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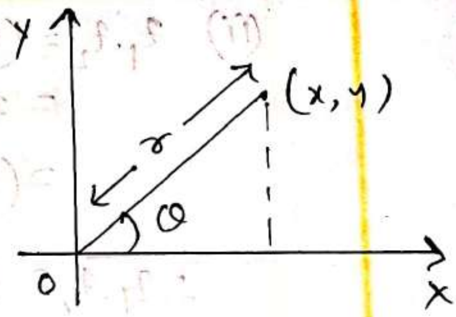
Complex variable:-



* $z = x + iy$
* x is real part $Re\{z\}$
  y is imaginary part $Im\{z\}$
* modulous of $z = |z| = r = \sqrt{x^2+y^2}$
* $x = r\cos\theta$
  $y = r\sin\theta$
* Polar form: $z = r(\cos\theta + i\sin\theta)$
* θ = argument of $z = \tan^{-1} y/x$ also called amplitude
* $z = r(\cos\theta + i\sin\theta) = re^{\theta i}$; $e^{\theta i} = \cos\theta + i\sin\theta$
* conjugate of $z = \bar{z} = z^* = x - iy$

Ex: Prove that-

(i) $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$

(ii) $\overline{z_1 z_2} = \bar{z}_1 . \bar{z}_2$

(iii) $z\bar{z} = |z|^2$

(iv) $z_1\bar{z}_2 + \bar{z}_1\bar{z}_2 = 2Re\{z_1\bar{z}_2\}$

Solⁿ:-

(i) $z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$

$\therefore z_1 + z_2 = (x_1+x_2) + i(y_1+y_2)$

$\therefore \overline{z_1+z_2} = (x_1+x_2) - i(y_1+y_2)$

$\bar{z}_1 + \bar{z}_2 = x_1 - iy_1 + x_2 - iy_2$

$= (x_1+x_2) - i(y_1+y_2)$

$= \overline{z_1+z_2}$

(Showed)

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(ii) $z_1 . z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$= x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2$

$= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$

$\therefore \overline{z_1 . z_2} = (x_1x_2 - y_1y_2) - i(x_1y_2 + x_2y_1)$

$\therefore \bar{z}_1 . \bar{z}_2 = (x_1 - iy_1).(x_2 - iy_2)$

$= x_1x_2 - ix_1y_2 - iy_1x_2 - y_1y_2$

$= (x_1x_2 - y_1y_2) - i(x_1x_2 + y_1y_2)$

$= \overline{z_1 . z_2}$ (Proved)

(iii) $z\bar{z} = (x+iy)(x-iy)$

$= x^2 + y^2$

$= (\sqrt{x^2+y^2})^2$

$= |z|^2$

(Proved)

(iv) $z_1 . \bar{z}_2 = (x_1 + iy_1)(x_2 - iy_2)$

$= x_1x_2 - ix_1y_2 + ix_2y_1 + y_1y_2$

$= (x_1x_2 + y_1y_2) - i(x_1y_2 - x_2y_1)$

$\overline{z_1 . \bar{z}_2} = (x_1x_2 + y_1y_2) + i(x_1y_2 - x_2y_1)$

$\therefore z_1\bar{z}_2 + \overline{z_1\bar{z}_2} = 2(x_1x_2 + y_1y_2) = 2Re\{z_1\bar{z}_2\}$ (Proved)

Ex- Prove that - (i) $|z_1 z_2| = |z_1||z_2|$

(ii) $|z_1 + z_2| \leq |z_1| + |z_2|$

Soln:- (i) $z_1 z_2 = (x_1 + iy_1)(x_2 + iy_1)$

$= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$

$\therefore |z_1 z_2| = \{(x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2\}^{1/2}$

$= \{x_1^2x_2^2 + y_1^2y_2^2 - 2x_1x_2y_1y_2 + x_1^2y_2^2 + x_2^2y_1^2 + 2x_1x_2y_1y_2\}^{1/2}$

$= \{x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + x_2^2y_1^2\}^{1/2}$

$|z_1||z_2| = (x_1^2 + y_1^2)^{1/2}(x_2^2 + y_2^2)^{1/2}$

$= \{(x_1^2 + y_1^2)(x_2^2 + y_2^2)\}^{1/2}$

$= \{x_1^2x_2^2 + x_1^2y_2^2 + y_1^2x_2^2 + y_1^2y_2^2\}^{1/2}$

$= |z_1 z_2|$ (Proved)

(ii) wee need to show that -

$$\sqrt{(x_1+x_2)^2 + (y_1+y_2)^2} \leq \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2}$$

Squaring -

$$(x_1+x_2)^2 + (y_1+y_2)^2 \leq x_1^2 + y_1^2 + x_2^2 + y_2^2 + 2\sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}$$

$$\Rightarrow x_1x_2 + y_1y_2 \leq \sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}$$

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Squaring again:-

$$x_1^2x_2^2+y_1^2y_2^2+2x_1x_2y_1y_2 \leq x_1^2x_2^2+x_1^2y_2^2+x_2^2y_1^2+y_1^2y_2^2$$

$$\Rightarrow 2x_1x_2y_1y_2 \leq x_1^2y_2^2+x_2^2y_1^2$$

$$\Rightarrow 0 \leq (x_1y_2-x_2y_1)^2$$, which is true.

$\therefore |z_1+z_2| \leq |z_1|+|z_2|$ (proved)

<u>Ex</u> | Find the value of (i) $|e^z|$ (ii) $|e^{iz}|$

<u>Soln:-</u>

(i) $e^z = e^{x+iy} = e^x.e^{iy} = e^x(\cos y + i\sin y)$

$\therefore |e^z| = \sqrt{(e^x\cos y)^2+(e^x\sin y)^2}$

$= e^x$ (Ans.)

(ii) $e^{iz} = e^{ix-y} = e^{-y}e^{ix} = e^{-y}(\cos x + i\sin x)$

$\therefore |e^{iz}| = e^{-y}$ (Ans.)

* Modulous is always a real number.

Ex- If $z = 6e^{\pi i/3}$; evaluate $|e^{iz}|$.

Soln:-

$e^{iz} = e^{i6e^{\pi i/3}}$

$= e^{i6(\cos\pi/3 + i\sin\pi/3)}$

$= e^{i6(1/2 + i\sqrt{3}/2)}$

$= e^{(3i - 3\sqrt{3})}$

$= e^{3i} \cdot e^{-3\sqrt{3}}$

$= e^{-3\sqrt{3}}(\cos 3 + i\sin 3)$

$\therefore |e^{iz}| = e^{-3\sqrt{3}}(\cos^2 3 + \sin^2 3)^{1/2}$

$= e^{-3\sqrt{3}}$

(Ans.)

Ex- Prove that, $z\bar{z} = |z|^2$.

Ex- Prove that (a) $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

(b) $\arg(z_1/z_2) = \arg z_1 - \arg z_2$

Soln:-

(a)

Let, $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$; $\arg z_1 = \theta_1$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$; $\arg z_2 = \theta_2$

$\therefore z_1 z_2 = r_1 r_2(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)$

$= r_1 r_2\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}$

$\therefore \arg(z_1 z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2$ (Proved)

(b)

$$z_1/z_2 = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$= \frac{r_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}$$

$$= r_1/r_2 \{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\}$$

$\therefore \arg(z_1/z_2) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2$

(Proved)

Ex- Prove that, $\arg \bar{z} = -\arg z$.

Soln-

$$\bar{z} = r(\cos\theta - i\sin\theta)$$
$$= r\{\cos(-\theta) + i\sin(-\theta)\}$$

$\therefore \arg \bar{z} = -\theta$

$= -\arg z$

(Proved)

Ex- Prove that (1) $|z_1 + z_2 + z_3| \leq |z_1| + |z_2| + |z_3|$

(2) $|z_1 - z_2| \geq |z_1| - |z_2|$

Soln-

(1) $|z_1 + z_2 + z_3| = |(z_1 + z_2) + z_3|$

$\leq |z_1 + z_2| + |z_3| \leq |z_1| + |z_2| + |z_3|$

(Proved)

(2)

$z_1 = x_1 + iy_1$
$z_2 = x_2 + iy_2$
$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$

We need to show that -

$$\sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \geq \sqrt{x_1^2+y_1^2} - \sqrt{x_2^2+y_2^2}$$

Squaring both sides -

$$(x_1-x_2)^2+(y_1-y_2)^2 \geq (x_1^2+y_1^2)+(x_2^2+y_2^2) - 2\sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}$$

$$\Rightarrow x_1^2+x_2^2-2x_1x_2+y_1^2+y_2^2-2y_1y_2 \geq x_1^2+x_2^2+y_1^2+y_2^2 - 2\sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}$$

$$\Rightarrow -2(x_1x_2+y_1y_2) \geq -2\sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}$$

$$\Rightarrow (x_1x_2+y_1y_2) \leq \sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}$$

Squaring again -

$$x_1^2x_2^2+y_1^2y_2^2+2x_1x_2y_1y_2 \leq x_1^2x_2^2+y_1^2y_2^2+x_1^2y_2^2+x_2^2y_1^2$$

$$\Rightarrow 2x_1x_2y_1y_2 \leq x_1^2y_2^2+x_2^2y_1^2$$

$$\Rightarrow 0 \leq x_1^2y_2^2 - 2x_1y_2 \cdot y_1x_2 + y_1^2x_2^2$$

$$\therefore 0 \leq (x_1y_2 - y_1x_2)^2$$

Which is true.

$$\therefore |z_1 - z_2| \geq |z_1| - |z_2|$$

(Proved)

Ex- Prove that $|\bar{z}| = |z|$

Soln:-

$z = x + iy$

$\therefore |z| = (x^2+y^2)^{1/2}$

$\bar{z} = x - iy$

$\therefore |\bar{z}| = (x^2+y^2)^{1/2}$

$\therefore |\bar{z}| = |z|$ (Proved)

Ex- Find the modulous and argument. Also express in polar form.

