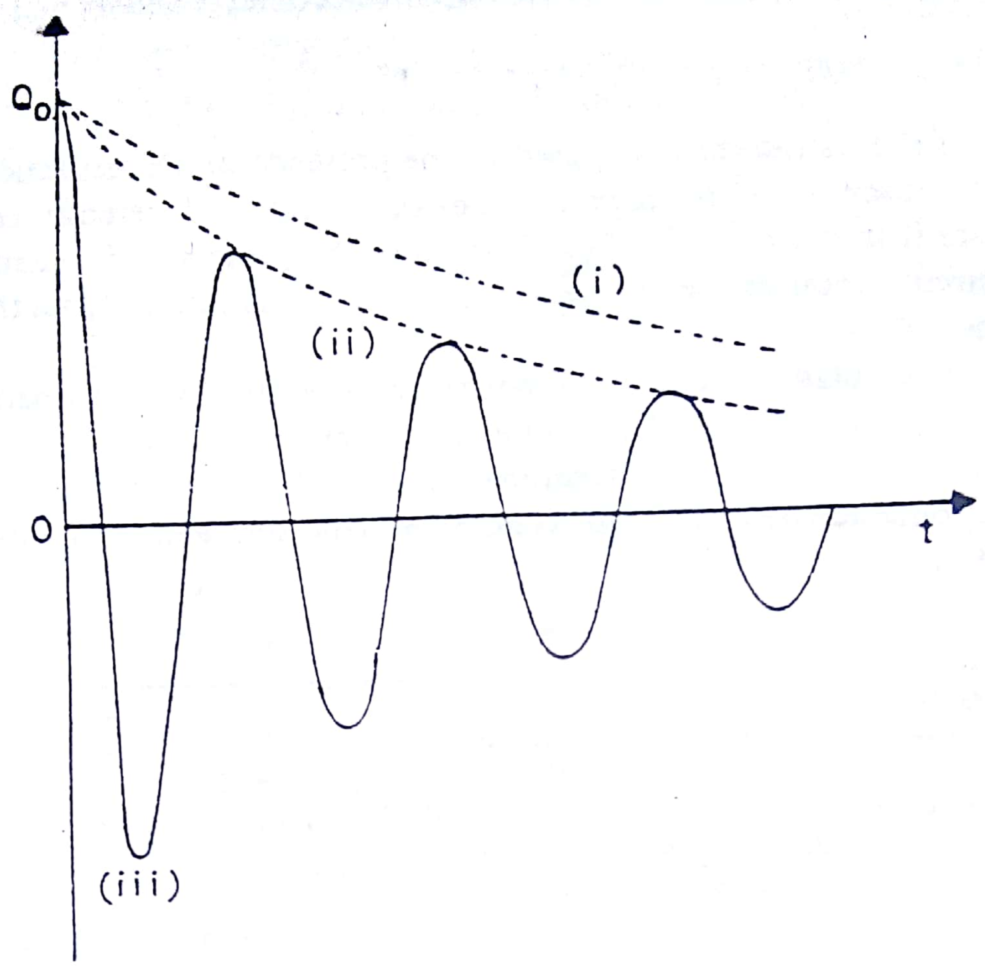


Fig. 3.6 The charge of a discharging condenser plotted against time. The cases include (i) high damping, (ii) critically damped, and (iii) oscillatory motion

the circuit is said to be critically damped. The charge on the condenser decays exponentially with time, Fig. 3.6(ii).

Case iii Oscillatory motion. When the condition

$$\frac{R_2}{4L^2} < \frac{1}{LC}$$

is fulfilled, the Eq. (3.38) becomes

$$q = \exp\left(-\frac{Rt}{2L}\right)\left[C_1 \exp\left(i\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\,t\right) + C_2 \exp\left(-i\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\,t\right)\right] \tag{3.39}$$

The discharge is oscillatory and is given by

$$q = q_0 e^{-Rt/2L} \sin\left(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\,t + \phi_0\right) \tag{3.40}$$

written out of analogy with Eq. (3.13). The amplitude of oscillations decays exponentially with time, Fig. 3.6(iii) and this angular frequency of damped oscillation is given by

$$\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \tag{3.41}$$

When $R = 0$, the frequency of the LC circuit becomes

$$\frac{1}{\sqrt{LC}} \tag{3.42}$$

The decay of the oscillations is caused by the presence of the resistance alone, since in its absence (when $R = 0$), the amplitude will remain constant. Furthermore it is evident from Eq. (3.41) that the presence of resistance in the LCR circuit, reduces the frequency of damped oscillations from the value ω_0, i.e. when $R = 0$.

There is a semblance between the equation of motion of a damped simple harmonic oscillator, Eq. (3.5) and an electrical circuit containing L, C and R, capable of damped simple harmonic electrical oscillations, Eq. (3.37). The correspondence between these systems may be summed up as displayed in Table 3.1.

Table 3.1

Mechanical system	*Electrical system*
Inertia, m	Inductance, L
Mechanical resistance, C	Electrical resistance, R
$\frac{1}{\text{Stiffness, } S} = \frac{\text{displacement}}{\text{force}}$	Capacity $C = \frac{\text{charge}}{\text{PD}}$
Displacement, x	Electrical charge, q
Velocity, $\frac{dx}{dt}$	Current, $\frac{dq}{dt}$
Acceleration $\frac{d^2x}{dt^2}$	Rate of current change $\frac{d^2q}{dt^2}$
Force applied by spring, Sx	Voltage across C, $\frac{q}{C}$
Dragg force, $C\dot{x}$	Voltage across R, RI
Force accelerating mass, $m\ddot{x}$	Voltage across $L = L\ddot{q}$
Potential energy, $\frac{1}{2}Sx$	Electric energy, $\frac{1}{2}\frac{q^2}{C}$
Kinetic energy, $\frac{1}{2}m\dot{x}^2$	Magnetic energy, $\frac{1}{2}L\dot{q}^2$

Ex. 3.7 After how long a time will the charge oscillations decay to half amplitude if $L = 10$ mH, $C = 1.0$ μf and $R = 0.1$ ohm? Calculate the frequency of the damped oscillations.

Solution

The amplitude of the oscillations will decrease to half when the amplitude factor $e^{-Rt/2L}$ has the value 1/2, i.e.

$$\frac{1}{2} = \exp\left(-\frac{RT}{2L}\right)$$

Therefore

$$T = \frac{2L}{R}\log 2$$

$$= \frac{2\times(10\times 10^{-3})(0.69)}{0.10}$$

$$= 0.14 \text{ s}$$

The angular frequency of the damped oscillation is

$$\sqrt{\frac{1}{LC}-\left(\frac{R}{2L}\right)^2}=\sqrt{\left[\frac{1}{(10\times 10^{-3})(1.0\times 10^{-6})}-\left(\frac{0.10}{2\times 10\times 10^{-3}}\right)^2\right]}$$

$$= \sqrt{(10^8 - 25)}$$

$$\simeq 10^4 \text{ radians/s}$$

It is obvious that in this case, the resistance has negligible effect on the frequency, in view of its small value.

Ex. 3.8 A condenser of capacity 1 μf, an inductance of 0.2 henry and a resistance of 800 ohms are in series. Is the circuit oscillatory?

Solution

The circuit will be oscillatory if $\frac{1}{LC} > \frac{R^2}{4L^2}$

Here
$$\frac{1}{LC} = \frac{1}{0.2\times 1\times 10^6} = 5\times 10^6$$

$$\frac{R^2}{4L^2} = \frac{(800)^2}{4\times(0.2)^2} = 4\times 10^6$$

As $\frac{1}{LC} > \frac{R^2}{4L^2}$, the circuit is oscillatory.

The frequency of oscillation $= \frac{1}{2\pi}\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}$

$$= \frac{1}{2\pi}\sqrt{5\times 10^6 - 4\times 10^6}$$

$$= \frac{10^3}{2\pi} \text{ Hz}$$

Ex. 3.9 In an oscillatory circuit $L = 0.2$ H, $C = 0.0012\,\mu$f. Find the maximum value of the resistance so that the circuit may oscillate.

Solution

The circuit will oscillate if

$$\frac{1}{LC} = \frac{R^2}{4L^2}$$

or
$$R^2 = \frac{4L}{C}$$

$$R = \sqrt{\frac{4L}{C}} = \sqrt{\frac{4\times 0.2}{0.0012\times 10^{-6}}}$$

$$= 2.58\times 10^4\ \Omega$$

Thus the discharge will be oscillatory if the resistance does not exceed the value $2.58\times 10^4\ \Omega$.

Ex. 3.10 Deduce the frequency and quality factor for a circuit with $L = 2\times 10^{-3}$ H, $C = 5\times 10^{-6}$ F and $R = 0.2$ ohm.

Solution

Frequency of oscillation

$$= \frac{1}{2\pi}\sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}$$

$$= \frac{1}{2\pi}\sqrt{\left[\frac{1}{2\times 10^{-3}\times 5\times 10^{-6}} - \frac{(0.2)^2}{4\times(2.10^{-3})^2}\right]}$$

$$= \frac{10^4}{2\pi} = 1.59\times 10^3 \text{ Hz}$$

$$\text{Quality factor} = \frac{L\omega}{R}$$

$$= \frac{2\times 10^{-3}\times 10^4}{0.2}$$

$$= 100$$

3.4.2 Electromagnetic Damping. Eddy Currents

A moving coil galvanometer consists of a current-carrying rectangular coil on an axis in a magnetic field. The magnetic field is provided by a permanent magnet, so shaped that the moving coil experiences the same magnitude of the field at all orientations. The steady current to be measured, produces a torque which is proportional to the current. The coil rotates under the electromagnetic torque and comes to an equilibrium position where the turning torque is balanced by the restoring torque due to the stiffness of the suspension.

The damping of the moving part in a galvanometer, apart from external artificial agency (like a shunt of low resistance connected in parallel) arises due to the following two causes:

(i) The damping due to the viscosity of the air. The damping force is approximately proportional to the angular velocity of the system but is usually negligibly small.

(ii) *Electromagnetic damping*: When the suspended coil of a galvanometer rotates in a strong magnetic field, it is resisted in open circuit by (i) viscous drag of air and mechanical friction and (ii) induced currents in the neighbouring conductors. The open circuit damping couple is proportional to the angular speed $|d\theta/dt|$, according to the law of electromagnetic induction and this is represented by $-b_e\, d\theta/dt$ where b_e is the *damping coefficient*. When the circuit is closed, there is an additional damping provided by the induced currents in the coil. The closed circuit damping is universally proportional to the total resistance of the circuit and is given by $-\psi/R\; d\theta/dt$, where ψ involves the area of the coil, magnetic flux etc. Denoting the angular displacement from its new equilibrium position by θ, we get

$$I\frac{d^2\theta}{dt^2} = -C\theta - b_e\frac{d\theta}{dt} - \frac{\psi}{R}\frac{d\theta}{dt}$$

where I is the moment of inertia of the vibrating system and C is the restoring couple per unit twist of the suspension. It becomes

$$\frac{d^2\theta}{dt^2} + \frac{1}{I}\left(b_e + \frac{\psi}{R}\right)\frac{d\theta}{dt} + \omega_0^2\theta = 0 \tag{3.43}$$

which is analogous to the Eq. (3.5).

Since $\omega_0 = \sqrt{\frac{C}{I}}$ and putting $\frac{1}{I}\left(b_e + \frac{\psi}{R}\right) = \gamma_e$ the solution of Eq. (3.43) out of analogy with the solution of Eq. (3.9) becomes

$$\theta = C_1 \exp\left[\left(-\frac{\gamma_e}{2} + \sqrt{\frac{\gamma_e^2}{4} - \omega_0^2}\right)t\right] + C_2 \exp\left[\left(-\frac{\gamma_e}{2} - \sqrt{\frac{\gamma_e^2}{4} - \omega_0^2}\right)t\right]$$

where C_1 and C_2 are undetermined constants to be determined from the initial conditions.

Three cases arise:

Case i, Dead-beat motion. If the damping is high such that the factor

$$\frac{\gamma_e^2}{4} > \omega_0^2$$

we have two real roots for the equation of angular displacement. Calling these roots α and β, one gets

$$\begin{aligned}\theta &= C_1 e^{-\alpha t} + C_2 e^{-\beta t} \\ &= C_1 \exp\left[\left(-\frac{\gamma_e}{2} + \sqrt{\frac{\gamma_e^2}{4} - \omega_0^2}\right)t\right] \\ &\quad + C_2 \exp\left[\left(-\frac{\gamma_e}{2} - \sqrt{\frac{\gamma_e^2}{4} - \omega_0^2}\right)t\right]\end{aligned}$$

Obviously, decays exponentially without any change of direction since the motion is non-oscillatory. This is the case of deadbeat motion.

