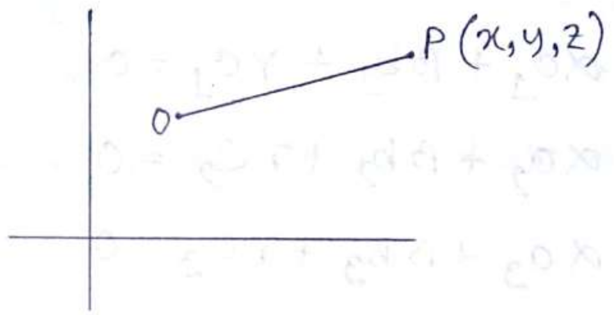
04-02-2018 : 1E : Sunday

A vector is a quantity having both magnitude and direction such as displacement, velocity, force and acceleration.



$\vec{OP} = x\hat{i} + y\hat{j} + z\hat{k}$

→ If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three vector, $\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c}$ is called linear combination.

→ If $\vec{a}, \vec{b}$ and $\vec{c}$ is linearly dependent, $\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = 0$ where, $\alpha, \beta, \gamma \neq 0$

→ If $\vec{a}, \vec{b}$, and $\vec{c}$ is linearly independent, $\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = 0$ where, $\alpha = \beta = \gamma = 0$

→ If $\vec{a}, \vec{b}$ and $\vec{c}$ is linearly dependent, they are co-planar means they will be in one plane.

→ If $\vec{a}, \vec{b}$ and $\vec{c}$ is linearly independent, they are non-coplanar.

Let,

$$\vec{a} = (a_1, a_2, a_3)$$
$$\vec{b} = (b_1, b_2, b_3)$$
$$\vec{c} = (c_1, c_2, c_3)$$

Now,

$$\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = 0$$

$$\Rightarrow \alpha (a_1, a_2, a_3) + \beta (b_1, b_2, b_3) + \gamma (c_1, c_2, c_3) = (0,0,0)$$

$$\Rightarrow (\alpha a_1 + \beta b_1 + \gamma c_1 , \alpha a_2 + \beta b_2 + \gamma c_2 , \alpha a_3 + \beta b_3 + \gamma c_3) = (0,0,0)$$

equalling the co-efficient of each axis,

$$\alpha a_1 + \beta b_1 + \gamma c_1 = 0$$

$$\alpha a_2 + \beta b_2 + \gamma c_2 = 0$$

$$\alpha a_3 + \beta b_3 + \gamma c_3 = 0$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{vmatrix} \alpha \\ \beta \\ \gamma \end{vmatrix} = 0$$

- If the determinants of $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is zero,

  then the solution will be non-zero and $(\vec{a}, \vec{b}, \vec{c})$ will be dependent.

- If the determinants of $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is non-zero,

  then the solution will be zero and $(\vec{a}, \vec{b}, \vec{c})$ will be independent.

- If the no. of variable > the no. of equation, the solution is non-zero

- If the no. of variable = the no. of equation, the solution is zero.

→ Show that the three vectors $\vec{A} = 2\hat{i} + \hat{j} - 3\hat{k}$, $\vec{B} = \hat{i} - 4\hat{k}$ and $\vec{C} = 4\hat{i} + 3\hat{j} - \hat{k}$ are linearly dependent. Determinate the relation between them.

Solution:

$$\vec{A} = 2\hat{i} + \hat{j} - 3\hat{k}$$
$$\vec{B} = \hat{i} - 4\hat{k}$$
$$\vec{C} = 4\hat{i} + 3\hat{j} - \hat{k}$$

$$\begin{vmatrix} 2 & 1 & -3 \\ 1 & 0 & -4 \\ 4 & 3 & -1 \end{vmatrix} = 2(0+12) - 1(-1+16) - 3(3-0)$$
$$= 24 - 15 - 9 = 0$$

As the determinant is zero, the three vectors will be linearly dependent.

$$\alpha\vec{A} + \beta\vec{B} + \gamma\vec{C} = 0$$
$$\Rightarrow \alpha(2,1,-3) + \beta(1,0,-4) + \gamma(4,3,-1) = 0$$
$$\cancel{\Rightarrow (2\alpha + \beta + 4\gamma), \alpha + 3\gamma, (-3\alpha - 4\beta - \gamma)}$$
$$\Rightarrow (2\alpha + \beta + 4\gamma, \alpha + 3\gamma, -3\alpha - 4\beta - \gamma) = (0,0,0)$$

which implies,
$$2\alpha + \beta + 4\gamma = 0$$
$$\alpha + 3\gamma = 0$$
$$-3\alpha - 4\beta - \gamma = 0$$

the cofficient matrix. of the system.

$$C = \begin{vmatrix} 2 & 1 & 4 \\ 1 & 0 & 3 \\ -3 & -4 & -1 \end{vmatrix} \xrightarrow{R_{21}} \begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & 4 \\ -3 & -4 & -1 \end{vmatrix}$$

$$\xrightarrow[R_{31}(3)]{R_{21}(-2)} \begin{vmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & -4 & 8. \end{vmatrix} \xrightarrow{R_{32}(4)} \begin{vmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{vmatrix}$$

This is the echolon form of the matrix.
hence the rank of the matrix, $\rho(c) = 2$
since, $\rho(c) < n$, the above system has infinitely many solution. and the number of free variable $= n - \rho(c) = 3-2 = 1$

Let, γ be the freee variable. and $\gamma = K$ where K is any integer number.

the equivalent system.

$$\alpha + 3\gamma = 0 \quad \therefore \alpha = -3K$$
$$\beta - 2\gamma = 0 \quad \therefore \beta = 2K$$
$$\gamma = K$$

$$\therefore -3K\vec{A} + 2K\vec{B} + k\vec{C} = 0$$

$$\therefore 3\vec{A} - 2\vec{B} - \vec{C} = 0$$

05-02-2018 : 2A : Monday

→ Dot and Cross product :

- Dot product :

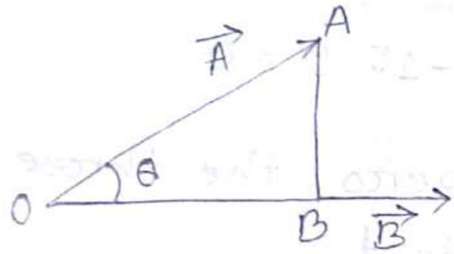
The dot product of two vectors $\vec{A}$ and $\vec{B}$, denoted by $\vec{A}\cdot\vec{B}$, is defined as the the product of the magnitudes of $\vec{A}$ and $\vec{B}$ and the cosine of the angle θ between them.

$$\vec{A}\cdot\vec{B} = AB\cos\theta$$

$$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$$

Unit vector :

A unit vector is a vector having unit magnitude. If $\vec{A}$ is a vector with magnitude and $A \neq 0$, then the unit vector of $\vec{A}$, $\vec{a} = \frac{\vec{A}}{A}$



$OA = A =$ Value of $\vec{A}$

$OB =$ projection on $\vec{B}$ of $\vec{A}$

$$\cos\theta = \frac{OB}{OA} \Rightarrow OB = OA\cos\theta \Rightarrow OB = A\cos\theta$$

$$\vec{A}\cdot\vec{B} = AB\cos\theta$$

$$\Rightarrow \vec{A}\cdot\vec{B} = OB \cdot B$$

$$\Rightarrow OB = \frac{\vec{A}\cdot\vec{B}}{B}$$

$$\Rightarrow OB = \vec{A}\cdot\frac{\vec{B}}{B}$$

$$\Rightarrow OB = \vec{A}\cdot\vec{b}$$

projection of $\vec{A}$ on $\vec{B} = \vec{A} \cdot \vec{b} = \vec{A} \cdot \frac{\vec{B}}{B}$

projection of $\vec{B}$ on $\vec{A} = \vec{B} \cdot \vec{a} = \vec{B} \cdot \frac{\vec{A}}{A}$

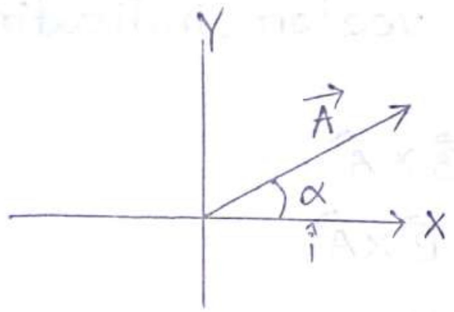
$\hat{i}$ = ~~components~~ component vector of any vector in the x-axis

$\hat{j}$ = " " of " " " " y-axis

$\hat{k}$ = " " of " " " " z-axis

$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$

$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{i} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{j} = \hat{k} \cdot \hat{i} = \hat{i} \cdot \hat{k} = 0$



$\vec{A} \cdot \hat{i} = A|\hat{i}| \cos\alpha = A\cos\alpha$

$\therefore \cos\alpha = \frac{\vec{A} \cdot \hat{i}}{A} = \vec{a} \cdot \hat{i}$

$\vec{A} \cdot \hat{j} = A|\hat{j}| \cos\beta = A\cos\beta$

$\therefore \cos\beta = \frac{\vec{A} \cdot \hat{j}}{A} = \vec{a} \cdot \hat{j}$

$\vec{A} \cdot \hat{k} = A|\hat{k}| \cos\gamma = A\cos\gamma$

$\therefore \cos\gamma = \frac{\vec{A} \cdot \hat{k}}{A} = \vec{a} \cdot \hat{k}$

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- Cross product:

The cross product of $\vec{A}$ and $\vec{B}$ is a vector $\vec{C} = \vec{A} \times \vec{B}$. The magnitude of $\vec{A} \times \vec{B}$ is defined as the product of the magnitudes of $\vec{A}$ and $\vec{B}$ and the sine of the angle θ between them. The direction of the vector $\vec{C} = \vec{A} \times \vec{B}$ is perpendicular to the plane of $\vec{A}$ and $\vec{B}$ and such that $\vec{A}$, $\vec{B}$ and $\vec{C}$ form a right-handed system.

In symbols,

$$\vec{A} \times \vec{B} = AB \sin\theta \, \hat{n}$$

where $\hat{n}$ is a unit vector indicating the direction of $\vec{A} \times \vec{B}$.

$$\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$$

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$

$$\hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{i} = -\hat{k}$$

$$\hat{j} \times \hat{k} = \hat{i}, \quad \hat{k} \times \hat{j} = -\hat{i}$$

$$\hat{k} \times \hat{i} = \hat{j}, \quad \hat{i} \times \hat{k} = -\hat{j}$$

→ Find the projection of vector $\vec{A} = \hat{i} - 2\hat{j} + \hat{k}$ on the vector $\vec{B} = 4\hat{i} - 4\hat{j} + 7\hat{k}$.

Solution:

∴ The projection, $\vec{A} \cdot \vec{b} = \dfrac{\vec{A} \cdot \vec{B}}{B}$

$$= \frac{(\hat{i} - 2\hat{j} + \hat{k}) \cdot (4\hat{i} - 4\hat{j} + 7\hat{k})}{\sqrt{16 + 16 + 49}}$$

$$= \frac{4 + 8 + 7}{9} = \frac{19}{9}$$

→ Find the work done in moving an object along a vector $\vec{r} = 3\hat{i}+2\hat{j}-5\hat{k}$ if the applied force is $\vec{F}=2\hat{i}-\hat{j}-\hat{k}$

Solution:

$$\text{work done} = \vec{F}\cdot\vec{r}$$
$$= (2\hat{i}-\hat{j}-\hat{k})\cdot(3\hat{i}+2\hat{j}-5\hat{k})$$
$$= 6-2+5$$
$$= 9$$

work done along a path $= \int_C F\cdot dr$ Here, C is the curve or path

→ Determine the unit vector perpendicular to the plane of $\vec{A} = 2\hat{i}-6\hat{j}-3\hat{k}$ and $\vec{B} = 4\hat{i}+3\hat{j}-\hat{k}$.

Solution: Method-1:-

$\vec{A}\times\vec{B}$ is a vector perpendicular to the plane of $\vec{A}$ and $\vec{B}$.

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -6 & -3 \\ 4 & 3 & -1 \end{vmatrix}$$
$$= \hat{i}(6+9) - \hat{j}(-2+12) + \hat{k}(6+24)$$
$$= 15\hat{i} - 10\hat{j} + 30\hat{k}$$

$$\text{unit vector of } \vec{A}\times\vec{B} = \frac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|}$$
$$= \frac{15\hat{i}-10\hat{j}+30\hat{k}}{\sqrt{225+100+900}}$$
$$= \frac{1}{7}(3\hat{i}-2\hat{j}+6\hat{k})$$

Method - 2 :-

Let, perpendicu~~unit~~lar vector, $\vec{c} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\vec{A} \cdot \vec{c} = 2x - 6y - 3z = 0 \quad \text{------------} \quad (i)$$

$$\vec{B} \cdot \vec{c} = 4x + 3y - z = 0 \quad \text{------------} \quad (ii)$$

$$\frac{x}{6+9} = \frac{y}{-12+2} = \frac{z}{6+24} = p \quad [\text{Let}]$$

$\Rightarrow x = 15p,\ y = -10p,\ z = 30p$

$\therefore \vec{c} = 15p\hat{i} - 10p\hat{j} + 30p\hat{k}$

$$\therefore \text{unit vector} = \frac{\vec{c}}{c}$$

$$= \frac{15p\hat{i} - 10p\hat{j} + 30p\hat{k}}{\sqrt{225p^2 + 100p^2 + 900p^2}}$$

$$= \cancel{5p\,(15\hat{i} - 10\hat{j}}$$

$$= \frac{5p(3\hat{i} - 2\hat{j} + 6\hat{k})}{35p}$$

$$= \frac{1}{7}(3\hat{i} - 2\hat{j} + 6\hat{k})$$

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→ Prove that the area of a parallelogram with sides $\vec{A}$ and $\vec{B}$ is $|\vec{A}\times\vec{B}|$.

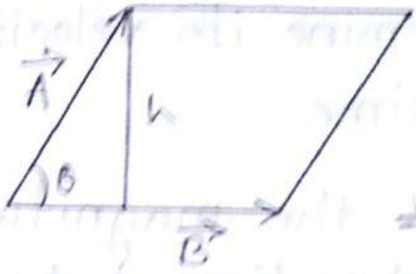
Solution:

Area of parallelogram

$= hB$

$= A\sin\theta\, B$

$= |\vec{A}\times\vec{B}|$



$\sin\theta = \frac{h}{A}$

$\Rightarrow h = A\sin\theta$

11-02-2018 : 2E : Sunday

Vector differentiation

$\vec{R}(u)$ is a vector which is a function of u. The differentiation of $\vec{R}(u)$ is $\frac{d\vec{R}}{du}$. If $\vec{R}(x,y,z)$ is vector which is a function of x, y and z, then the partial differentiation of $\vec{R}(x,y,z)$ is $\frac{\partial\vec{R}}{\partial x}, \frac{\partial\vec{R}}{\partial y}$ and $\frac{\partial\vec{R}}{\partial z}$.

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$ surface, $s = x^2+y^2+z^2$

$d\vec{r}\cdot d\vec{r} = dx^2 + dy^2 + dz^2 = ds^2$

$|d\vec{r}|^2 = ds^2$

$|d\vec{r}| = ds$

→ A particle moves along a curve whose parametric equations are $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$ where t is the time.

i) Determine its velocity and acceleration at any time.

ii) Find the magnitudes of the velocity and acceleration at $t=0$.

Solution:

We know,

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$= e^{-t}\hat{i} + 2\cos 3t\,\hat{j} + 2\sin 3t\,\hat{k}$$

i)

$$\text{Velocity}, \vec{v} = \frac{d\vec{r}}{dt}$$
$$= \frac{d}{dt}\left(e^{-t}\hat{i} + 2\cos 3t\,\hat{j} + 2\sin 3t\,\hat{k}\right)$$
$$= -e^{-t}\hat{i} - 6\sin 3t\,\hat{j} + 6\cos 3t\,\hat{k}$$

$$|\vec{v}| = \sqrt{e^{-2t} + 36\sin^2 3t + 36\cos^2 3t}$$

$$\Rightarrow v = \sqrt{e^{-2t} + 36}$$

$$\text{Acceleration}, \vec{a} = \frac{d\vec{v}}{dr}$$
$$= \frac{d}{dr}\left(-e^{-t}\hat{i} - 6\sin 3t\,\hat{j} + 6\cos 3t\,\hat{k}\right)$$
$$= e^{-t}\hat{i} - 18\cos 3t\,\hat{j} - 18\sin 3t\,\hat{k}$$

$$|\vec{a}| = \sqrt{e^{-2t} + 324\cos^2 3t + 324\sin^2 3t}$$

$$\Rightarrow a = \sqrt{e^{-2t} + 324}$$

ii)

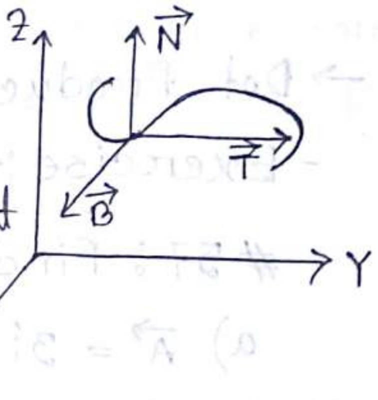
At $t = 0$,

$$v = \sqrt{e^0 + 36} = \sqrt{37}$$

$$a = \sqrt{e^0 + 324} = \sqrt{325}$$

Differential Geometry

If C is a curve defined by the function $\vec{r}(u)$, then $\frac{d\vec{r}}{du}$ is a vector in the direction of ~~the~~ the tangent to C. Here, $\vec{T}$ is the tangent to C, which is the unit tangent vector.



$$\therefore \frac{d\vec{r}}{ds} = \vec{T}$$

The curvature of the curve is $\frac{d\vec{T}}{ds}$ which is equal to $k\vec{N}$

$$\therefore \frac{d\vec{T}}{ds} = k\vec{N}$$

$\rho = \frac{1}{k}$ where, ρ is called the radius of curvature.

Here, $\vec{N}$ is a unit vector in this normal direction which is called unit normal vector. k is called the ~~called~~ curvature of C at the specified point.

$\vec{B} = \vec{T} \times \vec{N}$

where, $\vec{B}$ is called Bi-normal vector of the curve which direction is perpendicular to the plane of $\vec{T}$ and $\vec{N}$.

Frenet-Serret formula:

i) $\frac{d\vec{T}}{ds} = k\vec{N}$ ii) $\frac{d\vec{N}}{ds} = \tau \vec{B} - k\vec{T}$

iii) $\frac{d\vec{B}}{ds} = -\tau \vec{N}$

12-02-2018: 3A: Monday

→ Dot Product:

- Exercise:-

#57: Find the angle between:

a) $\vec{A} = 3\hat{i} + 2\hat{j} - 6\hat{k}$ and $\vec{B} = 4\hat{i} - 3\hat{j} + \hat{k}$

b) $\vec{C} = 4\hat{i} - 2\hat{j} + 4\hat{k}$ and $\vec{D} = 3\hat{i} - 6\hat{j} - 2\hat{k}$

Solution:

a) $\vec{A} \cdot \vec{B} = AB\cos\theta$

$\Rightarrow \cos\theta = \frac{\vec{A} \cdot \vec{B}}{AB}$

$\Rightarrow \cos\theta = \frac{(3\hat{i} + 2\hat{j} - 6\hat{k}) \cdot (4\hat{i} - 3\hat{j} + \hat{k})}{\sqrt{9+4+36}\sqrt{16+9+1}}$

$\Rightarrow \theta = \cos^{-1}\left(\frac{0}{7\sqrt{26}}\right)$

$\therefore \theta = 90^\circ$

b)

$$\vec{C} \cdot \vec{D} = CD\cos\theta$$

$$\Rightarrow \cos\theta = \frac{\vec{C}\cdot\vec{D}}{CD}$$

$$\Rightarrow \cos\theta = \frac{(4\hat{i}-2\hat{j}+4\hat{k})\cdot(3\hat{i}-6\hat{j}-2\hat{k})}{\sqrt{16+4+16}\,\sqrt{9+36+4}}$$

$$\Rightarrow \cos\theta = \frac{16}{42}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{8}{21}\right) \cancel{=6}$$

#59: Find the acute angles which the line joining the points (1, −3, 2) and (3, −5, 1) makes with the co-ordinate axes.

Solution:

Let, the vector of the line joining the points $\vec{A} = (3-1)\hat{i} + (-5+3)\hat{j} + (1-2)\hat{k} = 2\hat{i} - 2\hat{j} - \hat{k}$

Again, α, β and γ be the angles which $\vec{A}$ makes with the positive x, y and z axes respectively

$$\vec{A}\cdot\hat{i} = A|\hat{i}|\cos\alpha$$

$$\Rightarrow (2\hat{i}-2\hat{j}-\hat{k})\cdot\hat{i} = \sqrt{4+4+1}\ \cos\alpha$$

$$\Rightarrow \cos\alpha = \frac{2}{3} \quad \therefore \alpha = \cos^{-1}\left(\frac{2}{3}\right)$$

$$\vec{A}\cdot\hat{j} = A|\hat{j}|\cos\beta$$

$$\Rightarrow (2\hat{i}-2\hat{j}-\hat{k})\cdot\hat{j} = 3\cos\beta$$

$$\Rightarrow \cos\beta = -\frac{2}{3}$$

$$\therefore \beta = \cos^{-1}\left(-\frac{2}{3}\right)$$

$$\vec{A}\cdot\hat{k} = A|\hat{k}|\cos\gamma$$

$$\Rightarrow (2\hat{i}-2\hat{j}-\hat{k})\cdot\hat{k} = 3\cos\gamma$$

$$\Rightarrow \cos\gamma = -\frac{1}{3}$$

$$\therefore \gamma = \cos^{-1}\left(-\frac{1}{3}\right)$$

60: Find the direction cosines of the line ~~joing~~ joining the points $(3,2,-4)$ and $(1,-1,2)$.

Solution:

Let, the direction cosines of the line is (l, m, n) and direction ratio of the line is (a, b, c)

~~a~~ If,

$$a = (3-1) = 2$$
$$b = (2+1) = 3$$
$$c = (-4-2) = -6$$

$$\therefore l = \frac{a}{\sqrt{a^2+b^2+c^2}} = \frac{2}{\sqrt{4+9+36}} = \frac{2}{7}$$

$$m = \frac{b}{\sqrt{a^2+b^2+c^2}} = \frac{3}{\sqrt{4+9+36}} = \frac{3}{7}$$

$$n = \frac{c}{\sqrt{a^2+b^2+c^2}} = \frac{-6}{\sqrt{4+9+36}} = -\frac{6}{7}$$

$\therefore$ direction cosines $\left(\frac{2}{7}, \frac{3}{7}, -\frac{6}{7}\right)$

If

$a = (1-3) = -2$

$b = (-1-2) = -3$

$c = (2+4) = 6$

$$\therefore l = \frac{a}{\sqrt{a^2+b^2+c^2}} = \frac{-2}{\sqrt{4+9+36}} = -\frac{2}{7}$$

$$m = \frac{b}{\sqrt{a^2+b^2+c^2}} = \frac{-3}{\sqrt{4+9+36}} = -\frac{3}{7}$$

$$n = \frac{c}{\sqrt{a^2+b^2+c^2}} = \frac{6}{\sqrt{4+9+36}} = \frac{6}{7}$$

$\therefore$ direction cosines $\left(-\frac{2}{7}, -\frac{3}{7}, \frac{6}{7}\right)$

63: Find the projection of the vector $2\hat{i}-3\hat{j}+6\hat{k}$ on the vector $\hat{i}+2\hat{j}+2\hat{k}$.

Solution:

$$\text{projection} = \frac{(2\hat{i}-3\hat{j}+6\hat{k})\cdot(\hat{i}+2\hat{j}+2\hat{k})}{\sqrt{1+4+4}}$$

$$= \frac{2-6+12}{3}$$

$$= \frac{8}{3}$$

64: Find the projection of the vector $4\hat{i}-3\hat{j}+\hat{k}$ on the line passing through the points $(2,3,-1)$ and $(-2,-4,3)$.

Solution:

The line passing through the points $(-2-2)\hat{i}+(-4-3)\hat{j}+(3+1)\hat{k} = -4\hat{i}-7\hat{j}+4\hat{k}$

$$\therefore \text{projection} = \frac{(4\hat{i}-3\hat{j}+\hat{k})(-4\hat{i}-7\hat{j}+4\hat{k})}{\sqrt{16+49+16}}$$

$$= \frac{-16+21+4}{9}$$

$$= \frac{9}{9}$$

$$= 1.$$

65: If $\vec{A} = 4\hat{i}-\hat{j}+3\hat{k}$ and $\vec{B} = -2\hat{i}+\hat{j}-2\hat{k}$, find a unit vector perpendicular to both $\vec{A}$ and $\vec{B}$.

Solution:

$\vec{A}\times\vec{B}$ is perpendicular to both $\vec{A}$ and $\vec{B}$.

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 3 \\ -2 & 1 & -2 \end{vmatrix}$$

$$= \hat{i}(2-3)-\hat{j}(-8+6)+\hat{k}(4-2)$$

$$= -\hat{i}+2\hat{j}+2\hat{k}$$

$$\therefore |\vec{A}\times\vec{B}| = \sqrt{1+4+4} = 3$$

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$$\therefore \text{unit vector} = \frac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|}$$

$$= \frac{-\hat{i}+2\hat{j}+2\hat{k}}{3}$$

$$= -\frac{1}{3}(\hat{i}-2\hat{j}-2\hat{k})$$

82: Find the area of a parallelogram having diagonals $\vec{A} = 3\hat{i}+\hat{j}-2\hat{k}$ and $\vec{B} = \hat{i}-3\hat{j}+4\hat{k}$

Solution:

Area of the parallelogram

$$= |\vec{A}\times\vec{B}| \times \frac{1}{2}$$

$$= \sqrt{4+196+100} \times \frac{1}{2}$$

$$= 10\sqrt{3} \times \frac{1}{2}$$

$$= 5\sqrt{3}$$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix}$$

$$= \hat{i}(4-6) - \hat{j}(12+2) + \hat{k}(-9-1)$$

$$= -2\hat{i} - 14\hat{j} - 10\hat{k}$$

83: Find the area of a triangle with vertices at $(3,-1,2)$, $(1,-1,-3)$ and $(4,-3,1)$

Solution:

Let, the vertices be $P(3,-1,2)$, $Q(1,-1,-3)$ and $R(4,-3,1)$

$$\cancel{\vec{PR}=}\ \vec{PQ} = (1-3)\hat{i} + (-1+1)\hat{j} + (-3-2)\hat{k} = -2\hat{i} - 5\hat{k}$$

$$\vec{PR} = (4-3)\hat{i} + (-3+1)\hat{j} + (1-2)\hat{k} = \hat{i} - 2\hat{j} - \hat{k}$$

$\therefore$ Area of triangle $= \frac{1}{2} \left| \vec{PQ} \times \vec{PR} \right|$

$$= \frac{1}{2} \left| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 0 & -5 \\ 1 & -2 & -1 \end{vmatrix} \right|$$

$$= \frac{1}{2} \left| \hat{i}(0-10) - \hat{j}(2+5) + \hat{k}(4-0) \right|$$

$$= \frac{1}{2} \left| -10\hat{i} - 7\hat{j} + 4\hat{k} \right|$$

$$= \frac{1}{2} \sqrt{100+49+16}$$

$$= \frac{1}{2} \sqrt{165}$$

\# 90: Find the volume of the parallelepiped whose edges are represented by $\vec{A} = 2\hat{i} - 3\hat{j} + 4\hat{k}$, $\vec{B} = \hat{i} + 2\hat{j} - \hat{k}$ and $\vec{C} = 3\hat{i} - \hat{j} + 2\hat{k}$.

Solution:

The volume of the parallelepiped

$$= \left| \vec{A} \cdot (\vec{B} \times \vec{C}) \right| = \left| \vec{B} \cdot (\vec{C} \times \vec{A}) \right| = \left| \vec{C} \cdot (\vec{A} \times \vec{B}) \right|$$

$$= \left| \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix} \right|$$

$$= \left| 2(4-1) + 3(2+3) + 4(-1-6) \right|$$

$$= \left| 6 + 15 - 28 \right|$$

$$= |-7| = 7$$

92: Find the constant a such that the vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}+2\hat{j}-3\hat{k}$ and $3\hat{i}+a\hat{j}+5\hat{k}$ are coplanar.

Solution:

The vectors will be coplanar if

$$\begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & a & 5 \end{vmatrix} = 0$$

$$\Rightarrow 2(10+3a)+1(5+9)+1(a-6)=0$$

$$\Rightarrow 20+6a+14+a-6=0$$

$$\Rightarrow 7a+28=0$$

$$\Rightarrow a=-\frac{28}{7}$$

$$\therefore a=-4$$

100: Prove that, $(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D})+(\vec{B}\times\vec{C})\cdot(\vec{A}\times\vec{D})+(\vec{C}\times\vec{A})\cdot(\vec{B}\times\vec{D})=0$

Solution:

$$L.H.S. = (\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D})+(\vec{B}\times\vec{C})\cdot(\vec{A}\times\vec{D})+(\vec{C}\times\vec{A})\cdot(\vec{B}\times\vec{D})$$

$$= \cancel{(\vec{A}\times\vec{B})}\,\vec{X}\cdot(\vec{C}\times\vec{D})+\vec{Y}\cdot(\vec{A}\times\vec{D})+\vec{Z}\cdot(\vec{B}\times\vec{D})$$

Let, $\vec{A}\times\vec{B}=\vec{X}$, $\vec{B}\times\vec{C}=\vec{Y}$, $\vec{C}\times\vec{A}=\vec{Z}$

$$=(\vec{X}\times\vec{C})\cdot\vec{D}+(\vec{Y}\times\vec{A})\cdot\vec{D}+(\vec{Z}\times\vec{B})\cdot\vec{D}$$

$$[\because \vec{A}\cdot(\vec{B}\times\vec{C})=(\vec{A}\times\vec{B})\cdot\vec{C}]$$

$$= \{(\vec{A}\times\vec{B})\times\vec{C}\}\cdot\vec{D} + \{(\vec{B}\times\vec{C})\times\vec{A}\}\cdot\vec{D}$$
$$+ \{(\vec{C}\times\vec{A})\times\vec{B}\}\cdot\vec{D}$$

$$= \{\vec{B}\cdot(\vec{A}\cdot\vec{C}) - \vec{A}(\vec{B}\cdot\vec{C})\}\cdot\vec{D} + \{\vec{C}\cdot(\vec{A}\cdot\vec{B}) - \vec{B}(\vec{A}\cdot\vec{C})\}\cdot\vec{D}$$
$$+ \{\vec{A}\cdot(\vec{B}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})\}\cdot\vec{D}$$

[Spiegel, Page-28, Prob-47 (b)]

$$= (\vec{B}\cdot\vec{D})\cdot(\vec{A}\cdot\vec{C}) - (\vec{A}\cdot\vec{D})\cdot(\vec{B}\cdot\vec{C}) + (\vec{C}\cdot\vec{D})\cdot(\vec{A}\cdot\vec{B})$$
$$- (\vec{B}\cdot\vec{D})(\vec{A}\cdot\vec{C}) + (\vec{A}\cdot\vec{D})\cdot(\vec{B}\cdot\vec{C}) - (\vec{C}\cdot\vec{D})\cdot(\vec{A}\cdot\vec{B})$$

$= 0 =$ R.H.S.