(1) $-5+i5$ (2) $-3i$ (3) i (4) $-1+\sqrt{3}i$

(5) $-2\sqrt{3}-2i$ (6) $\sqrt{5}-i$

Soln:-

(1)

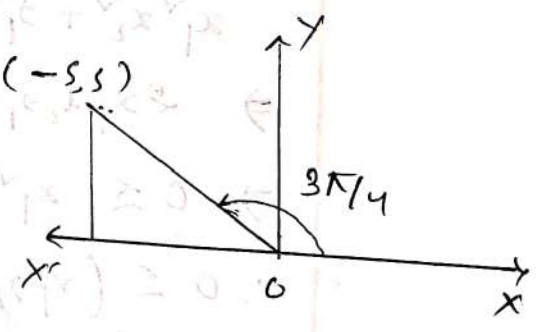
Modulous, $r = \{(-5)^2+5^2\}^{1/2} = 5\sqrt{2}$

arg, $\theta = \tan^{-1} 5/-5$

$= \pi - \tan^{-1} 1$

$= \pi - \pi/4$

$\therefore \theta = 3\pi/4$



Polar form $= 5\sqrt{2}e^{3\pi i/4}$

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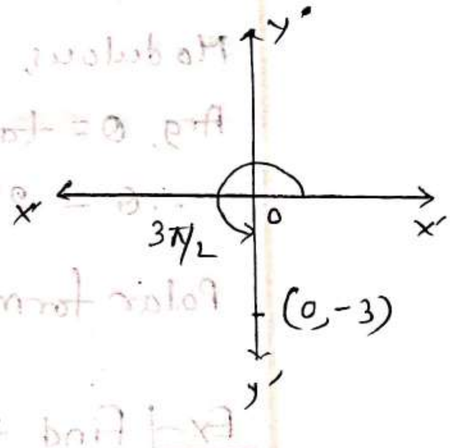
(2) Modulous, $r = 3$

Arg, $\theta = \tan^{-1} -3/0$

$= 2\pi - \tan^{-1} 3/0$

$= 2\pi - \pi/2$

$\therefore \theta = 3\pi/2$

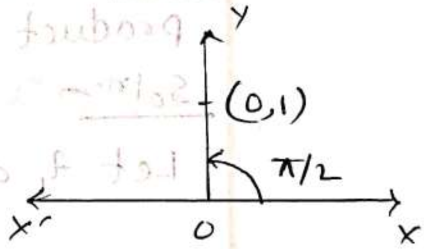


Polar form $= 3e^{3\pi i/2}$

(3) Modulous, $r = 1$

Arg, $\theta = \tan^{-1} 1/0 = \pi/2$

Polar form $= e^{\pi i/2}$



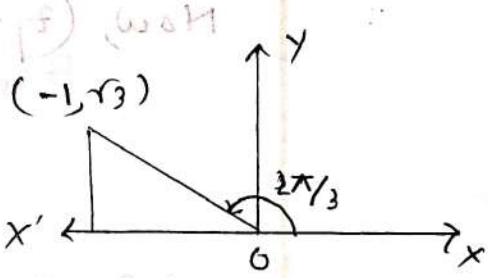
(4) Modulous, $r = \sqrt{1^2 + 3} = 2$

Arg, $\theta = \tan^{-1} \sqrt{3}/-1$

$= \pi - \tan^{-1} \sqrt{3}$

$= \pi - \pi/3$

$\therefore \theta = 2\pi/3$



~~M~~ Polar form $= 2e^{2\pi i/3}$

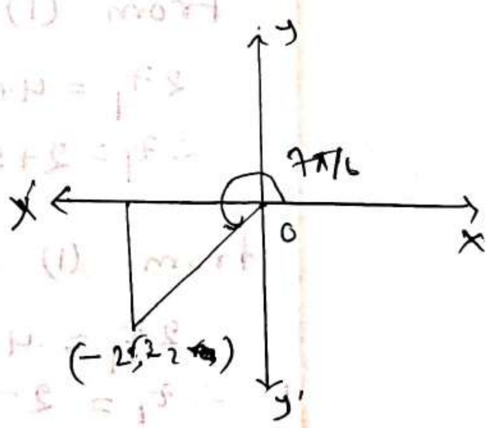
(5) Modulous, $r = 4$

Arg, $\theta = \tan^{-1} -2/-2\sqrt{3}$

$= \pi + \tan^{-1} 1/\sqrt{3}$

$= \pi + \pi/6$

$= 7\pi/6$



Polar form $= 4e^{7\pi i/6}$

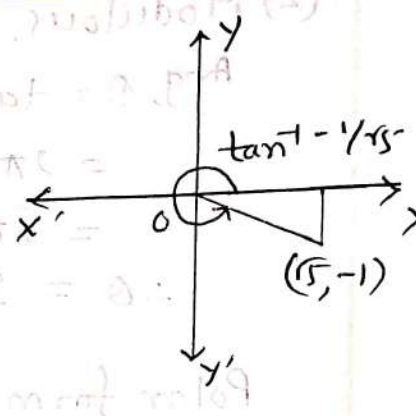
(6)

Modulous, $r = \sqrt{6}$

Arg, $\theta = \tan^{-1} -1/\sqrt{5}$

$\therefore \theta = 2\pi - \tan^{-1} 1/\sqrt{5}$

Polar form, $= \sqrt{6}\, e^{(2\pi - \tan^{-1} 1/\sqrt{5})i}$



Ex-| Find two complex numbers whose sum is 4 and product is 8.

Soln:-

Let z_1 and z_2 be two complex numbers; So-

$z_1 + z_2 = 4$ ——— (1)

$z_1 z_2 = 8$ ——— (2)

Now, $(z_1 - z_2)^2 = (z_1 + z_2)^2 - 4z_1 z_2$

$= 16 - 32$

$= -16$

$= 16i^2$

$\therefore z_1 - z_2 = 16i$ ——— (3) or $z_1 - z_2 = -16i$ ——— (4)

From (1) and (3) we get -

$2z_1 = 4 + 4i$ and $2z_2 = 4 - 4i$

$\therefore z_1 = 2 + 2i$ $\therefore z_2 = 2 - 2i$

From (1) and (4) we get -

$2z_1 = 4 - 4i$ and $2z_2 = 4 + 4i$

$\therefore z_1 = 2 - 2i$ $\therefore z_2 = 2 + 2i$

(Ans.)

Ex: If $z = x+iy$, prove that $|x|+|y| \leq \sqrt{2}\,|x+iy|$

Soln:-

$z = x+iy$

$\therefore |z| = \sqrt{x^2+y^2}$

$\Rightarrow |z|^2 = x^2+y^2$

$\Rightarrow 2|z|^2 = 2x^2+2y^2$

$\Rightarrow 2|z|^2 = x^2+y^2+x^2+y^2$

$\Rightarrow 2|z|^2 \geq x^2+y^2+2xy$ $[(x-y)^2 \geq 0$ or, $x^2+y^2 \geq 2xy]$

$\Rightarrow 2|z|^2 \geq |x|^2+|y|^2+2|x||y|$

$\Rightarrow 2|z|^2 \geq \{|x|+|y|\}^2$

$\therefore \sqrt{2}\,|x+iy| \geq |x|+|y|$ (Proved)

Ex: If $z = 6e^{\pi i/6}$ then find $|e^{iz}|$.

Soln:-

$e^{iz} = e^{i6e^{\pi i/6}}$

$= e^{i6(\cos\pi/6 + i\sin\pi/6)}$

$= e^{i6(\sqrt{3}/2 + i/2)}$

$= e^{(3\sqrt{3}i - 3)}$

$= e^{3\sqrt{3}i} \cdot e^{-3}$

$\therefore |e^{iz}| = e^{-3}$

(Ans.)

Ex- Prove that, $z^{-1} = \frac{\bar{z}}{|z|^2}$

Soln:-

Let, $z = x + iy$

$\therefore \bar{z} = x - iy$

$\therefore |z|^2 = x^2 + y^2$

$$\therefore \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2}$$

$$= \frac{x - iy}{(x+iy)(x-iy)}$$

$$= \frac{1}{x+iy}$$

$$= \frac{1}{z}$$

$$= z^{-1}$$

(Proved)

Ex–1 (a) Find the modulous and argument of $z = 2 + 2\sqrt{3}\,i$

(b) Express $z = 2 + 2\sqrt{3}\,i$ in polar form.

Soln:–

(a) Modulous, $r = |z|$

$= \sqrt{2^2 + (2\sqrt{3})^2}$

$= 4$

Argument, $\theta = \tan^{-1} y/x$

$= \tan^{-1} 2\sqrt{3}/2$

$= \tan^{-1} \sqrt{3}$

$= \pi/3$

(b) $r = 4$

$\theta = \pi/3$

$\therefore$ ~~$x = 4\cos\pi/3$~~ ~~$y = 4\sin\pi/3$~~ $\quad \therefore$ Polar form $= re^{i\theta} = 4e^{i\pi/3}$

(Ans.)

Ex–1 Find the modulous, argument and polar form of $z = -\sqrt{6} - \sqrt{2}\,i$

Soln:–

$r = |z| = \sqrt{(-\sqrt{6})^2 + (-\sqrt{2})^2}$

$= 2\sqrt{2}$

$\theta = \tan^{-1} y/x = \tan^{-1} -\sqrt{2}/-\sqrt{6} = \pi + \tan^{-1} \sqrt{2}/\sqrt{6}$

$= \pi + \tan^{-1} 1/\sqrt{3} = \pi + \pi/6 = 7\pi/6$

$\therefore$ Polar form $= 2\sqrt{2}\, e^{i 7\pi/6}$ (Ans)

⊛ State and prove De Moivre's theorem:-

Statement:-

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$; where n is any positive integer

Proof:-

Let, $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

and, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

So, $z_1 z_2 = r_1 r_2 (\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)$

$= r_1 r_2 [(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)]$

$= r_1 r_2 [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$ —— (i)

A generalitation of (1)-

$z_1 z_2 \dots z_n = r_1 r_2 \dots r_n [\cos(\theta_1+\theta_2+\dots+\theta_n) + i\sin(\theta_1+\theta_2+\dots+\theta_n)]$

If, $z_1 = z_2 = \dots = z_n$ ~~ant~~

$z^n = r^n(\cos n\theta + i\sin n\theta)$ —— (2)

We use principle of mathematical induction.

Assume that the result is true for the particular positive integer k i.e.- assume-

$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

Then multiplying both sides by $(\cos\theta + i\sin\theta)$, we find –

$$(\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta) = (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$$

$$\therefore (\cos\theta + i\sin\theta)^{k+1} = \{\cos(k+1)\theta + i\sin(k+1)\theta\} \quad [\text{By } (1)]$$

Thus, if the result is true for $n=k$, then it is also true for $n=k+1$. But since the result is clearly true for $n=1$, it must also be true for $n=1+1=2$ and $n=2+1=3$ etc., and so must be true for all positive integers.

(Proved)

* The result is equivalent to the statement –

$$(e^{i\theta})^n = e^{in\theta}$$

Roots of complex members:-

From de Moivre's theorem, we can show that if n is positive integer-

$$z^{1/n} = r^{1/n}\left\{\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right\} ; k = 0, \pm 1, \pm 2, \dots$$

Provided, $z \neq 0$

Ex- Prove that $e^{i\theta} = e^{i(\theta+2k\pi)}$; $k = 0, \pm 1, \pm 2, \dots$

Soln:-

$$e^{i(\theta+2k\pi)} = \cos(\theta+2k\pi) + i\sin(\theta+2k\pi)$$
$$= \cos\theta + i\sin\theta$$
$$= e^{i\theta}$$

(Proved)

Ex- Find the roots of $z^5+32=0$ and locate these values in complex plane.