Case ii, Critically damped motion. When $\frac{\gamma_e^2}{4} = \omega_0^2$, the galvanometer is said to be critically damped. In this case the coil after deflection comes to rest in a minimum of time and the direction of motion never changes.

Case iii, Light damping.

Ballistic motion: When $\gamma_e^2/4 < \omega_0^2$, both the roots α and β become imaginary and the solution become

$$= \exp\left(-\frac{\gamma_e t}{2}\right)\left[C_1 \exp\left[i\sqrt{\omega_0^2 - \frac{\gamma_e^2}{4}}\,t\right] + C_2 \exp\left[-i\sqrt{\omega_0^2 - \frac{\gamma_e^2}{4}}\,t\right]\right.$$

which can be put in the form

$$= C_0 \exp(-\gamma_e t/2) \sin(qt + \phi_0)$$

where
$$q = \sqrt{\omega_0^2 - \frac{\gamma_e^2}{4}}$$

$$= \sqrt{\frac{C}{I} - \frac{1}{4I^2}\left(b_e + \frac{\psi}{R}\right)^2}$$

The motion is oscillatory with the frequency

$$= \frac{1}{2\pi}\sqrt{\frac{C}{I} - \frac{1}{4I^2}\left(b_e + \frac{\psi}{R}\right)^2}$$

and the amplitude of motion is a decaying function of time. Thus for the motion to be ballistic the factor $\frac{\gamma_e}{2} = \frac{(b_e + \psi/R)}{2I}$ must be small. This requires that

(i) I should be large,
(ii) ψ should be small and the coil may be wound on a non-conducting frame like wood or paper,
(iii) R should be large, and
(iv) The em rotational resistance b_e, should be small and the suspension to be fine.

Obviously the requirements for making the galvanometer ballistic are reverse of the conditions for making it dead-beat.

The total width γ for the system will include the mechanical width γ_m in addition to the electromagnetic width γ_e. However γ_m is always much smaller than γ_e. The value of damping current depends on the magnitude of the external resistance connected across the terminals of the galvanometer and may become great when the shunt resistance is low. The larger the induced current, the larger will be b_e and γ_e.

However, the total resistance cannot be made smaller than the resistance of the coil. If that is not too large, one can choose a suitable external resistance so as to fulfill the requirement of critical damping and make the instrument dead-beat.

3.4.3 Eddy Currents and Their Applications

So far we have considered the currents resulting from induced emfs, confined to well-defined paths provided by the wires and external resistances. However, in many electrical equipments there are masses of metal moving in a magnetic field, with the consequence that induced currents circulate in the body of the metal. These circulatory currents are called eddy currents and are always proportional to $d\theta/dt$. The eddy currents find numerous practical applications and these are:

Dead-beat galvanometer The moving coil galvanometer, the ammeter and the voltmeter have their coils wound on a metallic frame made of copper or aluminium and the induced eddy currents in the frame, make the motion dead-beat. In the case of ballistic galvanometer, the damping is to be reduced and so the coil is wound on a non-metallic frame like paper or bamboo.

Induction Furnace The production of eddy current results in the loss of energy in the form of thermal heating. This phenomenon is utilized in the construction of induction furnace for the production of very high temperatures due to the eddy currents. The heating effect is proportional to the square of frequency of the current, and so high frequencies are preferred.

Diathermy Another application of eddy currents pertains to the deep heat treatment or 'Diathermy' in hospitals. A coil of wire is wound around the hand, foot or any other part of the body and high frequency current (50 MHz) is passed through it. The changes in flux induce eddy currents in the conducting tissues of the body and the resulting heat is able to penetrate locally and sufficiently deep into the affected part.

Speedometer It consists of a magnet placed inside an aluminium drum which is carefully balanced and held in position by a hair spring. When the magnet spins around its axis according to the speed of the vehicle, its motion sets up eddy currents in the drum, which oppose the relative motion between the inside magnet and the outside drum. As the magnet continues to spin, the drum drags through an angle which is proportional to the speed of the vehicle.

Electric Brakes A drum is provided to rotate with the axle when the train is in motion. When a train is to be stopped, magnetic field is applied to the drum and the eddy currents in the drum produce an opposing torque thus bringing the train to an instantaneous stop.

Induction Motor It is based on the principle of a rotating magnetic field. A metallic cylinder is placed in a rotating magnetic field and the eddy currents set up in the cylinder try to reduce the relative motion between the cylinder and the field. However, since the magnetic field continues to rotate, the only way to reduce the relative motion is by setting into rotation the metallic cylinder about its axis. That is how the induction motor works.

3.4.4 Collision Damping. Ionosphere and Metals

In the first chapter, we discussed the case of plasma vibrations in the case of ionosphere as well as of conduction electrons in metals and calculated the frequencies at which the electrons perform free vibrations in these cases. The electrons in the plasma all move together with the same displacement $x(t)$ and this collective displacement under the influence of the electric field is superimposed on the more vigorous random thermal motion. Any electron has a probability of undergoing a collision with another randomly moving particle, and getting scattered in a random direction not performing any vibration later on.

The rate at which electrons leak away from the vibration is proportional to the number of electrons present at any time, i.e.

$$-\frac{dN}{dt} \propto N$$

or

$$-\frac{dN}{dt} = \frac{N}{\tau_c} \tag{3.44}$$

where τ_c is the collision lifetime. The fraction colliding per unit time is given by $1/\tau_c$ and thus τ_c is the average length of time for which an electron performs a free vibration before getting lost as a result of collision. From Eq. (3.44), we get on integration

$$\int \frac{dN}{N} = -\int \frac{dt}{\tau_c}$$

$$\log N = -\frac{t}{\tau_c} + c \tag{3.45}$$

Thus the collision damping takes the form of a steady draining away of electrons from the plasma vibrations. The collision width γ_c, being the reciprocal of the collision lifetime is

$$\gamma_c = \frac{1}{\tau_c} \tag{3.46}$$

Obviously γ_c is evaluated from τ_c, the collision lifetime.

Estimation of Collision Width for D and F2 Layers of Ionosphere In the case of ionosphere, the electrons will suffer collisions with the neutral molecules. The number of collisions per unit time is proportional to the number of molecules per unit volume N_m, and their thermal speed V_{th} ($V_{th} \gg dx/dt$). The molecules are performing Brownian motion and as such their random molecular speeds V_{th} are much larger than their net velocities in any particular direction.

Therefore

$$\gamma_c \propto N_m V_{th}$$

or

$$\gamma_c = \sigma_c N_m V_{th} \tag{3.47}$$

where the constant of proportionality σ_c, is called the collision cross-section. It has the dimensions of area and represents the effective area offered by each molecule to the impinging electron.

Thus

$$\sigma_c = \frac{\gamma_c}{N_m V_{th}} = \frac{1}{\tau_c} \frac{1}{N_m V_{th}}$$

$$= \frac{L^3 \times T}{T \times L} = L^2$$

For a perfect gas, the number of molecules per unit volume is given by

$$N_m = \frac{P}{kT} \tag{3.48}$$

where P is the pressure and T the absolute temperature. The estimate of V_{th} is based on the principle of equipartition of energy by equating the average kinetic energy of an electron at temperature T to the value $3/2\, kT$. Strictly one should use Fermi-Dirac statistics for estimating the average kinetic energy of an electron. However at an elevated temperature, the Fermi-Dirac statistics reduces approximately to the Maxwellian statistics.

Thus $$\frac{1}{2} m_e V_{th}^2 \approx \frac{3}{2} kT$$

or $$V_{th} \approx \left(\frac{3kT}{m_e}\right)^{1/2} \tag{3.49}$$

The collision cross-section can be determined experimentally and its value is of the order of 10^{-19} cm². Thus from Eq. (3.47), one gets, after substituting for V_{th} from Eq. (3.49) putting the values $m_e = 9.11 \times 10^{-31}$ kg; $k = 1.38 \times 10^{-23}$ J K^{-1}

$$\gamma_c = \left(\frac{3\sigma_c^2}{m_e k}\right)^{1/2} P T^{-1/2}$$

$$= (4.9 \times 10^7 \text{ kg}^{-1} \text{ m sK}^{1/2})\, P T^{-1/2} \tag{3.50}$$

The pressure P in the D layer of ionosphere is about 0.5 Nm^{-2} and T is approximately 200°K. These figures given $\gamma_c \approx 2 \times 10^6$ s^{-1}. As estimated earlier plasma frequency from the D layer is $\omega_0 \approx 2 \times 10^6$ s^{-1}.

Thus $$Q = \frac{\omega_0}{\gamma_c} \sim 1 \text{ for the } D \text{ layer.}$$

Applying the analysis to the F_2 layer in the ionosphere, it is much more rarefied with $P \sim 10^{-10}$ Nm^{-2} or less and much hotter than the D layer with $T \sim 2000$ K. Thus $\gamma_c \approx 10^{-4}$ s^{-1} for the case of F_2 layer. Since the plasma frequency for the F_2 layer $\omega_0 \approx 10$ MHz we get $Q \gg 1$, implying thereby that the damping in the F_2 layer is very light ($\gamma \ll \omega_0$).

Estimation of Collision Width in Metals The conduction electrons in a metal will get displaced under the influence of the impressed field and undergo collisions with the positive ion lattice. If the frequency of these collisions is high, the metal is relatively a poor conductor of electricity as characterised by low electrical conductivity. The value of collision width γ_c can be estimated from the measured conductivity σ.

The electrons will acquire a drift velocity v_{dr} under the electric field E set up in the metal and the conductivity of the metal σ is given by

$$\sigma = \frac{Nev_{dr}}{E} \tag{3.51}$$

where N is the conduction electron density. According to Ohm's law the current density J is given by $J = \sigma E$, where σ is the conductivity and E the electric field. The current density $\mathbf{J} = NeV$. The electron on suffering a collision gets its velocity in the drift direction randomized. Between two successive collisions, the electron acquires an acceleration Ee/m_e due to the field E and if τ_c is the collision lifetime, the drift velocity v_{dr} is given by

$$v_{dr} \approx \left(\frac{Ee}{m_e}\right)\tau_c \tag{3.52}$$

Now the collision width γ_c being the reciprocal of the collision lifetime, is given by

$$\gamma_c = \frac{1}{\tau_c}$$

From Eq. (3.51), we have $\sigma \approx \frac{Ne}{E} \cdot \frac{Ee}{m_e}\tau_c \approx \frac{Ne^2}{m_e}\tau_c$

Therefore, putting the values: $m_e = 9.11 \times 10^{-31}$ kg; $e = 1.60 \times 10^{-19}$ C one gets

$$\gamma_c \approx \frac{Ne^2}{m_e \sigma}$$

$$= (2.8 \times 10^{-8}\ \mathrm{C^2\ kg^{-1}})\frac{N}{\sigma} \tag{3.53}$$

Assuming the values for copper

$$N = 8.4 \times 10^{28}\ \mathrm{m^{-3}},$$

$$\sigma = 5.8 \times 10^{7}\ \Omega^{-1}\ \mathrm{m^{-1}},$$

we get the collision width for copper,

$$\gamma_c \approx 4 \times 10^{13}\ \mathrm{s^{-1}}$$

Now as the plasma frequency for Cu as estimated earlier is $\omega_0 \approx 2 \times 10^{16}\ \mathrm{s^{-1}}$, we get $Q \approx 500$. Obviously the plasma vibrations are lightly damped in metals.

3.4.5 Friction Damping

It was shown in Sec. 3.1 that, in general, the damping force is proportional to the velocity of the vibrating body. However, there are situations when the drag force F_d will be one due to the friction between dry metal parts, rather than an essentially viscous force proportional to the velocity of the oscillating body. The simplest assumption is that a frictional drag force has a constant magnitude F_d, independent of the speed but its direction always opposing the motion. The equation of motion of a mass m becomes for motion towards the $+x$ axis,

$$m\ddot{x} = -Sx - F_{sl} \tag{3.54}$$

and for motion towards the $-x$-axis,

$$m\ddot{x} = -Sx + F_{sl} \tag{3.55}$$

where F_{sl} is the sliding friction force which is assumed to be constant.

These equations can be shown to have semblance to the equations of motion for undamped free vibration than to the damped motion, through a coordinate shift transformation.

Calling
$$x_r \equiv x - \frac{F_{sl}}{S}$$

$$x_l \equiv x + \frac{F_{sl}}{S}$$

The above equations become

$$\ddot{x}_r + \omega_0^2 x_r = 0 \tag{3.56}$$

$$\ddot{x}_l + \omega_0^2 x_l = 0 \tag{3.57}$$

where $\omega_0 = \sqrt{S/m}$ is the natural frequency of the system.

The amplitudes to the right and left x_r and x_l respectively, vary harmonically with angular frequency ω_0. The motion of mass m is harmonic in both the half cycles (either to the right or left), but is centred off-centre, a distance F_{sl}/S to the left for motion to the right and the same distance F_{sl}/S to the right for the motion to the left.

