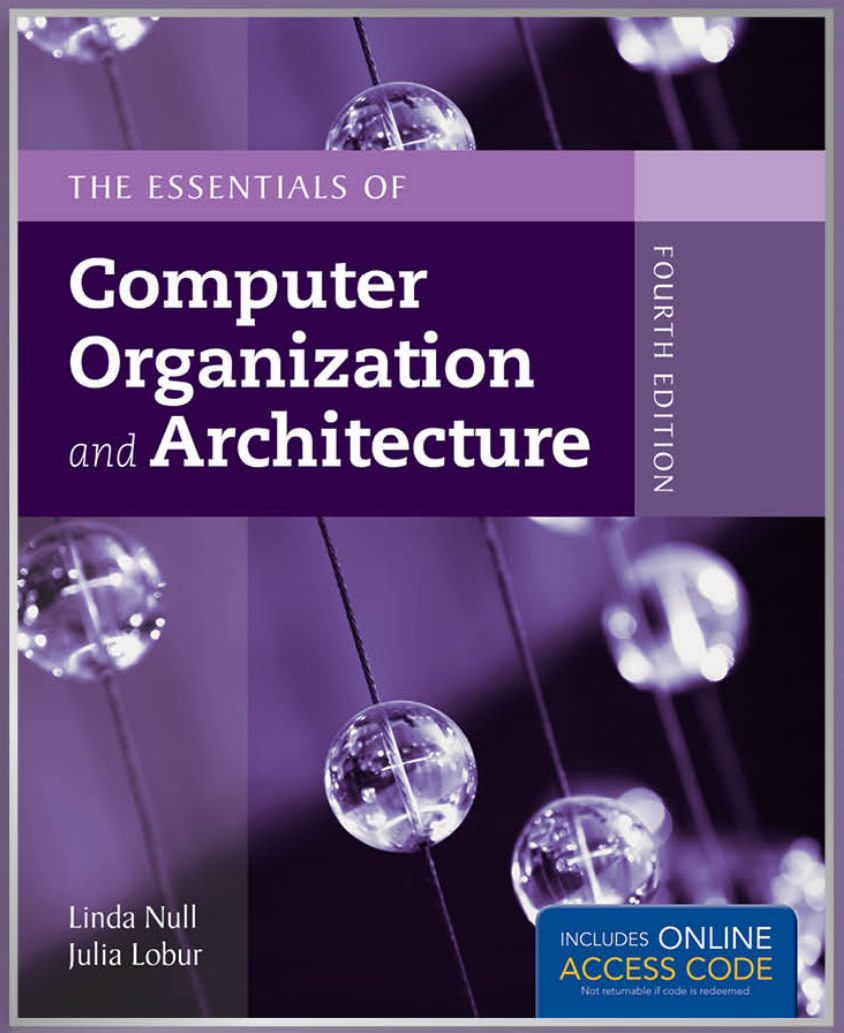
Chapter 2

Data Representation in Computer Systems



2.3 Decimal to Binary Conversions

- Fractional values can be approximated in all base systems.
- Unlike integer values, fractions do not necessarily have exact representations under all radices.
- The quantity ½ is exactly representable in the binary and decimal systems, but is not in the ternary (base 3) numbering system.
- The quantity 0.1 is exactly representable in the decimal system, but is not in the binary numbering system.

2.3 Decimal to Binary Conversions

- Fractional decimal values have nonzero digits to the right of the *decimal point*.
- Fractional values of other radix systems have nonzero digits to the right of the *radix point*.
- Numerals to the right of a radix point represent negative powers of the radix:

$$0.47_{10} = 4 \times 10^{-1} + 7 \times 10^{-2}$$

$$0.11_2 = 1 \times 2^{-1} + 1 \times 2^{-2}$$

$$= \frac{1}{2} + \frac{1}{4}$$

$$= 0.5 + 0.25 = 0.75$$

2.3 Decimal to Binary Conversions

- As with whole-number conversions, you can use either of two methods: a subtraction method and an easy multiplication method.
- The subtraction method for fractions is identical to the subtraction method for whole numbers. Instead of subtracting positive powers of the target radix, we subtract negative powers of the radix.
- We always start with the largest value first, r^{-1}, where r is our radix, and work our way along using larger negative exponents.