Soln.-

$$z^5 = -32$$
$$\Rightarrow z^5 = 32(\cos\pi + i\sin\pi)$$
$$\Rightarrow z^5 = 2^5\left\{\cos(\pi+2k\pi) + i\sin(\pi+2k\pi)\right\} \quad \text{if, } k = 0, 1, 2, \dots$$
$$\therefore z = 2\left\{\cos\frac{\pi+2k\pi}{5} + i\sin\frac{\pi+2k\pi}{5}\right\}$$
$$= 2e^{i\frac{\pi+2k\pi}{5}} ; \text{ if } k = 0, 1, 2, 3, 4$$

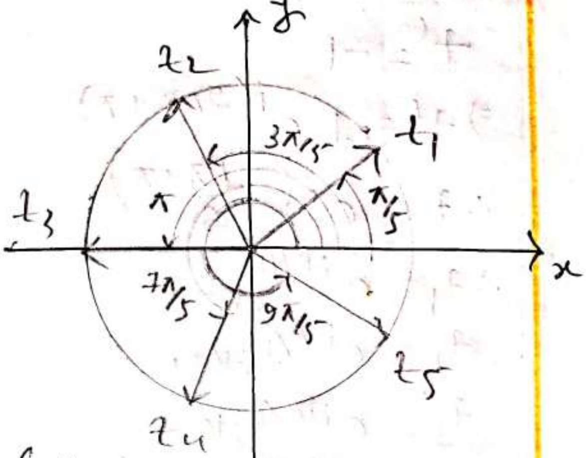
$z_1 = 2e^{i\pi/5}$; $k=0$

$z_2 = 2e^{i3\pi/5}$; $k=1$

$z_3 = 2e^{i\pi}$; $k=2$

$z_4 = 2e^{i7\pi/5}$; $k=3$

$z_5 = 2e^{i9\pi/5}$; $k=4$



Ex- Find the roots of the following-

(i) $z^2 + \pi^2 = 0$

(ii) $z^6 + 1 = 0$

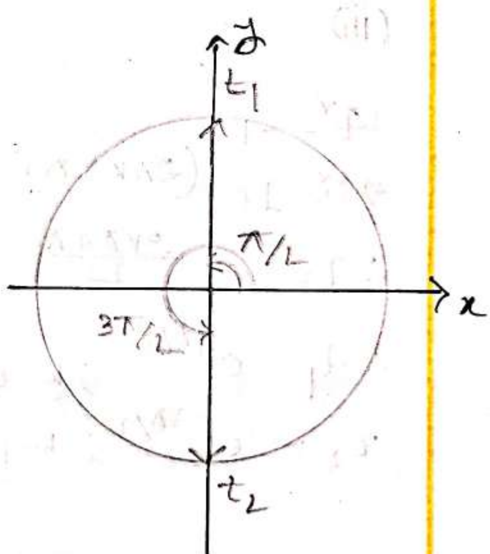
(iii) $z^2 + 1 = 0$

Soln:-

(i)
$$z^2 = -\pi^2$$
$$\Rightarrow z^2 = \pi^2 e^{i(2k\pi + \pi)}$$
$$\therefore z = \pi e^{i\frac{2k\pi+\pi}{2}}$$

$z_1 = \pi e^{i\pi/2}$; $k=0$

$z_2 = \pi e^{i3\pi/2}$; $k=1$



N.B. :- Exam-এ steps skip করার যাবে না।

(ii)

$z^6 = -1$

$\Rightarrow z^6 = 1.e^{i(2\pi k + \pi)}$

$\therefore z = e^{i\left(\frac{2k\pi+\pi}{6}\right)}$

$\therefore z_1 = e^{i\pi/6}$; $k=0$

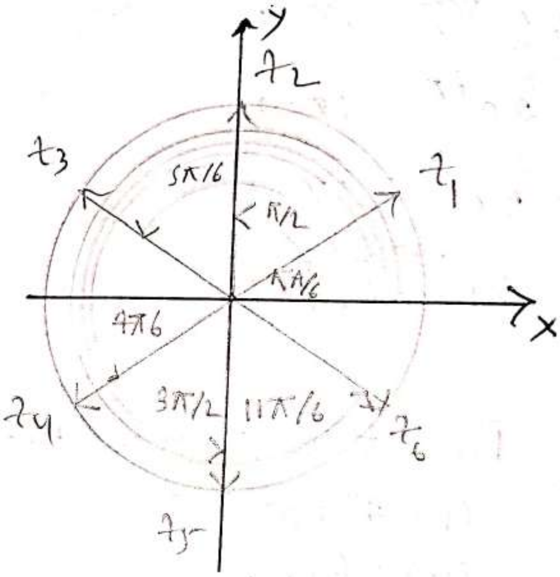
$z_2 = e^{i\pi/2}$; $k=1$

$z_3 = e^{i5\pi/6}$; $k=2$

$z_4 = e^{i7\pi/6}$; $k=3$

$z_5 = e^{i3\pi/2}$, $k=4$

$z_6 = e^{i11\pi/6}$; $k=5$



(iii)

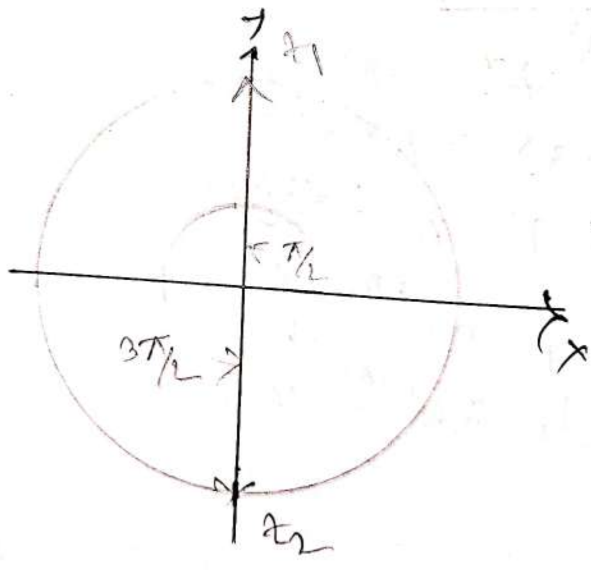
$z^2 = -1$

$\Rightarrow z^2 = 1e^{i(2\pi k + \pi)}$

$\therefore z = e^{i\frac{2k\pi+\pi}{2}}$

$\therefore z_1 = e^{i\pi/2}$; $k=0$

$z_2 = e^{i3\pi/2}$; $k=1$



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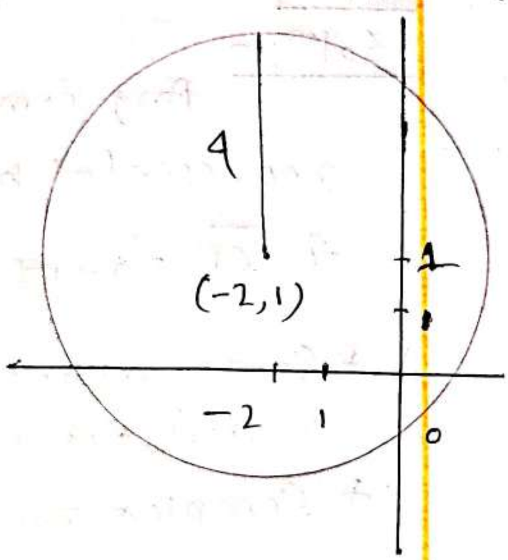
Ex- Find the eqn for circle of radius 4 with centre at $(-2,1)$ at complex plane.

Soln:-

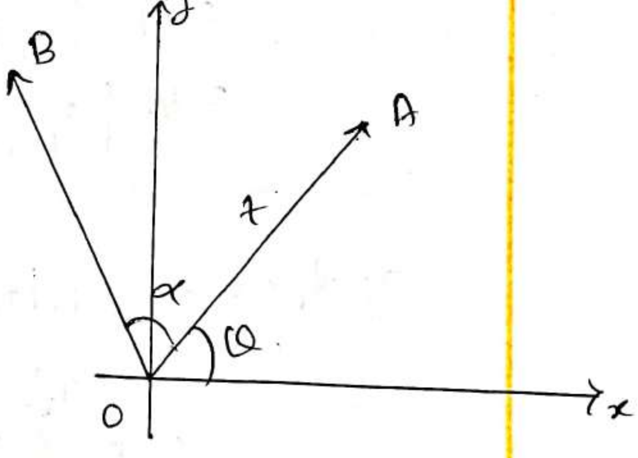


The centre can be represented by the complex number $-2+i$, if z is any point on the circle, the distance from z to $-2+i$ is $|z-(-2+i)|=4$

Or, $|z+2-i|=4$ is the required eqn.

Ex- Given a complex number z. interprete geometrically $ze^{i\alpha}$, where α is real.

Soln:-



Let $z=re^{i\theta}$ be represented graphically by a vector OA in figure. Then $ze^{i\alpha}=re^{i\theta}\cdot e^{i\alpha}=re^{i(\theta+\alpha)}$ is the vector represented by OB. Hence, multiplication of a vector z by $e^{i\alpha}$ amounts to rotating z anticlockwise through angle α.

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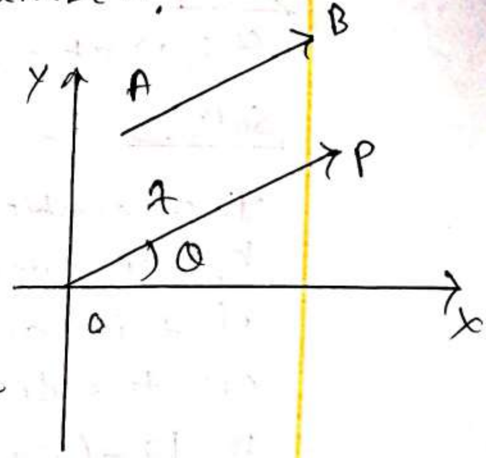
Ex- Vector interpretation of complex number.

Soln.-

Any complex number $z = x+iy$ represented by $\vec{OP}$.



$z = \vec{OP} = x+iy = \vec{AB}$

* Same magnitude and same directed vectors are equal.
* Complex numbe is a vector.

