Linear partial diff. eq^n

Farhan Shakib
1603084

Origin of partial diff. eq^n:-

Rule-1:-

Derivation of p.d.e. by elemination of arbitrary constants.

Ex- Find a partial diff. eq^n by eleminating a and b from the equation $z = ax+by+a^2+b^2$

Soln:-

Given, $z = ax+by+a^2+b^2$ ———— (i)

Diff. (i) partially wrt x and y we get-

$\frac{\partial z}{\partial x} = a \Rightarrow p = a \quad \therefore a = p$

And, $\frac{\partial z}{\partial y} = b \Rightarrow q = b \quad \therefore b = q$

Putting values in (i) we get-

$z = px + qy + p^2 + q^2$ (Ans.)

Ex- $z = (x-a)^2 + (y-b)^2$

Soln:-

Given, $z = (x-a)^2 + (y-b)^2$ ———— (1)

Diff. (1) p. wrt. x and y -

$\frac{\partial z}{\partial x} = 2(x-a) \quad \therefore (x-a) = p/2$

And, $\frac{\partial z}{\partial y} = 2(y-b) \quad \therefore (y-b) = q/2$

Putting in (1) we get - $z = p^2/4 + q^2/4$ (Ans.)

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Ex-| $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$

Solⁿ:-

Given- $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ —(1)

Diff. (1) p. wrt. x, y –

$2x/a^2 + 2z/c^2 \, \partial z/\partial x = 0$ ~~—(2)~~ $\therefore c^2x + a^2z \, \partial z/\partial x = 0$ —(2)

And, $2y/b^2 + 2z/c^2 \, \partial z/\partial y = 0$ ~~—(3)~~ $\therefore c^2y + b^2z \, \partial z/\partial y = 0$ —(3)

Diff. (2) and (3) p. wrt. x and y respectively—

$c^2 + a^2(\partial z/\partial x)^2 + a^2z \, \partial^2 z/\partial x^2 = 0$ —(4)

And, $c^2 + b(\partial z/\partial y)^2 + a^2z \, \partial^2 z/\partial y^2 = 0$ —(5)

From (2) –

$c^2 = -a^2z/x \, \partial z/\partial x$ —(6)

From (3) –

$c^2 = -a^2z/y \, \partial z/\partial y$ —(7)

From (4) and (6) –

$-a^2z/x \, \partial z/\partial x + a^2(\partial z/\partial x)^2 + a^2z \, \partial^2 z/\partial x^2 = 0$

$\Rightarrow -z/x \, \partial z/\partial x + (\partial z/\partial x)^2 + z \, \partial^2 z/\partial x^2 = 0$

$\Rightarrow zx \, \partial z/\partial x - x(\partial z/\partial x)^2 - zx \, \partial^2 z/\partial x^2 = 0$ (Ans.)

Similarly from (5) and (7) –

$zy \, \partial z/\partial y - y(\partial z/\partial y)^2 - zy \, \partial^2 z/\partial y^2 = 0$ (Ans.)

Rule-2:–

Derivation of P.D.E. by the elimination of arbitrary f^n ϕ from the equation $\phi(u,v)=0$, where u and v are f^ns of x, y and z

Proof:–

Given, $\varphi(u,v)=0$ ——— (1)

We treat z as dependant variable and x, y as independent variable–

$\therefore \partial z/\partial x = p,\ \partial z/\partial y = q,\ \partial x/\partial y = 0,\ \partial y/\partial x = 0$

Diff. (1) P. w.r.t. x–

$$\frac{\partial \varphi(u,v)}{\partial x} = \frac{\partial \varphi}{\partial u}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right) + \frac{\partial \varphi}{\partial v}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right)$$

$$= \frac{\partial \varphi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial \varphi}{\partial v}\left(\frac{\partial v}{\partial x} + q\frac{\partial v}{\partial z}\right)$$

$$\therefore \frac{\frac{\partial \varphi}{\partial u}}{\frac{\partial \varphi}{\partial v}} = \frac{-\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right)}{\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}} \quad \text{——— (2)}$$

Similarly, diff. (1) P. w.r.t. y–

$$\frac{\frac{\partial \varphi}{\partial u}}{\frac{\partial \varphi}{\partial v}} = \frac{-\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right)}{\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}} \quad \text{——— (3)}$$

From (2) and (3)–

$$\frac{\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}}{\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}} = \frac{\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}}{\frac{\partial u}{\partial y} + q\frac{\partial v}{\partial z}}$$

Or, $Pp + Qq = R$

where,

$$P = \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y}$$

$$Q = \frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}$$

$$R = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

Ex-1 $f(x+y+z,\ x^2+y^2+z^2) = 0$

Soln:-

Given, $f(x+y+z,\ x^2+y^2+z^2) = 0$ —— (1)