2.3 Decimal to Binary Conversions

- **The calculation to the right is an example of using the subtraction method to convert the decimal 0.8125 to binary.**
  - Our result, reading from top to bottom is:

$$0.8125_{10} = 0.1101_2$$

  - Of course, this method works with any base, not just binary.

$$\begin{array}{rll} 0.8125 & & \\ -\ 0.5000 & = 2^{-1} \times & 1 \\ \hline 0.3125 & & \\ -\ 0.2500 & = 2^{-2} \times & 1 \\ \hline 0.0625 & & \\ -\ 0 & = 2^{-3} \times & 0 \\ \hline 0.0625 & & \\ -\ 0.0625 & = 2^{-4} \times & 1 \\ \hline 0 & & \end{array}$$

2.3 Decimal to Binary Conversions

- **Using the multiplication method to convert the decimal 0.8125 to binary, we multiply by the radix 2.**
  - The first product carries into the units place.

```
  .8125
×    2
------
1.6250
```

2.3 Decimal to Binary Conversions

- **Converting 0.8125 to binary . . .**
  - Ignoring the value in the units place at each step, continue multiplying each fractional part by the radix.

```
  .8125
  ×   2
------
1.6250

  .6250
  ×   2
------
1.2500

  .2500
  ×   2
------
0.5000
```

2.3 Decimal to Binary Conversions

- **Converting 0.8125 to binary . . .**
  - You are finished when the product is zero, or until you have reached the desired number of binary places.
  - Our result, reading from top to bottom is:

    $0.8125_{10} = 0.1101_2$
  - This method also works with any base. Just use the target radix as the multiplier.

```
  .8125
  × 2
1.6250

  .6250
  × 2
1.2500

  .2500
  × 2
0.5000

  .5000
  × 2
1.0000
```

2.5 Floating-Point Representation

- The signed magnitude, one's complement, and two's complement representation that we have just presented deal with integer values only.
- Without modification, these formats are not useful in scientific or business applications that deal with real number values.
- Floating-point representation solves this problem.

2.5 Floating-Point Representation

- If we are clever programmers, we can perform floating-point calculations using any integer format.
- This is called *floating-point emulation*, because floating point values aren't stored as such, we just create programs that make it seem as if floating-point values are being used.
- Most of today's computers are equipped with specialized hardware that performs floating-point arithmetic with no special programming required.

2.5 Floating-Point Representation

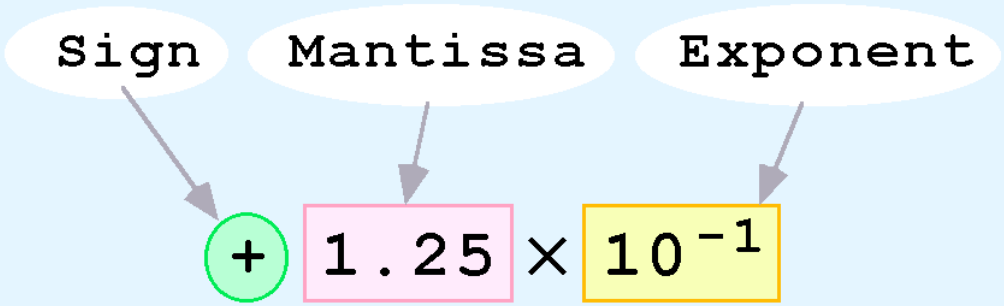
- Floating-point numbers allow an arbitrary number of decimal places to the right of the decimal point.
  - For example: 0.5 × 0.25 = 0.125
- They are often expressed in scientific notation.
  - For example:

