Math 2101 Date: 08-06-15

T-MHU(D_1)

Fourier Series

Periodic function:- The function $f(x)$ of variable x is said to be periodic if there exist ~~an~~ non-zero number T independent of x such that, $f(T+x) = f(x)$ holds for all values of x.

Ex: $f(x) = \sin x$ is a periodic function for period 2π.

Even function:- If a function is such that, $f(-x) = f(x)$ then the function is said to be even function.

Ex: $f(x) = x^2$ is an even function.

For even function $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

Odd function:- If a function is such that, $f(-x) = -f(x)$ then the function is said to be odd function.

Ex: $f(x) = x^3$ is an odd function

For odd function $\int_{-a}^{a} f(x)\,dx = 0$

Fourier series:- Under certain condition of the function in a period $-\pi \leq x \leq \pi$ can be represent as a trigonometric series,
$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ which is known as Fourier series with the Fourier constant or co-efficient a_0, a_n and b_n.

■ Determination of the co-efficient of Fourier series (with period 2π, $-\pi \leq x \leq \pi$).

We consider that the fourier series is uniformly convergent in $-\pi \leq x \leq \pi$ of $f(x)$

Hence, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ — (i)

Integrating eqn (i) in $(-\pi, \pi)$

$$\int_{-\pi}^{\pi} f(x)\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty}\left(\int_{-\pi}^{\pi} a_n \cos nx\,dx + \int_{-\pi}^{\pi} b_n \sin nx\,dx\right)$$

$$= \frac{a_0}{2}\left[x\right]_{-\pi}^{\pi} + \sum_{n=1}^{\infty} a_n\left[\frac{\sin nx}{n}\right]_{-\pi}^{\pi} + b_n\left[-\frac{\cos nx}{n}\right]_{-\pi}^{\pi}$$

$$= \frac{a_0}{2}\cdot 2\pi + 0 + 0$$

$$= \pi a_0$$

$\therefore a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

Now multiplying eqn (i) with $\cos nx$ and integrating in $(-\pi, \pi)$.

$$\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi}\cos nx\,dx + \sum_{n=1}^{\infty}\left(\int_{-\pi}^{\pi} a_n \cos^2 nx\,dx + \int_{-\pi}^{\pi} b_n \sin nx \cos nx\,dx\right)$$

$$= 0 + a_n\pi + 0$$

or, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

Again Multiplying eqn (i) by $\sin nx$ and integrating in $(-\pi, \pi)$

$$\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi}\sin nx\,dx + \sum_{n=1}^{\infty}\left(\int_{-\pi}^{\pi} a_n \cos nx \sin nx\,dx + \int_{-\pi}^{\pi} b_n \sin^2 nx\,dx\right)$$

$$= 0 + 0 + b_n\pi$$

or, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

Rules:- If $f(x)$ is an even function

$$a_0 = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx \neq 0$$

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx \neq 0$$

$$b_n = 0$$

If $f(x)$ is an odd function then,

$a_0 = 0$, $a_n = 0$ and $b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$

Fourier series with period $2L$:

In addition to the fourier series with period 2π can be develop to a fourier series with period $2L$ where L is a constant.

Let us consider a function $\phi(y)$ which is integrable.

also let $y = \frac{Lx}{\pi}$ ——— (i)

Then $\phi\left(\frac{Lx}{\pi}\right)$ is a function of period 2π.

$\therefore f(x) = \phi\left(\frac{Lx}{\pi}\right) = \phi(y)$

$y = \frac{Lx}{\pi}$ $\quad x = \frac{\pi y}{L}$

Now the fourier series ~~for period~~ $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$ will be

converted to the fourier series for $\phi(y)$ as

$$\phi(y) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{ny\pi}{L} + b_n \sin \frac{ny\pi}{L} \right)$$

Hence the fourier co-efficient are given by -

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{L} \int_{-L}^{L} \phi(y)\,dy$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx = \frac{1}{L} \int_{-L}^{L} \phi(y) \cos \frac{n\pi y}{L}\,dy$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx = \frac{1}{L} \int_{-L}^{L} \phi(y) \sin \frac{n\pi y}{L}\,dy$$

MHU (Math) (E1) Date: 09-06-15

\# State and prove persevals theorem?

Statement:- If the fourier series of a function $f(x)$ is uniformly converts to in the interval $(-L, L)$ then,

$$\frac{1}{L}\int_{-L}^{L}\{f(x)\}^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Proof:- We have the fourier series in $(-L, L)$ is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \frac{\sin n\pi x}{L}\right) \quad (i)$$

Multiplying eqn by $f(x)$ and integrating over $(-L, L)$

$$\therefore \int_{-L}^{L}\{f(x)\}^2 dx = \frac{a_0}{2}\int_{-L}^{L} f(x) dx + \sum_{n=1}^{\infty}\left(\int_{-L}^{L} a_n f(x)\cos\frac{n\pi x}{L}dx + \int_{-L}^{L} b_n f(x)\sin\frac{n\pi x}{L}dx\right)$$

But, $\int_{-L}^{L} f(x) dx = La_0$, $\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = La_n$

and $\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx = Lb_n$

$$\therefore \int_{-L}^{L}\{f(x)\}^2 dx = \frac{a_0^2}{2}L + L\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

$\therefore \frac{1}{L}\int_{-L}^{L}\{f(x)\}^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$ (proved)

What is half range cosine series? Expan the half range cosine series.

Half range cosine series:- The part which contains only the cosine part of a fourier series is known as half range fourier series. Its range is $(0,\pi)$ which is half of $(-\pi,\pi)$ of fourier series. The cosine term remain in even function.

Expansion:- The cosine series in the range $(0,\pi)$ in $f(x)$ is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{---} \quad (i)$$

Now Integrating equation (i) with respect to x over $(0,\pi)$

$$\int_0^{\pi} f(x)\,dx = \frac{a_0}{2}\int_0^{\pi} dx + \sum_{n=1}^{\infty} a_n \int_0^{\pi}\cos nx\,dx$$

$$= \frac{a_0}{2}\cdot\pi + 0$$

$$\therefore a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

Multiplying eq^n (i) by $\cos nx$ and intergrating w.r.t. x over $(0,\pi)$

$$\int_0^\pi f(x)\cos nx\,dx = \frac{a_0}{2}\int_0^\pi \cos nx\,dx + \sum_{n=1}^{\alpha} a_n \int_0^\pi \cos^2 nx\,dx$$

$$= \frac{a_0}{2}\left[\frac{\cancel{x}\sin n\cancel{n}x}{n}\right]_0^\pi + \sum_{n=1}^{\alpha} \frac{a_n}{2}\int_0^\pi 2\cos^2 nx\,dx$$

$$= 0 + \sum_{n=1}^{\alpha} \frac{a_n}{2}\int_0^\pi (1+\cos 2nx)\,dx$$

$$= \sum_{n=1}^{\alpha} \frac{a_n}{2}\left\{\int_0^\pi dx + \int_0^\pi \cos 2nx\right.$$

$$= \frac{a_n}{2}\pi$$

$$\therefore a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx$$

Hence in the range $(0,\pi)$ the cosine series is

$$f(x) = \frac{1}{\pi}\int_0^\pi f(x)\,dx + \frac{2}{\pi}\sum_{n=1}^{\alpha}\int_0^\pi f(x)\cos^2 nx\,dx$$

What is half range sine series? Expand sine series.

Half range sine series:- The part which contains only the sine part of a fourier series is called half range ~~fo~~ sine series. It's range is $(0,\pi)$ which is half of $(-\pi,\pi)$ of fourier series. The sine term remain in odd function.

Expansion:- The sine function in the range $(0,\pi)$ of $f(x)$ is —

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin nx \text{ ——— (i)}$$

Integrating eqn (i) w.r.t x over $(0,\pi)$

$$\int_0^{\pi} f(x)\,dx = \frac{a_0}{2}\int_0^{\pi} dx + \sum_{n=1}^{\infty}\int_0^{\pi} b_n \sin nx\,dx$$

$$= \frac{a_0}{2}\left[x\right]_0^{\pi} + \sum_{n=1}^{\infty} b_n \left[\cancel{x}\frac{\cos nx}{n}\right]_0^{\pi}$$

$$= \frac{a_0}{2}\pi + 0$$

$$\therefore a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \text{ ——— (ii)}$$

Multiplying eqn (i) by $\sin nx$ and integrating over $(0,\pi)$

$$\int_0^{\pi} f(x)\sin nx\,dx = \frac{a_0}{2}\int_0^{\pi}\sin nx\,dx + \sum_{n=1}^{\infty} a_n\int_0^{\pi}\sin^2 nx\,dx$$

$$= \frac{a_0}{2}\left[-\frac{\cos nx}{n}\right]_0^{\pi} + \sum_{n=1}^{\infty} \frac{a_n}{2}\int_0^{\pi} 2\sin^2 nx\,dx$$

$$= a_0 + \sum_{n=1}^{\infty} \frac{a_n}{2}\int_0^{\pi}(1-\cos 2nx)\,dx$$

$$= \frac{a_n}{2}\sum_{n=1}^{\infty}\left\{\int_0^{\pi} dx - \int_0^{\pi}\cos 2nx\,dx\right\}$$

$$= \frac{a_n}{2}\cdot\pi$$

$$\therefore a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Hence the sine series in the range $(0,\pi)$ is

$$f(x) = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx + \sum_{n=1}^{\infty}\frac{2}{\pi}\int_0^{\pi} f(x)\sin^2 nx\,dx$$

Expand the fourier series for the function $f(x)$ within $-\pi \le x \le \pi$.