4

Forced Vibrations and Resonance

In the earlier chapters, we considered the phenomena of free vibrations (damped or undamped), as these were solely maintained by the energy stored up in the system with no subsequent supply of energy except at the start for setting up the vibrations. The amplitude of damped vibrations decays with time and after some time the system comes to rest since there is a continual dissipation of energy. However, these vibrations can be kept up, provided there is a periodic input of energy by the application of an outside driving force. When the frequencies of the driving and driven systems are not the same, the natural frequency of the oscillator dies out soon and it begins to oscillate with the frequency of the impressed periodic force. The vibrations so kept up are called *forced vibrations*. However, when the frequency of the driving force coincides with the natural frequency of the driven system, it gives rise to the phenomena of resonance. This goes on building up either the displacement or velocity amplitude and the driven system keeps on receiving a certain amount of average power till any of these characteristics assumes the maximum value.

The forced vibrations are exemplified by a person carrying on a telephonic conversation with another. His voice sound exerts pressure on the diaphragm of the microphone which is set into forced vibrations. The person plays the role of the driving system and the link between him and the microphone (driven system) is called the *acoustical mechanical link*.

A radioset receives the radiowaves and these after detection and amplification produce electrical oscillations in the output circuit of the amplifier. These electrical oscillations deliver energy to the loudspeaker which, in turn, vibrates and produces sound. The link here between the driving (radio signal) and the driven system (loudspeaker) is an electro-mechanical one.

4.1 A FORCED OSCILLATOR

Let a mechanical oscillator of mass m, stiffness S and resistance R_m be driven by a sinusoidal force $F = F_0 \sin \omega t$, Fig. 4.1.

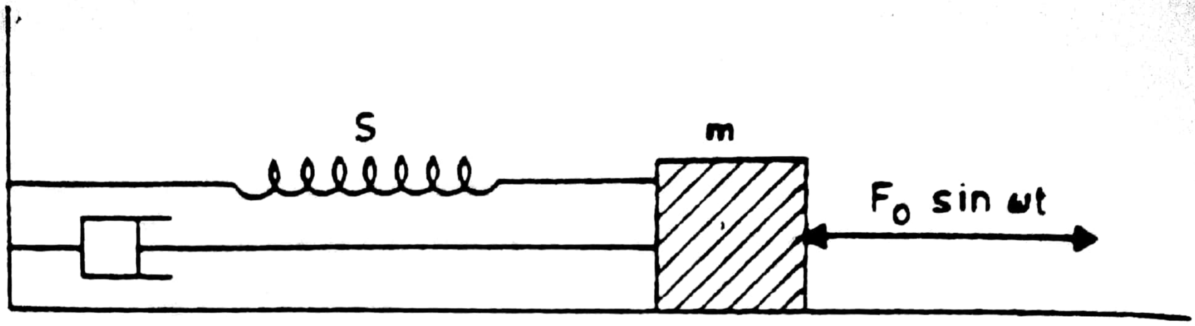


Fig. 4.1 A mechanical damped oscillator driven by a force $F_0 \sin \omega t$

The equation of motion in this case becomes

$$m\frac{d^2x}{dt^2} = F_0 \sin \omega t - R_m \frac{dx}{dt} - Sx$$

or
$$\frac{d^2x}{dt^2} = \frac{F_0}{m} \sin \omega t - \frac{R_m}{m}\frac{dx}{dt} - \frac{S}{m}x$$

Putting
$$\frac{F_0}{m} = f_0,$$

$$\frac{R_m}{m} = 2r$$

and
$$\omega_0 = \sqrt{\frac{S}{m}}$$

we get
$$\frac{d^2x}{dt^2} = f_0 \sin \omega t - 2r\frac{dx}{dt} - \omega_0^2 x$$

Let us solve this equation by the differential operator method.

Introducing $D \equiv \frac{d}{dt}$, we get

$$(D^2 + 2rD + \omega_0^2)\, x = f_0 \sin \omega t$$

The complete solution of this equation will consist of the sum of the complementary function (the solulion with the RHS equal to zero) and the particular integral.

The homogeneous part of the equation is

$$(D^2 + 2rD + \omega_0^2)\, x = 0 \tag{4.2}$$

The two roots of the characteristic equation are

$$D = \frac{-2r \pm \sqrt{4r^2 - 4\omega_0^2}}{2}$$

$$= -r \pm i\sqrt{\omega_0^2 - r^2} \tag{4.3}$$

and accordingly the complementary function is

$$\text{CF} = e^{-rt}[Ae^{i\sqrt{\omega_0^2 - r^2}\,t} + Be^{-i\sqrt{\omega_0^2 - r^2}\,t}]$$
$$= e^{-rt}[(A + B)\cos \sqrt{\omega_0^2 - r^2}\,t + i(A - B)\sin \sqrt{\omega_0^2 - r^2}\,t]$$
$$= e^{-rt}[C_0 \sin qt \cos \varphi_0 + C_0 \cos qt \sin \varphi_0]$$
$$= C_0 e^{-rt} \sin (qt + \varphi_0) \tag{4.4}$$

where $$C_0 = [\{i(A - B)\}^2 + \{(A + B)\}^2]^{1/2} = [4AB]^{1/2}$$

$$q = \sqrt{\omega_0^2 - r^2}$$

$$\tan \varphi_0 = \frac{A + B}{i(A - B)} = -i\frac{A + B}{A - B}$$

The particular integral is given by

$$\text{PI} = \frac{1}{D^2 + 2rD + \omega_0^2} f_0 \sin \omega t \tag{4.5}$$

$$= \frac{(D^2 + \omega_0^2) - 2rD}{(D^2 + \omega_0^2)^2 - 4r^2D^2} f_0 \sin \omega t$$

$$= f_0 \frac{D^2 - 2rD + \omega_0^2}{(\omega_0^2 - \omega^2)^2 + 4r^2\omega^2} \sin \omega t$$

$$= f_0 \frac{-\omega^2 \sin \omega t - 2r\omega \cos \omega t + \omega_0^2 \sin \omega t}{(\omega_0^2 - \omega^2) + 4r^2 \omega^2}$$

$$= f_0 \frac{(\omega_0^2 - \omega^2) \sin \omega t - 2r\omega \cos \omega t}{(\omega_0^2 - \omega_2)^2 + 4r^2 \omega^2}$$

$$= h \sin \omega t \cos \psi - h \cos \omega t \sin \psi$$

$$= h \sin (\omega t - \psi). \tag{4.6}$$

where $$h \cos \psi = \frac{f_0(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + 4r^2\omega^2}$$

$$h \sin \psi = \frac{2r\omega f_0}{(\omega_0^2 - \omega^2)^2 + 4r^2\omega^2}$$

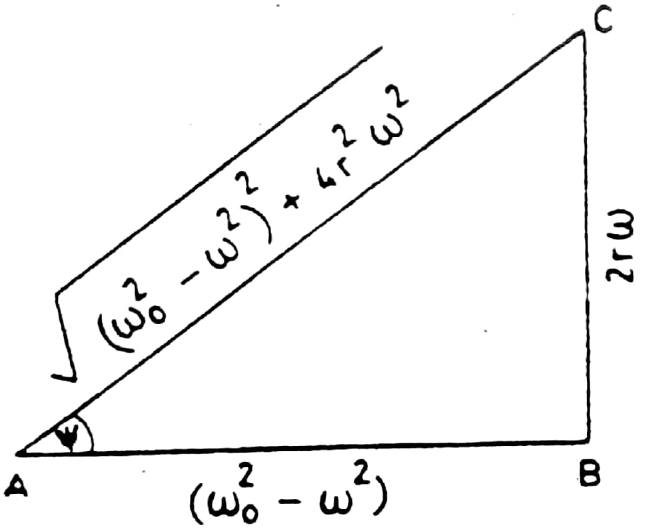


Fig. 4.2 Acoustic-Impedance triangle

so that $$h = \frac{f_0}{[(\omega_0^2 - \omega^2)^2 + 4r^2\omega^2]^{1/2}}$$

$$\tan \psi = \frac{2r\omega}{(\omega_0^2 - \omega^2)} \tag{4.8}$$

The complete solution, therefore, is

$$x = b \sin(\omega t - \psi) + C_0 e^{-rt} \sin(qt + \varphi_0) \tag{4.9}$$

The part of the complete solution given by the complementary function Eq. (4.4) is the *transient term* since it dies away with time as e^{-rt}. In the transient state, the oscillator oscillates neither with its natural frequency nor with the frequency of the impressed force. The second term is called the *steady state* term and describes the behaviour of the system after the transient term has decayed and ceased to be effective; although initially both the terms contribute to the solution. In the steady state the oscillator oscillates with the frequency of the external force.

Rewriting the steady state solution Eq. (4.6)

$$x = b \sin(\omega t - \psi)$$

and substituting for b, we get

$$x = \frac{f_0 \sin(\omega t - \psi)}{[(\omega_0^2 - \omega^2)^2 + 4r^2\omega^2]^{1/2}}$$

$$= \frac{F_0 \sin(\omega t - \psi)}{[(m\omega_0^2 - m\omega^2)^2 + 4r^2m^2\omega^2]^{1/2}}$$

$$= \frac{F_0 \sin(\omega t - \psi)}{\omega\left[R_m^2 + \left(m\omega - \frac{S}{\omega}\right)^2\right]^{1/2}} \qquad \left(\because R_m = 2rm, \; m\omega_0^2 = S\right)$$

Thus

$$b = \frac{F_0}{\omega\left[R_m^2 + \left(m\omega - \frac{S}{\omega}\right)^2\right]^{1/2}} \tag{4.10}$$

Defining Z_m, the mechanical impedance as

$$Z_m = \left[R_m^2 + \left(m\omega - \frac{S}{\omega}\right)^2\right]^{1/2} \tag{4.11}$$

X_m, the mechanical reactance as

$$X_m = \left(m\omega - \frac{S}{\omega}\right) \tag{4.12}$$

and R_m, the mechanical resistance as

$$R_m = 2rm \tag{4.13}$$

one gets

$$Z_m^2 = X_m^2 + R_m^2 \tag{4.14}$$

Accordingly, Eq. (4.10) becomes

$$b = \frac{F_0}{\omega Z_m} \tag{4.15}$$

and the expression for displacement x takes the form

$$x = \frac{F_0}{\omega Z_m} \sin(\omega t - \psi) \tag{4.16}$$

The phase in the steady state is defined completely wrt the driving force. Depending on the relative magnitudes of the driving and the natural frequencies ω and ω_0, respectively, three cases arise:

Case i (*For low driving frequency*, i.e. $\omega \ll \omega_0$) Under this condition from Eq. (4.8), one gets

$$\tan\psi = \frac{2r\omega}{(\omega_0^2 - \omega^2)} \to 0 \tag{4.8}$$

showing thereby that the driving force and the resulting displacement are in the same phase.

The amplitude b is given by

$$b = \frac{f_0}{[(\omega_0^2 - \omega^2)^2 + 4r^2\omega^2]^{1/2}} \tag{4.7}$$

$$= \frac{f_0}{\omega_0^2} = \frac{F_0/m}{S/m} = \frac{F_0}{S}$$

The amplitude at low driving frequencies thus depends on the driving force and the restoring force constant S. Such a system is said to be restoring force controlled.

Case ii (*Resonance*, i.e. $\omega = \omega_0$) The amplitude will become maximum at resonance and is given by

$$b_{max} = \frac{f_0}{2r\omega}$$

and the phase ψ is given by

$$\tan\psi = \infty \quad \text{so that} \quad \psi = \frac{\pi}{2}$$

The resonance response depends upon damping, the amplitude b_{max} at resonance is inversely proportional to r. If there is no damping, i.e. $r = 0$, then b_{max} becomes infinite. However, in reality, some damping is always present.

It may be remarked that in the presence of damping the maximum amplitude is attained at a frequency which is slightly less than ω_0. It may be easily shown by equating the derivative of b wrt ω equal to zero. Therefore

$$\frac{db}{d\omega} = \frac{d}{d\omega}\left[\frac{f_0}{[\omega_0^2 - \omega^2)^2 + 4r^2\omega^2]^{1/2}}\right]$$

$$= -f_0 \frac{(-4\omega)(\omega_0^2 - \omega^2) + 8r^2\omega}{2[(\omega_0^2 - \omega^2)^2 + 4r^2\omega^2]^{3/2}}$$

$$= 0$$

This is possible only when

$$-4\omega(\omega_0^2 - \omega^2) + 8r^2\omega = 0$$

or

$$\omega_0^2 - \omega^2 - 2r^2 = 0$$

$$\omega = \sqrt{\omega_0^2 - 2r^2}$$

$$= \omega_0 \sqrt{1 - \frac{R_m^2}{2m^2\omega_0^2}} \tag{4.17}$$

Thus the frequency at which the b_{max} occurs is slightly less than ω_0. However, if the damping is small, i.e. r is small, then this decrease can be neglected.

