$\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4....}}}}$

$1-1+1-1+1........=?$

$\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4....}}}}$

Discrete mathematics

The Foundations: Logic and Proofs

$\exists_{x\in\Re}\exists_{y\in\Re}(x=y)$

$\forall_x(\Re \mid x)$

$\sum_{x=1}^{\infty}\frac{1}{x}=?$

$\sum_{x=1}^{\infty}x=?$

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Nested Quantifiers

Section 1.5

Section Summary

- Nested Quantifiers
- Order of Quantifiers
- Translating from Nested Quantifiers into English
- Translating Mathematical Statements into Statements involving Nested Quantifiers.
- Translated English Sentences into Logical Expressions.
- Negating Nested Quantifiers.

Nested Quantifiers

- "**Every real number has an inverse**" is

$$\forall x \exists y (x + y = 0)$$

where the domains of x and y are the real numbers.

- We can also think of nested propositional functions:

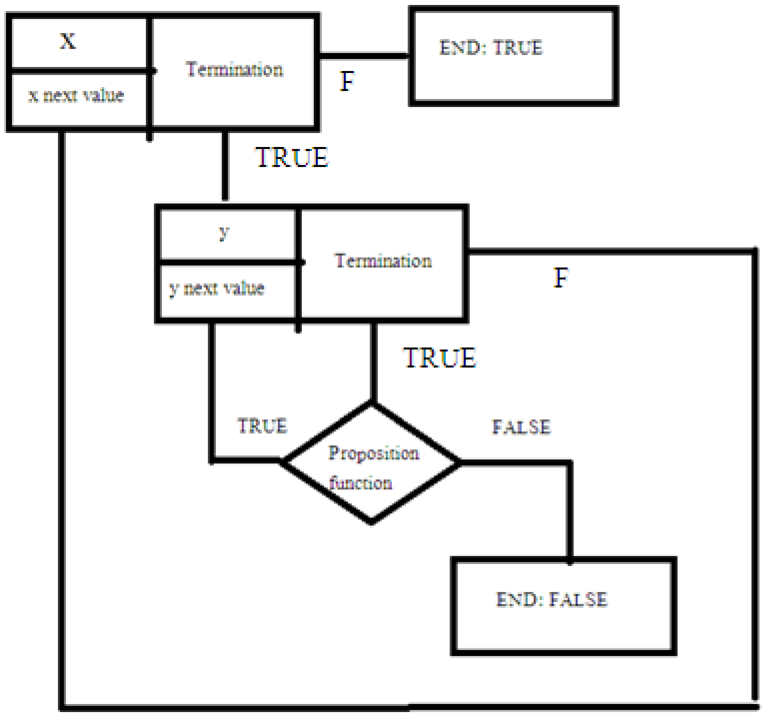
$\forall x \exists y (x + y = 0)$ can be viewed as $\forall x\, Q(x)$

where $Q(x)$ is $\exists y\, P(x, y)$

where $P(x, y)$ *is* $(x + y = 0)$

Thinking of Nested Quantification

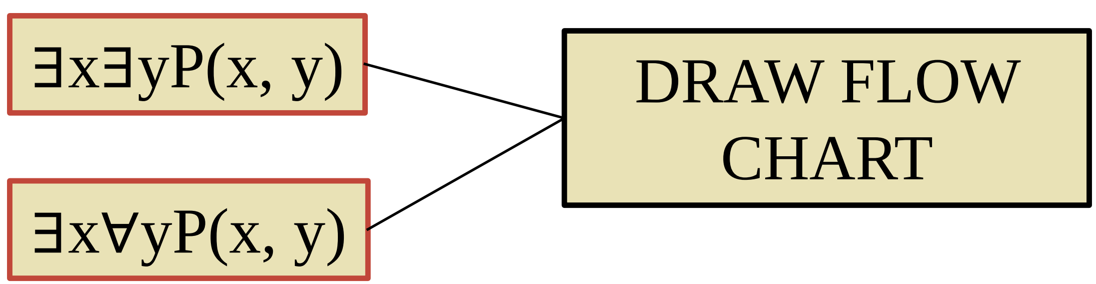
- Nested Loops
  - To see if $\forall x \forall y P\ (x,y)$ is true, loop through the values of x:
    - At each step, loop through the values for y.
    - If for some pair of x and y, $P(x,y)$ is false, then $\forall x\ \forall y P(x,y)$ is false and both the outer and inner loop terminate. Flag:=0
  - If Flag = 1
    - $\forall x\ \forall y\ P(x,y)$ is true.



Thinking of Nested Quantification

- Nested Loops
  - To see if $\forall x \exists y P(x,y)$ is true, loop through the values of x:
    - At each step, loop through the values for y.
    - The inner loop ends when a pair x and y is found such that $P(x, y)$ is true.
    - If no y is found such that $P(x, y)$ is true the outer loop terminates as $\forall x \exists y P(x,y)$ has been shown to be false.