Let, $f(u,v) = 0$ —— (2)

where, $u = x+y+z$

$v = x^2+y^2+z^2$

Diff. (2) p. wrt. x —

$$\partial f/\partial u\left(\partial u/\partial x + p\,\partial u/\partial z\right) + \partial f/\partial v\left(\partial v/\partial x + q\,\partial v/\partial z\right) = 0$$

$$\Rightarrow \partial f/\partial u\,(1+p) + \partial f/\partial v\,(2x + 2pz)$$

$$\therefore \frac{\partial f/\partial u}{\partial f/\partial v} = \frac{-(2x+2pz)}{1+p} \quad \text{—— (3)}$$

Similarly, $$\frac{\partial f/\partial u}{\partial f/\partial v} = \frac{-(2y+2qz)}{1+q} \quad \text{—— (4)}$$

From (3) and (4) —

$$\frac{2x+2pz}{1+p} = \frac{2y+2qz}{1+q}$$

$$\Rightarrow \frac{x+pz}{1+p} = \frac{y+qz}{1+q}$$

$$\Rightarrow x+pz+qx+pqz = y+qz+py+pqz$$

$$\Rightarrow (z-y)p+(x-z)q = y-x$$

$$\therefore (y-z)p+(z-x)q = (x-y)$$

(Ans.)

$$\boxed{\begin{array}{c}f(u,v) \text{ can be written as} \\ u=f(v)\end{array}}$$

Soln of the above problem:-

Given equation -

$$(x+y+z) = f(x^2+y^2+z^2) \quad \text{——— (1)}$$

Diff. (1) partially wrt. x -

$$\cancel{1+f} \quad 1+p = f'(x^2+y^2+z^2)(2x+2zp) \quad \text{——— (2)}$$

Similarly,

$$1+q = f'(x^2+y^2+z^2)(2z+2zq) \quad \text{——— (3)}$$

(2) ÷ (3)

$$\frac{1+p}{1+q} = \frac{x+pz}{y+qz}$$

$$\therefore (y-z)p+(z-x)q = (x-y)$$

(Ans.)

Ex-1 $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$ ——— (1)

Soln:-

Diff. (1) p. w.r.t. x —

$$p = 2f'\left(\frac{1}{x} + \log y\right)\left(-\frac{1}{x^2}\right)$$

$$\therefore \frac{1}{2}p = -\frac{1}{x^2} f'\left(\frac{1}{x} + \log y\right) \text{ ——— (2)}$$

Diff. (1) p. w.r.t. y —

$$q = 2y + 2f'\left(\frac{1}{x} + \log y\right) \cdot \frac{1}{y}$$

$$\therefore \frac{q - 2y}{2} = \frac{1}{y} f'\left(\frac{1}{x} + \log y\right) \text{ ——— (3)}$$

(2) ÷ (3)

$$\frac{p}{q - 2y} = -\frac{y}{x^2}$$

$$\Rightarrow px^2 = 2y^2 - qy$$

$$\therefore x^2 p + yq = 2y^2$$

(Ans)

Solving partial diff. eqⁿ:-

Lagrange's method:-

* $Pp + Qq = R$ is known as Lagrange's eqⁿ.

~~Rule-I:-~~

(1) Put the given linear p.diff. eqn. of the first order in the standard form:- $Pp + Qq = R$ —— (i)

(2) Write down Lagrange's auxiliary equations for (1):-

$$dx/P = dy/Q = dz/R \quad \text{—— (ii)}$$

(3) Solve (ii). $u = c_1$, $v = c_2$

(4) The general solⁿ (or integral) of (i) is then written in $\phi(u,v) = 0$ or, $u = \phi(v)$ or, $v = \phi(u)$

Ex-1 $xp + yq = z$ —— (1)

Rule-I:-

Suppose that one of the variables are either absent or cancels out from any two fractions of given eqn.- $dx/P = dy/Q = dz/R$. Then an integral can be obtained by the usual methods. The same process can be repeated with another two fractions

Solⁿ:-

Given, $(x)p + (y)q = (z)$ —— (2)

Lagrange's auxiliary equation-

$$dx/P = dy/Q = dz/R$$

$$\Rightarrow dx/x = dy/y = dz/z \quad \text{—— (3)}$$

Taking first two ratios-

$$dx/x = dy/y$$

Integrating-

$$\ln x = \ln y + \ln c_1$$

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$\therefore c_1 = x/y = u$

Taking last two ratios –

$$dy/y = dz/z$$

Integrating –

$$\ln y = \ln z + \ln c_2$$

$$\therefore c_2 = y/z = v$$

$\therefore$ General soln of the eqn. is –

$$f(x/y, y/z) = 0 \quad \text{(Ans.)}$$

Rule – 2:–

Suppose that one integral of $dx/P = dy/Q = dz/R$ --(1) is known by using Rule-1. But another ~~root~~ cannot be obtained from Rule 1. Then one integral known is used to obtain the another integral.