$\therefore$ L.H.S = R.H.S. (Proved)

→ Vector differentiation:

- Example:

#1: If $\vec{R}(u) = x(u)\hat{i} + y(u)\hat{j} + z(u)\hat{k}$ where x, y, z are differentiable functions of a scalar u. Prove that $\frac{d\vec{R}}{du} = \frac{dx}{du}\hat{i} + \frac{dy}{du}\hat{j} + \frac{dz}{du}\hat{k}$

Solution:

$$\text{L.H.S.} = \frac{d\vec{R}}{du} = \lim_{\Delta u \to 0} \frac{\vec{R}(u+\Delta u) - \vec{R}(u)}{\Delta u}$$

$$= \lim_{\Delta u \to 0} \frac{\{x(u+\Delta u)\hat{i} + y(u+\Delta u)\hat{j} + z(u+\Delta u)\hat{k}\} - \{x(u)\hat{i} + y(u)\hat{j} + z(u)\hat{k}\}}{\Delta u}$$

$$= \lim_{\Delta u \to 0} \frac{x(u+\Delta u) - x(u)}{\Delta u}\hat{i} + \frac{y(u+\Delta u) - y(u)}{\Delta u}\hat{j} + \frac{z(u+\Delta u) - z(u)}{\Delta u}\hat{k}$$

$$= \frac{dx}{du}\hat{i} + \frac{dy}{du}\hat{j} + \frac{dz}{du}$$

= R.H.S.

∴ L.H.S. = R.H.S. (Proved)

#2: Given $\vec{R} = \sin t\,\hat{i} + \cos t\,\hat{j} + t\hat{k}$, find

a) $\frac{d\vec{R}}{dt}$, b) $\frac{d^2\vec{R}}{dt^2}$, c) $\left|\frac{d\vec{R}}{dt}\right|$ d) $\left|\frac{d^2\vec{R}}{dt^2}\right|$

Solution:

a) $$\frac{d\vec{R}}{dt} = \frac{d}{dt}(\sin t)\hat{i} + \frac{d}{dt}(\cos t)\hat{j} + \frac{d}{dt}(t)\hat{k}$$

$$= \cos t\,\hat{i} - \sin t\,\hat{j} + \hat{k}$$

b) $$\frac{d^2\vec{R}}{dt} = \frac{d}{dt}(\cos t)\hat{i} - \frac{d}{dt}(\sin t)\hat{j} + \frac{d}{dt}(1)\hat{k}$$

$$= -\sin t\,\hat{i} - \cos t\,\hat{j}.$$

c) $$\left|\frac{d\vec{R}}{dt}\right| = \sqrt{\cos^2 t + \sin^2 t + 1} = \sqrt{2}$$

d) $$\left|\frac{d^2\vec{R}}{dt^2}\right| = \sqrt{\sin^2 t + \cos^2 t} = 1$$

\# 4: A particle moves along the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$ where t is the time. Find the components of its velocity and acceleration at time $t = 1$ in the direction $\hat{i} - 3\hat{j} + 2\hat{k}$.

Solution:

$$\vec{r} = 2t^2\,\hat{i} + (t^2 - 4t)\,\hat{j} + (3t - 5)\,\hat{k}$$

Velocity, $\frac{d\vec{r}}{dt} = \frac{d}{dt}(2t^2)\,\hat{i} + \frac{d}{dt}(t^2 - 4t)\,\hat{j} + \frac{d}{dt}(3t - 5)\,\hat{k}$

$$= 4t\,\hat{i} + (2t - 4)\,\hat{j} + 3\,\hat{k}$$

at time, $t = 1$

$$\text{velocity} = 4\hat{i} - 2\hat{j} + 3\hat{k}$$

unit vector in direction $\hat{i} - 3\hat{j} + 2\hat{k}$

$$= \frac{\hat{i} - 3\hat{j} + 2\hat{k}}{\sqrt{1 + 9 + 4}} = \frac{1}{\sqrt{14}}(\hat{i} - 3\hat{j} + 2\hat{k})$$

Component of velocity in the given direction,

$$= \frac{1}{\sqrt{14}}(\hat{i} - 3\hat{j} + 2\hat{k}) \cdot (4\hat{i} - 2\hat{j} + 3\hat{k})$$

$$= \frac{1}{\sqrt{14}}(4 + 6 + 6)$$

$$= \frac{16}{\sqrt{14}} = \frac{8\sqrt{14}}{7}$$

$$\cancel{= \frac{8\sqrt{2}}{\sqrt{7}}}$$

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Acceleration, $\frac{d^2\vec{r}}{dt} = \frac{d}{dt}(4t)\,\hat{i} + \frac{d}{dt}(2t-4)\,\hat{j} + \frac{d}{dt}(3)\,\hat{k}$

$= 4\hat{i} + 2\hat{j}$

Component of the acceleration in the given direction,

$= \frac{1}{\sqrt{14}}(\hat{i} - 3\hat{j} + 2\hat{k})\cdot(4\hat{i} + 2\hat{j})$

$= \frac{1}{\sqrt{14}}(4-6)$

$= -\frac{2}{\sqrt{14}}$

$= -\frac{\sqrt{14}}{7}$

#5: A curve C is defined by parametric equation $x = x(s)$, $y = y(s)$, $z = z(s)$, where s is the inverse of length of C measured from a fixed point on C. If $\vec{r}$ is the position vector of any point on C, show that $\frac{d\vec{r}}{ds}$ is a unit vector tangent to C.

Solution:

The vector, $\frac{d\vec{r}}{ds} = \frac{dx}{ds}\,\hat{i} + \frac{dy}{ds}\,\hat{j} + \frac{dz}{ds}\,\hat{k}$ is tangent to the curve $x = x(s)$, $y = y(s)$, $z = z(s)$.

$\therefore \left|\frac{d\vec{r}}{ds}\right| = \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2}$

$= \sqrt{\frac{(dx)^2 + (dy)^2 + (dz)^2}{(ds)^2}}$

$= \sqrt{\frac{(ds)^2}{(ds)^2}}$ $\quad [\because (ds)^2 = (dx)^2 + (dy)^2 + (dz)^2]$

$= 1$

As, the $\left|\frac{d\vec{r}}{ds}\right| = 1$.

So $\frac{d\vec{r}}{ds}$ is a unit vector tangent to C.
(Showed)

#6:

a) Find the unit tangent vector to any point on the curve $x = t^2+1$, $y = 4t-3$, $z = 2t^2-6t$.

b) Determine the unit tangent at the point where $t=2$.

Solution:

a)

The curve vector, $\vec{r} = (t^2+1)\hat{i} + (4t-3)\hat{j} + (2t^2-6t)\hat{k}$

A tangent vector to the curve

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}(t^2+1)\hat{i} + \frac{d}{dt}(4t-3)\hat{j} + \frac{d}{dt}(2t^2-6t)\hat{k}$$

$$= 2t\hat{i} + 4\hat{j} + (4t-6)\hat{k}$$

$$\therefore \left|\frac{d\vec{r}}{dt}\right| = \sqrt{(2t)^2 + 4^2 + (4t-6)^2}$$

The unit tangent vector, $\vec{T} = \frac{d\vec{r}}{ds}$

$$= \frac{\frac{d\vec{r}}{dt}}{\frac{ds}{dt}}$$

$\left[\because \left|\frac{d\vec{r}}{dt}\right| = \frac{ds}{dt}\right]$ $$= \frac{d\vec{r}}{dt} \times \frac{1}{\left|\frac{d\vec{r}}{dt}\right|}$$

$$\therefore \vec{T} = \frac{2t\,\hat{i} + 4\hat{j} + (4t-6)\hat{k}}{\sqrt{(2t)^2 + 4^2 + (4t-6)^2}}$$

b)

At . $t = 2$

the unit tangent vector, $\vec{T} = \dfrac{(2\times2)\hat{i} + 4\hat{j} + \{(4\times2)-6\}\hat{k}}{\sqrt{4^2 + 4^2 + 2^2}}$

$$= \frac{4\hat{i} + 4\hat{j} + 2\hat{k}}{\sqrt{36}}$$

$$= \frac{1}{3}(2\hat{i} + 2\hat{j} + \hat{k})$$

~~# 7 : #~~

\# 8 : If $\vec{A} = 5t^2\,\hat{i} + t\,\hat{j} - t^3\hat{k}$ and $\vec{B} = \sin t\;\hat{i} - \cos t\;\hat{j}$.

find a) $\frac{d}{dt}(\vec{A}\cdot\vec{B})$, b) $\frac{d}{dt}(\vec{A}\times\vec{B})$, c) $\frac{d}{dt}(\vec{A}\cdot\vec{A})$

Solution:

a) $\frac{d}{dt}(\vec{A}\cdot\vec{B})$

$$= \vec{A}\,\frac{d\vec{B}}{dt} + \vec{B}\,\frac{d\vec{A}}{dt}$$

$$= (5t^2\,\hat{i} + t\hat{j} - t^3\hat{k})\cdot\frac{d}{dt}(\sin t\;\hat{j} - \cos t\;\hat{j})$$

$$+ (\sin t\,\hat{i} - \cos t\;\hat{j})\,\frac{d}{dt}(5t^2\hat{i} + t\,\hat{j} - t^3\hat{k})$$

$$= (5t^2\,\hat{i} + t\hat{j} - t^3\hat{k})\cdot(\cos t\;\hat{i} + \sin t\;\hat{j})$$

$$+ (\sin t\;\hat{i} - \cos t\;\hat{j})\cdot(10t\,\hat{i} + \hat{j} - 3t^2\hat{k})$$

$$= 5t^2\cos t + t\sin t + 10t\sin t - \cos t$$

$$= (5t^2 - 1)\cos t + 11t\sin t$$

b)

$$\frac{d}{dt}(\vec{A}\times\vec{B})$$

$$= \vec{A}\times\frac{d\vec{B}}{dt} - \vec{B}\times\frac{d\vec{A}}{dt}$$

$$= \vec{A}\times\frac{d}{dt}(\sin t\,\hat{i} - \cos t\,\hat{j}) - \vec{B}\times\frac{d}{dt}(5t^2\hat{i} + t\hat{j} - t^3\hat{k})$$

$$= (5t^2\hat{i} + t\hat{j} - t^3\hat{k})\times(\cos t\,\hat{i} + \sin t\,\hat{j})$$
$$- (\sin t\,\hat{i} - \cos t\,\hat{j})\times(10t\,\hat{i} + \hat{j} - 3t^2\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5t^2 & t & -t^3 \\ \cos t & \sin t & 0 \end{vmatrix} - \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sin t & -\cos t & 0 \\ 10t & 1 & -3t^2 \end{vmatrix}$$

$$= \hat{i}(0 + t^3\sin t) - \hat{j}(0 + t^3\cos t) + \hat{k}(5t^2\sin t - t\cos t)$$
$$- \{ i(3t^2\cos t - 0) - \hat{j}(-3t^2\sin t - 0) + \hat{k}(\sin t + 10t\cos t) \}$$

$$= t^3\sin t\,\hat{i} - t^3\cos t\,\hat{j} + (5t^2\sin t - t\cos t)\hat{k} - 3t^2\cos t\,\hat{i}$$
$$- 3t^2\sin t\,\hat{j} - (\sin t + 10t\cos t)\hat{k}$$

$$= (t^3\sin t - 3t^2\cos t)\hat{i} - (t^3\cos t + 3t^2\sin t)\hat{j}$$
$$+ \{(5t - 1)\sin t - 11t\cos t\}\hat{k}.$$

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c) $\frac{d}{dt}(\vec{A}\cdot\vec{A})$

$$= \vec{A}\,\frac{d\vec{A}}{dt} + \vec{A}\,\frac{d\vec{A}}{dt}$$

$$= 2\cdot\vec{A}\cdot\frac{d}{dt}(5t^2\,\hat{i} + t\,\hat{j} - t^3\hat{k})$$

$$= 2\cdot(5t^2\,\hat{i} + t\,\hat{j} - t^3\hat{k})\cdot(10t\,\hat{i} + \hat{j} - 3t^2\hat{k})$$

$$= 2\cdot(50t^3 + t + 3t^5)$$

$$= 100t^3 + 2t + 6t^5$$

#9: If $\vec{A}$ has constant magnitude show that $\vec{A}$ and $\frac{d\vec{A}}{dt}$ are perpendicular provided $\left|\frac{d\vec{A}}{dt}\right| \neq 0$

Solution:

As, $\vec{A}$ has constant magnitude, $\vec{A}\cdot\vec{A}$ = constant.

$$\therefore \frac{d}{dt}(\vec{A}\cdot\vec{A}) = \vec{A}\frac{d}{dt}(\vec{A}) + \vec{A}\frac{d\vec{A}}{dt} = 2\cdot\vec{A}\,\frac{d\vec{A}}{dt}\ \cancel{=0}$$

~~Since~~ $\frac{d}{dt}(\vec{A}\cdot\vec{A}) = 0$ [$\because \vec{A}\cdot\vec{A}$ = constant]

$$\Rightarrow 2\vec{A}\cdot\frac{d\vec{A}}{dt} = 0$$

$$\therefore \vec{A}\cdot\frac{d\vec{A}}{dt} = 0$$

~~A~~ Since the dot product of $\vec{A}$ and $\frac{d\vec{A}}{dt}$ is 0, $\vec{A}$ and $\frac{d\vec{A}}{dt}$ are perpendicular.

#10: Prove $\frac{d}{du}(\vec{A}\cdot\vec{B}\times\vec{C}) = \vec{A}\cdot\vec{B}\times\frac{d\vec{C}}{du} + \vec{A}\cdot\frac{d\vec{B}}{du}\times\vec{C} + \frac{d\vec{A}}{du}\cdot\vec{B}\times\vec{C}$ where $\vec{A}, \vec{B}, \vec{C}$ are differentiable functions of a scalar u.

Solution:

$$\text{L.H.S.} = \frac{d}{du}(\vec{A}\cdot\vec{B}\times\vec{C})$$

$$= \vec{A}\cdot\frac{d}{du}(\vec{B}\times\vec{C}) + \frac{d\vec{A}}{du}\cdot(\vec{B}\times\vec{C})$$

$$= \vec{A}\cdot\left(\vec{B}\times\frac{d\vec{C}}{du} + \frac{d\vec{B}}{du}\times\vec{C}\right) + \frac{d\vec{A}}{du}\cdot(\vec{B}\times\vec{C})$$

$$= \vec{A}\cdot\vec{B}\times\frac{d\vec{C}}{du} + \vec{A}\cdot\frac{d\vec{B}}{du}\times\vec{C} + \frac{d\vec{A}}{du}\cdot\vec{B}\times\vec{C}$$

$= \text{R.H.S.}$

$\therefore$ L.H.S. = R.H.S. (Proved)

#11: Evaluate: $\frac{d}{dt}\left(\vec{V}\cdot\frac{d\vec{V}}{dt}\times\frac{d^2\vec{V}}{dt^2}\right)$

Solution:

$$\frac{d}{dt}\left(\vec{V}\cdot\frac{d\vec{V}}{dt}\times\frac{d^2\vec{V}}{dt^2}\right)$$

$$= \vec{V}\cdot\frac{d}{dt}\left(\frac{d\vec{V}}{dt}\times\frac{d^2\vec{V}}{dt^2}\right) + \frac{d\vec{V}}{dt}\cdot\frac{d\vec{V}}{dt}\times\frac{d^2\vec{V}}{dt^2}$$

$$= \vec{V}\left(\frac{d\vec{V}}{dt}\times\frac{d^3\vec{V}}{dt^3} + \frac{d^2\vec{V}}{dt^2}\times\frac{d^2\vec{V}}{dt^2}\right) + 0$$

$$= \vec{V}\cdot\frac{d\vec{V}}{dt}\times\frac{d^3\vec{V}}{dt^3} + \vec{V}\cdot\frac{d^2\vec{V}}{dt^2}\times\frac{d^2\vec{V}}{dt^2}$$

$$= \vec{V}\cdot\frac{d\vec{V}}{dt}\times\frac{d^3\vec{V}}{dt^3}$$

\# 12: A particle moves so that its position vector is given by $\vec{r} = \cos \omega t \, \hat{i} + \sin \omega t \, \hat{j}$ where ω is a constant. Show that

a) the velocity $\vec{v}$ of the particle is perpendicular to $\vec{r}$

b) the acceleration $\vec{a}$ is directed toward the origin and has magnitude proportional to the distance from the origin.

c) $\vec{r} \times \vec{v}$ = a constant vector.

Solution:

a) $\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(\cos \omega t)\,\hat{i} + \frac{d}{dt}(\sin \omega t)\,\hat{j}$

$$= -\omega \sin \omega t \, \hat{i} + \omega \cos \omega t \, \hat{j}$$

$$\vec{r} \cdot \vec{v} = (\cos \omega t \, \hat{i} + \sin \omega t \, \hat{j}) \cdot (-\omega \sin \omega t \, \hat{i} + \omega \cos \omega t \, \hat{j})$$

$$= -\omega \sin \omega t \cos \omega t + \omega \sin \omega t \cos \omega t$$

$$= 0$$

As, the dot product of $\vec{r}$ and $\vec{v}$ is zero, $\vec{v}$ is perpendicular to $\vec{r}$.

b)

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \frac{d\vec{v}}{dt}$$

$$= \frac{d}{dt}(-\omega \sin \omega t)\hat{i} + \frac{d}{dt}(\omega \cos \omega t)\hat{j}$$

$$= -\omega^2 \cos \omega t \,\hat{i} - \omega^2 \sin \omega t \,\hat{j}$$

$$= -\omega^2(\cos \omega t \,\hat{i} + \sin \omega t \,\hat{j})$$

$$= -\omega^2 \vec{r}$$

The acceleration is opposite to the direction of $\vec{r}$. It is directed toward the origin. Its magnitude is proportional to $|\vec{r}|$ which is the distance from the origin.

c) $\vec{r} \times \vec{v}$

$$= (\cos \omega t \,\hat{i} + \sin \omega t \,\hat{j}) \times (-\omega \sin \omega t \,\hat{i} + \omega \cos \omega t \,\hat{j})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos \omega t & \sin \omega t & 0 \\ -\omega \sin \omega t & \omega \cos \omega t & 0 \end{vmatrix}$$

$$= \hat{i} \times 0 - \hat{j} \times 0 + \hat{k}(\omega \cos^2 \omega t + \omega \sin^2 \omega t)$$

$$= \omega \hat{k} = \text{a constant vector}$$

$\therefore \vec{r} \times \vec{v}$ is a constant vector. (showed)

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#13: Prove : $\vec{A}\times\frac{d^2\vec{B}}{dt^2} - \frac{d^2\vec{A}}{dt^2}\times\vec{B} = \frac{d}{dt}\left(\vec{A}\times\frac{d\vec{B}}{dt} - \frac{d\vec{A}}{dt}\times\vec{B}\right)$

Solution:

$$L.H.S. = \vec{A}\times\frac{d^2\vec{B}}{dt^2} - \frac{d^2\vec{A}}{dt^2}\times\vec{B}$$

$$= \vec{A}\times\frac{d^2\vec{B}}{dt^2} + \frac{d\vec{A}}{dt}\times\frac{d\vec{B}}{dt} - \frac{d\vec{A}}{dt}\times\frac{d\vec{B}}{dt} - \frac{d^2\vec{A}}{dt^2}\times\vec{B}$$

$$= \left(\vec{A}\times\frac{d^2\vec{B}}{dt^2} + \frac{d\vec{A}}{dt}\times\frac{d\vec{B}}{dt}\right) - \left(\frac{d\vec{A}}{dt}\times\frac{d\vec{B}}{dt} + \frac{d^2\vec{A}}{dt^2}\times\vec{B}\right)$$

$$= \frac{d}{dt}\left(\vec{A}\times\frac{d\vec{B}}{dt}\right) - \frac{d}{dt}\left(\frac{d\vec{A}}{dt}\times\vec{B}\right)$$

$$= \frac{d}{dt}\left(\vec{A}\times\frac{d\vec{B}}{dt} - \frac{d\vec{A}}{dt}\times\vec{B}\right)$$

$= R.H.S.$

$\therefore L.H.S. = R.H.S.$ (Proved)

#14: show that : $\vec{A}\cdot\frac{d\vec{A}}{dt} = A\frac{dA}{dt}$

Solution:

$$\vec{A}\cdot\vec{A} = A^2$$

$$\Rightarrow \frac{d}{dt}(\vec{A}\cdot\vec{A}) = \frac{d}{dt}(A^2)$$

$$\Rightarrow 2\vec{A}\frac{d\vec{A}}{dt} = 2A\frac{dA}{dt}$$

$$\therefore \vec{A}\frac{d\vec{A}}{dt} = A\frac{dA}{dt} \text{ (showed)}$$

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\# 15: If $\vec{A} = (2x^2y - x^4)\hat{i} + (e^{xy} - y\sin x)\hat{j} + (x^2\cos y)\hat{k}$

find $\frac{\partial A}{\partial x}, \frac{\partial A}{\partial y}, \frac{\partial^2 A}{\partial x^2}, \frac{\partial^2 A}{\partial y^2}, \frac{\partial^2 A}{\partial x \partial y}, \frac{\partial^2 A}{\partial x \partial y}$

Solution:

$$\frac{\partial A}{\partial x} = \frac{\partial}{\partial x}(2x^2y - x^4)\hat{i} + \frac{\partial}{\partial x}(e^{xy} - y\sin x)\hat{j} + \frac{\partial}{\partial x}(x^2\cos y)\hat{k}$$

$$= (4xy - 4x^3)\hat{i} + (ye^{xy} - y\cos x)\hat{j} + (2x\cos y)\hat{k}$$

$$\frac{\partial A}{\partial y} = \frac{\partial}{\partial y}(2x^2y - x^4)\hat{i} + \frac{\partial}{\partial y}(e^{xy} - y\sin x)\hat{j} + \frac{\partial}{\partial y}(x^2\cos y)\hat{k}$$

$$= 2x^2\hat{i} + (xe^{xy} - \sin x)\hat{j} \cancel{+x^2 co} - x^2\sin y\,\hat{k}$$

$$\frac{\partial^2 A}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial A}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}(4xy - 3x^3)\hat{i} + \frac{\partial}{\partial x}(ye^{xy} - y\cos x)\hat{j} + \frac{\partial}{\partial x}(2x\cos y)\hat{k}$$

$$= (4y - 12x^2)\hat{i} + (y^2e^{xy} + y\sin x)\hat{j} + (2\cos y)\hat{k}$$

$$\frac{\partial^2 A}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial A}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}(2x^2)\hat{i} + \frac{\partial}{\partial y}(xe^{xy} - \sin x)\hat{j} - \frac{\partial}{\partial y}(x^2\sin y)\hat{k}$$

$= 0 + x^2 e^{xy}\hat{j} - x^2\cos y\,\hat{k} = x^2 e^{xy}\hat{j} - x^2\cos y\,\hat{k}$

$$\frac{\partial^2 A}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial A}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}(2x^2)\hat{i} + \frac{\partial}{\partial x}(xe^{xy} - \sin x)\hat{j} - \frac{\partial}{\partial x}(x^2\sin y)\hat{k}$$

$$= 4x\hat{i} + (xye^{xy} + e^{xy} - \cos x)\hat{j} - (2x\sin y)\hat{k}$$

$$\frac{\partial^2 A}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial A}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}(4xy - 4x^3) + \frac{\partial}{\partial y}(ye^{xy} - y\cos x)\hat{j}$$

$$+ \frac{\partial}{\partial y}(2x\cos y)\hat{k}$$

$$= 4x\hat{i} + (xye^{xy} + e^{xy} - \cos x)\hat{j} - (2x\sin y)\hat{k}$$

16: If $\phi(x,y,z) = xy^2z$ and $\vec{A} = xz\hat{i} - xy^2\hat{j} + yz^2\hat{k}$ find $\frac{\partial^3}{\partial x^2 \partial z}(\phi A)$ at the point $(2,-1,1)$.

solution:

$$\phi\vec{A} = (xy^2z)(xz\hat{i} - xy^2\hat{j} + yz^2\hat{k})$$

$$= x^2y^2z^2\hat{i} - x^2y^4z\hat{j} + xy^3z^3\hat{k}$$

$$\frac{\partial}{\partial z}(\phi\vec{A}) = \frac{\partial}{\partial z}(x^2y^2z^2)\hat{i} - \frac{\partial}{\partial z}(x^2y^4z)\hat{j} + \frac{\partial}{\partial z}(xy^3z^3)\hat{k}$$

$$= 2x^2y^2z\,\hat{i} - x^2y^4\hat{j} + 3xy^3z^2\hat{k}$$

$$\frac{\partial^2}{\partial x \partial z}(\phi \vec{A}) = \frac{\partial}{\partial x}(2x^2y^2z)\hat{i} - \frac{\partial}{\partial x}(x^2y^4)\hat{j} + \frac{\partial}{\partial x}(3xy^3z^2)\hat{k}$$

$$= 4xy^2z\,\hat{i} - 2xy^4\,\hat{j} + 3y^3z^2\,\hat{k}$$

$$\frac{\partial^3}{\partial x^2 \partial z}(\phi \vec{A}) = \frac{\partial}{\partial x}(4xy^2z)\hat{i} - \frac{\partial}{\partial x}(2xy^4)\hat{j} + \frac{\partial}{\partial x}(3y^3z^2)\hat{k}$$

$$= 4y^2z\,\hat{i} - 2y^4\,\hat{j}$$

at the point $(2,-1,1)$

$$\frac{\partial^3}{\partial x^2 \partial z}(\phi \vec{A}) = 4\times(-1)^2\times 1\ \hat{i} - 2\times(-1)^4\,\hat{j} = 4\hat{i} - 2\hat{j}$$

#18: Prove the Frenet-Serret formulae:

a) $\frac{d\vec{T}}{ds} = k\vec{N}$, b) $\frac{d\vec{B}}{ds} = -\tau\vec{N}$, c) $\frac{d\vec{N}}{ds} = \tau\vec{B} - k\vec{T}$

Solution:

a) As, $\vec{T}$ is the unit tangent vector, $\vec{T}\cdot\vec{T} = 1$.

$\therefore \vec{T}\cdot\frac{d\vec{T}}{ds} = 0$. $\frac{d\vec{T}}{ds}$ is perpendicular to $\vec{T}$.

If $\vec{N}$ is a unit vector in the direction $\frac{d\vec{T}}{ds}$,

$\frac{d\vec{T}}{ds} = k\vec{N}$ where k is the curvature.

b) Let, $\vec{B} = \vec{T}\times\vec{N}$

$$\Rightarrow \frac{d\vec{B}}{ds} = \vec{T}\times\frac{d\vec{N}}{ds} + \frac{d\vec{T}}{ds}\times\vec{N}$$

$$= \vec{T}\times\frac{d\vec{N}}{ds} + k\vec{N}\times\vec{N}$$

$$= \vec{T}\times\frac{d\vec{N}}{ds} \quad [\vec{N}\times\vec{N} = 0 \because \vec{N} \text{ is a unit vector}]$$

$$\vec{T}\cdot\frac{d\vec{B}}{ds} = \vec{T}\cdot\vec{T}\times\frac{d\vec{N}}{ds}$$

Since $\frac{d\vec{B}}{ds}$ is in the plane of $\vec{T}$ and $\vec{N}$ and is perpendicular to $\vec{T}$, then it is parralled to $\vec{N}$.

$$\therefore \frac{d\vec{B}}{ds} = -\tau\vec{N}$$

where τ is the torsion and $\sigma = \frac{1}{\tau}$ is the radius of torsion.

c) We know, $\vec{N} = \vec{B}\times\vec{T}$ [$\because$ $\vec{T}$, $\vec{N}$ and $\vec{B}$ form a right-handed system.]

$$\Rightarrow \frac{d\vec{N}}{ds} = \vec{B}\times\frac{d\vec{T}}{ds} + \frac{d\vec{B}}{ds}\times\vec{T}$$

$$\Rightarrow \frac{d\vec{N}}{ds} = \vec{B}\times k\vec{N} + (-\tau\vec{N})\times\vec{T}$$

$$\Rightarrow \frac{d\vec{N}}{ds} = k(\vec{B}\times\vec{N}) - \tau(\vec{N}\times\vec{T})$$

$$\Rightarrow \frac{d\vec{N}}{ds} = -k\vec{T} + \tau\vec{B}$$

$$\therefore \frac{d\vec{N}}{ds} = \tau\vec{B} - k\vec{T}.$$

Q#19: The space curve $x = 3\cos t$, $y = 3\sin t$, $z = 4t$.

find: a) the unit tangent $\vec{T}$

b) the principal normal $\vec{N}$, curvature k and radius of curvature ρ

c) the binormal $\vec{B}$, torsion τ and radius of torsion σ.

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Solution:

a) The curve, $\vec{r} = 3\cos t\,\hat{i} + 3\sin t\,\hat{j} + 4t\,\hat{k}$

$$\frac{d\vec{r}}{dt} = -3\sin t\,\hat{i} + 3\cos t\,\hat{j} + 4\hat{k}$$

$$\frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right| = \sqrt{9\sin^2 t + 9\cos^2 t + 4^2}$$

$$= \sqrt{9+16} = 5$$

$$\vec{T} = \frac{d\vec{r}}{ds} = \frac{\frac{d\vec{r}}{dt}}{\frac{ds}{dt}} = \frac{1}{5}(-3\sin t\,\hat{i} + 3\cos t\,\hat{j} + 4\hat{k})$$

b)

$$\frac{d\vec{T}}{dt} = \frac{d}{dt}\left(-\frac{3}{5}\sin t\,\hat{i} + \frac{3}{5}\cos t\,\hat{j} + \frac{4}{5}\hat{k}\right)$$

$$= -\frac{3}{5}\cos t\,\hat{i} - \frac{3}{5}\sin t\,\hat{j}$$

$$\frac{d\vec{T}}{ds} = \frac{\frac{d\vec{T}}{dt}}{\frac{ds}{dt}} = \frac{1}{25}(-3\cos t\,\hat{i} - 3\sin t\,\hat{j})$$

$$\frac{d\vec{T}}{ds} = k\vec{N}$$

$$\Rightarrow \left|\frac{d\vec{T}}{ds}\right| = k|\vec{N}|$$

$\Rightarrow \left|\frac{d\vec{T}}{ds}\right| = k \cdot 1$ [$\because$ As $\vec{N}$ is an unit vector, so $|\vec{N}| = 1$]

$$\Rightarrow k = \sqrt{\frac{1}{625}(9\cos^2 t + 9\sin^2 t)}$$

$$\Rightarrow k = \sqrt{\frac{9}{625}}$$

$\therefore k = \frac{3}{25}$ $\qquad \therefore \rho = \frac{1}{k} = \frac{25}{3}$

$$\frac{d\vec{T}}{ds} = k\vec{N}$$

$$\Rightarrow \vec{N} = \frac{1}{k}\frac{d\vec{T}}{ds}$$

$$= \frac{25}{3} \times \frac{3}{25}\left(-\cos t\,\hat{i} - \sin t\,\hat{j}\right)$$

$$= -\cos t\,\hat{i} - \sin t\,\hat{j}$$

c) $\vec{B}\ \cancel{\times} = \vec{T} \times \vec{N}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{3}{5}\sin t & \frac{3}{5}\cos t & \frac{4}{5} \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

$$\cancel{= \hat{i}\left(-\frac{4}{5}\sin t\right) - \hat{j}\left(\frac{3\sin^2 t}{5} + \frac{3\cos^2 t}{5}\right) + \hat{k}}$$

$$= \hat{i}\left(\frac{4}{5}\sin t\right) - \hat{j}\left(\frac{4}{5}\cos t\right) + \hat{k}\left(\frac{3\sin^2 t}{5} + \frac{3\cos^2 t}{5}\right)$$

$$= \frac{4\sin t}{5}\hat{i} - \frac{4\cos t}{5}\hat{j} + \frac{3}{5}\hat{k}$$

$$\frac{d\vec{B}}{dt} = \frac{4\cos t}{5}\hat{i} + \frac{4\sin t}{5}\hat{j}$$

$$\frac{d\vec{B}}{ds} = \frac{\frac{d\vec{B}}{dt}}{\frac{ds}{dt}} = \frac{1}{5}\left(\frac{4\cos t}{5}\hat{i} + \frac{4\sin t}{5}\hat{j}\right)$$

$$= \frac{4}{25}\left(\cancel{\cos t}\cos t\,\hat{i} + \sin t\,\hat{j}\right)$$

$$\tau\vec{N} = -\frac{d\vec{B}}{ds}$$

$$\Rightarrow \tau(\cos t\,\hat{i} + \sin t\,\hat{j}) = -\frac{4}{25}(\cos t\,\hat{i} + \sin t\,\hat{j})$$

$$\Rightarrow \tau = \frac{4}{25} \qquad \rho = \frac{1}{\tau} = \frac{25}{4}$$

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20. Prove that the radius of curvature of the ~~th~~ curve with parametric equations $x = x(s)$, $y = y(s)$, $z = z(s)$ is given by $\rho = \left[\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2\right]^{-\frac{1}{2}}$

Solution:

The curve, $\vec{r} = x(s)\hat{i} + y(s)\hat{j} + z(s)\hat{k}$

$$\vec{T} = \frac{d\vec{r}}{ds} = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k}$$

$$\frac{d\vec{T}}{ds} = \frac{d^2x}{ds^2}\hat{i} + \frac{d^2y}{ds^2}\hat{j} + \frac{d^2z}{ds^2}\hat{k}$$

$$\Rightarrow k\vec{N} = \frac{d^2x}{ds^2}\hat{i} + \frac{d^2y}{ds^2}\hat{j} + \frac{d^2z}{ds^2}\hat{k}$$

$$\Rightarrow k = \sqrt{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2}$$

$$\Rightarrow \frac{1}{\rho} = \left[\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2\right]^{\frac{1}{2}}$$

$$\Rightarrow \rho = \left[\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2\right]^{-\frac{1}{2}}$$

(Prove)

21. Show that $\frac{d\vec{r}}{ds} \cdot \frac{d^2\vec{r}}{ds^2} \times \frac{d^3\vec{r}}{ds^3} = \frac{\tau}{\rho^2}$

Solution:

$$\frac{d\vec{r}}{ds} = \vec{T}, \quad \frac{d^2\vec{r}}{ds^2} = \frac{d\vec{T}}{ds} = k\vec{N}.$$

$$\frac{d^3\vec{r}}{ds^3} = \frac{d}{ds}(k\vec{N}) = k\frac{d\vec{N}}{ds} + \frac{dk}{ds}\vec{N}$$

$$= k(\tau\vec{B} - k\vec{T}) + \frac{dk}{ds}\vec{N}$$

$$= k\tau\vec{B} - k^2\vec{T} + \frac{dk}{ds}\vec{N}$$

$$L.H.S. = \frac{d\vec{r}}{ds} \cdot \frac{d^2\vec{r}}{ds^2} \times \frac{d^3\vec{r}}{ds^3}$$

$$= \vec{T} \cdot k\vec{N} \times \left(k\tau\vec{B} - k^2\vec{T} + \frac{dk}{ds}\vec{N}\right)$$

$$= \vec{T} \cdot \left(k^2\tau\, \vec{N}\times\vec{B} - k^3\, \vec{N}\times\vec{T} + k\frac{dk}{ds}\, \vec{N}\times\vec{N}\right)$$

$$= \vec{T}\cdot\left(k^2\tau\,\vec{T} + k^3\,\vec{B}\right)$$

$$= k^2\tau\,\vec{T}\cdot\vec{T} + k^3\,\vec{T}\cdot\vec{B}$$

$$= k^2\tau$$

$$= \frac{\tau}{\rho^2} = R.H.S.$$

$\therefore$ L.H.S. = R.H.S. (showed)

22. Given the space curve $x=t,\ y=t^2,\ z=\frac{2}{3}t^3$. find

a) the curvature k,

b) the torsion τ.