Case iii (*For high driving frequencies*, i.e. $\omega \gg \omega_0$) Under this condition, the amplitude of the resulting vibration

$$b = \frac{f_0}{[(\omega_0^2 - \omega^2)^2 + 4r^2\omega^2]^{1/2}}$$

becomes

$$b = \frac{f_0}{\sqrt{(\omega^4 + 4r^2\omega^2)}} \approx \frac{f_0}{\omega^2}$$

as r is a small quantity. Since maximum amplitude depends on mass of oscillator, the system is referred to as mass controlled.

The phase is given by

$$\tan \psi = \frac{2r\omega}{(\omega_0^2 - \omega^2)} \approx -\frac{2r}{\omega} \rightarrow -0$$

or $\qquad \psi = \pi$

Thus with the increase of frequency ω of the impressed force, the amplitude goes on decreasing and the phase tends towards π.

The characteristics of the forced motion are summarised as follows:

(i) The maximum amplitude of the displacement is $\frac{F_0}{R_m\omega_0}$

(ii) The displacement lags behind the driving force $F_0 \sin \omega t$ by an angle ψ, which increases continuously from zero at $\omega = 0$ to π (in the limit $\omega \rightarrow \infty$), acquiring the value $\pi/2$ at precisely the frequency $\omega = \omega_0$.

The dependence of the amplitude and the phase angle upon the frequency ω of the driving force is shown in Fig. 4.3 (a) and (b). The sharpness of rise of the two curves depends on the magnitude of r, the damping factor.

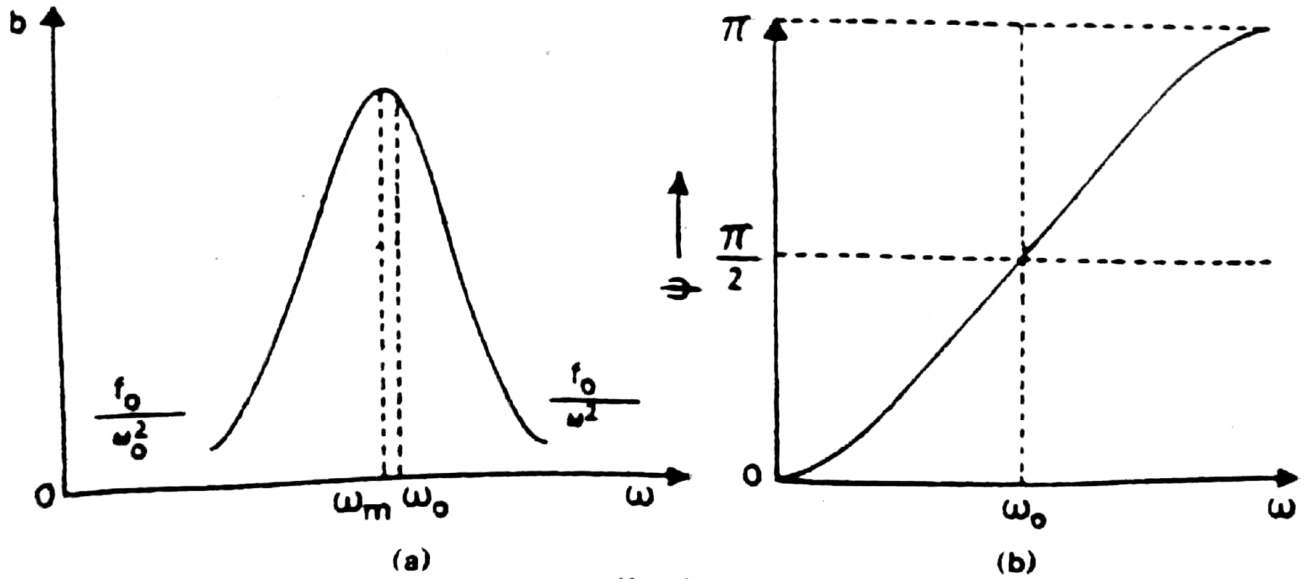


Fig. 4.3 (a) Dependence of the amplitude upon the frequency ω of the driving force; (b) Phase difference ψ as a function of ω

4.2 VELOCITY VERSUS DRIVING FORCE FREQUENCY

The velocity of the forced oscillation in the steady state is

$$v = \dot{x}$$

$$= \frac{F_0}{Z_m} \cos(\omega t - \psi) = \frac{F_0}{Z_m} \sin\left(\omega t + \frac{\pi}{2} - \psi\right) \quad (418)$$

Thus it is leading the applied force by a phase angle $(\pi/2 - \psi)$.

The amplitude of the velocity is $\dfrac{F_0}{Z_m}$

The velocity amplitude F_0/Z_m will have the maximum value when the mechanical impedance has the lowest value $Z_m = R_m$ and is a real quantity with zero reactance. This will be so when $m\omega - S/\omega = 0$ or $\omega = \sqrt{S/m}$ and the corresponding frequency ω_0 is the frequency of velocity resonance. At $\omega = \omega_0$, $\psi = \pi/2$ therefore the velocity and the force are in phase. The height of the peak will depend on the value of resistance R_m, which is the only effective term in the impedance.

The expressions for the velocity v, and $\tan\psi$

$$v = \frac{F_0}{Z_m} \cos(\omega t - \psi)$$

$$= \frac{F_0}{Z_m} \sin\left(\omega t - \psi + \frac{\pi}{2}\right)$$

$$\tan\psi = \frac{2r\omega}{(\omega_0^2 - \omega^2)}$$

show that ψ will be positive when $\omega < \omega_0$ and the velocity will lead the displacement by $\pi/2$. Thus at resonance when displacement lags behind the driving force by $\pi/2$, the velocity is in phase with the driving force. The variation of velocity amplitude with driving frequency will have the same shape as the displacement amplitude as a function of the driving frequency, Fig. 4.3(a).

4.3 POWER SUPPLIED TO THE FORCED OSCILLATOR BY THE DRIVING FORCE

In order to sustain the steady state oscillations of the forced oscillator, energy has to be supplied by the driving force in order to compensate the losses suffered due to the presence of the resistance. We shall show below that in the steady state the amplitude and phase of a driven oscillator will adjust themselves in such a manner that the average power supplied by the driving force is just compensating the losses due to the frictional term.

Now the instantaneous power P supplied is

$$P = \text{force} \times \text{velocity}$$

$$= F_0 \sin\omega t \times \frac{F_0}{Z_m} \cos(\omega t - \psi)$$

$$= \frac{F_0^2}{Z_m} \sin\omega t \cos(\omega t - \psi)$$

Average power supplied, P_{av} is

$$P_{av} = \frac{\text{Power supplied in one cycle}}{\text{Period of the cycle}}$$

$$= \frac{1}{T}\int_0^T P\,dt$$

$$= \frac{F_0^2}{Z_m T}\int_0^T \sin\omega t \cos(\omega t - \psi)\,dt$$

$$= \frac{F_0^2}{Z_m T}\int_0^T \sin\omega t (\cos\omega t \cos\psi + \sin\omega t \sin\psi)\,dt$$

$$= \frac{F_0^2}{Z_m T}\left[\int_0^T \frac{\sin 2\omega t}{2}\cos\psi\,dt + \int_0^T\left(\frac{1-\cos 2\omega t}{2}\right)\sin\psi\,dt\right]$$

$$= \frac{F_0^2}{Z_m T}\,\frac{T}{2}\sin\psi$$

because
$$\int_0^T \frac{\sin 2\omega t}{2}\cos\psi\,dt = 0$$

Thus
$$P_{av} = \frac{F_0^2}{2Z_m}\sin\psi$$

Substituting the value of $\sin\psi$ from Fig. (4.2), i.e.

$$\sin\psi = \frac{b(2r\omega)}{f_0}$$

and making use of the values of b from (4.15) we get

$$\sin\psi = \frac{R_m}{Z_m}$$

Thus the average power,

$$P_{av} = \frac{F_0^2 R_m}{2Z_m^2} \qquad (4.19)$$

Since R_m, the mechanical resistance, is resistive force per unit velocity, the total resistive force is $R_m\dot{x}$ and the rate of work done by the resistive force is

$$(R_m\dot{x})\dot{x} = R_m\dot{x}^2$$

$$= R_m\frac{F_0^2}{Z_m^2}\cos^2(\omega t - \psi).$$

which agrees with Eq. (4.19), proving thereby that the power supplied is equal to the power dissipated against the frictional forces.

The average power absorbed

$$P_{av} = \frac{F_0^2}{2Z_m}\sin\psi$$

will have its maximum when $\sin\psi = 1$, that is, when $\psi = \pi/2$ and $\omega_0^2 - \omega^2 = 0$ or $\omega = \omega_0$. The force and the velocity are then in phase and $Z_m = R_m$.

Thus

$$P_{av}\,(\text{maximum}) = \frac{F_0^2}{2R_m} \tag{4.20}$$

The variation of P_{av} with ω is plotted in Fig. 4.4, and this determines the response of the oscillator to the driving force. The height of peak is determined by R_m, since this is the only term effective at $\omega = \omega_0$ and the location of the peak is also at $\omega = \omega_0$. The frequency ω_0 is the frequency of velocity resonance since maximum power absorption takes places at it.

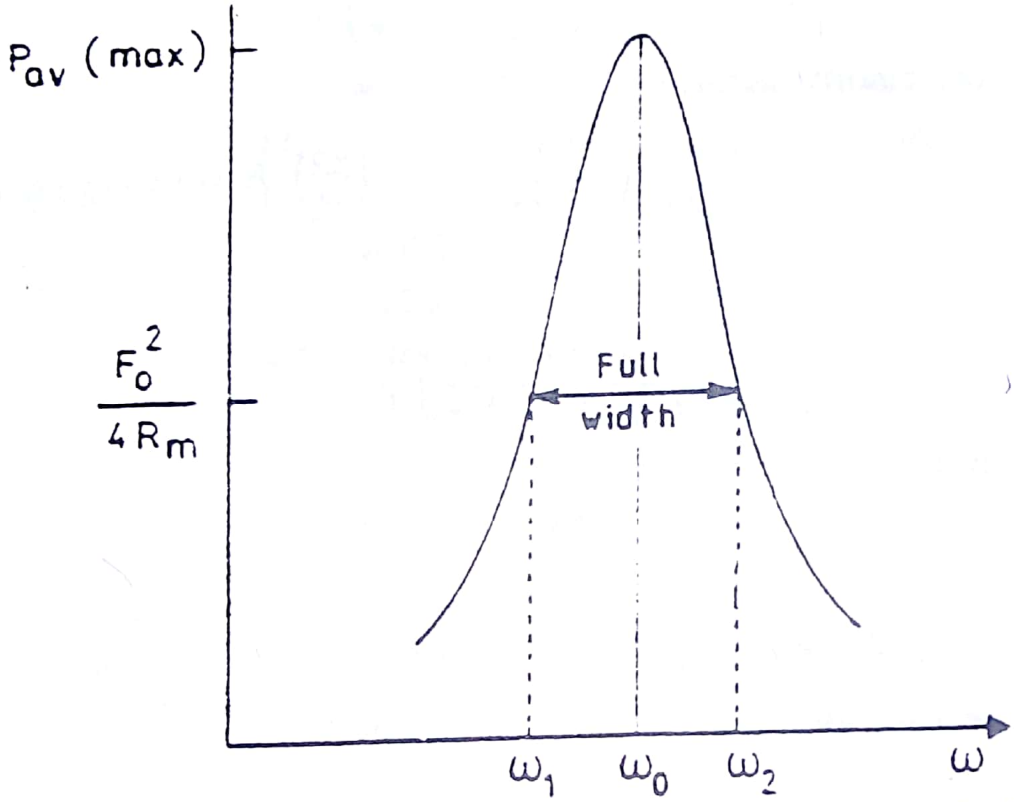


Fig. 4.4 P_{av} versus ω of the driving force $(\omega_2 - \omega_1)$ is the bandwidth of the resonance curve

4.4 QUALITY FACTOR OF A FORCED OSCILLATOR

4.4.1 In Terms of Energy Decay

Analogous to the case of a damped oscillator, the quality factor of a forced oscillator can be defined in terms of energy decay as

$$Q = 2\pi \frac{\text{Average energy stored}}{\text{Energy dissipated per cycle}}$$

$$= 2\pi \frac{E_{av}}{T \times P_{av}}$$

The total energy E at any time

$$= \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}Sx^2$$

$$= \frac{1}{2} m \left(\frac{F_0}{Z_m} \right)^2 \cos^2 (\omega t - \psi) + \frac{1}{2} m \omega_0^2 b^2 \sin^2 (\omega t - \psi)$$

Putting the values,

$$\langle \cos^2 (\omega t - \psi) \rangle = \tfrac{1}{2}$$

$$\langle \sin^2 (\omega t - \psi) \rangle = \tfrac{1}{2}$$

and

$$b = \frac{F_0}{\omega Z_m} \tag{4.10}$$

the average energy

$$E = \frac{1}{4} m \left(\frac{F_0}{Z_m} \right)^2 \left\{ 1 + \left(\frac{\omega_0}{\omega} \right)^2 \right\}$$

Therefore the quality factor,

$$Q = 2\pi \frac{\frac{1}{4} m \left(\frac{F_0}{Z_m} \right)^2 \left\{ 1 + \left(\frac{\omega_0}{\omega} \right)^2 \right\}}{\frac{2\pi}{\omega} \frac{F_0^2 R_m}{2 Z_m^2}}$$

$$= \frac{\omega}{2} \frac{m}{R_m} \left\{ 1 + \left(\frac{\omega_0}{\omega} \right)^2 \right\} \tag{4.21}$$

At resonance $\omega = \omega_0$, therefore,

$$Q = \frac{m \omega_0}{R_m}$$

4.4.2 Q in Terms of Absorption Band-width

The Q of a vibration is also defined by the expression

$$Q = \frac{\omega_0}{\omega_2 - \omega_1} \tag{4.22}$$

where ω_0 is the resonance frequency of the driving force and ω_1 and ω_2 are the two frequencies respectively below and above ω_0 at which the average power drops to one half of the maximum values, Fig. 4.4. The frequency difference $(\omega_2 - \omega_1)$ is the full width of the response curve at half maximum power and is called the bandwidth. Thus

$$Q = \frac{\text{Frequency at resonance}}{\text{Full width at half-maximum power}}$$

Obviously, Q measures the sharpness of tuning and is appropriately called the figure of merit of the oscillator.