Ex– Solve $z(z^2 + xy)(px - qy) = x^4$ ______ (1)

Soln:–

$$zx(z^2+xy)p - yz(z^2+xy)q = x^4 \quad \text{______ (2)}$$

According to L.A.E. –

$$\frac{dx}{zx(z^2+xy)} = \frac{dy}{-yz(z^2+xy)} = \frac{dz}{x^4}$$

From first two ratios—

$$dx/x = -dy/y$$

Integrating—

$$\ln x = -\ln y + \ln c_1$$

$$\Rightarrow \ln x + \ln y = \ln c_1$$

$$\therefore xy = c_1 \quad \text{———————— (3)}$$

From 1st and 3rd ratios—

$$\frac{dx}{xz(z^2+xy)} = \frac{dz}{x^4}$$

$$\Rightarrow \frac{dx}{z(z^2+c_1)} = \frac{dz}{x^3} \quad [\text{From (3)}]$$

$$\Rightarrow x^3 dx = (z^3 + zc_1) dz$$

Integrating—

$$\frac{1}{4}x^4 = \frac{1}{4}z^4 + \frac{1}{2}z^2 c_1 + c$$

$$\Rightarrow x^4 - z^4 - 2z^2 xy = c_2 \quad \text{—— (4)}$$

From (3) and (4), the required general solution is—

$$\varphi(xy,\ x^4 - z^4 - 2xyz^2) = 0$$

(Ans.)

* Rule-3:-

Let P_1, Q_1 and R_1 be functions of x, y and z. Then, by a well known principle of algebra, each function in $dx/P = dy/Q = dz/R$ will be equal to — (1)

$$\frac{P_1dx + Q_1dy + R_1dz}{PP_1 + QQ_1 + RR_1} \quad \text{———} (2)$$

If $PP_1 + QQ_1 + RR_1 = 0$, then we know that the numerator of (2) is also zero. This gives, $P_1dx + Q_1dy + R_1dz = 0$, which can be integrated to give $u_1(x, y, z) = c_1$. The method may be repeated to get another integral $u_2(x, y, z) = c_2$. P_1, Q_1, R_1 are called multipliers. As a special case, these can be constants also. Sometimes, only one integral is possible by use of multiple by use of multipliers. In such cases, second integral should be obtained by using Rule-1 or Rule-2.

Ex-| $z(x+y)p + z(x-y)q = x^2+y^2$ ——— (1)

Soln:-

According to L.A.E.-

$$\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{x^2+y^2} \text{ ———} (2)$$

Choosing $x, -y, -z$ and $y, x, -z$ as multipliers-

$$\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{x^2+y^2} = \frac{xdx - ydy - zdz}{0} = \frac{ydx + xdy - zdz}{0}$$

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From fourth ratio –

$xdx - ydy - zdz = 0$

Integrating –

$\frac{x^2}{2} + \frac{y^2}{2} - \frac{z^2}{2} = \frac{c}{2}$

$\therefore x^2 - y^2 - z^2 = c_1 = u_1$

* From last ratio –

$ydx + xdy - zdz = 0 \Rightarrow d(xy) - zdz = 0$

Integrating –

$\cancel{xy} + xy - \frac{z^2}{2} = c$

$\therefore 2xy - z^2 = c_2 = u_2$

The required general eqn is given by –

$\phi(x^2 - y^2 - z^2, 2xy - z^2) = 0$

Ex – Solve – $x(y^2 - z^2)p - y(z^2 + x^2)q = z(x^2 + y^2)$ —— (1)

Soln: –

Using LAE –

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{-y(z^2 + x^2)} = \frac{dz}{z(x^2 + y^2)} \quad \text{—— (2)}$$

choosing x, y, z and $1/x, -1/y, -1/z$ –

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{-y(z^2 + x^2)} = \frac{dz}{z(x^2 + y^2)} = \frac{xdx + ydy + zdz}{0}$$

$$= \frac{1/x\,dx - 1/y\,dy - 1/z\,dz}{0}$$

From fourt fraction-

$$xdx+ydy+zdz=0$$

Integrating -

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=\frac{c}{2}$$

$$\therefore x^2+y^2+z^2=c_1=u_1$$

From fifth ratio-

$$\frac{1}{x}dx-\frac{1}{y}dy-\frac{1}{z}dz=0$$

Integrating-

$$\ln x-\ln y-\ln z=\ln c_2$$

$$\Rightarrow \ln \frac{x}{yz}=\ln c_2$$

$$\therefore \frac{x}{yz}=c_2=u_2$$

$\therefore$ Hence, the required general sol^n-

$$\varphi\left(x^2+y^2+z^2, \frac{x}{yz}\right)=0$$

(Ans.)

Rule-4:-

Let P_1, Q_1 and R_1 be functions of x, y, z. Then by a well-known principle of algebra, each fraction of LAE will be equal to —

$$\frac{P_1 dx + Q_1 dy + R_1 dz}{PP_1 + QQ_1 + RR_1} \quad \text{— (1)}$$

numerator

denominator

Suppose the numerator of (1) is exact diff. of denominator of (1). Then (1) can be combined with a suitable fraction in LAE to give an integral. However in some problems, another set of multiples P_2, Q_2, R_2 so chosen that the fraction

$$\frac{P_2 dx + Q_2 dy + R_2 dz}{PP_2 + QQ_2 + RR_2} \quad \text{— (2)}$$

is such that its numerator is exact diff. of denominator. Fractions (1) and (2) are then combined to give an integral. This method may be repeated in some problems to get another integral.

