(i) $\oint_C f(z)\,dz = 0$ [Cauchy's theorem]

(ii) $f(a) = \dfrac{1}{2\pi i}\oint_C \dfrac{f(z)}{z-a}\,dz$

(iii) $f'(a) = \dfrac{1}{2\pi i}\oint_C \dfrac{f(z)}{(z-a)^2}\,dz$

(iv) $f^n(a) = \dfrac{n!}{2\pi i}\oint_c \dfrac{f(z)}{(z-a)^{n+1}}\,dz$

Cauchy's formula

(v) $\dfrac{1}{2\pi i}\oint_c F(z)\,dz = \lim_{z\to a} \dfrac{1}{(m-1)!}\dfrac{d^{m-1}}{dz^{m-1}}\left\{(z-a)^m f(z)\right\}$

(vi) Harmonic equation : $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ &

$\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$

(vii) Cauchy Riemann equation :- $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$

and $-\dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x}$

(viii) For analyticity : $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$

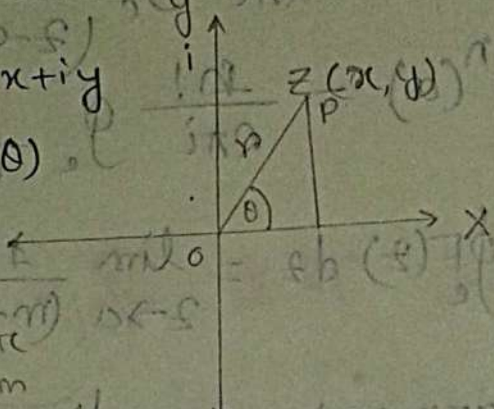
Complex Variables

Complex variable :- Any variable which contains complex number is called complex ~~nu~~ variable.

Ex: $z = a+ib$; Here z is a complex variable.

$OP = a+ib = re^{i\theta} = x+iy$

$= r(\cos\theta + i\sin\theta)$



$x = r\cos\theta$
$y = r\sin\theta$ } Polar form

θ is called amplitude or argument of z

$|z| = r = \sqrt{x^2+y^2}$

Modulus of the equation of complex variable is the radious of a circles.

$x^2+1=0 \Rightarrow x^2 = -1 \Rightarrow x^2 = i^2$

$\Rightarrow x = \sqrt{-1} = i$

Find $|e^z|$ if $z = x+iy$.

Sol^n: we have $z = x+iy$

$\therefore |e^z| = |e^{x+iy}| = |e^x . e^{iy}|$

$= |e^x . (\cos y + i\sin y)|$

$= e^x\sqrt{\sin^2 y + \cos^2 y}$

$= e^x$

If $z = x+iy$; then Find $|e^{iz}|$.

Sol^n: we have, $z = x+iy$.

$$|e^{iz}| = |e^{i(x+iy)}| = |e^{ix+i^2y}| = |e^{ix-y}|$$

$$= |e^{ix}\cdot e^{-y}| = |e^{ix}\cdot e^{-y}| = e^{-y}\sqrt{\cos^2x+\sin^2x} = e^{-y}$$

If $z = 6e^{\pi i/3}$; evaluate $|e^{iz}|$.

Sol^n:- $e^{iz} = e^{i\,6e^{\pi i/3}} = e^{i6(\cos\pi/3 + i\sin\pi/3)}$

$$= e^{i6\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)} = e^{3i-3\sqrt{3}} = e^{3i}\cdot e^{-3\sqrt{3}}$$

$$= e^{-3\sqrt{3}}(\cos 3 + i\sin 3)$$

$$\therefore |e^{iz}| = \sqrt{(e^{-3\sqrt{3}})^2\cdot(\cos^2 3+\sin^2 3)}$$

$$= e^{-3\sqrt{3}}$$

Complex conjugate :- A complex conjugate of a complex number $z = a+ib$ is $\bar{z} = a-ib$ while $|z\bar{z}| = |z|^2$

Prove that, $z\bar{z} = |z|^2$; Conjugate, $\bar{z} = x-iy$

$z = x+iy$

$|z| = \sqrt{x^2+y^2}$

Proof: $|z\bar{z}| = |(x+iy)(x-iy)|$

$$= |x^2 + ixy - ixy + y^2|$$

$$= |x^2 + y^2| = \sqrt{(x^2+y^2)^2}$$

$$= \left\{\sqrt{x^2+y^2}\right\}^2 = |z|^2$$

Prove that, (i) $\text{Arg}(z_1 z_2) = \arg z_1 + \arg z_2$

(ii) $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

(i) **Sol^n:-** let, $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ where $\arg z_1 = \theta_1$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ where $\arg z_2 = \theta_2$

$\therefore z_1 z_2 = r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$$= r_1 r_2 (\cos\theta_1\cos\theta_2 + i\sin\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i^2\sin\theta_1\sin\theta_2)$$

$$= r_1 r_2 \{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}$$

$\therefore \arg|z_1 z_2| = \theta_1 + \theta_2$

$= \arg z_1 + \arg z_2$ [Proved]

(ii) **Sol^n:-** $\arg(z_1/z_2) = \arg z_1 - \arg z_2$

let, $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ where $\arg z_1 =$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ where $\arg z_2 = \theta_2$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\{\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\}$$

$$\therefore \arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2$$

Prove that, $\arg \bar{z} = -\arg z$. where, $z = x+iy = re^{i\theta}$

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Prove that (i) $|z_1+z_2| \leq |z_1| + |z_2|$

(ii) $|z_1+z_2+z_3| \leq |z_1| + |z_2| + |z_3|$

09 (iii) $|z_1 - z_2| \geq |z_1| - |z_2|$

08 (iv) $|\bar{z}| = |z|$

(i) Sol^n:- let, $z_1 = x_1 + iy_1 \quad z_2 = x_2 + iy_2$

$$\therefore |z_1| = \sqrt{x_1^2+y_1^2} \qquad |z_2| = \sqrt{x_2^2+y_2^2}$$

$$\therefore z_1 + z_2 = (x_1+x_2) + i(y_1+y_2)$$

$$\Rightarrow |z_1+z_2| = \sqrt{(x_1+x_2)^2+(y_1+y_2)^2}$$

$$\Rightarrow |z_1+z_2|^2 = (x_1+x_2)^2+(y_1+y_2)^2$$

$$= x_1^2+x_2^2+2x_1x_2+y_1^2+y_2^2+2y_1y_2$$

$$= (x_1^2+y_1^2)+(x_2^2+y_2^2)+2(x_1x_2+y_1y_2)$$

$$= |z_1|^2+|z_2|^2+2\sqrt{(x_1x_2+y_1y_2)^2}$$

$$= |z_1|^2+|z_2|^2+2\sqrt{x_1^2x_2^2+y_1^2y_2^2+2x_1x_2y_1y}$$

$$\Rightarrow |z_1+z_2|^2 \leq |z_1|^2+|z_2|^2+2\sqrt{x_1^2x_2^2+y_1^2y_2^2+x_1^2y_2^2+x_2^2}$$

$$[\because (x_1y_2-x_2y_1)^2 \geq 0]$$

$$[\therefore x_1^2y_2^2+x_2^2y_1^2 \geq 2x_1x_2y_1y_2]$$

$$\Rightarrow |z_1+z_2|^2 \leq |z_1|^2+|z_2|^2+2\sqrt{x_1^2(x_2^2+y_2^2)+y_1^2(x_2^2+y_2^2)}$$

$$\leq |z_1|^2+|z_2|^2+2|z_1||z_2|$$

$$\leq \{|z_1|+|z_2|\}^2$$

$$\therefore |z_1+z_2| \leq |z_1|+|z_2| \quad \text{[Proved]}$$

(ii) Sol^n:- let, $z_1 = x_1+iy_1 \quad \therefore |z_1| = \sqrt{x_1^2+y_1^2}$

$z_2 = x_2+iy_2 \quad \therefore |z_2| = \sqrt{x_2^2+y_2^2}$

$z_3 = x_3+iy_3 \quad \therefore |z_3| = \sqrt{x_3^2+y_3}$

$\therefore$ NOW, $|z_1+z_2+z_3| = |z_1+(z_2+z_3)| \leq |z_1|+|z_2+z_3|$

From ~~eo~~ proof (i) we get,

$\cancel{|z_1|+|z_2+z_3| \leq |z_1|+|z_2|+|z_3|}$

$$|z_1+z_2+z_3| \leq |(z_1+z_2)+z_3|$$
$$\leq |z_1+z_2|+|z_3|$$
$$\leq |z_1|+|z_2|+|z_3|$$

) Sol^n :- let, $z_1 = x_1+iy_1 \quad \therefore |z_1| = \sqrt{x_1^2+y_1^2}$

$z_2 = x_2+iy_2 \quad \therefore |z_2| = \sqrt{x_2^2+y_2^2}$

$\therefore z_1-z_2 = (x_1-x_2)+i(y_1-y_2)$

$$\therefore |z_1-z_2|^2 = (x_1-x_2)^2+(y_1-y_2)^2$$
$$= x_1^2+x_2^2-2x_1x_2+y_1^2+y_2^2-2y_1y_2$$
$$= (x_1^2+y_1^2)+(x_2^2+y_2^2)-2(x_1x_2+y_1y_2)$$
$$|z_1-z_2|^2 \geq |z_1|^2+|z_2|^2-2\sqrt{(x_1x_2+y_1y_2)^2}$$
$$\geq |z_1|^2+|z_2|^2-2\sqrt{x_1^2x_2^2+y_1^2y_2^2+2x_1x_2y_1y_2}$$
$$\geq |z_1|^2+|z_2|^2-2\sqrt{x_1^2x_2^2+y_1^2y_2^2+x_1^2y_2^2+x_2^2y_1^2}$$
$$\geq |z_1|^2+|z_2|^2-2|z_1||z_2|$$
$$\geq \{|z_1|-|z_2|\}^2$$

$\therefore |z_1-z_2| \geq |z_1|-|z_2|$ [Proved]

(iv) Let, $z = x+iy$ $\bar{z} = x-iy$

$|z| = \sqrt{x^2+y^2}$ $|\bar{z}| = \sqrt{x^2+y^2}$

$\therefore |\bar{z}| = |z|$ [Proved]

Another proof of Question (i) :-

$$(|z_1|+|z_2|)^2 = (|z_1|)^2 + (|z_2|)^2 + 2|z_1||z_2|$$

$$(|z_1|+|z_2|)^2 \geq z_1^2 + z_2^2 + 2z_1z_2$$

$$(|z_1|+|z_2|)^2 \geq (z_1+z_2)^2$$

$$(|z_1|+|z_2|)^2 \geq (|z_1+z_2|)^2$$

$$\therefore |z_1+z_2| \leq |z_1| + |z_2|$$

(2)

Prove that, $\overline{z_1+z_2} = \bar{z_1} + \bar{z_2}$

Sol^n :- Let, $z_1 = x_1+iy_1$ and $z_2 = x_2+iy_2$

$$\text{L.H.S} = \overline{(x_1+iy_1)+(x_2+iy_2)}$$

$$= \overline{(x_1+x_2)+i(y_1+y_2)}$$

$$= (x_1+x_2) - i(y_1+y_2)$$

$$= (x_1-iy_1) + (x_2-iy_2)$$

$$= \bar{z_1} + \bar{z_2} \text{ [Proved]}$$

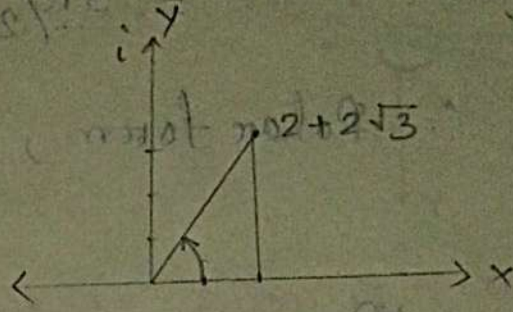
田 Find the modulus and argument. And also express in polar form.

(13) (07)

(i) $2+2\sqrt{3}i$ (ii) $-5+5i$ (iii) $-\sqrt{6}-\sqrt{2}i$ (iv) $-3i$ (v) i

(i) Sol^n:- Modulus, $r=\sqrt{4+12}=4$

Arg, $\theta = \tan^{-1}\frac{2\sqrt{3}}{2} = \pi/3$



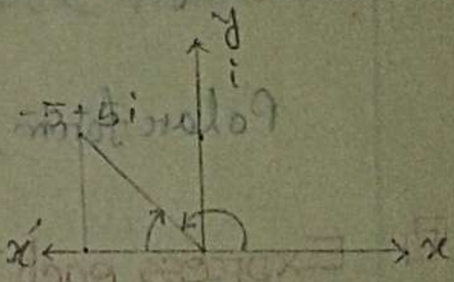
Polar form $= 4e^{\pi i/3}$

(ii) Sol^n:- Modulus, $r=\sqrt{25+25}=5\sqrt{2}$

arg, $\theta = -\tan^{-1}\frac{5}{+5} = -45^\circ$

$= 180^\circ - 45^\circ = 135^\circ$

$= 3\pi/4$



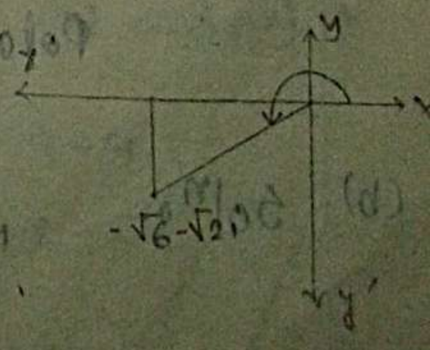
Polar form $= 5\sqrt{2}\,e^{3\pi i/4}$

(iii) Sol^n:- modulus, $r=\sqrt{6+2}=2\sqrt{2}$

arg, $\theta = \tan^{-1} \sin^{-1}\left(\frac{-\sqrt{2}}{2\sqrt{2}}\right)$

$= \sin^{-1}\sin 210^\circ$

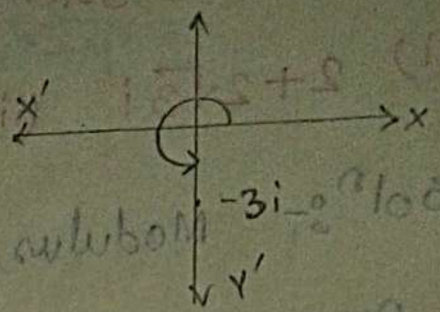
$= 7\pi/6$



Polar form $= 2\sqrt{2}\,e^{7\pi i/6}$

(iv) Solⁿ :- Modulus, $r = \sqrt{9} = 3$

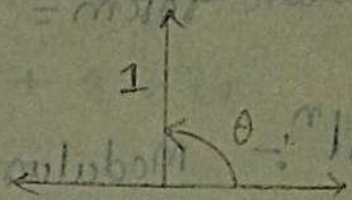
arg, $\theta = \sin^{-1}\left(\frac{-3}{3}\right) = 270°$
$= 3\pi/2$



$\therefore$ Polar form, $= 3e^{3\pi i/2}$

(v) Solⁿ :- Modulus : $r = \sqrt{1^2} = 1$

arg, $\theta = \sin^{-1}\left(\frac{1}{1}\right) = \pi/2$

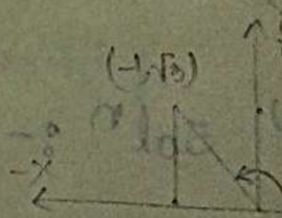


Polar form, $z = e^{\pi i/2}$

☐ Express each of the following complex number in polar form : (a) $-1+\sqrt{3}i$ (b) $-2\sqrt{3}-2i$ (c) $\sqrt{5}-i$

Solⁿ:- (a) Modulus : $r = \sqrt{1+3} = 2$

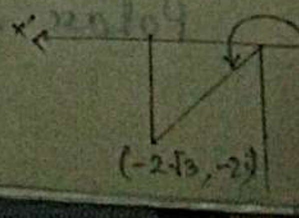
Arg, $\theta = \sin^{-1}\left(\frac{\cancel{-2}\sqrt{3}}{2\sqrt{3}}\right) = \cancel{210°}\ 120° = \cancel{7\pi/6}$
$= 2\pi/3$



$\therefore$ Polar form $= \cancel{2e^{7\pi i/6}}$
$= 2e^{2\pi i/3}$

(b) Solⁿ :- Modulus, $r = \sqrt{(2\sqrt{3})^2 + (2)^2}$
$= 4$

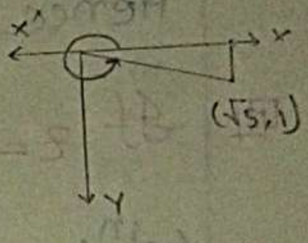
Arg, $\theta = \sin^{-1}\left(\frac{-2\sqrt{3}}{4}\right) = 7\pi/6$



$\therefore$ Polar form, $z = 4e^{7\pi i/6}$

(c) Sol^n:- Modulus, $r = \sqrt{(\sqrt{5})^2 + (1)^2} = \sqrt{6}$

arg, $\theta = \sin^{-1}\left(\frac{-1}{\sqrt{6}}\right)$



(12)

Find two complex number whose sum is 4 and whose product is 8.