Hence show that, $\pi^2/6 = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

Solⁿ: Given function $f(x) = x^2$

Since $f(-x) = (-x)^2 = x^2 = f(x)$

$\therefore$ The given function is an even function.

For even function the Fourier series is.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} a_n \cos nx \quad \text{---} (i)$$

For even function.

$$a_0 = 2/\pi \int_0^{\pi} f(x)\, dx$$

$$= 2/\pi \int_0^{\pi} x^2\, dx$$

$$= 2/\pi \left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2\pi^2}{3}$$

and, $a_n = 2/\pi \int_0^{\pi} f(x) \cos nx\, dx$

$$= 2/\pi \int_0^{\pi} x^2 \cos nx\, dx$$

by integration by parts —

$$= \frac{2}{\pi}\left[x^2 \frac{\sin nx}{n}\right]_0^{\pi} - \frac{2}{\pi}\int_0^{\pi} 2x \frac{\sin nx}{n}\, dx$$

$$= 0 - \frac{4}{n\pi}\left[\frac{-x\cos nx}{n}\right]_0^{\pi} + \frac{4}{n\pi}\int_0^{\pi} \frac{-\cos nx}{n}\, dx$$

$$= -\frac{4}{n^2\pi}\left[\cancel{\pi\cos n\pi} \; -\pi\cos n\pi + 0\right] - \frac{4}{n^2\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi}$$

$$= \frac{4\cos n\pi}{n^2} + 0$$

$$= \frac{4(-1)^n}{n^2}$$

putting these values in equation (i)

$$f(x) = \frac{1}{2}\cdot\frac{2\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

$$x^2 = \frac{\pi^2}{3} + 4\left(\frac{-\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \cdots\right)$$

putting $x = \pi$

$$\pi^2 = \frac{\pi^2}{3} - 4\left(\frac{\cos\pi}{1^2} - \frac{\cos 2\pi}{2^2} + \frac{\cos 3\pi}{3^2} - \cdots\right)$$

or, $$\pi^2 - \pi^2/3 = -4\left(\frac{\cancel{\cos\pi}-1}{1^2} - \frac{1}{2^2} + \frac{-1}{3^2} - \cdots\right)$$

or, $$\frac{2\pi^2}{3} \times \frac{1}{4} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

or, $$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + 1/3^2 + \cdots$$ (proved)

\# Expand the fourier series for the $f(x) = \begin{cases} -k, & \pi \le x \le 0 \\ k, & 0 \le x \le \pi \end{cases}$

show that $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \cdots$

Sol^n: Given function $f(x) = \left\{\begin{array}{ll} -k, & -\pi \le x \le 0 \\ k, & 0 \le x \le \pi \end{array}\right\}$ ⟵ (i)

The Fourier series for the following function

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\alpha} (a_n \cos nx + b_n \sin nx)$$

Now, $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} -k\,dx + \int_{0}^{\pi} k\,dx\right.$$

$$= 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} -k\cos nx\,dx + \int_{0}^{\pi} k\cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{-k\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + k\left[\frac{\sin nx}{n}\right]_{0}^{\pi}\right\}$$

$$= 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} -k\sin nx\,dx + \int_{0}^{\pi} k\sin nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{k\left[\frac{\cos nx}{n}\right]_{-\pi}^{0} - k\left[\frac{\cos nx}{n}\right]_{0}^{\pi}\right\}$$

$$= \frac{k}{n\pi}\left\{1 - \cos n\pi - \cos n\pi + 1\right\}$$

$$= \frac{k}{n\pi}\, 2(1 - \cos n\pi)$$

$$= \frac{2k}{n\pi}\left\{1 - (-1)^n\right\}$$

putting these values in equation (i)

$$f(x) = \frac{2k}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \sin nx$$

$$\Rightarrow f(x) = 2k/\pi \left\{ \frac{2\sin x}{1} + 0 + \frac{2\sin 3x}{3} + 0 + \frac{2\sin 5x}{5} + \cdots \right\}$$

putting $x = \pi/2$ then,

$$f(\pi/2) = \frac{4k}{\pi} \left\{ \sin \pi/2 + \frac{1}{3}\sin 3\pi/2 + \frac{1}{5}\sin 5\pi/2 + \cdots \right\}$$

$$\Rightarrow k = \frac{4k}{\pi} \left\{ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \right\}$$

$$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

(proved.)

Expand Fourier series in the interval $-2<x<2$ when $f(x) = \begin{cases} 2 ; & -2<x<0 \\ x ; & 0<x<2 \end{cases}$

Soln:

The Fourie series for the ~~the~~ given function

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n x \pi}{2} + b_n \sin\frac{n x \pi}{2} \right) \quad \text{(i)}$$

Now, $$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\left\{ \int_{-2}^{0} 2\,dx + \int_{0}^{2} x\,dx \right\}$$

$$= \frac{1}{2}\left\{ 2\,[x]_{-2}^{0} + \left[\frac{x^2}{2}\right]_0^2 \right\}$$

$$= \frac{1}{2}\{4+2\} = 6/2 = 3$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left\{\int_{-2}^{0} 2\cdot\cos\frac{n\pi x}{2}\,dx + \int_{0}^{2} x\cos\frac{n\pi x}{2}\,dx\right.$$

$$= \left[\frac{\sin\frac{n\pi x}{2}}{n\pi/2}\right]_{-2}^{0} + \frac{1}{2}\left[\frac{x\sin\frac{n\pi x}{2}}{n\pi/2}\right]_{0}^{2} - \frac{1}{2}\int_{0}^{2}\frac{\sin\frac{n\pi x}{2}}{n\pi/2}\,dx$$

$$= 0 + 0 - \frac{1}{2}\left[\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right]_{0}^{2}$$

$$= \cancel{\frac{4}{n^2\pi^2}}\ \frac{2}{n^2\pi^2}\left[\cos n\pi - \cos 0\right]$$

$$= \frac{2}{n^2\pi^2}\left\{(-1)^n - 1\right\}$$

again $b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx$

$$= \frac{1}{2}\left\{\int_{-2}^{0} 2\sin\frac{n\pi x}{2}\,dx + \int_{0}^{2} x\sin\frac{n\pi x}{2}\,dx\right\}$$

$$= \left[\frac{-\cos\frac{n\pi x}{2}}{n\pi/2}\right]_{-2}^{0} + \frac{1}{2}\left[\frac{-x\cos\frac{n\pi x}{2}}{n\pi/2}\right]_{0}^{2} - \frac{1}{2}\int_{0}^{2}\frac{\cancel{\sin\frac{n\pi x}{2}}\ \cos\frac{n\pi x}{2}}{}$$

$$= \cancel{\frac{2}{n\pi}\left[1 \quad \cos 2n\pi\right] + 0} - \frac{1}{2}\left[\frac{\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right]_0^2$$

$$= \cancel{\frac{-2}{n\pi} + 0}$$

$$= \frac{2}{n\pi}\left[1 - \cos n\pi\right] + \frac{1}{2}\cdot\frac{2}{n\pi}\left[-2\cos n\pi - 0\right] - 0$$

$$= 0 + \frac{2}{n\pi}(-1)$$

$$= \frac{-2}{n\pi}$$

putting these values in eq^n (1)

$$f(x) = 3/2 + \sum_{n=1}^{\infty}\left[\frac{2}{n^2\pi^2}\left\{(-1)^n - 1\right\}\cos\frac{n\pi x}{2} - \frac{2}{n\pi}\sin\frac{n\pi x}{2}\right]$$

Expand the fourier series for the given function $f(x) = x^2$ and show that $\pi^2/6 = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2}$.