Ex- Solve $(y^2+z^2-x^2)p - 2xyq = -2xz$

or, $(x^2-y^2-z^2)p + 2xyq = 2zx$ — (1)

Soln:-

Using LAE —

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2zx} \quad \text{— (ii)}$$

Choosing x, y, z as multipliers –

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2zx} = \frac{xdx+ydy+zdz}{x(x^2+y^2+z^2)}$$

From second and third ratio –

$$\frac{dy}{2xy} = \frac{dz}{2zx}$$

$$\Rightarrow \frac{dy}{y} = \frac{dz}{z}$$

Integrating –

$$\ln y = \ln z + \ln c_1$$

$$\therefore c_1 = y/z = u_1$$

From last two integrals –

$$\frac{dz}{2zx} = \frac{xdx+ydy+zdz}{x(x^2+y^2+z^2)}$$

$$\Rightarrow \frac{dz}{z} = \frac{2xdx+2ydy+2zdz}{(x^2+y^2+z^2)}$$

$$\Rightarrow \frac{dz}{z} = \frac{d(x^2+y^2+z^2)}{x^2+y^2+z^2}$$

Integrating – $\ln z + \ln c_1 = \ln(x^2+y^2+z^2)$

$$\therefore c_2 = (x^2+y^2+z^2)/z = u_2$$

The required general solⁿ is –

$$\phi\left(y/z, (x^2+y^2+z^2)/z\right) = 0$$

(Ans.)

Ex-1 Solve- $(x^2-yz)p+(y^2-zx)q=(z^2-xy)$ ——— (1)

Soln:-

Using LAE-

$$\frac{dx}{x^2-yz}=\frac{dy}{y^2-zx}=\frac{dz}{z^2-xy} \quad \text{———} (2)$$

Using $1,-1,0$; $0,1,-1$; x,y,z and $1,1,1$ as multipliers-

$$\frac{dx}{x^2-yz}=\frac{dy}{y^2-zx}=\frac{dz}{z^2-xy}=\frac{dx-dy}{(x-y)(x+y+z)}=\frac{dy-dz}{(y-z)(x+y+z)}$$
$$=\frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx}=\frac{xdx+ydy+zdz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

From 4th and 5th ratios-

$$\frac{dx-dy}{x-y}=\frac{dy-dz}{y-z}$$

Integrating-

$$\ln(x-y)=\ln(y-z)+\ln c_1$$

$$\therefore c_1=\frac{x-y}{y-z}=u_1$$

From 6th and 7th ratios-

$$\frac{dx+dy+dz}{1}=\frac{xdx+ydy+zdz}{x+y+z}$$

$$\Rightarrow xdx+ydy+zdz=(x+y+z)(dx+dy+dz)$$

Integrating :-

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=\frac{u^2}{2}+\frac{c}{2}$$

$$\Rightarrow x^2+y^2+z^2=u^2+c$$

$$\Rightarrow c=-2(xy+yz+zx)$$

$$\therefore c_2 = xy+yz+zx$$

$\therefore$ The required general solⁿ :-

$$\phi\left(\frac{x-y}{y-z},\ xy+yz+zx\right)=0$$

Integral surfaces passing through a given curve:-

Method-1:-

Let $Pp + Qq = R$ ———— (1) be the given curve.
Let its auxiliary equations give the following two independent solutions-

$$u(x,y,z) = c_1 \text{ and } v(x,y,z) = c_2 \quad \text{———— (2)}$$

Suppose, we wish to obtain the integral surface which passes through the curve whose eqn in parametric form

$$x = x(t),\ y = y(t),\ z = z(t) \quad \text{———— (3)}$$

Where t is a parameter. Then (2) may be expressed as -

$$u[x(t), y(t), z(t)] = c_1$$
$$v[x(t), y(t), z(t)] = c_2 \quad \text{———— (4)}$$

We eleminate single parameter t from the equation (4) and get a Relation involving c_1 and c_2.
Finally, we replace c_1, c_2 using (2) and obtain the required integral surface.

Method-2:-

Let $Pp+Qq=R$ ——— (1) be the given eq^n. Let its LAEs give the following two integrals (independent) -

$$u(x,y,z)=c_1 \text{ and } v(x,y,z)=c_2 \quad \text{——— (2)}$$

Suppose, we wish to obtain the integral surface passing through the curve which is determined by the two equations -

$$\phi(x,y,z)=0,\ \psi(x,y,z)=0 \quad \text{——— (3)}$$

We eleminate x,y,z ~~from~~ the equations (2) and (3) and obtain a relation between c_1 and c_2. Finally replace c_1 and c_2 by $u(x,y,z)$ and $v(x,y,z)$ respectively and obtain desired integral surface.