Sol^n:- Let z_1 and z_2 be two complex number. So we have

$$z_1 + z_2 = 4 \text{ ---- (i)}$$
$$z_1 z_2 = 8 \text{ ---- (ii)}$$

Now, $(z_1 - z_2)^2 = (z_1 + z_2)^2 - 4z_1 z_2 = 16 - 32 = -16$

$\Rightarrow (z_1 - z_2)^2 = 16i^2$

$\therefore z_1 - z_2 = 4i$ ——— (iii)

Or, $z_1 - z_2 = -4i$ ——— (iv)

From eq^n (i) & (iii) we get, From (ii) and (iv) $\Rightarrow$

$2z_1 = 4 + 4i$ and $2z_2 = 4 - 4i$

$\Rightarrow z_1 = 2 + 2i$ $z_2 = 2 - 2i$

Again, from (i) and (iv) we get,

$2z_1 = 4 - 4i \Rightarrow z_1 = 2 - 2i$

and $2z_2 = 4+4i$

$\therefore z_2 = 2+2i$

Hence the complex numbers are $2+2i$ and $2-2$

If $z = x+iy$, Prove that $|x|+|y| \leq \sqrt{2}\,|x+iy|$

Sol^n:- we have,

$z = x+iy \quad \therefore |z| = \sqrt{x^2+y^2}$

$\Rightarrow |z|^2 = x^2+y^2 \Rightarrow 2|z|^2 = 2x^2+2y^2 = x^2+y^2+x^2$

$\Rightarrow 2|z|^2 \geq x^2+y^2+2xy$ [$\because (x-y)^2 \geq 0$

$\Rightarrow 2|x+iy|^2 \geq |x|^2+|y|^2+2|x||y|$ OR, $x^2+y^2 \geq 2xy$]

$\Rightarrow 2|x+iy|^2 \geq \{|x|+|y|\}^2$

$\Rightarrow \sqrt{2}|x+iy| \geq |x|+|y|$ [Proved]

If $z = 6e^{\pi i/6}$ then find $|e^{iz}|$.

Sol^n:- $e^{iz} = e^{i6e^{\pi i/6}}$

$= e^{i6(\cos \pi/6 + i\sin \pi/6)}$

$= e^{i6(\sqrt{3}/2 + i\frac{1}{2})}$

$= e^{3\sqrt{3}i + i^2 3} = e^{3\sqrt{3}i - 3}$

$= e^{-3} \cdot e^{3\sqrt{3}i} = e^{-3}(\cos 3\sqrt{3} + i\sin 3$

$$\therefore |e^{iz}| = \sqrt{(e^{-3})^2 \cdot (\cos^2 3\sqrt{3} + \sin^2 3\sqrt{3})}$$

$$= e^{-3} \text{ (Ans)}$$

Prove that, $z^{-1} = \frac{\bar{z}}{|z|^2}$

Soln:- Let, $z = x+iy$ $\quad \therefore \bar{z} = x-iy$

$$\therefore \frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

$$\Rightarrow \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$$

$$\Rightarrow \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)}$$

$$\Rightarrow \frac{1}{x+iy} = \frac{1}{x+iy}$$ [Proved]

State and prove De Moivre's theorem.

Theorem :- De Moivere's theorem states that, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ where n is any positive integer.

Proof :- Let, $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and

$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

Now, $z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 + i\sin$

$= r_1 r_2 \{(\cos\theta_1 + i\sin\theta_1) \times (\cos\theta_2 + i\sin\theta_2)\}$

$= r_1 r_2 \{(\cos\theta_1\cos\theta_2 + i\sin\theta_1\cos\theta_2 + i\sin\theta_2 \cos$

$+ i^2 \sin\theta_1 \sin\theta_2)\}$

$= r_1 r_2 \{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin$

$\cos\theta_2 + \sin\theta_2\cos\theta_1)\}$

$= r_1 r_2 \{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}$

If $z_1, z_2, z_3 \cdots\cdots z_n = r_1, r_2, r_3 \cdots\cdots r_n \{$

$\cos(\theta_1+\theta_2+\theta_3+\cdots+\theta_n) + i\sin(\theta_1+\theta_2+\theta_3+\cdots+\theta_n$

— (1)

Let, $z_1 = z_2 = z_3 + \cdots = z_n = z$

$r_1 = r_2 = r_3 \cdots \cdots = r_n = r$

$\theta_1 = \theta_2 = \theta_3 = \cdots \cdots = \theta_n = \theta$

From eq^n (1)

$$z^n = r^n (\cos n\theta + i \sin n\theta)$$

$\Rightarrow r^n (\cos n\theta + i \sin n\theta) = r^n (\cos\theta + i \sin\theta)^n$

$\Rightarrow (\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta$ [Proved]

Roots of complex number :-

From De moivere's theorem we can show that if n is positive integer

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} \left\{ \cos\left(\frac{\theta + 2k\pi}{n}\right) + i \sin\left(\frac{\theta + 2k\pi}{n}\right) \right\}$$

where, $k = 0, 1, 2, 3 \cdots n-1$

Prove that, $e^{i\theta} = e^{i(\theta + 2k\pi)}$; where, $k = 0, \pm 1, \pm 2$

Sol^n:- $e^{i(\theta + 2k\pi)} = \cos(\theta + 2k\pi) + i \sin(\theta + 2k\pi)$

$= \cos\theta + i \sin\theta = e^{i\theta}$

Find the roots of $z^5+32=0$ and locate the value of z^5 in complex plane.

Soln :-

$$z^5+32=0 \Rightarrow z^5=-32$$

$$\Rightarrow z^5 = 32(\cos\pi + i\sin\pi)$$

$$\Rightarrow z^5 = 2^5(\cos\pi + i\sin\pi) = 2^5\{\cos(\pi+2k\pi) + i\sin(\pi+2k\pi)\}$$

$$\Rightarrow z = 2\left\{\cos\left(\frac{\pi+2k\pi}{5}\right) + i\sin\left(\frac{\pi+2\pi k}{5}\right)\right\}$$

$$\Rightarrow z = 2e^{i\left(\frac{\pi+2k\pi}{5}\right)}$$

$\therefore z_1 = 2e^{\pi i/5}$ if $k=0$

$z_2 = 2e^{3\pi/5\, i}$ if $k=1$

$z_3 = 2e^{\pi i}$ if $k=2$

$z_4 = 2e^{7\pi/5\, i}$ if $k=3$

$z_5 = 2e^{9\pi/5\, i}$ if $k=4$

Find the roots of the following equation.

(a) $z^2+\pi^2=0$ (b) $z^6+1=0$ (c) $z^2+1=0$ (d) $z^4+16=0$

(e) $z^4+a^4=0$ (f) $z^4+1=0$

(a) $z^2+\pi^2=0 \Rightarrow z^2=-\pi^2 \Rightarrow z^2=\pi^2 i^2 \Rightarrow z=\pm \pi i$

(b) $z^6+1=0 \Rightarrow z^6=-1 \Rightarrow z^6=\cos\pi+i\sin\pi$

$\Rightarrow z^6=\cos(\pi+2k\pi)+i\sin(\pi+2k\pi)$

$\Rightarrow z=\cos\left(\frac{\pi+2k\pi}{6}\right)+i\sin\left(\frac{\pi+2k\pi}{6}\right)$

$\Rightarrow z=e^{i\left(\frac{\pi+2k\pi}{6}\right)}$

$z_1=e^{\pi i/6}$ if $k=0$

$z_2=e^{\pi/2\, i}$ if $k=1$

$z_3=e^{5\pi/6\, i}$ if $k=2$

$z_4=e^{7\pi i/6}$ if $k=3$

$z_5=e^{3\pi/2\, i}$ if $k=4$

$z_6=e^{11\pi i/6}$ if $k=5$

(e) $z^4+a^4=0 \Rightarrow z^4=-a^4 \Rightarrow z^4=a^4(\cos\pi+i\sin\pi)$

$\Rightarrow z^4=a^4\{\cos(\pi+2k\pi)+i(\sin(\pi+2k\pi)\}$

$$\Rightarrow z^4 = a^4 \left\{ \cos(\pi + 2k\pi) + i \sin(\pi + 2k\pi) \right\}$$

$$\Rightarrow z = a \left\{ \cos\left(\frac{\pi + 2k\pi}{4}\right) + i \sin\left(\frac{\pi + 2k\pi}{4}\right) \right\}$$

$$\Rightarrow z = a e^{i\left(\frac{\pi + 2k\pi}{4}\right)}$$

$$\Rightarrow z_1 = a e^{\pi i/4} \quad \text{if } k=0$$

$$z_2 = a e^{3\pi i/4} \quad \text{if } k=1$$

$$z_3 = a e^{5\pi i/4} \quad \text{if } k=2$$

$$z_4 = a e^{7\pi i/4} \quad \text{if } k=3$$

(d) $z^4 + 16 = 0 \Rightarrow z^4 = -16 \Rightarrow z^4 = -2^4$

$$\Rightarrow z^4 = 2^4 (\cos\pi + i \sin\pi)$$

$$\Rightarrow z^4 = 2^4 \left\{ \cos(\pi + 2k\pi) + i \sin(\pi + 2k\pi) \right\}$$

$$\Rightarrow z = 2 \left\{ \cos\left(\frac{\pi + 2k\pi}{4}\right) + i \sin\left(\frac{\pi + 2k\pi}{4}\right) \right\}$$

$$\Rightarrow z = 2 e^{i\left(\frac{\pi + 2k\pi}{4}\right)}$$

$$z_1 = 2e^{\pi i/4} \quad \text{if } k=0$$

$$z_2 = 2e^{3\pi/4} \quad \text{if } k=1$$

$$z_3 = 2e^{5\pi/4} \quad \text{if } k=2$$

$$z_4 = 2e^{\frac{7\pi}{4}i} \quad \text{if } k=3$$

(e) $z^2+1=0 \Rightarrow z^2=-1 \Rightarrow z^2 = \cos\pi + i\sin\pi$

$$\Rightarrow z^2 = \cos(\pi+2k\pi) + i\sin(\pi+2\pi k)$$

$$\Rightarrow z = \cos\left(\frac{\pi+2k\pi}{2}\right) + i\sin\left(\frac{\pi+2\pi k}{2}\right)$$

$$\Rightarrow z = e^{i\left(\frac{\pi+2k\pi}{2}\right)}$$

$$z_1 = e^{\pi/2\, i} \quad \text{if } k=0$$

$$z_2 = e^{\frac{3\pi}{2}i} \quad \text{if } k=\underline{1}$$

$$\cancel{z_3 = e^{\frac{5\pi}{2}i}}$$

z

(f) $z^4+1=0 \Rightarrow z^4=-1 \Rightarrow z^4 = \cos\pi + i\sin\pi$

$$\Rightarrow z^4 = \cos(\pi+2\pi k) + i\sin(\pi+2k\pi)$$

$$\Rightarrow z = \cos\left(\frac{\pi+2\pi k}{4}\right) + i\sin\left(\frac{\pi+2k\pi}{4}\right)$$

$$\Rightarrow z = e^{i\left(\frac{\pi+2\pi k}{4}\right)}$$

$$z_1 = e^{\frac{\pi}{4}i} \quad \text{if } k=0$$

$$z_2 = e^{3\pi/4\, i} \quad \text{if } k=1$$

$$z_3 = e^{5\pi/4\, i} \quad \text{if } k=2$$

$$z_4 = e^{7\pi/4\, i} \quad \text{if } k=3$$

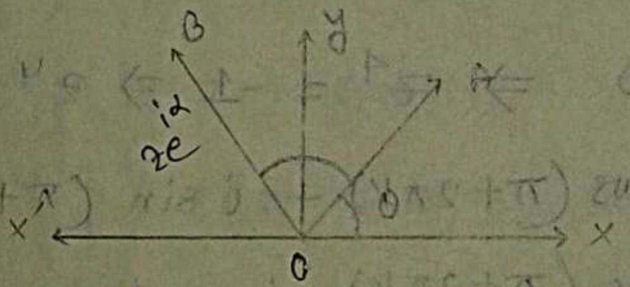
Find an equation for a circle of radius 4 with center at $(-2, 1)$ in complex plane.

\# The center can be represented by the complex number $(-2+i)$. If z is any point on the circle Then the distance form z to $(-2+i)$ is

$$|z-(-2+i)| = 4$$

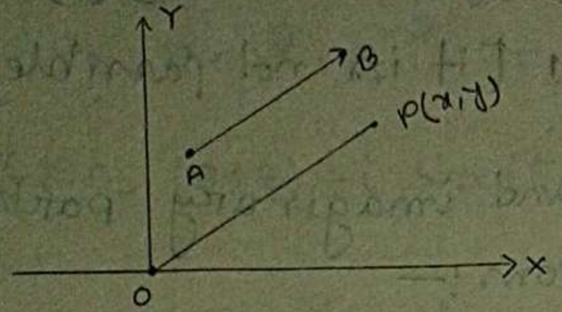
$\Rightarrow |z+2-i| = 4$ is the required equation.

Given a complex number z_0 interpret geometrically $ze^{i\alpha}$ where α is real.



Sol: let, $z = re^{i\theta}$ be represented by graphically by a vector OA in figure. Then $ze^{i\alpha} = re^{i\theta} \cdot e^{i\alpha}$ $= re^{i(\theta+\alpha)}$ is the vector represented by OB. Hence multiplication of a vector z by $e^{i\alpha}$ amount to ratating z anticlockwise through angle α.

Vector interpretation of complex number :–



$OP = x+iy$

$OP = AB = x+iy$

because they are same length, magnitude and direction.

Dot and cross product of complex number :–

$$z_1 \cdot z_2 = |z_1||z_2|\cos\theta = x_1x_2 + y_1y_2$$

$= Re\{\bar{z}_1 z_2\} = \frac{1}{2}\{\bar{z}_1 z_2 + z_1\bar{z}_2\}$ where θ is the angle between z_1 and z_2.