Sol^n: Given function $f(x) = x^2$

Given function is an even function

for even function the fourier series

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{(i)}$$

for odd function $a_0 = 2/\pi \int_0^{\pi} f(x)\, dx$

$$= 2/\pi \int_0^{\pi} x^2\, dx = 2/\pi \left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2\pi^2}{}$$

and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$

$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\, dx$

by integration by parts

$= \frac{2}{\pi}\left[\frac{x^2 \sin nx}{n}\right]_0^{\pi} - \frac{2}{\pi}\int_0^{\pi} \frac{2x \sin nx}{n}\, dx$

$= 0 - \frac{4}{n\pi}\left[\frac{-x\cos nx}{n}\right]_0^{\pi} - \frac{4}{n\pi}\int_0^{\pi} -\frac{\cos nx}{n}\, dx$

$= \frac{4}{n\pi}\left[\frac{\pi\cos n\pi}{n}\right] + \frac{4}{n^2\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi}$

$= \frac{4\cos n\pi}{n^2} + 0$

$= \frac{4(-1)^n}{n^2}$

putting these values in equation (i)

$f(x) = \frac{1}{2}\,\frac{2\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx$

$x^2 = \frac{\pi^2}{3} + 4\left(\frac{-\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \cdots\right)$

putting $x = \pi$

$\pi^2 = \frac{\pi^2}{3} + 4\left(-\frac{\cos\pi}{1^2} + \frac{\cos 2\pi}{2^2} + \frac{\cos 3\pi}{3^2} + \cdots\right)$

$\frac{2\pi^2}{3}\times\frac{1}{4} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2}$

$\therefore \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$ (proved)

Statistics
Class test

Date: 08-03-16

Quantile Mean $Q_i = l + \frac{\left(\frac{Ni}{4} - c\right)}{f} \times h$

Mean $= \frac{\sum_{i=1}^{n} f_i x_i}{N}$

Mood $= l + \frac{f_m - f_1}{2f_m - 2f_1 - f_2} \times h$

standard deviation $= \frac{\sum f_i x_i^2}{N} - \bar{x}^2$

Co-efficient of variation $= \frac{\sigma \times 100}{\bar{x}}$

$\bar{x}_A = \frac{\sum x_i}{N} = \frac{610}{10} = 61$

$\bar{x}_B = \frac{\sum x_i}{N} = \frac{705}{10} = 70.5$

$\sigma_A^2 = \frac{\sum x_i^2}{N} - \bar{x}^2$

$= \frac{33416}{10} - (61)^2$

$= 52.6$

$\sigma_A = 7.25$

$\sigma_B^2 = \frac{\sum x_i^2}{N} - \bar{x}^2$

$= \frac{51921}{10} - (70.5)^2$

$= 221.85$

$\sigma_B = 14.89$

Co efficient of variation of A $= \frac{\sigma_A \times 100}{61} = 11.89$

" " " of B $= \frac{\sigma_B \times 100}{70.5} = 21.12$

$$= \frac{1}{2\pi}\int_{-c}^{c} f(t)\left\{ \frac{\pi}{c}\sum_{-\infty}^{\infty} \cos\frac{n\pi}{c}(t-x)\right\} dt \quad \text{—} (3)$$

Let us assume that, c increases indefinitely so that we may write $n\pi/c = u$ and $\pi/c = du$.

This assumption gives $\lim_{c\to\infty}\left\{ \frac{\pi}{c}\sum_{-\infty}^{\infty} \cos\frac{n\pi}{c}(t-x)\right\} =$

$$= \int_{-\infty}^{\infty} \cos u(t-x)\, du$$

$$= 2\int_{0}^{\infty} \cos(t-x)u\, du \quad \text{—} (4)$$

Using (4) in (3) $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\left\{ 2\int_{0}^{\infty} \cos u(t-x)\, du\right\} dt$

$$= \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\cos u(t-x)\, du\, dt$$

(prov

Fourier Sine and Cosine integrals:—

It states that, sine integral is, $f(x) = \frac{2}{\pi}\int_{0}^{\infty} \sin ux\, du \int_{0}^{\infty} f(t)\sin ut\, dt$

and cosine integral is $f(x) = \frac{2}{\pi}\int_{0}^{\infty} \cos ux\, du \int_{0}^{\infty} f(t)\cos ut\, dt$

Proof:— We know Fourier integral theorem is,

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\cos u(t-x)\, du\, dt \quad \text{—} (1)$$

$$= \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\left(\cos ut \cos ux + \sin ut \sin ux\right) du\, dt$$

Math-2101
T-MHU (E_3)

Date:- 02-08-15

Prob2: $a_0 y^n + a_1 y^{n-1} + \cdots + a_n y = F(x)$

□ Determine Fouriers sine and cosine transform of $f(x) = e^{-ux}$

Solution:- we know, Fourier sine transform,

$$f(v) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(t) \sin vt \, dt$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ut} \sin vt \, dt \quad \left[\because f(x) = e^{-ux} \text{ so, } g(t) = e^{-ut}\right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{v}{u^2+v^2} \quad \left[\text{by laplace Integral transform } \int_0^\infty e^{-at} \sin bt \, dt = \frac{b}{a^2+b^2},\ \int_0^\infty e^{-at} \cos bt \, dt = \frac{a}{a^2+b^2}\right]$$

The reciprocal relation is given by,

$$g(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(v) \sin vt \, dt$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \cdot \frac{v}{u^2+v^2} \sin vt \, dt$$

$$= \frac{2}{\pi} \int_0^\infty \frac{v \sin vt}{u^2+v^2} \, dt.$$

we know, Fourier cosine transformation same as like before. just cosine insted of sine.

Prob. Determine the Fourier transform of $e^{-x^2/2}$ / or Find the cosine transform of : $e^{-x^2/2}$.

Sol^n: We know, Fourier cosine transform,

$$g(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos xt \, dt$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-t^2/2} \cos xt \, dt$$

Let, $I = \int_0^\infty e^{-t^2/2} \cos xt \, dt$ ——— (i)

$$\Rightarrow \frac{\partial I}{\partial x} = -\int_0^\infty t e^{-t^2/2} \sin xt \, dt$$

$$= -\left[\sin xt \, e^{-t^2/2}\right]_0^\infty - \int_0^\infty \cos xt \cdot x e^{-t^2/2} dt$$

$$= 0 - x \int_0^\infty e^{-t^2/2} \cos xt \, dt$$

$$= -xI$$

$$\Rightarrow \frac{dI}{I} = -x \, dx$$

$$\Rightarrow \log I = -\frac{x^2}{2} + \log c$$

$$\Rightarrow \log I = \log e^{-x^2/2} + \log c$$

$$\Rightarrow I = c e^{-x^2/2}$$ ——— (2)

when $x = 0$ then from equation (i)

$$I = \int_0^{\alpha} e^{-t^2/2}\, dt$$

$$= \int_0^{\alpha} e^{-z^2} \sqrt{2}\, dz$$

$$= \sqrt{2} \int_0^{\alpha} e^{-z^2} dz$$

$\frac{t}{\sqrt{2}} = z$

$t = \sqrt{2} z$

or $dt = \sqrt{2}\, dz$

$$= \sqrt{2} \cdot \frac{\sqrt{\pi}}{2}$$

$$= \sqrt{\pi/2}$$

Hence from (2) $I = C = \sqrt{\pi/2}$ [when $x=0$]

Putting these in eqn (ii)

$$I = \sqrt{\pi/2} \cdot e^{-x^2/2}$$

Therefore, we get,

$$g(x) = \sqrt{\frac{2}{\pi}} \sqrt{\pi/2}\, e^{-x^2/2}$$

$$= e^{-x^2/2}$$

The reciprocal relation is,

$$f(t) = \sqrt{2/\pi} \int_0^{\alpha} g(x) \cos xt\, dx$$

$$\Rightarrow e^{-t^2/2} = \sqrt{2/\pi} \int_0^{\alpha} e^{-x^2/2} \cos xt\, dx$$

$$\Rightarrow e^{-t^2/2} = \sqrt{2/\pi} \int_0^{\alpha} e^{-t^2/2} \cos xt\, dt$$

T- MMR

(B4)

Prob 1 Solve the differential eqⁿ $(D^2+4)y = \sin 3x$

T- MHU

(D4)

Date: 08-08-15

If $F(x) = e^{-|x|}$, find the Fourier transformation where $-\alpha \le x \le \alpha$,

Solⁿ: We have Fourier Integral transform,

$$f(\omega) = \int_{-\alpha}^{\alpha} F(u)\, e^{-i\omega u}\, du$$

$$= \int_{-\alpha}^{0} e^{u} \cdot e^{-i\omega u}\, du + \int_{0}^{\alpha} e^{-u} e^{-i\omega u}\, du$$

$$= \left[\frac{e^{(1-i\omega)u}}{1-i\omega}\right]_{-\alpha}^{0} + \left[\frac{e^{-(1+i\omega)u}}{-(1+i\omega)}\right]_{0}^{\alpha}$$

$$= \frac{1}{1-i\omega} + \frac{1}{1+i\omega}$$

$$= \frac{2}{1+\omega^2}$$

Also the reciprocal relation given by,

$$F(x) = \frac{1}{2\pi}\int_{-\alpha}^{\alpha} f(\omega)\, e^{i\omega x}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\alpha}^{\alpha} \frac{2}{1+\omega^2}\, e^{i\omega x}\, d\omega \quad \text{Ans}$$

C.T 14

Prob. # If $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in the range 0 to 2π, show that,

$$f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$

Sol^n: We know Fourier series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{--- (i)}$$

here; $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

$$= \frac{1}{\pi}\int_0^{2\pi} \left(\frac{\pi^2}{4} - \frac{\pi x}{2} + \frac{x^2}{4}\right) dx$$

$$= \frac{1}{\pi}\Big[$$

$$= \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{\pi^2}{4} \cos nx\,dx - \frac{1}{\pi}\int_0^{2\pi} \frac{\pi x}{2} \cos nx\,dx + \frac{1}{\pi}\int_0^{2\pi} \frac{x^2}{4} \cos nx\,dx$$

=

=

$$= \frac{1}{n^2}$$

রোকন

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \cdots$$

$$=$$

$$= 0$$

Putting these values of a_0, a_n, b_n in eqn (i)

Find the Fourier series in the expansion of a function represented by $f(x) = 0$ if $-\pi < x < 0$

Homework.