Solution:

a) The curve, $\vec{r} = t\hat{i} + t^2\hat{j} + \frac{2t^3}{3}\hat{k}$

$$\frac{d\vec{r}}{dt} = \hat{i} + 2t\,\hat{j} + 2t^2\hat{k}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{1^2 + (2t)^2 + (2t^2)^2}$$

$$\Rightarrow \frac{ds}{dt} = \sqrt{1 + 4t^2 + 4t^4}$$

$$= \sqrt{(1+2t^2)^2}$$

$$= 1 + 2t^2$$

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$$\vec{T} = \frac{\frac{d\vec{r}}{dt}}{\frac{ds}{dt}} = \frac{\hat{i}+2t\hat{j}+2t^2\hat{k}}{1+2t^2}$$

$$\frac{d\vec{T}}{dt} = \frac{(1+2t^2)(2\hat{j}+4t\hat{k}) - (\hat{i}+2t\hat{j}+2t^2\hat{k})(4t)}{(1+2t^2)^2}$$

$$= \frac{(2+4t^2)\hat{j} + (4t+8t^3)\hat{k} - 4t\hat{i} - 8t^2\hat{j} - 8t^3\hat{k}}{}$$

$$= \frac{-4t\hat{i} + (2-4t^2)\hat{j} + 4t\hat{k}}{(1+2t^2)^2}$$

$$\frac{d\vec{T}}{ds} = \frac{d\vec{T}/dt}{ds/dt} = \frac{-4t\hat{i} + (2-4t^2)\hat{j} + 4t\hat{k}}{(1+2t^2)^3}$$

$$\Rightarrow k\vec{N} = \frac{-4t\hat{i} + (2-4t^2)\hat{j} + 4t\hat{k}}{(1+2t^2)^3}$$

$$\Rightarrow k|\vec{N}| \cancel{\frac{4t\hat{i}+(2-8t^2)\hat{j}+4t\hat{k}}{(1+2t^2)^3}} = \frac{\sqrt{(-4t)^2 + (2-4t^2)^2 + (4t)^2}}{(1+2t^2)^3}$$

$$\Rightarrow k = \frac{\sqrt{16t^2 + 4 - 16t^2 + 16t^4 + 16t^2}}{(1+2t^2)^3}$$

$$\Rightarrow k = \frac{\sqrt{4(1+2t^2)^2}}{(1+2t^2)^3}$$

$$\Rightarrow k = \frac{2}{(1+2t^2)^2}$$

b) $$N = \frac{1}{k}\frac{d\vec{T}}{ds} = \frac{(1+2t^2)^2}{2} \times \frac{-4t\hat{i} + (2-4t^2)\hat{j} + 4t\hat{k}}{(1+2t^2)^3}$$

$$= \frac{-2t\hat{i} + (1-2t^2)\hat{j} + 2t\hat{k}}{1+2t^2}$$

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$$\vec{B} = \vec{T} \times \vec{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{1}{1+2t^2} & \frac{2t}{1+2t^2} & \frac{2t^2}{1+2t^2} \\ \frac{-2t}{1+2t^2} & \frac{1-2t^2}{1+2t^2} & \frac{2t}{1+2t^2} \end{vmatrix}$$

$$= \frac{1}{(1+2t^2)^2} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2t & 2t^2 \\ -2t & 1-2t^2 & 2t \end{vmatrix}$$

$$= \frac{1}{(1+2t^2)^2} \left\{ \hat{i}\left(4t^2 - 2t^2 + 4t^4\right) - \hat{j}\left(2t + 4t^3\right) + \hat{k}\left(1 - 2t^2 + 4t^2\right) \right\}$$

$$= \frac{(2t^2+4t^4)\hat{i} - (2t+4t^3)\hat{j} + (1+2t^2)\hat{k}}{(1+2t^2)^2}$$

$$= \frac{2t^2(1+2t^2)\hat{i} - 2t(1+2t^2)\hat{j} + 1(1+2t^2)\hat{k}}{(1+2t^2)^2}$$

$$= \frac{(1+2t^2)(2t^2\hat{i} - 2t\hat{j} + \hat{k})}{(1+2t^2)^2}$$

$$= \frac{2t^2\hat{i} - 2t\hat{j} + \hat{k}}{1+2t^2}$$

$$\frac{d\vec{B}}{dt} = \frac{(1+2t^2)(4t\hat{i} - 2\hat{j}) - (2t^2\hat{i} - 2t\hat{j} + \hat{k})(4t)}{(1+2t^2)^2}$$

$$\frac{d\vec{B}}{ds} = \frac{d\vec{B}/dt}{ds/dt} = \frac{(4t + 8t^3 - 8t^3)\hat{i} + (-2 - 4t^2 + 8t^2)\hat{j} - 4t\hat{k}}{(1+2t^2)^3}$$

$$\Rightarrow -\tau\vec{N} = \frac{4t\hat{i} + (4t^2 - 2)\hat{j} - 4t\hat{k}}{(1+2t^2)^3}$$

$$\Rightarrow -\tau\left[-\frac{2t\hat{i}-(1-2t^2)\hat{j} - 2t\hat{k}}{1+2t^2}\right] = \frac{2\{2t\hat{i}-(1-2t^2)\hat{j}-2t\hat{k}\}}{(1+2t^2)^3}$$

$$\Rightarrow \tau = \frac{2\{2t\hat{i}-(1-2t^2)\hat{j}-2t\hat{k}\}}{(1+2t^2)^3} \times \frac{1+2t^2}{2t\hat{i}-(1-2t^2)\hat{j}-2t\hat{k}}$$

$$\Rightarrow \tau = \frac{2}{(1+2t^2)^2}$$

Exercise:

\# 32: Find the velocity and acceleration of a particle which moves along the curve $x = 2\sin 3t$, $y = 2\cos 3t$, $z = 8t$ at any time $t > 0$. Find the magnitude of the velocity and acceleration.

Solution:

The curve, $\vec{r} = 2\sin 3t\,\hat{i} + 2\cos 3t\,\hat{j} + 8t\,\hat{k}$

$$\vec{v} = \frac{d\vec{r}}{dt} = 6\cos 3t\,\hat{i} - 6\sin 3t\,\hat{j} + 8\hat{k}$$

$$\therefore |\vec{v}| = \sqrt{36\cos^2 3t + 36\sin^2 3t + 64}$$

$$= \sqrt{36+64} = 10$$

$$\vec{a} = \frac{d\vec{v}}{dr} = -18\sin 3t\,\hat{i} - 18\cos 3t\,\hat{j}$$

$$= \sqrt{324\sin^2 3t + 324\cos^2 3t}$$

$$= \sqrt{324} = 18$$

35: If $\vec{A} = \sin u\,\hat{i} + \cos u\,\hat{j} + u\hat{k}$, $\vec{B} = \cos u\,\hat{i} - \sin u\,\hat{j} - 3\hat{k}$ and $\vec{C} = 2\hat{i} + 3\hat{j} - \hat{k}$. find $\frac{d}{du}\{\vec{A}\times(\vec{B}\times\vec{C})\}$ at $u = 0$.

Solution:

$$\vec{B}\times\vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos u & -\sin u & -3 \\ 2 & 3 & -1 \end{vmatrix}$$

$$= \hat{i}(\sin u + 9) - \hat{j}(-\cos u + 6) + \hat{k}(3\cos u + 2\sin u)$$

$$= (\sin u + 9)\hat{i} + (\cos u - 6)\hat{j} + (3\cos u + 2\sin u)\hat{k}$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sin u & \cos u & u \\ \sin u + 9 & \cos u - 6 & 3\cos u + 2\sin u \end{vmatrix}$$

$$= \hat{i}(3\cos^2 u + 2\sin u\cos u - u\cos u + 6u) - \hat{j}(3\sin u\cos u + 2\sin^2 u - u\sin u - 9u) + \hat{k}(\sin u\cos u - 6\sin u - \sin u\cos u - 9\cos u)$$

$$= \hat{i}(3\cos^2 u + 2\sin u\cos u - u\cos u + 6u) - \hat{j}(3\sin u\cos u + 2\sin^2 u - u\sin u - 9u) + \hat{k}(-6\sin u - 9\cos u)$$

$$\therefore \frac{d}{du}\{\vec{A}\times(\vec{B}\times\vec{C})\} = \hat{i}(-6\sin u\cos u\ \cancel{-6\sin u} + 2\cos^2 u - 2\sin^2 u - \cos u + u\sin u + 6) - \hat{j}(-3\sin^2 u + 3\cos^2 u + 4\sin u\cos u - \sin u - u\cos u - 9) + \hat{k}(-6\cos u + 9\sin u)$$

at $u=0$.

$$\frac{d}{du}\{\vec{A}\times(\vec{B}\times\vec{C})\} = \hat{i}(2-1+6)-\hat{j}(3-9)+\hat{k}(-6)$$

$$= 7\hat{i}+6\hat{j}-6\hat{k}$$

\# 38: If $\frac{d^2\vec{A}}{dt^2} = 6t\hat{i}-24t^2\hat{j}+4\sin t\,\hat{k}$, find $\vec{A}$. Given that $\vec{A} = 2\hat{i}+\hat{j}$ and $\frac{d\vec{A}}{dt} = -\hat{i}-3\hat{k}$ at $t=0$

Solution:

$$\frac{d^2\vec{A}}{dt^2} = 6t\hat{i}-24t^2\hat{j}+4\sin t\,\hat{k}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{d\vec{A}}{dt}\right) = 6t\hat{i}-24t^2\hat{j}+4\sin t\,\hat{k}$$

$$\Rightarrow d\left(\frac{d\vec{A}}{dt}\right) = (6t\hat{i}-24t^2\hat{j}+4\sin t\,\hat{k})\,dt$$

By integrating.

$$\int d\left(\frac{d\vec{A}}{dt}\right) = \int(6t\hat{i}-24t^2\hat{j}+4\sin t\,\hat{k})\,dt$$

$$\Rightarrow \frac{d\vec{A}}{dt} = 3t^2\hat{i}-8t^3\hat{j}-4\cos t\,\hat{k}+c_1 \quad ----(i)$$

where c_1 is an integrating constant

at $t=0$,

$$\frac{d\vec{A}}{dt} = -4\hat{k}+c_1$$

$$\Rightarrow -\hat{i}-3\hat{k} = -4\hat{k}+c_1$$

$$\Rightarrow c_1 = -\hat{i}+\hat{k}$$

putting the value of c_1 in equation (i)

$$\frac{d\vec{A}}{dt} = 3t^2\hat{i} - 8t^3\hat{j} - 4\cos t\,\hat{k} - \hat{i} + \hat{k}$$

$$\Rightarrow \frac{d}{dt}(\vec{A}) = (3t^2-1)\hat{i} - 8t^3\hat{j} - (4\cos t - 1)\hat{k}$$

$$\Rightarrow d\vec{A} = \{(3t^2-1)\hat{i} - 8t^3\hat{j} - (4\cos t - 1)\hat{k}\}\,dt$$

By integrating

$$\int d\vec{A} = \int \{(3t^2-1)\hat{i} - 8t^3\hat{j} - (4\cos t - 1)\hat{k}\}\,dt$$

$$\Rightarrow \vec{A} = (t^3-t)\hat{i} - 2t^4\hat{j} - (4\sin t - t)\hat{k} + c_2 \text{ --- (ii)}$$

where c_2 is integrating constant.

at $t=0$

$$\vec{A} = c_2$$

$$\Rightarrow c_2 = 2\hat{i} + \hat{j}$$

putting the value of c_2 in equation (ii)

$$\vec{A} = (t^3-t)\hat{i} - 2t^4\hat{j} - (4\sin t - t)\hat{k} + 2\hat{i} + \hat{j}$$

$$\therefore \vec{A} = (t^3-t+2)\hat{i} + (1-2t^4)\hat{j} + (t - 4\sin t)\hat{k}$$

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47: Find, a) the unit tangent $\vec{T}$.

b) the curvature k, c) the principal normal $\vec{N}$

d) the binormal $\vec{B}$, e) the torsion τ for the space curve $x = t - \frac{t^3}{3}$, $y = t^2$, $z = t + \frac{t^3}{3}$

Solution:

a) The curve, $\vec{r} = \left(t - \frac{t^3}{3}\right)\hat{i} + t^2\hat{j} + \left(t + \frac{t^3}{3}\right)\hat{k}$

$$\frac{d\vec{r}}{dt} = (1-t^2)\hat{i} + 2t\hat{j} + (1+t^2)\hat{k}$$

$$\frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right| = \sqrt{1-2t^2+t^4+4t^2+1+2t^2+t^4}$$

$$= \sqrt{2+4t^2+2t^4} = \sqrt{2(t^2+1)^2}$$

$$= \sqrt{2}(1+t^2)$$

$$\vec{T} = \frac{d\vec{r}}{ds} = \frac{d\vec{r}/dt}{ds/dt} = \frac{(1-t^2)\hat{i} + 2t\hat{j} + (1+t^2)\hat{k}}{\sqrt{2}(1+t^2)}$$

b) $$\frac{d\vec{T}}{dt} = \frac{d}{dt}\left(\frac{(1-t^2)\hat{i} + 2t\hat{j} + (1+t^2)\hat{k}}{\sqrt{2}(1+t^2)}\right)$$

$$\Rightarrow k\vec{N} = \frac{\sqrt{2}(1+t^2)(-2t\hat{i} + 2\hat{j} + 2t\hat{k}) - \cancel{\sqrt{2}(1+t^2)}\, 2\sqrt{2}t\,\{(1-t^2)\hat{i} + 2t\hat{j} + (1+t^2)\hat{k}\}}{2(1+t^2)^2}$$

$$\Rightarrow k\vec{N} = \frac{-2t(1+t^2)\hat{i} + 2(1+t^2)\hat{j} + 2t(1+t^2)\hat{k} - 2t(1-t^2)\hat{i} - 4t^2\hat{j} - 2t(1+t^2)\hat{k}}{\sqrt{2}(1+t^2)^2}$$

$$\Rightarrow k\vec{N} = \frac{(-2t-2t^3-2t+2t^3)\hat{i} + (2+2t^2-4t^2)\hat{j} + (2t+2t^3-2t-2t^3)\hat{k}}{\sqrt{2}(1+t^2)^2}$$

$$\Rightarrow k\vec{N} = \frac{-4t\hat{i} + (2-2t^2)\hat{j}}{\sqrt{2}(1+t^2)^2}$$

$$\Rightarrow k|\vec{N}| = \frac{\sqrt{16t^2+4-8t^2+4t^4}}{\sqrt{2}(1+t^2)^2}$$

$$\Rightarrow k = \frac{\sqrt{4+8t^2+4t^4}}{\sqrt{2}(1+t^2)^2}$$

$$\Rightarrow k = \frac{\sqrt{4(1+t^2)^2}}{\sqrt{2}(1+t^2)^2}$$

$$= \frac{2(1+t^2)}{\sqrt{2}(1+t^2)^2}$$

$$= \frac{\sqrt{2}}{1+t^2}$$

$$\frac{d\vec{T}}{ds} = \frac{d\vec{T}/dt}{ds/dt}$$

$$\Rightarrow k\vec{N} = \frac{\{-4t\hat{i} + (2-2t^2)\hat{j}\}\cancel{(1+t^2)}}{\sqrt{2}(1+t^2)^2\sqrt{2}(1+t^2)}$$

$$\Rightarrow k|\vec{N}| = \frac{\sqrt{16t^2+4-8t^2+4t^4}}{2(1+t^2)^3}$$

$$\Rightarrow k = \frac{\sqrt{4(1+t^2)^2}}{2(1+t^2)^3}$$

$$\Rightarrow k = \frac{2(1+t^2)}{2(1+t^2)^3} \quad \therefore k = \frac{1}{(1+t^2)^2}$$

c)
$$\frac{d\vec{T}}{ds} = \kappa \vec{N}$$

$$\Rightarrow \vec{N} = \frac{1}{\kappa} \frac{d\vec{T}/dt}{ds/dt}$$

$$= \frac{(1+t^2)^2 \{-4t\hat{i} + (2-2t^2)\hat{j}\}}{2.(1+t^2)^3}$$

$$= \frac{-4t\hat{i} + (2-2t^2)\hat{j}}{2(1+t^2)}$$

$$= \frac{-2t\hat{i} + (1-t^2)\hat{j}}{(1+t^2)}$$

d)
$$\vec{B} = \vec{T} \times \vec{N}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{1-t^2}{\sqrt{2}(1+t^2)} & \frac{2t}{\sqrt{2}(1+t^2)} & \frac{1+t^2}{\sqrt{2}(1+t^2)} \\ -\frac{2t}{1+t^2} & \frac{1-t^2}{1+t^2} & 0 \end{vmatrix}$$

$$= \frac{1}{\sqrt{2}(1+t^2)^2} \left[\hat{i}(-1+t^4) - \hat{j}(2t+2t^3) + \hat{k}\{(1-t^2)^2 + 4t^2\} \right]$$

$$= \frac{1}{\sqrt{2}(1+t^2)^2} \left[(t^4-1)\hat{i} - 2t(1+t^2)\hat{j} + (1+t^2)^2\hat{k} \right]$$

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$$= \frac{1}{\sqrt{2}(1+t^2)^2}\,(1+t^2)\left\{(t^2-1)\hat{i} - 2t\hat{j} + (1+t^2)\cdot\hat{k}\right]$$

$$= \frac{1}{\sqrt{2}(1+t^2)}\left[(t^2-1)\hat{i} - 2t\hat{j} + (1+t^2)\hat{k}\right]$$

e)

$$\frac{d\vec{B}}{dt} = \frac{\sqrt{2}(1+t^2)(2t\hat{i} - 2\hat{j} + 2t\hat{k}) - \left\{(t^2-1)\hat{i} - 2t\hat{j} + (1+t^2)\hat{k}\right\} 2\sqrt{2}t}{2(1+t^2)^2}$$

$$= \frac{(2t+2t^3)\hat{i} - (2+2t^2)\hat{j} + (2t+2t^3)\hat{k} - (2t^3-2t)\hat{i} + 4t^2\hat{j} - (2t+2t^3)\hat{k}}{\sqrt{2}(1+t^2)^2}$$

$$= \frac{(2t+2t^3-2t^3+2t)\hat{i} + (-2-2t^2+4t^2)\hat{j} + (2t+2t^3-2t-2t^3)\hat{k}}{\sqrt{2}(1+t^2)^2}$$

$$= \frac{4t\hat{i} + (2t^2-2)\hat{j}}{\sqrt{2}(1+t^2)^2}$$

$$\frac{d\vec{B}}{ds} = \frac{d\vec{B}/dt}{ds/dt}$$

$$\Rightarrow -\tau\vec{N} = \frac{4t\hat{i} + (2t^2-2)\hat{j}}{\sqrt{2}(1+t^2)^2\,\sqrt{2}(1+t^2)} \Rightarrow -\tau\vec{N} = \frac{-\left\{-4t\hat{i} - (2t^2-2)\hat{j}\right\}}{2(1+t^2)^3}$$

$$\cancel{\Rightarrow \tau|\vec{N}| = \frac{\sqrt{16t^2+(2t^2-2)^2}}{2(1+t^2)^3}} \Rightarrow \tau|\vec{N}| = \frac{\sqrt{16t^2+4t^4-8t^2+4}}{2(1+t^2)^3}$$

$$\cancel{\Rightarrow \tau = \frac{\sqrt{16t^2+4t^4-8t^2+4}}{2(1+t^2)^3}} \Rightarrow \tau = \frac{\sqrt{4(1+t^2)^2}}{2(1+t^2)^3}$$

$$\Rightarrow \tau = \frac{2(1+t^2)}{2(1+t^2)^3}$$

$$\therefore \tau = \frac{1}{(1+t^2)^2}$$

49: Find κ and τ for the space curve $x=t$, $y=t^2$, $z=t^3$ called the twisted cubic.

Solution:

The curve, $\vec{r} = t\hat{i} + t^2\hat{j} + t^3\hat{k}$

$$\frac{d\vec{r}}{dt} = \hat{i} + 2t\hat{j} + 3t^2\hat{k}$$

$$\frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right| = \sqrt{1^2+4t^2+9t^4} = \sqrt{9t^4+4t^2+1}$$

$$\vec{T} = \frac{d\vec{r}}{ds} = \frac{d\vec{r}/dt}{ds/dt}$$

$$= \frac{\hat{i}+2t\hat{j}+3t^2\hat{k}}{\sqrt{9t^4+4t^2+1}}$$

$$\frac{d\vec{T}}{dt} = \frac{\left(\sqrt{9t^4+4t^2+1}\right)(2\hat{j}+6t\hat{k}) - \dfrac{(\hat{i}+2t\hat{j}+3t^2\hat{k})(36t^3+8t)}{2\sqrt{9t^4+4t^2+1}}}{9t^4+4t^2+1}$$

$$= \frac{2(9t^4+4t^2+1)(2\hat{j}+6t\hat{k}) - (\hat{i}+2t\hat{j}+3t^2\hat{k})(36t^3+8t)}{2(9t^4+4t^2+1)^{\frac{3}{2}}}$$

$$= \cancel{-\hat{i} + (36t^4+16t^2}$$

$$\cancel{= (36t^3+8t)\hat{i} + (36t^4+16t^2+4-72t^4-16t^2)\hat{j} + (}$$

$$= \frac{2\left[-(18t^3+4t)\hat{i} + (18t^4+8t^2+2-36t^4-8t^2)\hat{j} + (54t^5+24t^3+6t-54t^5-12t^3)\hat{k}\right]}{2(9t^4+4t^2+1)^{\frac{3}{2}}}$$

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$$= \frac{-(18t^3+4t)\hat{i} + (2-18t^4)\hat{j} + (12t^3+6t)\hat{k}}{(9t^4+4t^2+1)^{\frac{3}{2}}}$$

$$\frac{d\vec{T}}{ds} = \frac{d\vec{T}/dt}{ds/dt}$$

$$\Rightarrow k\vec{N} = \frac{-(18t^3+4t)\hat{i} + (2-18t^4)\hat{j} + (12t^3+6t)\hat{k}}{(9t^4+4t^2+1)^{\frac{3}{2}}(9t^4+4t^2+1)^{\frac{1}{2}}}$$

$$\Rightarrow k\cdot|\vec{N}| = \frac{\sqrt{324t^8+144t^4+16t^2+4-72t^4+324t^8+144t^6+144t^4+36t^2}}{(9t^4+4t^2+1)^2}$$

$$\Rightarrow k = \frac{\sqrt{324t^8+468t^6+216t^4+52t^2+4}}{(9t^4+4t^2+1)^2}$$

$$= \frac{\sqrt{4(81t^8+117t^6+54t^4+13t^2+1)}}{(9t^4+4t^2+1)^2}$$

$$= \frac{2\sqrt{(9t^4+4t^2+1)(9t^4+9t^2+1)}}{(9t^4+4t^2+1)^2}$$

$$= \frac{2\sqrt{9t^4+9t^2+1}}{(9t^4+4t^2+1)^{\frac{3}{2}}}$$

$$\frac{d\vec{T}}{ds} = \kappa\vec{N}$$

$$\Rightarrow \vec{N} = \frac{1}{\kappa}\,\frac{d\vec{T}/dt}{ds/dt}$$

$$= \frac{(9t^4+4t^2+1)^{\frac{3}{2}}}{2\sqrt{9t^4+9t^2+1}} \times \frac{-(18t^3+4t)\hat{i}+(2-18t^4)\hat{j}+(12t^3+6t)\hat{k}}{(9t^4+4t^2+1)^2}$$

$$= \frac{-(18t^3+4t)\hat{i}+(2-18t^4)\hat{j}+(12t^3+6t)\hat{k}}{2\sqrt{(9t^4+9t^2+1)(9t^4+4t^2+1)}}$$

$$\vec{B} = \vec{T}\times\vec{N}$$

$$= \frac{1}{2(9t^4+4t^2+1)\sqrt{9t^4+9t^2+1}}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2t & 3t^2 \\ -(18t^3+4t) & (2-18t^4) & (12t^3+6t) \end{vmatrix}$$

$$= \frac{\hat{i}(24t^4+12t^2-6t^2+54t^6)-\hat{j}(12t^3+6t+54t^5+12t^3) + \hat{k}(2-18t^4+36t^4+8t^2)}{2(9t^4+4t^2+1)\sqrt{9t^4+9t^2+1}}$$

$$= \frac{2\left[\cancel{2}(27t^6+12t^4+3t^2)\hat{i}-(27t^5+12t^3+3t)\hat{j}+(9t^4+4t^2+1)\hat{k}\right]}{2(9t^4+4t^2+1)\sqrt{9t^4+9t^2+1}}$$

$$= \frac{3t^2(9t^4+4t^2+1)\hat{i} - 3t(9t^4+4t^2+1)\hat{j} + (9t^4+4t^2+1)\hat{k}}{(9t^4+4t^2+1)\sqrt{9t^4+9t^2+1}}$$

$$= \frac{3t^2\hat{i} - 3t\hat{j} + \hat{k}}{\sqrt{9t^4+9t^2+1}}$$

$$\frac{d\vec{B}}{dt} = \frac{\sqrt{9t^4+9t^2+1}\,(6t\hat{i} - 3\hat{j}) - \frac{(3t^2\hat{i} - 3t\hat{j} + \hat{k})(36t^3+18t)}{2\sqrt{9t^4+9t^2+1}}}{9t^4+9t^2+1}$$

$$= \frac{2(9t^4+9t^2+1)(6t\hat{i} - 3\hat{j}) - (3t^2\hat{i} - 3t\hat{j} + \hat{k})(36t^3+18t)}{2(9t^4+9t^2+1)^{\frac{3}{2}}}$$

$$= \frac{2[\hat{i}(54t^5+54t^3+6t-54t^5-27t^3) + \hat{j}(-27t^4-27t^2-3+54t^4+27t^2) - (18t^3+9t)\hat{k}}{2(9t^4+9t^2+1)^{\frac{3}{2}}}$$

$$= \frac{(27t^3+6t)\hat{i} + (27t^4-3)\hat{j} - (18t^3+9t)\hat{k}}{(9t^4+9t^2+1)^{\frac{3}{2}}}$$

$$\frac{d\vec{B}}{ds} = \frac{d\vec{B}/dt}{ds/dt}$$

$$\Rightarrow -\tau\vec{N} = \frac{-\left\{-(27t^3+6t)\hat{i} - (27t^4-3)\hat{j} + (18t^3+9t)\hat{k}\right\}}{(9t^4+9t^2+1)^{\frac{3}{2}}(9t^4+4t^2+1)^{\frac{1}{2}}}$$

$$\Rightarrow \tau|\vec{N}| = \frac{\sqrt{729t^6+324t^4+36t^2+729t^8-162t^4+9 + 324t^6+324t^4+81t^2}}{(9t^4+9t^2+1)^{\frac{3}{2}}(9t^4+4t^2+1)^{\frac{1}{2}}}$$

$$\Rightarrow \tau = \frac{\sqrt{729t^8+1053t^6+486t^4+117t^2+9}}{(9t^4+9t^2+1)^{\frac{3}{2}}(9t^4+4t^2+1)^{\frac{1}{2}}}$$

$$= \frac{\sqrt{9(81t^8+117t^6+54t^4+13t^2+1)}}{(9t^4+9t^2+1)^{\frac{3}{2}}(9t^4+4t^2+1)^{\frac{1}{2}}}$$

$$= \frac{3\sqrt{9t^4+9t^2+1}\sqrt{9t^4+4t^2+1}}{(9t^4+9t^2+1)^{\frac{3}{2}}(9t^4+4t^2+1)^{\frac{1}{2}}}$$

$$= \frac{3}{9t^4+9t^2+1}$$

#54: Prove that the curvature of the space curve $\vec{r} = \vec{r}(t)$ is given numerically by $\kappa = \dfrac{|\dot{r} \times \ddot{r}|}{|\dot{r}|^3}$ where dots denote differentiation respect to t.