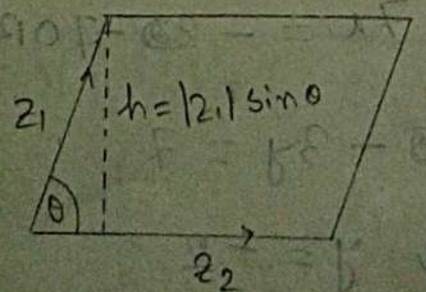
Cross product of complex number :–

$$z_1 \times z_2 = |z_1||z_2|\sin\theta = x_1y_2 - y_1y_2 = Im\{\bar{z}_1 z_2\}$$

$= 1/2i\{\bar{z}_1 z_2 - z_1 \bar{z}_2\}$

(2) Prove that the, area of a parallelogram having sides, z_1 and z_2 is $|z_1 \times z_2|$

Sol^n:–



Area of a parallelogr

$= (\text{base}) \times (\text{hight})$

$= (|z_2|) \times (|z_1|\sin\theta)$

$= |z_1 \times z_2|$

Explain the falloey :- $-1 = \sqrt{(-1)(-1)} = \sqrt{1} = 1$

Hence ; $-1 = 1$ [it is not possible]

Find the real and imaginary parts of the followings equations :-

(i) $2x - 3iy + 4ix - 2y - 5 - 10i = (x+y+2) - (y-x+3)i$

(ii) $f(z) = ze^{iz}$ (iii) $f(z) = \sqrt{z}$ (iv) $\frac{-1+\sqrt{3}i}{2}$

(i) $2x - 3iy + 4ix - 2y - 5 - 10i = (x+y+2) - i(y-x+3)$

$\Rightarrow (2x-2y-5) - i(3y - 4x + 10) = (x+y+2) - i(y-x+3)$

Equating real and imaginary pard,

$2x - 2y - 5 = x + y + 2$

$\Rightarrow x - 3y = 7$ —— (i)

And, $3y - 4x + 10 = y - x + 3$

$\Rightarrow 2y - 3x = -7$ —— (ii)

$2(i) + 3(ii) \Rightarrow -7x = -7$ OR, $x = 1$

From (i) $1 - 3y = 7$

OR, $y = -2$

(ii) $f(z) = ze^{iz}$

$= re^{i\theta} \cdot e^{iz} = re^{i(\theta+z)}$

$= r\{\cos(\theta+z) + i\sin(\theta+z)\}$

$\therefore x = r\cos(\theta+z)$

$y = r\sin(\theta+z)$

(iii) $f(z) = \sqrt{z} = \sqrt{re^{i\theta}} = \sqrt{r}\, e^{i\theta/2}$

$= \sqrt{r}\{\cos\theta/2 + i\sin\theta/2\}$

$\therefore x = \sqrt{r}\cos\theta/2 \; ; \; y = \sqrt{r}\sin\theta/2$

(iv) $\dfrac{-1+\sqrt{3}i}{2} = -\frac{1}{2} + i\dfrac{\sqrt{3}}{2}$

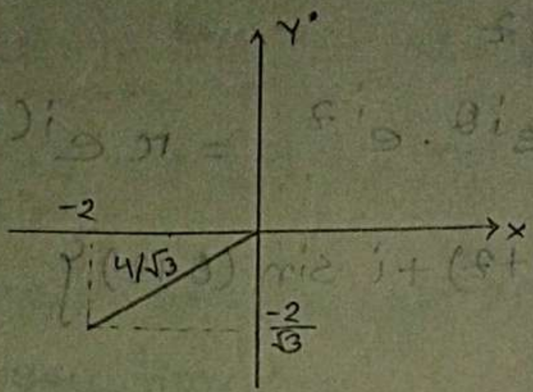
$\therefore x = -1/2 \; ; \; y = \dfrac{\sqrt{3}}{2}$

(2)

Find the modulus and argument of $\dfrac{-2}{1+i\sqrt{3}}$

Modulus, $r = \sqrt{(-2/1)^2 + (-2/\sqrt{3})^2} = \sqrt{4+\frac{4}{3}}$

$= \sqrt{\frac{16}{3}} = \dfrac{4}{\sqrt{3}}$



$$\theta = \tan^{-1} \frac{-2/\sqrt{3}}{-2} = \sin^{-1} \frac{-2/\sqrt{3}}{4/\sqrt{3}} = \sin^{-1}\left(-\frac{1}{2}\right)$$

$$= 7\pi/6$$

Function, limit, Continuity

Function :- $w = f(z)$ is a function. Two types function

(i) Single valued function
(ii) Multiple valued function

(i) Single valued function :- If only one value of w corresponds to each value of z, we say that w is a single valued function of z. Ex: $w = f(z) = z^2$ is a single valued function.

(ii) Multiple valued function :- If more than one value of w corresponds to each value of z, we say that w is a multiple valued function of z. Ex: $w = f(z) = \sqrt{z}$ is a multiple valued function.

Limit :- Let $f(z) \Rightarrow$ be defined and single valued, we say that the number l is the limit of $f(z) \to z_0$ and with $\lim_{z \to z_0} f(z) = l$ if for any positive number ϵ. we can find another positive number δ (depends on ϵ). Such that $|f(z) - l| < \epsilon \Leftrightarrow |z - z_0| < \delta$

Prove that $\lim_{z \to 0} \frac{\bar{z}}{z}$ does not exists.

Soln :- $\lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{(x,y) \to 0} \frac{x - iy}{x + iy} = \lim_{\substack{x \to 0 \\ y = 0}} \frac{x}{x} = 1$

$[\text{ if } y = 0]$

And similarly $\lim\limits_{z\to 0}\frac{\bar{z}}{z} = \lim\limits_{(x,y)\to 0}\frac{x-iy}{x+iy}$

$$= \lim_{\substack{x=0\\ y\to 0}} \frac{-iy}{iy} = -1 \quad [\text{if } x=0]$$

Since the two limits does not same results so that the limit does not exists.

Ex. $\lim\limits_{z\to i}\left\{(z-i)\frac{e^{imz}}{(z^2+1)}\right\}$

$$= \lim_{z\to i}\left\{\frac{(z-i)(z+i)e^{imz}}{(z^2+1)(z+i)}\right\}$$

$$= \lim_{z\to i}\left\{\frac{(z^2+1)e^{imz}}{(z^2+1)(z+i)}\right\}$$

$$= \lim_{z\to i}\left\{\frac{e^{imz}}{z+i}\right\}$$

$$= \frac{e^{i.m.i}}{i+i} = \frac{e^{-m}}{2i} = \frac{1}{e^m 2i} \quad \text{(Ans)}$$

Ex. $\lim\limits_{z\to ae^{\pi i/4}}\left\{(z-ae^{i\pi/4})\frac{1}{z^4+a^4}\right\}$.

$$= \lim_{z\to ae^{\pi i/4}}\left\{(z-ae^{i\pi/4})\frac{1}{z^4-(ae^{\pi i/4})^4}\right\}$$

$$= \lim_{z \to ae^{\pi i/4}} \left\{ (z - a^{i\pi/4}) \frac{1}{(z^2)^2 - \{(ae^{i\pi/4})^2\}^2} \right\}$$

$$= \lim_{z \to ae^{\pi i/4}} \left\{ \frac{(z - a^{i\pi/4}) \times 1}{\{z^2 + (ae^{i\pi/4})^2\} \{(z - ae^{i\pi/4})(z + ae^{i\pi/4})\}} \right\}$$

$$= \lim_{z \to ae^{i\pi/4}} \left\{ \frac{1}{z^2 + (ae^{i\pi/4})^2 \times (z + ae^{i\pi/4})} \right\}$$

$$= \frac{1}{a^2 e^{i\pi/2} + a^2 e^{i\pi/2}} \times \frac{1}{ae^{\pi i/4} + ae^{\pi i/4}}$$

$$= \frac{1}{2a^2 e^{i\pi/2} \times 2ae^{\pi i/4}}$$

$$= \frac{1}{4a^3 e^{3\pi i/4}} \quad \text{(Ans)}$$

$\boxed{\text{Q}}$

$$\lim_{z \to e^{\pi i/6}} \left\{ (z - e^{\pi i/6}) \frac{1}{z^6 + 1} \right\}$$

$$= \lim_{z \to e^{\pi i/6}} \left\{ \frac{1}{6z^5} \right\}$$

$$= \frac{1}{6e^{5\pi i/6}} = \frac{1}{6} e^{-5\pi/6 \, i} \quad \text{(Ans)}$$

Continuity :- $f(z)$ be continous at $z = z_0$ if

(i) $\lim_{z \to z_0} f(z) = l$ must exists.

(ii) $f(z_0)$ must exists i.e $f(z)$ is defined at $z = z_0$

(iii) $l = f(z_0)$

If $f(z) = \begin{cases} z^2 & ; z \neq i \\ 0 & ; z = i \end{cases}$ Is the function $f(z)$ continous at $z = i$? If not redefined the function to be continous.

Sol^n:- $\lim_{z \to i} f(z) = \lim_{z \to i} z^2 = i^2 = -1$ exists.

(ii) $f(z_0) = f(i) = 0$ exists

(iii) $l \neq f(z_0)$ i.e $-1 \neq 0$

So the function $f(z)$ does not continous at z

If we define $f(z) = z^2$ for all values of z is continous at $z = i$.

(11)

$f(z) = \begin{cases} \dfrac{z^2+4}{z-2i} & ; z \neq 2i \\ 3+4i & ; z = 2i \end{cases}$ If not redefine the function to be continous.

Sol^n:- (i) $\lim_{z\to 2i} \dfrac{z^2+4}{z-2i}$

$= \dfrac{(z+2i)\,(z-2i)}{(z-2i)} = z+2i = 4i$

(ii) $f(z_0) = f(2i) = 3+4i$

(iii) Here $l \neq f(z_0)$ i.e $4i \neq 3+4i$

$\therefore$ The $f(z)$ in not continous at $z=2i$

Redefine:- If we define $f(z) = \dfrac{z^2+4}{z-2i}$ for all value of z.

If $f(z) = \begin{cases} z^2+2z & ; z\neq i \\ 3+2i & ; z=i \end{cases}$, Is the function continous at $z=i$? If not, redefine the function to be continous.

Sol^n:- (i) $\lim_{z\to i} (z^2+2z) = i^2+2i = -1+2i$

(ii) $f(z_0) = f(i) = 3+2i$

(iii) Here $l \neq f(z_0)$ i.e $-1+2i \neq 3+2i$

$\therefore$ The $f(z)$ is not continous at $z = i$;

Redefine :-

If we define $f(z) = z^2 + 2z$ for all values of z.

Branch point :-

☐ Prove that the zeros of (i) $\sin z$ (ii) $\cos z$ are all real and find them.

(i) $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i} = 0$ OR, $e^{iz} = \dfrac{1}{e^{iz}} \Rightarrow e^{2iz} = 1$

$\Rightarrow \sin z = e^{2k\pi i}$

$\Rightarrow z = k\pi$; i.e $z = 0, \pm\pi, \pm 3\pi \cdots$

(ii) $\cos z = \dfrac{e^{iz} + e^{-iz}}{2i} = 0 \Rightarrow e^{2iz} = -1 = e^{(2k+1)\pi i}$

$\therefore z = (k + \tfrac{1}{2})\pi$ i,e $z = \pm \pi/2, \pm 3\pi/2 \cdots$

☐ Evaluate:- (a) $\lim\limits_{z \to \frac{-a+\sqrt{a^2-b^2}}{b}} \left\{ \left(z - \frac{-a+\sqrt{a^2-b^2}}{b}\right) \frac{2}{bz^2+2aiz-b} \right.$

(b) $\lim\limits_{z \to \pi i} \dfrac{d}{dz} \left\{ (z-\pi i)^2 \dfrac{e^z}{(z-\pi i)^2 (z+\pi i)^2} \right\}$

(i) $\lim\limits_{z \to \frac{-a+\sqrt{a^2-b^2}}{b}} \left\{ \left(z - \frac{-a+\sqrt{a^2-b^2}}{b}\right) \left(\frac{2}{bz^2+2aiz-b}\right) \right\}$

$= \lim\limits_{z \to \frac{-a+\sqrt{a^2-b^2}}{b}} \left\{ \dfrac{2}{2bz+2ai} \right\}$

$$= \lim_{z \to \frac{-a+\sqrt{a^2-b^2}}{b}} \left\{ \frac{1}{bz+ai} \right\}$$

$$= \frac{1}{\cancel{b} \times \frac{-a+\sqrt{a^2-b^2}}{\cancel{b}} + ai}$$

$$= \frac{1}{-a+\sqrt{a^2-b^2}+ai}$$

(ii) $$\lim_{z \to \pi i} \frac{d}{dz} \left\{ (z-\pi i)^2 \frac{e^z}{(z-\pi i)^2 (z+\pi i)^2} \right\}$$

$$= \lim_{z \to \pi i} \frac{d}{dz} \left\{ \frac{e^z}{(z+\pi i)^2} \right\}$$

$$= \lim_{z \to \pi i} \left\{ e^z \times -2(z+\pi i)^{-3} + (z+\pi i)^{-2} e^z \right\}$$

$$= e^{\pi i} (-2) \frac{1}{(\pi i + \pi i)^3} + \frac{e^{\pi i}}{(\pi i + \pi i)^2}$$

$$= \frac{e^{\pi i}}{(\pi i + \pi i)^2} \left\{ \frac{-2}{\pi i + \pi i} + 1 \right\}$$

$$= \frac{e^{\pi i}}{(2\pi i)^2}\left\{\frac{-\cancel{2}}{\cancel{2}\pi i}+1\right\}$$

$$= \frac{e^{\pi i}}{4\pi^2 i^2}\left\{\frac{-1+\pi i}{\pi i}\right\}$$

$$= \frac{1}{4\pi^2(-1)}(\cos\pi + i\sin\pi)\left\{\frac{-1+\pi i}{\pi i}\right\}$$

$$= \frac{\pi i - 1}{4\pi^3 i} = \frac{\pi i + i^2}{4\pi^3 i} = \frac{\pi + i}{4\pi^3} \text{ (Ans)}$$

CT-Question solve (10 series)

1) Evaluate $\oint_C \frac{dz}{z(z^2+9)}$ where C is the circle $|z|=1$

Sol^n:- $\oint_C \frac{\frac{1}{z^2+9}}{z-0}\,dz$

$f(z) = 1/z^2+9$ $\quad a=0$

$f(a) = 1/9$

$$\therefore \oint_C \frac{dz}{z(z^2+9)} = 2\pi i \times 1/9$$

$$= \frac{2\pi i}{9}$$

□ Evaluate $\int \bar{z}\, dz$ along the curve $y = x$ from $(0,0)$ to $(1,1)$

<u>Sol^n:-</u>

$$y = x \quad \therefore \quad dy = dx$$

limits: $x = 0 \quad y = 0$

$\qquad\quad x = 1 \quad y = 1$

$$\int \bar{z}\, dz = \int (x - iy)(dx + i\,dy)$$

$$= \int (x\,dx - iy\,dx + i\,x\,dy + y\,dy)$$

$$= \int (x\,dx + y\,dy) - i\,(x\,dy \cancel{+} - y\,dx)$$

$$= \int (x\,dx + x\,dx) - i\,(x\,dy - x\,dy)$$

$$= \int_0^1 2x\,dx$$

$$= \cancel{2}\,\frac{x^2}{\cancel{2}}\Big|_0^1 = \cancel{0}\ 1$$

10

(b) Prove that $u = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic. Find a function v such that $u + iv$ is analytic.

Soln:-

$$\frac{\partial u}{\partial x} = 2x - 2y - 2 \qquad \frac{\partial u}{\partial y} = -2y - 2x + 3$$

$$\frac{\partial^2 u}{\partial x^2} = 2 \qquad \frac{\partial^2 u}{\partial y^2} = -2$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = 2 - 2 = 0$$

So that the function is harmonic.

* Since $u + iv$ is analytic. So that,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x - 2y - 2 \quad \text{--- (i)}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y + 2x - 3 \quad \text{--- (ii)}$$

Integrating (i) w.r. to y and keeping x const

$$\therefore \int dv = \int (2x - 2y - 2)\, dy$$

$$\Rightarrow v = 2xy - y^2 - 2y + F(x) \quad \text{--- (3)}$$

$$\frac{\partial}{\partial x}\left(2xy - y^2 - 2y + F(x)\right) = 2y + 2x - 3$$

$$\Rightarrow 2y + F'(x) = 2y + 2x - 3$$

$$\Rightarrow F'(x) = 2x - 3$$

$$\therefore f(x) = x^2 - 3x + C$$

$$\therefore v = 2xy - y^2 - 2y + x^2 - 3x + C$$

Complex differentiation

Definition of Derivatives: If $f(z)$ is single valued in some region R, of the z plane, the derivative of $f(z)$ is defined as $f'(z) = \lim_{\Delta z \to 0} \dfrac{f(z+\Delta z) - f(z)}{\Delta z}$ provided the limit exists, independent of the manner in which $\Delta z \to 0$.

Analytic function: If the derivative $f'(z)$ exists at a point z of a region R, then $f(z)$ is said to be analytic in R.

Example: Test the analyticity of the function $f(z) = \bar{z}$

Sol^n:- By definition $f'(z) = \dfrac{f(z+\Delta z) - f(z)}{\Delta z}$, if provide that the limit exists, independent of the manner in which $\Delta z = \Delta x + i\Delta y \to 0$ thus

$$f'(z) = \frac{d f(z)}{dz} = \lim_{\Delta z \to 0} \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\overline{(x+iy) + (\Delta x + i\Delta y)} - \overline{(x+iy)}}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{x - iy + \Delta x - i\Delta y - (x - iy)}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

~~If $\Delta y = 0$, the required limit is $\lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1$~~

~~If $\Delta x = 0$, the required limit is $\lim_{\Delta y \to 0} \frac{\Delta x}{\Delta x} = 1$~~

If $\Delta y = 0$, the required limit is $\lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$

If $\Delta x = 0$, the required limit is $\lim_{\Delta y \to 0} \frac{-i\Delta y}{+i\Delta y} = -1$

Since the limit depends on the manner in which $\delta z \to 0$, the derivative does not exist.