    $0.125 = 1.25 \times 10^{-1}$

    $5{,}000{,}000 = 5.0 \times 10^{6}$

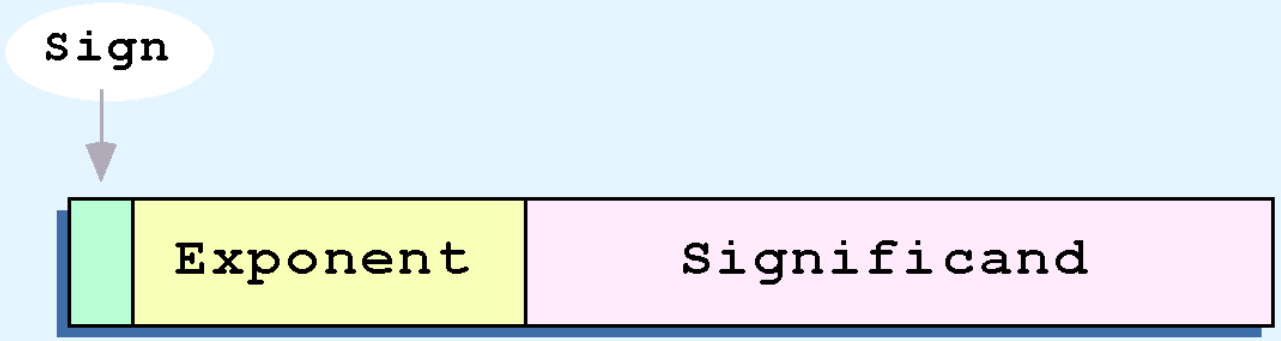
2.5 Floating-Point Representation

- Computers use a form of scientific notation for floating-point representation
- Numbers written in scientific notation have three components:



2.5 Floating-Point Representation

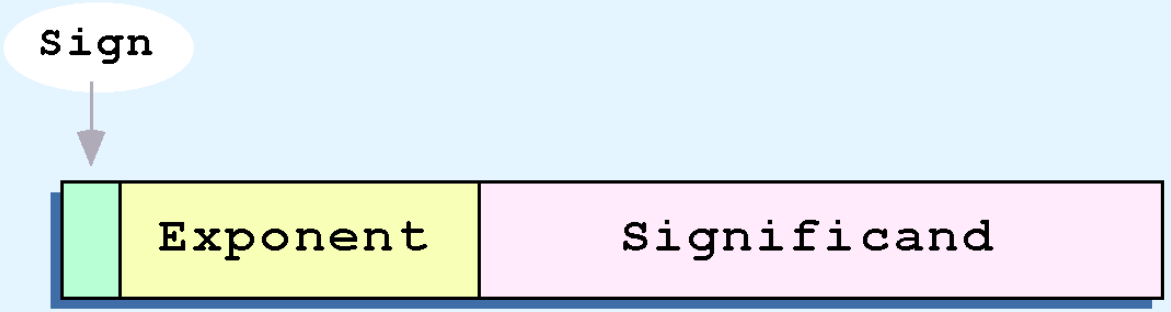
- Computer representation of a floating-point number consists of three fixed-size fields:



- This is the standard arrangement of these fields.

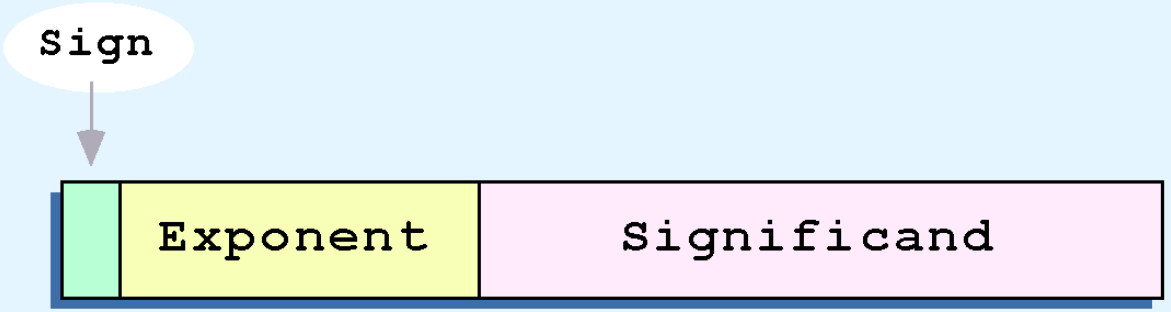
Note: Although "significand" and "mantissa" do not technically mean the same thing, many people use these terms interchangeably. We use the term "significand" to refer to the fractional part of a floating point number.