Solution:

$$\dot{r} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds} \times \frac{ds}{dt} = \vec{T}\dot{s}$$

$$|\dot{r}| = \left|\frac{d\vec{r}}{dt}\right| = \frac{ds}{dt} = \dot{s}$$

$$\ddot{r} = \frac{d}{dt}(\dot{r}) = $$

$$= \ddot{s}\vec{T} + \dot{s}\frac{d\vec{T}}{dt}$$

$$= \ddot{s}\vec{T} + \dot{s}\frac{d\vec{T}}{ds} \times \frac{ds}{dt}$$

$$= \ddot{s}\vec{T} + (\dot{s})^2 \kappa \vec{N} \quad \left[\because \frac{ds}{dt} = \dot{s} \ \& \ \frac{d\vec{T}}{ds} = \kappa\vec{N}\right]$$

$$\dot{r} \times \ddot{r} = \begin{vmatrix} \vec{T} & \vec{N} & \vec{B} \\ \dot{s} & 0 & 0 \\ \ddot{s} & (\dot{s})^2\kappa & 0 \end{vmatrix}$$

$$= (\dot{s})^3 \kappa \vec{B}$$

$$\therefore |\dot{r} \times \ddot{r}| = (\dot{s})^3 \kappa \, |\vec{B}| = (\dot{s})^3 \kappa$$

$$\therefore \kappa = \frac{|\dot{r} \times \ddot{r}|}{(\dot{s})^3} = \frac{|\dot{r} \times \ddot{r}|}{|\dot{r}|^3}. \text{ (Proved)}$$

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Ex #55:

a) Prove that $\tau = \frac{\dot{r} \cdot (\ddot{r} \times \dddot{r})}{|\dot{r} \times \ddot{r}|^2}$ for the space curve $\vec{r} = \vec{r}(t)$.

b) If the parameter t is the arc length s. Show that $\tau = \frac{\frac{d\vec{r}}{ds} \cdot \frac{d^2\vec{r}}{ds^2} \times \frac{d^3\vec{r}}{ds^3}}{\left(\frac{d^2r}{ds^2}\right)^2}$

Solution:

a)
$$\dot{r} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds} \times \frac{ds}{dt} = \vec{T}\dot{s}$$

$$\ddot{r} = \ddot{s}\vec{T} + \dot{s}\frac{d\vec{T}}{dt}$$

$$= \ddot{s}\vec{T} + \dot{s}\frac{d\vec{T}}{ds} \times \frac{ds}{dt}$$

$$= \ddot{s}\vec{T} + (\dot{s})^2 k\vec{N}$$

$$\dddot{r} = \dddot{s}\vec{T} + \ddot{s}\frac{d\vec{T}}{dt} + 2\dot{s}\ddot{s}k\vec{N} + (\dot{s})^2 k\frac{d\vec{N}}{dt}$$

$$= \dddot{s}\vec{T} + \ddot{s}\frac{d\vec{T}}{ds} \times \frac{ds}{dt} + 2k\dot{s}\ddot{s}\vec{N} + (\dot{s})^2 k\frac{d\vec{N}}{ds} \times \frac{ds}{dt}$$

$$= \dddot{s}\vec{T} + k\dot{s}\ddot{s}\vec{N} + 2k\dot{s}\ddot{s}\vec{N} + k(\dot{s})^3(\tau\vec{B} - k\vec{T})$$

$$= \left\{\dddot{s} - k^2(\dot{s})^3\right\}\vec{T} + 3k\dot{s}\ddot{s}\vec{N} + k\tau(\dot{s})^3\vec{B}$$

$$\ddot{\vec{r}} \times \dddot{\vec{r}} = \begin{vmatrix} \vec{T} & \vec{N} & \vec{B} \\ \ddot{s} & (\dot{s})^2 k & 0 \\ \dddot{s} - k^2(\dot{s})^3 & 3\dot{s}\ddot{s}k & k\tau(\dot{s})^3 \end{vmatrix}$$

$$= \vec{T}\left\{(\dot{s})^5 k^2 \tau\right\} - \vec{N}\left\{(\dot{s})^3 \ddot{s} k \tau\right\}$$
$$+ \vec{B}\left[3\dot{s}(\ddot{s})^2 k - (\dot{s})^2 k\left\{\dddot{s} - k^2(\dot{s})^3\right\}\right]$$

$$\dot{\vec{r}} \cdot (\ddot{\vec{r}} \times \dddot{\vec{r}}) = k^2(\dot{s})^6 \tau - 0 - 0$$

$$\Rightarrow \tau = \frac{\dot{\vec{r}} \cdot (\ddot{\vec{r}} \times \dddot{\vec{r}})}{k^2(\dot{s})^6}$$

$$\Rightarrow \tau = \frac{\dot{\vec{r}} \cdot (\ddot{\vec{r}} \times \dddot{\vec{r}})}{|\dot{\vec{r}} \times \ddot{\vec{r}}|^2} \quad \text{(Proved)}$$

b) $$\text{R.H.S.} = \frac{\frac{d\vec{r}}{ds} \cdot \frac{d^2\vec{r}}{ds^2} \times \frac{d^3\vec{r}}{ds^3}}{\left(\frac{d^2\vec{r}}{ds^2}\right)^2}$$

$$= \frac{\vec{T} \cdot \left(\frac{d\vec{T}}{ds} \times \frac{d^2\vec{T}}{ds^2}\right)}{\left(\frac{d\vec{T}}{ds}\right)^2}$$

$$= \frac{\vec{T} \cdot \left\{k\vec{N} \times \frac{d}{ds}(k\vec{N})\right\}}{(k\vec{N})^2}$$

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$$= \frac{\vec{T}\cdot k^2\{\vec{N}\times(\tau\vec{B}-k\vec{T})\}}{(k\vec{N})^2}$$

$$= \frac{\vec{T}\cdot k^2[\tau(\vec{N}\times\vec{B})-k(\vec{N}\times\vec{T})]}{k^2(\vec{N}\cdot\vec{N})}$$

$$= \frac{\vec{T}\cdot k^2[\tau\vec{T}+k\vec{B}]}{k^2}$$

$$= \frac{k^2\tau(\vec{T}\cdot\vec{T})+k^3(\vec{T}\cdot\vec{B})}{k^2}$$

$$= \frac{k^2\tau}{k^2}$$

$$= \tau$$

$= R.H.S.$

$\therefore L.H.S. = R.H.S.$ (showed)

#58: Find the torsion of the curve

$x = \frac{2t+1}{t-1}$, $y = \frac{t^2}{t-1}$, $z = t+2$.

Solution:

The curve, $\vec{r} = \left(\frac{2t+1}{t-1}\right)\hat{i} + \left(\frac{t^2}{t-1}\right)\hat{j} + (t+2)\hat{k}$

$$\frac{d\vec{r}}{dt} = -\frac{3}{(t-1)^2}\hat{i} + \frac{t^2-2t}{(t-1)^2}\hat{j} + \hat{k}$$

$$s = \left|\frac{d\vec{r}}{dt}\right| = \sqrt{\frac{9}{(t-1)^4} + \frac{(t^2-2t)^2}{(t-1)^4} + 1}$$

$$= \underline{\hspace{6cm}}$$

18-02-2018 : 3E : Sunday

→ Show that the acceleration $\vec{a}$ of a particle which travels along a space curve with velocity v is given by

$$\vec{a} = \frac{dv}{dt}\vec{T} + \frac{v^2}{\rho}\vec{N}$$

where $\vec{T}$ is the unit tangent vector to the space curve $\vec{N}$ is the unit principal normal and ρ is the radius of the curvature.

Solution:

Velocity, $\vec{v}$ = magnitude of $|\vec{v}|$ multiplied by unit tangent vector $\vec{T}$

$$\Rightarrow \vec{v} = v\vec{T}$$

Differentiating,

$$\frac{d\vec{v}}{dt} = \frac{dv}{dt}\vec{T} + v\frac{d\vec{T}}{dt}$$

$$= \frac{dv}{dt}\vec{T} + v\frac{d\vec{T}}{ds}\cdot\frac{ds}{dt}$$

$$= \frac{dv}{dt}\vec{T} + v\left|\frac{d\vec{r}}{dt}\right|\frac{d\vec{T}}{ds}$$

$$= \frac{dv}{dt}\vec{T} + v^2\kappa\vec{N}$$

$$\therefore \vec{a} = \frac{dv}{dt}\vec{T} + \frac{v^2}{\rho}\vec{N} \text{ (showed)}$$

Ex:-66: A particle moves along the curve $\vec{r} = (t^3 - 4t)\hat{i} + (t^2 + 4t)\hat{j} + (8t^2 - 3t^3)\hat{k}$, where t is the time. Find the magnitudes of the tangential and normal components of its acceleration when $t=2$.

Solution:

Here, $$\dot{\vec{r}} = (3t^2 - 4)\hat{i} + (2t + 4)\hat{j} + (16t - 9t^2)\hat{k}$$

$$\ddot{\vec{r}} = 6t\hat{i} + 2\hat{j} + (16 - 18t)\hat{k}$$

we know,

$$\frac{v^2}{\rho} = \kappa v^2$$

$$= \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3} \times v^2 \quad \left[\kappa = \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3}\right]$$

$$= \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{v^3} \times v^2 \quad [\,|\dot{\vec{r}}| = v\,]$$

$$= \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{v}$$

at $t = 2$, $$\dot{\vec{r}} = 8\hat{i} + 8\hat{j} - 4\hat{k}$$

$$\ddot{\vec{r}} = 12\hat{i} + 2\hat{j} - 20\hat{k}$$

$$\dot{\vec{r}} \times \ddot{\vec{r}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 8 & 8 & -4 \\ 12 & 2 & -20 \end{vmatrix}$$

$$= \hat{i}(-160 + 8) - \hat{j}(-160 + 48) + \hat{k}(16 - 96)$$

$$= -152\hat{i} + 112\hat{j} - 80\hat{k}$$

$$|\dot{\vec{r}} \times \ddot{\vec{r}}| = \sqrt{(-152)^2 + 112^2 + (-80)^2}$$

$$= 24\sqrt{73}$$

$$v = \sqrt{9t^4 - 24t^2 + 16 + 4t^2 + 16t + 16 + 256t^2 - 288t^3 + 81t^4}$$

$$= \sqrt{90t^4 - 288t^3 + 236t^2 + 16t + 32}$$

$$\frac{dv}{dt} = \frac{360t^3 - 864t^2 + 472t + 16}{2\sqrt{90t^4 - 288t^3 + 236t^2 + 16t + 32}}$$

at $t = 2$,

$$v = 12, \quad \frac{dv}{dt} = 16$$

$$\frac{v^2}{\rho} = \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{v} = \frac{24\sqrt{73}}{12} = 2\sqrt{73}$$

Ex:- 67: If a particle has velocity v and acceleration a along a space curve, prove that the radius of curvature of its path is given numerically ~~of~~ by $\rho = \frac{v^3}{|\vec{v} \times \vec{a}|}$

Solution:

we know,

$$k = \frac{1}{\rho}$$

$$\Rightarrow v^2 k = \frac{v^2}{\rho}$$

$$\Rightarrow v^2 \frac{|\vec{v} \times \vec{a}|}{v^3} = \frac{v^2}{\rho} \quad \left[\because k = \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3} = \frac{|\vec{v} \times \vec{a}|}{v^3}\right]$$

$\Rightarrow \rho = \dfrac{v^3}{|\vec{v}\times\vec{a}|}$

19-02-2018 : 4A : Monday

→ Gradient, divergence and curl :

– vector differential operator, ∇

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \quad , \quad \phi(x,y,z) \cancel{= \phi dx\hat{i} + \phi dy\hat{j} + \phi dz\hat{k}}$$

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)\cdot\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$$

$$= \nabla\phi \cdot dr$$

$$= |\nabla\phi|\cdot|dr|\cos\theta$$

If, $\theta = 0^\circ$ $\qquad\qquad$ $\theta = 90^\circ$

$d\phi = |\nabla\phi|\cdot|dr|$ $\qquad\qquad$ $\Rightarrow d\phi = 0$

Em: 10

→ Find the directional derivative of $\phi = x^2yz + 4xz^2$ at $(1,-2,-1)$ in the direction $2\hat{i} - \hat{j} - 2\hat{k}$

Solution:

$$\nabla\phi = \frac{\partial}{\partial x}(x^2yz + 4xz^2)\hat{i} + \frac{\partial}{\partial y}(x^2yz + 4xz^2)\hat{j} + \frac{\partial}{\partial z}(x^2yz + 4xz^2)\hat{k}$$

$$= (2xyz + 4z^2)\hat{i} + (x^2z)\hat{j} + (x^2y + 8xz)\hat{k}$$

$\nabla\phi$ at $(1,-2,1)$

$$\nabla\phi = (4+4)\hat{i} - \hat{j} + (-2-8)\hat{k} = 8\hat{i} - \hat{j} - 10\hat{k}$$

Let, $\vec{A} = 2\hat{i} - \hat{j} - 2\hat{k}$

$$\frac{\vec{A}}{A} = \frac{2\hat{i}-\hat{j}-2\hat{k}}{\sqrt{4+1+4}} = \frac{1}{3}(2\hat{i}-\hat{j}-2\hat{k})$$

$$\nabla\phi \cdot \frac{\vec{A}}{A} = (8\hat{i}-\hat{j}-10\hat{k})\cdot\left(\frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}\right)$$

$$= \frac{16}{3} + \frac{1}{3} + \frac{20}{3}$$

$$= \frac{16+1+20}{3} = \frac{37}{3}$$

$\boxed{\text{Em: 12}}$

→ Find the angle between the surface $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at the point $(2,-1,2)$.

Solution:

Let,

$\phi_1 = x^2+y^2+z^2-9$ $\quad \vec{\nabla}\phi_1 = 2x\hat{i}+2y\hat{j}+2z\hat{k}$

$\phi_2 = x^2+y^2-z-3$ $\quad \vec{\nabla}\phi_2 = 2x\hat{i}+2y\hat{j}-\hat{k}$

$\vec{\nabla}\phi_1$ and $\vec{\nabla}\phi_2$ at $(2,-1,2)$

$$\vec{\nabla}\phi_1 = \cancel{4x-2y\hat{j}}\ 4\hat{i}-2\hat{j}+4\hat{k}$$

$$\vec{\nabla}\phi_2 = 4\hat{i}-2\hat{j}-\hat{k}$$

If the angle between ϕ_1 and ϕ_2 is θ then

~~cos θ~~

$$\cos\theta = \frac{\vec{\nabla}\phi_1\cdot\vec{\nabla}\phi_2}{|\vec{\nabla}\phi_1||\vec{\nabla}\phi_2|} = \frac{16+4-4}{\sqrt{16+4+16}\sqrt{16+4+1}} = \frac{16}{6\sqrt{21}}$$

$$= \frac{8}{3\sqrt{21}}$$

$$\therefore \theta = \cos^{-1}\left(-\frac{8}{3\sqrt{21}}\right)$$

26-02-2018 : 4E : Monday

→ Curl of a vector :

If $\vec{\nabla}$ is a vector then $\nabla \times \vec{\nabla}$ is the curl of a vector.

If $\vec{\nabla} \times \vec{\nabla} = 0$ then the vector is ~~irrotation~~ irrotational.

$\boxed{\text{Em: 29}}$

→ Prove : $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = -\vec{\nabla}^2 \vec{A} + \vec{\nabla}(\vec{\nabla} \cdot \vec{A})$

Solution:

$$\text{L.H.S.} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A})$$

$$= \vec{\nabla} \times \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \vec{\nabla} \times \left[\hat{i}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) + \hat{j}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) + \hat{k}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\right]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} & \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} & \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\right]$$
$$+ \hat{j}\left[\frac{\partial}{\partial z}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\right]$$
$$+ \hat{k}\left[\frac{\partial}{\partial x}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\right]$$

$$= \hat{i}\left[\frac{\partial^2 A_2}{\partial x \partial y} - \frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2} + \frac{\partial^2 A_3}{\partial x \partial z}\right]$$
$$+ \hat{j}\left[\frac{\partial^2 A_3}{\partial y \partial z} - \frac{\partial^2 A_2}{\partial z^2} - \frac{\partial^2 A_2}{\partial x^2} + \frac{\partial^2 A_1}{\partial x \partial y}\right]$$
$$+ \hat{k}\left[\frac{\partial^2 A_1}{\partial x \partial z} - \frac{\partial^2 A_3}{\partial x^2} - \frac{\partial^2 A_3}{\partial y^2} + \frac{\partial^2 A_2}{\partial y \partial z}\right]$$

$$= \left(-\frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2}\right)\hat{i} + \left(-\frac{\partial^2 A_2}{\partial z^2} - \frac{\partial^2 A_2}{\partial x^2}\right)\hat{j} + \left(-\frac{\partial^2 A_3}{\partial x^2} - \frac{\partial^2 A_3}{\partial y^2}\right)\hat{k}$$
$$+ \left(\frac{\partial^2 A_2}{\partial x \partial y} + \frac{\partial^2 A_3}{\partial x \partial z}\right)\hat{i} + \left(\frac{\partial^2 A_3}{\partial y \partial z} + \frac{\partial^2 A_1}{\partial x \partial y}\right)\hat{j} + \left(\frac{\partial^2 A_1}{\partial x \partial z} + \frac{\partial^2 A_2}{\partial y \partial z}\right)\hat{k}$$

$$= \left(-\frac{\partial^2 A_1}{\partial x^2} - \frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2} + \frac{\partial^2 A_1}{\partial x^2}\right)\hat{i} + \left(-\frac{\partial^2 A_2}{\partial x^2} - \frac{\partial^2 A_2}{\partial y^2} - \frac{\partial^2 A_2}{\partial z^2} + \frac{\partial^2 A_2}{\partial y^2}\right)\hat{j}$$
$$+ \left(-\frac{\partial^2 A_3}{\partial x^2} - \frac{\partial^2 A_3}{\partial y^2} - \frac{\partial^2 A_3}{\partial z^2} + \frac{\partial^2 A_3}{\partial z^2}\right)\hat{k} + \left(\frac{\partial^2 A_2}{\partial x \partial y} + \frac{\partial^2 A_3}{\partial x \partial z}\right)\hat{i}$$
$$+ \left(\frac{\partial^2 A_1}{\partial x \partial y} + \frac{\partial^2 A_3}{\partial y \partial z}\right)\hat{j} + \left(\frac{\partial^2 A_1}{\partial x \partial z} + \frac{\partial^2 A_2}{\partial y \partial z}\right)\hat{k}$$

$$= -\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left(A_1\hat{i}+A_2\hat{j}+A_3\hat{k}\right)+\hat{i}\left(\frac{\partial^2 A_1}{\partial x^2}+\frac{\partial^2 A_2}{\partial x\partial y}+\frac{\partial^2 A_3}{\partial x\partial z}\right)$$
$$+\hat{j}\left(\frac{\partial^2 A_1}{\partial x\partial y}+\frac{\partial^2 A_2}{\partial y^2}+\frac{\partial^2 A_3}{\partial x\partial z}\right)+\hat{k}\left(\frac{\partial^2 A_1}{\partial x\partial z}+\frac{\partial^2 A_2}{\partial y\partial z}+\frac{\partial^2 A_3}{\partial z^2}\right)$$

$$= -\vec{\nabla}^2\vec{A}+\hat{i}\frac{\partial}{\partial x}\left(\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)+\hat{j}\frac{\partial}{\partial y}\left(\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)$$
$$+\hat{k}\frac{\partial}{\partial z}\left(\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)$$

$$= -\vec{\nabla}^2\vec{A}+\vec{\nabla}\left(\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)$$
$$= -\vec{\nabla}^2\vec{A}+\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) = R.H.S.$$

$\therefore$ L.H.S. = R.H.S. (Proved)

$\boxed{\text{Ex:84}}$ $\rightarrow$ Prove that the vector $\vec{A} = 3y^4z^2\hat{i}+4x^3z^2\hat{j}-3x^2y^2\hat{k}$ is solenoidal.

Solution: If $\vec{\nabla}\cdot\vec{A}=0$, then it is solenoidal

$$L.H.S. = \vec{\nabla}\cdot\vec{A}$$
$$= \frac{\partial}{\partial x}(3y^4z^2)+\frac{\partial}{\partial y}(4x^3z^3) - \frac{\partial}{\partial z}(3x^2y^2)$$
$$= 0+0+0$$
$$= 0$$
$$= R.H.S.$$

$\therefore$ L.H.S. = R.H.S. (Proved)

Ex: 104

$\longrightarrow$ If $\vec{A}$ & $\vec{B}$ are irrotational, prove that $\vec{A}\times\vec{B}$ is solenoidal.

Solution:

As $\vec{A}$ & $\vec{B}$ are irrotational,

$$\vec{\nabla}\times\vec{A}=0\ ,\ \vec{\nabla}\times\vec{B}=0$$

If $\vec{\nabla}\cdot(\vec{A}\times\vec{B})=0$, it is solenoidal.

$$\text{L.H.S.}=\vec{\nabla}\cdot(\vec{A}\times\vec{B})$$

$$=\vec{B}\cdot(\vec{\nabla}\times\vec{A})-\vec{A}\cdot(\vec{\nabla}\times\vec{B})$$

$$=0-0$$

$$=0=\text{R.H.S.}$$

$\therefore$ L.H.S. = R.H.S. (Proved)

Ex: 102

$\longrightarrow$ Show that $\vec{A}=(6xy+z^3)\hat{i}+(3x^2-z)\hat{j}+(3xz^2-y)\hat{k}$ is irrotational. Find ϕ such that $\vec{A}=\vec{\nabla}\phi$.

Solution:

If $(\vec{\nabla}\times\vec{A})=0$, then it is irrotational.

L.H.S. =

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy+z^3 & 3x^2-z & 3xz^2-y \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(3xz^2-y) - \frac{\partial}{\partial z}(3x^2-z)\right] + \hat{j}\left[\frac{\partial}{\partial z}(6xy+z^3) - \frac{\partial}{\partial x}(3xz^2-y)\right] + \hat{k}\left[\frac{\partial}{\partial x}(3x^2-z) - \frac{\partial}{\partial y}(6xy+z^3)\right]$$

$$= \hat{i}(-1+1) + \hat{j}(3z^2-3z^2) + \hat{k}(6x-6x)$$

$$= 0 = R.H.S.$$

$\therefore$ L.H.S. = R.H.S. (showed)

We know,

$$\frac{d\phi}{d\vec{r}} = \vec{\nabla}\phi$$

$$\Rightarrow d\phi = \vec{\nabla}\phi \, d\vec{r}$$

$$= \vec{A} \, d\vec{r}$$

$$= \left[(6xy+z^3)\hat{i} + (3x^2-z)\hat{j} + (3xz^2-y)\hat{k}\right] (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= (6xy+z^3)dx + (3x^2-z)dy + (3xz^2-y)dz$$

$$= z^3dx + 3xz^2dz + 6xydx + 3x^2dy - zdy - ydz$$

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By integrating,

$$\phi = xz^3 + 3x^2y - yz + c$$

Example : Chapter 4

$\boxed{\text{Em:-01}}$: If $\phi(x,y,z) = 3x^2y - y^3z^2$, find $\nabla\phi$ at the point $(1,-2,-1)$

Solution:

$$\vec{\nabla}\phi = \frac{\partial}{\partial x}(3x^2y - y^3z^2)\hat{i} + \frac{\partial}{\partial y}(3x^2y - y^3z^2)\hat{j} + \frac{\partial}{\partial z}(3x^2y - y^3z^2)\hat{k}$$

$$= 6xy\,\hat{i} + (3x^2 - 3y^2z^2)\hat{j} - 2y^3z\,\hat{k}$$

$\vec{\nabla}\phi$ at $(1,-2,-1)$,

$\cancel{-12\hat{i} - 9\hat{j}}$

$-12\hat{i} - 9\hat{j} - 16k$

$\boxed{\text{Em:-03}}$: Find $\nabla\phi$ if (a) $\phi = \ln|\vec{r}|$, (b) $\phi = \frac{1}{|\vec{r}|}$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Then $|\vec{r}| = \sqrt{x^2+y^2+z^2}$

$\therefore \phi = \ln|\vec{r}|$

$= \frac{1}{2}\ln(x^2+y^2+z^2)$

(a)

$$\therefore \vec{\nabla}\phi = \frac{1}{2}\vec{\nabla}\ln(x^2+y^2+z^2)$$

$$= \frac{1}{2}\left[\frac{\partial}{\partial x}\ln(x^2+y^2+z^2)\hat{i} + \frac{\partial}{\partial y}\ln(x^2+y^2+z^2)\hat{j} + \frac{\partial}{\partial z}\ln(x^2+y^2+z^2)\hat{k}\right]$$

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$$= \frac{1}{2}\left[\frac{2x}{x^2+y^2+z^2}\hat{i} + \frac{2y}{x^2+y^2+z^2}\hat{j} + \frac{2z}{x^2+y^2+z^2}\hat{k}\right]$$

$$= \frac{1}{2}\ \frac{2(x\hat{i}+y\hat{j}+z\hat{k})}{x^2+y^2+z^2} = \frac{\vec{r}}{|\vec{r}|^2}$$

(b) $\vec{\nabla}\phi = \vec{\nabla}\left(\frac{1}{|\vec{r}|}\right)$

$$= \vec{\nabla}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-\frac{1}{2}}\hat{i} + \frac{\partial}{\partial y}(x^2+y^2+z^2)^{-\frac{1}{2}}\hat{j} + \frac{\partial}{\partial z}(x^2+y^2+z^2)^{-\frac{1}{2}}\hat{k}$$

$$= -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}2x\hat{i} \cancel{+} -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}2y\hat{j} - \frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}2z\hat{k}$$

$$= -\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}(x\hat{i}+y\hat{j}+z\hat{k})$$

$$= -\frac{\vec{r}}{|\vec{r}|^3}$$

$\boxed{\text{Em:-05}}$: Show that $\vec{\nabla}\phi$ is a vector perpendicular to the surface $\phi(x,y,z)=c$ where c is a constant.

Solution:

Let, $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$ be the position vector to any point $P(x,y,z)$ on the surface. Then $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$ lies in the tangent plane to the surface at P.

But $d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = 0$

$$\Rightarrow \left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right)\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right) = 0$$

$$\therefore \vec{\nabla}\phi \cdot d\vec{r} = 0$$

so, $\vec{\nabla}\phi$ is pendiculan to $d\vec{r}$ and therefore to the surface.

$\boxed{\text{Em :- 6}}$: Find a unit normal to the surface $x^2y + 2xz = 4$ at the point $(2,-2,3)$

Solution:

$$\vec{\nabla}\phi = \frac{\partial}{\partial x}(x^2y + 2xz)\hat{i} + \frac{\partial}{\partial y}(x^2y + 2xz)\hat{j} + \frac{\partial}{\partial z}(x^2y + 2xz)\hat{k}$$

$$= (2xy + 2z)\hat{i} + x^2\hat{j} + 2x\hat{k}$$

at $(2,-2,3)$ point

$$-2\hat{i} + 4\hat{j} + 4\hat{k}$$

unit normal to the surface

$$= \frac{-2\hat{i} + 4\hat{j} + 4\hat{k}}{\sqrt{4+16+16}} = -\frac{2}{6}\hat{i} + \frac{4}{6}\hat{j} + \frac{4}{6}\hat{k}$$

$$= -\frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$$

$\boxed{Ex-07}$: Find an equation for the tangent plane to the tangent plane to the surface $2xz^2-3xy-4x=7$ at the point $(1,-1,2)$.

Solution: Let,

$$\phi = 2xz^2-3xy-4x-7$$

$$\vec{\nabla}\phi = (2z^2-3y-4)\hat{i} - 3x\hat{j} + 4xz\hat{k}$$

at $(1,-1,2)$ point

$$\vec{\nabla}\phi = 7\hat{i}-3\hat{j}+8\hat{k}$$

The equation of a plane passing through a point whose position vector is r_0 and which is perpendicular to the normal N is $(r-r_0)\cdot N=0$. Then required equation.

$$[(x\hat{i}+y\hat{j}+z\hat{k})-(\hat{i}-\hat{j}+2\hat{k})]\cdot(7\hat{i}-3\hat{j}+8\hat{k})=0$$

$$\Rightarrow [(x-1)\hat{i}+(y+1)\hat{j}+(z-2)\hat{k}](7\hat{i}-3\hat{j}+8\hat{k})=0$$

$$\therefore 7(x-1)-3(y+1)+8(z-2)=0$$

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Em:- 09 : Show that the greatest rate of change of ϕ i.e. the maximum directional derivative takes place in the direction of and has the magnitude of the vector $\vec{\nabla}\phi$.

Solution:

$$\frac{d\phi}{ds} = \vec{\nabla}\phi \frac{d\vec{r}}{ds}$$ is the projection of $\vec{\nabla}\phi$ in the direction $\frac{dr}{ds}$. This projection will be a maximum when $\vec{\nabla}\phi$ and $\frac{d\vec{r}}{ds}$ have the same direction. Then the maximum value of $\frac{d\phi}{ds}$ takes place in the direction of $\vec{\nabla}\phi$ and its magnitude is $|\vec{\nabla}\phi|$.

Em:- 11 :

ⓐ In what direction from the point $(2,1,-1)$ is the directional derivative of $\phi = x^2yz^3$ a maximum

ⓑ What is the magnitude of this maximum?

Solution:

ⓐ $$\vec{\nabla}\phi = \frac{\partial}{\partial x}(x^2yz^3)\hat{i} + \frac{\partial}{\partial y}(x^2yz^3)\hat{j} + \frac{\partial}{\partial z}(x^2yz^3)\hat{k}$$

$$= 2xyz^2\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k}$$

at $(2,1,-1)$ point

$$-4\hat{i} - 4\hat{j} + 12\hat{k}$$

This is the direction of the directional derivative.

(b) The magnitude of this maximum is

$$|\vec{\nabla}\phi| = \sqrt{16+16+144} = \sqrt{176} = 4\sqrt{11}$$

$\boxed{\text{Em:-17}}$: Prove that $\vec{\nabla}^2\left(\frac{1}{|\vec{r}|}\right) = 0$

Solution:

$$L.H.S. = \vec{\nabla}^2\left(\frac{1}{|\vec{r}|}\right)$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}\,2x$$

$$= -x(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{\partial}{\partial x}\left\{-x(x^2+y^2+z^2)^{-\frac{3}{2}}\right\}$$

$$= -\left(-\frac{3x}{2}\right)(x^2+y^2+z^2)^{-\frac{5}{2}}\,2x - (x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$= 3x^2(x^2+y^2+z^2)^{-\frac{5}{2}} - (x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$= \frac{3x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} - \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$= \frac{3x^2 - x^2 - y^2 - z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

$$= \frac{2x^2 - y^2 - z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

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$$\frac{\partial^2}{\partial y^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

$$\frac{\partial^2}{\partial z^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

$$\therefore \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} + \frac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} + \frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

$= 0$

$= R.H.S.$

$\therefore$ L.H.S. = R.H.S. (Proved)

$\boxed{\text{Em:-18}}$: Prove that

(a) $\vec{\nabla}\cdot(\vec{A}+\vec{B}) = \vec{\nabla}\cdot\vec{A} + \vec{\nabla}\cdot\vec{B}$

(b) $\vec{\nabla}\cdot(\phi\vec{A}) = (\vec{\nabla}\phi)\cdot\vec{A} + \phi(\vec{\nabla}\cdot\vec{A})$

Solution:

(a) L.H.S. $= \vec{\nabla}\cdot(\vec{A}+\vec{B})$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left[(A_1+B_1)\hat{i} + (A_2+B_2)\hat{j} + (A_3+B_3)\hat{k}\right]$$

$$= \frac{\partial}{\partial x}(A_1+B_1) + \frac{\partial}{\partial y}(A_2+B_2) + \frac{\partial}{\partial z}(A_3+B_3)$$

$$= \frac{\partial}{\partial x} A_1 + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} + \frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z}$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(A_1\hat{i} + A_2\hat{j} + A_3\hat{k}) + \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)$$
$$(B_1\hat{i} + B_2\hat{j} + B_3\hat{k})$$

$$= \vec{\nabla}\cdot\vec{A} + \vec{\nabla}\cdot\vec{B}$$

= R.H.S.

$\therefore$ L.H.S. = R.H.S. (Proved)

(b) L.H.S. = $\vec{\nabla}(\phi\vec{A})$

$$= \vec{\nabla}(\phi A_1\hat{i} + \phi A_2\hat{j} + \phi A_3\hat{k})$$

$$= \frac{\partial}{\partial x}(\phi A_1) + \frac{\partial}{\partial y}(\phi A_2) + \frac{\partial}{\partial z}(\phi A_3)$$

$$= \frac{\partial \phi}{\partial x} A_1 + \phi\frac{\partial A_1}{\partial x} + \frac{\partial \phi}{\partial y} A_2 + \phi\frac{\partial A_2}{\partial y} + \frac{\partial \phi}{\partial z} A_3 + \phi\frac{\partial A_3}{\partial z}$$

$$= \frac{\partial \phi}{\partial x} A_1 + \frac{\partial \phi}{\partial y} A_2 + \frac{\partial \phi}{\partial z} A_3 + \phi\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)$$

$$= \cancel{\phi}\left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right)(A_1\hat{i} + A_2\hat{j} + A_3\hat{k})$$
$$+ \phi\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(A_1\hat{i} + A_2\hat{j} + A_3\hat{k})$$

$$= (\vec{\nabla}\phi)\vec{A} + \phi(\vec{\nabla}\cdot\vec{A})$$

= R.H.S.