Ex:- Find the integral surface of $x^2p+y^2q+z^2=0$, which passes through the hyperbola $xy=x+y,\ z=1$

Sol^n:-

Given, $x^2p+y^2q+z^2=0$

or, $x^2p+y^2q=-z^2$ ——— (1)

Given curve is -

$$x+y=xy$$
$$z=1 \quad \text{——— (2)}$$

Using LAE-

$$\frac{dx}{x^2}=\frac{dy}{y^2}=\frac{dz}{-z^2}$$

Using 1st and 3rd fractions—

$$dx/x^2 = -dz/z^2$$

Integrating—

$$-1/x = +1/z - c_1$$

$$\Rightarrow 1/x + 1/z = c_1 \quad \text{—— (3)}$$

Using 2nd and 3rd fractions—

$$dy/y^2 = -dz/z^2$$

Integrating—

$$-1/y = +1/z - c_2$$

$$\therefore 1/y + 1/z = c_2 \quad \text{—— (4)}$$

(3) + (4)

$$1/x + 1/y + 2/z = c_1 + c_2$$

$$\Rightarrow x+y/xy + 2/z = c_1 + c_2$$

$$\Rightarrow xy/xy + 2/1 = c_1 + c_2 \quad [\text{from } (2)]$$

$$\Rightarrow 1 + 2 = c_1 + c_2$$

$$\Rightarrow c_1 + c_2 = 3$$

$$\Rightarrow 1/x + 1/z + 1/y + 1/z = 3 \quad [\text{using (3) and (4)}]$$

$$\Rightarrow 1/x + 1/y + 2/z = 3$$

$$\Rightarrow 2xy + yz + zx = 3xyz$$

(Ans.)

Non-linear PDE

Charpit's method:-

(1) Transfer all terms of given eq^n to LHS and denote the entire eq^n by f.

(2) Write down the Charpit's A.E.

$$\frac{dp}{\partial f/\partial x + p\,\partial f/\partial z} = \frac{dq}{\partial f/\partial y + q\,\partial f/\partial z} = \frac{dz}{-p\,\partial f/\partial p - q\,\partial f/\partial q} = \frac{dx}{-\partial f/\partial p} = \frac{dy}{-\partial f/\partial q}$$

(3) Determine $\partial f/\partial x$, $\partial f/\partial y$ and put is CAE.

(4) Select two proper fractions so that the resulting integral may come out to be the simplest relation involving at least one of p and q.

(5) The simplest relation of step-4 is solved along with the given equation to determine p and q. put these values of p and q in $dz = p\,dx + q\,dy$ which on integration gives the complete integral of the given eq^n.

Ex- $px + qy = pq$

Sol^n:-

Here given eq^n is $f(x,y,z,p,q) \equiv px + qy - pq = 0$ ——— (1)

Using CAE-

$$\frac{dx}{-(x-q)} = \frac{dy}{-(y-q)} = \frac{dz}{-p(x-q) - q(y-p)} = \frac{dp}{p}$$

$$= \frac{dq}{q} \quad \text{——— (2)}$$

Taking first two ratios–

$$\frac{dp}{p} = \frac{dq}{q}$$

Integrating–

$\ln p = \ln q + \ln a$

$\therefore p = aq$ ——— (3)

Putting value of p in (1)–

$ax + y = aq$

$\therefore q = (ax+y)/a$

$\therefore p = ax+y$

Hence, $dz = p\,dx + q\,dy$

$\Rightarrow dz = (ax+y)dx + \frac{ax+y}{a}dy$

$\Rightarrow a\,dz = a(ax+y)\cancel{(ax+}dx + (ax+y)dy$

$\Rightarrow a\,dz = (ax+y)(a\,dx+dy)$

$\Rightarrow a\,dz = t\,dt \quad [t = ax+y]$

Integrating –

$az = t^2/2 + b$

$\therefore az = \frac{(ax+y)^2}{2} + b$

(Ans.)

Ex – $p^2 - y^2 q = y^2 - x^2$ ——— (1)

Soln: –

The given eqn – $f(x, y, z, p, q) = p^2 - y^2 q - y^2 - x^2 = 0$

Using CAE –

$$\frac{dp}{2x} = \frac{dq}{-(1+q)2y} = \frac{dz}{-p(2p) - q(-y^2)} = \frac{dx}{-2p} = \frac{dy}{y^2}$$

Taking first and fourth fractions –

$$\frac{dp}{2x} = \frac{dx}{-2p}$$

$\Rightarrow \quad p\,dp = -x\,dx$

Integrating –

$$\frac{p^2}{2} = -\frac{x^2}{2} + \frac{a^2}{2}$$

$\therefore \; p = \sqrt{a^2 - x^2}$ ——— (2)

Putting value in (1) –

$$a^2 - x^2 - y^2 q - y^2 - x^2 = 0$$

$$\therefore q = \frac{a^2 - y^2}{y^2}$$

Hence, $dz = \sqrt{a^2 - x^2}\,dx + \dfrac{a^2 - y^2}{y^2}\,dy$

$\Rightarrow dz = \sqrt{a^2 - x^2}\,dx + \left(\dfrac{a^2}{y^2} - 1\right)dy$

Integrating –

$$z = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} - \frac{a^2}{y} - y + b$$

(Ans.)

