1E

1-E

Vector Analysis

→ M.R. Spiegel

Vector:

Scalar

Unit vector

Scalar product: The dot or scalar ~~p~~ products of two vectors $\vec{A}$ and $\vec{B}$ denoted by $\vec{A}\cdot\vec{B}$ is defined as $\vec{A}\cdot\vec{B} = AB\cos\theta$ where θ is the angle between $\vec{A}$ & $\vec{B}$ and $0 \le \theta \le \pi$.

If $\vec{A}\cdot\vec{B} = 0$ then $\vec{A}$ and $\vec{B}$ are ~~pe~~ perpendicular.

$\Rightarrow x^2+y^2+z^2=1$ --- (I)

Since $\vec{A} \perp \vec{C}$

So, $\vec{A}\cdot\vec{C} = 0$

$\Rightarrow 2x+3y-z=0$ --- (II)

Again Since $\vec{B} \perp \vec{C}$

So, $\vec{B}\cdot\vec{C}=0$

$3x-2y+z=0$ ---- (III)

Solving (I), (II) & (III)

$$\frac{x}{1} = \frac{y}{-5} = \frac{z}{-13}$$

$\therefore y=-5x \quad z=-13x$

From (I) $x^2 = \frac{1}{195}$

$x = \pm\frac{1}{\sqrt{195}}$

$y = \pm\frac{-5}{\sqrt{195}}$

$z = \pm\frac{-13}{\sqrt{195}}$

Hence the required unit

vector $= \vec{c} = \pm \dfrac{(\hat{i} - 5\hat{j} - 13\hat{k})}{\sqrt{195}}$ Ans

2E

2-E

Show that $\hat{i}+\hat{j}+\hat{k}$, $\hat{i}-\hat{k}$ and $\hat{i}-2\hat{j}+\hat{k}$ are mutually orthogonal (লম্ব)

Find x, y and z if $\hat{i}+\hat{j}+2\hat{k}$, $-\hat{i}+z\hat{k}$ and $2\hat{i}+x\hat{j}+y\hat{k}$ are mutually orthogonal

Sol^n:

$$\vec{a} = \hat{i}+\hat{j}+\hat{k}$$

$$\vec{b} = \hat{i}-\hat{k}$$

$$\vec{c} = \hat{i}-2\hat{j}+\hat{k}$$

$$\vec{a}\cdot\vec{b} = (\hat{i}+\hat{j}+\hat{k})\cdot(\hat{i}-\hat{k}) = 1-1=0$$

$$\vec{b}\cdot\vec{c} = (\hat{i}-\hat{k})(\hat{i}-2\hat{j}+\hat{k}) = 1-1=0$$

$$\vec{c}\cdot\vec{a} = (\hat{i}-2\hat{j}+\hat{k})(\hat{i}+\hat{j}+\hat{k}) = 1-2+1=0$$

Since $\bar{a}\cdot\bar{b} = \bar{b}\cdot\bar{c} = \bar{c}\cdot\bar{a} = 0$

So the three vectors are mutually orthogonal.

Again let,

$$\vec{A} = \hat{i} + \hat{j} + 2\hat{k}$$
$$\vec{B} = -\hat{i} + z\hat{k}$$
$$\vec{C} = 2\hat{i} + x\hat{j} + y\hat{k}$$

Since $\vec{A}$, $\vec{B}$ & $\vec{C}$ are mutually orthogonal

So $\vec{A} \cdot \vec{B} = 0$

$\Rightarrow (\hat{i} + \hat{j} + 2\hat{k})(-\hat{i} + z\hat{k}) = 0$

$\Rightarrow -1 + 2z = 0$

$\Rightarrow z = 1/2$

$\vec{B} \cdot \vec{C} = 0$

$\Rightarrow (-\hat{i} + z\hat{k})(2\hat{i} + x\hat{j} + y\hat{k}) = 0$

$\Rightarrow -2 + yz = 0$ | $\Rightarrow yz = 2$

$\therefore \cancel{y = 4}$ | $\Rightarrow y = 2/z = 2 \times 2 = 4$

$\therefore \vec{C} \cdot \vec{A} = 0$

$\cancel{x = -10}$

$\Rightarrow (2\hat{i} + x\hat{j} + y\hat{k}) \cdot (\hat{i} + \hat{j} + 2\hat{k}) = 0$

$\Rightarrow 2 + x + 2y = 0$

$\Rightarrow 2 + 8 = -x \therefore x = -10$

4 Find the vector perpendicular to $\hat{i}+2\hat{k}$ and $\hat{i}+\hat{j}-\hat{k}$ and the area of the triangle with two vectors as adjacent sides.

Soln:

Let $\vec{A} = \hat{i}+2\hat{k}$

$\vec{B} = \hat{i}+\hat{j}-\hat{k}$

Now any vector perpendicular to $\vec{A}$ & $\vec{B}$ is $\vec{A}\times\vec{B}$

$$A\times B = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2 \\ 1 & 1 & -1 \end{vmatrix} = -2\hat{i}+3\hat{j}+\hat{k}$$ Ans

$$\text{required area} = \frac{1}{2}\times(|\vec{A}\times\vec{B}|)$$

$$= \frac{1}{2}\sqrt{4+9+1}$$

$$= \frac{1}{2}\sqrt{14}$$ Ans

* If 3 vectors $(\vec{a}, \vec{b}, \vec{c})$ are coplaner then $\vec{a}\cdot(\vec{b}\times\vec{c})$

If three vectors $\vec{a} = \hat{i} - \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{c} = \lambda\hat{i} - \hat{j} + 2\hat{k}$ are $\boxed{\text{coplanar}}$. Find the value of λ.

Soln

Given,

$$\vec{a} = \hat{i} - \hat{j} + \hat{k}$$
$$\vec{b} = 2\hat{i} + \hat{j} - \hat{k}$$
$$\vec{c} = \lambda\hat{i} - \hat{j} + 2\hat{k}$$

If $\vec{a}, \vec{b}, \vec{c}$ are coplanar then $\frac{1}{6}[a\ b\ c] = 0$

$$\Rightarrow \vec{a}\cdot(\vec{b}\times\vec{c}) = 0$$

$$\Rightarrow (\hat{i} - \hat{j} + \hat{k}) \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ \lambda & -1 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow (\hat{i} - \hat{j} + \hat{k}) \cdot ((\lambda-1)\hat{i} - (2\lambda+\lambda)\hat{j} + (-2-\lambda)\hat{k}) = 0$$

$$\Rightarrow \lambda - 1 + 2\lambda + \lambda - 2 - \lambda = 0$$

$$\Rightarrow 3\lambda - 3 = 0$$

$$= \lambda = 1 \text{ Ans}$$

* Volume of a tetrahedron $= \frac{1}{6}\vec{a}\cdot(\vec{b}\times\vec{c})$

Find the ~~valve~~ volume of a tetrahedron whose one vertex is at origin and other three vectors are $(3,2,1)$ $(2,3,-1)$ & $(-1,2,3)$

Soln:

Let,

$$\vec{a} = 3\hat{i}+2\hat{j}+\hat{k}$$

$$\vec{b} = 2\hat{i}+3\hat{j}-\hat{k}$$

$$\vec{c} = -\hat{i}+2\hat{j}+3\hat{k}$$

We know, volume of a tetrahedron is $V = \frac{1}{6}[a\,b\,c]$

$$= \frac{1}{6}\ \vec{a}\cdot(\vec{b}\times\vec{c})$$

$$= \frac{1}{6}(3\hat{i}+2\hat{j}+\hat{k})\cdot\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & -1\\ -1 & 2 & 3\end{vmatrix}$$

$$= \frac{1}{6}(3\hat{i}+2\hat{j}+\hat{k})(11\hat{i}-5\hat{j}+7\hat{k})$$

$$= \frac{1}{6}(33-10+7) = \frac{1}{6}30 = 5$$

4. Find the volume of a tetrahedron whose ~~vee~~ verties are $P(3,4,5)$, $A(2,1,1)$, $B(2,1,5)$ and $C(1,4,2)$

Let,

$$\vec{P} = 3\hat{i} + 4\hat{j} + 5\hat{k}$$

$$\vec{A} = 2\hat{i} + \hat{j} + \hat{k}$$

$$\vec{B} = 2\hat{i} + \hat{j} + 5\hat{k}$$

$$\vec{C} = \hat{i} + 4\hat{j} + 2\hat{k}$$

Now, $\vec{PA} = \vec{A} - \vec{P} = -\hat{i} - 3\hat{j} - 4\hat{k}$

$$\vec{PB} = \vec{B} - \vec{P} = -\hat{i} - 3\hat{j}$$

$$\vec{PC} = \vec{C} - \vec{P} = -2\hat{j} - 3\hat{k}$$

We know the volume of a tetrahedron is $V = \frac{1}{6} \vec{PA} \cdot (\vec{PB} \times \vec{PC})$

$$= \frac{1}{6}(-\hat{i}-3\hat{j}-4\hat{k}) \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -3 & 0 \\ -2 & 0 & -3 \end{vmatrix}$$

$$= \frac{1}{6}(-\hat{i}-3\hat{j}-4\hat{k}) \cdot (9\hat{i}-3\hat{j}-6\hat{k})$$

$$= \frac{1}{6}(-9+9+24)$$

$$= \frac{1}{6} 24$$

$$= 4$$

3E

3-E

Show that the vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ form the sides of a right angle triangle.

Soln :

Let, $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$

$\vec{b} = \hat{i} - 3\hat{j} - 5\hat{k}$

$\vec{c} = 3\hat{i} - 4\hat{j} - 4\hat{k}$

$|\vec{a}| = \sqrt{4+1+1} = \sqrt{6}$

$|\vec{b}| = \sqrt{1+9+25} = \sqrt{35}$

$|\vec{c}| = \sqrt{9+16+16} = \sqrt{41}$

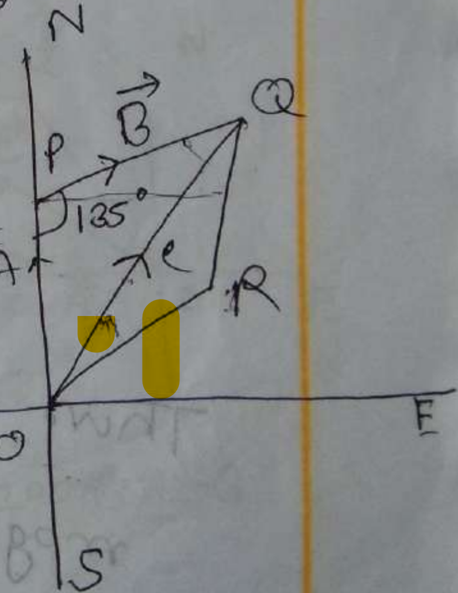
$\therefore |\vec{a}|^2 + |\vec{b}|^2 = 6 + 35 = 41 = |\vec{c}|^2$

Hence vectors form a right angled triangle.

An automobile travels 3km due north, then 5km northeast. Represent these displacement graphically and determine the resultant displacement analytically.

Soln :

Let $\overline{OP}$ or $\vec{A}$ represents displacement of 3 km due north ; vector $\overrightarrow{PQ}$ or $\vec{B}$ represent displacement 5 km northeast vector $\overrightarrow{OQ}$ or ω $\vec{C}$ represents the resultant displacement.



From the Δ OPQ, by law of cosine

$$C^2 = A^2 + B^2 - 2AB\cos\angle OPQ$$

$= 3^2 + 5^2 - 2 \times 3 \times 5 \cos 135$

$= 55.21$

$C = 7.43$

Also by the law of Sine $\frac{A}{\sin \angle OQP} = \frac{C}{\sin \angle OPQ}$

$\Rightarrow \frac{A \sin \angle OPQ}{C} = \sin \angle OQP$

$\Rightarrow \frac{3 \sin 135}{7.43} = \sin \angle OQR$

$= 0.2855$

$\therefore \angle OQR = 16^\circ 35'$

Thus the vector $\vec{C}$ has magnitude 7.43 km and direction $(45^\circ + 16^\circ 35') = 61^\circ 35'$ north of east.

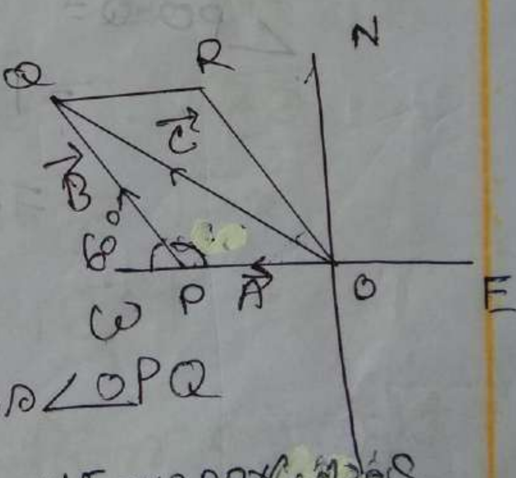
** H.W # An Aeroplane travels 200 km due west and then 150 km 60° north of west. Represent these displacements graphically and determine the resultant displacement ~~analogies~~ analytically.

magnitude 304.1 km

direction 25.17° north of west

Solⁿ:

From figure we have the laws of Cosine



$c^2 = A^2 + B^2 - 2AB\cos\angle OPQ$

$= (200)^2 + (150)^2 - 2\times150\times200\times\cos120°$

$c = 304.138$ km

Ans

Also from the sine :

$$\frac{A}{\sin\angle OQP} = \frac{C}{\sin\angle OPQ}$$

$$\sin\angle OQP = \frac{A \sin\angle OPQ}{C}$$

$$= \frac{200 \times \sin 120^\circ}{304.138}$$

$$\angle OQP = 34.715$$

$$\angle POQ = 180 - (\angle OPQ + \angle OQP)$$

$$= 180 - (120 + 34.715)$$

$$= 25.285$$

Ans

Show that vectors $\vec{a} = 3\hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = \hat{i} - 3\hat{j} + 5\hat{k}$, $\vec{c} = 2\hat{i} + \hat{j} - 4\hat{k}$ form a right angled triangle.

Soln:

Let

$\vec{AB} = \vec{a} = 3\hat{i} - 2\hat{j} + \hat{k}$

$\vec{BC} = \vec{b} = \hat{i} - 3\hat{i} + 5\hat{k}$

$\vec{AC} = \vec{c} = 2\hat{i} + \hat{j} - 4\hat{k}$

$|\vec{AB}|^2 = \sqrt{9+4+1} = \sqrt{14}$

$|\vec{BC}|^2 = \sqrt{1+9+25} = \sqrt{35}$

$|\vec{AC}|^2 = \sqrt{4+1+16} = \sqrt{21}$

$|\vec{AC}|^2 + |\vec{AB}|^2 = |\vec{BC}|^2$

Hence vectors form a right angle triangle.

Vector product: Cross or vector product of two vectors $\vec{A}$ and $\vec{B}$ denoted by $\vec{A} \times \vec{B}$ is defined as $\vec{A} \times \vec{B} = AB\sin\theta\,\hat{n}$

where θ is the angle between $\vec{A}$ & $\vec{B}$ and $\boxed{0 \leq \theta \leq \pi}$, $\hat{n}$ is an unit vector.

If $\vec{A} \times \vec{B} = 0$ then $\vec{A}$ and $\vec{B}$ are parallel.

Using dot product find the unit vector perpendicular to both $\vec{A} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{B} = 3\hat{i} - 2\hat{j} + \hat{k}$

Soln:

Given, $\vec{A} = 2\hat{i} + 3\hat{j} - \hat{k}$

$\vec{B} = 3\hat{i} - 2\hat{j} + \hat{k}$

Let, $\vec{c} = x\hat{i} + y\hat{j} + z\hat{k}$ be the required vector

Since $\vec{c}$ is a unit vector

$\vec{c} \cdot \vec{c} = 1$

$\Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (x\hat{i} + y\hat{j} + z\hat{k}) = 1$

★★ # Prove that the vectors $\vec{A}=3\hat{i}+\hat{j}-2\hat{k}$, $\vec{B}=-\hat{i}+3\hat{j}+4\hat{k}$, $\vec{C}=4\hat{i}-2\hat{j}-6\hat{k}$ can form the sides of a triangle. Also find the length of the medians of the triangle.

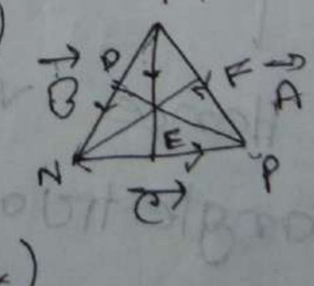
Sol^n: The vector can form a triangle if the sum of the two vectors equal to other vector.