So, $f(z) = \bar{z}$ is not analytic.

Cauchy - Rieman Equation :-

A necessary condition that,

$w = f(z) = u(x, y) + iv(x, y)$ be analytic in a re

R if u and v satisfy the cauchy Rieman equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Prove that a (i) Necessary and (ii) sufficient cond

that $w = f(z) = u(x, y) + iv(x, y)$ be analytic in a re

R is that the cauchy Rieman equation.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$ are satisfied in

Proof (i) : In order be for $f(z)$ to be analytic the limit, $f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)\} - \{u(x,y) + iv(x,y)\}}{\Delta x + i\Delta y} \quad \text{--- (1)}$$

must exists independent of the manner in which $\Delta z \to 0$

Case-I : when $\Delta y = 0$ then $\Delta x \to 0$, in this case eq^n (1) becomes

$$\lim_{\Delta x \to 0} \frac{\{u(x+\Delta x, y) + iv(x+\Delta x, y)\} - \{u(x,y) + iv(x,y)\}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[\left\{ \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} \right\} + i \left\{ \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} \right\} \right]$$

$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

Provided the practical derivative exists.

Case-II : when $\Delta x = 0$ then $\Delta y \to 0$, in this case eq^n (1) becomes

$$f'(z) = \lim_{\Delta y \to 0} \frac{\{u(x, y+\Delta y) + iv(x, y+\Delta y)\} - \{u(x,y) + iv(x,y)\}}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \left[\left\{ \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} \right\} + i \left\{ \frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y} \right\} \right]$$

$$= \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Now, $f(z)$ can not possibly be analytic unless the two limits are identical (case I & case II). Thus a necessary condition that $f(z)$ be analytic is,

$$-i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

So we have, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ [Proved]

Proof (ii) : Since $\frac{\partial u}{\partial x}$ and $\partial u/\partial y$ are supposed continuous we have, $\Delta u = u(x+\Delta x, y+\Delta y) - u(x,y)$

$$= \{u(x+\Delta x, y+\Delta y) - u(x, y+\Delta y)\} + \{u(x, y+\Delta y) - u(x,y)\}$$

$$= \left(\frac{\partial u}{\partial x} + e_1\right)\Delta x + \left(\frac{\partial u}{\partial y} + \eta_1\right)\Delta y$$

$$= \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial x}\Delta y + e_1 \Delta x + \eta_1 \Delta y$$

where $e_1 \longrightarrow 0$ and $\eta_1 \longrightarrow 0$ as $\Delta x \to 0$ and $\Delta y \to 0$

Similarly, since $\partial v/\partial x$ and $\partial v/\partial y$ are supposed ~~condition~~ continous, we have

$$\Delta v = (\partial v/\partial x + e_2)\Delta x + \left(\frac{\partial v}{\partial y} + \eta_2\right)\Delta y = \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}$$

$$+ e_2 \Delta x + \eta_2 \Delta y$$

where $e_2 \to 0$ and $\eta_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$. Then

$$\Delta w = \Delta u + i\Delta v = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Delta y$$

$$+ e\Delta x + \eta\Delta y \quad \text{——— (2)}$$

where, $e = e_1 + ie_2 \to 0$ and $\eta = \eta_1 + i\eta_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$

By the cauchy - Rieman equations, (2) can be written

$$\Delta w = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\Delta y + e\Delta x + \eta\Delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\Delta x + i\Delta y) + e\,\Delta x + \eta\,\Delta y$$

Then on dividing by $\Delta z = \Delta x + i\Delta y$ and taking the limit as $\Delta z \to 0$, we see that

$$\frac{dw}{dz} = f'(z) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

so that the derivative exists and is unique, i.e $f(z)$ is analytic in R.

Test the analyticity of the followings.

(i) $f(z) = ize^{-z}$ (ii) $f(z) = ze^{-\frac{z}{2}}$ (iii) $iz^2 + 2z$

(i) <u>Sol^n</u>: $f(z) = u(x,y) + iv(x,y) = ize^{-z}$

$$= i(x+iy)e^{-(x+iy)}$$

$$= (ix - y)e^{-x} \cdot e^{-iy}$$

$$= (ix - y)e^{-x}(\cos y - i\sin y)$$

$= e^{-x}(ix\cos y + x\sin y - y\cos y + iy\sin y)$

$= e^{-x}(x\sin y - y\cos y) + ie^{-x}(x\cos y + y\sin y)$

Now, $\frac{\partial u}{\partial x} = e^{-x}\sin y - e^{-x}x\sin y + e^{-x}y\cos y$

$\frac{\partial v}{\partial y} = -e^{-x}x\sin y + e^{-x}\sin y + e^{-x}y\cos y$

$= -xe^{-x}\sin y + e^{-x}(y\cos y + \sin y)$

$= e^{-x}\sin y - e^{-x}x\sin y + e^{-x}y\cos y$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, which satisfies cauchy Riema equation.

hence, $f(z) = ize^{-z}$ is analytic.

(ii) (ii) Sol^n:- $f(z) = ze^{-z}$

$f(z) = u(x,y) + iv(x,y) = ze^{-z}$

$= (x+iy)e^{-(x+iy)}$

$= (x+iy)e^{-x}.e^{-iy} = (x+iy)e^{-x}(\cos y - i\sin$

$= e^{-x}(x\cos y - ix\sin y + iy\cos y + y\sin y)$

$= e^{-x}(x\cos y + y\sin y) + ie^{-x}(y\cos y - x\sin y)$

$\therefore \frac{\partial u}{\partial x} = e^{-x}\cos y - e^{-x}x\cos y - e^{-x}y\sin y$

$\therefore \frac{\partial v}{\partial y} = e^{-x}\cos y - ye^{-x}\sin y - xe^{-x}\cos y$

Here, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, which satisfies cauchy Rieman equation. Hence $f(z) = ze^{-z}$ is analytic.

(iii) Soln: $f(z) = iz^2 + 2z$

$f(z) = u(x,y) + iv(x,y) = iz^2 + 2z$

$= i(x+iy)^2 + 2(x+iy)$

$= i(x^2 + i2xy - y^2) + 2x + i2y$

$= ix^2 - 2xy - iy^2 + 2x + i2y$

$\cancel{= (2x - 2xy) + i(x^2 - y^2)}$

$= (2x - 2xy) + i(x^2 - y^2 + 2y)$

$\therefore \frac{\partial u}{\partial x} = 2 - 2y$

$\therefore \frac{\partial v}{\partial y} = -2y + 2 = 2 - 2y$

Here, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, which satisfies cauchy Rieman equation. Hence, $f(z) = iz^2 + 2z$ is analytic.

Q. Find the analyticity of the function $f(z) = \frac{1+z}{1-z}$

Solⁿ: we know that,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\frac{1+z+\Delta z}{1-z-\Delta z} - \frac{1+z}{1-z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{1}+\cancel{z}+\Delta z-\cancel{z}-\cancel{z^2}-\cancel{z\Delta z}-\cancel{1}+\cancel{z}+\Delta z-\cancel{z}+\cancel{z^2}+\cdots}{(1-z)(1-z-\Delta z)(\Delta z)}$$

$$= \lim_{\Delta z \to 0} \frac{2\Delta z}{(1-z)(1-z-\Delta z)(\Delta z)}$$

$$= \lim_{\Delta z \to 0} \frac{2}{(1-z)(1-z-\Delta z)}$$

$$= \frac{2}{(1-z)^2}$$

which is independent of the manner in which $\Delta z \to 0$, provided $z \neq 1$.

So that the function is analytic.

$\boxed{09}$

Evaluate $\frac{1}{2\pi i}\oint \frac{e^{zt}}{z+1}\,dz$, $t>0$ where c is a circle $|z|=2$.

Solⁿ:- $f(z) = e^{zt}$ $a=-1$

$f(-1) = e^{-t}$

$$f(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-a}\,dz$$

$$\Rightarrow \frac{1}{2\pi i}\oint \frac{e^{zt}}{z+1}\,dz = \cancel{2\pi i \times}\, e^{-t}$$

Evaluate : $\frac{1}{2\pi i}\oint_c \frac{e^{zt}}{z^2+1}\,dz$

$z = \pm i$

$$\lim_{z\to i} (z-i)\frac{e^{zt}}{(z+i)(z-i)}$$

$$= e^{it}/2i$$

$$\lim_{z\to -i} (z+i)\frac{e^{zt}}{(z+i)(z-i)}$$

$$= \frac{e^{-it}}{-2i}$$

$$\frac{1}{2\pi i}\oint \frac{e^{zt}}{z^2+1} = \frac{e^{it}-e^{-it}}{2i} = \sin t$$

Laplacian: The operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is called the laplacian.

(a) **Harmonic function:** $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ & $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} =$ is called the harmonic function.

Prove that real and imaginary part of an analytic function $f(z)$ satisfy Laplacian's / Laplace's equation

Proof: Since the function $f(z) = u(x,y) + iv(x,y)$ be an analytic function. So it satisfies the cauchy's Rieman equation.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{--- (i)} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{--- (ii)}$$

Differentiating (i) and w.r. to x and Diff (ii) w.r. to y we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x . \partial y} \quad \text{--- (iii)}$$

and

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x . \partial y} \quad \text{--- (iv)}$$

By adding (iii) and (iv) we have,

$$\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} = 0$$

Similarly diff eq^n (i) w.r. to y and Diff. eq^n (ii) w.r. to x we can get the other laplace's equation.

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2} \quad \text{--- (v)} \qquad \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2} \quad \text{--- (vi)}$$

By substracting (v) & (vi) we have,

$$\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2}$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Hence we can say that, real and imaginary pa of an analytic function $f(z)$ satisfy laplace's eq

(10) # Show that (a) $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harm

(b) Find v such that $f(z) = u + iv$ is analytic

(c) Find also $f(z)$ in terms of z.

Solⁿ:- (a) Given, $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$$\therefore \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x$$

$$\frac{\partial^2 u}{\partial x^2} = 6x + 6 \quad \text{——— (i)}$$

$$\frac{\partial u}{\partial y} = -6xy - 6y$$

$$\frac{\partial^2 u}{\partial y^2} = \cancel{-6y - 6} - 6x - 6 \quad \text{——— (ii)}$$

From (i) + (ii), we get,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$ hence u is harmonic.

Sol^n (b) :- From cauchy Rieman equation we have

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x \quad \text{——— (iii)}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -(-6xy - 6y) = 6xy + 6y \quad \text{——— (iv)}$$

Intregrating equation (iii) w.r. to y and keeping x be constant, we have,

$$\int dv = \int (3x^2 - 3y^2 + 6x)\, dy$$

$$\Rightarrow v = 3x^2y - y^3 + 6xy + F(x) \quad \text{--- (v)}$$

where $F(x)$ is an arbitrary real function of x.

Now, substituting (v) into equation (iv) we have,

$$\frac{\partial}{\partial x}\{(3x^2y - y^3 + 6xy + F(x)\} = 6xy + 6y$$

$$\Rightarrow 6xy + 6y + F'(x) = 6xy + 6y$$

$$\Rightarrow F'(x) = 0$$

$$\Rightarrow F(x) = c \quad \text{[by integrating]}$$

Now, equation (v) becomes,

$$v = 3x^2y - y^3 + 6xy + c.$$

Sol^n:- (c) $f(z) = f(x+iy) = (x^3 - 3xy^2 + 3x^2 - 3y^2 + 1) + i(3x^2y - y^3 + 6xy + c)$

Putting $y = 0$; $f(x) = u(x,0) + iv(z,0)$ ——— (vi)

Replacing x by z,

$$f(z) = u(z,0) + iv(z,0) \quad \text{——— (vi)}$$

Now, $u(x,0) = x^3 + 3x^2 + 1 \qquad \therefore u(z,0) = z^3 + 3z^2 + 1$

$v(x,0) = c \qquad \therefore v(z,0) = c$

hence, from (vi),

$$f(z) = z^3 + 3z^2 + 1 + ic$$

Show that (i) $u = 2x(1-y)$ is harmonic (ii) Find v such that $f(z) = u + iv$ is analytic (iii) Find also $f(z)$ interm of z.

Sol^n (i) : Given $u = 2x - 2xy$

$$\therefore \frac{\partial u}{\partial x} = 2 - 2y \qquad \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{——— (i)}$$

$$\frac{\partial u}{\partial y} = -2x \qquad \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{——— (ii)}$$

Adding (i) & (ii), we have,

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} = 0,$$ hence it is harmonic.

(ii) Sol^n : From cauchy's Riemann equation we

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2 - 2x \quad \text{——— (iii)}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -(-2x) = 2x \quad \text{——— (iv)}$$

Integrating equation (iii) w.r.t. to y and keeping x constant,

$$\int dv = \int (2-2y)\,dy \Rightarrow v = 2y - y^2 + F(x) \quad \text{—— (v)}$$

where $F(x)$ is an arbitrary real function of x

Now substituting (v) in eqn (iv) we have,

$$\frac{\partial}{\partial x}\{2y - y^2 + F(x)\} = 2x$$

$$\Rightarrow \quad F'(x) = 2x$$

$$\Rightarrow \quad F(x) = x^2 + c \quad \text{[Integrating]}$$

$$\therefore v = 2y - y^2 + x^2 + c$$

(iii) Soln :- $f(z) = f(x+iy) = u + iv$

$$= (2x - 2xy) + i(x^2 - y^2 + 2y + c)$$

Putting $y = 0$

$$\therefore f(x) = \cancel{2x}\; u(x,0) + iv(x,0)$$

Replacing x by z,

$$f(z) = u(z,0) + iv(z,0) \quad \text{—— (vi)}$$

Now, $u(x,0) = 2x$ $\qquad \therefore u(z,0) = 2z$

$v(x,0) = x^2 + c$ $\qquad v(z,0) = z^2 + c$

$\therefore f(z) = 2z + i(z^2 + c)$

If $u_1(x,y) = \cancel{\partial y}\frac{\partial u}{\partial x}$ & $u_2(x,y) = \frac{\partial u}{\partial y}$ prove that

$f'(z) = u_1(z,0) - i\,u_2(z,0) = \left[\frac{\partial u}{\partial x}\right]_{\substack{x=z\\y=0}} - i\left[\frac{\partial u}{\partial y}\right]_{\substack{x=z\\y=0}}$

Sol^n:- Let, $f(z) = u + iv$, be analytic, so from Cauchy Riemann equation, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ —(i)

Now diff $f(z)$ w.r.to x we have

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \quad [\text{From eq}^n \text{ (i)}]$$

$\Rightarrow f'(x+iy) = u_1(x,y) - i\,u_2(x,y)$ —— (ii)

Putting $y=0$ in eq^n (ii) and replacing x by z we have the required result.

$$f'(z) = u_1(z,0) - i\,u_2(z,0)$$

$$= \left[\frac{\partial u}{\partial x}\right]_{\substack{x=z\\y=0}} - i\left[\frac{\cancel{\partial y}\,\partial u}{\partial y}\right]_{\substack{x=z\\y=0}} \quad \text{(Ans)}$$

If $v_1(x,y) = \frac{\partial v}{\partial y}$ and $v_2(x,y) = \frac{\partial v}{\partial x}$ prove that

$f'(z) = v_1(z,0) + i\,v_2(z,0) = \left[\frac{\partial v}{\partial y}\right]_{\substack{x=z\\y=0}} + i\left[\frac{\partial v}{\partial x}\right]_{\substack{x=z\\y=0}}$

Sol^n: Let $f(z) = u + iv$ be analytic so it satisfies cauchy's Riemann equation.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{—— (i)}$$

Now differentiating $f(z)$ w.r.t x

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} \cancel{+ i \frac{\partial u}{\partial y}}$$

$$= \frac{\partial v}{\partial y} + i \frac{\partial v}{\partial x} \quad [\text{using eq}^n \text{ (i)}]$$

$$\Rightarrow f'(z) = \frac{\partial v}{\partial y} + i \frac{\partial v}{\partial x}$$

$$\Rightarrow f'(x+iy) = v_1(x,y) + i v_2(x,y) \quad \text{—— (ii)}$$

Putting $y = 0$ and replacing x by z we have,

$$f'(z) = v_1(z,0) + i v_2(z,0) \quad \text{(Ans)}$$

If $\text{Im}\{f'(z)\} = 6x(2y-1)$ and $f(0) = 3-2i$; $f(1) = 6-5$

Find, $f(z)$ and also $f(1+i)$

Sol^n: $f(z) = u + iv$; Diff. w.r. to x

$$\therefore f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i \frac{\partial v}{\partial x}$$

$$= v_1(z,0) + i v_2(z,0) \quad \text{—— (i)}$$

$$\therefore \text{Im}\{f'(z)\} = \frac{\partial v}{\partial x} = 6x(2y-1) \quad \text{—— (ii)}$$

Intregreating eqn (ii) w.r. to x and keeping y constant

$$\int dv = \int (12xy - 6x)\, dx$$

$$\Rightarrow v = 6x^2y - 3x^2 + F(y) \quad \text{(iii)}$$

where $F(y)$ is an arbitrary imaginary function of y.