$\therefore$ L.H.S. = R.H.S. (Proved)

$\boxed{\text{Em:-19}}$: Prove $\vec{\nabla}\left(\frac{\vec{r}}{|\vec{r}|^3}\right)=0$

Solution:

$$\text{L.H.S.} = \vec{\nabla}\left(\frac{\vec{r}}{|\vec{r}|^3}\right)$$

$$= \vec{\nabla}\left(\vec{r}\cdot|\vec{r}|^{-3}\right)$$

$$= \left(\vec{\nabla}|\vec{r}|^{-3}\right)\vec{r} + |\vec{r}|^{-3}\left(\vec{\nabla}\cdot\vec{r}\right)$$

$$= -3|\vec{r}|^{-5}\vec{r}\cdot\vec{r} + |\vec{r}|^{-3}\cdot 3 \quad [\text{From Em: 4}]$$

$$= -3|\vec{r}|^{-5}|\vec{r}|^2 + 3|\vec{r}|^{-3}$$

$$= -3|\vec{r}|^{-3} + 3|\vec{r}|^{-3}$$

$$= 0$$

$$= \text{R.H.S.}$$

$\therefore$ L.H.S. = R.H.S. (Proved)

$\boxed{\text{Em:-22}}$: Determine the constant a so that the vector $\vec{V}=(x+3y)\hat{i}+(y-2z)\hat{j}+(x+az)\hat{k}$ is solenoidal.

Solution:

The vector, $\vec{V}$ is solenoidal, if $\vec{\nabla}\cdot\vec{V}=0$

$$\vec{\nabla}\cdot\vec{V}=0$$

$$\Rightarrow \frac{\partial}{\partial x}(x+3y)+\frac{\partial}{\partial y}(y-2z)+\frac{\partial}{\partial z}(x+az)=0$$

$$\Rightarrow 1+1+a=0$$

$$\therefore a=-2.$$

$\boxed{\text{Em:-23}}$: If $\vec{A} = xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$, find $\vec{\nabla}\times\vec{A}$ at the point $(1,-1,1)$.

Solution:

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(2yz^4) - \frac{\partial}{\partial z}(-2x^2yz)\right] - \hat{j}\left[\frac{\partial}{\partial x}(2yz^4) - \frac{\partial}{\partial z}(xz^3)\right] + \hat{k}\left[\frac{\partial}{\partial x}(-2x^2yz) - \frac{\partial}{\partial y}(xz^3)\right]$$

$$= \hat{i}\,[2z^4 + 2x^2y] - \hat{j}\,[0 - 3xz^2] + \hat{k}\,[-4xyz - 0]$$

$\cancel{= 2(x^2y}$

$$= 2(x^2y + z^4)\hat{i} + 3xz^2\hat{j} - 4xyz\hat{k}$$

at $(1,-1,1)$ point

$3\hat{j} + 4\hat{k}$

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Em:-24: If $\vec{A} = x^2y\,\hat{i} - 2xz\,\hat{j} + 2yz\,\hat{k}$, find curl curl $\vec{A}$.

Solution:

curl curl $\vec{A}$

$$= \vec{\nabla} \times (\vec{\nabla} \times \vec{A})$$

$$= \vec{\nabla} \times \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & -2xz & 2yz \end{vmatrix}$$

$$= \vec{\nabla} \times \left[\hat{i}(2z + 2x) - \hat{j}(0 - 0) + \hat{k}(-2z - x^2) \right]$$

$$= \vec{\nabla} \times \left[(2x + 2z)\hat{i} - (x^2 + 2z)\hat{k} \right]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x + 2z & 0 & -(x^2 + 2z) \end{vmatrix}$$

$$= \hat{i}(0 - 0) - \hat{j}(-2x - 2) + \hat{k}(0 - 0)$$

$$= (2x + 2)\,\hat{j}$$

Em:- 25 : Prove

(a) $\vec{\nabla}\times(\vec{A}+\vec{B}) = \vec{\nabla}\times\vec{A} + \vec{\nabla}\times\vec{B}$

(b) $\vec{\nabla}\times(\phi\vec{A}) = (\vec{\nabla}\phi)\times\vec{A} + \phi(\vec{\nabla}\times\vec{A})$

Solution:

(a) Let,

$$\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$$

$$\vec{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$$

$$\text{L.H.S.} = \vec{\nabla}\times(\vec{A}+\vec{B})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1+B_1 & A_2+B_2 & A_3+B_3 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(A_3+B_3) - \frac{\partial}{\partial z}(A_2+B_2)\right]\hat{i} + \hat{j}\left[\frac{\partial}{\partial z}(A_1+B_1) - \frac{\partial}{\partial x}(A_3+B_3)\right] + \left[\frac{\partial}{\partial x}(A_2+B_2) - \frac{\partial}{\partial y}(A_1+B_1)\right]\hat{k}$$

$$= \left[\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right]\hat{i} + \left[\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right]\hat{j} + \left[\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right]\hat{k}$$

$$+ \left[\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}\right]\hat{i} + \left[\frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial x}\right]\hat{j} + \left[\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}\right]\hat{k}$$

$= \vec{\nabla}\times\vec{A} + \vec{\nabla}\times\vec{B}$

$= R.H.S.$

$\therefore L.H.S. = R.H.S.$ (Proved)

(b) $L.H.S. = \vec{\nabla}\times(\phi\vec{A})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi A_1 & \phi A_2 & \phi A_3 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(\phi A_3) - \frac{\partial}{\partial z}(\phi A_2)\right]\hat{i} + \left[\frac{\partial}{\partial z}(\phi A_1) - \frac{\partial}{\partial x}(\phi A_3)\right]\hat{j}$$
$$+ \left[\frac{\partial}{\partial x}(\phi A_2) - \frac{\partial}{\partial y}(\phi A_1)\right]$$

$$= \left[\phi\frac{\partial A_3}{\partial y} + \frac{\partial\phi}{\partial y}A_3 - \phi\frac{\partial A_2}{\partial z} - \frac{\partial\phi}{\partial z}A_2\right]\hat{i}$$
$$+ \left[\phi\frac{\partial A_1}{\partial z} + \frac{\partial\phi}{\partial z}A_1 - \phi\frac{\partial A_3}{\partial x} - \frac{\partial\phi}{\partial x}A_3\right]\hat{j}$$
$$+ \left[\phi\frac{\partial A_2}{\partial x} + \frac{\partial\phi}{\partial z}A_2 - \phi\frac{\partial A_1}{\partial y} - \frac{\partial\phi}{\partial y}A_1\right]\hat{k}$$

$$= \phi\left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\hat{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\hat{k}\right]$$
$$+ \left[\left(\frac{\partial\phi}{\partial y}A_3 - \frac{\partial\phi}{\partial z}A_2\right)\hat{i} + \left(\frac{\partial\phi}{\partial z}A_1 - \frac{\partial\phi}{\partial x}A_3\right)\hat{j} + \left(\frac{\partial\phi}{\partial z}A_2 - \frac{\partial\phi}{\partial y}A_1\right)\hat{k}\right]$$

$= \phi(\vec{\nabla} \times \vec{A}) + (\vec{\nabla}\phi) \times \vec{A}$

$= R.H.S.$

$\therefore$ L.H.S. = R.H.S. (Proved)

$\boxed{Em:-26}$: Evaluate $\vec{\nabla} \cdot (\vec{A} \times \vec{r})$ if $\vec{\nabla} \times \vec{A} = 0$

Solution:

Let,

$$\vec{A} = A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\therefore \vec{A} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ x & y & z \end{vmatrix}$$

$$= \hat{i}(A_2 z - A_3 y) + \hat{j}(A_3 x - A_1 z) + \hat{k}(A_1 y - A_2 x)$$

$$\therefore \vec{\nabla} \cdot (\vec{A} \times \vec{r}) = \frac{\partial}{\partial x}(A_2 z - A_3 y) + \frac{\partial}{\partial y}(A_3 x - A_1 z) + \frac{\partial}{\partial z}(A_1 y - A_2 x)$$

$$= z\frac{\partial A_2}{\partial x} - y\frac{\partial A_3}{\partial x} + x\frac{\partial A_3}{\partial y} - z\frac{\partial A_1}{\partial y} + y\frac{\partial A_1}{\partial z} - x\frac{\partial A_2}{\partial z}$$

$$= x\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) + y\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) + z\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

$$= (x\hat{i} + y\hat{j} + z\hat{k})\left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\hat{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\hat{k}\right]$$

$$= \vec{r} \cdot (\vec{\nabla} \times \vec{A})$$

If $\vec{\nabla} \times \vec{A} = 0$ this reduce to zero.

$\boxed{\text{Em : 27}}$: Prove :

(a) $\vec{\nabla} \times (\vec{\nabla} \phi) = 0$ (b) $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$

Solution:

(a) $L.H.S. = \vec{\nabla} \times (\vec{\nabla} \phi)$

$$= \vec{\nabla} \times \left(\frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k} \right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \left[\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial y \partial z} \right] \hat{i} + \left[\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial x \partial z} \right] \hat{j} + \left[\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial x \partial y} \right] \hat{k}$$

$= 0$

$= R.H.S.$

$\therefore L.H.S. = R.H.S.$ (Proved)

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(b) $L.H.S. = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})$

$$= \vec{\nabla}\cdot\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \vec{\nabla}\cdot\left[\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)\hat{i}+\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)\hat{j}+\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)\hat{k}\right]$$

$$= \frac{\partial^2 A_3}{\partial x\partial y}-\frac{\partial^2 A_2}{\partial x\partial z}+\frac{\partial^2 A_1}{\partial y\partial z}-\frac{\partial^2 A_3}{\partial x\partial y}+\frac{\partial^2 A_2}{\partial x\partial z}-\frac{\partial^2 A_1}{\partial y\partial z}$$

$= 0$

$= R.H.S.$

$\therefore L.H.S. = R.H.S.$ (Proved)

$\boxed{\text{Em :- 30}}$: If $\vec{v} = \vec{\omega}\times\vec{r}$, prove $\vec{\omega} = \frac{1}{2}\,\text{curl}\,\vec{v}$ where $\vec{\omega}$ is a constant vector.

Solution:

curl $\vec{v}$

$= \vec{\nabla}\times\vec{v}$

$= \vec{\nabla}\times(\vec{\omega}\times\vec{r})$

$$= \vec{\nabla}\times\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

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$$= \vec{\nabla} \times \left[(\omega_2 z - \omega_3 y)\hat{i} + (\omega_3 x - \omega_1 z)\hat{j} + (\omega_1 y - \omega_2 x)\hat{k} \right]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= \hat{i}(\omega_1 + \omega_1) + \hat{j}(\omega_2 + \omega_2) + \hat{k}(\omega_3 + \omega_3)$$

$$= 2(\omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}) = 2\vec{\omega}$$

$\therefore$ curl $\vec{v} = 2\vec{\omega}$

$\therefore \vec{\omega} = \frac{1}{2} \times$ curl $\vec{v}$ (Proved)

Exercise : Chapter - 4

$\boxed{\text{Ex :- 45}}$: Find $\vec{\nabla} |\vec{r}|^3$

Solution: Let, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\therefore |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

$\therefore \vec{\nabla} |\vec{r}|^3$

$= \vec{\nabla} \cdot (x^2 + y^2 + z^2)^{\frac{3}{2}}$

$$= \hat{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{\frac{3}{2}} + \hat{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)^{\frac{3}{2}} + \hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)^{\frac{3}{2}}$$

$$= \hat{i}\left\{\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}}2x\right\} + \hat{j}\left\{\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}}2y\right\} + \hat{k}\left\{\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}}2z\right\}$$

$$= 3(x^2+y^2+z^2)^{\frac{1}{2}}(x\hat{i}+y\hat{j}+z\hat{k})$$

$$= 3|\vec{r}|\,\vec{r}.$$

$\boxed{\text{Ex : 85}}$: show that $\vec{A} = (2x^2+8xy^2z)\hat{i} + (3x^3y-3xy)\hat{j} - (4y^2z^2+2x^3z)\hat{k}$ is not solenoidal but $\vec{B} = xyz^2\vec{A}$ is solenoidal.

Solution:

If $\vec{\nabla}\cdot\vec{A} \neq 0$, then it is not solenoidal.

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial}{\partial x}(2x^2+8xy^2z) + \frac{\partial}{\partial y}(3x^3y-3xy) - \frac{\partial}{\partial z}(4y^2z^2+2x^3z)$$

$$= (4x+8y^2z) + (3x^3-3x) - \cancel{(8xz^2+2x^3)} - (8y^2z+2x^3)$$

$$\vec{\nabla}\cdot\vec{B} = \frac{\partial}{\partial x}(2x^3yz^2+8x^2y^3z^3) + \frac{\partial}{\partial y}(3x^4y^2z^2-3x^2y^2z^2)$$

$$- \frac{\partial}{\partial z}(4xy^3z^4+2x^4yz^3)$$

$$= 6x^2yz^2+16xy^3z^3+6x^4yz^2-6x^2yz^2$$

$$-16xy^3z^3-6x^4yz^3$$

$$= 0$$

$\therefore$ $\vec{B}$ is solenoidal but $\vec{A}$ is not.

$\boxed{\text{Ex:-91}}$: Evaluate $\vec{\nabla}\times\left(\frac{\vec{r}}{|\vec{r}|^2}\right)$

Solution:

$$\vec{\nabla}\times\left(\frac{\vec{r}}{|\vec{r}|^2}\right)$$

$$= \vec{\nabla}\left(|\vec{r}|^{-2}\,\vec{r}\right)$$

$$= \vec{\nabla}(|\vec{r}|^{-2})\times\vec{r} + |\vec{r}|^{-2}(\vec{\nabla}\times\vec{r})$$

$$= -2|\vec{r}|^{-4}(\vec{r}\times\vec{r}) + |\vec{r}|^{-2}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x & y & z\end{vmatrix}$$

$$= -2|\vec{r}|^{-4}\times 0 + |\vec{r}|^{-2}\left[\hat{i}\left(\frac{\partial z}{\partial y}-\frac{\partial y}{\partial z}\right)+\hat{j}\left(\frac{\partial x}{\partial z}-\frac{\partial z}{\partial x}\right) + \hat{k}\left(\frac{\partial y}{\partial x}-\frac{\partial x}{\partial y}\right)\right]$$

$$= 0 + |\vec{r}|^{-2}\times 0$$

$$= 0$$

$\boxed{\text{Ex:-92}}$: for what value of the constant a will ~~be~~ the vector $\vec{A} = (axy - z^3)\,\hat{i} + (a-2)x^2\,\hat{j} + (1-a)xz^2\,\hat{k}$ have its curl identically equal to zero?

Solution:

Curl identically equal to zero

$$\therefore \vec{\nabla}\times\vec{A} = 0$$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ axy - z^3 & (a-2)x^2 & (1-a)xz^2 \end{vmatrix} = 0$$

$$\Rightarrow \hat{i}(0-0) + \hat{j}\{-3z^2 - (1-a)z^2\} + \hat{k}\{2x(a-2) - ax\} = 0$$

$$\Rightarrow (-3z^2 - z^2 + az^2)\hat{j} + (2ax - 4 - ax)\hat{k} = 0$$

$$\Rightarrow (-4z^2 + az^2)\hat{j} + (ax-4)\hat{k} = 0$$

$$\Rightarrow (a-4)z^2\hat{j} + (ax-4)\hat{k} = 0$$

If the co-efficient of $\hat{i}, \hat{j}, \hat{k}$ are zero, the vector is zero

$$(a-4)z^2 = 0$$

$$\therefore a = 4$$

Ex:- 93: Prove curl $(\phi \text{ grad } \phi) = 0$

Solution:

$$\text{L.H.S.} = \text{curl}(\phi \text{ grad } \phi)$$

$$= \vec{\nabla} \times (\phi \vec{\nabla} \phi)$$

$$= \vec{\nabla} \times \left[\phi\left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right)\right]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi \frac{\partial \phi}{\partial x} & \phi \frac{\partial \phi}{\partial y} & \phi \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\left(\phi \frac{\partial \phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\phi \frac{\partial \phi}{\partial y}\right)\right] + \hat{j}\left[\frac{\partial}{\partial z}\left(\phi \frac{\partial \phi}{\partial x}\right) - \frac{\partial}{\partial x}\left(\phi \frac{\partial \phi}{\partial z}\right)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}\left(\phi \frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\phi \frac{\partial \phi}{\partial x}\right)\right]$$

$$= \hat{i}\left[\frac{\partial \phi}{\partial y} \cdot \frac{\partial \phi}{\partial z} + \phi \frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial \phi}{\partial z} \cdot \frac{\partial \phi}{\partial y} - \phi \frac{\partial^2 \phi}{\partial y \partial z}\right]$$

$$+ \hat{j}\left[\frac{\partial \phi}{\partial z} \cdot \frac{\partial \phi}{\partial x} + \phi \frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial \phi}{\partial x} \cdot \frac{\partial \phi}{\partial z} - \phi \frac{\partial^2 \phi}{\partial x \partial z}\right]$$

$$+ \hat{k}\left[\frac{\partial \phi}{\partial x} \cdot \frac{\partial \phi}{\partial y} + \phi \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial \phi}{\partial y} \cdot \frac{\partial \phi}{\partial x} - \frac{\partial^2 \phi}{\partial x \partial y}\right]$$

$= 0 = R.H.S.$

$\therefore$ L.H.S. = R.H.S. (Proved)

$\boxed{\text{Ex :- 98}}$: Prove $(\vec{V}\cdot\vec{\nabla})\vec{V} = \frac{1}{2}\vec{\nabla}V^2 - \vec{V}\times(\vec{\nabla}\times\vec{V})$

Solution:

$$R.H.S. = \frac{1}{2}\vec{\nabla}|\vec{V}|^2 - \vec{V}\times(\vec{\nabla}\times\vec{V})$$

$$= \frac{1}{2}2|\vec{V}|^0\vec{V} - \vec{V}\times\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \vec{V} - \vec{V}\times\left[\hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \hat{j}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\right]$$

$$= \vec{V} - \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ V_1 & V_2 & V_3 \\ \frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} & \frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x} & \frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \end{vmatrix}$$

$$= \vec{V} - \hat{i}\left(V_2\frac{\partial V_2}{\partial x} - V_2\frac{\partial V_1}{\partial y} - V_3\frac{\partial V_1}{\partial z} + V_3\frac{\partial V_3}{\partial x}\right)$$
$$- \hat{j}\left(V_3\frac{\partial V_3}{\partial y} - V_3\frac{\partial V_2}{\partial z} - V_1\frac{\partial V_2}{\partial x} + V_1\frac{\partial V_1}{\partial y}\right)$$
$$- \hat{k}\left(V_1\frac{\partial V_1}{\partial z} - V_1\frac{\partial V_3}{\partial x} - V_2\frac{\partial V_3}{\partial y} + V_2\frac{\partial V_2}{\partial z}\right)$$

$$= \vec{V} - \hat{i}\left(V_1\frac{\partial V_1}{\partial x} + V_2\frac{\partial V_2}{\partial x} + V_3\frac{\partial V_3}{\partial x}\right) - \hat{j}\left(V_1\frac{\partial V_1}{\partial y} + V_2\frac{\partial V_2}{\partial y} + V_3\frac{\partial V_3}{\partial y}\right)$$
$$- \hat{k}\left(V_1\frac{\partial V_1}{\partial z} + V_2\frac{\partial V_2}{\partial z} + V_3\frac{\partial V_3}{\partial z}\right) + \hat{i}\left(V_1\frac{\partial V_1}{\partial x} + V_2\frac{\partial V_1}{\partial y} + V_3\frac{\partial V_1}{\partial z}\right)$$
$$+ \hat{j}\left(V_1\frac{\partial V_2}{\partial x} + V_2\frac{\partial V_2}{\partial y} + V_3\frac{\partial V_2}{\partial z}\right) + \hat{k}\left(V_1\frac{\partial V_3}{\partial x} + V_2\frac{\partial V_3}{\partial y} + V_3\frac{\partial V_3}{\partial z}\right)$$

$$= \vec{\nabla} - \hat{i}\,(v_1 + 0 + 0) - \hat{j}\,(0 + v_2 + 0) - \hat{k}\,(0 + 0 + v_3)$$

$$+ \left\{ (v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}) \cdot \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \right\} (v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k})$$

$$= \vec{\nabla} - \vec{\nabla} + (\vec{\nabla} \cdot \vec{\nabla}) \cdot \vec{\nabla}$$

$$= (\vec{\nabla} \cdot \vec{\nabla}) \vec{\nabla}$$

$= R.H.S.$

$\therefore$ L.H.S. = R.H.S. (Proved)

$\boxed{\text{Ex:99}}$: Prove $\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$

Solution:

~~L.H.S. = $\vec{\nabla} \cdot (\vec{A} \times \vec{B})$~~

R.H.S. = $\vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$

$$= \vec{B} \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} - \vec{A} \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$= (B_1 \hat{i} + B_2 \hat{j} + B_3 \hat{k}) \cdot \left[\hat{i} \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) + \hat{j} \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) + \hat{k} \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \right] - (A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}) \cdot$$

$$\left[\hat{i} \left(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z} \right) + \hat{j} \left(\frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial x} \right) + \hat{k} \left(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y} \right) \right]$$

$$= B_1\frac{\partial A_3}{\partial y} - B_1\frac{\partial A_2}{\partial z} + B_2\frac{\partial A_1}{\partial z} - B_2\frac{\partial A_3}{\partial x} + B_3\frac{\partial A_2}{\partial x} - B_3\frac{\partial A_1}{\partial y}$$

$$- A_1\frac{\partial B_3}{\partial y} + A_1\frac{\partial B_2}{\partial z} - A_2\frac{\partial B_1}{\partial z} + A_2\frac{\partial B_3}{\partial x} - A_3\frac{\partial B_2}{\partial x} + A_3\frac{\partial B_1}{\partial y}$$

$$= A_2\frac{\partial B_3}{\partial x} + B_3\frac{\partial A_2}{\partial x} - A_3\frac{\partial B_2}{\partial x} - B_2\frac{\partial A_3}{\partial x} + A_3\frac{\partial B_1}{\partial y} + B_1\frac{\partial A_3}{\partial y}$$

$$- A_1\frac{\partial B_3}{\partial y} - B_3\frac{\partial A_1}{\partial y} + A_1\frac{\partial B_2}{\partial z} + B_2\frac{\partial A_1}{\partial z} - A_2\frac{\partial B_1}{\partial z} - B_1\frac{\partial A_2}{\partial z}$$

$$= \frac{\partial}{\partial x}(A_2 B_3) - \frac{\partial}{\partial x}(A_3 B_2) + \frac{\partial}{\partial y}(A_3 B_1) - \frac{\partial}{\partial y}(A_1 B_3)$$

$$+ \frac{\partial}{\partial z}(A_1 B_2) - \frac{\partial}{\partial z}(A_2 B_1)$$

$$= \frac{\partial}{\partial x}(A_2 B_3 - A_3 B_2) + \frac{\partial}{\partial y}(A_3 B_1 - A_1 B_3) + \frac{\partial}{\partial z}(A_1 B_2 - A_2 B_1)$$

$$= \vec{\nabla}\cdot(\vec{A}\times\vec{B})$$

$= L.H.S.$

$\therefore$ L.H.S. = R.H.S. (Proved)

Ex:- 100: Prove $\vec{\nabla}\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{A} - \vec{B}(\vec{\nabla}\cdot\vec{A}) - (\vec{A}\cdot\vec{\nabla})\cdot\vec{B} + \vec{A}(\vec{\nabla}\cdot\vec{B})$

Solution:

$$\text{R.H.S.} = (\vec{B}\cdot\vec{\nabla})\vec{A} - \vec{B}(\vec{\nabla}\cdot\vec{A}) - (\vec{A}\cdot\vec{\nabla})\cdot\vec{B} + \vec{A}(\vec{\nabla}\cdot\vec{B})$$

$$= \left(B_1\frac{\partial}{\partial x} + B_2\frac{\partial}{\partial y} + B_3\frac{\partial}{\partial z}\right)(A_1\hat{i} + A_2\hat{j} + A_3\hat{k})$$

$$- (B_1\hat{i} + B_2\hat{j} + B_3\hat{k})\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)$$

$$- \left(A_1\frac{\partial}{\partial x} + A_2\frac{\partial}{\partial y} + A_3\frac{\partial}{\partial z}\right)(B_1\hat{i} + B_2\hat{j} + B_3\hat{k})$$

$$+ (A_1\hat{i} + A_2\hat{j} + A_3\hat{k})\left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z}\right)$$

$$= \hat{i}\left(\cancel{B_1\frac{\partial A_1}{\partial x}} + B_2\frac{\partial A_1}{\partial y} + B_3\frac{\partial A_1}{\partial z} - \cancel{B_1\frac{\partial A_1}{\partial x}} - B_1\frac{\partial A_2}{\partial y} - B_1\frac{\partial A_3}{\partial z}\right.$$

$$\left. - \cancel{A_1\frac{\partial B_1}{\partial x}} - A_2\frac{\partial B_1}{\partial y} - A_3\frac{\partial B_1}{\partial z} + \cancel{A_1\frac{\partial B_1}{\partial x}} + A_1\frac{\partial B_2}{\partial y} + A_1\frac{\partial B_3}{\partial z}\right)$$

$$+ \hat{j}\left(B_1\frac{\partial A_2}{\partial x} + \cancel{B_2\frac{\partial A_2}{\partial y}} + B_3\frac{\partial A_2}{\partial z} - B_2\frac{\partial A_1}{\partial x} - \cancel{B_2\frac{\partial A_2}{\partial y}} - B_2\frac{\partial A_3}{\partial z}\right.$$

$$\left. - A_1\frac{\partial B_2}{\partial x} - \cancel{A_2\frac{\partial B_2}{\partial y}} - A_3\frac{\partial B_2}{\partial z} + A_2\frac{\partial B_1}{\partial x} + \cancel{A_2\frac{\partial B_2}{\partial y}} + A_2\frac{\partial B_3}{\partial z}\right)$$

$$+ \hat{k}\left(B_1\frac{\partial A_3}{\partial x} + B_2\frac{\partial A_3}{\partial y} + \cancel{B_3\frac{\partial A_3}{\partial z}} - B_3\frac{\partial A_1}{\partial x} - B_3\frac{\partial A_2}{\partial y} - \cancel{B_3\frac{\partial A_3}{\partial z}}\right.$$

$$\left. - A_1\frac{\partial B_3}{\partial x} - A_2\frac{\partial B_3}{\partial y} - \cancel{A_3\frac{\partial B_3}{\partial z}} + A_3\frac{\partial B_1}{\partial x} + A_3\frac{\partial B_2}{\partial y} + \cancel{A_3\frac{\partial B_3}{\partial z}}\right)$$

$$= \hat{i}\left(A_1\frac{\partial B_2}{\partial y} + B_2\frac{\partial A_1}{\partial y} - A_2\frac{\partial B_1}{\partial y} - B_1\frac{\partial A_2}{\partial y} - A_3\frac{\partial B_1}{\partial z} - B_1\frac{\partial A_3}{\partial z}\right.$$

$$\left. + A_1\frac{\partial B_3}{\partial z} + B_3\frac{\partial A_1}{\partial z}\right) + \hat{j}\left(A_2\frac{\partial B_3}{\partial z} + B_3\frac{\partial A_2}{\partial z} - A_3\frac{\partial B_2}{\partial z}\right.$$

$$\left. - B_2\frac{\partial A_3}{\partial z} - A_1\frac{\partial B_2}{\partial x} - B_2\frac{\partial A_1}{\partial x} + A_2\frac{\partial B_1}{\partial x} + B_1\frac{\partial A_2}{\partial x}\right)$$

$$+ \hat{k}\left(A_3\frac{\partial B_1}{\partial x} + B_1\frac{\partial A_3}{\partial x} - A_1\frac{\partial B_3}{\partial x} - B_3\frac{\partial A_1}{\partial x} - A_2\frac{\partial B_3}{\partial y}\right.$$

$$\left. - B_3\frac{\partial A_2}{\partial y} + A_3\frac{\partial B_2}{\partial y} + B_2\frac{\partial A_3}{\partial y}\right)$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(A_1B_2) - \frac{\partial}{\partial y}(A_2B_1) - \frac{\partial}{\partial z}(A_3B_1) + \frac{\partial}{\partial z}(A_1B_3)\right]$$

$$+ \hat{j}\left[\frac{\partial}{\partial z}(A_2B_3) - \frac{\partial}{\partial z}(A_3B_2) - \frac{\partial}{\partial x}(A_1B_2) + \frac{\partial}{\partial x}(A_2B_1)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(A_3B_1) - \frac{\partial}{\partial x}(A_1B_3) - \frac{\partial}{\partial y}(A_2B_3) + \frac{\partial}{\partial y}(A_3B_2)\right]$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(A_1B_2 - A_2B_1) - \frac{\partial}{\partial z}(A_3B_1 - A_1B_3)\right]$$

$$+ \hat{j}\left[\frac{\partial}{\partial z}(A_2B_3 - A_3B_2) - \frac{\partial}{\partial x}(A_1B_2 - A_2B_1)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(A_3B_1 - A_1B_3) - \frac{\partial}{\partial y}(A_2B_3 - A_3B_2)\right]$$

$$= \vec{\nabla}\times(\vec{A}\times\vec{B})$$

$= R.H.S.$

$\therefore$ L.H.S. = R.H.S. (Proved)

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$\boxed{\text{Ex:- 101}}$: Prove $\vec{\nabla}(\vec{A}\cdot\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{A} + (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{B}\times(\vec{\nabla}\times\vec{A}) + \vec{A}\times(\vec{\nabla}\times\vec{B})$