Let us try to express the quality factor in terms of the mechanical constants of the vibrating system. From Eqs. (4.19) and (4.20) for P_{av} and $(P_{av})_{max}$,

$$P_{av} = \tfrac{1}{2}(P_{av})_{max}$$

if

$$\frac{R_m^2}{2 Z_m^2} = \frac{1}{4} \tag{4.23}$$

$$Z_m^2 = 2R_m^2$$

or

$$R_m^2 + X_m^2 = 2R_m^2$$

or

$$X_m = \pm R_m$$

or

From Eq. (4.12), we get

$$m\omega - \frac{S}{\omega} = \pm R_m \tag{4.24}$$

By definition $\omega_2 > \omega_1$, therefore the two frequencies ω_1 and ω_2 at which the above condition is satisfied, are

$$m\omega_2 - \frac{S}{\omega_2} = R_m \tag{4.25}$$

and

$$m\omega_1 - \frac{S}{\omega_1} = -R_m \tag{4.26}$$

Eliminating S from Eq. (4.26) with the help of (4.25) one gets

$$m\omega_1 - \frac{\omega_2(m\omega_2 - R_m)}{\omega_1} = -R_m$$

or

$$m\omega_1^2 - m\omega_2^2 + \omega_2 R_m = -R_m\omega_1$$

or

$$m(\omega_1^2 - \omega_2^2) = -R_m(\omega_1 + \omega_2)$$

or

$$\omega_2 - \omega_1 = \frac{R_m}{m} \tag{4.27}$$

Substituting the value of $(\omega_2 - \omega_1)$ into Eq. (4.22), the expression for the quality factor of a mechanical oscillator becomes

$$Q = \frac{m\omega_0}{R_m} \tag{4.28}$$

which is the same as obtained by considering the energy decay.

Analogously, the quality factor Q of an electrical oscillator is defined as

$$Q = \frac{L\omega_0}{R}$$

where ω_0, the natural frequency is

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

The quality factor $Q = \omega_0 \tau$ where τ is the relaxation time. Thus

$$\tau = \frac{m}{R_m} \tag{4.29}$$

4.4.3 Q as Amplification Factor

There is yet another equivalent way of defining Q as the amplification factor of the displacement amplitude.

At low frequencies ($\omega \to 0$), the displacement amplitude b_0, when damping is small becomes,

$$b_0 = \frac{F_0}{S} = \frac{f_0}{\omega_0^2}$$

The displacement at resonance, b_{max}

$$b_{max} = \frac{f_0}{2r\omega_0} \quad (4.30)$$

Thus

$$\frac{b_{max}}{b_0} = \frac{\omega_0}{2r}$$

$$= \frac{m\omega_0}{R_m} \quad (4.31)$$

since

$$2r = \frac{R_m}{m}$$

It is obvious that the resonance displacement is Q times the displacement amplitude at low frequencies.

Ex. 4.1 If the Q value of a driven mechanical oscillator is high show that the width of the displacement reasonance curve is approximate $\sqrt{3}R_m/m$ where the width is measured between those frequencies where $x = x_{max}/2$.

Solution

Since Q of the oscillator is high, R_m is low and so $\omega_m \simeq \omega_0$. The amplitude of forced vibration is given by

$$b = \frac{f_0}{[(\omega_0^2 - \omega^2)^2 + 4r^2\omega^2]^{1/2}}$$

At resonance the driving frequency ω

$= \omega_0$, the natural frequency,

therefore

$$b_{max} = \frac{f_0}{2r\omega_0}$$

If ω_1 is the frequency, at which the amplitude is half of the maximum value at resonance, then

$$\frac{b_{max}}{2} = \frac{f_0}{4r\omega_0}$$

Therefore

$$(\omega_0^2 - \omega_1^2)^2 + 4r^2\omega_1^2 = 16r^2\omega_0^2$$

or

$$(\omega_0 - \omega_1)^2(\omega_0 + \omega_1)^2 + 4r^2\omega_1^2 = 16r^2\omega_0^2$$

Calling $(\omega_0 - \omega_1)$ = Half-width at full maximum, one gets

$$(\omega_0 + \omega_1) \approx 2\omega_0$$

$$4r^2\omega_1^2 \approx 4r^2\omega_0^2,$$

Therefore

$$\text{half-width} = \left[\frac{12r^2\omega_0^2}{4\omega_0^2}\right]^{1/2}$$

$$= [3r^2]^{1/2}$$

Therefore full-width $= 2\sqrt{3}r$

$$= \sqrt{3}\frac{R_m}{m}$$

Ex. 4.2 A simple harmonic oscillator is subjected to a sinusoidal driving force where frequency is altered but amplitude kept constant. It is found that the amplitude of the oscillator increases from 0.02 mm at very low driving frequency to 8.0 mm at a frequency of 200 Hz. Obtain the value for

(i) the quality factor
(ii) the relaxation time
(iii) the full-width of the resonance.

Solution

The amplitude of forced vibrations is

$$b = \frac{f_0}{[(\omega_0^2 - \omega^2)^2 + 4r^2\omega^2]^{1/2}}$$

When the driving frequency is low, i.e. $\omega \ll \omega_0$ and damping is small, one gets

$$b_{low} = \frac{f_0}{\omega_0^2}$$

At resonant frequency $\omega = \omega_0$, the amplitude is maximum and is given by

$$b_{max} = \frac{f_0}{2r\omega_0}$$

Thus $$Q = \frac{b_{max}}{b_{low}} = \frac{\omega_0}{2r} = \omega_0\tau$$

(i) The quality factor $Q = \dfrac{b_{max}}{b_{low}}$

$$= \frac{8.0}{.02} = 400$$

(ii) The relaxation time

$$= \frac{Q}{\omega_0} = \frac{400}{2\pi \times 200}$$

$$= 0.31 \text{ s}$$

(iii) The full-width of the resonance curve

$$2\sqrt{3}r = \frac{\sqrt{3}}{\tau}$$

$$= \frac{\sqrt{3}}{0.31}$$

$$= 5.59 \text{ radians/s}$$

Ex. 4.3 A driven anharmonic oscillator is subject to a driving force $F_0 \cos \omega t$. Solve the equation of motion for displacement in second approximation.

Solution

Consider the case of a mass attached to a spring, and assume that the restoring force of the spring varies nonlinearly with the displacement as

$$S(x)x = Kx \pm \epsilon x^3$$

where K and ϵ are positive constants. The positive sign on the quartic term corresponds to a 'hard' spring and stiffness increases with increasing displacement x. The negative sign corresponds to a 'soft' spring since in this case the stiffness decreases with increasing x. The equation of motion becomes

$$\ddot{x} = -\frac{Kx}{m} \mp \frac{\epsilon}{m} x^3 + \frac{F_0}{m} \cos \omega t$$

Further we make the assumption that $\epsilon \ll K$ in order to use the method of successive approximations to find a particular integral. Let the solution in the first approximation be assumed to be

$$x_1 = A \cos \omega t$$

Substituting it into the equation of motion, the equation for the second approximation is

$$\ddot{x}_2 = \frac{1}{m}(F_0 - KA) \cos \omega t \mp \frac{\epsilon A^3}{m} \cos^3 \omega t$$

Making use of the trigonometric identity

$$\cos^3 \theta = \frac{3}{4} \cos \theta + \frac{1}{4} \cos 3\theta,$$

we rewrite the above equation as

$$\ddot{x}_2 = \frac{1}{m}\left[\left(F_0 - KA \mp \frac{3\epsilon A^3}{4}\right) \cos \omega t \mp \frac{\epsilon A^3}{4} \cos 3\omega t\right]$$

Integrating it twice, we find the solution in the second approximation as

$$x_2 = \frac{1}{m\omega^2}\left[\left(KA - F_0 \pm \frac{3\epsilon A^3}{4}\right) \cos \omega t \pm \frac{\epsilon A^3}{36} \cos 3\omega t\right]$$

This is the valid solution provided ϵ is very small.

4.5 EXAMPLES OF RESONANCE DUE TO FORCED VIBRATIONS

The phenomenon of resonance is quite general and widespread in different branches of physics. Whenever a system is acted upon by an external action, which varies periodically with time, the response of the system as measured by its amplitude and phase, or the power absorbed, undergoes rapid changes, as the frequency of the external field of force passes through a certain range of values. The response is characterised by two parameters—a frequency ω_0

and the natural width of the driven system—and the resonance condition is said to be reached when the interaction between the driven and the driving systems has been maximised. The maximum amplitude occurs at or near ω_0 and the most marked changes occur over a range $\pm\Gamma$ wrt the maximum. It is proposed to extend the concept of resonance to other processes such as nuclear reactions, nuclear magnetic resonance, etc., in which there are favourable conditions for the transfer of energy from one system to another. As such the concept of energy resonance plays an important role in the description of many physical phenomenon. We treat some of them below.

4.5.1 Electrical Resonance

An electrical circuit consisting of an inductance (L) a capacitance (C) and a resistance (R) is connected to an external source of voltage $V_0 \sin \omega t$, Fig. 4.5. The circuit will work as a driven oscillator and the frictional loss which is the resistive loss in this case will be compensated by the supply of energy.

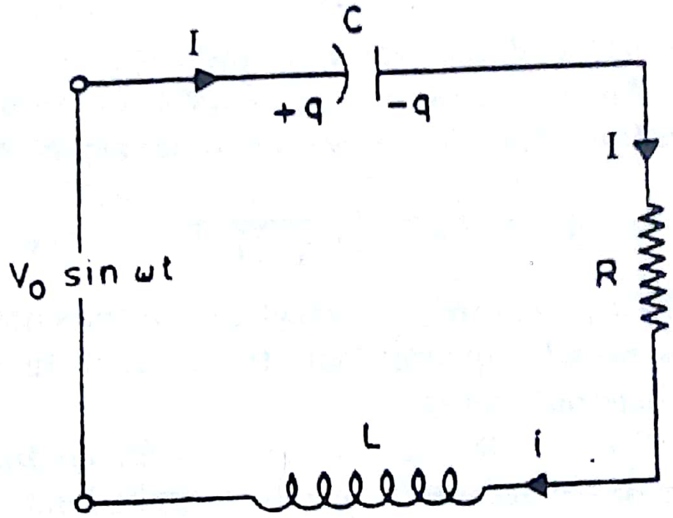


Fig. 4.5 An electrical circuit containing L, C and R in series driven by an external sinusoidal voltage $V_0 \sin \omega t$

If a current I is flowing at any instant, the voltage equation is

$$\frac{q}{C} + IR + L\frac{dI}{dt} = V_0 \sin \omega t.$$

Putting $I = dq/dt$ and dividing throughout by L, one gets

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = \frac{V_0}{L}\sin \omega t \tag{4.12}$$

This equation is identical to the equation of the forced oscillator (4.1), with q replacing x and R/L, $1/LC$ and V_0/L replacing $2r$, ω_0^2 and f_0, respectively. The steady state solution, therefore, out of analogy, is

$$q = \frac{\frac{V_0}{L}}{\sqrt{\left[\left(\frac{1}{LC} - \omega^2\right)^2 + \left(\frac{R\omega}{L}\right)^2\right]}}\sin(\omega t - \theta) \tag{4.33}$$

where $\tan\theta = \dfrac{\omega R/L}{\left(\dfrac{1}{LC} - \omega^2\right)} = \dfrac{\omega R/L}{(\omega_0^2 - \omega^2)} = \dfrac{R}{\left(\dfrac{1}{\omega c} - \omega L\right)}$

The current $I = \dfrac{dq}{dt}$

$$= \frac{V_0}{\sqrt{\left[R^2 + \left(L\omega - \dfrac{1}{C\omega}\right)^2\right]}} \cos(\omega t - \theta)$$

The denominator $\sqrt{R^2 + (L\omega - 1/C\omega)^2}$ which acts as the effective resistance is called the *impedance of the circuit* and its magnitude depends upon ω. Let us discuss the three cases.