Differentiating eqn (iii) with respect to y we get,

$$\frac{\partial v}{\partial y} = 6x^2 + F'(y) \quad \text{(iv)}$$

From (i) we have using (ii) & (iv)

$$f'(z) = 6z^2 + F'(0) - i6z$$

$$= 6z^2 - i6z + f_1 \quad [F'(0) = f_1]$$

so, $f(z) = 2z^3 - 3iz^2 + f_1 z + f_2$ — (v) [by integration]

Now, $f(0) = 3-2i$ implies that $3-2i = f_2$

Again, $f(1) = 6-5i$ " " $6-5i = 2-3i + f_1 + 3 - 2i$

$$\Rightarrow f_1 = 1$$

then, $f(z) = 2z^3 - 3iz^2 + z + 3 - 2i$

$$f(1+i) = 2(i+1)^3 - 3i(1+i)^2 + (1+i) + 3 - 2i$$

$$= 2(i^3 + 3i^2 + 3i + 1) - 3i - 3i^2 + 1 + i + 3 - 2i$$

$$= 2(-i - 3 + 3i + 1) - 3i + 3 + 1 + i + 3 - 2i$$

~~$8+8i+7$~~

$= 2(1+3i+3i^2-i) - 3i(1+2i+i^2)+1+i+3-$

$= 4i-4-3i+6+3i+1+i+3-2i$

$= 6+3i$ (Ans)

Singular point :-

A point at which $f(z)$ tends to be analytic called singular point.

08

Various types of singular point :-

(1) Isolated singular point :-

$|z-z_0|<d$, no singular point other than z_0

Ex : $f(z) = \frac{z}{z^2+4} = \frac{z}{(z+2i)(z-2i)}$

$z = \pm 2i$ isolated singularities.

(2) Poles : $\lim_{z\to z_0} (z-z_0)^n f(z) = A \neq 0$ then $z=z_0$ is called poles of order n.

If $n=1$ then it is called simple points.

Ex : $f(z) = \frac{z}{(z^2+4)^2} = \frac{z}{(z-2i)(z+2i)^2}$ has poles of order 2 at $z = \pm 2i$

3. Branch point :

Ex : $f(z) = (z-3)^{\frac{1}{2}}$ has a branch point at $z=3$

4. Removable singularity :

Ex : $f(z) = \frac{\sin z}{z}$ since $\lim_{z \to 0} \frac{\sin z}{z} = 1$

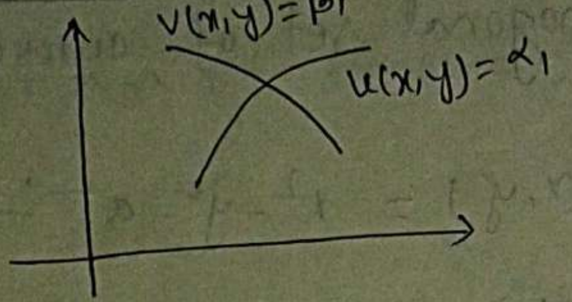
5. Essential singularities :-

Ex : $f(z) = e^{\frac{1}{z-2}}$ has an essential singularities. at $z = 2$

Orthogonal families :-

Let, $u(x,y) = \alpha$ and $v(x,y) = \beta$ where, u and v are the real and imaginary parts of an analytic function $f(z)$ and α and β are any constant represent two families of curves Prove that the families are orthogonal.

Proof : Consider any two points member of the respective families say $u(x,y) = \alpha$ and $v(x,y) = \beta$, where α and β are particular constant fig



Differentiating $u(x,y) = \alpha_1$ w.r. to x yeilds

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial x} = 0$$

Then the slope of $u(x,y) = \alpha_1$ is

$$dy/dx = -\frac{\partial u}{\partial x} \Big/ \frac{\partial u}{\partial y} = m_1$$

~~Similarly the slope is, using the cauchy-Riemann equation,~~ $\cancel{\partial u/\partial x \times \frac{\partial v}{\partial x} \Big/ \partial u/\partial y \times \frac{\partial v}{\partial y} =}$

Similarly, the sople of $v(x,y) = \beta_1$ is

$$dy/dx = +\frac{\partial v}{\partial x} \Big/ \partial v/\partial y = m_2$$

The product of the slope is, using cauchy-Rieman equations,

$$m_1 \times m_2 = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} \Big/ \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = -\frac{\partial v}{\partial y}\frac{\partial u}{\partial y} \Big/ \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = -1$$

Then the curve are orthogonal.

Find the orthogonal set of curves $x^2-y^2=a^2$

Solⁿ:- Let, $u(x,y) = x^2-y^2-a^2$ —— (i)

∴ According to Cauchy-Riemann equation

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x \quad \text{—— (ii)}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y \quad \text{—— (iii)}$$

Intregreting eqⁿ (ii) w.r.t y and keeping x constant we have,

$$\int dv = \int 2x\,dy$$

$$\Rightarrow v = 2xy + F(x) \quad \text{—— (iv)}$$

where $F(x)$ is a arbitary function of (x).

Putting this value in eqⁿ (iii)

$$\partial/\partial x\,[2xy + F(x)] = 2y$$

$$\Rightarrow 2y + F'(x) = 2y$$

$$\Rightarrow F'(x) = 0$$

By Integreting : $F(x) = c$

$\therefore$ Equation (iv) becomes,

$$v = 2xy + c$$

Now, $$m_1 \times m_2 = \frac{\partial u/\partial x \times \frac{\partial v}{\partial x}}{\partial u/\partial y \times \partial v/\partial y}$$

$$= \frac{2x \times 2y}{-2y \times 2x}$$

$$= -1$$

Since, $m_1 \times m_2 = -1$ then $v = 2xy + c$ is the required set of the ~~org~~ orthogonal.

Find the orthogonal projections of the family of curves in x, y plane defined by $e^{-x}(x \sin y - y \cos y) = \alpha$ where α is real constant.

Soln:- Let, $u(x, y) = e^{-x}(x \sin y - y \cos y) - \alpha$

According to the Cauchy-Riemann equation

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^{-x} \sin y - x e^{-x} \sin y + y e^{-x} \cos y \quad \text{(i)}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -(+e^{-x} x \cos y + y e^{-x} \sin y - e^{-x} \cos y) \quad \text{(ii)}$$

$$= -x\bar{e}^{x}\cos y - y\bar{e}^{x}\sin y + \bar{e}^{x}\cos y \quad \text{—(ii)}$$

Integreating eqn (i) w. r. to y and keeping x constant.

$$\int v\,dv = \int \left(\bar{e}^{x}\sin y - x\bar{e}^{x}\sin y + y\bar{e}^{x}\cos y\right) dy$$

$$\Rightarrow v = -\bar{e}^{x}\cos y + x\bar{e}^{x}\cos y + y\bar{e}^{x}\sin y + \bar{e}^{x}\cos y + f$$

$$= x\bar{e}^{x}\cos y + y\bar{e}^{x}\sin y + f(x) \quad \text{—(iii)}$$

Putting this value in eqn (ii)

$$\frac{\partial}{\partial x}\left[x\bar{e}^{x}\cos y + y\bar{e}^{x}\sin y + f(x)\right] = -x\bar{e}^{x}\cos y - y\bar{e}^{x}\sin y + \bar{e}^{x}\cos y$$

$$\Rightarrow \cancel{\bar{e}^{x}\cos y} - \cancel{x\bar{e}^{x}\cos y} - \cancel{y\bar{e}^{x}\sin y} + f'(x) = \cancel{-x\bar{e}^{x}\cos y} - \cancel{y\bar{e}^{x}\sin y} + \cancel{\bar{e}^{x}\cos y}$$

$$\Rightarrow f'(x) = 0$$

Intnegrating this, $f(x) = c$

eqn (iii) becomes, $v = x\bar{e}^{x}\cos y + y\bar{e}^{x}\sin y + c$

Now, $m_1 \times m_2 = \dfrac{\partial u/\partial x \times \frac{\partial v}{\partial x}}{\partial u/\partial y \times \frac{\partial v}{\partial y}}$

$$= \frac{(e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y)(-xe^{-x}\cos y - ye^{-x}\sin y + e^{-x}\cos y)}{(xe^{-x}\cos y + ye^{-x}\sin y - e^{-x}\cos y)(e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y)}$$

$$= -1$$

Since the production of the slopes of the two curves is -1 hence $v = xe^{-x}\cos y + ye^{-x}\sin y + c$ is the requiring orthogonal projections of the given curve.

(8)

Find the orthogonal trajectories of the family of curve $x^3y - xy^3 = c$

Soln:- $u = x^3y - xy^3 - c$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 3x^2y - y^3 \quad \text{--- (i)}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -(x^3 - 3xy^2) = -x^3 + 3xy^2 \quad \text{--- (ii)}$$

Integrating (i) with r. to y and keeping x constant.

$$\int dv = \int (3x^2y - y^3)\, dy$$

$$v = \frac{3}{2}x^2y^2 - \frac{y^4}{4} + f(x) \quad \text{(iii)}$$

$$\frac{\partial}{\partial x}\left\{\frac{3}{2}x^2y^2 - \frac{y^4}{4} + f(x)\right\} = -x^3 + 3xy^2$$

$$\Rightarrow 3xy^2 + f'(x) = -x^3 + 3xy^2$$

$$\Rightarrow f'(x) = -x^3$$

$$\therefore v = \frac{3}{2}x^2y^2 - y^4/4 - x^3$$

$$m_1 \times m_2 = \frac{\partial u/\partial x \times \partial v/\partial x}{\partial u/\partial y \times \partial v/\partial y}$$

$$= \frac{(3x^2y - y^3)\ (x^3 - 3xy^2)}{(-x^3 + 3xy^2)\ (3x^2y - y^3)}$$

$$= -1$$

Complex Integration :—

Definition :- The complex line integration define as

$$\int_c f(z)\,dz = \int_c (u+iv)(dx+i\,dy)$$

$$= \int_c (u\,dx - v\,dy) + i\int_c (v\,dx + u\,dy)$$

$\oint_c f(z)\,dz \rightarrow$ close curve integration.

05) Evaluate $\int \bar{z}\,dz$ from $z=0$ to $z=4+2i$ along the curve c given by $z=t^2+it$

Soln:- The limit point, $z=0$ then $t=0$

when, $z=4+2i$ then $t=2$

So the integral, $\int \bar{z}\,dz = \int_{t=0}^{2} (t^2-it)\,d(t^2+it)$

$$= \int_{t=0}^{2} (t^2-it)(2t+i)\,dt$$

$$= \int_{t=0}^{2} (2t^3 - 2it^2 + t^2 i + t)\,dt$$

$$= \frac{t^4}{2}\Big|_0^2 - \frac{2i}{3}t^3\Big|_0^2 + i\frac{t^3}{\cancel{4}3}\Big|_0^2 + \frac{t^2}{2}\Big|_0^2$$

$$= \frac{1}{2}\times 16 - \frac{2i}{3}\times 8 + \frac{i}{\cancel{4}}\times \cancel{8}^{2} + \cancel{\frac{1}{2}}\times \cancel{4}^{2}$$

$$= 8 - \frac{16i}{3} + 2i + 2 = 10 -$$

$$= \frac{t^4}{2}\Big|_0^2 - 2i\frac{t^3}{3}\Big|_0^2 + i\frac{t^3}{3}\Big|_0^2 + \frac{t^2}{2}\Big|_0^2$$

$$= \frac{t^4}{2}\Big|_0^2 - i\frac{t^3}{3}\Big|_0^2 + \frac{t^2}{2}\Big|_0^2$$

$$= \tfrac{1}{2}\times 16 - i\frac{8}{3} + 4/2$$

$$= 10 - \frac{8i}{3} \text{ (Ans)}$$

—x—

The line form $z=0$ to $z=2i$ and then the line form $z=2i$ to $z=4+2i$

Soln:- we know,

$$\int_c (x+iy)(dx+idy) = \int_c (xdx - ydy) + i\int_c (xdy + ydx)$$

~~for~~ for $\bar{z}$

$$\int_c (x-iy)(dx+idy) = \int_c (xdx+ydy) + i\int (xdy - ydx)$$

the line from $z=0$ to $z=2i$ is the same as the line from $(0,0)$ to $(0,2)$, so $x=0$ $dx=0$

$$\int_c (x-iy)(dx+idy) = \int_{y=0}^{2} y\,dy + i\int_{y=0}^{2} 0.dy - y.0$$

$$= y^2/2\Big|_0^2$$

$$= 2$$

Again the line form $z = 2i$ to $z = 4+2i$ is the same as this line from $(0,2)$ to $(4,2)$

$\therefore y = 2$, $dy = 0$

$$\therefore \int_C (x - iy)(dx + i\,dy) = \int_{x=0}^{4} x\,dx + i\int_{x=0}^{4} -2\,dx$$

$$= \tfrac{1}{2} \times 4^2 - 2i \times 4 = 8 - 8i$$

Then the required integral $= 2 + 8 - 8i$
$= 10 - 8i$

Evalute $\int_{(0,3)}^{(2,4)} (2y + x^2)\,dx + (3x - y)\,dy$ along (a) the parabola $x = 2t$, $y = t^2 + 3$ (b) straight lines from $(0,3)$ to $(2,3)$ then from $(2,3)$ to $(2,4)$ (c) a straight line from $(0,3)$ to $(2,4)$

Solⁿ:- (a) when, $x = 0$ $t = 0$ $x = 2t$ $y = t^2 + 3$

$x = 2$ $t = 1$ $dx = 2dt$ $dy = 2t\,dt$

$$\therefore \int_0^1 (2t^2 + 6 + 4t^2)\,2dt + (6t - t^2 - 3)\,2t\,dt$$

$$= \int_0^1 (4t^2 + 12 + 8t^2)\,dt + (12t^2 - 2t^3 - 6t)\,dt$$

$$\cancel{= \int_0^1 (-2t^3 + 24t^2 + 6t)\,dt}$$

$$= \int_0^1 (-2t^3 + 24t^2 - 6t + 12)\,dt$$

$$= -\frac{t^4}{2}\Big|_0^1 + 24\times\frac{t^3}{3}\Big|_0^1 - 6\,\frac{t^2}{2}\Big|_0^1 + 12t\Big|_0^1$$

$$= -1/2 + 8 - 3 + 12 = \frac{33}{2} \text{ (Ans)}$$

(b) Here the stright lines from $(0,3)$ to $(2,3)$

$\therefore y = 3,\ dy = 0$

$$\therefore \int_{(0,3)}^{(2,3)} (2y+x^2)\,dx + (3x-y)\,dy = \int_{x=0}^{2} (6+x^2)\,dx$$

$$+ (3x-3)\times 0$$

$$= \int_{x=0}^{2} (6+x^2)\,dx = 6x\Big|_0^2 + \frac{x^3}{3}\Big|_0^2$$

$$= 12 + 8/3 = \frac{44}{3} \text{ (Ans)}$$

Again from stright line $(2,3)$ to $(2,4)$

here, $x = 2 \quad dx = 0$

$$\int_{y=3}^{4} (2y+4)\times 0 + (3\times 2 - y)\,dy$$

$$= \int_3^4 (6-y)\,dy = 6y\Big|_3^4 - y^2/2\Big|_3^4$$

$$= 24 - 18 - \left(\frac{16}{2} - 9/2\right) = 6 - 7/2$$

$$= 5/2$$

$\therefore$ The required integral $= \frac{44}{3} + 5/2 = 103/6$ (Ans)

