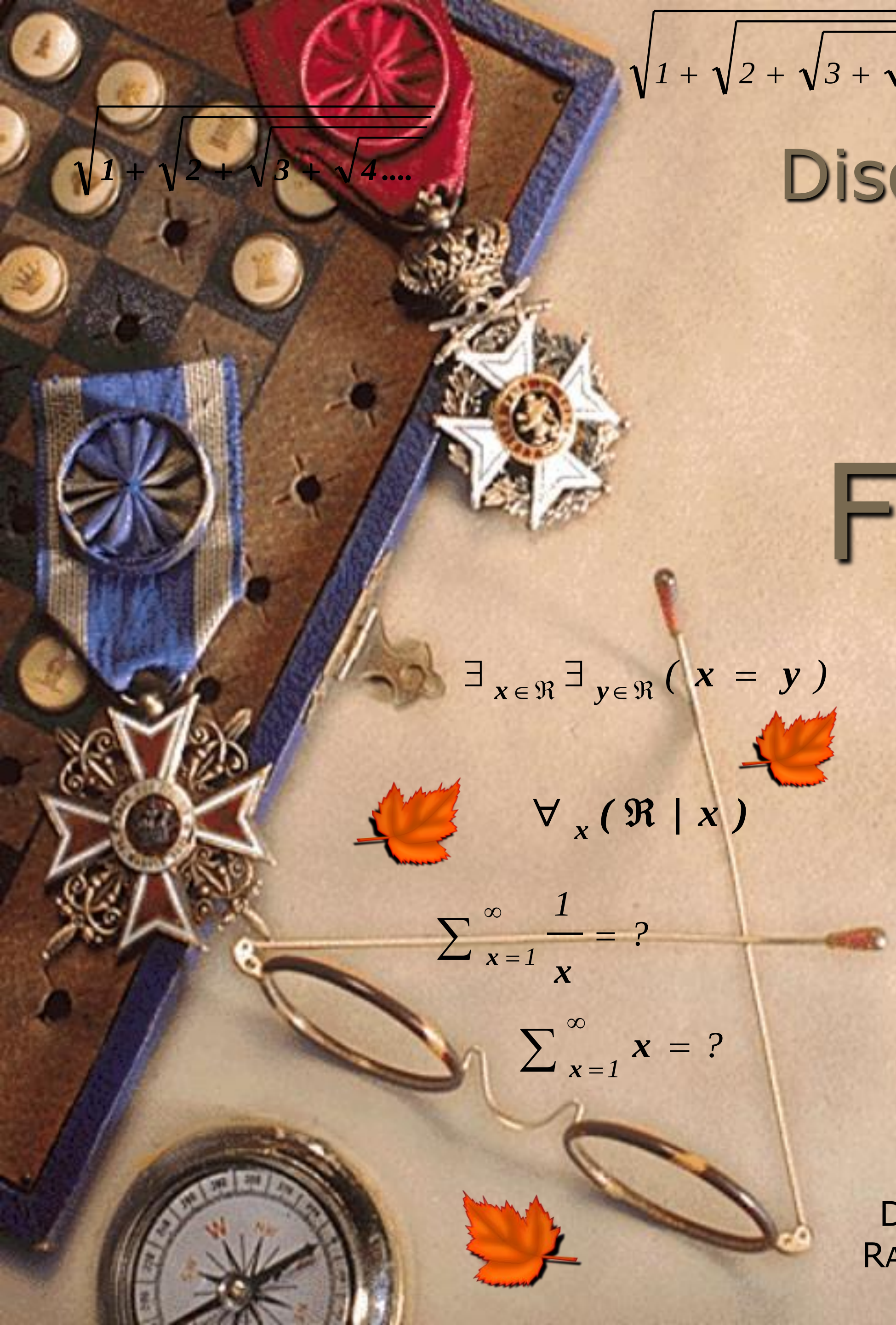

$$\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4\ldots}}}}$$

$$1-1+1-1+1\ldots\ldots = ?$$

$$\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4\ldots}}}}$$

Discrete mathematics

The Foundations: Logic and Proofs

$$\exists_{x\in\Re}\exists_{y\in\Re}(x = y)$$

$$\forall_{x}(\Re \mid x)$$

$$\sum_{x=1}^{\infty}\frac{1}{x} = ?$$

$$\sum_{x=1}^{\infty} x = ?$$

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Propositional Equivalences

Section 1.3

Section Summary

- Tautologies, Contradictions, and Contingencies.
- Logical Equivalence
  - Important Logical Equivalences
  - Showing Logical Equivalence
- Propositional Satisfiability
  - Sudoku Example

Tautologies, Contradictions, and Contingencies

- A **tautology** is a proposition which is always **true**.
  - Example: $p \vee \neg p$
- A **contradiction** is a proposition which is always **false**.
  - Example: $p \wedge \neg p$
- A **contingency** is a proposition which is neither a **tautology** nor a **contradiction**, such as p

p	¬p	p ∨¬p	p ∧¬p
T	F	T	F
F	T	T	F

Logically Equivalent

- Two **compound** propositions p and q are logically **equivalent** if **$p \leftrightarrow q$ is a tautology**.
- We write this as **$p \Leftrightarrow q$ or as $p \equiv q$** where *p* and *q* are compound propositions.
- This truth table shows that $\neg a \vee b$ is equivalent to $a \rightarrow b$.
    - p: $\neg a \vee b$
    - q: $a \rightarrow b$

a	b	$\neg a$	$\neg a \vee b$	$a \rightarrow b$	$p \leftrightarrow q$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