Dot product and cross product of complex:-

1. $z_1 . z_2 = |z_1||z_2| \cos\theta$

$= x_1x_2 + y_1y_2$

$= Re\{\bar{z}_1 z_2\}$

$= 1/2 \{z_1\bar{z}_1 + z_2\bar{z}_2\}$

where θ is angle between $z_1 = x_1+iy_1$, $z_2 = x_2+iy_2$

2. $z_1 \times z_2 = |z_1||z_2| \sin\theta$

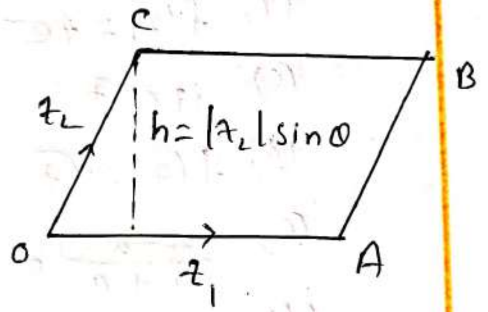
$= x_1y_2 - x_2y_1$

$= Im\{\bar{z}_1 z_2\}$

$= 1/2i \{\bar{z}_1 z_2 - z_1\bar{z}_2\}$

Ex-| Prove that the area of a parallelogram having sides z_1 and z_2 is $|z_1 \times z_2|$

Soln:-



The area of a parallelogram

$= $ Base $\times$ Height

$= OA \times h$

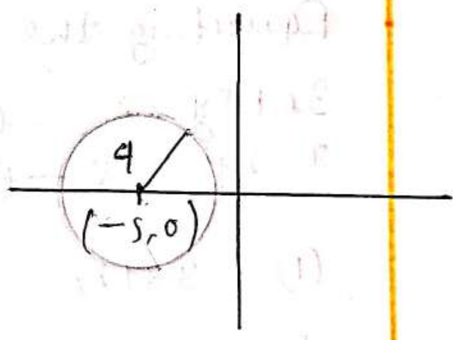
$= |z_1||z_2|\sin\theta$

$= |z_1 \times z_2|$

(Proved)

Ex-| Represent graphically the set of variables z in which $\frac{|z-3|}{|z+3|} = 2$.

Soln:-



$|z-3| = 2|z+3|$

$\Rightarrow |x-3+iy| = 2|x+3+iy|$

$\Rightarrow \sqrt{(x-3)^2+y^2} = 2\sqrt{(x+3)^2+y^2}$

$\Rightarrow (x-3)^2+y^2 = 4(x+3)^2+4y^2$

$\Rightarrow x^2+9-6x+y^2 = 4x^2+36+24x+4y^2$

$\Rightarrow 3x^2+30x+3y^2+27=0$

$\Rightarrow x^2+10x+y^2+9=0$

$\Rightarrow x^2+2.5.x+5^2+y^2 = 5^2-9$

$\Rightarrow (x+5)^2+y^2 = 16$

$\Rightarrow \sqrt{(x+5)^2+y^2} = 4$

$\Rightarrow |x+5+iy| = 4$

$\Rightarrow |z+5| = 4$

$\Rightarrow |z-(-5+0.i)| = 4$

which is a circle of radius 4 with centre at $(-5,0)$

Ex- Find the real and imaginary part of the following

(a) $3x + 2iy - ix + 5y = 7 + 5i$

(b) $f(z) = ze^{-z}$

(c) ize^{-z}

(d) $f(z) = \sqrt{z}$

(e) $\frac{z-a}{z+a}$

(f) $\left(\frac{-1+\sqrt{3}i}{2}\right)^3$

Soln:-

(a)

$(3x + 5y) + i(2y - x) = 7 + 5i$

Equating the real and the imaginary parts-

$3x + 5y = 7$ —— (1)

$x - 2y = -5$ —— (2)

(1) $-$ $3 \times$ (2)

$3x + 5y - 3x + 6y = 7 + 15$

$\Rightarrow 11y = 22$

$\therefore y = 2$

From (2)

$x - 4 = -5$

$\therefore x = -1$

$\therefore x = -1$

$y = 2$ (Ans.)

(b)

$f(z) = ze^{-z}$

$= re^{i\theta}\, e^{(-x-iy)}$

$= re^{i\theta} \cdot e^{-x} \cdot e^{-iy}$

$= re^{-x} e^{i(\theta - y)}$

$= re^{-x}\{\cos(\theta - y) + i\sin(\theta - y)\}$

$= re^{i\theta} e^{i^2 z}$

$= re^{i(\theta + iz)}$

$= r\{\cos(\theta + iz) + i\sin(\theta + iz)\}$

$\therefore x = r\cos(\theta + iz)$

$y = r\sin(\theta + iz)$

(c) ize^{-z}

$= ir\{\cos(\theta + iz) + i\sin(\theta + iz)\}$

$= r\{-\sin(\theta + iz) + i\cos(\theta + iz)\}$

$\therefore x = -r\sin(\theta + iz)$

$y = r\cos(\theta + iz)$

(f) $\left(\frac{-1+\sqrt{3}i}{2}\right)^3$

Let, $z_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

Modulous, $r = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$= 1$

Arg, $\theta = \tan^{-1}\frac{\sqrt{3}/2}{-1/2}$

$= \tan^{-1}\sqrt{3}/-1$

$= \pi - \tan^{-1}\sqrt{3}/1$

$= \pi - \pi/3 = 2\pi/3$

$\therefore z_1 = e^{2\pi i/3}$

$\therefore z_1^3 = (e^{2\pi i/3})^3$

$= e^{2\pi i}$

$= \cos 2\pi + i\sin 2\pi$

$= 1 + 0i$

$\therefore x = 1$

$y = 0$

(d) $f(z) = \sqrt{z}$

$z = re^{i\theta}$

$\therefore \sqrt{z} = \sqrt{r}\, e^{i\theta/2}$

$= \sqrt{r}\left(\cos\theta/2 + i\sin\theta/2\right)$

$\therefore x = \sqrt{r}\cos\theta/2$

$y = \sqrt{r}\sin\theta/2$

(e)

Ex- | Find the real and imaginary parts of following eqns -

(1) $2x - 3iy + 4ix - 2y - 5 - 10i = (x+y+2) - (y-x+3)i$

(2) $f(z) = ze^{iz}$

(3) $f(z) = \sqrt{z}$

(4) $\frac{-1+\sqrt{3}i}{2}$

Soln:-

(1)

$(2x-2y-5) - i(3y-4x+10) = (x+y+2) - i(y-x+3)$

Equating the real and imaginary parts-

$2x-2y-5 = x+y+2$

$\Rightarrow x-3y-7=0 \quad \therefore x-3y=7$ ——— (i)

And, $3y-4x+10 = y-x+3$

$\Rightarrow 3x-2y=7$ ——— (ii)

$2\times(i) - 3(ii)$

$2x-6y-9x+6y = 14-21$

$\Rightarrow -7x = -7$

$\therefore x = 1$

From (1) $1-3y=7$

$\Rightarrow -3y = 6$

$\therefore y = -2$

$\therefore x = 1$
$y = -2$

(Ans.)

(2)

$f(z) = ze^{iz}$

$= re^{i\theta}e^{iz}$

$= re^{i(z+\theta)}$

$= r[\cos(z+\theta) + i\sin(z+\theta)]$

$\therefore x = r\cos(z+\theta)$

$y = r\sin(z+\theta)$

(Ans.)

(3)

$f(z) = \sqrt{z}$

$= \sqrt{re^{i\theta}}$

$= \sqrt{r}e^{i\theta/2}$

$= \sqrt{r}(\cos\theta/2 + i\sin\theta/2)$

$x = \sqrt{r}\cos\theta/2$

$y = \sqrt{r}\sin\theta/2$

(Ans)

(4) $\dfrac{-1+\sqrt{3}i}{2} = -1/2 + i\sqrt{3}/2$ $\quad x = -1/2$

$y = \sqrt{3}/2$ (Ans)

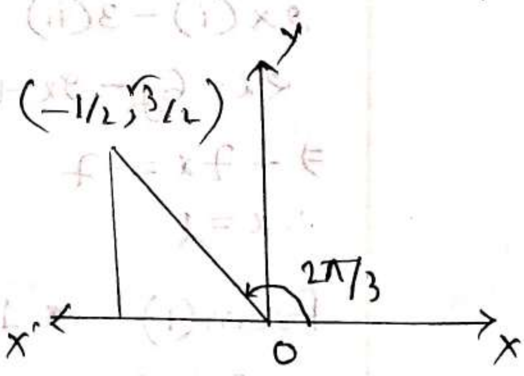
Ex- Find the modulous and argument of $\dfrac{-2}{1+i\sqrt{3}}$

Soln:-

$$\frac{-2}{1+i\sqrt{3}} = \frac{-2(1-i\sqrt{3})}{1+3}$$

$$= \frac{-1+i\sqrt{3}}{2}$$

$$= -1/2 + i\sqrt{3}/2$$



Modulous, $r = \sqrt{(-1/2)^2 + (\sqrt{3}/2)^2}$

$= \sqrt{1/4 + 3/4}$

$= 1$

Arg. $\theta = \tan^{-1}\dfrac{\sqrt{3}/2}{-1/2}$

$= 2\pi/3$

Ex- Find ~~comp~~ two complex number where sum is 4 and product is 8.

Ex- If $\omega = 3iz - z^2$ and $z = x+iy$. Find $|\omega|^2$.

Soln:-

$$\omega = 3i(x+iy) - (x+iy)^2$$
$$= 3xi - 3y - (x^2 + 2ixy - y^2)$$
$$= 3xi - 3y - x^2 - 2ixy + y^2$$
$$= (y^2 - x^2 - 3y) + i(3x - 2xy)$$
$$|\omega|^2 = (y^2 - x^2 - 3y)^2 + (3x - 2xy)^2$$
$$=$$

Ex- If $z = x+iy$, prove that $|x|+|y| \le \sqrt{2}\,|x+iy|$

Ex- Find the roots of (i) $z^4+16=0$ (ii) $z^4+a^4=0$

Soln:-

(ii) $z^4 = -a^4$

$= +a^4 e^{i(\pi+2k\pi)}$

$\therefore z = a e^{i\frac{\pi+2k\pi}{4}}$

$z_1 = a e^{\pi i/4}$; $k=0$

$z_2 = a e^{3\pi i/4}$; $k=1$

$z_3 = a e^{5\pi i/4}$; $k=2$

$z_4 = a e^{7\pi i/4}$; $k=3$

(Ans.)

(i) $z^4 = -16$

$\Rightarrow z^4 = 16 e^{i(2k\pi+\pi)}$

$\therefore z = 2 e^{i\frac{2k\pi+\pi}{4}}$

$\therefore z_1 = 2e^{\pi i/4}$; $k=0$

$\therefore z_2 = 2e^{3\pi i/4}$; $k=1$

$\therefore z_3 = 2e^{5\pi i/4}$; $k=2$

$\therefore z_4 = 2e^{7\pi i/4}$; $k=3$

(Ans.)

Function, limit and Continuity:-

Function:-

$w = f(z) = u(x,y) + iv(x,y) = u + iv$

(1) Single valued f^n:- If only one value of w corresponds to each value of z, we say that, w is single valued f^n of z.

Ex- $w = f(z) = z^2$ is a single valued f^n

(2) Multiple valued f^n:- If more than one value of w corresponds to each value of z, we say that w is a multiple valued f^n of z.