(c) A straight line joining $(0,3)$ to $(2,4)$ is,

$$\frac{x-0}{0-2} = \frac{y-3}{3-4}$$

$$\Rightarrow x = 2y - 6$$

$$\int_{y=3}^{4} \{2y + (2y-6)^2\} \, d(2y-6) + (6y - 18 - y) \, dy$$

$$= \int_{y=3}^{4} (2y + 4y^2 - 24y + 36) \, 2dy + (5y - 18) \, dy$$

$$= \int_{y=3}^{4} (4y + 8y^2 - 48y + 72) \, dy + (5y - 18) \, dy$$

$$= \int_{y=3}^{4} (8y^2 - 39y - 54) \, dy$$

$$= 8 \frac{y^3}{3} \Big|_3^4 - 39 \, y^2/2 \Big|_3^4 - 54y \Big|_3^4$$

$$= 8 \times \left(\frac{64}{3} - \frac{27}{3}\right) - 39\left(\frac{16}{2} - \frac{9}{2}\right) - 54(4-3)$$

$$= \frac{296}{3} - \frac{273}{2} - 54$$

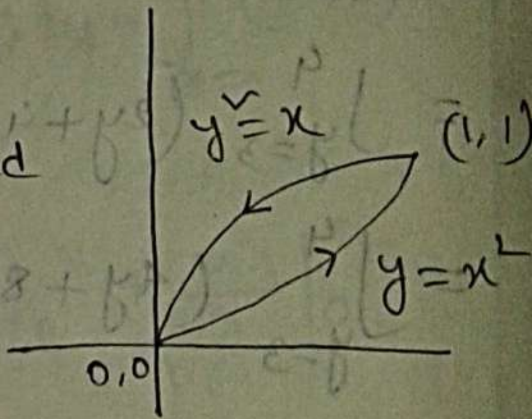
$$= 97/6$$

Green's theorem in the plane,

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Verify Green's theorem in the plane for $\oint_C (2xy - x^2)\,dx + (x + y^2)\,dy$ where C is the closed curve of the region boundend by $y = x^2$ and $y^2 = x$

Sol^n:- The plane curves $y = x^2$ and $y^2 = x$ intersect at $(0,0)$ and $(1,1)$. The positive direction in traversing C is as shown in fig.



Along $y = x^2$ the line integral

$$\int_{x=0}^{1} (2x^3 - x^2)\,dx + (x + x^4)\,2x\,dx$$

$$= \tfrac{1}{2} \times 1 - \tfrac{1}{3} + \tfrac{2}{3} + 2 \times \tfrac{1}{6}$$

$$= \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{2}{3} + \tfrac{1}{3}$$

$$= 7/6$$

Along $y^2 = x$ the line integral.

$$\int_{y=1}^{0} (2y^3 - y^4)\, 2y\, dy + (y^2 + y^2)\, dy$$

$$= \int_{y=1}^{0} (4y^4 - 2y^5 + 2y^2)\, dy$$

$$= -4/5 + 1/3 - 2/3 = -17/15$$

Hence the required line integral $= 7/6 - 17/15$

$= 1/30$

$$\circledast \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} (1-2x)\, dx\, dy$$

$P = 2xy - x^2$

$\frac{\partial P}{\partial y} = 2x$

$Q = x + y^2$

$\frac{\partial Q}{\partial x} = 1$

$$= \int_{x=0}^{1} (y - 2xy)\Big|_{x^2}^{\sqrt{x}}\, dx$$

$$= \int_{x=0}^{1} (\sqrt{x} - x^2 - 2x\sqrt{x} + 2x^3)\, dx$$

$$= \int_{x=0}^{1} (x^{1/2} - 2x^{3/2} - x^2 + 2x^3)\, dx$$

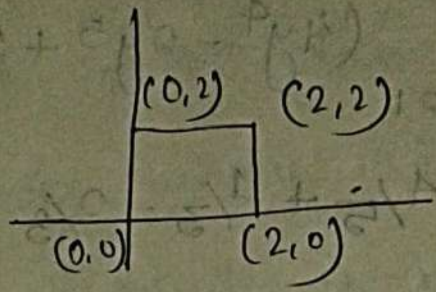
$$= \frac{x^{3/2}}{3/2}\Big|_0^1 - 2\,\frac{x^{5/2}}{5/2}\Big|_0^1 - \frac{x^3}{3}\Big|_0^1 + \frac{x^4}{2}\Big|_0^1$$

$= 2/3 - 4/5 - 1/3 + 1/2 = 1/30$. So Green's theorem is verified.

~~Verified the Green's theorem~~.

Verify Green's theorem in the plane $\oint_c (x^2-2xy)d$ $+ (y^2-x^3y)dy$ where c is a square with vertic $(0,0), (2,0), (2,2), (0,2)$.

Soln:-



For the line integral,

$(0,0)$ to $(2,0)$ $\quad y=0 \quad dy=0$

$$\therefore \int_0^2 (x^2-2x\times 0)\,dx + (0-0\times x^3)\times 0$$

$$= \int_0^2 (x^2)\,dx = \frac{x^3}{3}\Big|_0^2 = 8/3$$

For the line $(2,0)$ to $(2,2)$; $x=2 \quad dx=0$

$$\therefore \int_{y=0}^{2} (y^2-8y)\,dy = y^3/3\Big|_0^2 - 8\,y^2/2\Big|_0^2$$

$$= 8/3 - 16$$

For the line $(2,2)$ to $(0,2)$; $y=2 \quad dy=0$

$$\int_{x=2}^{0} (x^2-4x)\,dx = \frac{x^3}{3}\Big|_2^0 - 4\frac{x^2}{2}\Big|_2^0$$

$$= -8/3 + 8$$

And for the line $(0,2)$ to $(0,0)$; $x=0$ $dx=0$

$$\int_2^0 y^2\,dy = \frac{y^3}{3}\Big|_2^0 = -\frac{8}{3}$$

Hence the total line integral, $= \cancel{\frac{8}{3}} + \cancel{\frac{8}{3}} - 16 - \cancel{\frac{8}{3}} + 8 - \cancel{\frac{8}{3}} = -8$

* Here, $P = x^2 - 2xy$ $\quad \frac{\partial P}{\partial y} = -2x$

$Q = y^2 - x^3 y$ $\quad \frac{\partial Q}{\partial x} = -3x^2 y$

Now,
$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_{x=0}^{2}\int_0^2 (-3x^2 y + 2x)\,dx\,dy$$

$$= \int_{x=0}^{2} \left(-3x^2 \cdot \frac{y^2}{2} + 2xy\right)\Big|_0^2 dx$$

$$= \int_{x=0}^{2} (-6x^2 + 4x)\,dx$$

$$= \left(-6\frac{x^3}{3} + 4\frac{x^2}{2}\right)\Big|_0^2$$

$$= -16 + 8 = -8$$

So, Green's theorem is verified.

(10)(0

Cauchy Integral Theorem :

If $f(z)$ be analytic inside and on a simple closed curve c then,

$$\oint_c f(z)\,dz = 0$$

Proof:- Since $f(z)$ be analytic in the simple closed curve c, then from cauchy Riemann equation,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{—— (i)}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{—— (ii)}$$

Now By definition,

$$\oint_c f(z)\,dz = \oint_c (u+iv)(dx+i\,dy) = \oint_c (u\,dx - v\,dy) + i\oint_c (v\,dx + u\,dy) \quad \text{—— (iii)}$$

Appling Green's theorem in eq^n (iii) we get,

$$\oint_c f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$$= \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right) dx\,dy$$

$$= 0 + 0 = 0$$

$\therefore \oint_c f(z)\,dz = 0$ [Proved]

Prove that (i) $\oint_c dz = 0$ (ii) $\oint_c z\,dz = 0$

Solⁿ: (i) We know from Cauchy Riemann integration theorem, $\oint_c f(z)dz = 0$, where $f(z)$ is analytic.

Here, $f(z) = 1 = 1 + 0\cdot i$

$\therefore u = 1,\ v = 0$

$\frac{\partial u}{\partial x} = 0$ & $\frac{\partial v}{\partial y} = 0$

Since, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

So 1 is analytic inside and on a ~~st~~ simple closed curve.

$\therefore$ from Cauchy Riemann Intnegral theorem,

$$\oint_c 1\,dz = \oint_c dz = 0$$

(ii) Here, $z = x + iy$

$\therefore u = x$ & $v = y$

$\frac{\partial u}{\partial x} = 1$ $\frac{\partial v}{\partial y} = 1$, Hence, z is analytic.

So, from Cauchy Riemann Intnegration theorem

$$\oint_c z\,dz = 0.$$

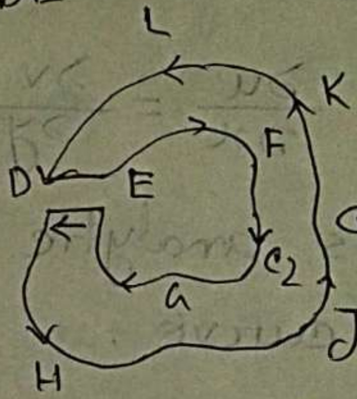
(12)

Let $f(z)$ be analytic in a region bounded by two simple closed curve C_1 an C_2 and also C_1 and C_2.

Prove that $\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$ where C_1 and C_2 are both traversed in the positive sense relative to their interior (anti clockwise).

Soln:- construct cross cut DE

By cauchy's theorem



$\int f(z)\,dz = 0$

DEFGEDHJKLD

$$\Rightarrow \oint_{DE} f(z)\,dz + \oint_{EFGE} f(z)\,dz + \oint_{ED} f(z)\,dz + \oint_{DHJKLD} f(z)\,dz = 0$$

$$\Rightarrow \int_{DHJKLD} f(z)\,dz = -\oint_{EFGE} f(z)\,dz$$

$$\Rightarrow \oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz \quad \text{[Proved]}$$

(68) Evalute $\oint_C \frac{dz}{z-a}$ where C is any simple closed curve and $z=a$ is (a) outside C (b) inside C

Solⁿ: (a) If a is outside c then $f(z) = 1/z-a$ is analytic everywhere inside and on c.

$f(z) = 1/_{z-a}$ is analytic so its integration is zero.

$$\therefore \oint_c \frac{dz}{z-a} = 0$$

(b) If $z=a$ is inside c, we construct a circle Γ of radius ϵ with center at $z=a$ then we can write

$$|z-a| = \epsilon \quad \& \quad z-a = \epsilon e^{i\theta} \quad \text{where } 0<\theta<2\pi$$

$$\Rightarrow z = a + \epsilon e^{i\theta}$$

$$\Rightarrow dz = i\epsilon e^{i\theta} d\theta$$

$$\therefore \oint_c \frac{dz}{z-a} = \oint_\Gamma \frac{dz}{z-a} = \int_0^{2\pi} \frac{i\epsilon e^{i\theta} d\theta}{\epsilon e^{i\theta}}$$

$$= \int_0^{2\pi} i\, d\theta = 2\pi i$$

Q Evaluate $\oint_c \frac{dz}{z-3}$

Solⁿ: $\oint_c \frac{dz}{z-3} = \oint_\Gamma \frac{dz}{z-3}$

where Γ is a circle of radius ϵ

$\therefore |z-3| = \epsilon$ and $z-3 = \epsilon e^{i\theta}$

$$\Rightarrow z = 3 + re^{i\theta}$$

$$\Rightarrow dz = i\,r e^{i\theta}\, d\theta$$

$$\therefore \oint_{\Gamma} \frac{dz}{z-3} = \oint_0^{2\pi} \frac{i\,r e^{i\theta}\, d\theta}{r e^{i\theta}}$$

$$= \oint_0^{2\pi} i\, d\theta = 2\pi i$$

— x —

(11)

Cauchy's Integral Formula:-

If $f(z)$ is analytic inside and its boundary C of a simply constructed region R, except at the point a inside C, then prove :-

(i) $$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\, dz$$

(ii) $$f'(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^2}\, dz$$ [For first derivative]

(iii) $$f^n(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz$$ [For nth derivative]

Proof:- The function $\frac{f(z)}{z-a}$ is analytic inside and on its boundary c, except at the point $z=a$ in fig 1. So we can write

$$\oint_c \frac{f(z)}{z-a} = \oint_\Gamma \frac{f(z)}{z-a}\, dz \quad \text{--- (1)}$$

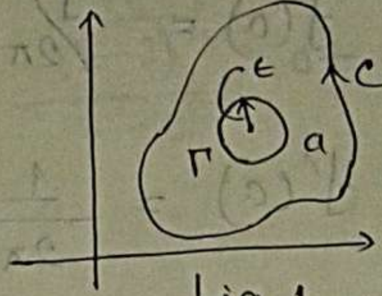


fig 1

where Γ is a circle of radius ϵ at the point a. Then for Γ we set,

$$|z-a| = \epsilon \quad \text{and} \quad z-a = \epsilon e^{i\theta} \quad [0 \leq \theta < 2\pi]$$

$$\Rightarrow z = a + \epsilon e^{i\theta}$$

$$\Rightarrow dz = i\epsilon e^{i\theta}\, d\theta$$

from eq^n (1)

$$\oint_c \frac{f(z)}{z-a}\, dz = \int_0^{2\pi} \frac{f(a+\epsilon e^{i\theta})(i\epsilon e^{i\theta}\, d\theta)}{\epsilon e^{i\theta}}$$

$$= i\int_0^{2\pi} f(a+\epsilon e^{i\theta})\, d\theta \quad \text{--- (2)}$$

Taking limit both sides as $\epsilon \to 0$

$$= \lim_{\epsilon \to 0} i\int_0^{2\pi} f(a+\epsilon e^{i\theta})\, d\theta$$

$$= i\int_0^{2\pi} f(a)\, d\theta$$

$$= i f(a) \int_0^{2\pi} d\theta = 2\pi i f(a)$$

$$\therefore f(a) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z-a} dz$$

$$f'(a) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{(z-a)^2} dz$$

$$f^n(a) = \frac{n!}{2\pi i} \oint_c \frac{f(z)}{(z-a)^{n+1}} dz \quad \text{where } n = 0, 1, 2 \cdots$$

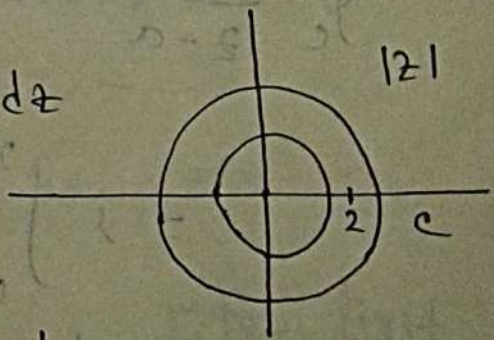
Evaluate :-

ⓐ $\frac{1}{2\pi i} \oint_c \frac{e^z}{z-2} dz$ where c is the circl

(i) $|z| = 3$ and (ii) $|z| = 1$

Soln :-

(i) we know, $f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-a} dz$



Here, $f(z) = e^z$

$a = 2$

$$\therefore f(a) = \frac{1}{2\pi i} \oint_c \frac{e^z}{z-2} dz$$

$$= e^2 \quad \text{(Ans)}$$

Since $z=2$ is insid the cincle c. So $f(z) = e^z$ is analyti exeept at the poi $z = 2$. and by cauchy's integral formula.

Evaluate :- $\oint_C \frac{1}{z(z-2)^4} dz$ where C is the circle $|z| = 1$.