$= 2x^3$ if $0 \le x < \pi$

$$a_0 = \frac{\pi^3}{2}, \quad a_n = \frac{6\pi(-1)^n}{n^2} - \frac{12}{n^4\pi}(-1)^n + \frac{12}{n^4\pi}$$

$$b_n = \frac{2\pi^2(-1)^{n+1}}{n} + \frac{12}{n^3}(-1)^n$$

Using Fourier transform solve the boundary value problem. $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$; $x, t > 0$

Soln: subject to the condition $u(0,t) = 0$; $u(x,0) = \begin{cases} 1, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$ and $u(x,t)$ is bounded.

Math-2101 Date: 09-08-15

T-MHU(E4)

Prob. □ Using Fourier transform solve the boundary value problem $\frac{\partial F}{\partial t} = \frac{\partial^2 F}{\partial x^2}, -\alpha < x < \alpha, t > 0$

$$F = f(x) \text{ when } t = 0$$

$$\text{and } \frac{\partial F}{\partial x} = 0 \text{ when } x = \pm\alpha$$

Soln: Using Fourier sine transform,

$$\int_{-\alpha}^{\alpha} \frac{\partial F}{\partial t} \sin px \, dx = \int_{-\alpha}^{\alpha} \frac{\partial^2 F}{\partial x^2} \cdot \sin px \, dx$$

$$\Rightarrow \frac{\partial}{\partial t} \int_{-\alpha}^{\alpha} F \sin px \, dx = \left[\frac{\partial F}{\partial x} \sin px\right]_{-\alpha}^{\alpha} - p \int_{-\alpha}^{\alpha} \frac{\partial F}{\partial x} \cos px \, dx$$

$$= 0 - p\left[F \cos px\right]_{-\alpha}^{\alpha} - p^2 \int_{-\alpha}^{\alpha} F \sin px \, dx$$

$$\Rightarrow \frac{\partial}{\partial t} \int_{-\alpha}^{\alpha} F \sin px \, dx = -p^2 \int_{-\alpha}^{\alpha} F \sin px \, dx$$

Let, $u = \int_{-\alpha}^{\alpha} F \sin px \, dx$ —— (i)

$$\therefore \frac{du}{dt} = -p^2 u$$

$$\Rightarrow \frac{du}{u} = \cancel{p^2 dt} - p^2 dt$$

$$\Rightarrow \log u = -p^2 t + \log k$$

$$\Rightarrow u(p,t) = k e^{-p^2 t} \quad \text{—— (ii)}$$

when, $t=0$ then, $u=k$

Hence $u(p,0)=\int_{-\alpha}^{\alpha} F(x,0)\sin px\,dx$ [From eq^n (i)]

$$\Rightarrow k=\int_{-\alpha}^{\alpha} f(x)\sin px\,dx \quad \text{—— (iii)}$$

Hence from eq^n (ii),

$$u=\int_{-\alpha}^{\alpha} f(x)\sin px\, e^{-p^2t}\,dx \quad \text{—— (iv)}$$

Therefore from eq^n (i) $u(p,t)=\int_{-\alpha}^{\alpha} F\sin px\,dx$

and its reciprocal relation is $F=\frac{2}{\pi}\int_{-\alpha}^{\alpha} u\sin px\,dp$

$$=\frac{2}{\pi}\int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha} f(x)\sin^2 px\, e^{-p^2t}\,dx\,dp$$ Ans

Prob. # Prove the solution of the boundary value problem $\frac{\partial u}{\partial t}=3\frac{\partial^2 u}{\partial x^2}$

$$u(0,t)=u(2,t)=0,\ t>0$$

$$u(x,0)=x;\ 0<x<2$$

is $u(x,t)=\sum_{n=1}^{\alpha}\frac{4(-1)^{n+1}}{n\pi}\sin\frac{n\pi x}{2}\,e^{-\frac{3}{4}n^2\pi^2t}$

Proof:- The given partial differential eqn is $\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2}$ — (i)

Taking the finite Fourier Sine transform of (i)

$$\int_0^2 \frac{\partial u}{\partial t} \sin\frac{n\pi x}{2}\, dx = \int_0^2 3\frac{\partial^2 u}{\partial x^2} \sin\frac{n\pi x}{2}\, dx \quad \text{— (ii)}$$

Let $u = u(n,t) = \int_0^2 U(x,t) \sin\frac{n\pi x}{2}\, dx$

$$\Rightarrow \frac{\partial u}{\partial t} = \int_0^2 \frac{\partial u}{\partial t} \sin\frac{n\pi x}{2}\, dx$$

$$= \int_0^2 3\frac{\partial^2 u}{\partial x^2} \sin\frac{n\pi x}{2}\, dx \quad [\text{using eqn } 2]$$

$$= 3\left[\frac{\partial u}{\partial x}\cdot \sin\frac{n\pi x}{2}\right]_0^2 - \frac{3n\pi}{2}\int_0^2 \cos\frac{n\pi x}{2}\frac{\partial u}{\partial x}\, dx$$

$$= 0 - \frac{3n\pi}{2}\left[U(x,t)\cos\frac{n\pi x}{2}\right]_0^2 - \frac{3n^2\pi^2}{4}\int_0^2 U(x,t)\sin\frac{n\pi x}{2}\, dx$$

$$\therefore \frac{\partial u}{\partial t} = -\frac{3n^2\pi^2}{4} u$$

$$\Rightarrow \cancel{\frac{\partial u}{\partial t}}\ \frac{du}{u} = -\frac{3n^2\pi^2}{4}\, dt$$

$$\Rightarrow u(n,t) = A\, e^{-\frac{3n^2\pi^2}{4}} \text{ ——— (iii)}$$

When $t=0$, $u(n,0) = A$ ——— (iv)

Now, $u(n,t) = \int_0^2 U(x,t) \sin\frac{n\pi x}{2}\, dx$

$$\Rightarrow u(n,0) = \int_0^2 x \sin\frac{n\pi x}{2}\, dx$$

$$=$$

$$= \frac{-4}{n\pi} \cdot \cos n\pi$$

from eqn (iv), $A = \frac{-4}{n\pi} \cdot \cos n\pi$

From eqn (iii)

$$u(n,t) = \frac{-4}{n\pi} \cos n\pi \cdot e^{-\frac{3n^2\pi^2}{4}t} \text{ ——— (v)}$$

$$\cancel{\Rightarrow \frac{\partial u}{u} = -\frac{3n^2\pi^2}{4}} dt$$

Taking inverse transform of A

$$U(x,t) = \frac{2}{2} \sum_{n=1}^{\infty} u(n,t) \sin\frac{n\pi x}{2}$$

$$= \sum_{n=1}^{\infty} - - - -$$

$$= \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi} \sin\frac{n\pi x}{2}\, e^{-\frac{3}{4}n^2\pi^2 t}$$

(proved)

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T- MHU (D5)

$\therefore y = C_0 + C_1 x + \left(\frac{C_0}{6} + \frac{C_1}{2}\right)x^3 + \frac{C_1}{12}x^4 + \left(\frac{C_0}{8} + \frac{3C_1}{8}\right)x^5 + \cdots = 0$

$y = C_0\left(1 + \frac{x^3}{6} + \frac{1}{8}x^5 + \cdots\right) + C_1\left(x + \frac{1}{2}x^3 + \frac{1}{12}x^4 + \cdots\right)$

Given that, $y'(0) = 6$

এই eqn কে differentiate করে $x = 0$ বসাতে হবে।

~~Partial D~~

Partial Differential equation

Partial differential equation (P.DE): An eqn involving one more partial derivatives is called PDE.

Example: $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$

Linear P.D.E:- A P.DE is said to be linear if it is of the 1st degree in partial derivatives.

example: (i) $x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 1$

ii) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y}$

always ask

$\{Z = (x, y)\}$

Prob # If $u(x,y,z) = c_1$ and $v(x,y,z) = c_2$ be any independent ~~son~~ sol^n of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ where P, Q, R are function of x, y, z then $\phi(u,v) = 0$ or $v = \phi(u)$ is a general sol^n of the lagrange Linear eq^n $Pp + Qq = R$.