Ex- $w = f(z) = \sqrt{z}$ is a multiple valued f^n.

Limit:-

Let $f(z)$ be defined and single valued. We say that the number 'l' is the limit of $f(z)$ at z approaches z_0 and write $\lim\limits_{z \to z_0} f(z) = l$ if for any positive number ϵ (however small) we can find another positive number δ (depend on ϵ) such that

$$|f(z) - l| < \epsilon \text{ whenever } |z - z_0| < \delta$$

Ex- Evaluate $\lim\limits_{z \to 0} \frac{\bar{z}}{z}$

Soln-

$$\lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{(x,y) \to 0} \frac{x - iy}{x + iy} = \lim_{(x,y=0) \to 0} \frac{x}{x} = 1 \text{ if } y = 0$$

$$\text{and } \lim_{(x=0,y) \to 0} \frac{-y}{y} = -1 \text{ if } x = 0$$

Since, the two approaches do not give the same answer, the limit does not exist.

Ex-| Evaluate – (i) $\lim_{z \to i} \left\{ (z-i) \dfrac{e^{imz}}{z^2+1} \right\}$

(ii) $\lim_{z \to ae^{\pi i/4}} \left\{ (z - ae^{\pi i/4}) \dfrac{1}{z^4+a^4} \right\}$

Soln:-

(i) $\lim_{z \to i} \left\{ (z-i) \dfrac{e^{imz}}{z^2+1} \right\}$

$= \lim_{z \to i} \left\{ \dfrac{(z-i)(z+i)\, e^{imz}}{(z^2+1)(z+i)} \right\}$

$= \lim_{z \to i} \left\{ \dfrac{(z^2+1)\, e^{imz}}{(z^2+1)(z+i)} \right\}$

$= \lim_{z \to i} \dfrac{e^{imz}}{z+i}$

$= \dfrac{e^{i.m.i}}{i+i}$

$= \dfrac{e^{-m}}{2i}$

$= \dfrac{1}{2e^{m}i}$ (Ans.)

(ii) $\lim\limits_{z \to ae^{\pi i/4}} \left\{ \left(z - ae^{\pi i/4}\right) \dfrac{1}{z^4 + a^4} \right\}$

$= \lim\limits_{z \to ae^{\pi i/4}} \left\{ \left(z - ae^{\pi i/4}\right) \dfrac{1}{z^4 - \left(ae^{\pi i/4}\right)^4} \right\}$

$= \lim\limits_{z \to ae^{\pi i/4}} \left\{ \left(z - ae^{\pi i/4}\right) \dfrac{1}{(z^2)^2 - \left\{\left(ae^{\pi i/4}\right)^2\right\}^2} \right\}$

$= \lim\limits_{z \to ae^{\pi i/4}} \left\{ \dfrac{z - ae^{\pi i/4}}{\left\{z^2 + \left(ae^{\pi i/4}\right)^2\right\}\left\{z^2 - \left(ae^{\pi i/4}\right)^2\right\}} \right\}$

$= \lim\limits_{z \to ae^{\pi i/4}} \left\{ \dfrac{1}{\left\{z^2 + \left(ae^{\pi i/4}\right)^2\right\}\left\{z + ae^{\pi i/4}\right\}} \right\}$

$= \dfrac{1}{\left(a^2 e^{\pi i/2} + a^2 e^{\pi i/2}\right)\left(2ae^{\pi i/4}\right)}$

$= \dfrac{1}{2a^2 e^{\pi i/2} \cdot 2ae^{\pi i/4}}$

$= \dfrac{1}{4a^3 e^{3\pi i/4}}$ (Ans.)

Ex-1 $\lim_{z \to e^{\pi i/6}} \left\{ (z - e^{\pi i/6}) \frac{1}{z^6+1} \right\}$

Soln:-

$\lim_{z \to e^{\pi i/6}} \frac{1}{6z^5}$ [La Hospital law]

$= \frac{1}{6 e^{5\pi i/6}}$ (Ans.)

* Continuity:-

$f(z)$ be continuous at $z=z_0$, if

(1) $\lim_{z\to z_0} f(z) = l$ must exist

(2) $f(z_0)$ must exist, i.e. $f(z)$ is defined at z_0

(3) $l = f(z_0)$

Ex If $f(z) = \begin{cases} z^2 & ; z\neq i \\ 0 & ; z=i \end{cases}$, is the f^n $f(z)$ continuous at $z=i$? If not, redefine the f^n $f(z)$ to be continuous.

Soln:-

$f(z)$ be continuous at $z=i$ if-

(1) $\lim_{z\to i} f(z) = \lim_{z\to i} z^2 = -1$ exists

(2) $f(z_0) = f(i) = 0$ exists

(3) $l \neq f(z_0)$ i.e. $0\neq -1$, so $f(z)$ is not continuous at $z=i$.

If we redefine the f^n $f(z)$ as $f(z) = \{ z^2$; for all values of z

Ex $f(z) = \begin{cases} \dfrac{z^2+4}{z-2i} & ; z\neq 2i \\ 3+4i & ; z=2i \end{cases}$ Is the f^n continuous at $z=2i$?

If not, redefine the f^n to be continuous.

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Soln:-

(1) $l = \lim_{z \to 2i} f(z)$

$= \lim_{z \to 2i} \frac{z^2+4}{z-2i}$

$= \lim_{z \to 2i} \frac{z^2-(2i)^2}{z-2i}$

$= \lim_{z \to 2i} (z+2i)$

$= 4i$

(2) $f(z_0) = f(2i) = 3+4i$

(3) $l \neq f(z_0)$

$\therefore$ $f(z)$ is not continuous at $z = 2i$

If we redefine $f(z) = \frac{z^2+4}{z-2i}$ for all values of z, then $f(z)$ will be continuous at $z = 2i$

Ex- If $f(z) = \begin{cases} z^2+2z & ; z \neq i \\ 3+2i & ; z = i \end{cases}$; Is $f(z)$ continuous at $z = i$? If not, redefine $f(z)$ to be continuous.

Soln:-

(1) $l = \lim_{z \to i} f(z)$

$= \lim_{z \to i} (z^2+2z)$

$= i^2+2i$

$= -1+2i$

(2) $f(z_0) = f(i)$

$= 3+2i$

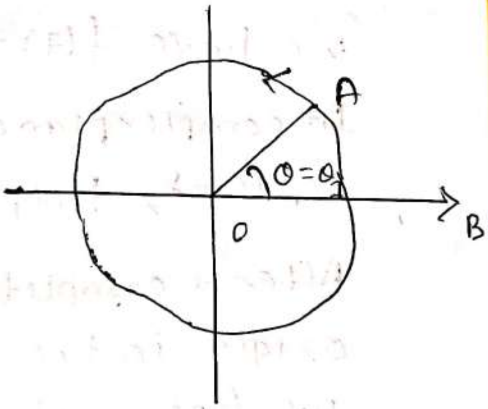
(3) $l \neq f(z_0)$

$\therefore$ $f(z)$ is not continuous at $z=i$.

If we redefine $f(z) = z^2+2z$ for all values of z, then $f(z)$ will be continuous.

1] Branch point and branch line:—

Suppose that, we are given the f^n $w = z^{1/2}$. Suppose further that we allow z to make a complete circuit around the origin starting from 'A'. We have $z = re^{i\theta}$ $\therefore$ $w = \sqrt{r}e^{i\theta/2}$ so that at A, $\theta = \theta_1$ and $z = \sqrt{r}e^{i\theta_1/2}$.



After a complete circuit back to A, $\theta = \theta_1 + 2\pi$ and $w = \sqrt{r}e^{i(\theta_1+2\pi)/2} = -\sqrt{r}e^{i\theta_1/2}$. Thus we have not achieved the same ~~result~~ value with which we started. However, by making a second complete circuit back to A, $\theta = \theta_1 + 4\pi$, $w = \sqrt{r}e^{i(\theta_1+4\pi)/2} = \sqrt{r}e^{i\theta_1/2}$ and then we do obtain the same value of w with which we started.

We can describe above by stating that if $0 \leq \theta < 2\pi$ we are on the one branch of the multiple valued ~~problem~~ f^n $z^{1/2}$, while if $2\pi \leq \theta < 4\pi$, we are on the other branch of the f^n.

It is clear that, each branch of the f^n is single valued. In order to keep the f^n single valued, we set up an artificial barrier such as OB where B is at infinity which we agree not to cross. This barrier is called branch line or branch cut. And point O is called a branch point.

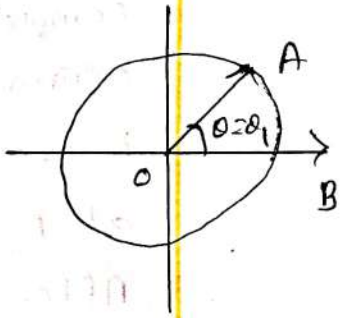
Ex- Prove that $f(z) = \ln z$ has a branch point at $z=0$

Soln-

we have $f(z) = \ln z = \ln(re^{i\theta}) = \ln r + i\theta$

In complexplane, for which $r=r_1$, $\theta=\theta_1$.

so, that, $\ln z_1 = \ln r_1 + i\theta_1$ in fig.



After a complete circuit about the origin in the anti-clockwise direction, we find ~~r~~eturning to z_1 that $r=r_1$, $\theta = \theta_1 + 2\pi$. So that $\ln z_1 = \ln r_1 + i(\theta_1 + 2\pi)$

So at $z=0$ is a branch point.

(Proved)

Ex- Find the branch point of the f^n-

(i) $f(z) = \ln(z-a)$

(ii) $f(z) = \sqrt{z^2+1}$

(iii) $f(z) = \ln(z-0)$

Ex: Prove that zeroes of (i) $f(z) = \sin z$ and $f(z) = \cos z$ are all real and find them.