Here $\vec{B}+\vec{C}=(-\hat{i}+3\hat{j}+4\hat{k})+(4\hat{i}-2\hat{j}-6\hat{k})$

$$=3\hat{i}+\hat{j}-2\hat{k}$$

$$=\vec{A}$$

Hence the vectors can form the sides of a triangle.



$$\vec{ME}=\frac{1}{2}(\vec{MN}+\vec{MP})$$

$$=\frac{1}{2}(\vec{B}+\vec{A})$$

$$=\frac{1}{2}(2\hat{i}+4\hat{j}+2\hat{k})$$

$$=\hat{i}+2\hat{j}+\hat{k}$$

$|\vec{ME}|=\sqrt{6}$ Ans

$$\overrightarrow{NF} = \frac{1}{2}(-\vec{B} + \vec{E})$$

$$= \frac{1}{2}(\hat{i} - 3\hat{j} - 4\hat{k} + 4\hat{i} - 2\hat{j} - 6\hat{k})$$

$$= 5/2\,\hat{i} - 5/2\,\hat{j} - 5\hat{k}$$

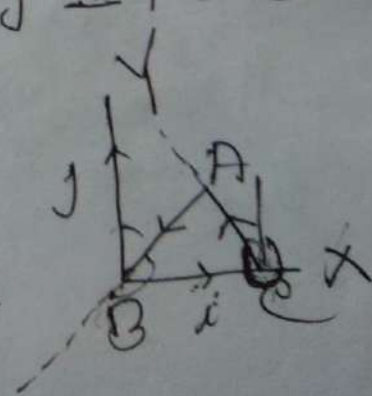
Ans

4E

4-E

A particle is subjected to forces 3 kg-wt, 4 kg-wt, 5 kg-wt respectively acting in the directions parallel to the edges $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$ of an equilateral ΔABC. Find the ([illegible]) resultant force acting on the particle.

$\Rightarrow$ Let, $\overrightarrow{BC}$ direction be parallel to $\hat{i}$ and $\hat{j} \perp$ to $\overrightarrow{BC}$ at B.



3 kg-wt force along $\overrightarrow{AB} = 3\left(\frac{-\hat{i}}{2} - \right.$

$\overrightarrow{AB} = 3(\hat{i}\cos 240^\circ + \hat{j}\cos 210^\circ)$

$= 3(-\hat{i}/2 - \sqrt{3}/2\,\hat{j}) = \frac{-3}{2}\hat{i} - \frac{3\sqrt{3}}{2}\hat{j}$

4 kg-wt force along $\overrightarrow{BC} = 4\hat{i}$

5 kg-wt " " $\overrightarrow{CA} = 5(\hat{i}\cos 240^\circ + \hat{j}\cos 330^\circ)$

$= -5/2\,\hat{i} + \frac{5\sqrt{3}}{2}\hat{j}$

Hence the total force $= \frac{-3\hat{i}}{2} - \frac{3\sqrt{3}}{2}\hat{j}$

$+ 4\hat{i} - \frac{5}{2}\hat{i} + \frac{5\sqrt{3}}{2}\hat{j}$

$= \sqrt{3}\,\hat{j}$

$= \sqrt{3}$ kg-wt $\perp$ to $\overrightarrow{BC}$

Linearly dependent and Independent vectors:

The vectors $\vec{A}, \vec{B}, \vec{C}, \ldots$ are called linearly dependent if we can find a set of scalars $a, b, c \ldots$ not all zero such that

infinite number vectors & Scalars (Scalars not zero case)

$$a\vec{A} + b\vec{B} + c\vec{C} + \cdots = 0$$

otherwise they are called linearly independent.

If $\hat{a}, \hat{b}, \hat{c}$ are ~~non~~ non-coplanar vectors determine whither the vectors

$$\vec{r_1} = 2\hat{a} - 3\hat{b} + \hat{c},$$

$$\vec{r_2} = 3\hat{a} - 5\hat{b} + 2\hat{c}$$

$$\vec{r_3} = 4\hat{a} - 5\hat{b} + \hat{c} \text{ are}$$

Linearly independent or dependent.

Soln:

Given,

$$\vec{r_1} = 2\hat{a} - 3\hat{b} + \hat{c}$$
$$\vec{r_2} = 3\hat{a} - 5\hat{b} + 2\hat{c}$$
$$\vec{r_3} = 4\hat{a} - 5\hat{b} + \hat{c}$$

Let, $x\vec{r_1} + y\vec{r_2} + z\vec{r_3} = 0$ where x, y, z are scalar

$$\Rightarrow x(2\hat{a} - 3\hat{b} + \hat{c}) + y(3\hat{a} - 5\hat{b} + 2\hat{c}) + z(4\hat{a} - 5\hat{b} + \hat{c}) = 0$$

$$\Rightarrow (2x + 3y + 4z)\hat{a} - (3x + 5y + 5z)\hat{b} + (x + 2y + z)\hat{c} = 0$$

non-coplaner হলে coefficient "0" হবে

Since $\hat{a}, \hat{b}, \hat{c}$ are non-coplaner

So, $2x + 3y + 4z = 0$ --- (i)

$3x + 5y + 5z = 0$ --- (ii)

$x + 2y + z = 0$ --- (iii)

By solving (ii) & (iii)

$$x = -5z$$
$$y = 2z$$

putting the values of x, y in (i) $-10z + 6z + 4z = 0$

$$\Rightarrow 0 = 0$$

Hence $\vec{r_1}, \vec{r_2}, \vec{r_3}$ are linearly dependent.

5E

20-03-17
5-DE

H.W

Determine whither the vectors are independent or dependent

(i) $\vec{A} = 2\hat{i} + \hat{j} - 3\hat{k}$ dependent
$\vec{B} = \hat{i} - 4\hat{k}$
$\vec{C} = 4\hat{i} + 3\hat{j} - \hat{k}$

(ii) $\vec{A} = \hat{i} - 3\hat{j} + 2\hat{k}$ Independent
$\vec{B} = 2\hat{i} - 4\hat{j} - \hat{k}$
$\vec{C} = 3\hat{i} + 2\hat{j} - \hat{k}$

i) Soln: Let Given,
$\vec{A} = 2\hat{i} + \hat{j} - 3\hat{k}$
$\vec{B} = \hat{i} - 4\hat{k}$
$C = 4\hat{i} + 3\hat{j} - \hat{k}$

Let, $\cancel{x\vec{A} + y\vec{z}}$ $x\vec{A} + y\vec{B} + z\vec{C} = 0$ where x, y, z are scalars.

$\therefore (2\hat{i} + \hat{j} - 3\hat{k})x + (\hat{i} - 4\hat{k})y + (4\hat{i} + 3\hat{j} - \hat{k})z = 0$

$\Rightarrow (2x + y + 4z)\hat{i} + (x + 3z)\hat{j} + (-3x - 4y - z)\hat{k} = 0$

As $\hat{i}, \hat{j}, \hat{k}$ are non-coplanar,

$2x + y + 4z = 0$ --- (i)

$x + 3z = 0$ --- (ii)

$-3x - 4y - z = 0$ --- (iii)

By solving (ii) & (iii)

$$\frac{x}{-1-(16)} = \frac{y}{-12-(-2)} = \frac{z}{-8-(-3)}$$

$$\Rightarrow \frac{x}{15} = \frac{y}{-10} = \frac{z}{-5}$$

$$\Rightarrow \frac{x}{-3} = \frac{y}{2} = \frac{z}{1}$$

$\therefore x = -3z$

$y = 2z$

Putting the value in eqn (ii) we get,

$-3z + 3z = 0$

$\Rightarrow 0 = 0$

Hence $\vec{A}, \vec{B}, \vec{C}$ are linearly dependent.

ii) Soln:

Given,

$$\vec{A} = \hat{i} - 3\hat{j} + 2\hat{k}$$
$$\vec{B} = 2\hat{i} - 4\hat{j} - \hat{k}$$
$$\vec{C} = 3\hat{i} + 2\hat{j} - \hat{k}$$

Let, $x\vec{A} + y\vec{B} + z\vec{C} = 0$, where x, y, z are scalars.

$\therefore x(\hat{i} - 3\hat{j} + 2\hat{k}) + y(2\hat{i} - 4\hat{j} - \hat{k}) + z(3\hat{i} + 2\hat{j} - \hat{k}) = 0$

$\Rightarrow (x + 2y + 3z)\hat{i} + (-3x - 4y + 2z)\hat{j} + (2x - y - z)\hat{k} = 0$

Since $\hat{i}, \hat{j}, \hat{k}$ are non coplaner,

$x + 2y + 3z = 0$ --- (i)

$-3x - 4y + 2z = 0$ --- (ii)

$2x - y - z = 0$ --- (iii)

By solving (ii) & (iii) we get,

$$\frac{x}{4-(-2)} = \frac{y}{4-3} = \frac{z}{3-(-8)}$$

$$\Rightarrow \frac{x}{6} = \frac{y}{1} = \frac{z}{11}$$

$\therefore x = 6y$, $z = 11y$

Putting the value of x, y in eqn (i) we get,

$$6y + 2y + 33y = 0$$

$$\Rightarrow 41y \neq 0$$

Hence the vectors are linearly independent.

A particle is acted on by constant force $3\hat{i}+2\hat{j}+5\hat{k}$ and $2\hat{i}+\hat{j}-3\hat{k}$ and is displaced from a point whose position vector is $2\hat{i}-\hat{j}-3\hat{k}$ to a point whose position vector $4\hat{i}-3\hat{j}+7\hat{k}$. Calculate the work done.

Soln:

Total force $\vec{F} = (3\hat{i}+2\hat{j}+5\hat{k}) + (2\hat{i}+\hat{j}-3\hat{k})$

$= 5\hat{i}+3\hat{j}+2\hat{k}$

Displacement $D = (4\hat{i}-3\hat{j}+7\hat{k})-(2\hat{i}-\hat{j}-3\hat{k})$

$= 2\hat{i}-2\hat{j}+10\hat{k}$

Work done $= F.D = (5\hat{i}+3\hat{j}+2\hat{k})(2\hat{i}-2\hat{j}+10\hat{k})$

$= 10-6+20$

$= 24$ Ans

Find a vector of magnitude 5 parallel to yz plane and perpendicular to $2\hat{i}+3\hat{j}+\hat{k}$.

Soln, Let, $\vec{A} = b\hat{j}+c\hat{k}$ any vector

parallel to yz plane

and $\vec{B} = 2\hat{i}+3\hat{j}+\hat{k}$

Since $\vec{A}$ is perpendicular to $\vec{B}$

$$\vec{A}\cdot\vec{B} = 0$$

$\Rightarrow (b\hat{j}+c\hat{k})(2\hat{i}+3\hat{j}+\hat{k}) = 0$

$\Rightarrow \cancel{2b}\ 3b+c=0$

$c = -3b$

$\therefore \vec{A} = b\hat{j} - 3b\hat{k}$

also $|\vec{A}| = \sqrt{b^2 + 9b^2}$

$= b\sqrt{10}$

According to question, $b\sqrt{10} = 5$

$b = \sqrt{5/2}$

$\therefore c = -3\sqrt{5/2}$

So the required vector

$\vec{A} = \sqrt{5/2}\,(\hat{j} - 3\hat{k})$

$\cancel{\frac{3\sqrt{10}}{\sqrt{10}}}$ 2

$\boxed{\frac{5}{\sqrt{10}} = \frac{\cancel{5}\,\sqrt{5}}{\sqrt{5}\times\sqrt{2}}}$

7D

7-D

H.W

Find a vector of magnitude 9 which is perpendicular to both vectors $4\hat{i}-\hat{j}+3\hat{k}$ and $-2\hat{i}+\hat{j}-2\hat{k}$

Sol^n: Let $\vec{A} = 4\hat{i}-\hat{j}+3\hat{k}$

$\vec{B} = -2\hat{i}+\hat{j}-2\hat{k}$

A vector perpendicular to both A and B is $\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 3 \\ -2 & 1 & -2 \end{vmatrix}$

$= \hat{i}(2-3) - \hat{j}(-8+6) + \hat{k}(4$

$= -\hat{i}+2\hat{j}+2\hat{k}$

As magnitude of $\vec{A}$ is 9

$\therefore m(-\hat{i}+2\hat{j}+2\hat{k}) = 9$

$\Rightarrow m\sqrt{4+4+1} = 9$

$\Rightarrow m\sqrt{9} = 9$

$\Rightarrow 3m = 9$

$m = 3$

$\therefore A = 3(-\hat{i}+2\hat{j}+2\hat{k})$

H.W

Find a unit vector parallel to xy plane and perpendicular to $4\hat{i}-3\hat{j}+\hat{k}$

(Ans: $\pm\frac{1}{5}(3\hat{i}+4\hat{j})$)

Soln : Let $\vec{A}=a\hat{i}+b\hat{j}$ be any vector parallel to xy plane.

and $\vec{B}=4\hat{i}-3\hat{j}+\hat{k}$

As $\vec{A}\perp\vec{B}$

$\therefore \vec{A}\cdot\vec{B}=0$

$\Rightarrow (a\hat{i}+b\hat{j})(4\hat{i}-3\hat{k}+\hat{k})=0$

$\Rightarrow 4a-3b=0$

$\therefore a=3/4\,b$

$\vec{A}=3/4b\,\hat{i}+b\hat{j}$

As $\vec{A}$ is unit vector,

$\sqrt{(3/4b)^2+b^2}=1$

$\Rightarrow \frac{9}{16}b^2+b^2=1$

$\Rightarrow \frac{9b^2+16b^2}{16}=1$

$\Rightarrow 25b^2=16$

$\Rightarrow b^2=16/25$

$\therefore b=\pm 4/5$

$\therefore a=\frac{3}{4}\times\frac{4}{5}$

$=\pm 3/5$

$$\therefore \vec{A} = \pm \frac{3}{5}\hat{i} \pm \frac{4}{5}\hat{j}$$

$$= \pm \frac{1}{5}(3\hat{i} \pm 4\hat{j})$$

Ans

H.W

If $\vec{A} = 4\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{B} = -2\hat{i} + \hat{j} - 2\hat{k}$, find a unit vector ∴ Ans ~~$\pm\frac{1}{5}(3\hat{i}+4\hat{j})$~~ ~~parallel~~ perpendicular to both $\vec{A}$ & $\vec{B}$

Hint → Any vector perpendicular to both A & B is $\vec{A} \times \vec{B}$ Ans: $\frac{1}{3}(-\hat{i}+2\hat{j}+2\hat{k})$

Solⁿ: Given,

$$\vec{A} = 4\hat{i} - \hat{j} + 3\hat{k}$$

$$\vec{B} = -2\hat{i} + \hat{j} - 2\hat{k}$$

$$\therefore \vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 3 \\ -2 & 1 & -2 \end{vmatrix}$$

$$= \hat{i}(2-3) - \hat{j}(-8+6) + \hat{k}(4-2)$$

$$= -\hat{i} + 2\hat{j} + 2\hat{k}$$

∴ unit vector $\hat{n} = \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|}$

$$= \frac{1}{\sqrt{4+4+1}}(-\hat{i}+2\hat{j}+2\hat{k})$$

$$= \frac{1}{3}(-\hat{i}+2\hat{j}+2\hat{k})$$

Ans

Find the projection of the vector $4\hat{i}-3\hat{j}+\hat{k}$ on the line passing through the point (2, 3, -1) and (-2, -4 3)

Sol^n:

Let $\vec{A} = 4\hat{i}-3\hat{j}+\hat{k}$

The vector passing through the point (2, 3, -1) and (-2, 4, 3) is

$$\vec{B} = -4\hat{i}-7\hat{j}+4\hat{k}$$

So the projection of $\vec{A}$ on $\vec{B}$ $= \dfrac{\vec{A}\cdot\vec{B}}{|\vec{B}|}$

$$= \frac{-16+21+4}{\sqrt{16+49+16}} = \frac{9}{\sqrt{81}} = \frac{9}{9} = 1$$

Ans

Vector Differentation:

Let $\vec{A}$ be any vector and $\vec{A}$ is a function of u and it can be written as

$$\vec{A}(u) = A_1(u)\hat{i} + A_2(u)\hat{j} + A_3(u)\hat{k}$$ then

its differentation is

$$\frac{d\vec{A}(u)}{du} = \frac{dA_1}{du}\hat{i} + \frac{dA_2}{du}\hat{j} + \frac{dA_3}{du}\hat{k}$$

Vector Differential operator

'del' $\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

Gradient of a scalar function

ϕ (say) is grad $\phi = \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$

Divergence of a vector $\vec{A}$ is

$$\operatorname{div}\vec{A} = \nabla\cdot\vec{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\vec{A}$$

□ Curl of a vector $\vec{A}$ is

$$\text{curl } \vec{A} = \nabla \times \vec{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \vec{A}$$

□ Gradient of a vector is always undefined.