2.5 Floating-Point Representation



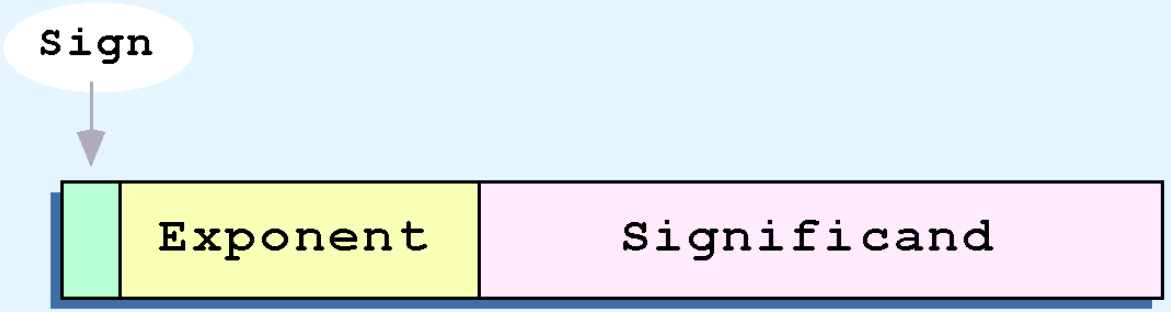
- The one-bit sign field is the sign of the stored value.
- The size of the exponent field, determines the range of values that can be represented.
- The size of the significand determines the precision of the representation.

2.5 Floating-Point Representation



- We introduce a hypothetical “Simple Model” to explain the concepts
- In this model:
  - A floating-point number is 14 bits in length
  - The exponent field is 5 bits
  - The significand field is 8 bits

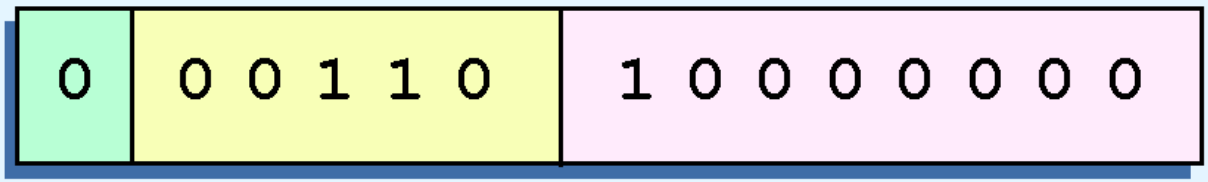
2.5 Floating-Point Representation



- The significand of a floating-point number is always preceded by an implied binary point.
- Thus, the significand always contains a fractional binary value.
- The exponent indicates the power of 2 to which the significand is raised.

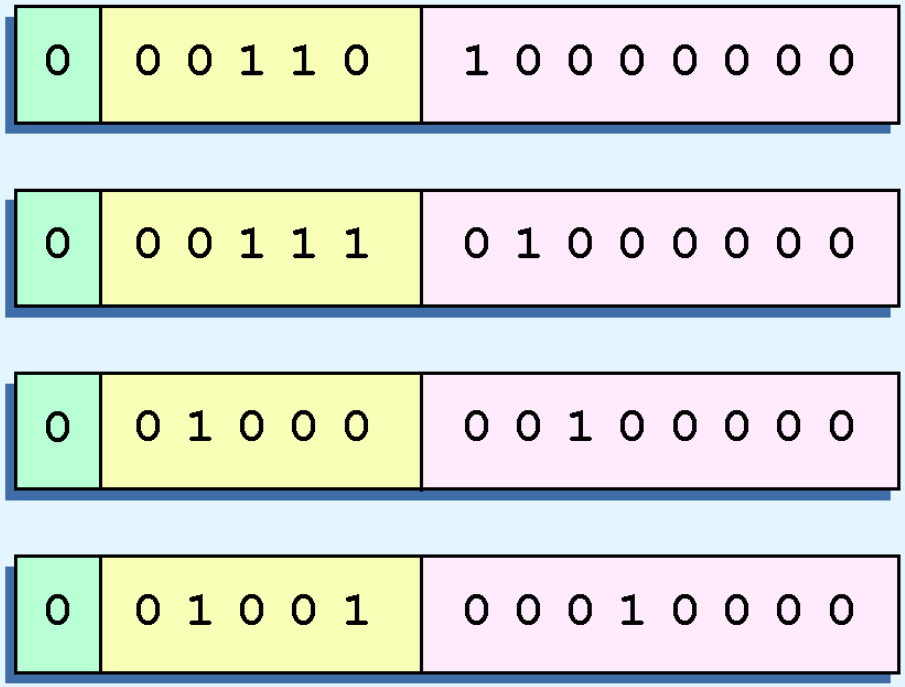
2.5 Floating-Point Representation

- Example:
  - Express 32_{10} in the simplified 14-bit floating-point model.
- We know that 32 is 2^5. So in (binary) scientific notation $32 = 1.0 \times 2^5 = 0.1 \times 2^6$.
- Using this information, we put 110 ($= 6_{10}$) in the exponent field and 1 in the significand as shown.