Sol^n: Let $u = u(x,y,z)$ and $v = v(x,y,z)$ be any two function, connected by $\phi(u,v) = 0$ —— (i)

Partially differentiating eq^n (i) w.r.t x and y respectively.

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial z}{\partial x}\frac{\partial u}{\partial z}\right) + \frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial z}{\partial x}\frac{\partial v}{\partial z}\right) = 0 \quad \text{—— (ii)}$$

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right) + \frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial z}{\partial y}\cdot\frac{\partial v}{\partial z}\right) = 0 \quad \text{—— (iii)}$$

Now eliminating $\frac{\partial\phi}{\partial u}$ and $\frac{\partial\phi}{\partial v}$ from (ii) and (iii)

$$\begin{vmatrix} \frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} & \frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} & \frac{\partial v}{\partial y} + p\frac{\partial v}{\partial z} \end{vmatrix} = 0$$

where, $\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$

$$\Rightarrow \left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) - \left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0$$

$\Rightarrow$

$$\Rightarrow p\left(\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z}\right) + q\left(\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x}\right) = \left(\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}\right) \text{ --- (iv)}$$

Let, $\lambda P = \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z}$

$\lambda R = \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}$

$\lambda Q = \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x}$

Hence from eq^n (iv) we get,

$$\lambda Pp + \lambda Qq = \lambda R$$

$$\Rightarrow Pp + Qq = R \text{ --- (v)}$$

which is the partial differential equation in p and q and free from arbitrary f^n $\phi(u,v)$. also which is a Lagranges linear eq^n.

Let $u(x,y,z) = c_1$ and $v(x,y,z) = c_2$ be two integrals of eqn (5).

Differentiating these with respect to x,y,z respectively we get.

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0 \quad \text{--- (6)}$$

and $$\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz = 0 \quad \text{--- (7)}$$

Solving eqn (6) and (7) we get.

$$\frac{dx}{\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z} - \frac{\partial v}{\partial y}\cdot\frac{\partial u}{\partial z}} = \frac{dy}{\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x} - \frac{\partial v}{\partial z}\cdot\frac{\partial u}{\partial x}} = \frac{dz}{\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial y}}$$

$$\Rightarrow \frac{dx}{\lambda P} = \frac{dy}{\lambda Q} = \frac{dz}{\lambda R}$$

$$\Rightarrow \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

which is the ~~f~~ required Lagranges auxilary equation.

Ex.1 $\quad px + qy = z$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Ques1. Solve the PDE $\frac{y^2 z}{x} p + xzq = y^2$

Soln:- Given eqn $\frac{y^2 z}{x} p + xzq = y^2$ —— (i)

The Lagranges auxilary eqn is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{xz} = \frac{dz}{y^2}$$

Taking 1st and 2nd ratios

$$\frac{x\,dx}{y^2 z} = \frac{dy}{xz}$$

$$\Rightarrow x^2 dx = y^2 dy$$

$$\Rightarrow x^3 - y^3 = c_1 \text{ (by integrating)} \quad \text{—— (ii)}$$

Taking 1st and 3rd ratio.

$$x\,dx = z\,dz$$

$$x^2 - z^2 = c_2 \quad \text{—— (iii)}$$

Hence the general solution is

$\phi(x^3 - y^3, x^2 - z^2) = 0$ where ϕ is an arbitary function.

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T-MHU (D_6) (Linear P.D.E)

Prob.1 Solve the PDE $(3x+y-z)p+(x+y-z)q=2(z-y)$

Sol^n: Given eq^n $(3x+y-z)p+(x+y-z)q=2(z-y)$

The Lagrange A.E is $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

$$\Rightarrow \frac{dx}{3x+y-z}=\frac{dy}{x+y-z}=\frac{dz}{2(z-y)}$$

Choosing 1, -3, -1 as multiplier.

$$\frac{dx}{3x+y-z}=\frac{dy}{x+y-z}=\frac{dz}{2(z-y)}=\frac{dx-3dy-dz}{}$$

i.e. $dx-3dy-dz=0$

$\Rightarrow x-3y-z=c_1$ —— (i) [by integration]

Again choosing 1, 1, -1 and 1, -1, 1 as multiplier

then $\frac{dx}{3x+y-z}=\frac{dy}{x+y-z}=\frac{dz}{2(z-y)}=\frac{dx+dy-dz}{4(x+y-z)}=\frac{dx-dy+dz}{2(x-y+z)}$

Taking 4th and 5th ratio,

$$\frac{1}{2}\frac{dx+dy-dz}{x+y-z}=\frac{dx-dy+dz}{x-y+z}$$

$$\Rightarrow \frac{1}{2}\log(x+y-z)=\log(x-y+z)+\log c_2 \text{ by (integ)}$$

$$\Rightarrow \frac{\sqrt{x+y-z}}{x-y+z}=c_2 \text{ —— (ii)}$$

Hence the General sol^n $\phi\left(x-3y-z, \frac{\sqrt{x+y-z}}{x-y+z}\right)=0$

Prob.02 Solve the P.D.E $(y+zx)p-(x+yz)q=x^2-y^2$

Given eq^n $(y+zx)p-(x+yz)q=x^2-y$

Now Lagrange's A.E is $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

$$\Rightarrow \frac{dx}{(y+zx)}=\frac{dy}{-(x+yz)}=\frac{dz}{x^2-y^2}$$

Choosing $y, x, 1$ as multiplier.

$$\frac{dx}{(y+zx)}=\frac{dy}{-(x+yz)}=\frac{dz}{(x^2-y^2)}=\frac{y\,dx+x\,dy+dz}{0}$$

i.e $y\,dx+x\,dy+dz=0$

$\Rightarrow d(xy)+dz=0$

$\Rightarrow xy+z=c_1$ —— (i)

Again Choosing $x, y, -z$ as multiplier

then $\frac{dx}{---}=\frac{dy}{---}=\frac{dz}{---}=\frac{x\,dx+y\,dy-z\,dz}{0}$

i.e $x\,dx+y\,dy-z\,dz=0$

$\Rightarrow x^2+y^2-z^2=0$ —— (ii)

Hence the general sol^n $\phi(xy+z, (x^2+y^2-z^2))=0$

Prob3 Find the integral surface of the linear PDE $x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$ which contain the line $x+y=0$, $z=1$

Sol^n: The Lagrange's auxilary eq^n is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)}$$

Choosing $(x, y, -1)$ as multiplier

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)} = \frac{x\,dx + y\,dy - dz}{0}$$

i.e $x\,dx + y\,dy - dz = 0$

$\Rightarrow x^2+y^2-2z = c_1$ —— (i) [by integration]

Again choosing x, y, z as divider

$$\Rightarrow \frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

i.e $\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$

$\Rightarrow xyz = c_2$ —— (ii) [by integrating]

Since (1) and (2) passes $x+y=0$, $z=1$ (i) + (ii) eq

$\therefore xy = c_2$ and $x^2+y^2-2 = c_1$

$(x-y)y \div (y-x)x$
$(x-y)y + (x-y)x$

Sol^n: The lagrange A.E $= \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\Rightarrow \frac{dx}{(x-y)y^2} = \frac{dy}{(y-x)x^2} = \frac{dz}{(x^2+y^2)z}$$

Choosing, $x^2, y^2, 0$ and $1, -1, 0$ as multiplier

$$\frac{dx}{(x-y)y^2} = \frac{dy}{(y-x)x^2} = \frac{dz}{(x^2+y^2)z} = \frac{x^2dx + y^2dy}{0}$$

$$= \frac{dx-dy}{(x-y)(x^2+y^2)}$$

i.e. $x^2dx + y^2dy = 0$

$\Rightarrow x^3 + y^3 = c_1$ —— (i) [by integration]

Taking 3rd and 5th ratio. we get;

$$\frac{dz}{z} = \frac{dx-dy}{x-y}$$

$\Rightarrow \log z = \log(x-y) + \log c_2$ —— (ii) [by integration]

$\Rightarrow \frac{z}{x-y} = c_2$ —— (ii)

Since, it passes through $xz = a^3$, $y=0$

from (i) $x^3 = c_1$ and

from (ii) $\frac{z}{x} = c_2$

(Non linear P.D.E)

$\Rightarrow \frac{xz}{x^2} = c_2$

$\Rightarrow \frac{a^3}{x^2} = c_2$

$\frac{a^9}{(x^3)^2} = c_2^3$

$\Rightarrow \frac{a^9}{c_1^2} = c_2^3$

For non-linear —

i) Given eqn $F(x,y,z,p,q) = 0$

then, $F() = () = 0$

then we have to apply Charpit's Auxilary equation —

Charpit's Auxilary equation —

$$\frac{dp}{\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}} = \frac{dz}{-p\frac{\partial F}{\partial p} - q\frac{\partial F}{\partial q}} = \frac{dx}{-\frac{\partial F}{\partial p}} = \frac{dy}{-\frac{\partial F}{\partial q}}$$

$$\Rightarrow \frac{dp}{F_x + pF_z} = \frac{dq}{F_y + qF_z} = \frac{dz}{-pF_p - qF_q} = \frac{dx}{-F_p} = \frac{dy}{-F_q} \quad \text{(i)}$$