(ii) Evaluate – $$\lim_{z \to \frac{-a+\sqrt{a^2-b^2}}{b}i} \left\{ \left(z - \frac{-a+\sqrt{a^2-b^2}}{b}i\right) \frac{2}{bz^2+2aiz-b} \right\}$$

Ans: $\frac{1}{\sqrt{a^2-b^2}}i$

(iii) $$\lim_{z \to \pi i} \frac{d}{dz} \left\{ (z-\pi i)^2 \frac{e^z}{(z^2+\pi^2)^2} \right\}$$

Soln:-

(i)

$\sin z = 0$

$\Rightarrow \frac{e^{iz} - e^{-iz}}{2i} = 0$

$\Rightarrow e^{iz} = e^{-iz}$

$\Rightarrow e^{iz} = 1/e^{iz}$

$\Rightarrow e^{2iz} = 1$

$\Rightarrow e^{2iz} = e^{2k\pi i}$

$\therefore z = k\pi$

i.e. $0, \pm\pi, \pm 2\pi, \ldots$

$\cos z = 0$

$\Rightarrow \frac{e^{iz} + e^{-iz}}{2} = 0$

$\Rightarrow e^{iz} = -e^{iz}$

$\Rightarrow e^{2iz} = -1$

$\Rightarrow e^{2iz} = e^{(2k+1)\pi i}$

$\therefore z = \left(\frac{2k+1}{2}\right)\pi = \left(k + \frac{1}{2}\right)\pi$

i.e. $\pm \pi/2, \pm 3\pi/2, \ldots$

(ii)

$$\lim_{z \to \frac{-a+\sqrt{a^2-b^2}}{b}} \left\{ \left(z - \frac{-a+\sqrt{a^2-b^2}}{b}\right) \frac{2}{bz^2+2aiz-b} \right\}$$

$$= \lim_{z \to \frac{-a+\sqrt{a^2-b^2}}{b}} \left\{ \frac{2}{2bz+2ai} \right\}$$

$$= \lim_{z \to \frac{-a+\sqrt{a^2-b^2}}{b}} \left\{ \frac{1}{bz+ai} \right\}$$

$$= \frac{1}{-ai+\sqrt{a^2-b^2}\,i+ai} \quad \cancel{\text{(Ans.)}} = \frac{1}{\sqrt{a^2-b^2}\,i} \quad \text{(Ans)}$$

(iii)

$$\lim_{z \to \pi i} \frac{d}{dz} \left\{ (z-\pi i)^2 \frac{e^z}{(z^2+\pi^2)^2} \right\}$$

$$= \lim_{z \to \pi i} \frac{d}{dz} \left\{ (z-\pi i)^2 \frac{e^z}{\{z^2-(\pi i)^2\}^2} \right\}$$

$$= \lim_{z \to \pi i} \frac{d}{dz} \left\{ (z-\pi i)^2 \frac{e^z}{(z-\pi i)^2 (z+\pi i)^2} \right\}$$

$$= \lim_{z \to \pi i} \frac{d}{dz} \left\{ \frac{e^z}{(z+\pi i)^2} \right\}$$

$$= \lim_{z \to \pi i} \frac{e^z (z+\pi i)^2 - e^z \cdot 2(z+\pi i)}{(z+\pi i)^4}$$

$$= \frac{e^{\pi i} \{(2\pi i)^2 - 2(2\pi i)\}}{(2\pi i)^4}$$

$$= \frac{e^{\pi i}\left(-4\pi^2 - 4\pi i\right)}{16\pi^4}$$

$$= \frac{-4\pi(\pi + i)}{16\pi^4} \times -1$$

$$= \frac{\pi + 1}{4\pi^3} \quad \text{(Ans.)}$$

Complex diff. and the Cauchy-Riemann eqn:-

Derivatives:-

If $f(z)$ is single valued in some region R of the z plane, the derivative of $f(z)$ is defined as

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

Provided that the limit exists independent of the manner in which $\Delta z \to 0$. In such case we say that $f(z)$ is differentiable at z.

Analytic f^n:-

If the derivative $f'(z)$ exists at all points of z of a region R, then $f(z)$ is said to be analytic in R and is referred to as an analytic f^n in R.

A necessary condition that $w = f(z) = u(x,y) + iv(x,y)$ be analytic in a region R if u and v satisfy the cauchy-riemann eqn:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Ex- Test the analyticity of $f(z) = \bar{z}$

Soln-

Method -I

$f(z) = u + iv = \bar{z} = x - iy$; Here $u(x,y) = x$
$v(x,y) = -y$

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From Cauchy-Riemann eqn -

$\frac{\partial u}{\partial x} = 1$ and $\frac{\partial v}{\partial y} = -1$; Since $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$, $f(z)$ is not analytic

(Ans.)

Method 2

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\bar{z} + \overline{\Delta z} - \bar{z}}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{x - iy + \Delta x - i\Delta y - x + iy}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y = 0}} \frac{\Delta x}{\Delta x} = 1 \text{ ; if } \Delta y = 0$$

And, $\lim_{\substack{\Delta x = 0 \\ \Delta y \to 0}} \frac{-\Delta y i}{\Delta y i} = -1$; if $\Delta x = 0$

Since, the two limits does not give same results, so $f(z) = \bar{z}$ is not analytic.

(Ans.)

Ex - State and prove cauchy - Riemann equations.

Soln -

Statement:- a necessary and sufficient condition that $w = f(z) = u(x,y) + iv(x,y)$ be analytic in a region R is that the Cauchy - Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ are satisfied in R where it is supposed that these partial derivatives are continuous in R.

Necessity:-

In order for $f(z)$ to be analytic, the limit

$$f(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)\} - \{u(x,y) + iv(x,y)\}}{\Delta x + i\Delta y} \quad \text{--- (1)}$$

Must exist independent of the matter in which Δz (Δx and Δy) approaches to zero.

We cosider two possible approachs

Case - I: $\Delta y = 0$, $\Delta x \to 0$; In this case, (1) becomes

$$\lim_{\Delta x \to 0} \left[\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x}\right] + i\left[\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}\right]$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Provided the partial derivative exists.

Case-II: $\Delta x = 0$, $\Delta y \to 0$. In this case, (1) becomes-

$$\lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}$$

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{(-i^2)}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Now, $f(z)$ cannot be analytic unless these two limits are identical. Thus a necessary condition that $f(z)$ be analytic is

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Or, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

⊛ Sufficiency:-

Since $\partial u/\partial x$ and $\partial u/\partial y$ are supposed continuous, we have,

$$\Delta u = u(x+\Delta x, y+\Delta y) - u(x,y)$$
$$= \{u(x+\Delta x, y+\Delta y) - u(x, y+\Delta y)\} + \{u(x, y+\Delta y) - u(x,y)\}$$
$$= (\partial u/\partial x + \epsilon_1)\Delta x + (\partial u/\partial y + \eta_1)\Delta y$$
$$= \partial u/\partial x\, \Delta x + \partial u/\partial y\, \Delta y + \epsilon_1 \Delta x + \eta_1 \Delta y$$

where $\epsilon_1 \to 0$ and $\eta_1 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$

~~Since~~, similarly, since $\partial v/\partial x$ and $\partial u/\partial x$ are supposed continuous, we have,

$$\Delta v = \left(\partial v/\partial x + \varepsilon_2\right)\Delta x + \left(\partial v/\partial y + \eta_2\right)\Delta y$$

$$= \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \varepsilon_2 \Delta x + \eta_2 \Delta y$$

where $\varepsilon_2 \to 0$ and $\eta_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$

~~By the cauchy-Riemann equations~~

$$\Delta w = \Delta u + i\Delta v$$

$$= \left(\partial u/\partial x + i\,\partial v/\partial x\right)\Delta x + \left(\partial u/\partial y + i\,\partial v/\partial y\right)\Delta y + \varepsilon \Delta x + \eta \Delta y$$

where $\varepsilon = \varepsilon_1 + i\varepsilon_2 \to 0$ and $\eta = \eta_1 + i\eta_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$ (2)

By the cauchy-riemann equations (2) can be written

$$\Delta w = \left(\partial u/\partial x + i\,\partial v/\partial x\right)\Delta x + \left(-\partial v/\partial x + i\,\partial u/\partial x\right)\Delta y + \varepsilon \Delta x + \eta \Delta y$$

$$= \left(\partial u/\partial x + i\,\partial v/\partial x\right)\Delta x + \left(i^2\,\partial v/\partial x + i\,\partial u/\partial x\right)\Delta y + \varepsilon \Delta x + \eta \Delta y$$

$$= \left(\partial u/\partial x + i\,\partial v/\partial x\right)\Delta x + i\left(\partial u/\partial x + i\,\partial v/\partial x\right)\Delta y + \varepsilon \Delta x + \eta \Delta y$$

$$\therefore \Delta w = \left(\partial u/\partial x + i\,\partial v/\partial x\right)\left(\Delta x + i\Delta y\right) + \varepsilon \Delta x + \eta \Delta y$$

Then on dividing by $\Delta x = \Delta x + i\Delta y$ and taking the limit as $\Delta x \to 0$, we see that

$$\frac{dw}{dx} = f'(z) = \lim_{x \to 0} \frac{\Delta w}{\Delta x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

So that the ~~derivative~~ derivative exists and its unique i.e, $f(z)$ is analytic in R.

Ex- Test the analyticity of the following-

(i) $f(z) = ize^{-z}$ (ii) $f(z) = ze^{-z}$ (iii) $iz^2 + 2z$

Solⁿ:-

(i)

$$
\begin{aligned}
f(z) &= ize^{-z} \\
&= i(x+iy)e^{-(x+iy)} \\
&= (ix-y).e^{-x}e^{-iy} \\
&= (ix-y)e^{-x}(\cos y - i\sin y) \\
&= e^{-x}(ix\cos y + x\sin y - y\cos y + iy\sin y) \\
&= e^{-x}\{(x\sin y - y\cos y) + i(x\cos y + y\sin y)\}
\end{aligned}
$$

$\therefore u + iv = e^{-x}(x\sin y - y\cos y) + ie^{-x}(x\cos y + y\sin y)$

$u = e^{-x}(x\sin y - y\cos y)$
$v = e^{-x}(x\cos y + y\sin y)$

Now, $\partial u/\partial x = -e^{-x}(x\sin y - y\cos y) + e^{-x}(\sin y + 0)$
$= e^{-x}(-x\sin y + y\cos y + \sin y)$

$\partial v/\partial y = e^{-x}(-x\sin y + y\cos y + \sin y)$

$\therefore \partial u/\partial x = \partial v/\partial y$, which satisfies the cauchy-Riemann equation.

Hence, $f(z) = ize^{-z}$ is analytic.

(Ans.)

(ii)

$$f(z) = ze^{-z}$$
$$= (x+iy)e^{-(x+iy)}$$
$$= (x+iy)e^{-x}(\cos y - i\sin y)$$
$$= e^{-x}(x\cos y - ix\sin y + iy\cos y + y\sin y)$$
$$= e^{-x}\{(x\cos y + y\sin y) + i(-x\sin y + y\cos y)\}$$

$\therefore u+iv = e^{-x}(x\cos y + y\sin y) + ie^{-x}(-x\sin y + y\cos y)$

$u = e^{-x}(x\cos y + y\sin y)$
$v = e^{-x}(-x\sin y + y\cos y)$

Now, $\partial u/\partial x = -e^{-x}x\cos y - e^{-x}y\sin y + e^{-x}\cos y$

$\partial v/\partial y = -e^{-x}x\cos y - e^{-x}y\sin y + e^{-x}\cos y$

$\partial u/\partial x = \partial v/\partial y$, which satisfies the cauchy-Riemann equation.

Hence, $f(z) = ze^{-z}$ is analytic

(Ans.)