Given $x^2y + y^2z$ = constant. Find the unit normal vector at $(1,1,0)$ $\text{Ans } \left(\frac{1}{\sqrt{6}}(2\hat{i}+\hat{j}+\hat{k})\right)$

Solⁿ: Normal vector to the surface is $\nabla(x^2y + y^2z)$

$$= \left(\frac{\partial \hat{i}}{\partial x} + \frac{\partial \hat{j}}{\partial y} + \frac{\partial \hat{k}}{\partial z}\right)(x^2y + y^2z)$$

$$= 2xy\hat{i} + (x^2 + 2yz)\hat{j} + y^2\hat{k}$$

at $(1,1,0)$ Normal vector

$$= 2\hat{i} + \hat{j} + \hat{k}$$

So the required unit vector is

$$= \frac{2\hat{i} + \hat{j} + \hat{k}}{\sqrt{4+1+1}} = \frac{1}{\sqrt{6}}(2\hat{i} + \hat{j} + \hat{k}) \quad \text{Ans}$$

7E

7-E

Find the velocity and acceleration of a particle which moves along the curve $x = 2\sin 3t$, $y = 2\cos 3t$, $z = 8t$ at any time $t>0$. Find the magnitude of velocity and acceleration.

Solⁿ: Let, the position vector of the particle, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$= 2\sin 3t\,\hat{i} + 2\cos 3t\,\hat{j} + 8t\,\hat{k}$

$\therefore$ velocity, $\vec{v} = \frac{d\vec{r}}{dt} = 6\cos 3t\,\hat{i} - 6\sin 3t\,\hat{j} + 8\hat{k}$ Ans

acceleration,

$\vec{a} = \frac{d\vec{v}}{dt} = -18\sin 3t\,\hat{i} - 18\cos 3t\,\hat{j}$

Ans

Magnitude of $\vec{v} = \sqrt{6^2\cos^2 3t + 6^2\sin^2 3t + 8^2}$

$= \sqrt{6^2+8^2}$

$= \sqrt{64+36} = \sqrt{100} = 10$ Ans

and, $|\vec{a}| = \sqrt{(-18\sin 3t)^2 + (-18\cos 3t)^2}$

magnitude of acceleration $= 18$ Ans

A particle moves along the curve $x = t^3+1$, $y = t^2$, $z = 2t+5$ where t is the time. Find the components of its velocity and acceleration at $t=1$ in the direction $2\hat{i}+3\hat{j}+6\hat{k}$

Sol^n: Let the position vector of the particle, $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$

$= (t^3+1)\hat{i} + t^2\hat{j} + (2t+5)\hat{k}$

$\therefore$ velocity $\vec{v} = \frac{d\vec{r}}{dt} = 3t^2\hat{i} + 2t\hat{j} + 2\hat{k}$

$\therefore$ acceleration $\vec{a} = \frac{d\vec{v}}{dt} = 6t\hat{i} + 2\hat{j}$

$36+9+4$

Component of $\vec{A}$ vector along $\vec{B}$ vector is $\vec{A}\cdot\frac{\vec{B}}{|\vec{B}|}$

at $t=1$, $\vec{v} = 3\hat{i}+2\hat{j}+2\hat{k}$

$$\vec{a} = 6\hat{i}+2\hat{j}$$

component of velocity $\vec{v}$ along $2\hat{i}+3\hat{j}+6\hat{k}$ $= (3\hat{i}+2\hat{j}+2\hat{k})\cdot\frac{2\hat{i}+3\hat{j}+6\hat{k}}{\sqrt{2^2+3^2+6^2}}$

$$= \frac{(3\hat{i}+2\hat{j}+2\hat{k})\cdot(2\hat{i}+3\hat{j}+6\hat{k})}{7}$$

$$= \frac{6+6+12}{7}$$

$$= \frac{24}{7}$$

Ans

Component of acceleration $\vec{a}$ along $2\hat{i}+3\hat{j}+6\hat{k}$ $= (6\hat{i}+2\hat{j})\frac{(2\hat{i}+3\hat{j}+6\hat{k})}{\sqrt{2^2+3^2+6^2}}$

$$= \frac{(6\hat{i}+2\hat{j})\cdot(2\hat{i}+3\hat{j}+6\hat{k})}{7}$$

$$= \frac{12+6}{7}$$

$$= \frac{18}{7}$$

Ans

Find a unit vector tangent vector to any point on the curve $x = a\cos\omega t$, $y = a\sin\omega t$, $z = bt$. here a, b, ω are constant

Soln. Let the position vector of the point $= \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$= \cancel{a\sin\omega t\,\hat{i}} + b$
$= a\cos\omega t\,\hat{i} + a\sin\omega t\,\hat{j} + bt\,\hat{k}$

tangent vector $\vec{T} = \frac{d\vec{r}}{dt}$
$= -a\omega\sin\omega t\,\hat{i} + a\omega\cos\omega t\,\hat{j} + b\hat{k}$

$\therefore$ unit tangent $= \frac{\vec{T}}{|\vec{T}|}$

$= \frac{-a\omega\sin\omega t\,\hat{i} + a\omega\cos\omega t\,\hat{j} + b\hat{k}}{\sqrt{(a\omega\sin\omega t)^2 + (a\omega\cos\omega t)^2 + b^2}}$

$= \frac{-a\omega\sin\omega t\,\hat{i} + a\omega\cos\omega t\,\hat{j} + b\hat{k}}{a^2\omega^2}$

H.W

A particle moves along the curve $x=2t^2$, $y=t^2-4t$, and $z=3t-5$ where t is the time. Find the components of its velocity and acceleration at $t=1$ in the direction $\hat{i}-3\hat{j}+2\hat{k}$. Ans $\frac{8\sqrt{14}}{7}$, $\frac{-\sqrt{14}}{7}$

Sol^n: Let, the position ~~of~~ vector of the point is $\vec{r}=x\hat{i}+y\hat{j}+\hat{k}z$

$$= 2t^2\hat{i}+(t^2-4t)\hat{j}+(3t-5)\hat{k}$$

$\therefore$ velocity $\vec{v}=\frac{d\vec{r}}{dt}=\cancel{2t}$
$4t\hat{i}+(2t-4)\hat{j}+3\hat{k}$

and acceleration $\vec{a}=\frac{d\vec{v}}{dt}$

$$=4\hat{i}+2\hat{j}$$

at $t=1$ $\vec{v} = 4\hat{i} - 2\hat{j} + 3\hat{k}$

and $\vec{a} = 4\hat{i} + 2\hat{j}$

$\therefore$ component of velocity $\vec{v}$ along $\hat{i} - 3\hat{j} + 2\hat{k}$ is $= (4\hat{i} - 2\hat{j} + 3\hat{k}) \cdot \frac{\hat{i} - 3\hat{j} + 2\hat{k}}{\sqrt{1+9+4}}$

$$= \frac{(4\hat{i} - 2\hat{j} + 3\hat{k})\cdot(\hat{i} - 3\hat{j} + 2\hat{k})}{\sqrt{14}}$$

$$= \frac{4+6+6}{\sqrt{14}}$$

$$= \frac{\cancel{14}\;16}{\sqrt{14}}$$

$$= \frac{16 \times \sqrt{14}}{14} = \frac{8\sqrt{14}}{7}$$ Ans

Component of acceleration $\vec{a}$ along $\hat{i} - 3\hat{j} + 2\hat{k}$ is $= \cancel{(4\hat{i} - 2\hat{j} + 3\hat{k})}$ $(4\hat{i} + 2\hat{j}) \cdot \frac{(\hat{i} - 3\hat{j} + 2\hat{k})}{\sqrt{1+9+4}}$

$$= \frac{4-6}{\sqrt{14}}$$

$$= \frac{-2\sqrt{14}}{14} = \frac{-\sqrt{14}}{7}$$ Ans

8D

8-D

17-~~07~~ 04-2014

The position vector of a particle at t is $\vec{r} = \cos(t-1)\hat{i} + \sinh(t-1)\hat{j} + \alpha t^3 \hat{k}$. Find the condition imposed on α by requiring that at $t=1$ the acceleration is normal to the position vector.

Solⁿ

Let,

Position vector,

Given $\vec{r} = \cos(t-1)\hat{i} + \sinh(t-1)\hat{j} + \alpha t^3 \hat{k}$

$$\frac{d\vec{r}}{dt} = -\sin(t-1)\hat{i} + \cosh(t-1)\hat{j} + 3\alpha t^2 \hat{k}$$

acceleration $\vec{a} = -\cos(t-1)\hat{i} + \sinh(t-1)\hat{j} + 6\alpha t \hat{k}$

at $t=1$ $\vec{a} = -\hat{i} + 6\alpha\hat{k}$

$\vec{r} = \hat{i} + \alpha\hat{k}$

Since $\vec{a}$ is normal to $\vec{r}$

So $\vec{a} \cdot \vec{r} = 0$

$\Rightarrow (-\hat{i} + 6\alpha\hat{k}) \cdot (\hat{i} + \alpha\hat{k}) = 0$

$\Rightarrow -1 + 6\alpha^2$

$\Rightarrow \alpha = \pm \frac{1}{\sqrt{6}}$

Ans

Directional derivative: The component of $\nabla\phi$ in the direction of a vector $\vec{d}$ is equal to $\nabla\phi \cdot \vec{d}$ and is called ~~diff~~ directional derivative of ϕ in the direction of $\vec{d}$

The temperature at any point in space is given by $T = xy + yz + zx$. Determine the derivative of T in the direction of $3\hat{i} - 4\hat{k}$ at $(1,1,1)$

Soln:

$$\nabla T = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(xy + yz + zx)$$

$$= (y+z)\hat{i} + \hat{j}(x+z) + \hat{k}(y+x)$$

at $(1,1,1)$ $\nabla T = 2\hat{i} + 2\hat{j} + 2\hat{k}$

So the directional derivative in the direction $3\hat{i} - 4\hat{k} =$

$$(2\hat{i} + 2\hat{j} + 2\hat{k}) \cdot \frac{(3\hat{i} - 4\hat{k})}{\sqrt{9+16}}$$

$$= \frac{-2}{5}$$ Ans

Find the directional derivative of $\mathrm{div}(\vec{u})$ at $(1,2,2)$ in the direction of the ~~other~~ outer normal to the sphere $x^2+y^2+z^2=9$ for $\vec{v}=x^4\hat{i}+y^4\hat{j}+z^4\hat{k}$

Soln: Here

$$\mathrm{div}\,\vec{v} = \nabla\cdot(x^4\hat{i}+y^4\hat{j}+z^4\hat{k})$$

$$= 4x^3+4y^3+4z^3$$

Here normal $= \nabla(x^2+y^2+z^2-9)$

$$= 2x\hat{i}+2y\hat{j}+2z\hat{k}$$

At $(1,2,2)$ outer normal $= 2\hat{i}+4\hat{j}+4\hat{k}$

Directional derivative $= \nabla(4x^3+4y^3+4z^3)$

$$= 12x^2\hat{i}+12y^2\hat{j}+12z^2\hat{k}$$

At $(1,2,2)$ the directional derivative $= 12\hat{i}+48\hat{j}+48\hat{k}$

The directional derivative in the direction of the outer normal

$$= (12\hat{i} + 48\hat{j} + 48\hat{k}) \cdot \left(\frac{2\hat{i} + 4\hat{j} + 4\hat{k}}{\sqrt{4+16+16}}\right)$$

$$= 68$$ Ans

Find the directional derivative of $f(x, y, z) = x^2 + xy + z^2$ of $A(1,-1,-1)$ in the direction of the line $\overrightarrow{AB}$ where $B(3,2,1)$ (Ans $\frac{1}{\sqrt{17}}$)

Soln: Here $f(x,y,z) = x^2 + yx + z^2$

$$\therefore \nabla \phi = (2x+y)\hat{i} + x\hat{j} + 2z\hat{k}$$

at $(1,-1,-1)$ $\nabla \phi$ is

$$= \hat{i} + \hat{j} - 2\hat{k}$$

$\vec{AB} = \vec{B} - \vec{A}$

$= 2\hat{i} + 3\hat{j} + 2\hat{k}$

$\therefore$ The directional derivative of the scalar function $f(x,y,z)$ in the direction of $\vec{AB}$

is $= \nabla\phi \cdot \hat{AB}$

$= (\hat{i}+\hat{j}-2\hat{k}) \cdot \dfrac{(2\hat{i}+3\hat{j}+2\hat{k})}{\sqrt{4+9+4}}$

$= \dfrac{2+3-4}{\sqrt{17}}$

$= \dfrac{1}{\sqrt{17}}$ Ans

H.W

Find the directional derivative of divergence of $f(x,y,z) = xy\hat{i} + xy^2\hat{j} + z^2\hat{k}$ at $(2,1,2)$ in the direction of ~~it~~ the ~~other~~ outer normal to the sphere $x^2+y^2+z^2=9$

Ans $13/3$

Sol^n : Divergence of $f(x,y,z)$ is

$= \nabla \cdot (xy\hat{i} + xy^2\hat{j} + z^2\hat{k})$

$= \cancel{y+}$

$= y + 2xy + 2z$

Outer normal to the sphere

$= \nabla (x^2+y^2+z^2-9)$

$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

Outer normal to the sphere at ~~to~~ $(2,1,2)$ is $= 4\hat{i} + 2\hat{j} + 4\hat{k}$

$\therefore$ directional derivative

$= \nabla\,(y+2xy+2z)$

$= 2y\hat{i}+\hat{j}+2x\hat{j}+2\hat{k}$

$= 2y\hat{i}+(2x+1)\hat{j}+2\hat{k}$

at $(2,1,2)$ the directional derivative is $= 2\hat{i}+5\hat{j}+2\hat{k}$

$\therefore$ Directional derivative along the outer normal

$$= (2\hat{i}+5\hat{j}+2\hat{k})\cdot\frac{(4\hat{i}+2\hat{j}+4\hat{k})}{\sqrt{16+4+16}}$$

$$= \frac{26}{\sqrt{36}}$$

$$= \frac{26}{6} = \frac{13}{3}$$ Ans

Show that the gradient field describing a motion is irrotational

⇒ Any vector is irrotational if its curl is zero.

Let a field is $f(x,y,z)$

$$\text{gradient } f(x,y,z) = \nabla f$$

$$= \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

curl ~~At~~ of $\nabla f = \nabla \times \nabla f$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \hat{i}\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right) + \hat{j}\left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right) + \hat{k}\left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)$$

$$= 0$$

8E

8-E

18-04-2017

Find the equation for the tangent plane to the surface $\cancel{x} = xz^2 + x^2y = z-1$ at the point $(1,-3,2)$

Sol^n: Let,

$$\varphi = xz^2 + x^2y - z + 1$$

Normal to the plane is

$$N = \nabla\varphi$$

$$= \nabla\left(xz^2 + x^2y - z + 1\right)$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(xz^2 + x^2y - z + 1\right)$$

$$= (z^2 + 2xy)\hat{i} + x^2\hat{j} + (2xz - 1)\hat{k}$$

$$= -2\hat{i} + \hat{j} + 3\hat{k} \text{ at } (1,-3,2)$$

Let, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of the tangent plane at (x, y, z)

at $(1, -3, 2)$, $\vec{r}_0 = \hat{i} - 3\hat{j} + 2\hat{k}$

The equation of a tangent plane is $= (\vec{r} - \vec{r}_0) \cdot \vec{N} = 0$

$\Rightarrow \left((x-1)\hat{i} + (y+3)\hat{j} + (z-2)\hat{k}\right) \cdot (-2\hat{i} + \hat{j} + 3\hat{k}) = 0$

$\Rightarrow 2x - y - 3z + 1 = 0$ Ans

HW

Find eqn of the tangent plane to the surface $z = x^2+y^2$ at $(2,-1,5)$

Ans $4x-2y-z=5$

Soln:

Let $\varphi = x^2+y^2-z$

Normal to the plane is

$$\vec{N} = \nabla\varphi$$

$$= \nabla(x^2+y^2-z)$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2+y^2-z)$$

$$= 2x\hat{i} + 2y\hat{j} - \hat{k}$$

$$= 4\hat{i} - 2\hat{j} - \hat{k} \text{ at } (2,-1,5)$$

Let, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of the tangent plane at ~~(2,-1,5)~~ (x,y,z)

At $(2,-1,5)$, $\vec{r}_0 = 2\hat{i} - \hat{j} + 5\hat{k}$

The equation of tangent plane is

$$(\vec{r}-\vec{r}_0)\cdot\vec{N} = 0$$

$$\Rightarrow \left((x-2)\hat{i} + (y+1)\hat{j} + (z-5)\hat{k}\right)\cdot \cancel{\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)} (4\hat{i} - 2\hat{j} - \hat{k}) = 0$$

$$\Rightarrow \cancel{1}$$

$$\Rightarrow 4x - 8 - 2y - 2 - z + 5 = 0$$

$$\Rightarrow 4x - 2y - z = 5$$

Ans

Prove that the vector

$\vec{A} = 3y^4z^2\hat{i} + 4x^3z^2\hat{j} - 3x^2y^2\hat{k}$ is

Solenoidal.