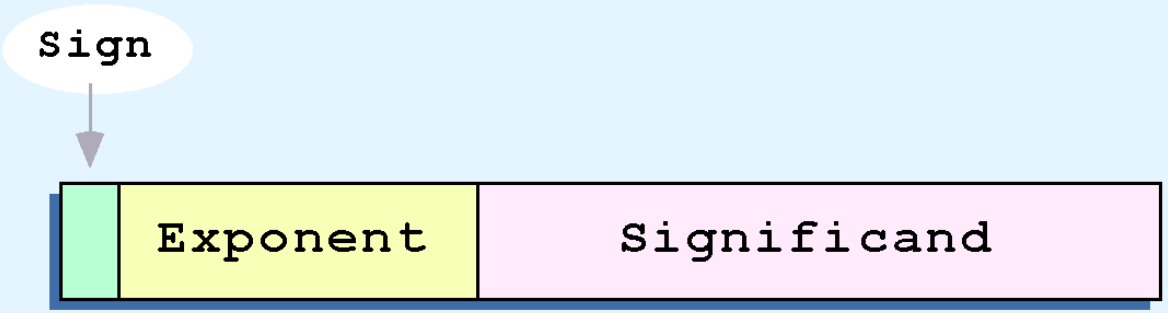


2.5 Floating-Point Representation

- The illustrations shown at the right are *all* equivalent representations for 32 using our simplified model
- Not only do these synonymous representations waste space, but they can also cause confusion.



2.5 Floating-Point Representation



- Another problem with our system is that we have made no allowances for negative exponents.

  We have no way to express 0.5 ($=2^{-1}$)! (Notice that there is no sign in the exponent field!)

All of these problems can be fixed with no changes to our basic model.

2.5 Floating-Point Representation

- To resolve the problem of synonymous forms, we will establish a rule that the first digit of the significand must be 1, with no ones to the left of the radix point.
- This process, called *normalization*, results in a unique pattern for each floating-point number.
  - In our simple model, all significands must have the form 0.1xxxxxxxx
  - For example, $4.5 = 100.1 \times 2^0 = 1.001 \times 2^2 = 0.1001 \times 2^3$. The last expression is correctly normalized.

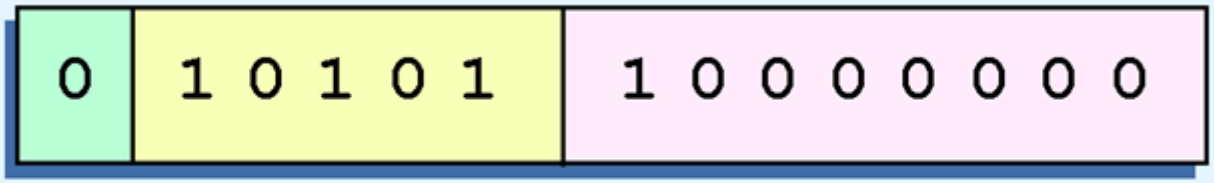
In our simple instructional model, we use no implied bits.

2.5 Floating-Point Representation

- To provide for negative exponents, we will use a *biased exponent*.
- A bias is a number that is approximately midway in the range of values expressible by the exponent. We subtract the bias from the value in the exponent to determine its true value.
  - In our case, we have a 5-bit exponent.
  - $2^{5-1}-1 = 2^4-1 = 15$
  - Thus will use 15 for our bias: our exponent will use *excess-15* representation.
- In our model, exponent values less than 15 are negative, representing fractional numbers.