Math-2101 Date: 14-10-15

T-MHU (D7)

Problem 1 Find the complete integral of the given PDE by Charpit's method?

$$p^2 - y^2 q = y^2 - x^2$$

Sol^n:

Let $F(x,y,z,p,q) = p^2 - y^2 q - y^2 + x^2$ —— (i)

Charpit's A.E is

$$\frac{dp}{\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}} = \frac{dz}{-p\frac{\partial F}{\partial p} - q\frac{\partial F}{\partial q}} = \frac{dx}{-\frac{\partial F}{\partial p}} = \frac{dy}{-\frac{\partial F}{\partial q}}$$

$$\Rightarrow \frac{dp}{2x+0} = \frac{dq}{-2yq-2y+0} = \frac{dz}{-2p^2+y^2 q} = \frac{dx}{-2p} = \frac{dy}{y^2}$$

From 1st and 4th ratio

$$\frac{dp}{2x} = \frac{dx}{-2p}$$

$$\Rightarrow p^2 + x^2 = c_1 \text{ —— (ii)}$$

Now solving eq^n (i) and (ii)

$$c_1 - y^2 q - y^2 = 0$$

$$\Rightarrow q = \frac{c_1 - y^2}{y^2} = \left(\frac{c_1}{y^2} - 1\right)$$

and $p = \sqrt{c_1 - x^2}$

$\frac{(c_1-x^2)^{1/2+1}}{2}$ $3/2$

We know,

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$\Rightarrow dz = p\,dx + q\,dy$$

$$\Rightarrow dz = \sqrt{c_1 - x^2}\, dx + \left(\frac{c_1}{y^2} - 1\right) dy$$

$$\Rightarrow z = \frac{x\sqrt{c_1-x^2}}{2} + \frac{c_1}{2}\sin^{-1}\frac{x}{\sqrt{c_1}} - \frac{c_1}{y} - y + k$$ (by inte)

which is the ~~follow~~ required eqn of the given equation.

Problem 02 Solve, $z^2(p^2z^2+q^2) = 1$ by charpits method.

Soln: Let $F(x,y,z,p,q) = p^2z^4 + z^2q^2 - 1 = 0$ — (i)

Charpits A.E is,

$$\frac{dp}{\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}} = \frac{dz}{-p\frac{\partial F}{\partial p} - q\frac{\partial F}{\partial q}} = \frac{dx}{-\frac{\partial F}{\partial p}} = \frac{dy}{-\frac{\partial F}{\partial q}}$$

$$\Rightarrow \frac{dp}{0 + p(4p^2z^3+2zq^2)} = \frac{dq}{0+q(4p^2z^3+2zq^2)} = \frac{dz}{-p\,2pz^4 - 2qz^2} = \frac{dx}{-2pz^4}$$

$$= \frac{dy}{-2qz^2}$$

From 1st & 2nd ratio

$$\frac{dp}{p} = \frac{dq}{q}$$

$\therefore p = cq$ —— (ii)

Now solving eqn (i) and (ii)

$$c^2q^2z^4 + z^2q^2 - 1 = 0$$

$\Rightarrow q = \frac{1}{z\sqrt{c^2z^2+1}}$ and $p = \frac{c}{z\sqrt{c^2z^2+1}}$

we know,

$$dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy$$

$$\Rightarrow dz = p\,dx + q\,dy$$

$$\Rightarrow dz = \frac{c}{z\sqrt{c^2z^2+1}}\,dx + \frac{1}{z\sqrt{c^2z^2+1}}\,dy$$

$$\therefore z = cx + y + k$$

$\Rightarrow \frac{1}{3c^2}(c^2z^2+1)^{3/2} = cx + y + k$ (by integrating)

which is the required complete sol^n of (1)

Problem 03 Solve $pxy + pq + qy = yz$ by Charpit's method.

Solⁿ: Let $F(x,y,z,p,q) = pxy + pq + qy - yz = 0$ —— (i)

Charpit A.E is,

$$\frac{dp}{\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}} = \frac{dz}{-p\frac{\partial F}{\partial p} - q\frac{\partial F}{\partial q}} = \frac{dx}{-\frac{\partial F}{\partial p}} = \frac{dy}{-\frac{\partial F}{\partial q}}$$

$$\Rightarrow \frac{dp}{py + p(-y)} = \frac{dq}{q + px - qy - z} = \frac{dz}{-pxy - p.q - pq - qy} = \frac{dx}{-xy-q} = \frac{dy}{-(p+y)}$$

From first ratio,

$dp = 0$

$\Rightarrow p = c$ —— (ii)

Now solving eqⁿ (i) and (ii)

$cxy + cq + qy - yz = 0$

$\Rightarrow q = \frac{yz - cxy}{c+y}$

We know,

$dz = \frac{\partial z}{\partial x}.dx + \frac{\partial z}{\partial y}.dy$

$\Rightarrow dz = p\,dx + q\,dy$

$\Rightarrow dz = c\,dx + \frac{y(z-cx)}{c+y}dy$

$\log \gamma \quad \log e^{\gamma} \quad \log_{\gamma} e \quad ke \quad \log. e = \frac{1}{\log}$

$$\Rightarrow \frac{dz - c\,dx}{z - cx} = \left(\frac{y}{c+y}\right) dy \quad \left(1 - \frac{c}{c+y}\right)$$

$$\Rightarrow \log(z - cx) = y - c\log(c+y) + \log k$$

$$\Rightarrow z = cx + ke^{y} (c+y)^{-c}$$ Ans:

Problem 04 Solve $16p^2z^2 + 9q^2z^2 + 4z^2 - 4 = 0$ by Charpits method and identify the surface.

Soln: Let, $F(x, y, z, p, q) = 16p^2z^2 + 9q^2z^2 + 4z^2 - 4 = 0$ — (i)

Charpits A.E is,

$$\frac{dp}{\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{y} + q\frac{\partial F}{\partial z}} = \frac{dz}{-p\frac{\partial F}{\partial p} - q\frac{\partial F}{\partial q}} = \frac{dx}{-\frac{\partial F}{\partial p}} = \frac{dy}{-\frac{\partial F}{\partial q}}$$

$$\Rightarrow \frac{dp}{32p^3z + 18pq^2z + 8pz} = \frac{dq}{32p^2qz + 18q^3z + 8qz} = \frac{dz}{-32p^2z^2 - 18q^2z^2}$$

$$= \frac{dx}{-32pz^2} = \frac{dy}{-18qz^2} = \frac{dx + 4p\,dz + 4z\,dp}{0}$$

i.e, $dx + 4p\,dz + 4z\,dp = 0$

$\Rightarrow dx + 4\,d(pz)$

$\Rightarrow x + 4pz = c$

$\therefore p = \frac{c - x}{4z}$ — (ii)

Using (2) and (1)

$$\frac{16(c-x)^2}{16z^2}z^2+9q^2z^2+4z^2-4=0$$

$$9q^2z^2=4-4z^2-(c-x)^2$$

$$\therefore q=\frac{2}{3z}\sqrt{1-z^2-\frac{1}{4}(c-x)^2}$$

Now $dz=p\,dx+q\,dy$

$$\Rightarrow dz=\frac{c-x}{4z}dx+\frac{2}{3z}\sqrt{1-z^2-\frac{1}{4}(c-x)^2}\,dy$$

$$\Rightarrow z\cdot dz-\frac{1}{4}(c-x)dx=\frac{2}{3}\sqrt{1-z^2-\frac{1}{4}(c-x)^2}\,dy$$

Math-2101 Date: 25-10-15

T-MMR (B8)

Bessel's Function

Problem $\frac{d}{dx}\left[x J_n(x) J_{n+1}(x)\right] = x\left[J_n^2(x) - J_{n+1}^2(x)\right]$

$$\frac{d}{dx}\left[x J_n(x) J_{n+1}(x)\right] = J_n(x) J_{n+1}(x) + x\left[\frac{d}{dx}\left(J_n(x) J_{n+1}(x)\right)\right]$$

$$= J_n(x) J_{n+1}(x) + x\left[J_n(x) J'_{n+1}(x) + J'_n(x) J_{n+1}(x)\right]$$

$$= J_n(x) J_{n+1}(x) + \left[\left(x J'_n(x)\right) J_{n+1}(x) + J_n^x + \left(J'_{n+1}(x)\right)\right] \text{ --- (i)}$$

$$\left\{x J'_n(x)\right\} = n J_n(x) - x J_{n+1}(x) \text{ --- (ii)}$$

$$\left\{x J'_{n+1}(x)\right\} / \; x J'_n(x) = -n J_n(x) + x J_{n-1}(x) \text{ --- (iii)}$$