⇒ The vector will be Solenoidal if its divergent is zero

i.e $\nabla \cdot \vec{A} = 0$

So, $\nabla \cdot \vec{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(3y^4z^2\hat{i} + 4x^3z^2\hat{j} - 3x^2y^2\hat{k}\right)$

$= \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial z}(-3x^2y^2)$

$= \cancel{9x^3z^2 +}$

$= 0 + 0 + 0$

$= 0$

Hence $\vec{A}$ is Solenoidal.

H.W

Prove that the vector $\vec{A} = (2x^2+8xy^2z)\hat{i} + (3x^3y-3xy)\hat{j} - (4y^2z^2+2x^3z)\hat{k}$ is not soleroidal but $\vec{B} = xyz^2\vec{A}$ is soleroidal.

Sol^n Let, $\vec{A} = (2x^2+8xy^2z)\hat{i} + (3x^3y-3xy)\hat{j} - (4y^2z^2+2x^3z)\hat{k}$

$$\vec{B} = xyz^2\vec{A}$$

divergent $\nabla\cdot\vec{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left((2x^2+8xy^2z)\hat{i} + (3x^3y-3xy)\hat{j} - (4y^2z^2+2x^3z)\hat{k}\right)$

$= 4x + \cancel{8y^2z + 9x^2\,3x^3 - 3x - 8y^2z}$

$\cancel{-2x^3}$

$= \cancel{x + x^3 - 8y^2z + 8y}$

$= 4x + 8y^2z + 3x^3 - 3x - 8y^2z - 2x^3$

$= x + x^3 + 8y^2z - 8y^2z$

$\neq 0$

$\therefore \vec{A}$ is not solenoidal

$\vec{B} = xyz^2\vec{A}$

$= xyz^2\left\{(2x^2+8xy^2z)\hat{i} + (3x^3y-3xy)\hat{j} - (4y^2z^2+2x^3z)\hat{k}\right\}$

$$= (2x^3yz^2 + 8x^2y^3z^3 \cancel{+ xyz^3})\hat{i} + (3x^4y^2z^2 - 3x^2y^2z^2)\hat{j} - (4xy^3z^4 + 2x^4yz^3)\hat{k}$$

$$\therefore \nabla \cdot \vec{B} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \{(2x^3yz^2 + 8x^2y^3z^3 \cancel{+ xyz^3})\hat{i} + (3x^4y^2z^2 - 3x^2y^2z^2)\hat{j} - (4xy^3z^4 + 2x^4yz^3)\hat{k}\}$$

$$= 6x^2yz^2 + \cancel{16xy^3z^3} \cancel{+ yz^3} + \cancel{12x^3}\, 6x^4yz^2 - 6x^2yz^2 - \cancel{16xy^3z^3} - 6x^4yz^2$$

$$= 0$$

Hence $\vec{B}$ is Solenoidal Am

Show that $\vec{E} = \dfrac{\vec{r}}{r^2}$ is irrotational.

$\Rightarrow$ E will be irrotational if

$$\nabla \times \vec{E} = 0$$

Let, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\therefore r^2 = x^2 + y^2 + z^2$$

$$\therefore E = \frac{\vec{r}}{r^2} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{x^2+y^2+z^2}$$

$$= \frac{x}{x^2+y^2+z^2}\hat{i} + \frac{y}{x^2+y^2+z^2}\hat{j} + \frac{z}{x^2+y^2+z^2}\hat{k}$$

Now,
$$\nabla \times \vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{x^2+y^2+z^2} & \frac{y}{x^2+y^2+z^2} & \frac{z}{x^2+y^2+z^2} \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}\left(\frac{z}{x^2+y^2+z^2}\right) - \frac{\partial}{\partial z}\left(\frac{y}{x^2+y^2+z^2}\right)\right\}$$
$$+ \hat{j}\left\{\frac{\partial}{\partial z}\left(\frac{x}{x^2+y^2+z^2}\right) - \frac{\partial}{\partial x}\left(\frac{z}{x^2+y^2+z^2}\right)\right\}$$
$$+ \hat{k}\left\{\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2+z^2}\right) - \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2+z^2}\right)\right\}$$

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$= \hat{i}\left\{\frac{-2yz}{(x^2+y^2+z^2)^2} + \frac{2yz}{(x^2+y^2+z^2)^2}\right\} +$

$\hat{j}\left\{\frac{-2xz}{(x^2+y^2+z^2)^2} + \frac{2xz}{(x^2+y^2+z^2)^2}\right\} + \hat{k}\left\{\frac{-2xy}{(x^2+y^2+z^2)^2} + \frac{2xy}{(x^2+y^2+z^2)^2}\right\}$

$= \hat{i}\cdot 0 + \hat{j}\cdot 0 + \hat{k}\cdot 0$

$= 0$

Hence $\vec{E}$ is irrotational

(Shown)

~~N.Q~~ # Show that $\vec{v} = \frac{-x\hat{i} - y\hat{j}}{\sqrt{x^2+y^2}}$ is a sink field (curl must be equal to zero)

$\Rightarrow$ A vector $\vec{v}$ will be sink field if $\nabla \times \vec{v} = 0$

Soln: Given $\vec{v} = \frac{-x\hat{i} - y\hat{j}}{\sqrt{x^2+y^2}}$

$= \frac{-x}{\sqrt{x^2+y^2}}\hat{i} - \frac{y}{x^2+y^2}\hat{j}$

$$\therefore \nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cancel{\frac{-x}{\sqrt{x^2+y^2}}} & \cancel{\frac{-y}{\sqrt{x^2+y^2}}} & 0 \end{vmatrix}$$

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-x}{\sqrt{x^2+y^2}} & \frac{-y}{\sqrt{x^2+y^2}} & 0 \end{vmatrix}$$

$$= \hat{i}\left\{0 - \frac{\partial}{\partial z}\left(\frac{-y}{\sqrt{x^2+y^2}}\right)\right\} + \hat{j}\left\{\frac{\partial}{\partial z}\left(\frac{-x}{\sqrt{x^2+y^2}}\right) - 0\right\} + \hat{k}\left\{\frac{\partial}{\partial x}\left(\frac{-y}{\sqrt{x^2+y^2}}\right) - \frac{\partial}{\partial y}\left(\frac{-x}{\sqrt{x^2+y^2}}\right)\right\}$$

$$= \hat{i}(0-0) + \hat{j}(0-0) + \hat{k}\left\{(-y)\left(-\tfrac{1}{2}\right)(x^2+y^2)^{-3/2}(2x) - (-x)\left(-\tfrac{1}{2}\right)(x^2+y^2)^{-3/2}\, 2y\right\}$$

$$= 0 + 0 + \hat{k}\left(\frac{1}{2}2xy\frac{1}{(x^2+y^2)^{3/2}} - \frac{1}{2}2xy\frac{1}{(x^2+y^2)^{3/2}}\right)$$

$= 0 \quad \therefore \vec{V}$ is a sink field

(Shocon)

$\nabla \times \vec{A} = 0$
$\nabla \times \vec{B} = 0$

$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B}(\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$

H.W # If $\vec{A}$ and $\vec{B}$ are irrotational then prove that $\vec{A} \times \vec{B}$ is solenoidal.

Sol^n: Let $\vec{A}$ and $\vec{B}$ is a vector and they are irrotational

$\therefore \nabla \times \vec{A} = 0$ --- (i)

$\nabla \times \vec{B} = 0$ --- (ii)

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$$
$$= \vec{B} \cdot 0 - \vec{A} \cdot 0$$
$$= 0 - 0$$
$$= 0$$

$\therefore \vec{A} \times \vec{B}$ is solenoidal.

9-D

Vector Integration:

Formula for surface integration

$$\rightarrow \iint_S \vec{A} \cdot \hat{n}\, dS = \iint_R \vec{A}\,\hat{n} \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$= \iint_R \vec{A}\,\hat{n} \frac{dx\,dz}{|\hat{n}\cdot\hat{j}|} = \iint_R \vec{A}\,\hat{n} \frac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$$

Divergence theorem:

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\vec{F}\, dv$$

9D

9-D

Vector Integration:

Formula for surface integration

$$\rightarrow \iint_S \vec{A} \cdot \hat{n} \, dS = \iint_R \vec{A}\,\hat{n} \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$= \iint_R \vec{A}\,\hat{n} \frac{dx\,dz}{|\hat{n}\cdot\hat{j}|} = \iint_R \vec{A}\,\hat{n} \frac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$$

Divergence theorem:

$$\iint_S \vec{F} \cdot \hat{n} \, ds = \iiint_V \nabla \cdot \vec{F} \, dv$$

Evolvate $\iint_S \vec{A}\cdot\hat{n}\,ds$ where

$\vec{A} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$ and

S is that part of the plane $2x + 3y + 6z = 12$ which is located in the first octant

Soln

$\Rightarrow$ we know,

$$\iint_S \vec{A}\cdot\hat{n}\,ds = \iint_R \vec{A}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

Normal vector to the surface

$$\nabla(2x + 3y + 6z - 12) = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

So unit normal, $\hat{n} = \dfrac{2\hat{i} + 3\hat{j} + 6\hat{k}}{\sqrt{4+9+36}}$

$$= \frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k})$$

$\therefore \vec{A} \cdot \hat{n} = (18z\hat{i} - 12\hat{j} + 3y\hat{k}) \cdot \left(\frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k})\right)$

$$= \frac{36z - 36 + 18y}{7}$$

$$\therefore \iint_R \vec{A} \cdot \hat{n} \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|}$$

$$= \iint_R \frac{36z - 36 + 18y}{7} \frac{dx\,dy}{|6/7|}$$

$$= \frac{1}{6} \iint_R (36z - 36 + 18y)\, dx\,dy$$

$$= \frac{1}{6} \iint_R (72 - 12x - 18y - 36 + 18y)\, dx\,dy$$

60

42

$\frac{\ \ }{2}$

$\therefore 2x + 3y + 6z = 12$

$6z = 12 - 2x - 3y$

$\frac{12 - 6z}{3}$ $\cancel{(4 - 2y)}$

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y limit $\frac{3}{x} = 0$
x limit $y = 0, z = 0$

$$= \frac{1}{6} \iint_R (36 - 12x)\, dx\, dy$$

$$= \int_{x=0}^{6} \int_{y=0}^{\frac{12-2x}{3}} (6 - 2x)\, dx\, dy$$

$$= \int_{x=0}^{6} \left[6y - 2xy \right]_{y=0}^{\frac{12-2x}{3}} dx$$

$$= \frac{4}{3} \int_{x=0}^{6} (18 - 9x + x^2)\, dx$$

$$= 24 \text{ Ans}$$

$24 - 4x - \frac{2x}{3}(12-2x)$

$$\left[18x - \frac{9x^2}{2} + \frac{x^3}{3} \right]_0^6$$

$24 - 4x - 4x \frac{3}{3}$

$24 - 4x - 2x$

$\frac{24x + 4x^2}{3}$

$24 - 4x - 8x^2$

$6 \frac{12-2x}{3} - 2x \frac{12-2x}{3}$

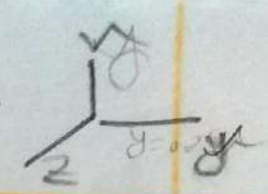
$24 - 4x - 2x \frac{12-2x}{3}$

$$= \frac{72 - 12x - 24x + 4x^2}{3} = \frac{72 - 12x - 24x + 4x^2}{3}$$

$$= \frac{72 - 36x + 4x^2}{3} = \frac{4x^2 - 36x + 72}{3}$$

$\frac{4}{3}(18 - 9x + x^2)$ $\frac{4}{3}(x^2 - 9x + 18)$

$4/3\,(36 - 72 + 18)$



H.W.# Evalute $= \iint_S \vec{A}\cdot\hat{n}\, ds$ where $\vec{A} = z\hat{i} + x\hat{j} - 3y^2 z\hat{k}$ and S is the surface of the cylinder $x^2+y^2=16$ included in the first octant & between $z=0$ and $z=5$

(Ans: 90)

Soln: $\iint_S \vec{A}\cdot\hat{n}\, ds = \iint_R \vec{A}\cdot\hat{n}\,\frac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$

$\hat{n}$ = Normal vector to the surface of cylinder $= \nabla(x^2+y^2-16)$

$= 2x\hat{i} + 2y\hat{j}$

$\therefore$ Normal unit vector $\hat{n} = \frac{2x\hat{i}+2y\hat{j}}{\sqrt{4x^2+4y^2}}$

$= \frac{x\hat{i}+y\hat{j}}{\sqrt{x^2+y^2}}$

$$= \frac{x\hat{i} + y\hat{j}}{\sqrt{16}} \quad [\because x^2+y^2=16]$$

$$= \frac{x\hat{i} + y\hat{j}}{4}$$

$$\therefore \vec{A}\cdot\hat{n} = (z\hat{i} + x\hat{j} - 3y^2z\hat{k})\cdot\left(\frac{x\hat{i}+y\hat{j}}{4}\right)$$

$$= \frac{1}{4}(xz + xy)$$

$$\hat{n}\cdot\hat{i} = \frac{x\hat{i}+y\hat{j}}{4}\cdot\hat{i}$$

$$= \frac{x}{4}$$

$$\iint_S \vec{A}\cdot\hat{n}\,ds = \iint_R \vec{A}\cdot\hat{n}\frac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$$

$$= \iint_R \frac{\frac{4}{4}(xz+xy)}{x}\,dy\,dz$$

~~$x^2+y^2=16$~~
~~$x^2=16$ $\therefore x=4$~~

$$=\int_0^5\int_0^4 (y+z)\,dy\,dz$$

$$=\int_0^5\left[\frac{y^2}{2}+zy\right]_0^4 dz$$

$$=\int_0^5 (8+4z)\,dz$$

$$=\left[8z+\frac{4z^2}{2}\right]_0^5$$

$$=40+50$$

$$=90$$

Ans

9E

9-E

Evalute $\iint_S \phi \cdot \hat{n}\, dS$ where

$\phi = \frac{3}{8}xyz$ and S is the surface of $x^2+y^2=16$ include in the first octant $z=0$ and $z=5$

<u>Sol^n</u> Normal to the surface $= \nabla(x^2+y^2-16)$

$$= 2x\hat{i}+2y\hat{j}$$

unique normal $\hat{n} = \frac{2x\hat{i}+2y\hat{j}}{\sqrt{4x^2+4y^2}}$

$$= \frac{x\hat{i}+y\hat{j}}{4}$$

We know,

$$\iint_S \phi \cdot \hat{n}\, ds = \iint_R \phi\, \hat{n}\, \frac{dx\,dz}{|\hat{n}\cdot\hat{j}|}$$

$$= \iint_R \frac{3}{8} r y z \left(\frac{x\hat{i}+y\hat{j}}{4}\right)\frac{dx\,dz}{y/4}$$

$$= \frac{3}{8}\int_{r=0}^{4}\int_{z=0}^{5} \left(r^2 z\,\hat{i} + rz\sqrt{16-r^2}\,\hat{j}\right) dx\,dz$$

$$= \frac{3}{8}\int_0^4 \left[r^2\frac{z^2}{2}\hat{i} + \frac{rz^2}{2}\sqrt{16-r^2}\,\hat{j}\right]_0^5$$

$$= \frac{3}{8}\times\frac{25}{2}\int_0^4 r^2\hat{i} + r\sqrt{16-r^2}\,\hat{j}\,dr$$

$$= 100\hat{i} + 100\hat{j}$$ Ans

$$\frac{75}{16}\left[\frac{r^3}{3}\hat{i}\right]_0^4 + \int_4^0 \frac{75}{16}\sqrt{z^2}\,(-2dz)\,\hat{j}$$

Let, $16-r^2=z^2$, $2z\,dz=-2r\,dr$, $r\,dr=-z\,dz$

r	0	4
z	4	0

$$= \frac{75}{16}\times\frac{64}{3}\hat{i} - \int_4^0 \frac{75}{16}\left[\frac{z^3}{3}\right]\hat{j} = 100\hat{i} - (-100\hat{j}) = 100\hat{i}+100\hat{j}$$

★ # If $\vec{F} = y\hat{i} + (x - 2xz)\hat{j} - xy\hat{k}$

evalute $\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS$ ~~which~~

where S is the surface of

$x^2 + y^2 + z^2 = a^2$ above xy plane

Sol^n:

Normal to surface $= \nabla(x^2 + y^2 + z^2 - a^2)$

$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$\therefore$ unit normal vector $= \dfrac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4(x^2 + y^2 + z^2)}}$

$= \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{a}$

Also $\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x-2xz & -xy \end{vmatrix}$

$= x\hat{i} + y\hat{j} - 2z\hat{k}$

We know, $\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds$

$= \iint_R (\nabla \times \vec{F}) \cdot \hat{n} \dfrac{dx\,dy}{|\hat{n} \cdot \hat{k}|}$

$= \iint_R (x\hat{i} + y\hat{j} - 2z\hat{k}) \cdot \left(\dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{a}\right) \dfrac{dx\,dy}{z/a}$

$$= \int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{x^2+y^2-2z^2}{\sqrt{a^2-x^2-y^2}}\, dx\, dy$$