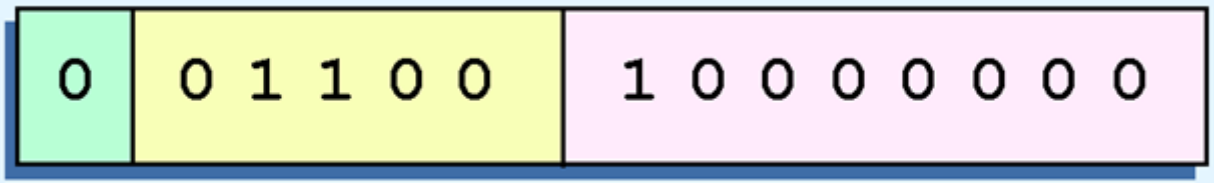
2.5 Floating-Point Representation

- Example:
  - Express 32_{10} in the revised 14-bit floating-point model.
- We know that $32 = 1.0 \times 2^5 = 0.1 \times 2^6$.
- To use our excess 15 biased exponent, we add 15 to 6, giving 21_{10} ($=10101_2$).
- So we have:



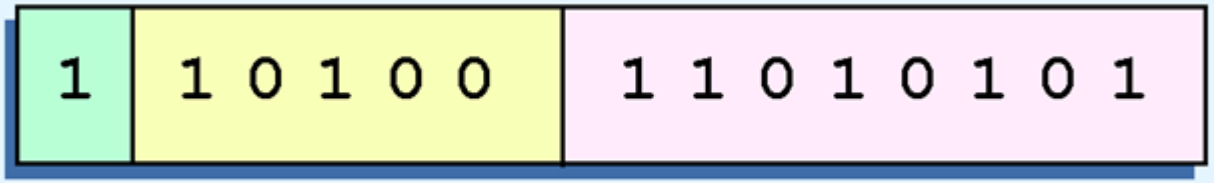
2.5 Floating-Point Representation

- Example:
  - Express 0.0625_{10} in the revised 14-bit floating-point model.
- We know that 0.0625 is 2^{-4}. So in (binary) scientific notation $0.0625 = 1.0 \times 2^{-4} = 0.1 \times 2^{-3}$.
- To use our excess 15 biased exponent, we add 15 to -3, giving 12_{10} (=01100_2).



2.5 Floating-Point Representation

- Example:
  - Express -26.625_{10} in the revised 14-bit floating-point model.
- We find $26.625_{10} = 11010.101_2$. Normalizing, we have: $26.625_{10} = 0.11010101 \times 2^5$.
- To use our excess 15 biased exponent, we add 15 to 5, giving 20_{10} ($=10100_2$). We also need a 1 in the sign bit.



2.5 Floating-Point Representation

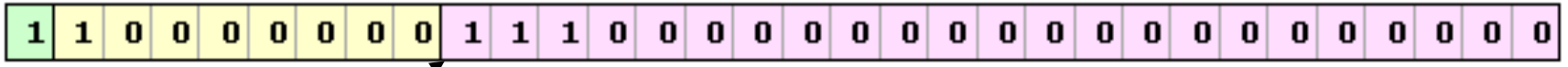
- The IEEE has established a standard for floating-point numbers
- The IEEE-754 *single precision* floating point standard uses an 8-bit exponent (with a bias of 127) and a 23-bit significand.
- The IEEE-754 *double precision* standard uses an 11-bit exponent (with a bias of 1023) and a 52-bit significand.

2.5 Floating-Point Representation

- In both the IEEE single-precision and double-precision floating-point standard, the significant has an implied 1 to the LEFT of the radix point.
  - The format for a significand using the IEEE format is: 1.xxx…
  - For example, $4.5 = .1001 \times 2^3$ in IEEE format is $4.5 = 1.001 \times 2^2$. The 1 is implied, which means is does not need to be listed in the significand (the significand would include only 001).

2.5 Floating-Point Representation

- Example: Express -3.75 as a floating point number using IEEE single precision.
- First, let's normalize according to IEEE rules:
  - $-3.75 = -11.11_2 = -1.111 \times 2^1$
  - The bias is 127, so we add 127 + 1 = 128 (this is our exponent)



(implied)

  - Since we have an implied 1 in the significand, this equates to

$-(1).111_2 \times 2^{(128-127)} = -1.111_2 \times 2^1 = -11.11_2 = -3.75.$

2.5 Floating-Point Representation

- Using the IEEE-754 single precision floating point standard:
  - An exponent of 255 indicates a special value.
    - If the significand is zero, the value is ± infinity.
    - If the significand is nonzero, the value is NaN, "not a number," often used to flag an error condition (such as the square root of a negative number and division by zero).
- Using the double precision standard:
  - The "special" exponent value for a double precision number is 2047, instead of the 255 used by the single precision standard. Most FPUs use only the double precision standard.