Ex- $q = 3p^2$ ______ (1)

Soln:-

Given equation $f(x,y,z,p,q) \equiv 3p^2 - q = 0$

Using CAE -

$$\frac{dp}{0} = \frac{dq}{0} = \frac{dz}{-6p^2+q} = \frac{dx}{-6p} = \frac{dy}{1}$$

Taking the 1st fraction-

$dp = 0$

Integrating - $p = a$

Putting value in (1) - $q = 3a^2$

Hence, $dz = a\,dx + 3a^2 dy$

Integrating - $z = ax + 3a^2 y + b$ (Ans)

Ex- $z^2(p^2z^2 + q^2) = 1$

Soln:-

The given eqn - $f(x,y,z,p,q) \equiv p^2z^4 + q^2z^2 - 1 = 0$ ___ (1)

Using - CAE -

$$\frac{dp}{p(4p^2z^3+2zq^2)} = \frac{dq}{q(4p^2z^3+2zq^2)} = \frac{dz}{-2p^2z^4-2q^2z^2} = \frac{dx}{-2pz^4} = \frac{dy}{-2qz^2}$$

Using first two fractions -

$\frac{dp}{p} = \frac{dq}{q} \Rightarrow$ Integrating - $\ln p = \ln q + \ln a \therefore p = qa$

Putting value in (1)-

$$q^2a^2z^4 + q^2z^2 = 1$$

$$\Rightarrow q^2z^2(a^2z^2+1) = 1$$

$$\Rightarrow q^2 = 1/(a^2z^2+1)$$

$$\therefore q = \frac{1}{z\sqrt{(a^2z^2+1)}}$$

$$\therefore p = \frac{a}{z\sqrt{a^2z^2+1}}$$

Hence, $dz = \frac{adx}{z\sqrt{a^2z^2+1}} + \frac{dy}{z\sqrt{a^2z^2+1}}$

$$\Rightarrow dz = \frac{adx+dy}{z\sqrt{a^2z^2+1}}$$

$$\Rightarrow adx + dy = z\sqrt{a^2z^2+1}\,dz$$

$$\Rightarrow adx + dy = (1/a^2)\,\cancel{t.tdt}\,t.t^2dt$$

Let, $t^2 = a^2z^2+1$

$2tdt = 2a^2zdz$

$\therefore tdt/a^2 = zdz$

Integrating-

$$\Rightarrow ax+y+b = (1/a^2)\frac{\cancel{t^{3/2}}\ t^3}{\cancel{3/2}\ 3}$$

$$\therefore ax+y+b = (1/3a^2)(a^2z^2+1)^3$$

(Ans)

□ Satandard form:- [of Charpit's method]

Standard form - 1:- (Only p and q are present.)

Consider eqⁿs of the form -

$$\boxed{f(p,q)=0} \text{ ——— } (1)$$

Using CAE -

$$\frac{dp}{0} = \frac{dq}{0} = \frac{dz}{-p\,\partial f/\partial p - q\,\partial f/\partial q} = \frac{dx}{-\partial f/\partial p} = \frac{dy}{-\partial f/\partial q}$$

Taking 1st ratio - $dp=0$

Integrating, $p=a$ ——— (2)

Taking 2nd ratio - $dq=0$

Integrating, $q=b$ ——— (3)

Substituting (1) - $f(a,b)=0$ ——— (4)

Now, $dz = p\,dx + q\,dy \Rightarrow dz = a\,dx + b\,dy$

Integrating - $z = ax+by+c$ ——— (5)

Now, solving (4) we find $b=f(a)$

$\therefore\ z = ax + f(a)y + c$ ——— (6)

which has two arbitrary constants a and c.

Diff. (6) P. wrt. a and c

$0 = x + f'(a)y$ and $0=1$ which is meaningless.

So, standard form (1) has no singular solution.

To find general integral of (1), we take $c=\phi(a)$ being an arbitrary fⁿ.

Ex– $\cancel{pq = k}$ ———— (1)

Soln:–

~~Using @ Since (1) is of form $f(p,q)=0$~~

~~$\therefore p = a,\ q = k/p = a/p = b$~~

From (6) –

$$z = ax + f(a)y + \phi(a) \quad \text{———— (7)}$$

Diff. (7) P. w.r.t. a –

$$0 = x + f'(a)y + \phi'(a) \quad \text{———— (8)}$$

Eleminating a ~~from~~ between (7) and (8) we get the general soln of (1)

Ex– ~~pq~~ $pq = k$ ———— (1)

Soln:–

Since (1) is of form $f(p,q)=0$

$\therefore p = a,\ q = k/a$

$$\therefore z = ax + (k/a)y + c \quad \text{———— (2)}$$

Putting $c = \phi(a)$ –

$$z = ax + (k/a)y + \phi(a) \quad \text{———— (3)}$$

Diff. (3) p. w.r.t. a –

$$0 = x - (k/a^2)y + \phi'(a) \quad \text{———— (4)}$$

(An)