Solution :

$$\text{R.H.S.} = (\vec{B}\cdot\vec{\nabla})\vec{A} + (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{B}\times(\vec{\nabla}\times\vec{A}) + \vec{A}\times(\vec{\nabla}\times\vec{B})$$

$$= \left(B_1\frac{\partial}{\partial x} + B_2\frac{\partial}{\partial y} + B_3\frac{\partial}{\partial z}\right)(A_1\hat{i} + A_2\hat{j} + A_3\hat{k})$$

$$+ \left(A_1\frac{\partial}{\partial x} + A_2\frac{\partial}{\partial y} + A_3\frac{\partial}{\partial z}\right)(B_1\hat{i} + B_2\hat{j} + B_3\hat{k})$$

$$+ \vec{B}\times\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} + \vec{A}\times\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$= \left(B_1\frac{\partial}{\partial x} + B_2\frac{\partial}{\partial y} + B_3\frac{\partial}{\partial z}\right)(A_1\hat{i} + A_2\hat{j} + A_3\hat{k})$$

$$+ \left(A_1\frac{\partial}{\partial x} + A_2\frac{\partial}{\partial y} + A_3\frac{\partial}{\partial z}\right)(B_1\hat{i} + B_2\hat{j} + B_3\hat{k})$$

$$+ \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_1 & B_2 & B_3 \\ \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} & \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} & \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \end{vmatrix}$$

$$+ \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ \frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z} & \frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial x} & \frac{\partial B_2}{\partial x} - \frac{\partial A_1}{\partial y} \end{vmatrix}$$

$$= \hat{i}\Big(B_1 \frac{\partial A_1}{\partial x} + B_2 \cancel{\frac{\partial A_1}{\partial y}} + B_3 \cancel{\frac{\partial A_1}{\partial z}} + A_1 \frac{\partial B_1}{\partial x} + A_2 \cancel{\frac{\partial B_1}{\partial y}} + A_3 \cancel{\frac{\partial B_1}{\partial z}}$$

$$+ B_2 \frac{\partial A_2}{\partial x} - B_2 \cancel{\frac{\partial A_1}{\partial y}} - B_3 \cancel{\frac{\partial A_1}{\partial z}} + B_3 \frac{\partial A_3}{\partial x} + A_2 \frac{\partial B_2}{\partial x} - A_2 \cancel{\frac{\partial B_1}{\partial y}}$$

$$- A_3 \cancel{\frac{\partial B_1}{\partial z}} + A_3 \frac{\partial B_3}{\partial x}\Big) + \hat{j}\Big(B_1 \cancel{\frac{\partial A_2}{\partial x}} + B_2 \frac{\partial A_2}{\partial y} + B_3 \cancel{\frac{\partial A_2}{\partial z}}$$

$$+ A_1 \cancel{\frac{\partial B_2}{\partial x}} + A_2 \frac{\partial B_2}{\partial y} + A_3 \cancel{\frac{\partial B_2}{\partial z}} + B_3 \frac{\partial A_3}{\partial y} - B_3 \cancel{\frac{\partial A_2}{\partial z}} - B_1 \cancel{\frac{\partial A_2}{\partial x}}$$

$$+ B_1 \frac{\partial A_1}{\partial y} + A_3 \frac{\partial B_3}{\partial y} - A_3 \cancel{\frac{\partial B_2}{\partial z}} - A_1 \cancel{\frac{\partial B_2}{\partial x}} + A_1 \frac{\partial B_1}{\partial y}\Big)$$

$$+ \hat{k}\Big(B_1 \cancel{\frac{\partial A_3}{\partial x}} + B_2 \cancel{\frac{\partial A_3}{\partial y}} + B_3 \frac{\partial A_3}{\partial z} + A_1 \cancel{\frac{\partial B_3}{\partial x}} + A_2 \cancel{\frac{\partial B_3}{\partial y}} + A_3 \frac{\partial B_3}{\partial z}$$

$$+ B_1 \frac{\partial A_1}{\partial z} - B_1 \cancel{\frac{\partial A_3}{\partial x}} - B_2 \cancel{\frac{\partial A_3}{\partial y}} + B_2 \frac{\partial A_2}{\partial z} + A_1 \frac{\partial B_1}{\partial z} - A_1 \cancel{\frac{\partial B_3}{\partial x}}$$

$$- A_2 \cancel{\frac{\partial B_3}{\partial y}} + A_3 \frac{\partial B_2}{\partial z}\Big)$$

$$= \hat{i}\left[\frac{\partial}{\partial x}(A_1 B_1) + \frac{\partial}{\partial x}(A_2 B_2) + \frac{\partial}{\partial x}(A_3 B_3)\right]$$

$$+ \hat{j}\left[\frac{\partial}{\partial y}(A_1 B_1) + \frac{\partial}{\partial y}(A_2 B_2) + \frac{\partial}{\partial y}(A_3 B_3)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial z}(A_1 B_1) + \frac{\partial}{\partial z}(A_2 B_2) + \frac{\partial}{\partial z}(A_3 B_3)\right]$$

$$= \hat{i}\frac{\partial}{\partial x}(A_1 B_1 + A_2 B_2 + A_3 B_3) + \hat{j}\frac{\partial}{\partial y}(A_1 B_1 + A_2 B_2 + A_3 B_3)$$

$$+ \hat{k}\frac{\partial}{\partial z}(A_1 B_1 + A_2 B_2 + A_3 B_3)$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(A_1 B_1 + A_2 B_2 + A_3 \cdot B_3)$$

$$= \vec{\nabla}(\vec{A} \cdot \vec{B})$$

$= L.H.S.$

$\therefore L.H.S. = R.H.S.$ (Proved)

27-02-2018 : 5A : Tuesday

→ Vector integration:

$\vec{R}(u)$ is vector where u is scaler. The integration $\vec{R}(u)$ is $\int \vec{R}(u)\,du$

$$= \int A_1\,du\,\hat{i} + \int A_2\,du\,\hat{j} + \int A_3\,du\,\hat{k} + \text{const}$$

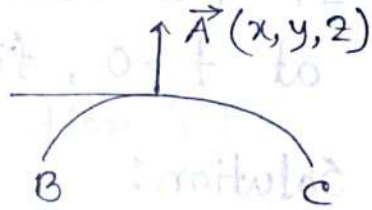
$\vec{R}(t) = t^2\hat{i} + 2t\hat{j} + \hat{k}$

$$\Rightarrow \int \vec{R}(t) = \int t^2\,dt\,\hat{i} + \int 2t\,dt\,\hat{j} + \int dt\,\hat{k}$$

$$= \frac{t^3}{3}\hat{i} + t^2\hat{j} + t\hat{k} + \text{const}$$

→ Line integration:

Line integration is the integration of the component of the unit tangent vector of $\vec{A}$.



the unit tangent vector of $\vec{A}$, $\vec{T} = \frac{d\vec{r}}{ds}$

the component of the unit tangent vector of $\vec{A}$

$= \vec{A}\cdot\frac{d\vec{r}}{ds}$

$$\therefore \text{Line integration} = \int_C \vec{A}\cdot\frac{d\vec{r}}{ds}\cdot ds$$

$$= \int_C \vec{A}\cdot d\vec{r}$$

Em:1
$\rightarrow$ If $\vec{R}(u) = (u-u^2)\hat{i} + 2u^3\hat{j} - 3\hat{k}$, find $\int \vec{R}(u)\,du$.

Solution:

$$\int \vec{R}(u)\,du$$

$$= \int (u-u^2)\,du\,\hat{i} + \int (2u^3)\,du\,\hat{j} - \int 3\,du\,\hat{k}$$

$$= \left(\frac{u^2}{2} - \frac{u^3}{3}\right)\hat{i} + \frac{u^4}{2}\hat{j} - 3u\hat{k} + c$$

where, c is an integrating constant.

Em:2
$\rightarrow$ The acceleration of a particle at any time $t \geq 0$ is given by $\vec{a} = \frac{d\vec{v}}{dt} = 12\cos 2t\,\hat{i} - 8\sin 2t\,\hat{j} + 16t\,\hat{k}$. If the velocity, $\vec{v}$ and displacement, $\vec{r}$ are zero at $t=0$, find $\vec{v}$ & $\vec{r}$ at any time.

Solution:

$$\vec{a} = \frac{d\vec{v}}{dt} = 12\cos 2t\,\hat{i} - 8\sin 2t\,\hat{j} + 16t\,\hat{k}$$

$$\Rightarrow d\vec{v} = (12\cos 2t)dt\,\hat{i} - (8\sin 2t)dt\,\hat{j} + (16t)\,dt\,\hat{k}$$

$\therefore$ By integrating.

$$\int d\vec{v} = \int (12\cos 2t)\,dt\,\hat{i} - \int (8\sin 2t)\,dt\,\hat{j} + \int (16t)\,dt\,\hat{k}$$

$$\Rightarrow v = 6\sin 2t\,\hat{i} + 4\cos 2t\,\hat{j} + 8t^2\hat{k} + c_1 \quad \text{---} \quad (i)$$

$$\Rightarrow \frac{d\vec{r}}{dt} = 6\sin 2t\,\hat{i} + 4\cos 2t\,\hat{j} + 8t^2\hat{k} + c_1$$

$$\Rightarrow d\vec{r} = (6\sin 2t)\,dt\,\hat{i} + (4\cos 2t)\,dt\,\hat{j} + (8t^2)\,dt\,\hat{k} + c_1 dt$$

By integrating,

$$\int d\vec{r} = \int (6\sin 2t)dt\,\hat{i} + \int (4\cos 2t)dt\,\hat{j} + \int (8t^2)dt\,\hat{k} + \int c_1\,dt$$

$$\Rightarrow \vec{r} = -3\cos 2t\,\hat{i} + 2\sin 2t\,\hat{j} + \frac{8t^3}{3}\hat{k} + c_1 t + c_2 \quad \text{--- (ii)}$$

at $t=0$, (i) & (ii) no. equation.

$$0 = 4\hat{j} + c_1 \quad \therefore c_1 = -4\hat{j}$$

$$0 = -3\hat{i} + c_2 \quad \therefore c_2 = 3\hat{i}$$

putting the value of c_1 in equation (i)

$$\vec{v} = 6\sin 2t\,\hat{i} + 4\cos 2t\,\hat{j} + 8t^2\hat{k} \cancel{+} - 4\hat{j}$$

$$\therefore \vec{v} = 6\sin 2t\,\hat{i} + (4\cos 2t - 4)\hat{j} + 8t^2\hat{k}$$

putting the value of c_1 and c_2 in equation (ii)

$$\vec{r} = -3\cos 2t\,\hat{i} + 2\sin 2t\,\hat{j} + \frac{8t^3}{3}\hat{k} \cancel{+} - 4t\hat{j} + 3\hat{i}$$

$$\therefore \vec{r} = (3 - 3\cos 2t)\hat{i} + (2\sin 2t - 4t)\hat{j} + \frac{8t^3}{3}\hat{k}$$

$\boxed{\text{Em: 6}}$ If $\vec{A} = (3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$, evaluate $\int_C \vec{A}\cdot d\vec{r}$ from $(0,0,0)$ to along the following path C.

i) $x = t$, $y = t^2$, $z = t^3$

ii) the straight line from $(0,0,0)$ to $(1,0,0)$, then $(1,1,0)$ and then $(1,1,1)$

i)

$$\int_C \vec{A}\cdot d\vec{r} = \int_C \left\{(3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}\right\}\cdot(dx\hat{i}+dy\hat{j}+dz\hat{k})$$

$$= \int_C \left\{(3x^2+6y)\,dx - (14yz)\,dy + (20xz^2)\,dz\right\}$$

$$= \int_0^1 \left\{(3t^2+6t^2)\,dt - (14t^5)\,2t\,dt + (20t^7)\,3t^2dt\right\}$$

$$= \int_0^1 (9t^2 - 28t^6 + 60t^9)\,dt$$

$$= \left[3t^3 - 4t^7 + 6t^{10}\right]_0^1$$

$$= 3-4+6 = 5.$$

ii) The straight line $(0,0,0)$ to $(1,0,0)$

$y=0,\ z=0,\ dy=0,\ dz=0$ and only x varies from 0 to 1.

$$\int_C \vec{A}\cdot d\vec{r} = \int_C \left\{(3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}\right\}\cdot(dx\hat{i}+dy\hat{j}+dz\hat{k})$$

$$= \int_C \left\{(3x^2+6y)\,dx - (14yz)\,dy + (20xz^2)\,dz\right\}$$

$$= \int_0^1 (3x^2+6y)\,dx - \int_0^0 (14yz)\,dy + \int_0^0 (20xz^2)\,dz$$

$$= \left[x^3\right]_0^1 + 0 + 0$$

$$= 1$$

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The straight line from $(1,0,0)$ to $(1,1,0)$

$x=1$, $z=0$, $dx=0$, $dz=0$ and only y varies from 0 to 1

$$\int_C \vec{A}\cdot d\vec{r} = \int_C \left\{(3x^2+6y)dx - (14yz)dy + (20xz^2)dz\right\}$$

$$= \int_1^1 (3x^2+6y)dx - \int_0^1 (14yz)dy + \int_0^0 (20xz^2)dz$$

$$= 0$$

The straight line from $(1,1,0)$ to $(1,1,1)$

$x=1$, $y=1$, $dx=0$, $dy=0$ and only z varies from 0 to 1

$$\int_C \vec{A}\cdot d\vec{r} = \int_C \left\{(3x^2+6y)dx - (14yz)dy + (20xz^2)dz\right\}$$

$$= \int_1^1 (3x^2+6y)dx - \int_1^1 (14yz)dy + \int_0^1 (20xz^2)dz$$

$$= \int_0^1 (20z^2)\,dz$$

$$= \left[\frac{20z^3}{3}\right]_0^1$$

$$= \frac{20}{3}$$

Adding the straight line $= 1 + 0 + \frac{20}{3}$

$$= \frac{23}{3}$$

05-03-2018 : 5E : Monday :

Em:07 → Find the total work done in moving a particle in a force field given by $\vec{F} = 3xy\,\hat{i} - 5z\,\hat{j} + 10x\,\hat{k}$. Along the curve $x = t^2+1$, $y = 2t^2$, $z = t^3$ from $t=1$ to $t=2$.

Solution:

The curve, $\vec{r} = (t^2+1)\,\hat{i} + (2t^2)\,\hat{j} + t^3\,\hat{k}$

$$\Rightarrow d\vec{r} = (2t\,\hat{i} + 4t\,\hat{j} + 3t^2\,\hat{k})\,dt$$

∴ The work done

$$= \int_C \vec{F}\cdot d\vec{r}$$

$$= \int_C (3xy\,\hat{i} - 5z\,\hat{j} + 10x\,\hat{k})\cdot(2t\,\hat{i} + 4t\,\hat{j} + 3t^2\,\hat{k})\,dt$$

$$= \int_C \{(3xy)(2t) - (5z)(4t) + (10x)(3t^2)\}\,dt$$

$$= \int_1^2 \{(6t^4+6t^2)(2t) - (5t^3)(4t) + (10t^2+10)(3t^2)\}\,dt$$

$$= \int_1^2 (12t^5+12t^3-20t^4+30t^4+30t^2)\,dt$$

$$= \int_1^2 (12t^5+10t^4+12t^3+30t^2)\,dt$$

$$= \left[2t^6+2t^5+3t^4+10t^3\right]_1^2$$

$$= 128+64+48+80-2-2-3-10 = 303$$

Em: 08

→ If $\vec{F} = 3xy\,\hat{i} - y^2\hat{j}$, Evaluate $\int \vec{F}\cdot d\vec{r}$ where c is the curve in the xy plane $y = 2x^2$ from $(0,0)$ to $(1,2)$.

Solution:

As the curve is in xy plane,

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$$

$$y = 2x^2$$

$$\Rightarrow \frac{dy}{dx} = 4x \quad \therefore dy = 4x\,dx$$

$$\therefore \int_c \vec{F}\cdot d\vec{r}$$

$$= \int_c (3xy\,\hat{i} - y^2\hat{j})\cdot(dx\,\hat{i} + dy\,\hat{j})$$

$$= \int_0^1 (3xy)\,dx - \int_0^2 (y^2)\,dy$$

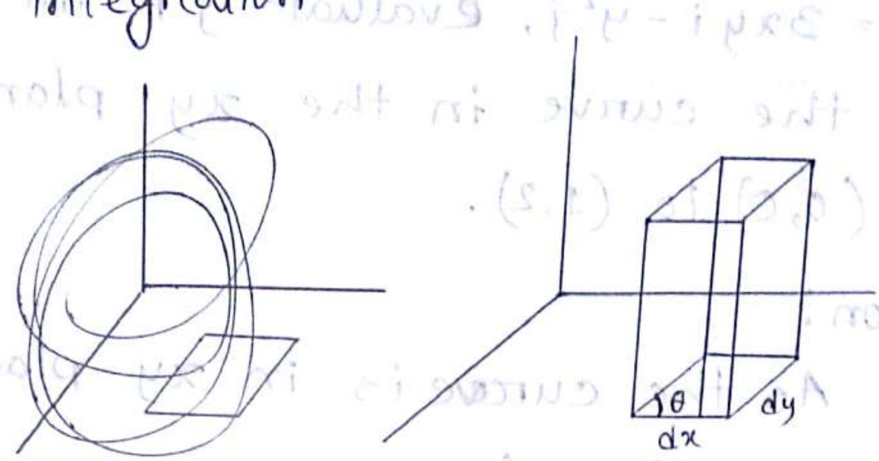
$$= \int_0^1 (6x^3)\,dx - \int_0^2 (y^2)\,dy$$

$$= \left[\frac{3x^4}{2}\right]_0^1 - \left[\frac{y^3}{3}\right]_0^2$$

$$= \frac{3}{2} - \frac{8}{3}$$

$$= \frac{9-16}{6} = -\frac{7}{6}$$

→ surface integration



surface integration $= \iint \vec{F} \cdot d\vec{s}$

$= \iint \vec{F} \cdot \hat{n} \cdot ds$

From figure,

projection $= ds \cos\theta$

$\Rightarrow dxdy = ds\,|\hat{n} \cdot \hat{k}|$

$\Rightarrow ds = \dfrac{dxdy}{|\hat{n} \cdot \hat{k}|}$

$\hat{n} \cdot \hat{k} = |\hat{n}| \cdot |\hat{k}| \cos\theta$

$\Rightarrow \cos\theta = \dfrac{\hat{n} \cdot \hat{k}}{|\hat{n}| \cdot |\hat{k}|}$

$\Rightarrow \cos\theta = \hat{n} \cdot \hat{k}$

[As $\hat{n}$ & $\hat{k}$ unit vector; $|\hat{n}| = |\hat{k}| = 1$]

$\therefore$ surface integration $= \iint \vec{F} \cdot \hat{n}\, \dfrac{dxdy}{|\hat{n} \cdot \hat{k}|}$

→ 20-03-2018 : EE : Tuesday

Em: 19 → Evaluate $\iint \vec{A} \cdot \hat{n}\, ds$ where $\vec{A} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$ and s is $2x + 3y + 6z = 12$ which is located in the first octant.

Solution:

Let, $\phi = 2x + 3y + 6z - 12$

$\vec{\nabla}\phi = \frac{\partial}{\partial x}(2x)\hat{i} + \frac{\partial}{\partial y}(3y)\hat{j} + \frac{\partial}{\partial z}(6z)\hat{k}$

$= 2\hat{i} + 3\hat{j} + 6\hat{k}$

$$|\vec{\nabla}\phi| = \sqrt{4+9+36} = \sqrt{49} = 7$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k})$$

As the surface is located at first octant, so it is in xy plane.

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$\therefore \hat{n}\cdot\hat{k} = \frac{6}{7}$$

$\therefore$ In xy plane the surface equation,

$$2x+3y=12$$

$$\therefore y = \frac{12-2x}{3}$$

If $y=0$ in the surface equation, $2x=12$

$$\therefore x=6$$

$$\therefore \iint \vec{A}\cdot\hat{n}\cdot ds$$

$$= \int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} (18z\hat{i}-12\hat{j}+3y\hat{k})\cdot\frac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k})\, dx\,dy\,\frac{7}{6}$$

$$= \frac{1}{6}\int_0^6\int_0^{\frac{12-2x}{3}} (36z-36+18y)\,dx\,dy$$

$2x+3y+6z=12$

$$\therefore z = \frac{12-2x-3y}{6}.$$

$$= \frac{1}{6}\int_0^6\int_0^{\frac{12-2x}{3}} \left\{36\left(\frac{12-2x-3y}{6}\right)-36+18y\right\}dx\,dy$$

$$= \frac{1}{6}\int_0^6\int_0^{\frac{12-2x}{3}} (72-12x-18y-36+18y)\,dx\,dy$$

$$= \frac{1}{6}\int_0^6\int_0^{\frac{12-2x}{3}} (36-12x)\,dy\,dx$$

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$$= \frac{1}{6}\int_0^6 \left[36y - 12xy\right]_0^{\frac{12-2x}{3}} dx$$

$$= \frac{1}{6}\int_0^6 (144 - 24x - 48x + 8x^2)\, dx$$

$$= \frac{1}{6}\int_0^6 (144 - 72x + 8x^2)\, dx$$

$$= \frac{1}{6}\left[144x - 36x^2 + \frac{8x^3}{3}\right]_0^6$$

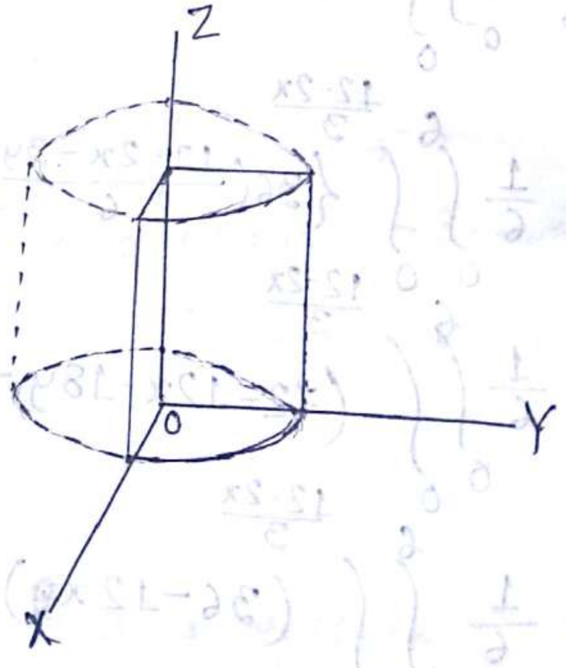
$$= \frac{1}{6}\left[6(144 - 216 + 96)\right]$$

$$= 24$$

Em: 20

→ Evaluate : $\iint \vec{A}\cdot\hat{n}\, ds$ where $\vec{A} = z\hat{i} + x\hat{j} - 3y^2 z\hat{k}$ and s is the surface of the cylineder $x^2+y^2=16$ include first octant between $z=0$ $z=5$

Solution:



Let, $\phi = x^2+y^2-16$

$$\nabla\phi = \frac{\partial x^2}{\partial x}\hat{i} + \frac{\partial y^2}{\partial y}\hat{j}$$

$$= 2x\hat{i} + 2y\hat{j}$$

$$|\nabla\phi| = \sqrt{4x^2+4y^2}$$

$$= 2\sqrt{x^2+y^2}$$

$$= 2\sqrt{16} = 8$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\hat{i}+2y\hat{j}}{8} = \frac{x\hat{i}+y\hat{j}}{4}$$

As surface, s project on xz, not xy plane.

so,

$$ds = \frac{dx\,dz}{|\hat{n}\cdot\hat{j}|}$$

In the xz plane surface equation

$$x^2 = 16$$

$$\therefore x = 4$$

$$\therefore \hat{n}\cdot\hat{j} = \frac{y}{4}$$

$$\therefore \iint \vec{A}\cdot\hat{n}\,ds$$

$$= \int_{x=0}^{4}\int_{z=0}^{5} (z\hat{i}+x\hat{j}-3y^2z\hat{k})\,\frac{1}{4}(x\hat{i}+y\hat{j})\,\frac{4}{y}\,dx\,dz$$

$$= \int_0^4\int_0^5 \left(\frac{xz+xy}{y}\right) dx\,dz$$

$$= \int_0^4\int_0^5 \left(\frac{xz}{y}+x\right) dx\,dz$$

$$= \int_0^5\int_0^4 \left(\frac{xz}{\sqrt{16-x^2}}+x\right) dx\,dz \quad \Bigg| \quad x^2+y^2=16 \;\therefore y=\sqrt{16-x^2}$$

$$\cancel{= \int_0 \left[xz\right]_0^4} = \int_0^5 \left[xz+\frac{x^2}{2}\right]_0^4 dz$$

$$= \int_0^5 (4z+8)\,dz = \cancel{\left[2z+8\right]_0^5}\; \left[2z^2+8z\right]_0^5$$

$$= 2\times5^2+8\times5 = 90$$

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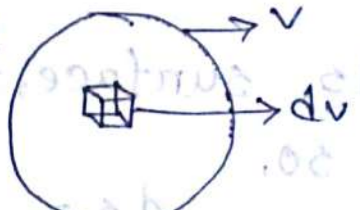
21-03-2018 : 7A : Wednesday

→ Volume integral :

Consider a closed surface in space enclosing a volume V.



Then,

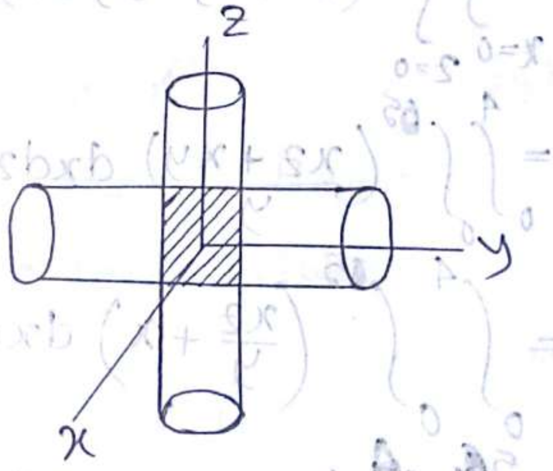
$$\iiint dv = \iiint dx\,dy\,dz$$

$\boxed{Em627}$

→ Find the volume of the region common to to the intersecting cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$.

Solution:

$$\iiint dv = \iiint dx\,dy\,dz$$



$x^2+y^2=a^2$

$\Rightarrow x^2 = a^2-y^2$

$\Rightarrow x^2 = a^2 \quad [\because y=0]$

$\Rightarrow x = \pm a$

$x = [0, a]$

$x^2+y^2=a^2$

$\Rightarrow y^2 = a^2-x^2$

$\Rightarrow y = \pm\sqrt{a^2-x^2}$

$\therefore y = [0, \sqrt{a^2-x^2}]$

$x^2+z^2=a^2$

$\Rightarrow z^2 = a^2-x^2$

$\Rightarrow z = \pm\sqrt{a^2-x^2}$

$\therefore z = [0, \sqrt{a^2-x^2}]$

In ~~x, y, z~~ the limit of x, y, z; only positive value ~~was~~ is taken because here the volume ~~the~~ of the first quadrant is being determined.

$$\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\int_{z=0}^{\sqrt{a^2-x^2}} dx\,dy\,dz$$

$$= \int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} [z]_0^{\sqrt{a^2-x^2}}\, dy\,dx$$

$$= \int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2}\, dy\,dx$$

$$= \int_{x=0}^{a} \sqrt{a^2-x^2}\,[y]_0^{\sqrt{a^2-x^2}}\, dx$$

$$= \int_0^a (a^2-x^2)\,dx$$

$$= \left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$= a^3 - \frac{a^3}{3}$$

$$= \frac{2a^3}{3}$$

$\therefore$ As every quadrant has two side, upper and lower, so the value of the quadrant need to be multiply by 8 to get the full volume.

$\therefore$ Volume $= 8 \times \frac{2a^3}{3}$

$= \frac{16a^3}{3}$

Em 126

→ Let, $\vec{F} = 2xz\hat{i} - x\hat{j} + y^2\hat{k}$. Evaluate $\iiint \vec{F}\,dv$ where V is the region bounded by the surfaces $x=0,\ y=0,\ y=6,\ z=x^2,\ z=4$.

Solution:

$$z = x^2$$

$$\Rightarrow x^2 = 4$$

$$\therefore x = 2$$

$$\therefore \iiint \vec{F}\,dv$$

$$= \int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} (2xz\hat{i} - x\hat{j} + y^2\hat{k})\,dx\,dy\,dz$$

$$= \int_{x=0}^{2}\int_{y=0}^{6} \left[xz^2\hat{i} - xz\hat{j} + y^2z\hat{k}\right]_{x^2}^{4} dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{6} \left[\left(16x\hat{i} - 4x\hat{j} + 4y^2\hat{k} - x^5\hat{i} + x^3\hat{j} - x^2y^2\hat{k}\right)\right] dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{6} \left[(16x - x^5)\hat{i} + (x^3 - 4x)\hat{j} + (4y^2 - x^2y^2)\hat{k}\right] dy\,dx$$

$$= \int_{x=0}^{2} \left[(16x - x^5) y\hat{i} + (x^3 - 4x) y\hat{j} + \left(\frac{4y^3}{3} - \frac{x^2y^3}{3}\right)\hat{k}\right]_0^6 dx$$

$$= \int_0^2 \left[(96x - 6x^5)\hat{i} + (6x^3 - 24x)\hat{j} + (288 - 72x^2)\hat{k}\right] dx$$

$$= \left[(48x^2 - x^6)\hat{i} + \left(\frac{3x^4}{2} - 12x^2\right)\hat{j} + (288x - 24x^3)\hat{k}\right]_0^2$$

$$= (192 - 64)\hat{i} + (24 - 48)\hat{j} + (576 - 192)\hat{k}$$

$$= 128\hat{i} - 24\hat{j} + 384\hat{k}$$

$\boxed{\text{Ex: 70}}$

$\Rightarrow$ If $\vec{F} = (2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$, evaluate $\iiint \vec{\nabla} \cdot \vec{F} \cdot dV$ where V is the closed region bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $2x + 2y + z = 4$.

Solution:

$$2x + 2y + z = 4$$

$$\therefore z = 4 - 2x - 2y$$

If, $z = 0$, the equation of the plane,

$$2x + 2y = 4$$

$$\Rightarrow y = 2 - x$$

If, $y = z = 0$, the equation of the plane,

$$2x = 4$$

$$\therefore x = 2$$

$$\therefore \iiint \vec{\nabla} \cdot \vec{F} \cdot dv$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} \left[\frac{\partial}{\partial x}(2x^2-3z) - \frac{\partial}{\partial y}(2xy) - \frac{\partial}{\partial z}(4x)\right] dx\,dy\,dz$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} (4x - 2x)\, dz\,dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x} \left[2xz\right]_0^{4-2x-2y} dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x} (8x - 4x^2 - 4xy)\, dy\,dx$$

$$= \int_{x=0}^{2} \left[8xy - 4x^2y - 2xy^2\right]_0^{2-x} dx$$

$$= \int_0^2 (16x - \cancel{16}8x^2 - 8x^2 + 4x^3 - 8x + 8x^2 - 2x^3)\, dx$$

$$= \int_0^2 (2x^3 - 8x^2 + 8x)\, dx = \left[\frac{x^4}{2} - \frac{8x^3}{3} + 4x^2\right]_0^2$$

$$= 8 - \frac{64}{3} + 16 = \frac{8}{3}$$

02-04-2018 : 8C : Monday

$\boxed{Ex:32}$ → The acceleration $\vec{a}$ of a particle at any time $t \geq 0$ is given by $\vec{a} = e^{-t}\hat{i} - 6(t+1)\hat{j} + 3\sin t\,\hat{k}$ if the velocity $\vec{v}$ and displacement $\vec{r}$ are zero at $t=0$, find $\vec{v}$ and $\vec{r}$ at any time.

Solution:

$$\vec{a} = e^{-t}\hat{i} - 6(t+1)\hat{j} + 3\sin t\,\hat{k}$$

By integrating,

$$\int \vec{a}\,dt = \int e^{-t}\hat{i}\,dt - \int 6(t+1)\hat{j}\,dt + \int 3\sin t\,\hat{k}\,dt$$

$$\Rightarrow \vec{v} = -e^{-t}\hat{i} - (3t^2+6t)\hat{j} - 3\cos t\,\hat{k} + c_1$$

$$\Rightarrow 0 = -\hat{i} - 3\hat{k} + c_1$$

$$\therefore c_1 = \hat{i} + 3\hat{k}$$

$$\therefore \vec{v} = -e^{-t}\hat{i} - (3t^2+6t)\hat{j} - 3\cos t\,\hat{k} + \hat{i} + 3\hat{k}$$

$$\Rightarrow \vec{v} = (1-e^{-t})\hat{i} - (3t^2+6t)\hat{j} + (3-3\cos t)\hat{k}$$

By integrating,

$$\int \vec{v}\,dt = \int (1-e^{-t})\hat{i}\,dt - \int (3t^2+6t)\hat{j}\,dt + \int (3-3\cos t)\hat{k}\,dt$$

$$\Rightarrow \vec{r} = (t+e^{-t})\hat{i} - (t^3+3t^2)\hat{j} + (3t-3\sin t)\hat{k} + c_2$$

$$\Rightarrow 0 = \hat{i} + c_2 \quad \therefore c_2 = -\hat{i}$$

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$$\therefore \vec{r} = (t + e^{-t})\hat{i} - (t^3 + 3t^2)\hat{j} + (3t - 3\sin t)\hat{k} - \hat{i}$$

$$= (t - 1 + e^{-t})\hat{i} - (t^3 + 3t^2)\hat{j} + (3t - 3\sin t)\hat{k}$$

Ex: 34

→ Evaluate: $\int_2^3 \vec{A} \cdot \frac{d\vec{A}}{dt}\, dt$ if $\vec{A}(2) = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{A}(3) = 4\hat{i} - 2\hat{j} + 3\hat{k}$.