(iii)

$f(z) = iz^2 + 2z$

$= i(x+iy)^2 + 2(x+iy)$

$= i(x^2 + 2ixy - y^2) + 2(x+iy)$

$= x^2 i - 2xy - iy^2 + 2x + 2iy$

$\therefore u + iv = (2x - 2xy) + i(x^2 - y^2 + 2y)$

$\therefore u = 2x - 2xy$

$v = x^2 - y^2 + 2y$

$\partial u/\partial x = 2 - 2y$

$\partial v/\partial y = 2 - 2y$

$\therefore \partial u/\partial x = \partial v/\partial y$, Which satisfies the Cauchy-Riemann equation.

Hence, $f(z) = iz^2 + 2z$ is analytic.

(Ans.)

Ex: Find the analiticity of the function $f(z) = \frac{1+z}{1-z}$

Soln:-

We know that,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\frac{1+z+\Delta z}{1-z-\Delta z} - \frac{1+z}{1-z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{1+z+\Delta z - z - z^2 - z\Delta z - 1 + z + \Delta z - z + z^2 + z\Delta z}{(1-z)(1-z-\Delta z)(\Delta z)}$$

$$= \lim_{\Delta z \to 0} \frac{2\Delta z}{(1-z-\Delta z)(1-z)(\Delta z)}$$

$$= \lim_{\Delta z \to 0} \frac{2}{(1-z)(1-z-\Delta z)}$$

$$= \frac{2}{(1-z)^2}$$

which is independent of the manner in which $\Delta z \to 0$, provided $z \neq 0$

So, the function is analytic.

(Ans.)

☐

⊛ Laplacian: $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is called laplacian.

⊛ Harmonic f^n: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

Ex: If $f(t) = u + iv$ is analytic in region R, prove that u and v are harmonic in R if they have continuous second partial derivatives in R.

Or, Prove that the real and imaginary points of an analytic f^n satisfy Laplace's equation.

Proof:-

If $f(t) = u + iv$ is analytic in R, then it satisfies the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{—} (1)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{—} (2)$$

Since, u and v have continuous second partial derivatives,-

Diff. (1) p. wrt. x, we get $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial y \partial x}$ — (3)

Diff. (2) p. wrt. y, we get $\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}$ — (4)

From (3) and (4) we get -

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \therefore u \text{ is harmonic}$$

Similarly diff. (1) p. wrt. y and (2) p. wrt. x, we get another Laplace's equation,

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad \therefore v \text{ is harmonic}$$

(Proved)

Ex– (a) Show that $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic.

(b) Find v such that $f(z) = u + iv$ is analytic.

(c) Also find $f(z)$ in terms of z.

Soln:–

(a)

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x \qquad \Big| \qquad \frac{\partial u}{\partial y} = -6xy - 6y$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = 6x + 6 \quad \text{—(1)} \qquad \Big| \qquad \therefore \frac{\partial^2 u}{\partial y^2} = -6x - 6 \quad \text{—(2)}$$

(1) + (2)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ which is a harmonic equation.

$\therefore$ u is harmonic. (Showed)

(b)

Since $f(z)$ be analytic, so from Cauchy-Riemann theorem–

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x \quad \text{—(3)}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 6xy + 6y \quad \text{—(4)}$$

Integrating (3) wrt y and taking x constant–

$$\int dv = \int (3x^2 - 3y^2 + 6x)\, dy$$

$\therefore v = 3x^2y - y^3 + 6xy + F(x)$ —(5) ; where $F(x)$ is an arbitrary real f^n of x

Diff (5) p. wrt. x —

$\cancel{\frac{\partial v}{\partial x} = 3y^2 + 6y + F'(x)}$ ~~— (6)~~ $\frac{\partial v}{\partial x} = 6xy + 6y + F'(x)$ — (6)

From ~~(4)~~ and (6) —

$\cancel{3y^2+6y+F'(x)=}$ $6xy + 6y + F'(x) = 6xy + 6y$

$\Rightarrow F'(x) = 0$

Integrating — $\therefore F(x) = c$; where c is any constant

Then (5) becomes, $v = 3x^2y - y^3 + 6xy + c$

(Ans.)

(c)

$f(z) = f(x+iy) = u(x,y) + iv(x,y)$

Putting $y=0$, $x=z$, $f(z+i0) = u(z,0) + i\,v(z,0)$

$\Rightarrow f(z) = u(z,0) + iv(z,0)$

$= z^3 + 3z^2 + 1 + i(c)$

$\therefore f(z) = z^3 + 3z^2 + 1 + ic$

(Ans.)

Method-2 :-

$f(z) = u(x,y) + iv(x,y)$

$= x^3 - 3xy^2 + 3x^2 - 3y^2 + 1 + i(3x^2y - y^3 + 6xy + c)$

$= x^3 - 3xy^2 + 3x^2 - 3y^2 + 1 + 3x^2yi - iy^3 + 6xyi + ic$

$=$

Ex- Show that (a) $u = 2x(1-y)$ is harmonic

(b) Find v such $f(z) = u + iv$ is analytic

(c) Find also $f(z)$ in terms of z.

Soln:- (a)

$\partial u/\partial x = 2(1-y)$

$\therefore \partial^2 u/\partial x^2 = 0$

$\partial u/\partial y = -2x$

$\therefore \partial^2 u/\partial y^2 = 0$

$\therefore \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$

$\therefore u$ is harmonic. (showed)

(b)

As $f(z)$ is analytic,

$\partial v/\partial y = \partial u/\partial x = 2(1-y)$ —— (1)

$\partial v/\partial x = -\partial u/\partial y = 2x$ —— (2)

Integrating (1) wrt y and keeping x const. -

$v = 2y - y^2 + F(x)$ —— (3)

Diff. (3) p. wrt x -

$\partial v/\partial x = F'(x)$

$\Rightarrow F'(x) = 2x$

Integrating-

$F(x) = x^2 + c$

$\therefore v = 2y - y^2 + x^2 + c$ (Ans.)

(c)

$f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$

Putting $x = z$ and $y = 0$

$f(z) = u(z,0) + i\,v(z,0)$

$\therefore f(z) = 2z + i\,(z^2 + c)$ (Ans.)

Ex: Prove that $u = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic. Find a function v such that $u + iv$ is analytic.

Soln:-

$\partial u/\partial x = 2x - 2y - 2 \quad \therefore \partial^2 u/\partial x^2 = 2$

$\partial u/\partial y = -2y - 2x + 3 \quad \therefore \partial^2 u/\partial y^2 = -2$

$\therefore \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$

Hence, u is harmonic.

Now, if $u + iv$ is analytic -

$\partial v/\partial x = -\partial u/\partial y = 2y + 2x - 3$ —— (1)

$\partial v/\partial y = \partial u/\partial x = 2x - 2y - 2$ —— (2)

Integrating (2) wrt y keeping x const. -

$v = 2xy - y^2 - 2y + F(x)$ —— (3)

Diff. (3) p.wrt. x -

$\partial v/\partial x = 2y + F'(x)$

$\Rightarrow 2y + 2x - 3 = 2y + F'(x)$

$\Rightarrow F'(x) = 2x - 3$

Integrating -

$F(x) = x^2 - 3x + c$

$\therefore v = 2xy - y^2 - 2y + x^2 - 3x + c$

(Ans.)

Ex-| If $u_1(x,y) = \partial u/\partial x$ and $u_2(x,y) = \partial u/\partial y$, then prove that $f'(z) = u_1(z,0) - i\,u_2(z,0)$

$$= \left[\partial u/\partial x\right]_{\substack{x=z\\y=0}} - i\left[\partial u/\partial y\right]_{\substack{x=z\\y=0}}$$

Solⁿ:-

We consider $f(z) = u+iv$ be analytic. So from Cauchy Riemann eqⁿ-

$$\partial u/\partial x = \partial v/\partial y \text{ and } \partial u/\partial y = -\partial v/\partial x \quad \text{--- (1)}$$

Since, $f(z) = u+iv$, so -

$$f'(z) = \partial u/\partial x + i\,\partial v/\partial x$$

$$= \partial u/\partial x - i\,\partial u/\partial y$$

$$\Rightarrow f'(x+iy) = \partial u/\partial x - i\,\partial u/\partial y$$

$$= u_1(x,y) - iu_2(x,y) \quad \text{--- (2)}$$

Putting $x=z$ and $y=0$ in (2) -

$$f'(z) = u_1(z,0) - iu_2(z,0)$$

$$= \left[\partial u/\partial x\right]_{\substack{x=z\\y=0}} - i\left[\partial u/\partial y\right]_{\substack{x=z\\y=0}}$$

(Proved)

Ex- If $v_1(x,y) = \partial v/\partial y$ and $v_2(x,y)$ then prove that
$f'(z) = v_1(z,0) + iv_2(z,0) = \left[\partial v/\partial y\right]_{\substack{x=z\\y=0}} + i\left[\partial v/\partial x\right]_{\substack{x=z\\y=0}}$

Solⁿ:-

We consider $f(z) = u+iv$ be analytic, so from Cauchy-Riemann eqⁿ -

$$\partial u/\partial x = \partial v/\partial y \text{ and } \partial u/\partial y = -\partial v/\partial x \quad \text{—— (1)}$$

Since, $f(z) = u+iv$, so,

$$f'(z) = \partial u/\partial x + i\,\partial v/\partial x$$
$$= \partial v/\partial y + i\,\partial v/\partial x$$
$$\Rightarrow f'(x+iy) = \partial v/\partial y + i\,\partial v/\partial x$$
$$= v_1(x,y) + iv_2(x,y) \quad \text{—— (1)}$$

Putting $y=0$ and $x=z$, we get -

$$f'(z) = v_1(z,0) + iv_2(z,0)$$
$$= \left[\partial v/\partial y\right]_{\substack{x=z\\y=0}} + i\left[\partial v/\partial x\right]_{\substack{x=z\\y=0}}$$

(Proved)

Ex-|Sed: If $Im\{f'(z)\} = 6x(2y-1)$ and $f(0) = 3-2i$, $f(1) = 6-5i$. Find $f(z)$ and $f(1+i)$

Soln:-

~~we know, $f'(z) = v_1(x,y) + v_2(x,y)$~~

we know, $f'(z) = v_1(z,0) + i v_2(z,0) = \partial v/\partial y + i\, \partial v/\partial x$ —— ①

Given, $Im\{f'(z)\} = \partial v/\partial x = 6x(2y-1)$ —— ②

Integrating ② wrt. x keeping y constant-

$$v = 3x^2(2y-1) + F(y) \quad \text{—— ③}$$

where $F(y)$ is an arbitrary imaginary f^n of y

Diff. ③ p. wrt. y we get-

$$\frac{\partial v}{\partial y} = 3x^2 \cdot 2 + F'(y) \quad \text{—— ④}$$

From ① we get using ② and ④-

$$f'(z) = 6z^2 + F'(0) + i\{6z(2\cdot0-1)\}$$

$$\Rightarrow f'(z) = 6z^2 + F'(0) - i6z$$

$$\Rightarrow f'(z) = 6z^2 + F_1 - i6z \quad \text{—— ⑤} \quad [\text{Let } F'(0) = F_1]$$

Integrating ⑤ we get-

$$f(z) = 2z^3 - i3z^2 + F_1 z + F_2 \quad \text{—— ⑥}$$

Now, $f(0) = F_2$

$\therefore F_2 = 3-2i$

Again $f(1) = \ldots$

$\Rightarrow 6-5i = 5-5i + F_1$

$\therefore F_1 = 1$

Thus, $f(t) = 2t^3 - 3it^2 + t + 3 - 2i$

(Ans.)