Eleminating a from (3) and (4) we get required gen. so

⊛ Standard form-2:- Clairaut eq^n

$$\boxed{z = px + qy + f(p,q)} \quad \text{——— (1)}$$

Using CAE-

$$\frac{dp}{0} = \frac{dq}{0} = \frac{dz}{-px-qy-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial p}} = \frac{dx}{-x-\frac{\partial f}{\partial p}} = \frac{dy}{-y-\frac{df}{dq}}$$

Using 1st and 2nd ratio-

$p=a$ and $q=b$

$\therefore dz = p\,dx + q\,dy$

$\therefore z = px+qy+c = ax+by+c$

$= ax+by+f(a,b)$

Ex- $z = px+qy+pq$ ——— (1)

Soln:-

The complete integral is-

$z = ax+by+ab$ ——— (2)

Singular integral:-

Diff. (1) w.r.t. a, b

$0 = x+b$, $0 = y+a$

$\therefore b = -x$, $\therefore a = -y$

$\therefore z = -xy - xy + xy = -xy$

$\therefore z = -xy$

General integral:-

Taking $b=\phi(a)$, where ϕ denotes an arbitrary function.

$$\therefore z = ax + \phi(a)y + a\phi(a) \text{ ——— } (3)$$

Diff. (3) p. wrt. a, -

$$0 = x + \phi'(a)y + \phi(a) - a\phi'(a) \text{ — } (4)$$

The general integral is obtained by eleminating a from (3) and (4)

⊛ Standard form - 3:- Only p, q, z present

$$\boxed{f(p,q,z)=0} \text{ ——— } (1)$$

Using CAE -

$$\frac{dp}{p\frac{\partial f}{\partial z}} = \frac{dq}{q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

Taking the first two ratios -

$$\frac{dp}{p} = \frac{dq}{q}$$

Integrating -

$$\cancel{p=aq} \quad \ln q = \ln p + \ln a$$

$$\therefore q = ap$$

Now, $dz = p\,dx + q\,dy$ | Let, $u = x + ay$

$= p\,dx + ap\,dy$ | $\therefore du = dx + a\,dy$

$= p(dx + a\,dy)$

$\Rightarrow dz = p\,du$

~~Integrating, $z = pu$~~

$\therefore p = \frac{dz}{du}$

$\therefore q = a\frac{dz}{du}$

$\therefore$ (1) becomes –

$$f\left(\frac{dz}{du}, a\frac{dz}{du}, z\right) = 0 \quad \text{———} \quad (2)$$

which is an ODE of ~~degree~~ order 1.

Solving (2) we get z as a function of u. Complete integration is obtained by replacing u with $(x+ay)$.

Working rule of solⁿ: –

(1) Let $u = x + ay$

(2) Replace p and q by $\frac{dz}{du}$ and $a\frac{dz}{du}$ in (1) and solve the resulting ODE by usual method.

(3) Replace u by $(x+ay)$

Ex- Find a complete integral of $9(p^2 z + q^2) = 4$

Soln:-

Given, $9(p^2 z + q^2) = 4$ ——— (1)

Let $u = x + ay$, $p = \frac{dz}{du}$, $q = a\frac{dz}{du}$

(1) becomes-

$$9\left[z\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2\right] = 4$$

$$\Rightarrow \left(\frac{dz}{du}\right)^2 = \frac{4}{9(a^2+z)}$$

$$\Rightarrow \frac{dz}{du} = \pm \frac{2}{3}\frac{1}{\sqrt{a^2+z}}$$

$$\Rightarrow \frac{du}{dz} = \pm \frac{3}{2}\sqrt{a^2+z}$$

$$\Rightarrow du = \pm \frac{3}{2}\sqrt{a^2+z}\, dz$$

Integrating-

$$u = \pm \frac{3}{2}\frac{(a^2+z)^{3/2}}{3/2} + b$$

$$\Rightarrow u = \pm (a^2+z)^{3/2} + b$$

$$\Rightarrow (u-b) = \pm (a^2+z)^{3/2}$$

$$\Rightarrow (u-b)^2 = (a^2+z)^3$$

$$\therefore (x+ay-b)^2 = (a^2+z)^3$$

(Ans.)