$n \to n+1$ in eq^n iii)

$$\left[x J'_{n+1}(x)\right] = -(n+1) J_{n+1}(x) + x J_n(x) \text{ --- (4)}$$

Orthogonality of Bessel Functions —

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0$$

where $y = J_n(x)$

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left[(\alpha x)^2-n^2\right]y=0 \quad \text{— (i)}$$

$$y = J_n(\alpha x)$$

$$\therefore\ x^2\frac{d^2z}{dx^2}+x\frac{dz}{dx}+\left[(x\beta)^2-n^2\right]z=0 \quad \text{— (ii)}$$

$$z = J_n(\beta x)$$

$$\int_0^1 x\, J_n(\alpha x)\, J_n(\beta x)\, dx = 0$$

if α, β be the roots of $J_n(x) = 0$

Multiplying (i) by z/x and (ii) by $-y/x$ and then adding we get.

$$x\left[z\frac{d^2y}{dx^2}-y\frac{d^2z}{dx^2}\right]+\left[z\frac{dy}{dx}-y\frac{dz}{dx}\right]+(\alpha^2-\beta^2)xyz=0 \quad \text{— (iii)}$$

$$\Rightarrow \frac{d}{dx}\left[x\left(z\frac{dy}{dx}-y\frac{dz}{dx}\right)+(\alpha^2-\beta^2)xyz=0\right.$$

$$(\beta^2-\alpha^2)\int_0^1 xyz\,dx = \left[x\left(z\frac{dy}{dx} - y\frac{dz}{dx}\right)\right]_0^1$$

$$= \left(z\frac{dy}{dx} - y\frac{dz}{dx}\right)$$

$$(\beta^2-\alpha^2)\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = \Big[\alpha J_n'(\alpha x) J_n(\beta x) - \beta J_n'(\beta x) J_n(\alpha x)\Big]_{x=1}$$

$$= \alpha J_n'(\alpha) J_n(\beta) - \beta J_n'(\beta) J_n(\alpha)$$

Problem 02. Show that.

$$\int_0^1 x J_n^2(\alpha x)\,dx = \frac{1}{2}\left[J_n'(\alpha)\right]^2$$

MHU (D_8) Date: 27-10-15

Problem 05 Find the complete and singular integral of

$$(p^2+q^2)y = qz$$

Soln: Let $f(x,y,z,p,q) = (p^2+q^2)y - qz = 0$ —— (i)

Then the Charpits A.E is,

$$\frac{dp}{\frac{\partial F}{\partial x}+p\frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{\partial y}+q\frac{\partial F}{\partial z}} = \frac{dz}{-p\frac{\partial F}{\partial p}-q\frac{\partial F}{\partial q}} = \frac{dx}{-\frac{\partial F}{\partial p}} = \frac{dy}{-\frac{\partial F}{\partial q}}$$

$$\Rightarrow \frac{dp}{-pq} = \frac{dq}{p^2+q^2-q^2} = \frac{dz}{-2p^2y-2q^2y+qz} = \frac{dx}{-2py} = \frac{dy}{-2qy+z}$$

From 1st and 2nd ratio,

$$\frac{dp}{-pq} = \frac{dq}{p^2+q^2-q^2}$$

or, $\frac{dp}{-q} = \frac{dq}{p}$

$\Rightarrow p^2+q^2 = a$ (constant) —— (ii)

Solving (i) and (ii)

$$q = \frac{ay}{z} \quad \text{and} \quad p = \frac{\sqrt{az^2-a^2y^2}}{z}$$

We know, $dz = p\,dx + q\,dy$

$$\Rightarrow dz = \frac{\sqrt{az^2-a^2y^2}}{z}dx + \frac{ay}{z}dy$$

$\Rightarrow \dfrac{az\,dz - a^2 y\,dy}{\sqrt{az^2 - a^2y^2}} = a\,dx$

$\Rightarrow \sqrt{az^2 - a^2y^2} = ax + b$

$\Rightarrow az^2 - a^2y^2 = (ax+b)^2$ —— (iii)

Diff. eq^n (3) w.r.t a and b respectively

$z^2 - 2ay^2 = 2x\,(ax+b)$ —— (iv)

and $0 = 2(ax+b)$

$\Rightarrow ax+b = 0$ —— (v)

yes pehle manforce

From (iv) $z^2 - 2ay^2 = 0$

$a = \dfrac{z^2}{2y^2}$

and From (v) $b = \dfrac{-xz^2}{2y^2}$

putting this value of a, b in eq^n (iii)

$\dfrac{z^4}{2y^2} - \dfrac{z^4}{4y^4}\cdot y^2 = 0$ Ans

$-2xy+p$

Problem 06 Find the Complete and Singular integral of

$$2xz - px^2 - 2qxy + pq = 0$$

Sol^n: Let $(F\ x,y,z,p,q) = 2xz - px^2 - 2qxy + pq = 0$

Charpit's A.E is –

$$\frac{dp}{\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}} = \frac{dz}{-p\frac{\partial F}{\partial p} - q\frac{\partial F}{\partial q}} = \frac{dx}{-\frac{\partial F}{\partial p}} = \frac{dy}{-\frac{\partial F}{\partial q}}$$

$$\frac{dp}{2z - 2px - 2qx + 2px} = \frac{dq}{-2qx + 2qx} = \frac{dz}{2qxy - pq + 2x^2p - pq} = \frac{dx}{x^2 + q} = \frac{dy}{2xy - p}$$

From 2nd ratio.

$dq = 0$

$\Rightarrow q = \text{constant} = a$ —— (ii)

$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$

$= p\,dx + q\,dy$

solving eq^n (i) and (ii)

$q = a,\ p = \frac{2x(z - ay)}{x^2 - a}$

We know $dz = p\,dx + q\,dy$

$$\Rightarrow dz = \frac{2x(z - ay)}{x^2 - a}dx + a\,dy$$

$$\Rightarrow \frac{dz - a\,dy}{z - ay} = \frac{2x}{x^2 - a}\cancel{dy}\,dx$$

$$\Rightarrow \log(z - ay) = \log(x^2 - a) + \log b$$

$$\therefore z = ay + b(x^2 - a) \text{ —— (iii)}$$

Diff. eqn (iii) with respect to a & b respectively

$0 = y - b$ and $0 = x^2 - a$

$\therefore b = y$ $\Rightarrow a = x^2$

Now From eqn (iii)

$z = x^2 y$ (answer)

H.W

Problem 06 → Solve $2z + p^2 + qy + 2y^2 = 0$ by Charpit's Method

Problem 07 → Solve $p^2 + q^2 = py - qx$ by Charpit's Method.

$ay + b(x^2 - a)$

$x^2 y + x^2 y (x^2 - x^2)$

$x^2 y$

Math-2101
T-MHU (Dg)

Date: 03-11-15

Two dimension polar co-ordinate

Solve: $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$

Solⁿ: $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$ —— (i)

Let. $u(r,\theta) = R(r)\,F(\theta)$ —— (ii)

$R''F + \frac{1}{r}R'F + \frac{1}{r^2}RF'' = 0$ —— (iii)

$\Rightarrow F\left(R'' + \frac{R'}{r}\right) = \frac{-RF''}{r^2}$

$\Rightarrow \frac{1}{R}\left(r^2R'' + rR'\right) = \frac{-F''}{F} = n^2$ (say)

$\therefore r^2R'' + rR' - n^2R = 0$ —— (iv)

and $F'' + n^2F = 0$ —— (v)

From eqⁿ (iv) $r^2R'' + rR' - n^2R = 0$
which is a linear homogeneous differential eqⁿ.

$r = e^z$

$\therefore \log r = \cancel{e^z}\, z$

$\Rightarrow \frac{dz}{dr} = \frac{1}{r}$

$$\frac{dR}{dr} = \frac{dR}{dz}\cdot\frac{dz}{dr} = \frac{1}{r}\frac{dR}{dz} \text{ ——— (vii)}$$

$$\Rightarrow r\frac{dR}{dr} = \frac{dR}{dz}$$

Diff. eqn (vii) w.r.t r.

$$\frac{d}{dr}\left(\frac{dR}{dr}\right) = \frac{d}{dr}\frac{1}{r}\left(\frac{dR}{dz}\right)$$

$$\Rightarrow \frac{d^2R}{dr^2} = \frac{d}{dz}\frac{1}{r}\left(\frac{dR}{dz}\right)\frac{dz}{dr}$$

$$= \left[\frac{1}{r}\cdot\frac{d^2R}{dz^2} + \frac{dR}{dz}\frac{d}{dz}\left(\frac{1}{r}\right)\right]\frac{1}{r}$$

$$= \left[\frac{1}{r}\frac{d^2R}{dz^2} + \frac{dR}{dz}\frac{d}{dr}\left(\frac{1}{r}\right)\frac{dr}{dz}\right]\frac{1}{r}$$

$$= \frac{1}{r^2}\frac{d^2R}{dz^2} - \frac{dR}{dz}\cdot\frac{1}{r^2}\cdot r\cdot\frac{1}{r}$$

$$= \frac{1}{r^2}\left(\frac{d^2R}{dz^2} - \frac{dR}{dz}\right)$$

$$\therefore r^2\frac{d^2R}{dr^2} = \frac{d^2R}{dz^2} - \frac{dR}{dz}$$

Again from eqn (4)

$$\frac{d^2R}{dz^2} - \frac{dR}{dz} + \frac{dR}{dz} - n^2R = 0$$

$$\Rightarrow \frac{d^2R}{dz^2} - n^2 R = 0$$

Auxilary equation $m^2 - n^2 = 0$

$\therefore m = \pm n$

Hence Complementary Sol^n —

$$R(r) = A_n e^{nz} + B_n e^{-nz}$$

$$= A_n r^n + B_n r^{-n} \quad \text{——— (viii)}$$

Also from eq^n (v)

$$F(\theta) = C_n \cos n\theta + D_n \sin n\theta \quad \text{——— (ix)}$$

Hence the General Solution of eq^n (i)

$$u = \left(A_n r^n + B_n r^{-n}\right)\left(C_n \cos n\theta + D_n \sin n\theta\right)$$ Ans.