Let $x = r\cos\theta$

$y = r\sin\theta$

$\therefore r = \sqrt{x^2+y^2}$

and $dx\,dy = r\,dr\,d\theta$

$$\therefore \iint (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, ds = \int_{r=0}^{a}\int_{\theta=0}^{2\pi} \frac{3(r^2\cos^2\theta + r^2\sin^2\theta) - 2a^2}{\sqrt{a^2 - r^2\cos^2\theta - r^2\sin^2\theta}}\, r\, dr\, d\theta$$

$\therefore x^2+y^2+z^2 = a^2$

$x^2+y^2 - 2a^2 + 2x^2 + 2y^2$

$3x^2+3y^2-2a^2$

$3(r^2\cos^2\theta + r^2\sin^2\theta) - 2a^2$

$$= \int_{r=0}^{a} \int_{\theta=0}^{2\pi} \frac{3r^2-2a^2}{\sqrt{a^2-r^2}} \, r\,dr\,d\theta$$

$$= \int_{r=0}^{a} \left[\frac{3r^2-2a^2}{\sqrt{a^2-r^2}} \, \theta \right]_0^{2\pi} r\,dr$$

$$= \int_{r=0}^{a} \frac{3r^2-2a^2}{\sqrt{a^2-r^2}} \, 2\pi \, r\,dr$$

$$= 2\pi \int_{r=0}^{a} \frac{3r^2}{\sqrt{a^2-r^2}} \, r\,dr - 2\pi \int_{r=0}^{a} \frac{2a^2}{\sqrt{a^2-r^2}} \, r\,dr$$

Let, $z = a^2 - r^2$

$dz = -2r\,dr$

$\therefore r\,dr = \frac{dz}{-2}$

r	0	a
z	a^2	0

$$= 6\pi \int_{a^2}^{0} \frac{a^2 - z}{\sqrt{z}} \left(\frac{dz}{-2} \right) - 4\pi \int_{a^2}^{0} \frac{a^2}{\sqrt{z}} \left(\frac{dz}{-2} \right)$$

$$= \left(\frac{6\pi}{2}\right)\int_{a^2}^{0}(a^2-z)z^{-\frac{1}{2}}dz + 2\pi\int_{a^2}^{0}a^2 z^{-\frac{1}{2}}dz$$

$$= -3\pi\int_{a^2}^{0}(a^2 z^{-1/2} - z^{1/2})dz + 2\pi\int_{a^2}^{0}a^2 z^{-1/2}dz$$

$$= -3\pi\left[a^2\frac{z^{1/2}}{\frac{1}{2}} - \frac{z^{3/2}}{3/2}\right]_{a^2}^{0} + 2\pi\left[a^2\frac{z^{\frac{1}{2}}}{\frac{1}{2}}\right]_{a^2}^{0}$$

$$= -3\pi\left(0 - 2a^2 a + \frac{2}{3}a^3\right) + 2\pi\left(-2a^3\right)$$

$$= -3\pi\left(\frac{2}{3}a^3 - 2a^3\right) - 4\pi a^3$$

$$= -2a^3\pi + 6a^3\pi - 4\pi a^3$$

$$= 0$$ Ans

10D

10-D

If $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ evalute $\iint_S \vec{F} \cdot \hat{n}\, ds$ where S is the surface of the cubic bounded by $x=0, x=1$ $y=0, y=1$ $z=0, z=1$

Soln

We know from divergence theorem

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$$

Now $\nabla \cdot \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(4xz\hat{i} - y^2\hat{j} + yz\hat{k}\right)$

$$= 4z - 2y + y$$

$$= 4z - y$$

Now $\iint_S \vec{F} \cdot n\, ds = \iiint_V (4z - y)\, dv$

$$= \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (4z - y)\, dx\, dy\, dz$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} \left[\frac{4}{2}z^2 - yz\right]_{z=0}^{1} dx\, dy$$

$$\int_{x=0}^{1} \left[2y - \frac{y^2}{2}\right]_{0}^{1} dx$$

$$= \frac{3}{2}$$

Evolute $\iint_S \vec{A}\cdot\hat{n}\,ds$ over the entire surface S of the region ~~located~~ bounded by the circle $x^2+z^2=9$ $x=0, y=0, z=0$ and $y=8$ if $\vec{A} = 6z\hat{i}+(2x+y)\hat{j} - x\hat{k}$

Sol^n^:

We know $\iint_S \vec{A}\,\hat{n}\,ds = \iiint_V \nabla\cdot\vec{A}\,dv$

$$= \int_{x=0}^{3}\int_{y=0}^{8}\int_{z=0}^{\sqrt{9-x^2}} 1\,dx\,dy\,dz$$

Here $\nabla\cdot\vec{A} = \left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\cdot\left(6z\hat{i}+(2x+y)\hat{j}-x\hat{k}\right)$

$= 1$

$$= \int_{x=0}^{3}\int_{y=0}^{8}\sqrt{9-x^2}\,dx\,dy$$

$$= \int_{r=0}^{3} 8\sqrt{9-r^2}\,dr$$

$$= 8\int_{r=0}^{3} \sqrt{9-r^2}\,dr$$

Let, $r = 3\sin\theta$

$dr = 3\cos\theta\, d\theta$

r	0	3
θ	0	$\pi/2$

$$= 8\int_{\theta=0}^{\pi/2} \cos^2\theta\, d\theta$$

$$= 18\pi$$

Am

Find the surface area of the plane $x+2y+2z=12$ cut off

a) $x=0, y=0, x=1, y=1$

b) $x=y=0$ and $x^2+y^2=16$

We know the surface area of the plane is $\iint_S dS = \iint_R \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$

Normal to the surface

$\nabla(x+2y+2z)$

$= \hat{i}+2\hat{j}+2\hat{k}$

So unit normal $\hat{n} = \frac{\hat{i}+2\hat{j}+2\hat{k}}{3}$

$\therefore \hat{n}\cdot\hat{k} = 2/3$

a) Now $\iint_S dS = \int_{x=0}^{1}\int_{y=0}^{1} \frac{dx\,dy}{2/3}$

$= 3/2$

Ans

b) Now $\iint_S dv = \int_{x=0}^{4}\int_{y=0}^{\sqrt{16-x^2}} \frac{dx\,dy}{2/3}$

$I = \frac{3}{2}\int_{x=0}^{4}\left[\sqrt{16-x^2}\,y\right]_0^1 dx$

$= \frac{3}{2}\int_{x=0}^{4}\sqrt{16-x^2}\,dx$ | Let $x=4\sin\theta$, $dx=4\cos\theta$

$$= \frac{3}{2}\int_{\theta=0}^{\cancel{\sin^{-1}1/3}\ \pi/2} 16\cos^2\theta\, d\theta$$

$$= \frac{3}{2}8\int_{0}^{\pi/2} (1+\cos 2\theta)\, d\theta$$

$$= \frac{24}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= 12\left(\pi/2 - \frac{\sin\pi}{2}\right)$$

$$= 6\pi$$

Ans

~~$= \frac{3}{4}\left(\frac{\pi}{2} + \frac{\sin\pi}{2}\right)$~~

~~$= \frac{3}{4}\left(\frac{\pi}{2}\right)$~~

~~$= \frac{3\pi}{8}$~~

□ Evalute $\iiint_V (2x+y)\,dv$ where

H.W

V is the closed ~~region reg~~ region bounded by the cylender $z=4-x^2$ and the planes $x=0, y=0, y=2$ and $z=0$

Sol^n: $\iiint_V (2x+y)\,dv = \int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{4-x^2} (2x+y)\, dz\,dy\,dx$

$8x+4y-2x^3-x^2y$

$$=\int_{x=0}^{2}\int_{y=0}^{2}\left[(2x+y)z\right]_0^{4-x^2} dx\,dy$$

$$=\int_{x=0}^{2}\int_{y=0}^{2}(2x+y)(4-x^2)\,dx\,dy$$

$$=\int_{x=0}^{2}\int_{y=0}^{2} 8x+4y-2x^3-x^2y\,dx\,dy$$

$$=\int_{x=0}^{2}\left[8xy+2y^2\pm 2x^3y\pm x^2y^2/2\right]_0^2 dx$$

$$=\int_{x=0}^{2}(16x+8\pm 4x^3\pm 2x^2)\,dx$$

$$=\left[16\frac{x^2}{2}+8x\pm x^4\pm\frac{2}{3}x^3\right]_0^2$$

$$=(8\times 4)+16\pm 16+\left(\frac{2}{3}\times 8\right)$$

$$=32+\cancel{16}\pm\cancel{16}\pm\frac{16}{3}$$

$$=$$

$$=80/3 \text{ Ans}$$

If $\vec{F} = (2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$

evaluate (a) $\iiint_V \nabla \cdot \vec{F}\, dv$ and

(b) $\iiint_V \nabla \times \vec{F}\, dv$ where v is

the closed region bounded by

the planes $x = y = z = 0$ and

$2x + 2y + z = 4$

Soln: Normal to the surface $= \nabla \cdot (2x + 2y + z)$

$= 2\hat{i} + 2\hat{j} + \hat{k}$

$\therefore$ Unit normal $\hat{n} = \dfrac{2\hat{i} + 2\hat{j} + \hat{k}}{3}$

$\nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left((2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}\right)$

$= 2x$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2-3z & -2xy & -4x \end{vmatrix}$$

$$= \hat{j} - 2y\hat{k}$$

a) $\iiint_V \nabla \cdot \vec{F} \, dV$

$$= \int_{x=0}^{2} \int_{y=0}^{2-x} \int_{z=0}^{4-2x-2y} 2x \, dx\,dy\,dz$$

$= 8/3$ Ans

$\boxed{\text{Ans}: 8/3(\hat{j}-\hat{k})}$

(b) $= \iiint_V \nabla \times \vec{F} \, dV$

Math 2113

a) $\iiint_V \nabla \cdot \vec{F}\, dv$

$= \int_{x=0}^{2} \int_{y=0}^{2-x} \int_{x=0}^{4-2x-2y} 2x \, dx\, dy\, dz$

$= \int_{x=0}^{2} \int_{y=0}^{2-x} \left[2xz\right]_0^{4-2x-2y} dx\, dy$

$= \int_{x=0}^{2} \int_{y=0}^{2-x} 2x(4-2x-2y)\, dx\, dy$

$= \int_{x=0}^{2} \int_{y=0}^{2-x} 8x - 4x^2 - 4xy \, dx\, dy$

$= \int_{x=0}^{2} \left[8xy - 4x^2y - 4x\frac{y^2}{2}\right]_0^{2-x} dx$

$= \int_{x=0}^{2} 8x(2-x) - 4x^2(2-x) - 2x(2-x)^2 \, dx$

$$= \int_{x=0}^{2} 16x - 8x^2 - 8x^2 + 4x^3 - 2x(4 - 4x + x^2)\, dx$$

$$= \int_{x=0}^{2} 16x - 16x^2 + 4x^3 - 8x + 8x^2 - 2x^3\, dx$$

$$= \int_{x=0}^{2} 8x - 8x^2 + 2x^3\, dx$$

$$= \left[8\frac{x^2}{2} - \frac{8x^3}{3} + 2\frac{x^4}{4} \right]_0^2$$

$$= \left\{ (4 \times 4) - \left(\frac{8}{3} \times 8\right) + \left(\frac{1}{2} \cdot 16\right) \right\}$$

$$= 16 - \frac{64}{3} + 8$$

$$= 24 - \frac{64}{3}$$

$$= \frac{72 - 64}{3}$$

$$= \frac{8}{3}$$ Ans

b) $\iiint_V \nabla\times\vec{F}\, dv$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} (\hat{j} - 2y\hat{k})\, dx\,dy\,dz$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} \hat{j}\, dx\,dy\,dz - \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} 2y\hat{k}\, dx\,dy\,dz$$

$$= \frac{8}{3}\hat{j} - \int_{x=0}^{2}\int_{y=0}^{2-x} \left[2y\hat{k}\cdot z\right]_0^{4-2x-2y} dx\,dy$$

$$= \frac{8}{3}\hat{j} - \int_{x=0}^{2}\int_{y=0}^{2-x} (8y - 4xy - 4y^2\,\hat{k})\, dx\,dy$$

[From (a)]

$$= \frac{8}{3}\hat{j} - \int_{x=0}^{2} \left[4y^2 - 2xy^2 - \frac{4}{3}y^3\hat{k}\right]_0^{2-x} dx$$

$$= \frac{8}{3}\hat{j} - \int_{x=0}^{2} 4(2-x)^2 - 2x(2-x)^2 - \frac{4}{3}(2-x)^3\hat{k}\; dx$$

$$= \frac{8}{3}\hat{j} - (2-x)^2 \int_{x=0}^{2} \left\{4 - 2x - \frac{4}{3}(2-x)\right\}\hat{k}\, dx$$

$$= \frac{8}{3}\hat{j} - (4-4x+x^2)\int_{x=0}^{2}\left(4-2x-\frac{8}{3}+\frac{4}{3}x\right)\hat{k}\, dx$$

$$= \frac{8}{3}\hat{j} - (4-4x+x^2)\int_{x=0}^{2}\left(\frac{4}{3}-\frac{2}{3}x\right)\hat{k}\, dx$$

$$= \frac{8}{3}\hat{j} - \int_{x=0}^{2}\left(\frac{16}{3} - \frac{8}{3}x - \frac{16x}{3} + \frac{8x^2}{3} + \frac{4}{3}x^2 - \frac{2}{3}x^3\right)\hat{k}\, dx$$

$$= \frac{8}{3}\hat{j} - \int_{x=0}^{2}\left(\frac{16}{3} - \frac{24}{3}x + \frac{12}{3}x^2 - \frac{2}{3}x^3\right)\hat{k}\, dx$$

$$= \frac{8}{3}\hat{j} - \left[\frac{16}{3}x - \frac{8x^2}{2} + \frac{4}{3}x^3 - \frac{2}{3}\cdot\frac{x^4}{4}\right]_0^2 \hat{k}$$

$$= \frac{8}{3}\hat{j} - \left\{\left(\frac{16}{3}\times 2\right) - (4\times 4) + \frac{32}{3} - \left(\frac{2}{3}\times\frac{16}{4}\right)\right\}\hat{k}$$

$$= \frac{8}{3}\hat{j} - \left(\frac{32}{3} - 16 + \frac{32}{3} - \frac{8}{3}\right)\hat{k}$$

$$= \frac{8}{3}\hat{j} - \frac{8}{3}\hat{k}$$

$$= 8/3\,(\hat{j} - \hat{k})$$

10E

10-E

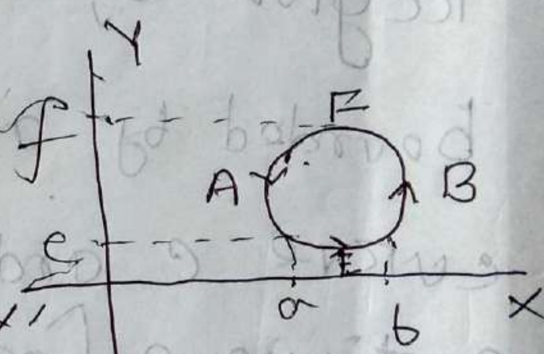
Green's theorem

Statement: If R is a closed region of the xy plane (xy) bounded by a simple closed curve C and if M and N are continuous function of x and y having continuous derivatives in R then

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Proof: Let c be a closed curve which has property that straight line parallel to the coordinate axes cuts c at most two points.

Let the eq^n of the curve AEB & AFB be $y = Y_1(x)$ and $y = Y_2(x)$ respectively.