Solution:

$$\frac{d}{dt}(\vec{A} \cdot \vec{A}) = \vec{A} \cdot \frac{d\vec{A}}{dt} + \frac{d\vec{A}}{dt} \cdot \vec{A}$$

$$\Rightarrow \frac{d}{dt}(\vec{A} \cdot \vec{A}) = 2\vec{A} \cdot \frac{d\vec{A}}{dt}$$

$$\therefore \vec{A} \cdot \frac{d\vec{A}}{dt} = \frac{1}{2}\frac{d}{dt}(\vec{A} \cdot \vec{A})$$

$$\therefore \int_2^3 \vec{A} \cdot \frac{d\vec{A}}{dt}\, dt = \int_2^3 \frac{1}{2}\frac{d}{dt}(\vec{A} \cdot \vec{A})\, dt \quad \text{[By integrating]}$$

$$= \frac{1}{2}\int_2^3 d(\vec{A} \cdot \vec{A})$$

$$= \frac{1}{2}\left[\vec{A} \cdot \vec{A}\right]_2^3$$

$$= \frac{1}{2}\left[\{\vec{A}(3) \cdot \vec{A}(3)\} - \{\vec{A}(2) \cdot \vec{A}(2)\}\right]$$

$$= \frac{1}{2}\left[(16 + 4 + 9) - (4 + 1 + 2)\right]$$

$$= \frac{1}{2}[29 - 9] = 10$$

Ex: 37

→ If $\vec{A} = (2y+3)\,\hat{i} + xz\,\hat{j} + (yz-x)\,\hat{k}$, evaluate $\int_C \vec{A}\cdot d\vec{r}$ along the following paths C:

a) $x = 2t^2,\ y = t,\ z = t^3$ from $t=0$ to $t=1$.

b) the straight line from $(0,0,0)$ to $(0,0,1)$ then to $(0,1,1)$ and then to $(2,1,1)$

c) the straight line joining $(0,0,0)$ and $(2,1,1)$.

Solution:

a) $$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$\int_C \vec{A}\cdot d\vec{r}$$

$$= \int_C \left[(2y+3)\,dx + xz\,dy + (yz-x)\,dz\right]$$

$$= \int_0^1 \left[(2t+3)\,4t\,dt + (2t^5)\,dt + (t^4-2t^2)\,3t^2\,dt\right]$$

$$= \int_0^1 \left(8t^2 + 12t + 2t^5 + 3t^6 - 6t^4\right)dt$$

$$= \left[\frac{8t^3}{3} + 6t^2 + \frac{t^6}{3} + \frac{3t^7}{7} - \frac{6t^5}{5}\right]_0^1$$

$$= \frac{8}{3} + 6 + \frac{1}{3} + \frac{3}{7} - \frac{6}{5}$$

$$= \frac{280+630+35+45-126}{105} = \frac{864}{105} = \frac{288}{35}$$

b) The straight lines from $(0,0,0)$ to $(0,0,1)$
$x=0, y=0, dx=0, dy=0$ only z varies from 0 to 1

$$\int_C \vec{A}\cdot d\vec{r} = \int_{(0,0,0)}^{(0,0,1)} \left[(2y+3)dx + (xz)dy + (yz-x)dz\right]$$

$$= \int_0^1 0\, dz \quad [\because dx = dy = 0]$$

$$= 0$$

The straight line from $(0,0,1)$ to $(0,01,1)$
$x=0, z=1, dx=0, dz=0$ only y varies from 0 to 1.

$$\int_C \vec{A}\cdot d\vec{r} = \int_{(0,0,1)}^{(0,1,1)} \left[(2y+3)dx + (xz)dy + (yz-x)dz\right]$$

$$= \int_0^1 0\, dy \quad [\because dx = dz = 0]$$

$$= 0$$

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The straight line from $(0,1,1)$ to $(2,1,1)$
$y=1, z=1, dy=0, dz=0$ only x varies from 0 to 2

$$\int_C \vec{A}\cdot d\vec{r} = \int_{(0,1,1)}^{(2,1,1)} \left[(2y+3)dx + (xz)dy + (yz-x)dz\right]$$

$$= \int_0^2 (2+3)dx \quad [\because y=1 \;\&\; dy=dz=0]$$

$$= [5x]_0^2$$

$$= 10$$

$\therefore$ Adding the straight lines $= 0+0+10 = 10$

c) The straight line joining $(0,0,0)$ and $(2,1,1)$ is given in parametric ~~fro~~ form by $x=2t$, $y=t$, $z=t$. Then,

$$\int_C \vec{A}\cdot d\vec{r} = \int_{(0,0,0)}^{(2,1,1)} \left[(2y+3)dx + (xz)dy + (yz-x)dz\right]$$

$$= \int_0^1 \left[(2t+3)2dt + (2t^2)dt + (t^2-2t)dt\right]$$

$$= \int_0^1 (4t+6+2t^2+t^2-2t)\,dt$$

$$= \int_0^1 (3t^2+2t+6)\,dt$$

$$= \left[t^3+t^2+6t\right]_0^1$$

$$= 1+1+6 = 8$$

$\boxed{\text{Ex: 58}}$

$\rightarrow$ Evaluate: $\iint_S \vec{A}\cdot\hat{n}\,ds$ for each of the following cases.

a) $\vec{A} = xy\hat{i} + 2x\hat{j} - z\hat{k}$ and s is the surface of the plane $2x+y=6$ in the first octant cut of by the plane $z=4$.

b) $\vec{A} = (x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ and S is the surface of the plane $2x+y+2z=6$ in the first octant.

Solution:

a) Let, $\phi = 2x+y-6$

$$\vec{\nabla}\phi = \frac{\partial}{\partial x}(2x)\hat{i} + \frac{\partial}{\partial y}(y)\hat{j}$$

$$= 2\hat{i} + \hat{j}$$

$$|\vec{\nabla}\phi| = \sqrt{4+1} = \sqrt{5}$$

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$\therefore \hat{n} = \dfrac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \dfrac{1}{\sqrt{5}}(2\hat{i}+\hat{j})$

As the surface is on ~~the~~ a plane xy in the first octant and cut off by the plane $z=4$, so the surface is on the xz plane.

$ds = \dfrac{dx\,dz}{|\hat{n}\cdot\hat{j}|}$

$\hat{n}\cdot\hat{j} = \dfrac{1}{\sqrt{5}}$

In the xz plane, the surface equation is,

$2x = 6$

$\therefore x = 3$

$\therefore \iint_S \vec{A}\cdot\hat{n}\cdot ds$

$= \int_{x=0}^{3}\int_{z=0}^{4} (y\hat{i}+2x\hat{j}-z\hat{k})\dfrac{1}{\sqrt{5}}(2\hat{i}+\hat{j})\dfrac{dx\,dz}{\frac{1}{\sqrt{5}}}$

$= \int_{x=0}^{3}\int_{z=0}^{4} (2y+2x)\,dx\,dz$

$= \int_{x=0}^{3}\left[2(6-2x)z+2xz\right]_0^4 dx \quad [\because 2x+y=6 \Rightarrow y=6-2x]$

$= \int_0^3 \left[12z-2xz\right]_0^4 dx = \int_0^3 (48-8x)\,dx$

$$= \left[48x - 4x^2\right]_0^3$$

$$= 144 - 36 = 108$$

b)

Let $\phi = 2x + y + 2z - 6$

$$\vec{\nabla}\phi = \frac{\partial}{\partial x}(2x)\hat{i} + \frac{\partial}{\partial y}(y)\hat{j} + \frac{\partial}{\partial z}(2z)\hat{k}$$

$$= 2\hat{i} + \hat{j} + 2\hat{k}$$

$$|\vec{\nabla}\phi| = \sqrt{4+1+4} = \sqrt{9} = 3$$

$$\therefore \hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{1}{3}(2\hat{i} + \hat{j} + 2\hat{k})$$

As the surface is in the first octant of xyz plane, so it is in xy plane.

$$ds = \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|}$$

$$2x + y + 2z = 6$$

$$\therefore z = \frac{6 - 2y - 2x}{2}$$

$$\hat{n} \cdot \hat{k} = \frac{2}{3}$$

In the xy plane the surface equation

$2x + y = 6$ when, $y = 0$

$\Rightarrow y = 6 - 2x$

$2x = 6$

$\therefore x = 3$

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$$\iint_S \vec{A}\cdot\hat{n}\, ds$$

$$= \int_{x=0}^{3}\int_{y=0}^{6-2x} \left[(x+y^2)\hat{i} - (2x)\hat{j} + (2yz)\hat{k}\right] \frac{1}{3}(2\hat{i}+\hat{j}+2\hat{k}) \frac{dx\,dy}{\frac{2}{3}}$$

$$= \frac{1}{2}\int_{x=0}^{3}\int_{y=0}^{6-2x} \left(2x+2y^2-2x+4yz\right) dx\,dy$$

$$= \int_{x=0}^{3}\int_{y=0}^{6-2x} \left\{ y^2 + 2y\left(\frac{6-y-2x}{2}\right)\right\} dy\,dx$$

$$= \int_{x=0}^{3}\int_{y=0}^{6-2x} \left(y^2+6y-y^2-2xy\right) dy\,dx$$

$$= \int_{x=0}^{3}\int_{y=0}^{6-2x} \left(6y-2xy\right) dy\,dx$$

$$= \int_{x=0}^{3} \left[3y^2 - xy^2\right]_0^{6-2x} dx$$

$$= \int_{x=0}^{3} \left[(3-x)(6-2x)^2\right] dx$$

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$$= \int_0^3 \left[(3-x)(36-24x+4x^2)\right] dx$$

$$= \int_0^3 (108 - 72x + 12x^2 - 36x + 24x^2 - 4x^3)\,dx$$

$$-4x^3$$

$$= \int_0^3 (108 - 108x + 36x^2)\,dx$$

$$= \left[108x - 54x^2 + 12x^3 - x^4\right]_0^3$$

$$= 324 - 486 + 324 - 81$$

$$= 81.$$

$\boxed{\text{Ex 6-38}}$: If $\vec{F} = (5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j}$, evaluate $\int \vec{F}\cdot d\vec{r}$ along the curve C in the xy plane, $y = x^3$ from the point $(1,1)$ to $(2,8)$.

Solution:

As the curve is in xy plane,

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$$

$$y = x^3$$

$$\Rightarrow dy = 3x^2\,dx$$

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$$\int_C \vec{F}\cdot d\vec{r}$$

$$= \int_{(1,1)}^{(2,8)} \left[(5xy-6x^2)\,dx + (2y-4x)\,dy\right]$$

$$= \int_1^2 (5x^4-6x^2)\,dx + \cancel{\int_1^8 (2y-4\sqrt[3]{y})\,dy} \int_1^2 (2x^3-4x)3x^2dx$$

$$= \left[x^5-2x^3\right]_1^2 + \int_1^2 (6x^5-12x^3)\,dx$$

$$= (32-16-1+2) + \left[x^6-3x^4\right]_1^2$$

$$= 17+64-48-1+3$$

$$= 35$$

$\boxed{\text{Ex :- 39}}$: If $\vec{F} = (2x+y)\hat{i} + (3y-x)\hat{j}$, evaluate $\int_C \vec{F}\cdot d\vec{r}$ where C is the curve in the xy plane consisting of the straight lines from $(0,0)$ to $(2,0)$ and then to $(3,2)$.

Solution: As the curve is in xy plane,

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$$

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The straight line from $(0,0)$ to $(2,0)$;
$y=0$, $dy=0$ only x varies from 0 to 2

$$\int_C \vec{F}\cdot d\vec{r}$$

$$= \int_{(0,0)}^{(2,0)} \left[(2x+y)\,dx + (3y-x)\,dy\right]$$

$$\cancel{= \int_0^2 (3y-0)\,dy \quad [\because dx=0]}$$

$$= \int_0^2 (2x-0)\,dx \quad [\because dy=0]$$

$$\cancel{= \left[\frac{3y^2}{2}\right]_0^2}$$

$$= \left[x^2\right]_0^2 = 4$$

$$\cancel{= 6}$$

The straight line from ~~(0,2)~~ $(2,0)$ to ~~(2,3)~~ $(3,2)$ is given in parametric form by $x = t$, $y = 2t$. Then,

$$\int_C \vec{F}\cdot d\vec{r}$$

$$= \int_{(2,0)}^{(3,2)} \left[(2x+y)\,dx + (3y-x)\,dy\right]$$

$$= \int_0^1 \left[(2t+2t)\,dt + (6t-t)\,2dt\right]$$

$= \int_0^1 (4t + 10t)\, dt$

$= \left[\frac{14t^2}{2}\right]_0^1$

$= 7$

Add the straight line $= 4+7 = 11$

~~40. f~~

$\boxed{\text{Ex :- 40}}$: Find the work done in moving a particle in the ~~force~~ force field $\vec{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$ along

a) the straight line from $(0,0,0)$ to $(2,1,3)$

b) the space curve $x = 2t^2$, $y = t$, $z = 4t^2 - t$ from $t = 0$ to $t = 1$

c) the curve defined by $x^2 = 4y$, $3x^3 = 8z$ from $x = 0$ to $x = 2$.

Solution:

a) The straight line from $(0,0,0)$ to $(2,1,3)$ is given in parametric form by $x=2t$, $y=t$, $z=3t$. Then, work done

$$\int_C \vec{F}\cdot d\vec{r}$$

$$= \int_{(0,0,0)}^{(2,1,3)} \left[(3x^2)dx + (2xz-y)\,dy + (z)\,dz\right]$$

$$= \int_0^1 \left[(12t^2)\,2dt + (12t^2-t)\,dt + (3t)\,3dt\right]$$

$$= \int_0^1 (24t^2 + 12t^2 - t + 9t)\,dt$$

$$= \int_0^1 (36t^2 + 8t)\,dt$$

$$= \left[12t^3 + 4t^2\right]_0^1$$

$$= 12 + 4 = 16$$

b)

$$x = 2t^2 \qquad y = t \qquad z = 4t^2 - t$$

$$\Rightarrow dx = 4t\,dt \qquad \Rightarrow dy = dt \qquad \Rightarrow dz = (8t-1)\,dt$$

$$\text{Work done} = \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_C \left[(3x^2)\,dx + (2xz - y)\,dy + (z)\,dz \right]$$

$$= \int_0^1 \left[(12t^4)\,4t\,dt + \cancel{8}\left(16t^4 - 4t^3 - t\right) dt + (4t^2 - t) \cdot (8t-1)\,dt \right]$$

$$= \int_0^1 \left(48t^5 + 16t^4 - 4t^3 - t + 32t^3 - 8t^2 - 4t^2 + t \right) dt$$

$$= \int_0^1 \left(48t^5 + 16t^4 + 28t^3 - 12t^2 \right) dt$$

$$= \left[8t^6 + \frac{16t^5}{5} + 7t^4 - 4t^3 \right]_0^1$$

$$= 8 + \frac{16}{5} + 7 - 4$$

$$= \frac{40 + 16 + 35 - 20}{5}$$

$$= \frac{71}{5} = 14.2$$

c)

$x^2 = 4y$ $\qquad$ $3x^3 = 8z$

$\cancel{\Rightarrow y = \frac{2^2}{4}}$ $\qquad$ $\cancel{\Rightarrow z = \frac{3\times 2^3}{8}}$

$\cancel{\therefore y = 1}$ $\qquad$ $\cancel{\therefore z = 3}$

$\Rightarrow 2x\,dx = 4dy$ $\qquad$ $\Rightarrow 9x^2 dx = 8dz$

$\therefore dy = \frac{x\,dx}{2}$ $\qquad$ $\Rightarrow dz = \frac{9x^2 dx}{8}$

work done

$$= \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_C \left[(3x^2)\,dx + (2xz - y)\,dy + (z)\,dz \right]$$

$$= \int_0^2 \left[(3x^2)\,dx + \left(\frac{3x^4}{4} - \frac{x^2}{4} \right) \frac{x\,dx}{2} + \left(\frac{3x^3}{8} \right) \frac{9x^2 dx}{8} \right]$$

$$= \int_0^2 \left(3x^2 + \frac{3x^5 - x^3}{8} + \frac{27x^5}{64} \right) dx$$

$$= \left[x^3 + \frac{x^6 - x^4}{32} + \frac{9x^6}{128} \right]_0^2$$

$$= \left[8 + \frac{128 - 16}{32} + \frac{9\times 64}{128} \right]$$

$$= 8 + \frac{7}{2} + \frac{9}{2} = \cancel{\frac{16 + 14 + 3}{4}} = \frac{16 + 7 + 9}{2}$$

$$= 16$$

42. $\boxed{\text{Ex:- 41}}$: Evaluate $\int \vec{F}\cdot d\vec{r}$ where $\vec{F} = (x-3y)\hat{i} + (y-2x)\hat{j}$ and C is the closed curve in the xy plane, $x = 2\cos t$, $y = 3\sin t$ from $t=0$ to $t = 2\pi$.

Solution:

$x = 2\cos t \qquad y = 3\sin t$

$\Rightarrow dx = -2\sin t\,dt \qquad \Rightarrow dy = 3\cos t\,dt$

$$\int \vec{F}\cdot d\vec{r}$$

$$= \int_C \left[(x-3y)\,dx + (y-2x)\,dy\right]$$

$$= \int_0^{2\pi} \left[(2\cos t - 9\sin t)\cdot(-2\sin t\,dt) + (3\sin t - 4\cos t)(3\cos t\,dt)\right]$$

$$= \int_0^{2\pi} (18\sin^2 t - 4\sin t\cos t + 9\sin t\cos t - 12\cos^2 t)\,dt$$

$$= \int_0^{2\pi} (18\sin^2 t + 5\sin t\cos t - 12\cos^2 t)\,dt$$

$$= \int_0^{2\pi} \left\{\frac{5}{2}(2\sin t\cos t) - 12(\cos^2 t - \sin^2 t) + 6\sin^2 t\right\} dt$$

$$= \int_0^{2\pi} \left\{ \frac{5}{2}\sin 2t - 12\cos 2t + 3\cdot(1-\cos 2t) \right\} dt$$

$$= \int_0^{2\pi} \left(\frac{5}{2}\sin 2t - 15\cos 2t + 3 \right) dt$$

$$= \left[-\frac{5}{4}\cancel{\sin}\cos 2t - \frac{15}{2}\,\cancel{\cos 2t}\,\sin 2t + 3t \right]_0^{2\pi}$$

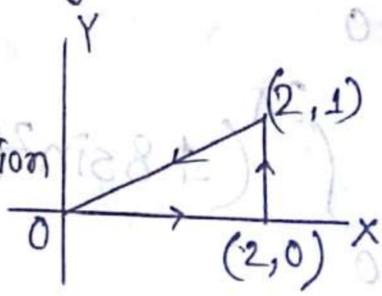
$$= \left[-\frac{5}{4} - 0 + 6\pi + \frac{5}{4} + 0 - 0 \right]$$

$$= 6\pi$$

$\boxed{\text{Ex:-43}}$:- If $\vec{F} = (2x+y^2)\hat{i} + (3y-4x)\hat{j}$, evaluate $\oint_C \vec{F}\cdot d\vec{r}$ around ~~the~~ triangle C of the figure

a) in the indicated direction

b) opposite to the indicated direction



solution:

a) The straight line from $(0,0)$ to $(2,0)$

$y=0$, $dy=0$, only x varies from 0 to 2.

$$\int \vec{F}\cdot d\vec{r} = \int_{(0,0)}^{(2,0)} \left[(2x+y^2)dx + (3y-4x)dy \right]$$

$$= \int_0^2 2x\,dx + 0 = \left[x^2\right]_0^2 = 4$$

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The straight line from $(2,0)$ to $(2,1)$

$x=2$, $dx=0$, only y varies from 0 to 1.

$$\int \vec{F}\cdot d\vec{r} = \int_{(2,0)}^{(2,1)} \left[(2x+y^2)dx + (3y-4x)dy\right]$$

$$= 0 + \int_0^1 (3y - 4\times 2)\,dy$$

$$= \left[\frac{3y^2}{2} - 8y\right]_0^1$$

$$= \frac{3}{2} - 8 = -\frac{13}{2}$$

The straight line from $(2,1)$ to $(0,0)$ is given in parametric form by $x = 2t$, $y = t$.

$$\int \vec{F}\cdot d\vec{r} = \int_{(2,1)}^{(0,0)} \left[(2x+y^2)dx + (3y-4x)dy\right]$$

$$= \int_1^0 \left[(\cancel{8}4t + t^2)2dt + (3t - 8t)\,dt\right]$$

$$= \int_1^0 (2t^2 + 8t - 5t)\,dt$$

$$= \int_1^0 (2t^2 + 3t)\,dt = \left[\frac{2t^3}{3} + \frac{3t^2}{2}\right]_1^0$$

$$= -\frac{2}{3} - \frac{3}{2}$$

$$= \frac{-4-9}{6} = -\frac{13}{6}$$

$\therefore$ Adding the ~~triang~~ lines

$= 4 - \frac{13}{2} - \frac{13}{6}$

$= \frac{24 - 39 - 13}{6} = -\frac{28}{6} = -\frac{14}{3}$

b)

The straight line from $(0,0)$ to $(2,1)$ is given in parametric form by $x = 2t$, $y = t$.

$$\int \vec{F} \cdot d\vec{r} = \int_{(0,0)}^{(2,1)} \left[(2x + y^2)\, dx + (3y - 4x)\, dy \right]$$

$$= \int_0^1 \left[(4t + t^2)\, 2dt + (3t - 8t)\, dt \right]$$

$$= \int_0^1 (8t + 2t^2 - 5t)\, dt$$

$$= \int_0^1 (2t^2 + 3t)\, dt$$

$$= \left[\frac{2t^3}{3} + \frac{3t^2}{2} \right]_0^1$$

$$= \frac{2}{3} + \frac{3}{2} = \frac{4+9}{6} = \frac{13}{6}$$

The straight line from $(2,1)$ to $(2,0)$

$x=2$, $dx=0$, only y varies from 1 to 0

$$\int \vec{F}\cdot d\vec{r} = \int_{(2,1)}^{(2,0)} \left[(2x+y^2)\,dx + (3y-4x)\,dy\right]$$

$$= 0 + \int_1^0 [3y-(4\times 2)]\,dy$$

$$= \int_1^0 (3y-8)\,dy = \left[\frac{3y^2}{2} - 8y\right]_1^0$$

$$= -\frac{3}{2} + 8 = \frac{13}{2}$$

The straight line from $(2,0)$ to $(0,0)$

$y=0$, $dy=0$, only x varies from 2 to 0

$$\int \vec{F}\cdot d\vec{r} = \int_{(2,0)}^{(0,0)} \left[(2x+y^2)\,dx + (3y-4x)\,dy\right]$$

$$= \int_2^0 2x\,dx$$

$$= \left[x^2\right]_2^0 = -4$$

Adding the lines $= \frac{13}{6} + \frac{13}{2} - 4$

$$= \frac{13+39-24}{6} = \frac{28}{6}$$

$$= \frac{14}{6}$$

$\boxed{\text{Ex-58}}$: ~~Evaluate~~ $\cancel{\iint_S \vec{A}\cdot\hat{n}\,ds}$ ~~for each~~ of ~~the following cases:~~

~~a) A~~

$\boxed{\text{Ex-59}}$: If $\vec{F} = 2y\hat{i} - z\hat{j} + x^2\hat{k}$ and s is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y=4$ and $z=6$, evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$

Solution:

Let,

$$\phi = y^2 - 8x$$

$$= -8x + y^2$$

$$\vec{\nabla}\phi = \frac{\partial}{\partial x}(-8x)\hat{i} + \frac{\partial}{\partial y}(y^2)\hat{j} = -8\hat{i} + 2y\hat{j}$$

$$\therefore |\vec{\nabla}\phi| = \sqrt{64+4y^2}$$

$$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{1}{\sqrt{64+4y^2}}\left(-8\hat{i} + 2y\hat{j}\right)$$

As the surface project on yz plane, not xy,

$$ds = \frac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$$

$$\hat{n}\cdot\hat{i} = -\frac{8}{\sqrt{64+4y^2}}$$

$$\iint \vec{F}\cdot\hat{n}\,ds$$

$$= \int_{y=0}^{4}\int_{z=0}^{6} \left(2y\hat{i} - z\hat{j} + x^2\hat{k}\right)\frac{(-8\hat{i}+2y\hat{j})}{\sqrt{64+4y^2}} \frac{dy\,dz}{-\frac{8}{\sqrt{64+4y^2}}}$$

$$= \int_{z=0}^{6}\int_{y=0}^{4} \left(\frac{-16y - 2yz}{-8}\right) dy\,dz$$

$$= \frac{1}{4}\int_{z=0}^{6}\int_{y=0}^{4} (8y + yz)\,dy\,dz$$

$$= \frac{1}{4}\int_{z=0}^{6}\left[4y^2 + \frac{y^2 z}{2}\right]_0^4 dz = \frac{1}{4}\int_0^6 (64 + 8z)\,dz$$

$$= \int_0^6 (16 + 2z)\,dz$$

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$$\phi = \left[16z + z^2\right]_0^6$$

$$= 96 + 36$$

$$= 132$$

$\boxed{\text{Ex :- 60}}$: Evaluate $\iint_S \vec{A}\cdot\hat{n}\, ds$ over the entire surface S of the region bounded by the cylinder $x^2+z^2=9$, $x=0$, $y=0$, $z=0$ and $y=8$, if $\vec{A} = 6z\,\hat{i} + (2x+y)\,\hat{j} - x\hat{k}$

Solution: Let,

$$\phi = x^2+z^2-9$$

$$\vec{\nabla}\phi = \frac{\partial}{\partial x} x^2\,\hat{i} + \frac{\partial}{\partial z} z^2\hat{k}$$

$$= 2x\hat{i} + 2z\,\hat{k}$$

$$|\vec{\nabla}\phi| = \sqrt{4x^2+4z^2} = 2\sqrt{x^2+z^2} = 2\sqrt{9} = 6$$

$$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{2(x\hat{i}+z\hat{k})}{6} = \frac{x\hat{i}+z\hat{k}}{3}$$

As the surface project on xy plane

$$ds = \frac{dx\,dy}{\hat{n}\cdot\hat{k}}$$

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$\hat{n}\cdot\hat{k} = \frac{2}{3}$

$\iint_S \vec{A}\cdot\hat{n}\, ds$

$= \int_{x=0}^{3}\int_{y=0}^{8} (6xz - xz)\frac{dx\,dy}{z}$

$= \int_{x=0}^{3}\int_{y=0}^{8} 5x\, dy\, dx$

$= \int_{x=0}^{3} \left[5xy\right]_0^8 dx$

$= \int_0^3 40x\, dx$

$= \left[40\frac{x^2}{2}\right]_0^3$

$= 20\times 3^2$

$= 180$

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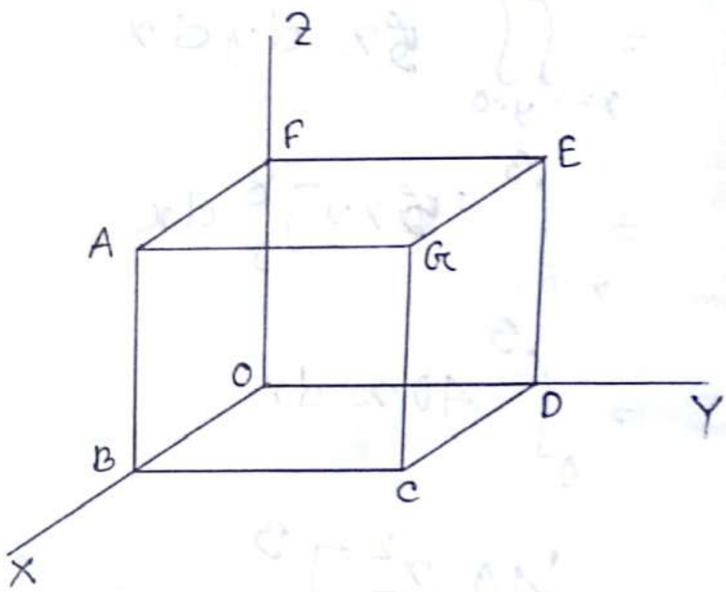
Ex:-61: Evaluate $\iint_S \vec{r}\cdot\hat{n}\,ds$ over:

a) the surface S of the unit cube bounded by the coordinate planes and the planes $x=1,\ y=1,\ z=1$.

b) the surface of a sphere of radius a with center at $(0,0,0)$

Solution:



a) surface ABCG:

$\hat{n} = \hat{i},\ x = 1,$

$\iint \vec{r}\cdot\hat{n}\,ds$

$= \int_{y=0}^{1}\int_{z=0}^{1} (x\hat{i}+y\hat{j}+z\hat{k})\cdot\hat{i}\ dy\,dz$

$= \int_0^1\int_0^1 x\,dy\,dz$

$= \int_0^1\int_0^1 dy\,dz = \int_0^1 [y]_0^1\,dz = \int_0^1 dz = [z]_0^1 = 1$

Surface ODEF:

$\hat{n} = -\hat{i},\ x = 0$

$\iint \vec{r}\cdot\vec{n}\,ds$

$= \int_0^1\int_0^1 -x\,dy\,dz = 0$

surface CDEG:

$\hat{n} = \hat{j}$, $y = 1$

$\iint \vec{r} \cdot \hat{n} \, ds$

$= \int_0^1 \int_0^1 y \, dx \, dz$

$= \int_0^1 \int_0^1 dx \, dz = 1$

surface ABOF:

$\hat{n} = -\hat{j}$, $y = 0$

$\iint \vec{r} \cdot \hat{n} \, ds = \int_0^1 \int_0^1 -y \, dx \, dz = 0$

surface AFEG:

$\hat{n} = \hat{k}$, $z = 1$

$\iint \vec{r} \cdot \hat{n} \, ds = \int_0^1 \int_0^1 z \, dx \, dy = \int_0^1 \int_0^1 dx \, dy = 1$

surface OBCD:

$\hat{n} = -\hat{k}$, $z = 0$

$\iint \vec{r} \cdot \hat{n} \, ds = \int_0^1 \int_0^1 -z \, dx \, dy = 0$

Adding surface $= 1 + 0 + 1 + 0 + 1 + 0$

$= 3$

b)

$\boxed{\text{Ex:- 62}}$: Evaluate $\iint \vec{A}\cdot\hat{n}\, ds$ over the entire surface of the region above the xy plane ~~bono~~ bounded by the cone $z^2 = x^2+y^2$ and the plane $z=4$, if $\vec{A} = 4xz\,\hat{i} + xyz^2\,\hat{j} + 3z\,\hat{k}$.

Solution:

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Ex :- 69 :- Evaluate $\iiint (2x+y)\, dV$, where V is the closed region bounded by the cylinder $z = 4-x^2$ and the planes $x=0, y=0, y=2$ and $z=0$

Solution:

$$z = 4-x^2$$
$$\Rightarrow 0 = 4-x^2$$
$$\Rightarrow x = 2$$

$$\iiint (2x+y)\, dv$$

$$= \int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{4-x^2} (2x+y)\, dx\, dy\, dz$$

$$= \int_0^2\int_0^2 \Big[2xz + yz \Big]_0^{4-x^2} dx\, dy$$

$$= \int_0^2\int_0^2 \left(8x - 2x^3 + 4y - yx^2\right) dy\, dx$$

$$= \int_0^2 \left[8xy - 2x^3y + 2y^2 - \frac{y^2x^2}{2} \right]_0^2 dx$$

$$= \int_0^2 \left(16x - 4x^3 + 8 - 2x^2\right) dx$$

$$= \left[8x^2 - x^4 + 8x - \frac{2x^3}{3} \right]_0^2$$

$$= 32 - 16 + 16 - \frac{16}{3}$$

$$= \frac{96-16}{3} = \frac{80}{3}$$

Example 6 Chapter 5

$\boxed{\text{Em:-03}}$: Evaluate $\int \vec{A}\times\frac{d^2\vec{A}}{dt}\,dt$

Solution:

$$\frac{d}{dt}\left(\vec{A}\times\frac{d\vec{A}}{dt}\right)$$

$$=\vec{A}\times\frac{d^2\vec{A}}{dt^2}+\frac{d\vec{A}}{dt}\times\frac{d\vec{A}}{dt}$$

$$=\vec{A}\times\frac{d^2\vec{A}}{dt^2}$$

$$\therefore \int\vec{A}\times\frac{d^2\vec{A}}{dt^2}\,dt$$

$$=\int\frac{d}{dt}\left(\vec{A}\times\frac{d\vec{A}}{dt}\right)dt$$

$$=\int d\left(\vec{A}\times\frac{d\vec{A}}{dt}\right)$$

$$=\vec{A}\times\frac{d\vec{A}}{dt}+C$$

$\boxed{\text{Em:-09}}$: Find the work done in moving a particle once around a circle C in the xy plane, if the circle has center at the origin and radius 3 and if the force field is given by

$$\vec{F}=(2x-y+z)\hat{i}+(x+y-z^2)\hat{j}+(3x-2y+4z)\hat{k}$$

Solution:

In the xy plane force field, $\vec{F}$ and movement of the particle, $d\vec{r}$,

$$\vec{F} = (2x-y)\hat{i} + (x+y)\hat{j} + (3x-2y)\hat{k}$$

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$$

$$\therefore \text{work done} = \int \vec{F}\cdot d\vec{r}$$

$$= \int (2x-y)\,dx + (x+y)\,dy$$

The parametric equations of the circle as $x = 3\cos t$, $y = 3\sin t$ where t varies from 0 to 2π.