Sol^n:- $\oint_C \frac{1}{z(z-2)^4} dz = \oint_C \frac{\frac{1}{(z-2)^4}}{z-0} dz$

since, $z=0$ in inside C

$\therefore$ let, $f(z) = \frac{1}{(z-2)^4}$

so $f(0) = \frac{1}{2^4}$; No $f(z)$ is analytic whi within C. where C is circle $|z| = 1$

$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z} dz = f(0)$

$\Rightarrow \oint_C \frac{1}{z(z-2)^4} dz = \frac{1}{2^4} \times 2\pi i = \frac{\pi i}{8}$

Evalute :- $\oint_C \frac{e^{2z}}{(z+1)^4} dz$ where C is the circle $|z| = 3$

Sol^n:- we know that,

$f^n(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz$

Here, $n = 3$ $[\because 4 = 3+1]$

$a = -1$

$f(z) = e^{2z}$

Now, $\oint_c \frac{e^{2z}}{(z+1)^4} = \frac{2\pi i}{3!} \times f'''(a)$

$f(z) = e^{2z}$ $\quad f''(z) = 4e^{2z}$

$f'(z) = 2e^{2z}$ $\quad f'''(z) = 8e^{2z}$

$f'''(a) = f'''(-1) = 8e^{-2}$

$\therefore \oint_c \frac{e^{2z}}{(z+1)^4} = \frac{2\pi i}{6} \times 8e^{-2}$

$= \frac{8\pi i e^{-2}}{3}$ (Ans)

(2) Evaluate :- $\oint_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$ where c is the circle $|z| = 3$

Sol^n:- we know that,

$$f(a) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z-a} dz$$

Here, $\oint_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$

Now, $\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$

$$\therefore \oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}\,dz = \oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{z-2} - \oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{z-1}\,dz$$

Now, $\oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{z-2}\,dz$

Here, $f(z) = \sin\pi z^2 + \cos\pi z^2$

$a = 2$

$$\therefore \oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{z-2} = 2\pi i \times f(a)$$

$$= 2\pi i \times f(2)$$

$$= 2\pi i \times \{\sin 4\pi + \cos 4\pi\}$$

$$= 2\pi i$$

Again, $\oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{z-1}\,dz$

Here, $f(z) = \sin\pi z^2 + \cos\pi z^2$

$a = 1$

$$\therefore \oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{z-1} = 2\pi i \times f(a)$$

$$= 2\pi i \times f(1)$$

$$= 2\pi i \times \{\sin\pi + \cos\pi\}$$

$$= -2\pi i$$

$$\therefore \oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-2)(z-1)}\,dz = 2\pi i - (-2\pi i)$$

$$= 4\pi i$$

Cauchy's Residual theorem:-

If $f(z)$ is analytic and on a simple closed c except for a pole of order m at $z=a$ inside c prove that (i)

$$\frac{1}{2\pi i} \oint_c f(z)\,dz = \lim_{z\to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \{(z-a)^m f(z)\}$$

(ii) If there are two poles at $z=a$ is $z=a_2$ inside c of order m_1 & m_2 respectively, Prove that,

$$\frac{1}{2\pi i} \oint_c F(z)\,dz = \lim_{z\to a_1} \frac{1}{(m_1-1)!} \frac{d^{m_1-1}}{dz^{m_1-1}} \{(z-a_1)^{m_1} F(z)\} + \lim_{z\to a_2} \frac{1}{(m_2-1)!} \frac{d^{m_2-1}}{dz^{m_2-1}} \{(z-a_2)^{m_2} F(z)\}$$ (ii)

(iii) In general of $F(z)$ has a number of poles inside c with residues R ... then

$$\oint_c F(z)\,dz = 2\pi i (R_1 + R_2 + \cdots)$$
$$= 2\pi i \,(\text{sum of rasidues})$$

(i) **Proof:-** If $F(z)$ has a pole of radius m at $z=a$, Then $F(z) = \frac{f(z)}{(z-a)^m}$ where $f(z)$ is analyti inside and on c and $f(a) \neq 0$.

Then by cauchy's Integral formula,

$$f^{m-1}(a) = \frac{(m-1)!}{2\pi i} \oint_c \frac{f(z)}{(z-a)^m}\,dz$$

Now, $\frac{1}{2\pi i}\oint_c F(z)\,dz = \frac{1}{2\pi i}\oint_c \frac{f(z)}{(z-a)^m}\,dz$

$= \frac{f^{m-1}(a)}{(m-1)!}$

$= \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\{f(z)\}$

$= \lim_{z\to a}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m F(z)\}$

Proof: ~~If there are two poles at $z = a_1$ and $z = a_2$ inside 'c' of orders m_1 & m_2 respectively.~~

Evalute: $\oint \frac{e^z}{(z^2+\pi^2)^2}\,dz$ where c is the circle, $|z| = 4$

Solⁿ: From cauchy's residual theorem,

$\frac{1}{2\pi i}\oint_c F(z)\,dz = \lim_{z\to a}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m F(z)\}$

Now, $\oint_c \frac{e^z}{(z^2+\pi^2)^2}\,dz = \oint_c \frac{e^z}{\{(z-\pi i)(z+\pi i)\}^2}\,dz$

This function is non analytic at $z = \pm\pi i$

or, $z = \pm\pi i$ inside c and are both of order 2

Residue at $z = \pi i$ is

$$\lim_{z\to\pi i}\frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}\left\{(z-\pi i)^2\frac{e^z}{(z-\pi i)^2(z+\pi i)^2}\right\}$$

$$=\lim_{z\to\pi i}\frac{d}{dz}\left\{\frac{e^z}{(z+\pi i)^2}\right\}$$

$$=\lim_{z\to\pi i}\left[\frac{(z+\pi i)^2e^z-2(z+\pi i)e^z}{(z+\pi i)^4}\right]$$

$$=\lim_{z\to\pi i}\left[\frac{e^z(z+\pi i)-2e^z}{(z+\pi i)^3}\right]$$

$$=\lim_{z\to\pi i}\left[\frac{e^z(z+\pi i-2)}{(z+\pi i)^3}\right]$$

$$=\frac{e^{\pi i}(\pi i+\pi i-2)}{(\pi i+\pi i)^3}=\frac{e^{\pi i}(2\pi i-2)}{(2\pi i)^3}$$

$$=\frac{2e^{\pi i}(\pi i-1)}{8\pi^3 i^3}$$

$$=\frac{e^{\pi i}(\pi i+i^2)}{4\pi^3 i^3}$$

$$=\frac{-1(\pi i+i^2)}{-4\pi^3 i}=\frac{i(\pi+i)}{4\pi^3 i}=\frac{\pi+i}{4\pi^3}\ \text{(Ans)}$$

Again at Residue $\Rightarrow z = -\pi i$

$$\lim_{z \to -\pi i} \frac{1}{1!} \frac{d}{dz} \left\{ (z+\pi i)^2 \frac{e^z}{(z-\pi i)^2 (z+\pi i)^2} \right\}$$

$$= \lim_{z \to -\pi i} \frac{d}{dz} \left\{ \frac{e^z}{(z-\pi i)^2} \right\} = \frac{\pi - i}{4\pi^3}$$

$$\therefore \oint_C \frac{e^z}{(z^2+\pi^2)^2} dz = 2\pi i \left[\text{Sum of Residues} \right]$$

$$= 2\pi i \left[\frac{\pi+i}{4\pi^3} + \frac{\pi - i}{4\pi^3} \right]$$

$$= 2\pi i \; \frac{2\pi}{4\pi^3} = i/\pi$$

(5) Evalute :- $\displaystyle 1/2\pi i \oint_C \frac{e^{zt}}{(z^2+1)^2} dz$ if $t > 0$ and c is the circle $|z| = 3$.

Soln:- Here, $\displaystyle \oint_C \frac{e^{zt}}{(z^2+1)^2} dz = \oint_C \frac{e^{zt}}{(z+i)^2 (z-i)^2} dz$ are at $z = \pm i$ inside c and are both order 2

residue at $z = +i$ is

$$\lim_{z \to i} \frac{1}{1!} \frac{d^{2-1}}{dz^{2-1}} \left\{ (z-i)^2 \frac{e^{zt}}{(z+i)^2 (z-i)^2} \right\}$$

$$= \lim_{z \to i} \frac{d}{dz} \left\{ \frac{e^{zt}}{(z+i)^2} \right\}$$

$$= \lim_{z \to i} \frac{te^{zt}(z+i)^2 - e^{zt}2(z+i)}{(z+i)^4}$$

$$= \lim_{z \to i} \frac{e^{zt}(tz+it-2)}{(z+i)^3}$$

$$= \frac{e^{it}(2it-2)}{2^3 i^3} = \frac{e^{it}(it-1)}{4i^3}$$

$$= \frac{e^{it}(it+i^2)}{-4i} = \frac{e^{it}(t+i)}{-4}$$

$$= \frac{te^{it}+ie^{it}}{-4}$$

Residue at $z=-i$

$$\lim_{z \to i} \frac{1}{1!} \frac{d}{dz} \left\{ \frac{e^{zt}}{(z-i)^2} \right\}$$

$$= \lim_{z \to -i} \left[\frac{te^{zt}(z-i)^2 - 2(z-i)e^{zt}}{(z-i)^4} \right]$$

$$= \lim_{z \to -i} \left[\frac{e^{zt}(tz-it-2)}{(z-i)^3} \right]$$

$$= \frac{e^{-it}(-2it-2)}{(-2i)^3}$$

$$= \frac{2e^{-it}(-ti-1)}{8i^3} = \frac{e^{-it}(-ti+i^2)}{4i}$$

$$= \frac{e^{-it}(-t+i)}{4} = \frac{-te^{it}+ie^{it}}{4}$$

By residue theorem :-

$$\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{(z^2+1)^2}\,dz = (\text{sum of residue})$$

$$= \frac{1}{2}\left(\frac{-te^{it}+ie^{it}-te^{-it}+ie^{-it}}{2}\right)$$

$$= \frac{1}{2}\left(\frac{-te^{it}+ie^{it}}{2} - t\,\frac{e^{it}+e^{-it}}{2}\right\}$$

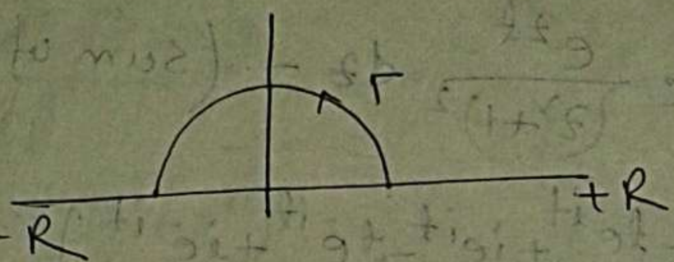
$$= \frac{1}{2}\left\{\frac{-i(e^{it}-e^{it})}{-2i^2} - t\,\frac{e^{it}+e^{it}}{2}\right\}$$

$$= \frac{1}{2}\left\{\frac{e^{it}-e^{-it}}{2i} - t\,\frac{e^{it}+e^{-it}}{2}\right\}$$

$$= \frac{1}{2}(\sin t - t\cos t) \quad (\text{Ans})$$

Theorem :- Consider the integral $I = \int_{-\infty}^{\infty} F(x)\,dx$ where $F(z)$ is a function that satisfies the following conditions.

(i) It is analytic in the upper half plane except at a finite number of poles.



(ii) It has no poles on the real axis.

(iii) $zF(z) \to 0$ (converge) uniformly as $|z| \to 0$ for $0 < \theta < \pi$

(iv) where x is real, $xF(x) \to 0$ (converge) as $x \to \pm\infty$ in such way that $\int_{-\infty}^{\infty} F(x)\,dx$ both converge. Then

$$I = \oint_C F(z)\,dz = \int_{-R}^{R} F(x)\,dx + \int_{\Gamma} F(x)\,dx$$

$$= 2\pi i \{\text{Sum of Residues}\}$$

$$\oint_C F(z)\,dz = \int_{-R}^{R} F(x)\,dx + \int_{\Gamma} F(z)\,dz$$

Taking limit $R \to \infty$

$$\oint_C F(z)\,dz = \int_{-\infty}^{\infty} F(x)\,dx + 0 = 2\pi i \left\{ \text{Sum of Residues} \right\}$$

(12) ~~Evaluate~~ Evaluate :- $\int_0^{\infty} \frac{dx}{x^4 + a^4}$ by contoure integration

Soln :- Consider the integral $\oint_C \frac{dz}{z^4 + a^4}$ where C is the closed contouer of the fig 1

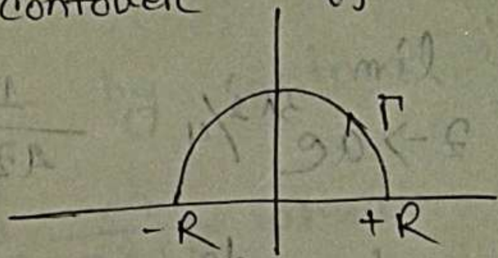


Fig 1

consisting the ~~times~~ line -R to R and the semicircular circle Γ, travesed in the anticlock wise. The function $f(z) = \frac{1}{z^4 + a^4}$ has simple poles $ae^{\pi i/4}$, $ae^{3\pi i/4}$, $ae^{5\pi i/4}$ and $ae^{7\pi i/4}$, But only first two poles $ae^{\pi i/4}$, $ae^{3\pi i/4}$ lies inside C i.e half plane.

The function F(z) clearly satisfies conditions of the theorem.

So the residue at $z = ae^{\pi i/4}$ is

$$\lim_{z \to ae^{\pi i/4}} \left\{ (z - ae^{\pi i/4}) \frac{1}{z^4 + a^4} \right\}$$

$$= \lim_{z \to ae^{\pi i/4}} \frac{1}{4z^3} = \frac{1}{4a^3} e^{-3\pi i/4}$$

And the residue at $z = ae^{3\pi i/4}$

$$\lim_{z \to ae^{3\pi i/4}} \left\{ (z - ae^{\pi 3 i/4}) \frac{1}{z^4 + a^4} \right\}$$

$$= \lim_{z \to ae^{3\pi i/4}} \frac{1}{4z^3} = \frac{1}{4a^3} e^{-9\pi i/4}$$

Thus, $\oint_C \frac{dz}{z^4 + a^4} = \int_{-R}^{R} \frac{dx}{x^4 + a^4} + \int_\Gamma \frac{dz}{z^4 + a^4}$

$= 2\pi i \left\{ \text{Sum of Resid}\text{ees} \right\}$

$= 2\pi i \left\{ \frac{1}{4a^3} e^{-3i\pi/4} + \frac{1}{4a^3} e^{-9i\pi/4} \right\}$

$= 2\pi i \times \frac{1}{4a^3} \times -i\sqrt{2} = \frac{\pi}{\sqrt{2}a^3}$

Taking the limit of both sides of (1)

$R \longrightarrow \infty$

$$\therefore \oint_C \frac{dz}{z^4+a^4} = \int_{-\infty}^{\infty} \frac{dx}{x^4+a^4} + 0 = \frac{\pi}{\sqrt{2}a^3}$$

$$\text{OR, } 2\int_0^{\infty} \frac{dx}{x^4+a^4} = \frac{\pi}{\sqrt{2}\,a^3}$$

$$\Rightarrow \int_0^{\infty} \frac{dx}{x^4+a^4} = \frac{\pi}{2\sqrt{2}a^3}$$

Evaluate :- $\int_0^{\infty} \frac{dx}{x^6+1}$ by contour integration.

Solⁿ:- Consider $\oint_C \frac{dz}{z^6+1}$ where C is the closed contour of fig 1, consisting of the line $-R$ to R and the semicircle Γ, traversed in the positive (anticlockwise) sense.

Since, $z^6+1=0$ when $z = e^{\pi i/6}, e^{3\pi i/6}, e^{5\pi i/6}, e^{7\pi i/6}, e^{9\pi i/6}$, these are simple poles of $1/(z^6+1)$. Only the poles $e^{\pi i/6}, e^{3\pi i/6}$ and $e^{5\pi i/6}$ lie within C. Then using L' Hospital rule.