(i) When $L\omega = 1/C\omega$ In this case the impedance in the circuit is the minimum and consequently the peak value of the current

$$I_0 = \frac{V_0}{R}$$

Moreover $\theta = \pi/2$ and the current in the circuit is in phase with the applied emf. If the corresponding frequency ω where this condition is attained is called ω_0, then the resonance frequency is given by

$$\frac{\omega}{2\pi} = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\frac{1}{\sqrt{LC}}$$

(ii) When $L\omega > 1/C\omega$, i.e., the net reactance in the circuit is inductive. Then $\tan\theta$, is a negative quantity and the current in the circuit will lag behind the external voltage.

(iii) When $L\omega < 1/C\omega$, i.e., the net reactance in the circuit is capacitative and $\tan\theta$ is a positive quantity. Thus this applied emf will lag behind the current in the circuit.

The variations of the resulting peak current and the phase versus the frequency of the impressed voltage are shown in the Fig. 4.6. The lower the value of R, the higher the value of peak current and sharper the resonance.

The current leads or lags the emf according as the value of ω is smaller or greater than ω_0.

At resonance, the potential differences across the inductance and the capacitor are equal and 180° out of phase and thus cancel out.

The applied emf is to overcome the resistance opposition only. In this case the voltage amplification is given by

$$= \frac{\text{Potential difference across the inductance}}{\text{Applied emf}}$$

$$= \frac{L\omega I}{RI} = \frac{L\omega}{R},$$

which is greater than unity. Thus a resonant circuit is capable of magnifying the impressed voltage. This consequence is of considerable importance

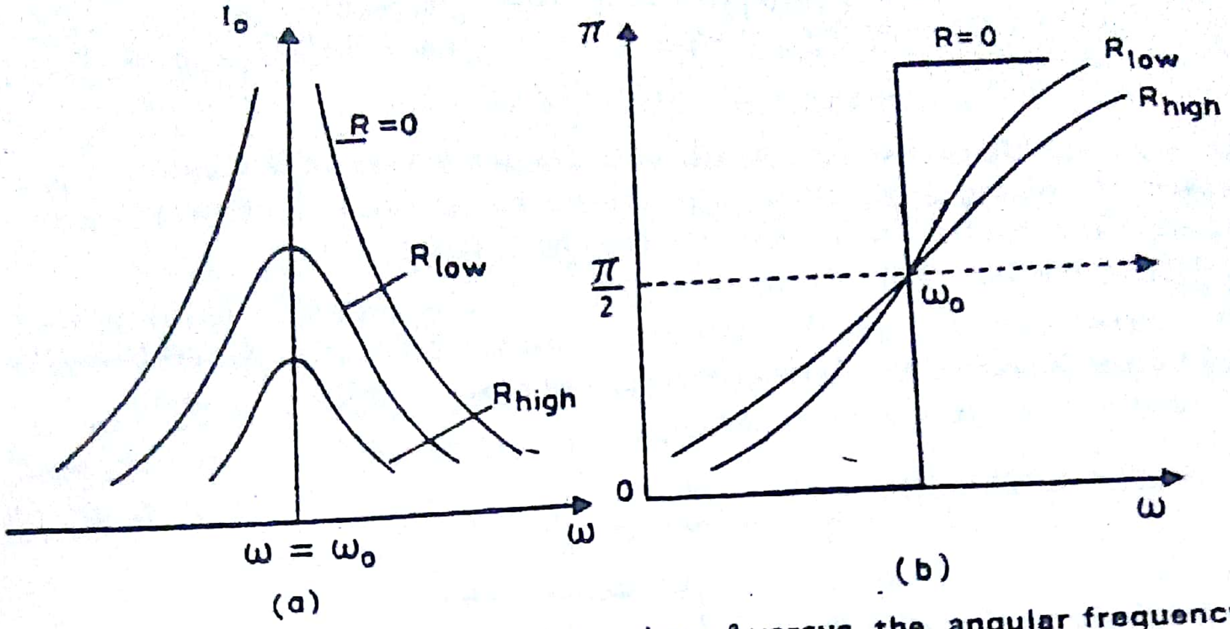


Fig. 4.6 The peak current I_0 and the phase θ versus the angular frequency ω of the applied emf

in radio reception which when tuned to a resonant frequency, will provide amplified voltage across the inductance or the capacitance.

Extending the analogy of the *LCR* circuit with the mechanical oscillator, one can define the quality factor Q for it. Thus from Eq. (4.21) one gets

$$Q = \frac{\omega}{2}\frac{L}{R}\left(1 + \left(\frac{\omega_0}{\omega}\right)^2\right) \tag{4.34}$$

which at resonance, $\omega = \omega_0$ becomes

$$Q = \frac{L\omega_0}{R}$$

Ex. 4.4 Find the natural frequency of a circuit containing inductance of 100 μH and a capacity of 0.001 μF. To which wavelength, its response will be maximum.

Solution

The natural frequency

$$= \frac{1}{2\pi\sqrt{LC}}$$

$$= \frac{1}{2\pi[100\times 10^{-6}\times 0.001\times 10^{-6}]^{1/2}}$$

$$= \frac{10^7}{2\pi\sqrt{10}}$$

$$= 5.033\times 10^5 \text{ Hz}$$

The wavelength corresponding to the natural frequency

$$= \frac{c}{\nu}$$

$$= \frac{3 \times 10^8}{5.035 \times 10^5}$$

$$= 595.5 \text{ m}$$

Ex. 4.5 An alternative emf $E_0 \sin \omega t$ is applied across an inductance L and capacity C, placed in parallel. Calculate the current at any instant. Deduce the condition under which electric resonance occurs.

Solution

Let current I flow into the junction and let I_1 be the current in the inductance L and I_2, the current in the condenser C, Fig. 4.7, then

$$I = I_1 + I_2$$

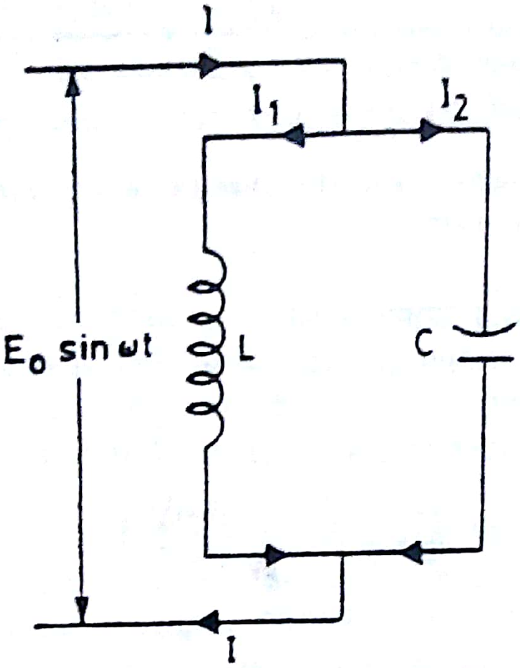


Fig. 4.7 Parallel resonant circuit

The whole of the applied emf acts across the inductance and we get

$$L \frac{dI_1}{dt} = E_0 \sin \omega t$$

Thus
$$I_1 = \frac{E_0}{L} \int \sin \omega t \, dt$$

$$= -\frac{E_0}{\omega L} \cos \omega t$$

The charge on the condenser is

$$q = CE_0 \sin \omega t$$

and
$$I_2 = \frac{dq}{dt} = \omega C E_0 \cos \omega t$$

Hence
$$I = I_1 + I_2$$

$$= (\omega C - 1/2L) E_0 \cos \omega t$$

When the frequency of the applied emf becomes equal to the natural frequency of the parallel circuit, the voltage across C is equal and opposite to the potential difference across the inductance and the current in the circuit reduces to zero. Thus

$$\omega C = \frac{1}{\omega L}$$

or

$$f = \frac{1}{2\pi}\frac{1}{\sqrt{LC}}$$

Such a circuit is called parallel resonant circuit. At resonance, the current in it becomes zero and the impedance infinite.

Ex. 4.6 Show that when a coil of inductance L and resistance R is attached to an external source of voltage, $E_0 \sin \omega t$, the average rate of consumption of energy is

$$\frac{\frac{1}{2}E_0^2 R}{(R^2 + L^2\omega^2)}$$

Solution

The current flowing through the circuit is

$$I = \frac{E_0}{(R^2 + L^2\omega^2)} \sin(\omega t - \varphi)$$

where

$$\varphi = \tan^{-}\left(\frac{R}{L\omega}\right)$$

The energy will be dissipated only in the resistance component, the average rate of energy consumption is

$$= \frac{\int_0^{2\pi/\omega} I^2 R\, dt}{\int_0^{2\pi/\omega} dt}$$

$$= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \frac{E_0^2 R}{R^2 + L^2\omega^2} \sin^2(\omega t - \phi)\, dt$$

$$= \frac{E_0^2 R\omega}{2\pi(R^2 + L^2\omega^2)} \frac{1}{2}\int_0^{2\pi/\omega} [1 - \cos 2(\omega t - \varphi)]\, dt$$

$$= \frac{E_0^2 R}{2(R^2 + L^2\omega^2)}$$

4.5.2 Optical Resonance

For a gas at low pressure, where the atoms are effectively isolated from each other, the spectrum consists of a set of discrete lines and it is assumed that atoms behave like tuned oscillators while emitting or absorbing radiation. However, a solid emits a continuum spectrum, since each atom is coupled strongly to its near neighbours. The coupling causes a change in the dynamical state of the electron chiefly emitting radiation in the visible or the near visible part of the radiation.

The electromagnetic radiation carries energy, linear and angular momenta and when the radiation reaction is included in the equation of motion of the emitter, the atom behaves as a damped simple harmonic oscillator. The excited atom emits the radiation when it undergoes a quantum jump from one state to another, the radiated energy being given by $E = h\nu$, where ν is the frequency of radiation. The emitted radiation on examination with an interferometer, is found to consist of wave trains of finite extension. The excited atoms decay according to the law $e^{-t/\tau}$ where τ is its lifetime. It is given by the length of the wave train divided by c, the velocity of light.

Fraunhofer observed experimentally for the first time the existence of dark lines in the solar spectrum and these constitute the most famous example of optical-resonance absorption. The dark lines at 5890 and 5896 Å are the absorption lines originating from sodium vapour, due to resonance absorption. The continuous radiation from the hot and dense matter near the sun's surface, is filtered selectively by the tenuous vapours of the solar atmosphere. The study of the spectral quality of the solar radiation serves as a clue to the constitution of steller matter.

4.5.3 Nuclear Resonance

The results of an experimental study of a nuclear radiation, are usually expressed either in the number of processes which take place under the conditions of the experiment or in terms of a cross-section for the reaction. The second alternative of expressing in terms of cross-section has the advantage since its value is independent of the flux of the incident particles and the density of the material used for target. The knowledge about the cross section is valuable in many practical applications such as artificial production of radionuclides. The subject of nuclear reactions abounds in a particularly important case of nuclear resonances. The inelastic scattering of protons by aluminium shows resonances and gives information about excited states of the compound Si^{28}. The reaction is described by the equation

$$_{13}Al^{27} + {_1H^1} \rightarrow [_{14}Si^{28}] \rightarrow {_{13}Al^{27*}} + {_1H^1}$$

$$_{13}Al^{27*} \rightarrow {_{13}Al^{27}} + \gamma$$

The asterisk shows the excited state of Al^{27}. Here the resonant system is the compound nucleus $_{14}Si^{28}$ and the variable of interaction is the energy of the projectile, $_1H^1$. When the total of the binding and the kinetic energies of the proton, takes the compound nucleus to one of its excited states, resonance is said to occur and the probability of capture of the proton is maximum at that energy. The response of the system is measured not in terms of the absorbed power but in terms of the probability of emission of γ-ray. The cross section in the neighbourhood of a single resonance level formed by the incident particle with zero angular momentum is given by Breit–Wigner formula (without proof)

$$\sigma(x, y) = \frac{\lambda^2}{4\pi} \frac{\Gamma_x \Gamma_y}{(E - E_0)^2 + \left(\frac{\Gamma}{2}\right)^2} \tag{4.36}$$

where λ is the de Broglie wavelength of the incident particle, E is the energy, E_0 is the energy at the peak of resonance and Γ, are the total widths at half maximum, Fig. 4.8.