Ex-] Find complete integral of $pz = 1 + q^2$ ——— (1)

Soln:-

which is of form $f(p, q, z) = 0$

Let $u = x + ay$ and $p = dz/du$, $q = a\,dz/du$ -

$$z \frac{dz}{du} = 1 + a^2 \left(\frac{dz}{du}\right)^2$$

$$\Rightarrow a^2 \left(\frac{dz}{du}\right)^2 - z\frac{dz}{du} + 1 = 0$$

$$\therefore \frac{dz}{du} = \frac{z \pm \sqrt{z^2 - 4a^2}}{2a^2}$$

$$\Rightarrow \frac{dz}{z \pm \sqrt{z^2 - 4a^2}} = \frac{du}{2a^2}$$

$$\Rightarrow \frac{\left(z \mp \sqrt{z^2 - 4a^2}\right) dz}{\left(z \pm \sqrt{z^2 - 4a^2}\right)\left(z \mp \sqrt{z^2 - 4a^2}\right)} = \frac{du}{2a^2}$$

$$\Rightarrow \frac{\left(z \mp \sqrt{z^2 - 4a^2}\right) dz}{4a^2} = \frac{du}{2a^2}$$

$$\Rightarrow \left(z \mp \sqrt{z^2 - 4a^2}\right) dz = 2du$$

Integrating-

$$\frac{z^2}{2} \mp \left[\frac{z}{2}\sqrt{z^2 - 4a^2} - \frac{4a^2}{2} \ln\left\{z + \sqrt{z^2 - 4a^2}\right\}\right] = 2u + b/2$$

$$\therefore z^2 \mp \left[z\sqrt{z^2 - 4a^2} - 4a^2 \ln\left\{z + \sqrt{z^2 - 4a^2}\right\}\right] = 4(x + ay) + b$$

(Ans)

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Ex - $(px+qy-z)^2 = 1+p^2+q^2$ ——— (1) [Standard form-2]

Soln:-

(1) is of form ~~f(x,y,z,p,~~ $z = px+qy+f(p,q)$

Rewriting we get-

$$px+qy-z = \pm\sqrt{1+p^2+q^2}$$

$$\therefore z = px+qy \pm \sqrt{1+p^2+q^2} \text{ ——— } (2)$$

Its complete integral is-

$$z = Ax+By \pm \sqrt{1+A^2+B^2} \text{ ——— } (3)$$

Taking + sign and $A = -a/c$, $B = -b/c$ -

$$z = -a/c\,x - b/c\,y + \left(1+a^2/c^2 + b^2/c^2\right)^{1/2}$$

$$\Rightarrow cz = -ax - by + c\left(1+a^2/c^2 + b^2/c^2\right)^{1/2}$$

$$\Rightarrow ax+by+cz = \left(a^2+b^2+c^2\right)^{1/2}$$

(Ans.)

Standard form-4:-

If variables are separable in such way-

$$\boxed{f(x,y,z,p,q) \equiv f_1(x,p) - f_2(y,q) = 0} \quad \text{———— (1)}$$

Using CAE-

$$\frac{dp}{\partial f_1/\partial x} = \frac{dq}{-\partial f_2/\partial y} = \frac{dz}{-p\left(\partial f_1/\partial p\right) + q\left(\partial f_2/\partial q\right)} = \frac{dx}{-\partial f_1/\partial p} = \frac{dy}{\partial f_2/\partial q}$$

From 1st and 4th ratio-

$$\frac{dp}{\partial f_1/\partial x} = \frac{\partial x}{-\partial f_1/\partial p}$$

$$\Rightarrow \frac{\partial f_1}{\partial x} dx + \frac{\partial f_1}{\partial p} dp = 0$$

$$\Rightarrow df_1 = 0$$

Integrating - $f_1 = a$

From (1)

$$f_1(x,p) = a = f_2(y,q) \quad \text{———— (2)}$$

From (2) solving p, q we can write-

$$p = F_1(x,a), \; q = F_2(y,a) \quad \text{———— (3)}$$

Now, $dz = F_1(x,a)\,dx + F_2(y,a)\,dy$

$$\therefore z = \int F_1(x,a)\,dx + \int F_2(y,a)\,dy + b$$

Ex Find the complete integral of : $x(1+y)p = y(1+x)q$

Soln:-

Given, $x(1+y)p = y(1+x)q$

$$\therefore \frac{x}{1+x}p = \frac{y}{1+y}q \quad \text{——— (1)}$$

which is of form $f_1(x,p) = f_2(y,q)$

Equating each side with arbitrary constant a -

$$\frac{1+x}{x}p = a = \frac{1+y}{y}q$$

$$\therefore p = \frac{x+1}{x}a, \quad q = \frac{y+1}{y}a$$

$$dz = a\left[\frac{x+1}{x}dx + \frac{y+1}{y}dy\right]$$

$$\Rightarrow dz = a\left[(1+1/x)dx + (1+1/y)dy\right]$$

$$\Rightarrow z = a\left[x + \ln x + y + \ln y\right] + b$$

$$\therefore z = a(x+y+\ln xy) + b \quad \text{(Ans.)}$$

Ex $p - 3x^2 = q^2 - y$

Soln:-

which is of form $f_1(x,p) = f_2(y,q)$

Equating each side with arbitrary const. a -

$$p - 3x^2 = a, \quad q^2 - y = a$$

$$\therefore p = a + 3x^2, \quad q = \sqrt{a+y}$$

$$dz = (a+3x^2)\,dx + \sqrt{a+y}\,dy$$

$$\Rightarrow z = ax + x^3 + \tfrac{2}{3}(a+y)^{3/2} + b$$

(Ans)