Math – 2101 Date: 04-11-15

T-MHU (E₉)

Problem Q3. #3: Laplaces eq^n in three dimensional Co-ordinate

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$$

Problem 03

Solve $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$

Sol^n: Given that, $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$ ——— (i)

Let, $u(x,y,z) = X(x)\, Y(y)\cdot Z(z)$

Then, $\frac{\partial u}{\partial x} = X'YZ \Rightarrow \frac{\partial^2 u}{\partial x^2} = X''YZ$

Similarly, $\frac{\partial^2 u}{\partial y^2} = XY''Z$, $\frac{\partial^2 u}{\partial z^2} = XYZ''$

Using these in eq^n (i) and dividing by XYZ

$$\frac{X''}{X}+\frac{Y''}{Y}+\frac{Z''}{Z}=0$$

$$\Rightarrow \frac{X''}{X}+\frac{Y''}{Y}=-\frac{Z''}{Z} \quad \text{——— (ii)}$$

The following three cases arises,

Case-I: when each ratio is equal to zero

then $X''=0$, $Y''=0$, $Z''=0$

in this case the general solⁿ is $X=Ax+B$, $Y=Cy+D$, $Z=Ez+F$

Hence the required solⁿ of (i) is,

$$u=(Ax+B)\cdot(Cy+D)(Ez+F)$$

Where, A, B, C, D, E, F are constant.

Case-II:- Let $\frac{X''}{X}=-n^2$; $\frac{Y''}{Y}=-m^2$ and $\frac{Z''}{Z}=p^2$

such that, $m^2+n^2=p^2$

Now, $X''+n^2X=0$ giving $X=A\cos nx+B\sin nx$

$Y''+m^2Y=0$ giving $Y=C\cos my+D\sin my$

also $Z''-p^2Z=0$ giving $Z=Ee^{pz}+Fe^{-pz}$

Hence the required solⁿ of (i) is

$$u=(A\cos nx+B\sin nx)(C\cos my+D\sin my)(Ee^{pz}+Fe^{-pz})$$

where, $p^2=m^2+n^2$, A, B, C, D, E, F are constant.

Case-III: Let $\frac{X''}{X} = n^2$, $\frac{Y''}{Y} = m^2$ and $\frac{Z''}{Z} = -p^2$

such that $m^2 + n^2 = p^2$

Now,

$X'' - n^2 X = 0$ giving $X = Ae^{nx} + Be^{-nx}$

$Y'' - m^2 Y = 0$ giving $Y = Ce^{my} + De^{-my}$

Also $Z'' + p^2 Z = 0$ giving $Z = E\cos pz + F\sin pz$

Hence the required sol^n of (i) is

$$u = (Ae^{nx} + Be^{-nx})(Ce^{my} + De^{-my})(E\cos pz + F\sin pz)$$

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Math-2101

T-MHU (E_{11})

Date: 23-11-15

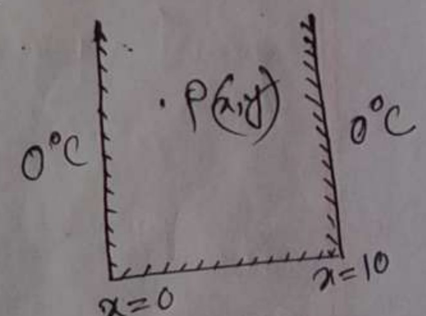
Problem 01

A rectangular plate with insulated surfaces is 10 cm wide and so long compare to it's width that it may be considered infinite in length without introducing an appreciable error. If the temperature along the short edge $y=0$ is given by $u(0,x) = 20x$; $0 < x \le 5$

$= 20(10-x)$; $5 < x < 10$

while the two long edges $x=0$ and $x=10$ as well as the other short edges are kept at $0°C$. Find the steel state temperature at (x,y) of the plate?

Soln:- In the steady state, the temperature $u(x,y)$ at any point $P(x,y)$ satisfy the equation,



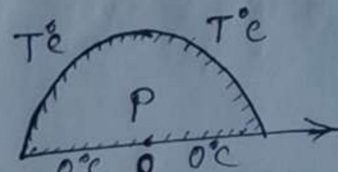
$$\frac{\partial^2 u}{\partial x^2} + \quad 0$$

(Math) D_{12} Date: 29-11-15
(MHU)

Problem 02

The diameter of a semicircular plate of radious a kept at $0°c$ and the temperature at the semi-circular boundary is $T°c$. Find the steady state temperature in the plate.

Solⁿ: Let the centre O of the semi-circular plate be the pole and the bounding diameter be as the initial line.



Let, $u(r,\theta)$ be the steady state temperature at any point $p(r,\theta)$ and u satisfies the equation –

$$r^2\frac{\partial^2 u}{\partial r^2}+r\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial \theta^2}=0 \quad (1)$$

The boundary coditions are:

i) $u(r,0)=0, \quad 0\le r\le a$

ii) $u(r,\pi)=0, \quad 0\le r\le a$

iii) $u(a,\theta)=T$

From condition ii) and iii) we have $u\to 0$ as $r\to 0$

The solⁿ of eqⁿ (1) $u=(c_1r^p+c_2r^{-p})(c_3\cos p\theta+c_4\sin p\theta)$ (2)

putting $u(r,0)=0$, $0=(c_1r^p+c_2r^{-p})c_3$

$\therefore c_3=0$

$\therefore$ The eqⁿ (2) becomes $u=(c_1r^p+c_2r^{-p})c_4\sin p\theta$ (iii)

Using $u(r,\pi)=0$ in eqⁿ (3) $0=(c_1r^p+c_2r^{-p})c_4\sin p\pi$

$\therefore \sin p\pi=0$

$\therefore p = n \quad [\because p\pi = n\pi]$

$\therefore$ eqn (3) becomes,

$$u = (c_1 r^n + c_2 r^{-n})\, c_4 \sin n\theta \quad \text{—— (4)}$$

Since $u = 0$ when $r = 0$, $0 = c_2 \quad \left[c_2 \frac{1}{r^n} = 0\right]$

eqn (4) beomes

$$u = c_1 c_4 r^n \sin n\theta$$

The general solution of (1) $u(r,\theta) = \sum_{n=1}^{\infty} b_n r^n \sin n\theta$ — (5)

putting $r = a$, & $u = T$ in (5)

$$T = \sum b_n a^n \sin n\theta$$

By Fourier half range sine series.

$$b_n a^n = \frac{2}{\pi}\int_0^{\pi} T \sin n\theta \, d\theta = \frac{2}{\pi} T \left[\frac{-\cos n\theta}{n}\right]_0^{\pi} \quad \frac{1}{r^n} = \frac{1}{0} =$$

$$= \frac{2T}{n\pi}\left[-(-1)^n + 1\right]$$

$\therefore b_n a^n = 0$ when n is even.

$b_n a^n = \frac{4T}{n\pi}$ when n is odd. $\Rightarrow b_n = \frac{4T}{n\pi a^n}$

Hence 5 becomes $u(r,\theta) = \frac{4T}{\pi} \sum_{n=1}^{\infty} \frac{\left(\frac{r}{a}\right)^n}{n} \sin n\theta$. when n is o

Spherical:- $\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\cot\phi}{r^2}\frac{\partial u}{\partial \phi} + \frac{1}{r^2 \sin^2\phi}\frac{\partial^2 u}{\partial \theta^2} =$

$\Rightarrow \frac{1}{r^2}\left[r^2 \frac{\partial^2 u}{\partial r^2} + 2r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \phi^2} + \cot\phi \frac{\partial u}{\partial \phi} + \frac{1}{\sin^2\phi}\frac{\partial^2 u}{\partial^2 \theta} = 0\right]$