2.5 Floating-Point Representation

Floating-Point Number	Single-Precision Representation
1.0	0 01111111 00000000000000000000000
0.5	0 01111110 00000000000000000000000
19.5	0 10000011 00111000000000000000000
-3.75	1 10000000 11100000000000000000000
Zero	0 00000000 00000000000000000000000
± Infinity	0/1 11111111 00000000000000000000000
NaN	0/1 11111111 any nonzero significand
Denormalized Number	0/1 00000000 any nonzero significand

Some Example IEEE-754 Single-Precision Floating-Point Numbers

2.5 Floating-Point Representation

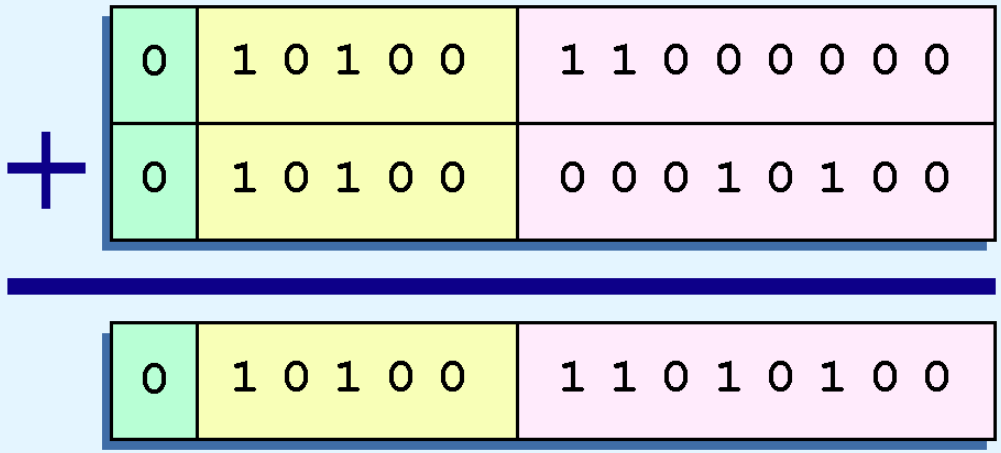
- Both the 14-bit model that we have presented and the IEEE-754 floating point standard allow two representations for zero.
  - Zero is indicated by all zeros in the exponent and the significand, but the sign bit can be either 0 or 1.
- This is why programmers should avoid testing a floating-point value for equality to zero.
  - Negative zero does not equal positive zero.

2.5 Floating-Point Representation

- Floating-point addition and subtraction are done using methods analogous to how we perform calculations using pencil and paper.
- The first thing that we do is express both operands in the same exponential power, then add the numbers, preserving the exponent in the sum.
- If the exponent requires adjustment, we do so at the end of the calculation.

2.5 Floating-Point Representation

- Example:
  - Find the sum of 12_{10} and 1.25_{10} using the 14-bit floating-point model.
- We find $12_{10} = 0.1100 \times 2^4$. And $1.25_{10} = 0.101 \times 2^1 = 0.000101 \times 2^4$.
- Thus, our sum is 0.110101×2^4.

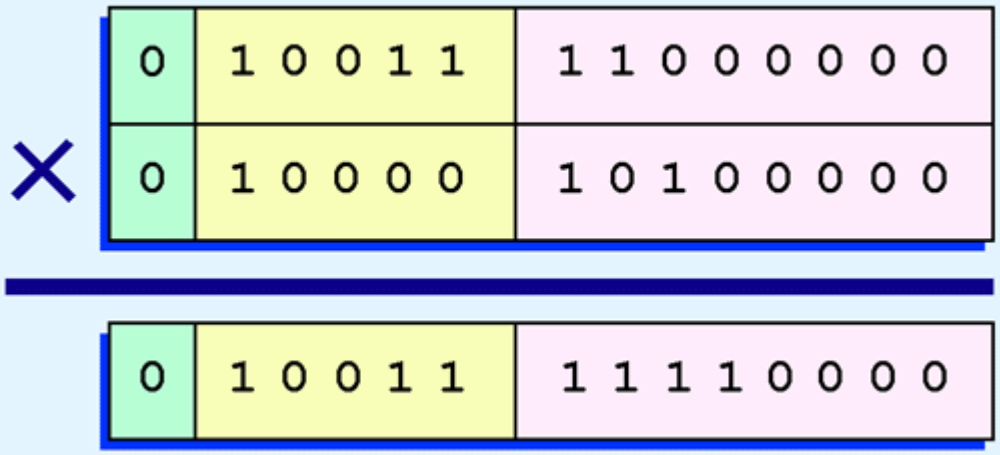


2.5 Floating-Point Representation

- Floating-point multiplication is also carried out in a manner akin to how we perform multiplication using pencil and paper.
- We multiply the two operands and add their exponents.
- If the exponent requires adjustment, we do so at the end of the calculation.