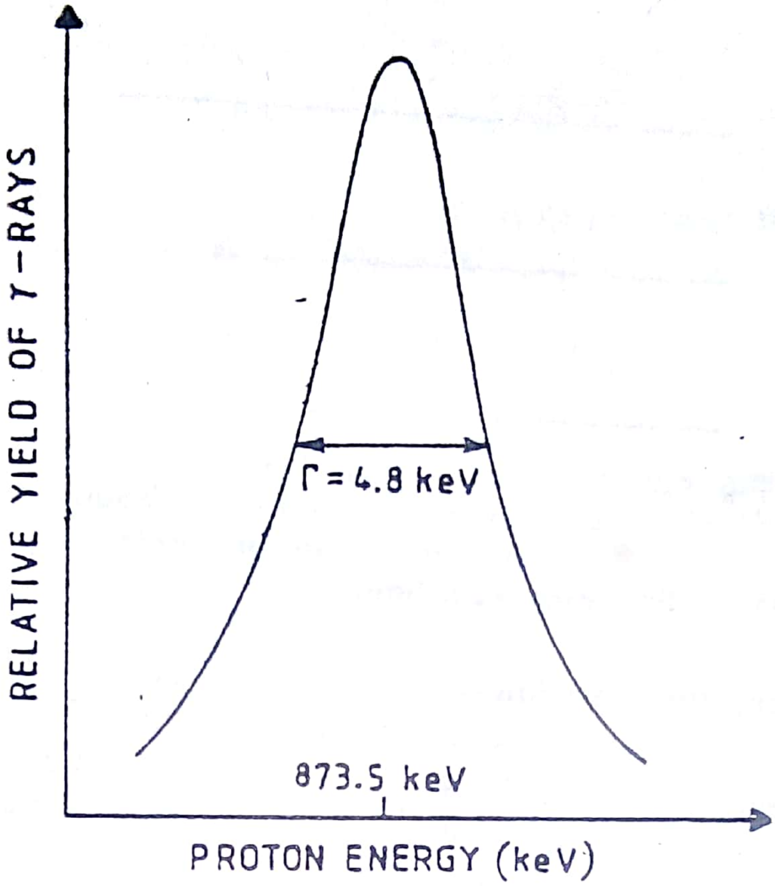


Fig. 4.8 Relative yield of γ-rays as a function of the proton energy in the $Al^{27}(p, \gamma)\,Si^{28}$ reaction

It has been shown that Al^{27} has 20 excited levels at energies between 0.84 MeV and 5.74 MeV above the ground state.

4.5.4 Nuclear Magnetic Resonance

This is a resonant process in which atomic nuclei behaving like tiny magnets, flip from one orientation to another on the absorption of electromagnetic radiation of a particular energy. An atomic nucleus is a quantized system and has a set of possible energy states when placed in an external magnetic field. The energy separations between these levels are well-defined and are given by the work done against the magnetic force in turning the magnet from one position to another. The energy of interaction of a magnetic moment $\boldsymbol{\mu}$ in an external field $\mathbf{B}$ is $-\boldsymbol{\mu}.\mathbf{B}$. For the sake of simplicity, considering the magnets to be independent and noninteracting, the magnetic moment is proportional to the total angular momentum J of the ion, atom or molecule. A dilute sample placed in a uniform field will have for each of its constituent paramagnetic ions, a system of $2J + 1$, evenly spaced levels, Fig. 4.9. Nuclear magnetic resonance is the phenomenon of inducing transitions among the $(2J + 1)$ magnetically split levels.

The flip from one orientation to another is caused by the absorption of electromagnetic radiation of a particular frequency. For the case of protons

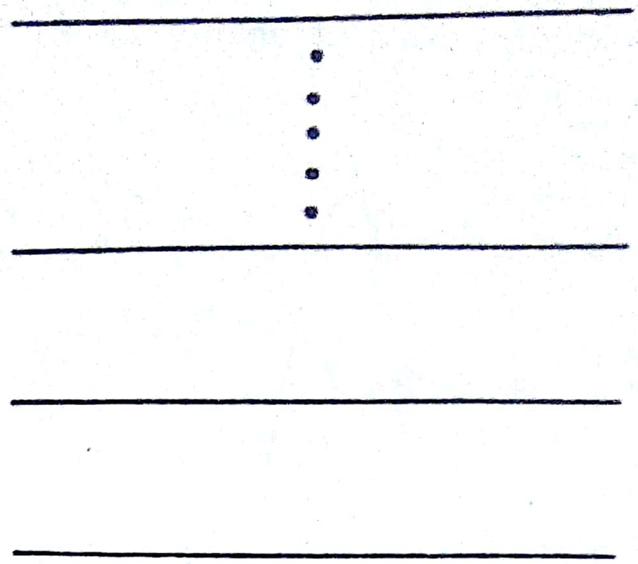


Fig. 4.9 (2J + 1) energy levels of a free paramagnetic ion of angular momentum J in an external magnetic field

in a field of 5 kg, the resonant frequency is about 21 MHz. If all the protons in the volume of the sample are made to flip, they will give rise to a detectable voltage signal in the pick-up coil. To achieve the resonance conditions, two options get recommended themselves. If the magnetic field is kept fixed, the electromagnetic radiation is scanned over a certain frequency range, till the required resonance condition is met. But what is normally done, is to keep the frequency fixed and vary the magnetic field

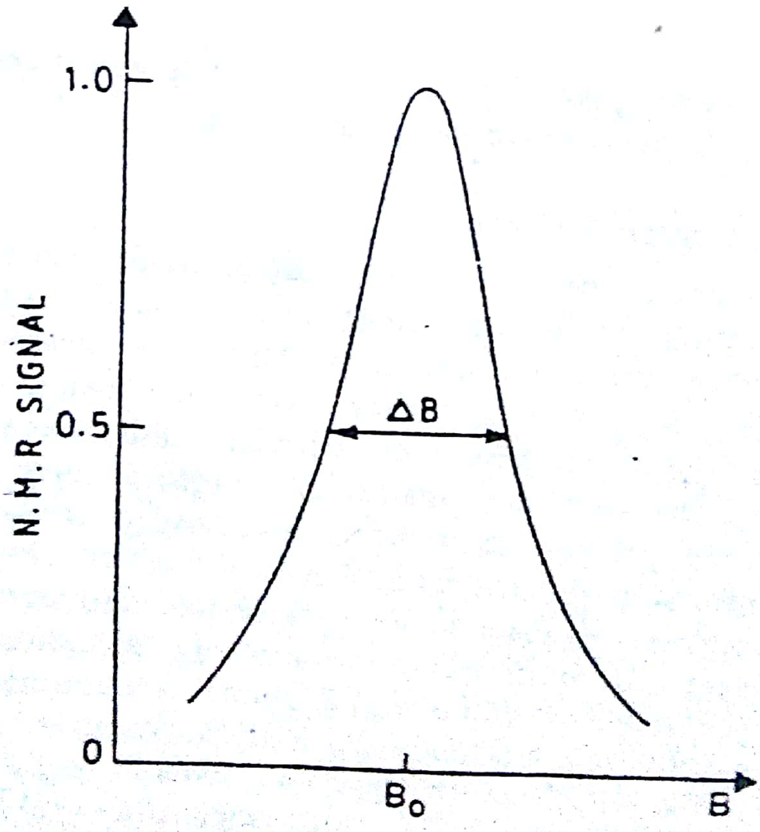


Fig. 4.10 NMR Signal of protons in water. The addition of a paramagnetic salt like $MnSO_4$ to the water produces a concentration of protons

till the condition of resonance absorption is reached. The magnetic resonance signal as a function of the magnetic field is expressed as

$$V(B) = \frac{V_0(\Delta B)^2}{(B - B_0)^2 + \left(\frac{\Delta B}{2}\right)^2} \tag{4.37}$$

where B_0 is the value of magnetic field at resonance and ΔB is the width of the resonance at half-maximum, Fig. 4.10.

MNZ CLASS LECTURE (9-12-2017)

8.12. Uses of Laue's Patterns

X-ray diffraction patterns are widely used for the following purposes :

(i) in determining the microscope structure of matter in solid state, as for example, in metallurgy for the study of mechanical processes of rolling, hardening and annealing.

(ii) in locating as well as determining angular relationship between different planes in a crystal lattice which are rich in atoms.

8.13. Bragg's X-ray Spectrometer

Soon after Laue's experiments, further study of X-ray diffraction by crystals was taken up by British father-and-son team of Sir W.H. Bragg and Sir Lawrence Bragg. For this purpose, they devised an X-ray spectrometer in which a crystal is used not as *transmission grating* (as in Laue's experiment) but as a *reflection grating*.

A schematic diagram of Bragg's X-ray spectrometer is shown in Fig. 8.19 whereas Fig. 8.20 gives a more detailed view. In Fig. 8.19, M represents collimator and IC stands for ionization chamber. As shown in Fig. 8.20, X-rays from a tube are collimated into a narrow beam by two fine slits S_1 and S_2 cut in two lead plates L_1 and L_2. Next, the beam is incident at a glancing angle of θ on the face of a crystal plate (of calcite, rock salt, NaCl or KCl etc.) which is mounted on the turntable T of the spectrometer. This turntable is capable of rotation about a vertical axis passing through its centre and its rotation can be read from the circular scale S.

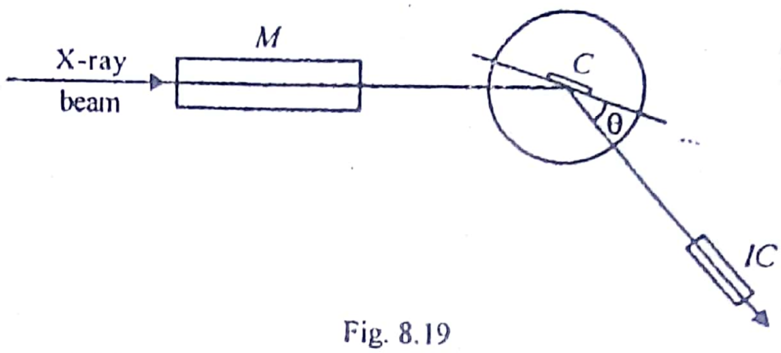


Fig. 8.19

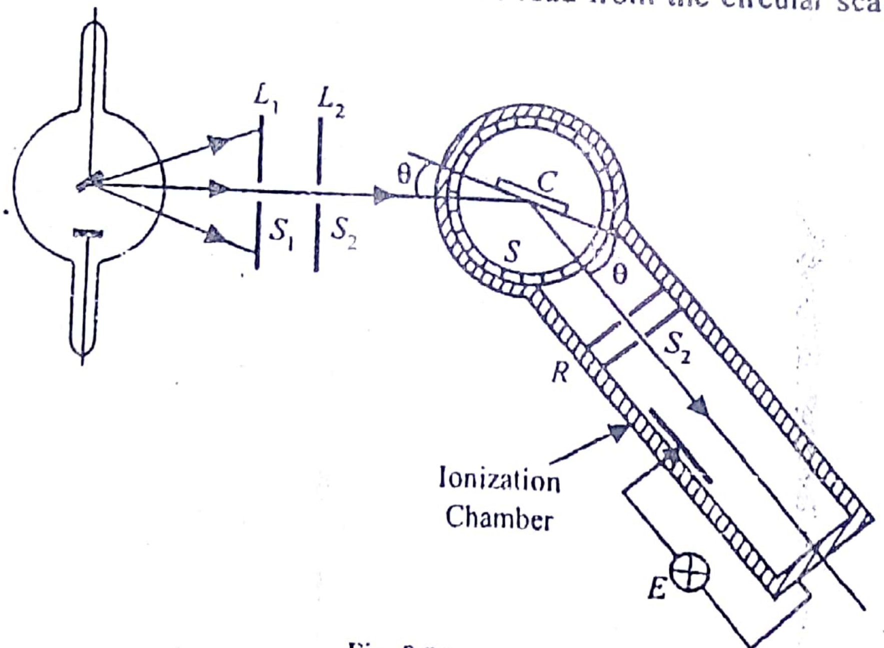


Fig. 8.20

Most of the incident beam passes straight through the crystal plate but some X-rays are scattered by the regularly-arranged atoms lying in different crystal planes. These scattered X-rays can be looked upon as having been reflected from the crystal planes particularly those which are rich in atoms (Art. 4.15). The reflected X-ray beam enters an ionization chamber carried by an arm R which is capable of rotation about the same axis as the turntable.