Residue at $e^{\pi i/6} = \lim\limits_{z \to e^{\pi i/6}} \left\{ (z - e^{\pi i/6}) \frac{1}{z^6+1} \right\}$

$= \lim\limits_{z \to e^{\pi i/6}} \frac{1}{6z^5} = \frac{1}{6e^{5\pi i/6}} = \frac{1}{6} e^{-5\pi i/6}$

Residue at, $e^{3\pi i/6} = \lim\limits_{z \to e^{3\pi i/6}} \left\{ (z - e^{3\pi i/6}) \frac{1}{z^6+1} \right\}$

$= \lim\limits_{z \to e^{3\pi i/6}} \frac{1}{6z^5} = \frac{1}{6} e^{-3 \times 5\pi i/6}$

$= \frac{1}{6} e^{-15\pi i/6} = \frac{1}{6} e^{-5\pi i/2}$

Residue at $e^{5\pi i/6} = \lim\limits_{z \to e^{5\pi i/6}} \left\{ (z - e^{5\pi i/6}) \frac{1}{z^6+1} \right\}$

$= \lim\limits_{z \to e^{5\pi i/6}} \frac{1}{6z^5} = \frac{1}{6} e^{-25\pi i/6}$

Thus, $\oint_C \frac{dz}{z^6+1} = 2\pi i \left\{ \frac{1}{6} e^{-5\pi i/6} + \frac{1}{6} e^{-5\pi i/2} + \frac{1}{6} e^{-25\pi i/6} \right\}$

$= \frac{2\pi}{3}$

i.e $\int_{-R}^{R} \frac{dx}{x^6+1} + \int_{\Gamma} \frac{dz}{z^6+1} = \frac{2\pi}{3}$

Taking limit of both sides of (1) as $R \to \infty$ ~~and~~

$$\int_{-\infty}^{\infty} \frac{dz}{z^6+1} = \int_{-\infty}^{\infty} \frac{dx}{x^6+1} + 0 = \frac{2\pi}{3}$$

$$\Rightarrow 2\int_0^{\infty} \frac{dx}{x^6+1} = \frac{2\pi}{3}$$

$$\Rightarrow \int_0^{\infty} \frac{dx}{x^6+1} = \pi/3$$

(11) (15)

Evalute : $\int_0^{\infty} \frac{dx}{1+x^4}$ by contour integration.

$1+z^4=0$ when $z = e^{\pi i/4}, e^{3\pi i/4}, e^{5\pi i/4}, e^{7\pi i/4}$

But only $e^{\pi i/4}$ and $e^{3\pi i/4}$ lie within C.

$\therefore$ Residue at, $e^{\pi i/4} = \lim_{z \to e^{\pi i/4}} \left\{ (z - e^{\pi i/4}) \frac{1}{(z^4+1)} \right\}$

$$= \lim_{z \to e^{\pi i/4}} \frac{1}{4z^3} = \frac{1}{4} e^{-3\pi i/4}$$

Residue at, $e^{3\pi i/4} = \lim_{z \to e^{3\pi i/4}} \left\{ (z - e^{3\pi i/4}) \frac{1}{z^4+1} \right\}$

$$= \lim_{z \to e^{3\pi i/4}} \frac{1}{4z^3} = \frac{1}{4} e^{-9\pi i/4}$$

$\therefore$ Thus $\oint_C \frac{dz}{z^4+1} = 2\pi i \left\{ \frac{1}{4} e^{-3\pi i/4} + \frac{1}{4} e^{-9\pi i/4} \right\}$

$= 2\pi i \times \frac{1}{4} \left\{ e^{-12\pi i/4} \right\} = \frac{\pi i}{2} \left\{ e^{-3\pi i} \right\}$

$= -\frac{\pi i}{2}$

Taking limit both sides. $R \to \infty$

$\int_{-\infty}^{\infty} \frac{dz}{z^4+1} = \int_{-\infty}^{\infty} \frac{dx}{x^4+1} + 0 = -\frac{\pi i}{2}$

$\Rightarrow 2\int_{0}^{\infty} \frac{dx}{x^4+1} = -\frac{\pi i}{2}$

$\Rightarrow \int_{0}^{\infty} \frac{dx}{x^4+1} = -\pi i/4$ A.

Thus $\oint_C \frac{dz}{z^4+1} = \frac{2\pi i}{4} \left\{ e^{-3\pi i/4} + e^{-9\pi i/4} \right\}$

$= \frac{\pi i}{2} \left\{ \cos\frac{-3\pi}{4} + i\sin\frac{-3\pi}{4} + \cos\frac{-9\pi}{4} + i\sin\frac{-9\pi}{4} \right\}$

$= \frac{\pi i}{2} \left\{ \cancel{\frac{-1}{\sqrt{2}}} - i\frac{1}{\sqrt{2}} + \cancel{\frac{1}{\sqrt{2}}} - i\frac{1}{\sqrt{2}} \right\}$

$= \frac{\pi i}{2} \left(-\frac{2i}{\sqrt{2}} \right) = \frac{\pi}{\sqrt{2}}$

Taking limit both sides,

$$\int_{-\infty}^{\infty} \frac{dz}{z^4+1} = \int_{-\infty}^{\infty} \frac{dx}{x^4+1} + 0 = \frac{\pi}{\sqrt{2}}$$

$$\Rightarrow 2\int_0^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}$$

$$\Rightarrow \int_0^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}$$

Evaluate :- $\int_0^{\infty} \frac{\cos mx}{x^2+1}\, dx$ by contour integ.

Consider, $\oint_C \frac{e^{imz}}{z^2+1}\, dz$ where C is the contour of fig 1. The integrand has simple poles at $z = \pm i$, but only $z = i$ lies within C.

Residue at $z = i$, $\lim_{z \to i} \left\{ (z-i) \frac{e^{imz}}{(z-i)(z+i)} \right\}$

$$= \frac{e^{-m}}{2i}$$

$$\oint_C \frac{e^{imz}}{z^2+1}\, dz = 2\pi i \left(\frac{e^{-m}}{2i} \right) = \pi e^{-m}$$

OR, $\int_{-\infty}^{\infty} \frac{e^{imx}}{x^2+1}\, dx + \int_{\Gamma} \frac{e^{imz}}{z^2+1}\, dz = \pi e^{-m}$

i.e, $\int_{-R}^{R} \frac{\cos mx}{x^2+1} dx + i \int_{-R}^{R} \frac{\sin mx}{x^2+1} + \int_{\Gamma} \frac{e^{imz}}{z^2+1}$

$= \pi e^{-m}$

$\Rightarrow 2\int_{0}^{R} \frac{\cos mx}{x^2+1} + \int_{\Gamma} \frac{e^{imz}}{z^2+1} = \pi e^{-m}$

Taking limit $R \to \infty$

$\therefore 2\int_{0}^{\infty} \frac{\cos mx}{x^2+1} = \pi e^{-m}$

$\Rightarrow \int_{0}^{\infty} \frac{\cos mx}{x^2+1} = \frac{\pi}{2} e^{-m}$

(12) Evaluate: $\int_{0}^{\infty} \frac{\cos mx}{(x^2+1)^2} dx$ by contour In

from Cauchy delta function

$\oint_{C} \frac{e^{imz}}{(z^2+1)^2} dz$ where C - - - -

- - - - - - - - - - -

The integral has poles $z = \pm i$ of order 2. But only $z = i$ lies in C.

Residue at $z = i$,

$$\lim_{z\to i} \left\{ \frac{1}{(2-1)!} \frac{d^{2-1}}{dz^{2-1}} \left\{ (z-i)^2 \frac{e^{imz}}{(z^2+1)^2} \right\} \right\}$$

$$\Rightarrow \lim_{z\to i} \frac{d}{dz} \left\{ e^{imz} \times \frac{(z-i)^2}{(z-i)^2 (z+i)^2} \right\}$$

$$\Rightarrow \lim_{z\to i} \frac{d}{dz} \frac{e^{imz}}{(z+i)^2}$$

$$\Rightarrow \lim_{z\to i} \frac{(z+i)^2 e^{imz} \times mi - e^{imz} \times 2(z+i)}{(z+i)^4}$$

$$\Rightarrow \lim_{z\to i} \frac{-4e^{-m} im - 2e^{-m} \times 2i}{(2i)^4}$$

$$\Rightarrow \frac{-i4e^{-m} m - 4i e^{-m}}{16}$$

$$\Rightarrow \frac{-4i(e^{-m} m + e^{-m})}{16} = \frac{e^{-m}(1+m)}{4i}$$

Thus $\oint_C \frac{e^{imz}}{(z^2+1)^2} dz = 2\pi i \{\text{Sum of Residues}\}$

$$= 2\pi i \times \frac{e^{-m}(1+m)}{4i}$$

$$= \frac{\pi e^{-m}(1+m)}{2}$$

$$\text{OR, } \oint_C \frac{e^{imz}}{(z^2+1)^2}\,dz = \int_{-R}^{R} \frac{e^{imx}}{(x^2+1)^2}\,dx + \int_\Gamma \frac{e^{im\ldots}}{(z^2+1\ldots}$$

$$= \frac{\pi e^{-m}(1+m)}{2}$$

Taking limit on both sides $R \to \infty$

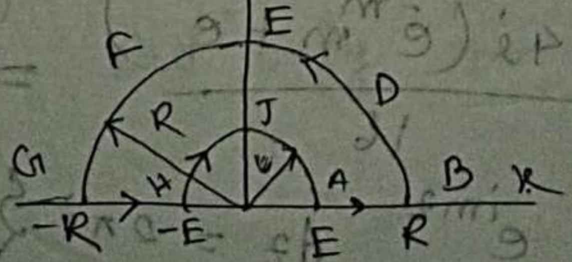
$$\therefore \oint_C \frac{e^{imz}}{(z^2+1)^2}\,dz = \int_{-\infty}^{\infty} \frac{e^{imx}}{(x^2+1)^2} + 0 = \frac{\pi e^{-m}(\ldots}{2}$$

$$\Rightarrow 2\int_0^\infty \frac{e^{imx}}{(x^2+1)^2}\,dx = \frac{\pi e^{-m}(1+m)}{2}$$

$$\Rightarrow \int_0^\infty \frac{e^{imx}}{(x^2+1)^2}\,dx = \frac{\pi e^{-m}(1+m)}{4}$$

(v) (18) Evaluate :- $\int_0^\infty \frac{\sin x}{x}\,dx$ by countour In.

Sol :-



Since, $z=0$ is outside C', we have

$$\oint_{C} \frac{e^{iz}}{z}\, dz = 0$$

OR, $\int_{-R}^{-E} \frac{e^{ix}}{x}\, dx + \int_{HJA} e^{iz}/z\, dz + \int_{E}^{R} \frac{e^{ix}}{x}\, dx + \int_{BDEFG} e^{iz}/z\, dz = 0$

Replacing x by $-x$ in the first integral and combining with the third integral, we find,

$$\int_{E}^{R} \frac{e^{ix}-e^{-ix}}{x}\, dx + \int_{HJA} \frac{e^{iz}}{z}\, dz + \int_{BDEFG} \frac{e^{iz}}{z}\, dz = 0$$

OR, $2i \int_{E}^{R} \frac{\sin x}{x}\, dx = -\int_{HJA} \frac{e^{iz}}{z}\, dz - \int_{BDEFG} \frac{e^{iz}}{z}\, dz$

Let, $E \to 0$ and $R \to \infty$, letting $z = Ee^{i\theta}$

$$-\lim_{E\to 0} \int_{\pi}^{0} \frac{e^{iEe^{i\theta}}}{Ee^{i\theta}}\, iEe^{i\theta}\, d\theta = -\lim_{E\to 0} \int_{\pi}^{0} ie^{iEe^{i\theta}}\, d\theta = \pi i$$

Since the limit can be taken under the integral sign,

$$\lim_{\substack{R\to\infty \\ E\to 0}} 2i \int_{E}^{R} \frac{\sin x}{x}\, dx = \pi i \quad \text{OR} \quad \int_{0}^{\infty} \frac{\sin x}{x}\, dx = \pi/2$$

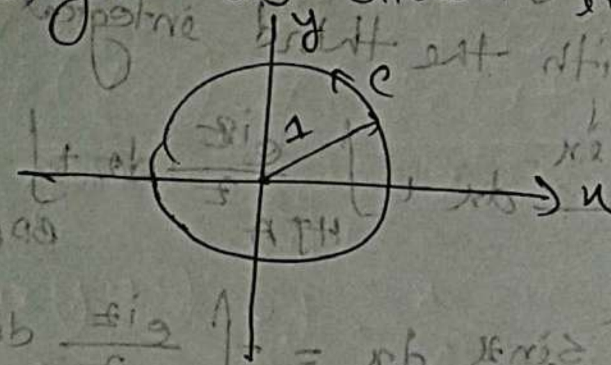
(12) Evaluate : $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta}$ by contour $\int_n$

Solⁿ: let, $z = e^{i\theta}$ Then $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i}$

$dz = ie^{i\theta}\,d\theta = iz\,d\theta$ so that,

$$\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \oint_c \frac{dz/iz}{a+b(z-z^{-1})/2i} = \oint \frac{2\,dz}{bz^2+2aiz-b}$$

where c is the circle of unit radius with centre at the origin as shown in fig



The poles of $\frac{2}{bz^2+2aiz-b}$ are obtained by solving $bz^2+2aiz-b=0$ and are given by

$$z = \frac{-2ai \pm \sqrt{-4a^2+4b^2}}{2b} = \frac{-ai \pm \sqrt{a^2-b^2}\,i}{b}$$

$$= \left\{\frac{-a+\sqrt{a^2-b^2}}{b}\right\}i \;;\; \left\{\frac{-a-\sqrt{a^2-b^2}}{b}\right\}i$$

only $\dfrac{-a \pm \sqrt{a^2-b^2}}{b} i$ lies inside C, since

$$\left|\frac{-a+\sqrt{a^2-b^2}}{b} i\right| = \left|\frac{\sqrt{a^2-b^2}-a}{b} * \frac{\sqrt{a^2-b^2}+a}{\sqrt{a^2-b^2}+a}\right|$$

$$= \left|\frac{b}{\sqrt{a^2-b^2}+a}\right| < 1 \text{ if } a > |b|$$

Residue at $z_1 = \dfrac{-a+\sqrt{a^2-b^2}}{b} i$

$$= \lim_{z \to z_1} (z-z_1) \frac{2}{bz^2+2aiz-b}$$

$$= \lim_{z \to z_1} \frac{2}{2bz+2ai} = \frac{1}{bz_1+ai} = \frac{1}{\sqrt{a^2-b^2}\, i}$$

$$\oint_C \frac{2\,dz}{bz^2+2aiz-b} = 2\pi i \left(\frac{1}{\sqrt{a^2-b^2}\, i}\right)$$

$$= \frac{2\pi}{\sqrt{a^2-b^2}}$$

Evalute : $\displaystyle\int_0^\infty \frac{\ln(x^2+1)}{x^2+1}\,dx$ by contour Int.

Consider, $\displaystyle\oint_C \frac{\ln(z^2+i)}{z^2+1}$

The only pole ~~inside~~ of $\ln(z+i)/z^2+1$ inside C is $z = i$

$$\lim_{z\to i} (z-i)\frac{\ln(z+i)}{(z+i)(z-i)} = \frac{\ln(2i)}{2i}$$

$$\oint_C \frac{\ln(z^2+i)}{z^2+1}\,dz = 2\pi i \times \frac{\ln(2i)}{2i}$$

$$= \pi \ln(2i) = \pi \ln 2 + \frac{1}{2}\pi$$

$$\int_{-R}^{R} \frac{\ln(x+i)}{x^2+1} + \int_\Gamma \frac{\ln(z+i)}{z^2+1} = \pi \ln 2 + \frac{1}{2}\pi^2 i$$

$$\Rightarrow \int_{-R}^{0} \frac{\ln(x+i)}{x^2+1} + \int_0^R \frac{\ln(x+i)}{x^2+1} + \int_\Gamma \frac{\ln(z+i)}{z^2+1}\,dz = \pi \ln 2 +$$

Replacing x by $-x$ in 1st integral

$$\therefore \int_0^R \frac{\ln(i-x)}{x^2+1} + \int_0^R \frac{\ln(i+x)}{x^2+1} + \int_\Gamma \frac{\ln(z+i)}{z^2+1}\,dz = \pi \ln 2 +$$

$$\ln(i-x) + \ln(x+i) = \ln(x^2+1) + \pi i$$

$$\Rightarrow \int_0^R \frac{\ln(x^2+1)}{x^2+1} + \int_0^R \frac{\pi i}{x^2+1} + \int_\Gamma \frac{\ln(z+i)}{z^2+1} = \pi \ln 2 + \frac{1}{2}\pi$$

$$\Rightarrow \lim_{R\to\infty} \int_0^R \frac{\ln(x^2+1)}{x^2+1}\,dx = \int_0^\infty \frac{\ln(x^2+1)}{x^2+1}\,dx = \pi \ln 2$$