  $\forall x \exists y\, P(x,y)$ is true if the outer loop ends after stepping through each x.



Order of Quantifiers

Examples:

1. Let $P(x,y)$ be the statement "$x + y = y + x$." Assume that U is the real numbers.
   – Then $\forall x\, \forall y P(x,y)$ **and** $\forall y\, \forall x P(x,y)$ have the same truth value.
2. Let $Q(x,y)$ be the statement "$x + y = 0$." Assume that U is the real numbers.
   – Then $\forall x \exists y Q(x,y)$ is true, but $\exists y\, \forall x Q(x,y)$ is false.

Draw Flow Chart or Program

Questions on Order of Quantifiers

Example 2: Let U be the real numbers,

Define $P(x,y)$: $x / y = 1$

What is the truth value of the following:

1. $\forall x \forall y P(x,y)$

   Answer: False

2. $\forall x \exists y P(x,y)$

   Answer: False

3. $\exists x \forall y\, P(x,y)$

   Answer: False

4. $\exists x \exists\, y\, P(x,y)$

   Answer: True

Quantifications of Two Variables

TABLE 1 Quantifications of Two Variables.

Statement	*When True?*	*When False?*
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

Translating Nested Quantifiers into English

Example 1: Translate the statement

$$\forall x \ (C(x) \lor \exists y \ (C(y) \land F(x, y)))$$

- $C(x)$ is "x has a computer,"
- $F(x,y)$ is "x and y are friends,"
- The domain for both x and y consists of all students in your school.

Solution: Every student in your school has a computer or has a friend who has a computer.

Example 2: Translate the statement

$$\exists x \ \forall y \ \forall z \ ((F(x, y) \land F(x,z) \land (y \neq z)) \rightarrow \neg F(y,z))$$

Solution: There is a student none of whose friends are also friends with each other.

Translating Mathematical Statements into Predicate Logic

Example : Translate "The sum of two positive integers is always positive" into a logical expression.

Solution:

- Rewrite the statement to make the implied quantifiers and domains explicit:

  "For every two integers, if these integers are both positive, then the sum of these integers is positive."

- Introduce the variables x and y, and specify the domain, to obtain:

  "For all positive integers x and y, x + y is positive."

- The result is:

  $\forall x \; \forall y \; ((x > 0) \land (y > 0) \rightarrow (x + y > 0))$

  where the domain of both variables consists of all integers

Translating English into Logical Expressions Example

Example: Use quantifiers to express the statement "There is **a woman** who has taken **a flight** on **every airline** in the world."

Solution:

1. Let ***P(w,f)*** be "*w* has taken *f* " and ***Q(f,a)*** be "*f* is a flight on *a* ."
2. The **domain** of *w* is all women, the **domain** of *f* is all flights, and the **domain** of *a* is all airlines.
3. Then the statement can be expressed as:

$$\exists_w \forall_a \exists_f \, (P(w,f) \wedge Q(f,a))$$

Negating Nested Quantifiers

Example 1: Recall the logical expression developed three slides back:

$$\exists w \forall a \exists f\, (P(w,f) \wedge Q(f,a))$$

Part 1: Use quantifiers to express the statement that "There does not exist a woman who has taken a flight on every airline in the world."

Solution: $\neg\exists w \forall a \exists f\, (P(w,f) \wedge Q(f,a))$

Part 2: Now use De Morgan's Laws to move the negation as far inwards as possible.

Solution:

$\neg\exists w \forall a \exists f\, (P(w,f) \wedge Q(f,a))$

$\forall w \neg \forall a \exists f\, (P(w,f) \wedge Q(f,a))$ by De Morgan's for $\exists$

$\forall w \exists a \neg \exists f\, (P(w,f) \wedge Q(f,a))$ by De Morgan's for $\forall$

$\forall w \exists a \forall f \neg (P(w,f) \wedge Q(f,a))$ by De Morgan's for $\exists$

$\forall w \exists a \forall f (\neg P(w,f) \vee \neg Q(f,a))$ by De Morgan's for $\wedge$.

Part 3: Can you translate the result back into English?

Solution:

"For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline"

Query???

$$\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4....}}}}$$

$$\exists_{x\in\Re}\exists_{y\in\Re}(x=y)=?$$

$$\sum_{x=1}^{\infty} x = ?$$

$$\sum_{x=1}^{\infty}\frac{1}{x} = ?$$

$$\forall_x(\Re\,|\,x)=?$$

$$\exists_{x\in\Re}\exists_{y\in\Re}(x=y)=?$$

$$\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4....}}}} = ?$$

$$1-1+1-1+1........=?$$

$$\sum_{x=1}^{\infty}\frac{1}{x} = ?$$