The turntable and arm R are so linked together that when the turntable (and hence the crystal) rotates through an angle θ, the arm R (and hence the ionization chamber) turns through double the angle, *i.e.*, 2θ. In this, way the beam is always reflected into the ionization chamber whatever its incident or glancing angle at the crystal surface. The ionization current produced by the reflected X-ray beam can be measured by a sensitive electrometer E (or the reflected beam can be recorded on a photographic film in which case the spectrometer is known as spectrograph).

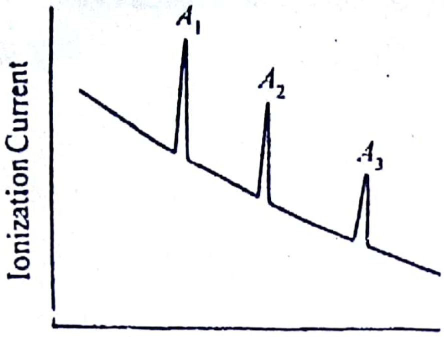


Fig. 8.21

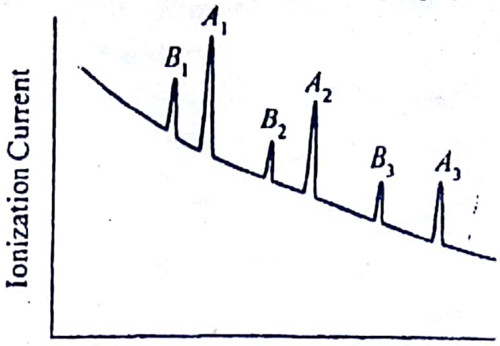


Fig. 8.22

By plotting the electrometer readings corresponding to different glancing angles, it is possible to delineate the X-ray diffraction spectrum. Fig. 8.21 shows such a spectrum when X-ray beam is monochromatic whereas Fig. 8.22 represents the case when the X-ray beam consists of two wavelengths λ_1 and λ_2.

8.14. Theory of Bragg's Diffraction

When a beam of monochromatic X-rays falls on a crystal, it is scattered by the individual atoms which are arranged in sets of parallel layers. Each atom becomes a source of scattered radiations. As already discussed in Art. 4.15, in every crystal, there are certain sets of planes which are particularly rich in atoms. The combined scattering of X-rays from these planes can be looked upon as *reflections* from these *planes*. Because, of this, Bragg *scattering* is usually referred to as Bragg *reflection* and these planes are known as *Bragg planes*. It is dues to the presence of such sets of parallel planes that a crystal acts as reflection grating.

For certain glancing angles, reflections from these sets of parallel planes are in phase with each other. Hence, different reflected X-rays reinforce each other so that the resulting reflection is exceptionally strong. Consequently, ionization current produced is also large. For some other values of θ, the X-ray reflection from different planes are antiphase with each other so that the resulting reflection is either zero or extremely feeble.

Obviously, as the angle of incidence or glancing angle is changed by rotating the turntable T of the spectrometer, a series of alternate maxima and minima of intensity are obtained.

8.15. Bragg's Law

Fig. 8.23 gives a 3-dimensional view of how a beam of monochromatic X-rays undergoes Bragg's reflection from different planes in a NaCl crystal. Fig. 8.24 gives a 2-dimensional view of the same diagram. It shows a beam of monochromatic X-rays incident at a glancing angle θ* on a set of parallel planes of NaCl crystal. The beam is successive layers rich in atoms. Ray No. 1 is *reflected* from atom A in plane 1 whereas Ray No. 2 is *reflected* from atom B lying in plane 2 immediately below atom A. Whether two reflected rays will be in phase or antiphase with each other will depend on their path difference. This path difference can

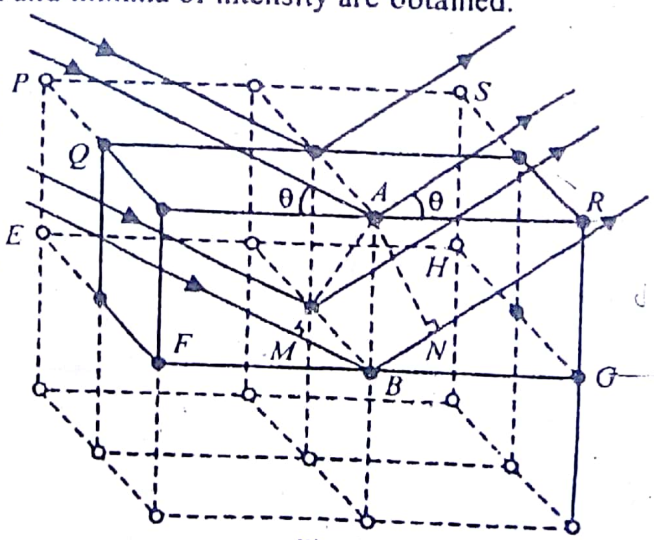


Fig. 8.23

* It may be noted that in Optics, it is usual to measure the angles of incidence and reflection between the rays and the normal to the surface. However, the Bragg angle is measured between the ray and crystal plane.

be found by drawing perpendiculars AM and AN on ray No. 2. Since the two rays travel the same distance from points A and N onwards, it is obvious that ray No. 2 travels an extra distance

$$= MB + BN$$

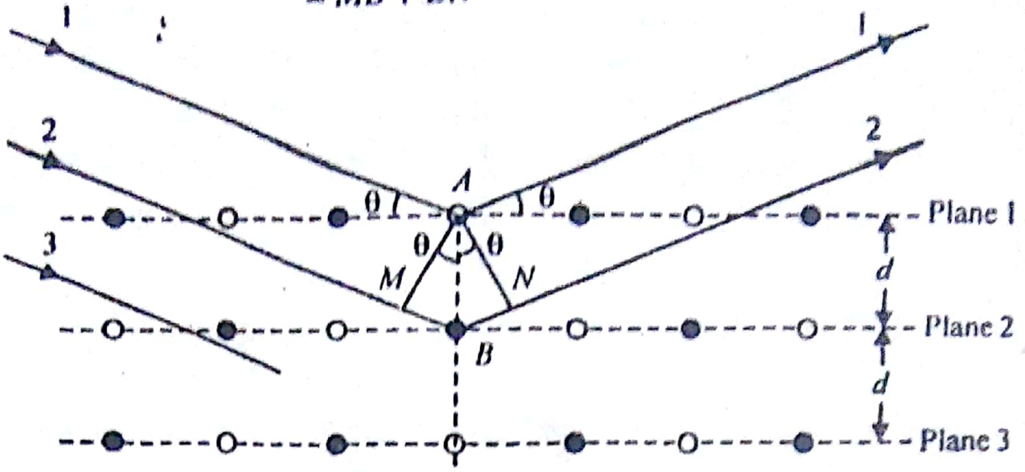


Fig. 8.24

Hence, the path difference between the two reflected beams is

$$= MB + BN = d\sin\theta + d\sin\theta = 2d\sin\theta$$

where d is the interplanar spacing, *i.e.*, vertical distance between two adjacent planes belonging to the same set.

The two reflected beams will be in phase with each other* if this path difference equals an integral multiple of λ and will be antiphase if it equals an odd multiple of $\lambda/2$.

Hence, the condition for producing maxima becomes

$$\boxed{2d\sin\theta = n\lambda}$$

where n = 1, 2, 3, etc. for the first-order, second-order and third-order maxima respectively. This relation is known as Bragg's Law.

Different directions in which intense reflections will be produced can be found by giving different values of θ in the above equation.

For the first maxima, $\sin\theta_1 = \lambda/2d$

For the second maxima, $\sin\theta_2 = 2\lambda/2d$

For the third maxima, $\sin\theta_3 = 3\lambda/2d$ etc.

As seen from Fig. 8.22, the intensities of maxima go on decreasing as the order of the spectrum increases. Moreover, it will be seen from Fig. 8.22 that the separation between maxima for λ_1 and λ_2 goes on increasing as higher orders are considered.

Bragg's Law may be used to find the wavelength λ of the X-rays if interplanar spacing d is known. Conversely, d may be computed if λ of X-rays is known from some other experiment. In

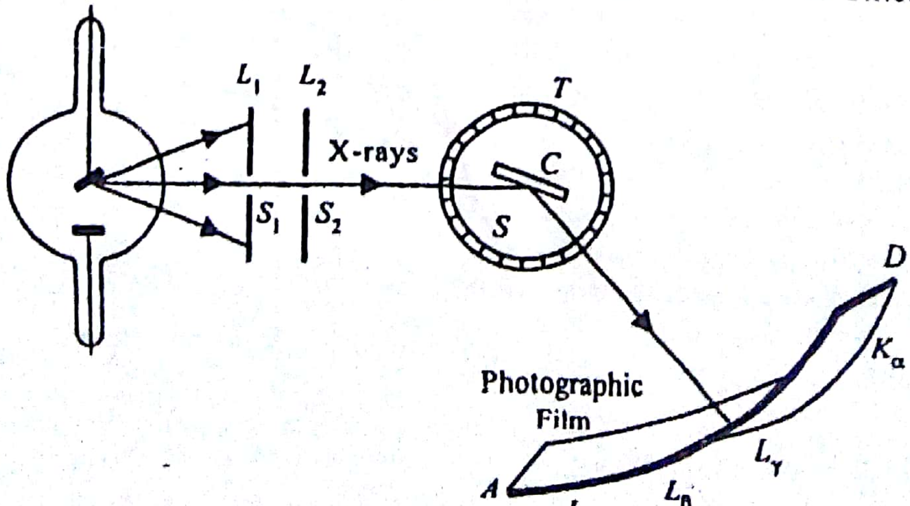


Fig. 8.25

*Same condition will apply to other parallel rays like ray No. 3 shown in the figure.

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modern X-ray spectrometers, the ionization chamber is replaced by a photographic plate set at right angles to the reflected beam as shown in Fig. 8.25. Such an instrument is known as X-ray spectrograph and takes much less time for obtaining the complete spectrum.

If we substitute the value of d in terms of the Miller indices of the planes for a cubic system, we get

$$2\frac{a}{\sqrt{h^2+k^2+l^2}}\sin\theta = n\lambda$$

or

$$\boxed{\sin^2\theta = \frac{\lambda^2}{4a^2}(h^2+k^2+l^2)} \qquad -n=1$$

8.16. Bragg's Law and Crystal Structure

Bragg used a KCl crystal (which is cubic) and found first maxima of reflected X-rays to occur at values of θ equal to 5·22°, 7·30° and 9·05° respectively when the three planes *BFGC*, *AFGD* and *AFH* (Fig. 4.26) are used in turn as reflecting planes.

Now, for first-order spectrum,

$$2d\sin\theta = \lambda \qquad (\because n=1)$$

$$\frac{1}{d} = \frac{2\sin\theta}{\lambda}$$

$$\therefore \quad \frac{1}{d_1}:\frac{1}{d_2}:\frac{1}{d_3} :: \sin 5{\cdot}22° : \sin 7{\cdot}30° : \sin 9{\cdot}05°$$

$$:: 0{\cdot}091 : 0{\cdot}127 : 0{\cdot}157$$

$$:: 1 : 1{\cdot}4 : 1{\cdot}73$$

These figures agree remarkably well with those derived geometrically in Art. 4.15 which confirms that KCl crystal has a simple cubic crystal structure.

Suppose in the case of some crystal, we get

$$\frac{1}{d_1}:\frac{1}{d_2}:\frac{1}{d_3} :: 1:\frac{1}{\sqrt{2}}:\sqrt{3}$$

then we are sure that is has a body-centred type structure (Art. 4.15). In this way, Bragg's Law can be utilized to analyse different types of crystal structures.

Example 8.10. *X-rays from a tube undergo first-order reflection at a glancing angle of 12° from the face of a calcite crystal. The grating space of the calcite is $3{\cdot}04 \times 10^8$ cm. Calculate the wavelength of the X-rays. At what angle will be third-order reflection take place from the crystal ?*

Solution. Here, $\theta = 12°$, $n = 1$, $d = 3{\cdot}04 \times 10^{-10}$ m

$$\therefore \quad 2 \times 3{\cdot}04 \times 10^{-10} \sin 12° = 1 \times \lambda$$

$$\therefore \quad \lambda = 0{\cdot}79 \times 10^{-10}\,\text{m} = \mathbf{0{\cdot}79\ Å}$$

For third-order reflection,

$$2 \times 3{\cdot}04 \times 10^{-10} \sin\theta_3 = 3 \times 0{\cdot}79 \times 10^{-10}$$

$$\therefore \quad \sin\theta_3 = 0{\cdot}3898;\ \theta_3 = 22{\cdot}9°.$$