$\therefore f(1+i) = 2(1+i)^3 - 3i(1+i)^2 + 1+i + 3-2i$

$= 2(1+3i+3i^2+i^3) - 3i(1+2i+i^2) + 4-i$

$= 2(1+3i -3 - i) - 3i(1+2i-1) + 4-i$

$= 2(2i-2) - 3i . 2i + 4 - i$

$= 4i - 4 + 6 + 4 - i$

$\therefore f(1+i) = 6+3i$

(Ans.)

□ Singular point:-

A point at which $f(z)$ fails to be analytic is called singular point.

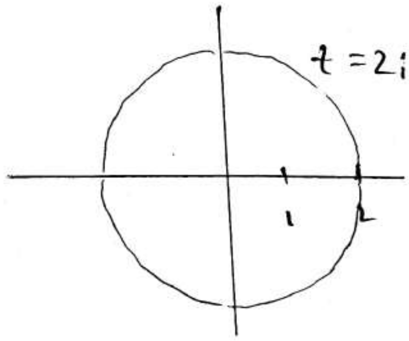
Ex- $w = f(z) = \frac{z}{z^2+4} = \frac{z}{(z+2i)(z-2i)}$

$\therefore z = \pm 2i$ is singular point.

⊛ Types of singular point:-

1. Isolated singularities:- The point $z = z_0$ is called isolated singularities if we can find $\delta > 0$ such that the circle $|z - z_0| = \delta$ enclose no singular point other than z_0.

Ex- $w = f(z) = \frac{z}{z^2+4}$ $\therefore z = \pm 2i$ is isolated singularities-



2. Poles:- If we can find a positive integer n such that $\lim_{z \to z_0} (z-z_0)^n f(z) = A \neq 0$, then $z = z_0$ is called a pole of order n. If $n=1$, z_0 is called simple poles.

Ex- $w = f(z) = \frac{z}{(z^2+4)^2}$

$= \frac{z}{(z+2i)^2(z-2i)^2}$ $\therefore z = \pm 2i$ is ploes of order 2

3. Branch point:-

Ex- $w = f(z) = \sqrt{z-3}$ Has a branch point at $z=3$

4. Removable singularity:- If $\lim_{z \to z_0} f(z)$ exists, then it is called removable singularity.

Ex- $w = f(z) = \frac{\sin z}{z}$ has a removable singularity at $z=0$ since $\lim_{z \to 0} \frac{\sin z}{z} = 1$

5. Essential singularity:- A singularity which is not isolated, poles, branchpoint or removable is called essential singularities.

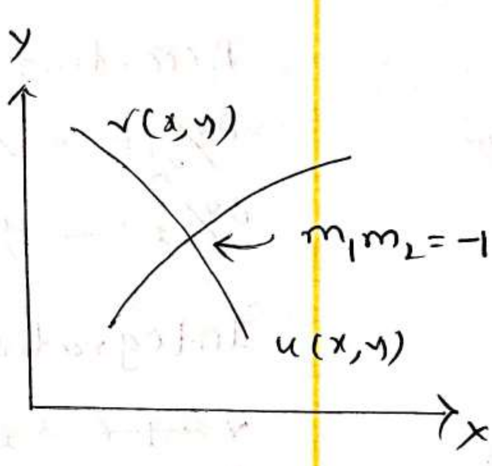
Ex- $f(z) = e^{1/(z-2)}$ has an essential singularity at $z=2$.

■ Orthogonal families:-

Let $u(x,y)=\alpha$ and $v(x,y)=\beta$, where u and v are real and imaginary parts respectively of an analytic f^n $f(z)$ and α and β are any constants, represent two families of curves.

Prove that the families are orthogonal.

Proof:-



Consider any two members of the respective families, say $u(x,y)=\alpha_1$ and $v(x,y)=\beta_1$.

where α_1 and β_1 are particular constant in fig.

Diff. $u(x,y)=\alpha_1$ w.r.t. p. x yelds-

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dx}=0$$, then the slope of $u(x,y)=\alpha_1$ is-

$$m_1 = dy/\partial x = -\frac{\partial u/\partial x}{\partial u/\partial y}.$$

Similarly the slope of $v(x,y)=\beta_1$ is-

$$m_2 = -\frac{\partial v/\partial x}{\partial v/\partial y}$$

The product of the slops -

$$m_1 \times m_2 = \frac{\partial u/\partial x}{\partial u/\partial y}\cdot\frac{\partial v/\partial x}{\partial v/\partial y}$$

$$= \frac{\partial u/\partial x}{\partial u/\partial y}\cdot\frac{-\partial u/\partial y}{\partial u/\partial x}$$

$$= -1$$

(Proved)

Ex-] Find the orthogonal trajectories of the family of the curves in the xy plane defined by

$$e^{x}(x\sin y - y\cos y) = \alpha \text{ where } \alpha \text{ is real const.}$$

Soln:-

Let, $u(x,y) = e^{-x}(x\sin y - y\cos y) - \alpha$

According to Cauchy-Riemann equations,

$$\partial v/\partial y = \partial u/\partial x = -e^{-x}x\sin y + e^{-x}y\cos y + e^{-x}\sin y \quad \text{——— (1)}$$

$$\partial v/\partial x = -\partial u/\partial y = -e^{-x}x\cos y - ye^{-x}\sin y + e^{-x}\cos y \quad \text{——— (2)}$$

Integrating (1) wrt. y keeping x const.

$\cancel{v = + e^{-x}x\cos y +}$

$$\int dv = \int(-e^{-x}x\sin y + e^{-x}y\cos y + e^{-x}\sin y)\,dy$$

$$\Rightarrow v = e^{-x}x\int -\sin y\,dy + e^{-x}\int y\cos y\,dy + e^{-x}\int \sin y\,dy$$

$$= e^{-x}\left[x\cos y + y\int\cos y\,dy - \int\left\{\frac{d}{dy}y\int\cos y\,dy\right\}dy - \cos y\right] + F(x)$$

$$= e^{-x}\left[x\cos y + y\sin y + \cos y - \cos y\right] + F(x)$$

$$\therefore v = e^{-x}x\cos y + e^{-x}y\sin y + F(x) \quad \text{——— (3)}$$

Diff. (3) p. wrt. x -

$$\partial v/\partial x = -e^{-x}x\cos y + e^{-x}\cos y - e^{-x}y\sin y + F'(x)$$

$$\Rightarrow -e^{-x}x\cos y + e^{-x}\cos y - e^{-x}y\sin y = -e^{-x}x\cos y + e^{-x}\cos y - e^{-x}y\sin y + F'(x)$$

$\Rightarrow F'(x)=0$

Integrating $F(x)=c$.

Putting values in (3) -

$v = xe^{-x}\cos y + ye^{-x}\sin y + c$

$$\text{Now, } m_1 \times m_2 = \frac{\partial u/\partial x \times \partial v/\partial x}{\partial u/\partial y \times \partial v/\partial y}$$

$$= \frac{(e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y)(-xe^{-x}\cos y - ye^{-x}\sin y + e^{-x}\cos y)}{(xe^{-x}\cos y + ye^{-x}\sin y - e^{-x}\cos y)(e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y)}$$

$= -1.$

Since the production of the slopes of the two curves is -1, hence $v = xe^{-x}\cos y + ye^{-x}\sin y + c$ is the required orthogonal projections of the given curve.

Ex] Find the orthogonal set of curves $x^2 - y^2 = a^2$.

Soln:-

Let, $u(x,y) = x^2 - y^2 - a^2$

According to Cauchy-Riemann equations-

$\partial v/\partial y = \partial u/\partial x = 2x$ ——— (1)

$\partial v/\partial x = \partial u/\partial y = 2y$ ——— (2)

Integrating (1) wrt y keeping x const.

$v = 2xy + F(x)$ ——— (3)

Diff. (3) partially wrt. x -

$\partial v/\partial x = 2y + F'(x)$

$\Rightarrow 2y = 2y + F'(x)$

$\therefore F'(x) = 0$

Integrating $F(x) = c$

Putting value in (3)

$v = 2xy + c$

How, $m_1 \times m_2 = \dfrac{\partial u/\partial x \times \partial v/\partial x}{\partial u/\partial y \times \partial v/\partial y}$

$= \dfrac{2x \times 2y}{-2y \times 2x}$

$= -1$

Since, $m_1 \times m_2 = -1$, then $v = 2xy + c$ is the required set of the orthogonal.

Ex-1 Find the orthogonal trajectories of the family of curve $x^3y - xy^3 = c$

Soln:-

$u = x^3y - xy^3 - c$

According to Cauchy-Riemann equations:-

$\partial v/\partial y = \partial u/\partial x = 3x^2y - y^3$ ——— (1)

$\partial v/\partial x = -\partial u/\partial y = -x^3 + 3y^2x = 3y^2x - x^3$ ——— (2)

Integrating (1) wrt y keeping x const.-

$v = \frac{3}{2}x^2y^2 - \frac{1}{4}y^4 + F(x)$ ——— (3)

Diff. (3) p. wrt. x -

$\partial v/\partial x = 3xy^2 + F'(x)$

$\Rightarrow -x^3 + 3xy^2 = 3xy^2 + F'(x)$

$\Rightarrow F'(x) = -x^3$

Integrating, $F(x) = -\frac{1}{4}x^4 + c$

Putting value in (3)

$v = \frac{3}{2}x^2y^2 - \frac{y^4}{4} - \frac{x^4}{4} + c$

How, $m_1 \times m_2 = \dfrac{\partial u/\partial x \times \partial v/\partial x}{\partial u/\partial y \times \partial v/\partial y}$

$= \dfrac{(3x^2y - y^3) \times (3xy^2 - x^3)}{-(3xy^2 - x^3) \times (3x^2y - y^3)}$

$= -1$

Since, $m_1 \times m_2 = -1$, Hence, $v = \frac{3}{2}x^2y^2 - \frac{1}{4}x^2 - \frac{1}{4}y^2 + c$ is the required set of the orthogonal.