If R is the region bounded by c we have

$$\iint_R \frac{\partial M}{\partial y}\, dx\, dy = \int_{x=a}^{b}\int_{y=Y_1}^{Y_2} \frac{\partial M}{\partial y}\, dx\, dy$$

$$= \int_{x=a}^{b} \left[M(x,y) \right]_{y=y_1}^{y=y_2} dx =$$

$$= \int_{a}^{b} \left[M(x, y_2) - M(x, y_1) \right] dx$$

$$= -\int_{b}^{a} M(x, y_2)\, dx - \int_{a}^{b} M(x, y_1)\, dx$$

$$= -\left[\int_{b}^{a} M(x, y_2)\, dx + \int_{a}^{b} M(x, y_1)\, dx \right]$$

$$\iint_R \frac{\partial M}{\partial y} dx\, dy = -\oint M\, dx \text{ --- (1)}$$

Again let the eqn of the EAF and EBF be $x = x_1(y)$ and $x = x_2(y)$.

then $\iint_R \frac{\partial N}{\partial x} dx dy = \int_{y=e}^{f} \int_{x=x_1}^{x_2} \frac{\partial N}{\partial x} dx dy$

$$= \int_{y=e}^{f} [N(x_2, y) - N(x_1, y)] dy$$

$$= \left\{ \int_e^f N(x_2, y) + \int_f^e N(x_1, y) \right\} dy \quad \text{(1)}$$

$$= \oint N dy \text{ ---- (2)}$$

Adding (1) and (2)

$$\oint M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

proved

Divergence theorem:

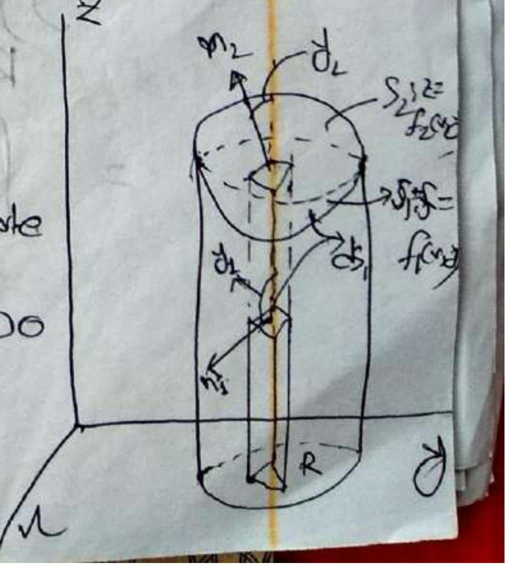
Statement: If V be the volume bounded by a closed surface ~~a~~ S ~~and b~~ and $\vec{A}$ is a vector function of position with continuous derivatives then

$$\iiint_V \nabla \cdot \vec{A}\, dV = \iint_S \vec{A} \cdot \hat{n}\, ds$$

$$= \oiint \vec{A} \cdot d\vec{S} .$$

H.W

Proof: Let S be a closed surface which is such that any line parallel to the coordinate axes cuts S in at most two points.



Assume the equation of the lower and upper positions, S_1 and S_2 to be $z = f_1(x,y)$ and $z = f_2(x,y)$ respectively.

Denote the projection of the surface on the xy plane by R. Consider

$$\iiint_V \frac{\partial A_3}{\partial z} dv = \iiint_V \frac{\partial A_3}{dz} dz\,dy\,dx =$$

$$\iint_R \left[\int_{z=f_1(x,y)}^{f_2(x,y)} \frac{\partial A_3}{\partial z} dz \right] dy\,dx$$

$$= \iint_R \left[A_3(x,y,z)\right]_{f_1(x,y,z)}^{f_2(x,y,z)} dy\,dz$$

$$= \iint_R \left[A_3(x,y,f_2) - A_3(x,y,f_1)\right] dy\,dx$$

For the ~~po~~ upper position S_2 $dy\,dx = \cos\gamma_2\, dS_2$

$= k\cdot n_2\, dS_2$ since the normal n_2 to S_2 makes an acute angle γ_2 with k.

For the lower position S_1 $dy\,dx = -\cos\gamma_1\, dS_1$

$= -k\cdot n_1\, dS_1$ since the normal n_1 to S_1 makes an obtuse angle γ_1 with k.

$$\iint_R A_3(x,y,f_2)\,dy\,dx = \iint_{S_2} A_3\, k\cdot n_2\, dS_2$$

$$\iint_R A_3(x,y,f_1)\,dy\,dx = -\iint_{S_1} A_3\, k\, n_1\, dS_1$$

and $$\iint_R A_3(x,y,f_2)\,dy\,dx - \iint_R A_3(x,y,f_1)\,dy\,dx$$

$$= \iint_{S_2} A_3\, k\cdot n_2\, dS_2 + \iint_{S_1} A_3\, k\cdot n_1\, dS_1$$

$$= \iint_S A_3\, k\, n\, dS$$

So that, (1) $$\iiint_V \frac{\partial A_3}{\partial z}\, dV = \iint_S A_3\, k\cdot n\, dS$$

Similarly by ~~pof~~ projecting S on the other coordinate planes

2) $$\iiint_V \frac{\partial A_1}{\partial x}\, dv = \iint_S A_1\, i\, n\, dS$$

3) $$\iiint_V \frac{\partial A_2}{\partial y}\, dv = \iint_S A_2\, j\, n\, dS$$

Adding ①, ② and ③

$$\iiint_V \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) dv$$

$$= \iint_S (A_1 i + A_2 j + A_3 k)\, n\, dS$$

$$\Rightarrow \iiint_V \nabla \cdot A\, dv = \iint_S \vec{A} \cdot n\, dS$$

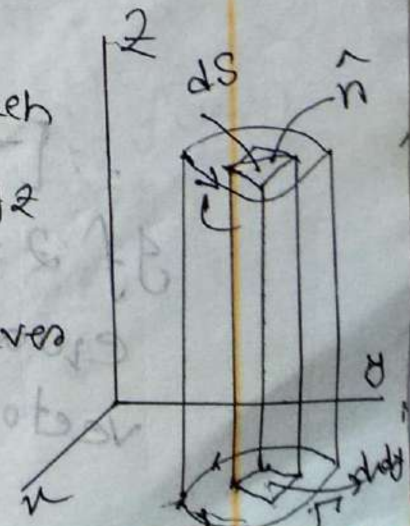
Stokes theorem:

Statement: The line integral of the tangential component of a vector $\vec{A}$ taken around a simple closed curve is equal to the surface integral of the normal component of the curl of $\vec{A}$ taken over any surface having C as its boundary.

i.e. $$\oint_C \vec{A}\, d\vec{r} = \iint_S (\nabla \times \vec{A}) \cdot \hat{n}\, ds$$

H.W. Proof: ~~Consider f~~

Let S be a surface which is such that its projection on the xy, yz and zx planes are regions bounded by simple closed curves as shown in figure.



Assume S to have representations
$z = f(x,y)$ or $x = g(y,z)$ or $y = h(x,z)$
We have to show that

$$\iint_S (\nabla \times \vec{A}) \cdot \hat{n} \, ds = \iint_S \left[\nabla \times [A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}]\right] \cdot \hat{n} \, ds$$

$$= \oint_C \vec{A} \cdot d\vec{r}$$

Consider first $\iint_S [\nabla \times A_1 \hat{i}] \, \hat{n} \, ds$

Since $\nabla \times A_1 \hat{i} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & 0 & 0 \end{vmatrix}$

$$= \frac{\partial A_1}{\partial z} \hat{j} - \frac{\partial A_1}{\partial y} \hat{k}$$

$$\therefore [\nabla \times A_1 \hat{i}] \cdot \hat{n} \, ds = \left(\frac{\partial A_1}{\partial z} \hat{n} \cdot \hat{j} - \frac{\partial A_1}{\partial y} \hat{n} \cdot \hat{k}\right) ds \quad \text{--- (1)}$$

If $z = f(x,y)$ is taken as the equation of S then the position vector to any point ~~of~~ of S is

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\Rightarrow \vec{r} = x\hat{i} + y\hat{j} + f(x,y)\hat{k}$

$\therefore \frac{\partial \vec{r}}{\partial y} = \hat{j} + \frac{\partial f}{\partial y}\hat{k}$

But $\frac{\partial \vec{r}}{\partial y}$ is a vector tangent to S and perpendicular with $\hat{n}$

Then $\hat{n}\cdot\frac{\partial \vec{r}}{\partial y} = \hat{n}\cdot\hat{j} + \frac{\partial f}{\partial y}\hat{n}\cdot\hat{k} = 0$

$$\Rightarrow \hat{n}\cdot\hat{j} = -\frac{\partial z}{\partial y}\hat{n}\cdot\hat{k} \quad \Big| \because z = f(x,y)$$

Putting this (1)

$$[\nabla \times A_1\hat{i}]\,\hat{n}\,dS = \left[-\frac{\partial A_1}{\partial z}\frac{\partial z}{\partial y}\hat{n}\cdot\hat{k} - \frac{\partial A_1}{\partial y}\cdot\hat{n}\cdot\hat{k}\right]dS$$

$$= -\left(\frac{\partial A_1}{\partial y} + \frac{\partial A_1}{\partial z}\frac{\partial z}{\partial y}\right)\hat{n}\cdot\hat{k}\,dS \quad \text{--- (2)}$$

Now, on S $A_1(x,y,z) = A_1(x,y,f(x,y))$
$= F(x,y)$ [Say]

So $\frac{\partial A_1}{\partial y} + \frac{\partial A_1}{\partial z}\cdot\frac{\partial z}{\partial y} = \frac{\partial F}{\partial y}$

Then from (2) $[\nabla \times A_1\hat{i}] \cdot \hat{n}\, dS = -\frac{\partial F}{\partial y} dx dy$

[$n \cdot k = 1$ because they are parallel]

~~Then from~~

Then $\iint_S [\nabla \times A_1\hat{i}] \cdot \hat{n}\, dS = \iint_R -\frac{\partial F}{\partial y} dx dy$

The projection of S on the xy plane

By Green's theorem

$$\iint_R -\frac{\partial F}{\partial y} dx dy = \int F dx$$

$$= \oint_C A_1 dx$$

Similarly $\iint_S (\nabla \times A_2\hat{j}) \cdot \hat{n}\, dS = \oint_C A_2 dy$ and

$$\iint_S (\nabla \times A_3\hat{k}) \cdot \hat{n}\, dS = \oint_C A_3 dz$$

Adding these $\iint_S (\nabla \times \vec{A}) \cdot \hat{n}\, dS = \oint_C \vec{A} \cdot d\vec{r}$

[Proved]

11D

11-D
13-05-2017

Verify Green's theorem in the plane for $\oint_C (3x^2-8y^2)dx+(4y-6xy)dy$ where C is the boundary of the region devided by (a) $y=\sqrt{x}$, $y=x^2$ (b) $x=0$, $y=0$, $x+y=1$

Let,

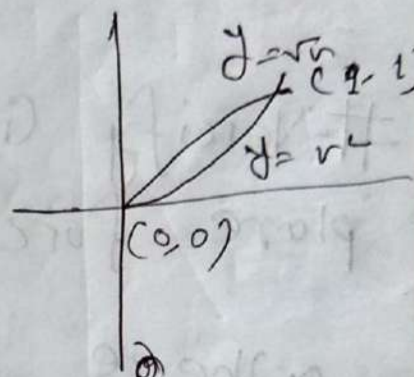
(a) for directly solving $y=\sqrt{x}$, $y=x^2$ we get $x=0$ $y=0$ ~~Along~~ and $x=1$ $y=1$. along $y=x^2$

$$\oint_C (3x^2-8y^2)dx+(4y-6xy)dy$$

$$=\int_0^1 (3x^2-8x^4)dx+(4x^2-6x^3).2x\,dx$$

$$= \int_0^1 (3x^2 - 8x^4 + 8x^3 - 12x^4)\,dx$$

$$= -1$$



Along $y = \sqrt{x}$

$$\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy = \int_1^0 (3x^2 - 8x)\,dx + (4\sqrt{x} - 6x\sqrt{x})\,\tfrac{1}{2}x^{-\frac{1}{2}}\,dx$$

$$= \int_1^0 (3x^2 - 8x + 2 - 3x)\,dx$$

$$= 5/2$$

$\cancel{3x^2 - 11x + 2}$

$3x^2 - 11x + 2$

$\left[x^3 - \frac{11x^2}{2} + 2x\right]_0^1$

$1 - \frac{11}{2} + 2$

$3 - \frac{11}{2} = \frac{6-11}{2} = \frac{-5}{2} - 0$

$\oint = 5/2 - 1 = 3/2$

By Green's theorem.

$$\oint_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy = \int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} \left\{ \frac{\partial(4y-6xy)}{\partial x} - \frac{\partial}{\partial y}(3x^2-8y^2) \right\} dx\,dy$$

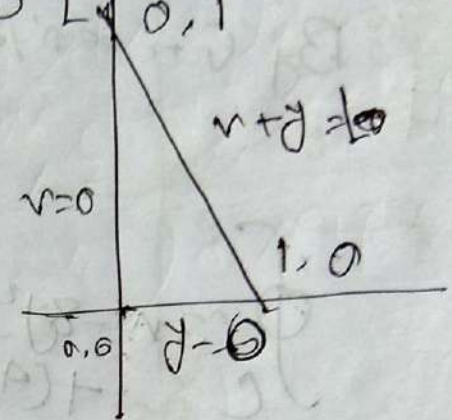
$$= \int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} (-6y+16y)\,dx\,dy$$

$$= \int_{x=0}^{1} 5[y^2]_{x^2}^{\sqrt{x}}\,dx = \int_{x=0}^{1} (5x-5x^4)\,dx = \int_{x=0}^{1} (5x-5x^4)\,dx = 3/2$$

Hence the Green's theorem is verified.

(b) Along $y=0$ $dy=0$

x varies from 0 to 1



$$\oint_C (3x^2-8y^2)dx + (4y-6xy)dy = \int_0^1 3x^2 dx = 1$$

Along $x+y=1$ $y=1-x$ $dy=-dx$

x varies from 1 to 0

$$\oint_C (3x^2-8y^2)dx + (4y-6xy)dy = \int_{x=1}^{0} \{3x^2-8(1-x)^2\}dx - \{4(1-x)-6x(1-x)\}dy$$

$$= \int_1^0 (-11x^2 + 26x - 12)\, dx$$

$$= 8/3$$

Along $x = 0$

$dx = 0$

$\boxed{y \text{ varies from } 1 \text{ to } 0}$

$$\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy = \int_1^0 4y\, dy = -2$$

Therefore the required integration

$$= 1 + 8/3 - 2 = 5/3$$

By Green's theorem

$$\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy = \int_{x=0}^{1} \int_{y=0}^{1-x} \left[\frac{\partial}{\partial x}(4y - 6xy) - \frac{\partial}{\partial y}(3x^2 - 8y^2) \right] dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x} 10y\, dy\, dx$$

$$= \int_{x=0}^{1} \cancel{4(1-x^2)\, dx} = 5/3 = \int_{x=0}^{1} 5(1-x)^2 dx$$

$$= \int_{x=0}^{1} \cancel{5 y^2 / dx} = \int_{x=0}^{1} (5-10x+5x^2)\, dx$$

$$= \left[5x - 5\frac{x^2}{2} + 5\frac{x^3}{3}\right]_{x=0}^{1} = 5/3$$

H.W # Evalute $\oint (x^2-2xy)\,dx + (x^2y+3)\,dy$

evalute the boundery of the region bounded by $y^2 = 8x$ and $x=2$ (a) directly (b) by using Green's theorem.

Ans $= 128/5$

Soln: Along $x=2$ $\therefore dx=0$

Solving $y=\pm 4$

$$\oint (x^2-2xy)\,dx + (x^2y+3)\,dy$$

$$= \int_{y=-4}^{y=4} (4y+3)\,dy = \cancel{24} \quad [\because dx=0]$$

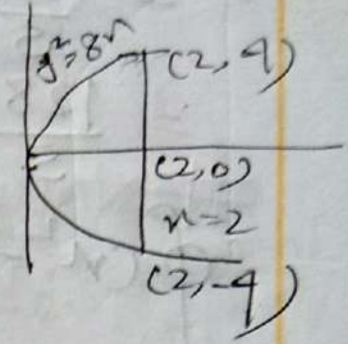
$$= \left[\frac{4y^2}{2} + 3y\right]_{-4}^{4} = \{(2\times16)+12 - (2\times16)+12\}$$

$$= 24$$

Along $y^2 = 8x$

$\cancel{dx} \Rightarrow 2y\,dy = 8\,dx$

$dx = \frac{y\,dy}{4}$



$\therefore \oint (x^2-2xy)\,dx + (xy^2+3)\,dy$

$= \int_4^{-4} \left\{\left(\frac{y^4}{64} - \frac{y^3}{4}\right)\frac{1}{4}y\,dy + \left(\frac{y^5}{64}+3\right)dy\right\}$

$= \int_4^{-4} \left(\frac{y^5}{256} - \frac{y^4}{16} + \frac{y^5}{64} + 3\right)dy$

$= \int_4^{-4}\left(\frac{5}{256}y^5 - \frac{y^4}{16} + 3\right)dy$

$= \left[\frac{5y^6}{256\times 6} - \frac{y^5}{16\times 5} + 3y\right]_4^{-4}$

$= \left[\frac{5}{1536}y^6 - \frac{y^5}{80} + 3y\right]_4^{-4}$

$= \left\{\left(\frac{5}{1536}\times 4096\right) + \frac{1024}{80} - 12\right\} - \left\{\left(\frac{5}{1536}\times 4096\right) - \frac{1024}{80} + 12\right\}$

$= \cancel{2}\ \frac{64}{5} - 12 + \frac{64}{5} - 12$

$= \frac{128}{5} - 24$

$\therefore \oint (x^2 - 2xy)\,dx + (x^2y + 3)\,dy = 24 + \frac{128}{5} - 24$

$= \frac{128}{5}$

Ans

b) By Green's theorem,

$\oint (x^2 - 2xy)\,dx + (x^2y+3)\,dy$

$= \int_{x=0}^{2} \int_{y=-\sqrt{8x}}^{\sqrt{8x}} \left\{ \frac{\partial}{\partial x}(x^2y+3) - \frac{\partial}{\partial y}(x^2 - 2xy) \right\} dx\,dy$

$= \int_{x=0}^{2} \int_{y=-\sqrt{8x}}^{\sqrt{8x}} (2xy + 2x)\,dx\,dy$

$$= \int_{x=0}^{2} \left[x y^2 + 2xy \right]_{-\sqrt{8x}}^{\sqrt{8x}} dx$$

$$= \int_{x=0}^{2} \left\{ (x \cdot 8x) + 2x\sqrt{8x} - (x \cdot 8x) - 2 \cdot x \cdot (-\sqrt{8x}) \right\} dx$$

$$= \int_{x=0}^{2} 4x\sqrt{8x}\, dx$$

$$= \int_{x=0}^{2} 4\sqrt{4 \times 2}\; x^{3/2}\, dx$$

$$= 8\sqrt{2} \int_{x=0}^{2} x^{3/2}\, dx$$

$$= \frac{8\sqrt{2} \times 2}{5} \left[x^{5/2} \right]_0^2$$

$$= \frac{16\sqrt{2}}{5} \left(2^{5/2} \right)$$

$$= \frac{16\sqrt{2}}{5} \left(2^5 \right)^{\frac{1}{2}}$$

$$= \frac{16\sqrt{2}}{5} (32)^{\frac{1}{2}}$$

$$= \frac{16 \times \sqrt{2} \times \sqrt{32}}{5}$$

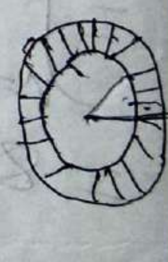
$$= \frac{16\sqrt{64}}{5}$$

$$= 128/5 \text{ Ans}$$

~~C[ ] [ ]~~

H.W

Verify Green's theorem in the plane for $\oint_C (2x - y^3)\,dx - xy\,dy$ where C is the boundary of the region bounded by the circle $x^2+y^2=1$ $x^2+y^2=9$

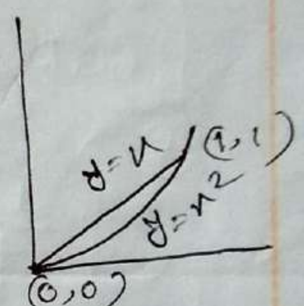


G.T.