2.5 Floating-Point Representation

- Example:
  - Find the product of 12_{10} and 1.25_{10} using the 14-bit floating-point model.
- We find $12_{10} = 0.1100 \times 2^4$. And $1.25_{10} = 0.101 \times 2^1$.
- Thus, our product is $0.0111100 \times 2^5 = 0.1111 \times 2^4$.
- The normalized product requires an exponent of $19_{10} = 10011_2$.



2.5 Floating-Point Representation

- No matter how many bits we use in a floating-point representation, our model must be finite.
- The real number system is, of course, infinite, so our models can give nothing more than an approximation of a real value.
- At some point, every model breaks down, introducing errors into our calculations.
- By using a greater number of bits in our model, we can reduce these errors, but we can never totally eliminate them.

2.5 Floating-Point Representation

- Our job becomes one of reducing error, or at least being aware of the possible magnitude of error in our calculations.
- We must also be aware that errors can compound through repetitive arithmetic operations.
- For example, our 14-bit model cannot exactly represent the decimal value 128.5. In binary, it is 9 bits wide:

$$10000000.1_2 = 128.5_{10}$$

2.5 Floating-Point Representation

- When we try to express 128.5_{10} in our 14-bit model, we lose the low-order bit, giving a relative error of:

$$\frac{128.5 - 128}{128.5} \approx 0.39\%$$

- If we had a procedure that repetitively added 0.5 to 128.5, we would have an error of nearly 2% after only four iterations.

2.5 Floating-Point Representation

- Floating-point errors can be reduced when we use operands that are similar in magnitude.
- If we were repetitively adding 0.5 to 128.5, it would have been better to iteratively add 0.5 to itself and then add 128.5 to this sum.
- In this example, the error was caused by loss of the low-order bit.
- Loss of the high-order bit is more problematic.

2.5 Floating-Point Representation

- Floating-point overflow and underflow can cause programs to crash.
- Overflow occurs when there is no room to store the high-order bits resulting from a calculation.
- Underflow occurs when a value is too small to store, possibly resulting in division by zero.

Experienced programmers know that it's better for a program to crash than to have it produce incorrect, but plausible, results.

2.5 Floating-Point Representation

- When discussing floating-point numbers, it is important to understand the terms *range, precision,* and *accuracy*.
- The range of a numeric integer format is the difference between the largest and smallest values that can be expressed.
- Accuracy refers to how closely a numeric representation approximates a true value.
- The precision of a number indicates how much information we have about a value

2.5 Floating-Point Representation

- Most of the time, greater precision leads to better accuracy, but this is not always true.
  - For example, 3.1333 is a value of pi that is accurate to two digits, but has 5 digits of precision.
- There are other problems with floating point numbers.
- Because of truncated bits, you cannot always assume that a particular floating point operation is associative or distributive.

2.5 Floating-Point Representation

- This means that we cannot assume:

$$(a + b) + c = a + (b + c) \text{ or}$$

$$a \times (b + c) = ab + ac$$

- Moreover, to test a floating point value for equality to some other number, it is best to declare a "nearness to x" epsilon value. For example, instead of checking to see if floating point x is equal to 2 as follows:

```
if (x == 2) …
```

it is better to use:

```
if (abs(x - 2) < epsilon) ...
```

(assuming we have epsilon defined correctly!)

2.5 Floating-Point Representation

type (in C)	size	range
`short`	16 bit	[-32768; 32767]
`int`	32 bit	[-2147483648; 2147483647]
`long long`	64 bit	[-9223372036854775808; 9223372036854775807]
`float`	32 bit	$\pm 10^{36}$, $\pm 10^{-34}$ (6 significant decimal digits)
`double`	64 bit	$\pm 10^{308}$, $\pm 10^{-324}$ (15 significant decimal digits)

End of Chapter 2