De Morgan's Laws

TABLE 2 **De Morgan's Laws.**

$\neg(p \wedge q) \equiv \neg p \vee \neg q$
$\neg(p \vee q) \equiv \neg p \wedge \neg q$

Augustus De Morgan
1806-1871

This truth table shows that De Morgan's Second Law holds.

p	q	¬p	¬q	(p∨q)	¬(p∨q)	¬p∧¬q
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Key Logical Equivalences

TABLE 6 **Logical Equivalences.**

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws

Key Logical Equivalences (*cont*)

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

More Logical Equivalences

TABLE 7 **Logical Equivalences Involving Conditional Statements.**

$p \rightarrow q \equiv \neg p \vee q$
$p \rightarrow q \equiv \neg q \rightarrow \neg p$
$p \vee q \equiv \neg p \rightarrow q$
$p \wedge q \equiv \neg(p \rightarrow \neg q)$
$\neg(p \rightarrow q) \equiv p \wedge \neg q$
$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$
$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$
$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$
$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

TABLE 8 **Logical Equivalences Involving Biconditional Statements.**

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Equivalence Proofs

Example: Show that $\neg(p \lor (\neg p \land q))$ is logically equivalent to $\neg p \land \neg q$

Solution:

$$\begin{array}{lcll}
\neg(p \lor (\neg p \land q)) & \equiv & \neg p \land \neg(\neg p \land q) & \text{by the second De Morgan law} \\
 & \equiv & \neg p \land [\neg(\neg p) \lor \neg q] & \text{by the first De Morgan law} \\
 & \equiv & \neg p \land (p \lor \neg q) & \text{by the double negation law} \\
 & \equiv & (\neg p \land p) \lor (\neg p \land \neg q) & \text{by the second distributive law} \\
 & \equiv & F \lor (\neg p \land \neg q) & \text{because } \neg p \land p \equiv F \\
 & \equiv & (\neg p \land \neg q) \lor F & \text{by the commutative law} \\
 & & & \text{for disjunction} \\
 & \equiv & (\neg p \land \neg q) & \text{by the identity law for } \mathbf{F}
\end{array}$$

Equivalence Proofs

Example: Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Solution:

$$
\begin{aligned}
(p \wedge q) \rightarrow (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) && \text{by truth table for } \rightarrow \\
&\equiv (\neg p \vee \neg q) \vee (p \vee q) && \text{by the first De Morgan law} \\
&\equiv (\neg p \vee p) \vee (\neg q \vee q) && \text{by associative and} \\
& && \text{commutative laws} \\
& && \text{laws for disjunction} \\
&\equiv T \vee T && \text{by truth tables} \\
&\equiv T && \text{by the domination law}
\end{aligned}
$$

Propositional Satisfiability

- A compound proposition is *satisfiable* if there is an assignment of truth values to its variables that make it true. When no such assignments exist, the compound proposition is *unsatisfiable*.
- A compound proposition is unsatisfiable if and only if its negation is a tautology.

Questions on Propositional Satisfiability

Example: Determine the satisfiability of the following compound propositions:

$$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$$
$$(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$
$$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$

Solution: Satisfiable. Assign **T** to *p, q,* and *r*.

Solution: Satisfiable. Assign **T** to *p* and ***F*** to *q*.

Solution: Not satisfiable. Check each possible assignment of truth values to the propositional variables and none will make the proposition true.

Notation

$$\bigvee_{j=1}^{n} p_j \text{ is used for } p_1 \vee p_2 \vee \ldots \vee p_n$$

Needed for the next example.

Sudoku

- A **Sudoku puzzle** is represented by a 9×9 grid made up of nine 3×3 subgrids, known as **blocks**. Some of the 81 cells of the puzzle are assigned one of the numbers 1,2, …, 9.
- The puzzle is solved by assigning numbers to each blank cell so that every row, column and block contains each of the nine possible numbers.
- Example

	2	9				4		
			5			1		
	4							
				4	2			
6							7	
5								
7			3					5
	1			9				
							6	

Encoding as a Satisfiability Problem

- Let $p(i,j,n)$ denote the proposition that is true when the number n is in the cell in the ith row and the jth column.
- There are $9{\times}9 \times 9 =$ **729** such propositions.
- In the sample puzzle $p(5,1,6)$ is true, but $p(5,j,6)$ is false for $j = 2,3,\ldots 9$

	2	9				4		
			5			1		
	4							
				4	2			
6							7	
5								
7			3					5
	1			9				
							6	

Encoding (cont)

- For each cell with a given value, assert $p(i,j,n)$, when the cell in row i and column j has the given value.
- Assert that every row contains every number.

$$\bigwedge_{i=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{j=1}^{9} p(i,j,n)$$

- Assert that every column contains every number.

$$\bigwedge_{j=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{i=1}^{9} p(i,j,n)$$

	2	9				4		
			5			1		
	4							
				4	2			
6							7	
5								
7			3					5
	1			9				
							6	

Query???

$$\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4\ldots}}}}$$

$$\exists_{x\in\Re}\exists_{y\in\Re}(x=y)=?$$

$$\sum_{x=1}^{\infty} x = ?$$

$$\sum_{x=1}^{\infty}\frac{1}{x}=?$$

$$\forall_{x}(\Re \mid x)=?$$

$$\exists_{x\in\Re}\exists_{y\in\Re}(x=y)=?$$

$$\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4\ldots}}}}=?$$

$$1-1+1-1+1\ldots\ldots\ldots=?$$

$$\sum_{x=1}^{\infty}\frac{1}{x}=?$$