Verify Green's theorem in the plane for $\oint_C (xy+y^2)\,dx + x^2\,dy$ where C is the closed curve of the region bounded by $y=x$ and $y=x^2$

Sol^n:

By solving $y=x$ and $y=x^2$ we get the intersection point $(0,0)$ and $(1,1)$.



Along $y=x^2$

$dy = 2x\,dx$

$\therefore \oint_C (xy+y^2)\,dx + x^2\,dy = \int_0^1 (x^3+x^4+2x^3)\,dx$

$= \int_0^1 3x^3 + x^4 \; dx$

$= \left[\frac{3x^4}{4} + \frac{x^5}{5}\right]_0^1$

$= \frac{3}{4} + \frac{1}{5} = \frac{19}{20}$

Along $y = x$

$dy = dx$

$$\oint_c (xy + y^2)\,dx + x^2\,dy = \int_1^0 (x^2 + x^2 + x^2)\,dx$$

$$= \int_1^0 3x^2\,dx$$

$$= [x^3]_1^0$$

$$= -1$$

$\therefore$ The required line integral

$$= \frac{19}{20} - 1$$

$$= -\frac{1}{20}$$

By Green's theorem,

$$\oint_c (xy + y^2)\,dx + x^2\,dy = \int_{x=0}^{1} \int_{y=x^2}^{x} \left\{\frac{\partial}{\partial x} x^2 - \frac{\partial}{\partial y}(xy + y^2)\right\} dx\,dy$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{x} (2x - x - 2y)\,dx\,dy$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{x} (x-2y)\, dy\, dx$$

$$= \int_{x=0}^{1} \left[xy - y^2\right]_{x^2}^{x} dx$$

$$= \int_{x=0}^{1} \left(\cancel{x^2} - \cancel{x^2} - x^3 + x^4\right) dx$$

$$= \int_{x=0}^{1} x^4 - x^3\, dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{5} - \frac{1}{4}$$

$$= \frac{4-5}{20}$$

$$= \frac{-1}{20}$$

Ans

Hence Green's theorem verified

Solve of Circle

Along $x^2+y^2=9$

$$\oint (2x-y^3)\,dx - xy\,dy$$

$$= \int_0^{2\pi} \{(6\cos\theta - 27\sin^3\theta)(-3\sin\theta\, d\theta) - 3\cos\theta \cdot 3\sin\theta \cdot 3\cos\theta\, d\theta\}$$

Let, $x = 3\cos\theta$
$y = 3\sin\theta$

$$= \int_0^{2\pi} \{81\sin^4\theta - 18\cos\theta\sin\theta - 27\cos^2\theta\sin\theta\} d\theta$$

$$= \int_0^{2\pi} \left\{\frac{81}{4}(1-\cos 2\theta)^2 - 9\sin 2\theta - 27\cos^2\theta\sin\theta\right\} d\theta$$

$$= \int_0^{2\pi} \frac{81}{4}\left(1+\cos^2 2\theta - 2\cos 2\theta\right) d\theta - \int_0^{2\pi} 9\sin 2\theta\, d\theta - 27\int_0^{2\pi} \sin\theta\cos^2\theta\, d\theta$$

$$= \int_0^{2\pi} \frac{81}{4}\, d\theta + \int_0^{2\pi} \frac{81}{4}\cos^2 2\theta\, d\theta - \int_0^{2\pi} \frac{81}{2}\cos 2\theta\, d\theta + 9\left[\frac{\cos 2\theta}{2}\right]_0^{2\pi} + 27\int_{1}^{1} z^2\, dz$$

$$= \frac{81}{4}[\theta]_0^{2\pi} + \int_0^{2\pi} \frac{81}{8}(1+\cos 4\theta)\,d\theta$$

$$- \frac{81}{2}\left[\frac{\sin 2\theta}{2}\right]_0^{2\pi} + 27\left[\frac{z^3}{\times 3}\right]_1^1$$

$$+ 9/2\ (1-1)$$

Let, $\cos\theta = z$

$\therefore dz = -\sin\theta\, d\theta$

z	1	1
θ	0	2π

$$= \frac{81}{4}2\pi + \frac{81}{8}2\pi + \frac{81}{8}\left[\frac{\sin 4\theta}{4}\right]_0^{2\pi} - \frac{81}{4}(0-0) + \frac{27}{2}(1-1)$$

$$= \frac{81}{2}\pi + \frac{81}{4}\pi + \frac{81}{32}(0-0)$$

$$= \frac{162\pi + 81\pi}{4}$$

$$= \frac{243\pi}{4}$$

Alain along $x^2+y^2=1$

$$\oint (2x-y^3)\,dx - xy\,dy = \int_0^{2\pi} \{(2\cos\theta - \sin^3\theta)(-\sin\theta\, d\theta) - \cos\theta\sin\theta\cos\theta\, d\theta\}$$

Let,

$x = \cos\theta$

$y = \sin\theta$

$$= \int_0^{2\pi} \{\sin^4\theta - 2\cos\theta\sin\theta - \cos^2\theta\sin\theta\} d\theta$$

$$= \int_0^{2\pi} \sin^4\theta \, d\theta - \int_0^{2\pi} \sin 2\theta \, d\theta - \int_0^{2\pi} \cos^2\theta \sin\theta \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{4}(1-\cos 2\theta)^2 d\theta + \left[\frac{\cos 2\theta}{2}\right]_0^{2\pi} + \int_1^1 z^2 dz$$

$$= \int_0^{2\pi} \frac{1}{4}(1+\cos^2 2\theta - 2\cos 2\theta) d\theta + \frac{1}{2}(1-1) + \left[\frac{z^3}{3}\right]_1^1$$

Let, $z = \cos\theta$
$dz = -\sin\theta \, d\theta$

z	1	1
θ	0	2π

$$= \int_0^{2\pi} \frac{1}{4} d\theta + \int_0^{2\pi} \frac{1}{4}\cos^2 2\theta \, d\theta - \int_0^{2\pi} \frac{1}{2}\cos 2\theta \, d\theta + \frac{1}{3}(1-1)$$

$$= \frac{1}{4} 2\pi + \frac{1}{8}\int_0^{2\pi} (1+\cos 4\theta) d\theta + \frac{1}{2}\left[\frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2}\pi + \frac{1}{8}2\pi + \frac{1}{8}\int_0^{2\pi} \cos 4\theta\, d\theta + \frac{1}{4}(0-0)$$

$$= \frac{1}{2}\pi + \frac{1}{4}\pi + \frac{1}{8}\left[\frac{\sin 4\theta}{4}\right]_0^{2\pi}$$

$$= \frac{2\pi+\pi}{4} + \frac{1}{32}(0-0)$$

$$= \frac{3\pi}{4}$$

$\therefore$ Thus the line integral closed by two circles is $= \frac{243}{4}\pi - \frac{3\pi}{4}$

$$= \frac{240}{4}\pi = 60\pi$$

By Green's theorem,

$$\oint_c (2x-y^3)dx - xy\,dy = \iint_R (-y+3y^2)\,dx\,dy$$

Along the circle $x^2+y^2=9$

$$= \int_{x=-3}^{3}\int_{y=-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (-y+3y^2)\,dx\,dy$$

$$= \int_{x=-3}^{3} \left[y^3 - \frac{y^2}{2} \right]_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dx$$

$$= \int_{x=-3}^{3} \left\{ (9-x^2)\sqrt{9-x^2} - \frac{9-x^2}{2} + (9-x^2)\sqrt{9-x^2} + \frac{9-x^2}{2} \right\} dx$$

$$= \int_{x=-3}^{3} 2(9-x^2)^{3/2} \, dx$$

$\therefore x = 3\cos\theta$

$\therefore dx = -3\sin\theta \, d\theta$

x	3	-3
θ	0	2π

$$= \int_{2\pi}^{0} 2(9 - 9\sin^2\theta)^{3/2} (-3\sin\theta \, d\theta)$$

$$= \int_{2\pi}^{0} 9^{3/2} (1-\sin^2\theta)^{3/2} (-3\sin\theta) \, d\theta$$

$$= \int_{2\pi}^{0} 27(\cos^2\theta)^{3/2} (-3\sin\theta \, d\theta)$$

$$= -81 \int_{2\pi}^{0} \cos^3\theta \sin\theta \, d\theta$$

$\cos\theta = z$

$\therefore dz = -\sin\theta \, d\theta$

θ	0	2π
z	1	1

$$\cancel{= 81\int_1^1}$$

$$= 4\int_0^3 (9-x^2)^{3/2}\,dx$$

Let $x = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$

$$= 4\cdot 3^3\cdot 3\int_0^{\pi/2} \cos^4\theta\, d\theta$$

$$= 4\times 81\ \frac{\Gamma 5/2\ \Gamma 1/2}{2\Gamma 3}$$

$$= 81\times 4\times \frac{3\pi}{16}$$

$$= \frac{243\pi}{4}$$

Again along the circle $x^2+y^2=1$

$$= \int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (-y+3y^2)\,dx\,dy$$

$$= \int_{x=-1}^{1} \left[-\frac{y^2}{2}+y^3\right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$$

$= 2\int_{-1}^{1} (1-x^2)^{3/2} dx$

$= 4\int_{0}^{1} (1-x^2)^{3/2} dx$

$= 4\int_{0}^{\pi/2} \cos^4\theta \, d\theta$ Let, $x = \sin\theta$, $dx = \cos\theta \, d\theta$

$= 4 \cdot \dfrac{\Gamma 5/2 \, \Gamma 1/2}{2\Gamma 3}$

$= 4$

$= 4 \times \dfrac{3\pi}{16}$

Hence the area is (by Green's theorem

$= \dfrac{243\pi}{4} - \dfrac{3\pi}{4}$

$= 60\pi$

11E

11-E
14052017

Verify the divergence theorem for $\vec{A} = 2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k}$ taken over the region in the first octant bounded by $y^2 + z^2 = 9$ and $x = 2$

Sol^n Volume integration $\iiint_V \nabla \cdot \vec{A}\, dv = \int_{z=0}^{3} \int_{y=0}^{\sqrt{9-z^2}} \int_0^2 (4xy - 2y + 8xz)\, dx\,dy\,dz$

$$= \int_{z=0}^{3} \int_{y=0}^{\sqrt{9-z^2}} (4y + 16z)\, dy\,dz$$

$$= \int_{z=0}^{3} \left\{ 2(9-z^2) + 16z\sqrt{9-z^2} \right\} dz$$

$$= \left[18z - \frac{2z^3}{3}\right]_{z=0}^{3} + \int_{t=9}^{0} -8\sqrt{t}\,dt$$

$$= 54 - 18 - 8 \times \frac{2}{3}\left[t^{3/2}\right]_{t=9}^{0} \quad \Big| \quad 9 - z^2 = t, \quad 2z\,dz = -dt$$

$= 180$

The surface S of the region is consists by the bone $S_1 (r=0)$ the ~~to~~ top $S_2 (r=2)$ and the convex ~~po~~ portion $(y^2+z^2) = 9$

So; Surface integration $\iint_S \vec{A}\,\hat{n}\,ds$

$$= \iint_{S_1} + \iint_{S_2}$$

$$+\iint_{S_3}$$

For the surface S_1 unit normal $\boxed{\hat{n}=-\hat{i}}$

and at $x=0$ then $\vec{A}=-y\hat{j}$

$$\therefore \iint_{S_1} \vec{A}\cdot\hat{n}\, dS_1 = 0$$

For surface S_2 unit normal $\hat{n}=\hat{i}$

and at $x=2$ $\vec{A} = 8y\hat{i} - y^2\hat{j} + z^2\hat{k}$

$$\therefore \vec{A}\cdot\hat{n} = 8y$$

$$\therefore \iint_{S_2} \vec{A}\cdot\hat{n}\, dS_2 = \int_{z=0}^{3}\int_{y=0}^{\sqrt{9-z^2}} 8y\, dy\, dz$$

$$= \int_{z=0}^{3} 4(9-z^2)\, dz$$

$$= \left[4\left(9z - \frac{z^3}{3}\right)\right]_{z=0}^{3} = 72$$

For surface S_3 unit normal

$$\hat{n} = \frac{2y\hat{j} + 2z\hat{k}}{\sqrt{4y^2+4z^2}}$$

$$= y/3\,\hat{j} + z/3\,\hat{k}$$

$$\vec{A}\cdot\hat{n} = -y^3/3 + 4/3\, xz^3$$

Let, $y = 3\cos\theta$ and $z = 3\sin\theta$

$\therefore dS_3 = 3\,d\theta\,dx$ and

$x \to 0, 2$ then $\theta \to 0, \pi/2$

So, $$\iint_{S_3} \vec{A}\cdot\hat{n}\,dS = \iint_R \left\{-y^3/3 + 4/3\, xz^3\right\} 3\,dx\,d\theta$$

$$= 3\int_{\theta=0}^{\pi/2}\int_{x=0}^{2}\left\{\frac{-27\cos^3\theta}{3} + \frac{4}{3}x\,27\sin^3\theta\right\} dx\,d\theta$$

$$= 3\int_{\theta=0}^{\pi/2} \left\{ -9r\cos^3\theta + \frac{36}{2} r^2 \sin^3\theta \right\} \Big]_{r=0}^{2} d\theta$$

$$= 3\int_{\theta=0}^{\pi/2} \left\{ -18\cos^3\theta + 72\cos^3\theta \right\} d\theta$$

$$= 108$$

$$\therefore \oiint_S \vec{A}\cdot\hat{n}\,ds = 0 + 72 + 108 = 180$$

Since, $\therefore$ Surface integral = volume integral

Hence the divergence theorem is verified.

Evalute $\iint_S \vec{r} \cdot \hat{n} \, ds$

where (a) S is the sphere of radius 2 with centre at (0,0,0)

(b) S is the surface of the cube bounded by $x=-1, y=-1, z=-1$; $x=1, y=1, z=1$

(c) S is the surface bounded by the paraboloid $z = 4-(x^2+y^2)$ and the xy plane

Ans (a) $32\pi^3$ (b) 24. (c) 24π

b) Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\therefore \iint_S \vec{r}\,\hat{n}\,dS = \iiint_V \nabla\cdot\vec{r}\,dv$$

$$= \iiint_V 3\,dv$$

$$= \int_{x=-1}^{1}\int_{y=-1}^{1}\int_{z=-1}^{1} 3\,dx\,dy\,dz$$

$$= \int_{x=-1}^{1}\int_{y=-1}^{1} [3z]_{-1}^{1}\,dx\,dy$$

$$= \int_{x=-1}^{1}\int_{y=-1}^{1} (3+3)\,dx\,dy$$

$$= \int_{x=-1}^{1} [6y]_{-1}^{1}\,dx$$

$$= \int_{x=-1}^{1} 12\,dx$$

$= [12x]_{-1}^{1}$

$= 12 - (-12)$

$= 24$

Ans

c) By divergence theorem,

$$\iint_S \vec{r} \cdot \hat{n}\, dS = \iiint_V \nabla \cdot \vec{r}\, dV$$

$$= \iiint_V 3 \; dx\, dy\, dz$$

Now in xy plane $z=0$

$\Rightarrow x^2 + y^2 = 4$

Let, $x = r\cos\theta$

$y = r\sin\theta$

and the limit $\theta : 0$ to 2π

limit of $r = 0$ to 2

limit of $z = 0$ to $4-(x^2+y^2) = 4-(r^2\cos^2\theta + r^2\sin^2\theta)$

$= 4 - r^2$

Now, $\iint_S \vec{r}\cdot\hat{n}\, ds = 3\int_{r=0}^{2}\int_{\theta=0}^{2\pi}\int_{z=0}^{4-r^2} r\,dr\,d\theta\,dz$

$= 3\int_{r=0}^{2}\int_{z=0}^{4-r^2}[r\theta]_0^{2\pi}\,dr\,dz$

$= 6\pi\int_{r=0}^{2}[rz]_0^{4-r^2}\,dr$

$= 6\pi\int_{r=0}^{2} r(4-r^2)\,dr$

$= 6\pi\int_{r=0}^{2} 4r - r^3\,dr$

$= 6\pi\left[\frac{4r^2}{2} - \frac{r^4}{4}\right]_0^2$

$= 6\pi(2\times2^2 - 4)$

$= 6\pi(8-4) = 24\pi$ Ans

$\cancel{\frac{4}{3}\pi r^3}$ $\cancel{\frac{4}{3}\pi 2^3}$ $\cancel{\frac{32}{3}\pi}$ $\cancel{V = \frac{4}{3}\pi r^3}$

a) $\iint_S \vec{r} \cdot \hat{n} \, dS$

$= \iiint_V \nabla \cdot (x\hat{i} + y\hat{j} + z\hat{k}) \, dV$

$= \iiint_V 3 \, dV$

$= 3V$

$= 3 \cdot \frac{4}{3} \pi r^3$

$= 4\pi \, 2^3$

$= 32\pi$

Ans

Extra

Verify the divergence theorem for $A = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ taken over the region bounded by $x^2+y^2=4$, $z=0$ and $z=3$.

Soln: volume integral $= \iiint_V \nabla \cdot A \, dv$

$$= \iiint_V \left[\frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)\right] dv$$

$$= \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{z=0}^{z=3} (4 - 4y + 2z) \, dx\, dy\, dz$$

$$= \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4z - 4yz + z^2\right]_0^3 dx\, dy$$

$$= \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (12 - 12y + 9) \, dx\, dy$$

$$= \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 21 - 12y \, dx\, dy$$

$$= \int_{x=-2}^{2} \left[2\theta y - 6y^2 \right]_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{x=-2}^{2} 2\theta \left(\sqrt{4-x^2} + \sqrt{4-x^2}\right) dx - \int_{x=-2}^{2} 6 \left(4-x^2 - 4 + x^2\right) dx$$

$$= \int_{x=-2}^{2} 2\theta \cdot 2\sqrt{4-x^2}\, dx - \int_{x=2}^{2} 0\, dx$$

$$= 4\theta \int_{x=2}^{2} \sqrt{4-x^2}\; dx$$

Let, $x = 2\sin\theta$

$\therefore dx = 2\cos\theta\, d\theta$

x	2	-2
θ	$\pi/2$	$-\pi/2$

$$= 4\theta \int_{-\pi/2}^{\pi/2} \sqrt{4 - 4\sin^2\theta}\; 2\cos\theta\, d\theta$$

$$= 4\theta \int_{-\pi/2}^{\pi/2} 4\cos^2\theta\, d\theta$$

$$= 4\theta \int_{-\pi/2}^{\pi/2} 2\left(1 + \cos 2\theta\right) d\theta$$

$$= 84 \int_{-\pi/2}^{\pi/2} 1 + \cos 2\theta\, d\theta$$

$$= 84 \left[\theta + \frac{\sin 2\theta}{2}\right]_{-\pi/2}^{\pi/2}$$

$$= 84\left(\pi/2 + 0 + \pi/2 + 0\right)$$

$$= 84 \cdot 2\frac{\pi}{2}$$

$$= 84\pi$$

Surface integral $= \iint_S \vec{A}\cdot\hat{n}\, ds$

$$= \iint_{S_1} \vec{A}\cdot\hat{n}\, dS_1 + \iint_{S_2} \vec{A}\cdot\hat{n}\, dS_2 + \iint_{S_3} \vec{A}\cdot\hat{n}\, dS_3$$

On

The surfaces S of the cylinder consists of a base $S_1 (z=0)$, the top $S_2 (z=3)$ and the convex portion $S_3 (x^2+y^2=4)$

On S_1 $(z=0)$, $n = -k$, $A = 4x\hat{i} - 2y^2\hat{j}$

$$\therefore \vec{A}\cdot\hat{n} = 0 \quad \therefore \iint_{S_1} \vec{A}\cdot\hat{n}\, dS_1 = 0$$

On S_2 $(z=3)$, $n = k$ $\quad A = 4x\hat{i} - 2y^2\hat{j} + 9\hat{k}$

$$\therefore A\cdot\hat{n} = 9. \quad \therefore \iint_{S_2} \vec{A}\cdot\hat{n}\, dS_2 = \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 9\, dx dy$$

$$= 9\int_{x=-2}^{2} \left[y\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= 9\int_{x=-2}^{2} 2\sqrt{4-x^2}\, dx$$

$$= 18\int_{x=-2}^{2} \sqrt{4-x^2}\, dx$$

Let,
$x = 2\sin\theta$
$\therefore dx = 2\cos\theta\, d\theta$

x	-2	2
θ	$-\pi/2$	$\pi/2$

$$= 18\int_{-\pi/2}^{\pi/2} 4\cos^2\theta\, d\theta$$

$$= 36\int_{-\pi/2}^{\pi/2} 1+\cos 2\theta\, d\theta$$

$$= 36\left[\theta + \frac{\sin 2\theta}{2}\right]_{-\pi/2}^{\pi/2}$$

$$= 36\left(\pi/2 + \pi/2\right)$$

$$= 36 \cdot \cancel{2}\frac{\pi}{\cancel{2}}$$

On S_3 $(x^2+y^2=4)$

Perpendicular to $\cancel{\nabla(x^2+y^2)}$ has the direction $x^2+y^2=4$

$$\nabla(x^2+y^2) = 2x\hat{i}+2y\hat{j}$$

$\therefore$ unit normal $\hat{n} = \dfrac{2x\hat{i}+2y\hat{j}}{\sqrt{4x^2+4y^2}}$

$$= \frac{x\hat{i}+y\hat{j}}{\sqrt{x^2+y^2}}$$

$$= \frac{1}{2}\, x\hat{i}+y\hat{j}$$

$$\therefore A.\hat{n} = (4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}).\frac{1}{2}(x\hat{i}+y\hat{j})$$

$$= 2x^2 - y^3$$

Let, $x = 2\cos\theta$, $y = 2\sin\theta$

$\therefore$ $dS_3 = 2\,d\theta\,dz$

$$\therefore \iint_S \vec{A}.\vec{n}\,dS = \int_{\theta=0}^{2\pi}\int_{z=0}^{3} 8\cos^2\theta - 8\sin^3\theta \; 2\,d\theta\,dz$$

$$= \int_{\theta=0}^{2\pi} (48\cos^2\theta - 48\sin^3\theta)\, d\theta$$

$$= \int_{0}^{2\pi} 48\cos^2\theta\, d\theta$$

$$= 48\pi$$

$\therefore$ Total surface integral $= 0 + 36\pi + 48\pi$

$= 84\pi$

Hence divergence theorem verified.

12D

12-D
20-05-2017

Verify stokes theorem for

$\vec{A} = (\cancel{x-y+z}\; y-z+2)\hat{i} + (yz+4)\hat{j} - xz\hat{k}$ where

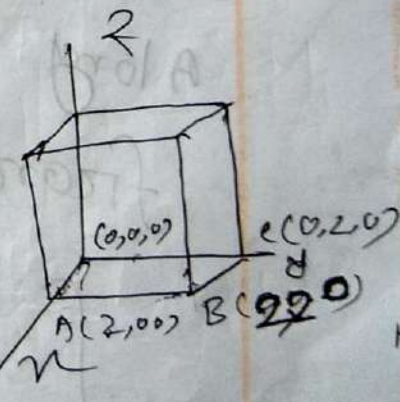
S is the surface of the cube

$x=0,\ y=0,\ z=0,\ x=2,\ y=2,\ z=0$ above

the xy plane.

$\Rightarrow$ we know stokes theorem is

$$\oint_C \vec{A}\, dr = \iint_S (\nabla \times \vec{A}) \cdot \hat{n}\, ds$$



Now $\oint_C \vec{A}\, dr = \oint_C (y-z+2)\,dx + (yz+4)\,dy - xz\,dz$

$= \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$

Along OA $y = z = 0$ and x varies 0 to 2

$$\therefore \int_{OA} = \int_{x=0}^{2} 2\,dx = 4$$

Along AB $x = 2, z = 0$, y varies from 0 to 2

$$\int_{AB} = \int_{0}^{2} 4\,dy = 8$$

Along BC $y = 2, z = 0$ x varies from 2 to 0

$$\int_{BC} = \int_{2}^{0} 4\,dx = -8$$

Along CO $x = 0, z = 0$ y varies from 2 to 0

y

$z = 0$ $\quad \hat{n} = \hat{k}$

$$\int_{C_3} = \int_2^0 4\,dy = -8$$

$$\therefore \oint_C \vec{A}\cdot d\vec{r} = 4+8-8-8 = -4$$

and to find $\iint_S (\nabla\times\vec{A})\cdot\hat{n}\, dS$,

$$\nabla\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z+2 & yz+4 & -xz \end{vmatrix}$$

$$= -y\hat{i} + (z-1)\hat{j} \cancel{+} -\hat{k}$$

$$\therefore (\nabla\times\vec{A})\cdot\hat{n} = \{-y\hat{i} + (z-1)\hat{j} \cancel{+} -\hat{k}\}\cdot\hat{n}$$

$$= -k\cdot\hat{n}$$

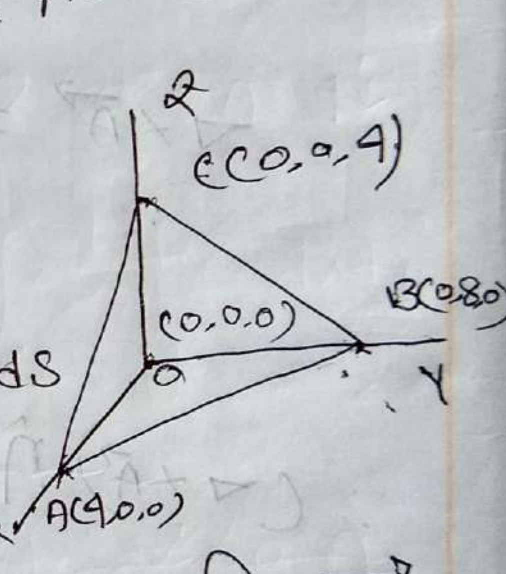
$$\therefore \iint_S (\nabla\times\vec{A})\cdot\hat{n}\, dS = \iint (-k\cdot\hat{n}) \frac{dx\,dy}{|n\cdot\hat{k}|}$$

$$= -\int_{x=0}^{2}\int_{y=0}^{2} dx\,dy = -4$$

Hence the Stokes theorem is verified

Verify Stockes theorem for $\vec{F} = xz\hat{i} - y\hat{j} + x^2y\hat{k}$ where S is the surface of the region bounded by $x=0$, $y=0$, $z=0$, $2x+y+2z=8$ which is not included in the ~~xz~~ xy plane.

Answer: $\frac{32}{3}$



Sol^n:

By Stocks theorem,

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\, dS$$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_{OC} \vec{F}\cdot d\vec{r} + \int_{CA} \vec{F}\cdot d\vec{r} + \int_{AO} \vec{F}\cdot d\vec{r}$$

$\cancel{\vec{F}\cdot dx =}$

Along OC $x=0$, $y=0$, z varies from 0 to 4

$$\therefore \int_{OC} \vec{F}\cdot d\vec{r} = \int_0^4 0\cdot dz = 0$$

Along CA ~~x=0,~~ y=0 & x veries from 0 to 4

$$\int_{CA} \vec{F}\cdot d\vec{r} = \int_{x=0}^{4} x(4-x)\,dx = \int_0^4 4x - x^2\,dx$$

$$= \left[4\frac{x^2}{2} - \frac{x^3}{3}\right]_0^4$$

$$= 2\times 16 - \frac{64}{3}$$

$$= \frac{96-64}{3} = \frac{32}{3}$$

$\frac{32}{3}$ 96

Along AO, y=0, z=0, x veries from 0 to 4.

$$\int_{AO} \vec{F}\cdot d\vec{r} = \int_{x=0}^{4} 0\,dx = 0$$

$$\therefore \oint_C \vec{F}\cdot d\vec{r} = 0 + 32/3 + 0 = 32/3$$

Now $\iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds = \iint_{OAB} (\nabla\times\vec{F})\cdot\hat{n}\,ds$

$+ \iint_{OBC} (\nabla\times\vec{F})\hat{n}\,ds + \iint_{ABC} (\nabla\times\vec{F})\cdot\hat{n}\,ds$

For the face OAB $z=0$, $n=\hat{k}$

$\therefore \iint_{OAB} (\nabla\times\vec{F})\cdot\hat{n}\,ds = \int_{x=0}^{4}\int_{y=0}^{8} 0\,dx\,dy$
$= 0$

For the face OBC $x=0$ $\hat{n}=-i$

$\therefore \iint_{OBC} (\nabla\times\vec{F})\cdot\hat{n}\,ds = \int_{y=0}^{8}\int_{z=0}^{4} -x^2\,dy\,dz$
$= 0$

For the face ABC, $dS = \dfrac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$

$$\nabla(2x+y+2z) = 2\hat{i}+\hat{j}+2\hat{k}$$

$$\therefore \hat{n} = \frac{2\hat{i}+\hat{j}+2\hat{k}}{\sqrt{4+1+4}} = \frac{2}{3}\hat{i}+\frac{1}{3}\hat{j}+\frac{2}{3}\hat{k}$$

$\therefore (\nabla\times\vec{F})\,\hat{n} = \{$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & -y & x^2 y \end{vmatrix}$$

$$\cancel{= x\hat{i} + xz}\,(x-2xy)\hat{j} + \hat{k}(0-0)$$

$$= x^2\hat{i} + (x-2xy)\hat{j}$$

$$\therefore (\nabla\times\vec{F})\cdot\hat{n} = \{x^2\hat{i} + (x-2xy)\hat{j}\} \cdot \{2/3\,\hat{i} + 1/3\,\hat{j} + 2/3\,\hat{k}\}$$

$$= \frac{2x^2\hat{i} + (x-2xy)\hat{j}}{3}$$

$$\therefore \hat{n}\cdot\hat{k} = 2/3$$

$$\iint_{ABC} (\nabla \times \vec{F}) \cdot \hat{n}\, ds = \iint_R \frac{2x^2+x-2xy}{3} \cdot \frac{dx\,dy}{2/3}$$

$$= \frac{1}{2} \int_{x=0}^{4} \int_{y=0}^{8-2x} (2x^2+x-2xy)\, dx\,dy$$

$$= \frac{1}{2} \int_{x=0}^{4} \left[2x^2 y + xy - xy^2\right]_0^{8-2x} dx$$

$$= \frac{1}{2} \int_{x=0}^{4} 2x^2(8-2x) + (8-2x)x - x(8-2x)^2\, dx$$

$$= \frac{1}{2} \int_{x=0}^{4} 16x^2 - 4x^3 + 8x - 2x^2 - x(64+4x^2-32x)\, dx$$

$$= \frac{1}{2} \int_{x=0}^{4} 16x^2 - 4x^3 + 8x - 2x^2 - 64x - 4x^3 + 32x^2\, dx$$

$$= \frac{1}{2}\int_{x=0}^{4} 46x^2 - \cancel{8x} - 8x^3 - 56x \, dx$$

$$= \frac{1}{2} \left[46\frac{x^3}{3} - 8\frac{x^4}{4} - 56\frac{x^2}{2}\right]_0^4$$

$$= \frac{1}{2}\left\{\frac{46}{3}(4^3) - 2(4^4) - 28\times 16\right\}$$

$$= \frac{1}{2}\left(\frac{2944}{3} - 512 - 448\right)$$

$$= \frac{1}{2}\,\frac{2944 - 1536 - 1344}{3}$$

$$= \frac{64}{2\times 3} = \frac{32}{3}$$

Hence Stoeks theorem verified.