So the line integral,

$$\int_0^{2\pi} (6\cos t - 3\sin t)(-3\sin t\,dt) + (3\cos t + 3\sin t)(3\cos t\,dt)$$

$$= \int_0^{2\pi} (-18\sin t\cos t + 9\sin^2 t + 9\cos^2 t + 9\sin t\cos t)\,dt$$

$$= \int_0^{2\pi} (9 - 9\sin t\cos t)\,dt$$

$$= \int_0^{2\pi} \left(9 - \frac{9}{2}\sin 2t\right) dt$$

$$= \left[9t - \frac{9}{4}\cos 2t\right]_0^{2\pi} = 18\pi - 0 - \frac{9}{4} + \frac{9}{4}$$

$$= 18\pi$$

Em :- 12 :

a) Show that $\vec{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ is a conservative force field.

b) Find the scalar potential.

c) Find the work done in moving an object in this field from $(1,-2,1)$ to $(3,1,4)$

Solution:

a) If $\vec{F}$ is a conservative force field, curl $\vec{F}$, $\vec{\nabla} \times \vec{F} = 0$.

$$L.H.S. = \vec{\nabla} \times \vec{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$$= \hat{i}(0-0) + \hat{j}(3z^2 - 3z^2) + \hat{k}(2x - 2x)$$

$$= 0$$

$$= R.H.S.$$

$\therefore$ L.H.S. = R.H.S. (showed)

b) we know

$\vec{F}\cdot d\vec{r} = \vec{\nabla}\phi \cdot d\vec{r}$

$= \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$

$= d\phi$

$\therefore d\phi = \vec{F}\cdot d\vec{r}$

$= (2xy+z^3)dx + x^2 dy + 3xz^2 dz$

$= (2xy\,dx + x^2 dy) + (z^3 dx + 3xz^2 dz)$

$\therefore d\phi = d(x^2y) + d(xz^3)$

By integrating,

$\phi = x^2y + xz^3 + c$

where, c is an integrating constant

c) work done

$= \int_{(1,-2,1)}^{(3,1,4)} \vec{F}\cdot d\vec{r}$

~~$= \int [(2xy+z^3)dx$~~

$= \int_{(1,-2,1)}^{(3,1,4)} \left[(2xy+z^3)dx + x^2 dy + 3xz^2 dz\right]$

$$= \int_{(1,-2,1)}^{(3,1,4)} \left[(2xy\,dx + x^2 dy) + (z^3 dx + 3xz^2 dz) \right]$$

$$= \int_{(1,-2,1)}^{(3,1,4)} \left[d(x^2 y) + d(xz^3) \right]$$

$$= \left[x^2 y + xz^3 \right]_{(1,-2,1)}^{(3,1,4)}$$

$$= 9 + 192 + 2 - 1$$

$$= 202$$

$\boxed{\text{Ex:-14}}$:

a) show that a necessary and sufficient condition that $F_1 dx + F_2 dy + F_3 dz$ be an exact differential is that $\vec{\nabla} \times \vec{F} = 0$ where $\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$.

b) show that $(y^2 z^3 \cos x - 4x^3 z)\,dx + 2z^3 y \sin x\,dy + (3y^2 z^2 \sin x - x^4)\,dz$ is an exact differential of a function ϕ and find ϕ.

Solution:

a) Let,

$f_1 dx + f_2 dy + f_3 dz$

$= d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$, an exact differential.

Since x, y and z ~~and~~ are independent variables

$f_1 = \frac{\partial \phi}{\partial x}, f_2 = \frac{\partial \phi}{\partial y}, f_3 = \frac{\partial \phi}{\partial z}$

$\therefore \vec{F} = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k} = \vec{\nabla}\phi$

$\therefore \vec{\nabla} \times \vec{F} = \vec{\nabla} \times \vec{\nabla}\phi = 0$

$\therefore f_1 dx + f_2 dy + f_3 dz = d\phi$, an exact differential (showed)

b) ~~If curl $\vec{F} = 0$, then $\vec{F}$ is~~

Let, $\vec{F} = (xy^2z^3 \cos x - 4x^3z)\hat{i} + 2z^3y \sin x\,\hat{j} + (3y^2z^2 \sin x - x^4)\hat{k}$

If curl $\vec{F} = 0$, then $\vec{F} \cdot d\vec{r}$ is an exact differential of a function ϕ

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y^2z^3 \cos x - 4x^3z) & 2z^3y \sin x & (3y^2z^2 \sin x - x^4) \end{vmatrix}$$

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$= \hat{i}\left(6yz^2\sin x - 6yz^2\sin x\right) + \hat{j}\left(3y^2z^2\cos x - 4x^3 - 3y^2z^2\cos x + 4x^3\right) + \hat{k}\left(2z^3y\cos x - 2yz^3\cos x\right)$

$= 0$

As, $\vec{\nabla}\times\vec{F} = 0$

$\vec{F}\cdot d\vec{r} = d\phi$

$\Rightarrow d\phi = \vec{F}\cdot d\vec{r}$

$\Rightarrow d\phi = (y^2z^3\cos x - 4x^3z)\,dx + 2z^3y\sin x\,dy + (3y^2z^2\sin x - x^4)\,dz$

$\Rightarrow d\phi = \left(y^2z^3\cos x\,dx + 2z^3y\sin x\,dy + 3y^2z^2\sin x\,dz\right) - \left(4x^3z\,dx + x^4\,dz\right)$

$\Rightarrow d\phi = d\left(y^2z^3\sin x\right) - d\left(x^4z\right)$

By integrating,

$\phi = y^2z^3\sin x - x^4z + c$

where, c is an integrating constant

$\boxed{Em:-16}$: If $\phi = 2xyz^2$, $\vec{F} = xy\hat{i} - z\hat{j} + x^2\hat{k}$ and C is the curve $x = t^2$, $y = 2t$, $z = t^3$ from $t=0$ to $t=1$ evaluate the line integrals ⓐ $\int \phi \, d\vec{r}$. ⓑ $\int \vec{F} \times d\vec{r}$

Solution:

ⓐ $\phi = 2xyz^2 = \cancel{2} 4t^9$

$\vec{r} = x\hat{i} + y\hat{j} + \cancel{k} z\hat{k}$

$= t^2\hat{i} + 2t\hat{j} + t^3\hat{k}$

$d\vec{r} = (2t\hat{i} + 2\hat{j} + 3t^2\hat{k})\, dt$

$\int \phi \cdot d\vec{r}$

$= \int_0^1 \cancel{8} (8t^{10}\hat{i} + 8t^9\hat{j} + 12t^{11}\hat{k})\, dt$

$= \left[\frac{8t^{11}}{11}\hat{i} + \frac{4t^{10}}{5}\hat{j} + t^{12}\hat{k} \right]_0^1 \cancel{dt}$

$= \frac{8}{11}\hat{i} + \frac{4}{5}\hat{j} + \hat{k}$

ⓑ $\vec{F} = xy\hat{i} - z\hat{j} + x^2\hat{k}$

$= 2t^3\hat{i} - t^3\hat{j} + t^4\hat{k}$

$d\vec{r} = (2t\hat{i} + 2\hat{j} + 3t^2\hat{k})\, dt$

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$\therefore \int \vec{F} \times d\vec{r}$

$$= \int_0^1 \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2t^3 & -t^3 & t^4 \\ 2t & 2 & 3t^2 \end{vmatrix} dt$$

$$= \int_0^1 \left[\hat{i}\left(-3t^5 - 2t^4\right) + \hat{j}\left(2t^5 - 6t^5\right) + \hat{k}\left(4t^3 + 2t^4\right)\right] dt$$

$$= \int_0^1 \left[-\hat{i}\left(3t^5 + 2t^4\right) - 4t^5 \hat{j} + \hat{k}\left(4t^3 + 2t^4\right)\right] dt$$

$$= \left[-\hat{i}\left(\frac{t^6}{2} + \frac{2t^5}{5}\right) - \frac{2t^6}{3}\hat{j} + \hat{k}\left(t^4 + \frac{2t^5}{5}\right)\right]_0^1$$

$$= -\hat{i}\left(\frac{1}{2} + \frac{2}{5}\right) - \frac{2}{3}\hat{j} + \left(1 + \frac{2}{5}\right)\hat{k}$$

$$= -\frac{9\hat{i}}{10} - \frac{2\hat{j}}{3} + \frac{7\hat{k}}{5}$$

$\boxed{\text{Em:-21}}$: Evaluate $\iint \phi \hat{n}\, ds$ where $\phi = \frac{3}{8}xyz$ and S is the surface of the cylinder $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$.

Solution:

Let,

$$u_y = x^2+y^2-16$$

$$\nabla u_y = 2x\hat{i} + 2y\hat{j}$$

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$$\hat{n} = \frac{\vec{\nabla}\psi}{|\vec{\nabla}\psi|} = \frac{2x\hat{i}+2y\hat{j}}{\sqrt{4x^2+4y^2}} = \frac{x\hat{i}+y\hat{j}}{4}$$

As the surface project on xz plane,

$$ds = \frac{dx\,dz}{|\hat{n}\cdot\hat{j}|}$$

~~Here, $\phi = \frac{3}{8}xz$~~

$$\hat{n}\cdot\hat{j} = \frac{y}{4}$$

$$\iint \phi\,\hat{n}\,ds$$

$$\cancel{= \iint \frac{3}{8}xz\,(x\hat{i}+}$$

$$= \int_{x=0}^{4}\int_{z=0}^{5} \frac{3}{8}xyz\,\frac{(x\hat{i}+y\hat{j})}{4}\,\frac{dx\,dz}{\frac{y}{4}}$$

$$= \int_0^4\int_0^5 \left(\frac{3x^2z}{8}\hat{i} + \frac{3xyz}{8}\hat{j}\right)dx\,dz$$

$$= \frac{3}{8}\int_0^4\int_0^5 \left[(x^2z)\hat{i} + (xz\sqrt{16-x^2})\hat{j}\right]dx\,dz$$

$$= \frac{3}{8}\int_0^4 \left[\left(\frac{x^2z^2}{2}\right)\hat{i} + \left(\frac{xz^2\sqrt{16-x^2}}{2}\right)\hat{j}\right]_0^5 dx$$

$$= \frac{3}{8}\int_0^4 \left[\frac{25x^2}{2}\hat{i} + \frac{25x\sqrt{16-x^2}}{2}\hat{j}\right]dx$$

$$= \frac{3}{8}\left[\frac{25x^3}{6}\hat{i} + \frac{25x^3}{6}\right]_0^4$$

$$= \frac{3}{8}\left(\frac{800}{3}\hat{i} + \frac{800}{3}\hat{j}\right)$$

$$= 100\hat{i} + 100\hat{j}$$

$\boxed{\text{Em:-22}}$: If $\vec{F} = y\hat{i} + (x - 2xz)\hat{j} - xy\hat{k}$, evaluate $\iint (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,ds$ where S is the surface of the sphere $x^2+y^2+z^2=a^2$ above the xy plane.

Solution:

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x-2xz & -xy \end{vmatrix}$$

$$= \hat{i}(-x+2x) + \hat{j}(0+y) + \hat{k}(1-2z-1)$$

$$= x\hat{i} + y\hat{j} - 2z\hat{k}$$

Let, $\phi = x^2+y^2+z^2-a^2$

$$\therefore \vec{\nabla}\phi = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}$$

As the surface on the xy plane

$$ds = \frac{dx\,dy}{\hat{n}\cdot\hat{k}}$$

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$\hat{n}\cdot\hat{k} = \frac{z}{a}$

$$\iint (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, ds$$

$$= \int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{x^2+y^2-2z^2}{z}\, dx\, dy$$

$$= \int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{3(x^2+y^2)-2a^2}{\sqrt{a^2-x^2-y^2}}\, dy\, dx \quad [\because x^2+y^2+z^2=a^2]$$

To evaluate the double integral, transform to polar cordinates (ρ,ϕ) where $x=\rho\cos\phi$, $y=\rho\sin\phi$ and $dy\,dx$ is replaced by $\rho\, d\phi\, d\rho$.

$$\int_{\phi=0}^{2\pi}\int_{\rho=0}^{a} \frac{3\rho^2-2a^2}{\sqrt{a^2-\rho^2}}\, \rho\, d\rho\, d\phi$$

$$= \int_{\phi=0}^{2\pi}\int_{\rho=0}^{a} \frac{3(\rho^2-a^2)+a^2}{\sqrt{a^2-\rho^2}}\, \rho\, d\rho\, d\phi$$

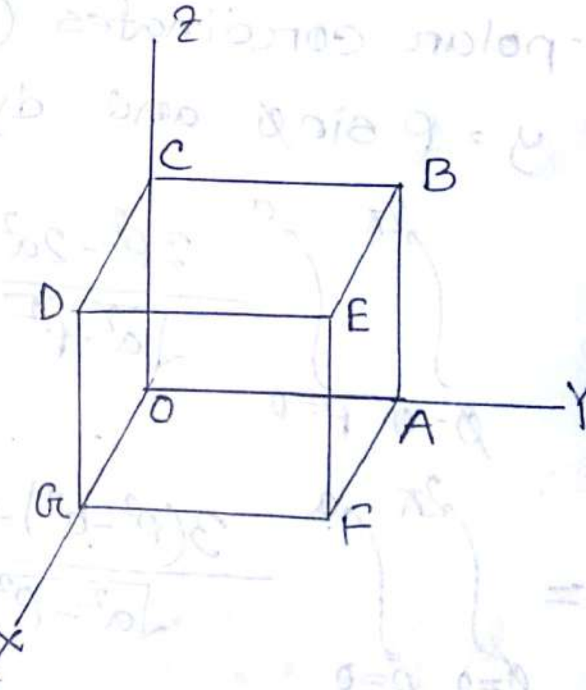
$$= \int_{\phi=0}^{2\pi}\int_{\rho=0}^{a} \left(-3\rho\sqrt{a^2-\rho^2} + \frac{a^2\rho}{\sqrt{a^2-\rho^2}}\right) d\rho\, d\phi$$

$$= \int_{\phi=0}^{2\pi} \left[(a^2-\rho^2)^{\frac{3}{2}} - a^2\sqrt{a^2-\rho^2}\right]_{\rho=0}^{a} d\phi$$

$$= \int_{\phi=0}^{2\pi} (a^3-a^3)\, d\phi = 0$$

$\boxed{\text{Ex 6-23}}$: If $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$, evaluate $\iint \vec{F}\cdot\hat{n}\,ds$ where S is the surface of the cube bounded by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$.

Solution:



~~In ABCO: $\hat{n}=$~~

In DEFG: $\hat{n} = \hat{i}$, $x=1$, then,

$$\iint \vec{F}\cdot\hat{n}\,ds$$

$$= \int_0^1\int_0^1 (4z\hat{i} - y^2\hat{j} + yz\hat{k})\cdot\hat{i}\, dy\, dz$$

$$= \int_0^1\int_0^1 4z\, dy\, dz = \int_0^1 \left[4yz\right]_0^1 dz = \int_0^1 4z\, dz$$

$$= \left[2z^2\right]_0^1$$

$$= 2$$

In ABCO: $\hat{n} = -\hat{i}$, $x=0$

$$\iint \vec{F}\cdot\hat{n}\,ds$$

$$\cancel{= \int_0^1\int_0^1 4z\,dy\,dz} = \int_0^1\int_0^1 (-y^2\hat{j} + yz\hat{k})(-\hat{i})\,dy\,dz$$

$$= 0$$

In ABEF: $\hat{n} = \hat{j}$, $y = 1$

$$\iint \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 (4xz\hat{i} - \hat{j} + z\hat{k})\,\hat{j}\,dx\,dz$$

$$= \int_0^1\int_0^1 -dx\,dz = -\int_0^1 [x]_0^1\,dz = -\int_0^1 dz = -1$$

In OGDC: $\hat{n} = -\hat{j}$, $y = 0$

$$\iint \vec{F}\cdot\hat{n}\,ds$$

$$= \int_0^1\int_0^1 (4xz\hat{i})\cdot(-\hat{j})\,dx\,dz = 0$$

In BCDE: $\hat{n} = \hat{k}$, $z = 1$

$$\iint \vec{F}\cdot\hat{n}\,ds$$

$$= \int_0^1\int_0^1 (4x\hat{i} - y^2\hat{j} + y\hat{k})\,\hat{k}\,dx\,dy = \int_0^1\int_0^1 y\,dx\,dy$$

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$$= \int_0^1 \left[\frac{y^2}{2}\right]_0^1 dx$$

$$= \frac{1}{2}\int_0^1 dx$$

$$= \frac{1}{2}$$

In AFGO: $\hat{n} = -\hat{k}$, $z = 0$

$$\iint \vec{F}\cdot\hat{n}\,ds$$

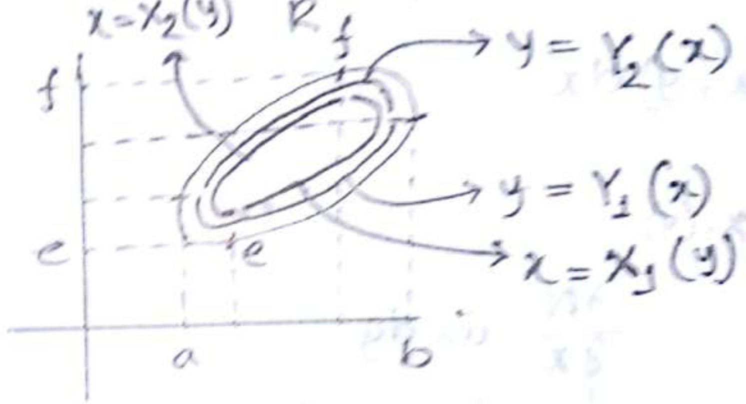
$$= \int_0^1\int_0^1 (-y^2\hat{j})(-\hat{k})\,dx\,dy = 0$$

04-04-2018 : 8E : Wednesday

→ Green's theorem in the plane

If R is a region in closed path C and M, N are continuous function in R then

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$



$$\int_{x=a}^{b} \int_{y=Y_1(x)}^{Y_2(x)} \frac{\partial M}{\partial y}\,dx\,dy$$

$$= \int_a^b \left\{ \int_{Y_1}^{Y_2} \frac{\partial M}{\partial y}\,dy \right\} dx$$

$$= \int_a^b \left[M(x,y) \right]_{Y_1}^{Y_2} dx$$

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$$= \int_a^b [M(x, Y_2) - M(x, Y_1)]\, dx$$

$$= -\int_b^a M(x, Y_2)\, dx - \int_a^b M(x, Y_1)\, dx$$

$$= -\left\{ \int_a^b M(x, Y_1)\, dx + \int_b^a M(x, Y_2)\, dx \right\}$$

$$= -\oint M(x, y)\, dx$$

$$\int_{y=e}^{f} \int_{x=X_1(y)}^{X_2(y)} \frac{\partial N}{\partial x}\, dx\, dy$$

$$= \int_e^f \left\{ \int_{X_1}^{X_2} \frac{\partial N}{\partial x}\, dx \right\} dy$$

$$= \int_e^f [N(x,y)]_{X_1}^{X_2}\, dy$$

$$= \int_e^f [N(X_2, y)]\, dy - \int_e^f [N(X_1, y)]\, dy$$

$$= \int_e^f [N(X_2, y)\, dy + \int_f^e [N(X_1, y)]\, dy$$

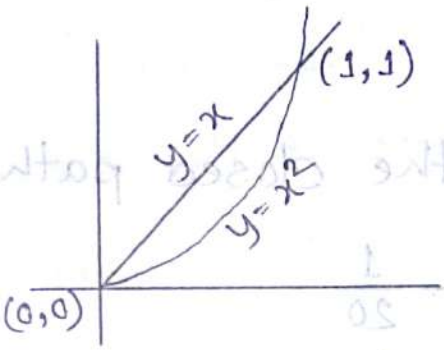
$$= \oint_c N(x, y)\, dy$$

$$\therefore \oint (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$$

$\boxed{Em:-02}$

→ Verify Green's Theorem for $\oint_C [(xy+y^2)\,dx + x^2\,dy]$ where C is the closed curve of region bounded by $y = x$, $y = x^2$

Solution:



Integration along the path $y = x^2$

$$\int_0^1 [(x^3+x^4)\,dx + 2x^3\,dx]$$

$$= \int_0^1 (3x^3+x^4)\,dx$$

$$= \left[\frac{3x^4}{4} + \frac{x^5}{5} \right]_0^1$$

$$= \frac{3}{4} + \frac{1}{5} = \frac{15+4}{20} = \frac{19}{20}$$

Integration along the path $y=x$

$$\int_1^0 \left[(x^2+x^2)\,dx + x^2\,dx\right]$$

$$= \int_1^0 [3x^2]\,dx = [x^3]_1^0 = -1$$

$\therefore$ Integration of the closed path

$$L.H.S. = \frac{19}{20} - 1 = -\frac{1}{20}$$

$$R.H.S. = \int_{x=0}^{1}\int_{y=x^2}^{x} \left\{ \frac{\partial(x^2)}{\partial x} - \frac{\partial(xy+y^2)}{\partial y} \right\} dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{x} (2x - x - 2y)\,dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{x} (x-2y)\,dx\,dy$$

$$= \int_0^1 \left[xy - y^2\right]_{x^2}^{x} dx$$

$$\cancel{= \int_0^1 (x^2 - x^2 - x^3 + x^4)\,dx}$$

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$$= \int_0^1 \left[x^2 - x^2 - x^3 + x^4\right] dx$$

$$= \int_0^1 \left[x^4 - x^3\right] dx = \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{5} - \frac{1}{4} = \frac{4-5}{20} = -\frac{1}{20}$$

$\therefore$ L.H.S. = R.H.S. (Proved)

$\boxed{\text{Ex:-37}}$: Verify Green's theorem in the plane for $\oint\left[(3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy\right]$ where C is the boundary of the region defined by

a) $y = \sqrt{x}$, $y = x^2$

b) $x = 0$, $y = 0$, $x + y = 1$

Solution:

a) Integration along the path $y = \sqrt{x}$

$$\int_1^0 \left[(3x^2 - 8x)\,dx + (4\sqrt{x} - 6x\sqrt{x})\frac{dx}{2\sqrt{x}}\right]$$

$$= \int_1^0 (3x^2 - 8x + 2 - 3x)\,dx$$

$$= \int_1^0 (3x^2 - 11x + 2)\,dx$$

$$= \left[x^3 - \frac{11x^2}{2} + 2x\right]_1^0$$

$$= -1 + \frac{11}{2} - 2 = \frac{5}{2}$$

Integrating along the path $y = x^2$

$$\int_0^1 \left[(3x^2 - 8x^4)\,dx + (4x^2 - 6x^3)\,2x\,dx\right]$$

$$= \int_0^1 (3x^2 - 8x^4 + 8x^3 - 12x^4)\,dx$$

$$= \int_0^1 (3x^2 + 8x^3 - 20x^4)\,dx$$

$$= \left[x^3 + 2x^4 - 4x^5\right]_0^1$$

$$= 1 + 2 - 4 = -1$$

Integration of the closed path

$$L.H.S. = \frac{5}{2} - 1 = \frac{3}{2}$$

$$R.H.S. = \int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} \left\{ \frac{\partial(4y-6xy)}{\partial x} - \frac{\partial(3x^2-8y^2)}{\partial y} \right\} dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} (-6y+16y)\,dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} (10y)\,dy\,dx$$

$$= \int_{0}^{1} \left[5y^2\right]_{x^2}^{\sqrt{x}} dx$$

$$= \int_{0}^{1} (5x-5x^4)\,dx$$

$$= \left[\frac{5x^2}{2} - \frac{5x^5}{5}\right]_0^1$$

$$= \frac{5}{2} - 1 = \frac{3}{2}$$

$\therefore$ L.H.S. = R.H.S. (verified)

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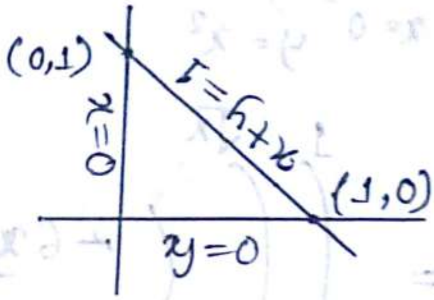
(b) Integration along the path $x = 0$

$$\int_1^0 [0 + (4y)\,dy]$$



$$= \left[2y^2\right]_1^0$$

$$= -2$$

Integration along the path $y = 0$

$$\int_0^1 [(3x^2)\,dx + 0]$$

$$= \left[x^3\right]_0^1 = 1$$

Integration along the path $x + y = 1$

$$\int_1^0 \left[\{3x^2 - 8(1-x)^2\}dx - (4 - 4x - 6x + 6x^2)\,dx\right]$$

$$= \int_1^0 (3x^2 - 8 + 16x - 8x^2 - 4 + 10x - 6x^2)\,dx$$

$$= \int_1^0 (-11x^2 + 26x - 12)\,dx$$

$= \frac{11}{3} - 13 + 12$

$= \frac{11-39+36}{3} = \frac{8}{3}$

Integration of the closed path

$L.H.S. = -2 + 1 + \frac{8}{3}$

$= \frac{-6+3+8}{3}$

$= \frac{5}{3}$

$$R.H.S. = \int_{x=0}^{1}\int_{y=0}^{1-x}\left[\frac{\partial(4y-6xy)}{\partial x} - \frac{\partial(3x^2-8y^2)}{\partial y}\right] dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x}\left[-6y+16y\right] dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x}(10y)\,dy\,dx.$$

$$= \int_{0}^{1}\left[5y^2\right]_0^{1-x} dx$$

$$= \int_{0}^{1} 5(1-x)^2\,dx = \int_{0}^{1}(5-10x+5x^2)\,dx$$

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$$= \left[5x - 5x^2 + \frac{5x^3}{3}\right]_0^1 = 5 - 5 + \frac{5}{3} = \frac{5}{3}$$

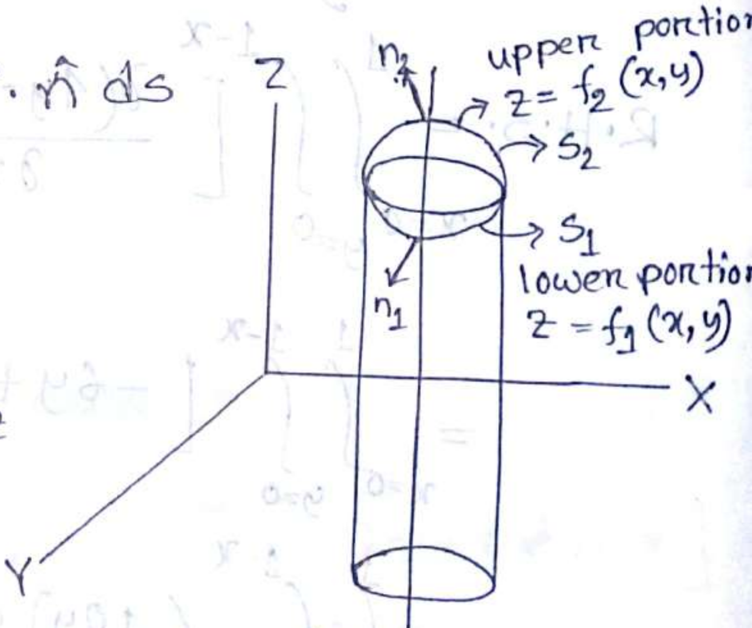
$\therefore$ L.H.S. = R.H.S. (Verified)

18-04-2018 : 9E : Wednesday

→ Divergence theorem

If v is a volume mounted by a surface S and $\vec{A}$ is any vector on the surface

$$\iiint \vec{\nabla}\vec{A}\, dv = \iint \vec{A}\cdot\hat{n}\, ds$$



$$\vec{\nabla}\cdot\vec{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

L.H.S. =

$$\iiint \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) dx\,dy\,dz$$

$$= \iiint \frac{\partial A_1}{\partial x}\, dx\,dy\,dz + \iiint \frac{\partial A_2}{\partial y}\, dx\,dy\,dz + \iiint \frac{\partial A_3}{\partial z}\, dx\,dy\,dz$$

$$\iiint \frac{\partial A_3}{\partial z}\, dx\,dy\,dz$$

$$= \iint \left\{ \int \frac{\partial A_3}{\partial z}\, dz \right\} dx\,dy$$

$$= \iint \left\{ \int_{f_1}^{f_2} \frac{\partial A_3}{\partial z} dz \right\} dx\,dy$$

$$= \iint \left[A_3(x,y,z) \right]_{f_1}^{f_2} dx\,dy$$

$$= \iint \left[A_3(x,y,f_2) - A_3(x,y,f_1) \right] dx\,dy$$

$$= \iint_{S_2} A_3(x,y,f_2)\, dx\,dy - \iint_{S_1} A_3(x,y,f_1)\, dx\,dy$$

$$ds_1 = \frac{dx\,dy}{\cos(180-\theta)}$$

$$\Rightarrow ds_1 = \frac{dx\,dy}{-\cos\theta}$$

$$\Rightarrow ds_1 = -\frac{dx\,dy}{\hat{n}_1 \cdot \hat{k}} \qquad \therefore dx\,dy = -\, ds_1 |\hat{n} \cdot \hat{k}|$$

$$ds_2 = \frac{dx\,dy}{\cos\theta}$$

$$\Rightarrow ds_2 = \frac{dx\,dy}{\hat{n}_2 \cdot \hat{k}}$$

$$\therefore dx\,dy = ds_2\, \hat{n}_2 \cdot \hat{k}$$

$$\therefore \iint_{S_2} A_3 (\hat{k} \cdot \hat{n}_2\, ds_2) + \iint_{S_1} A_3 (\hat{k} \cdot \hat{n}_1 \cdot ds_1)$$

$$= \iint_{S} A_3 \cdot \hat{k} \cdot \hat{n}\, ds$$

$$\iiint \frac{\partial A_1}{\partial x}\,dx\,dy\,dz \qquad \iiint \frac{\partial A_2}{\partial y}\,dx\,dy\,dz$$

$$= \iint A_1 \cdot \hat{i} \cdot \hat{n}\,ds \qquad = \iint A_2\,\hat{j} \cdot \hat{n}\,ds$$

$$\therefore \text{L.H.S.} = \iiint \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) dx\,dy\,dz$$

$$= \iint_S \left(A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k} \right) \hat{n}\,ds$$

$$= \iint_S \vec{A} \cdot \hat{n}\,ds$$

$$= \text{R.H.S.}$$

$\therefore$ L.H.S. = R.H.S. (Proved)

$\boxed{\text{Em:-17}}$: Evaluate $\iint_S \vec{F} \cdot \hat{n}\,ds$ where $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ and S is the surface of the cube bounded by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$.

Solution:

$$\iint \vec{F} \cdot \hat{n}\,ds$$

$$= \iiint \vec{\nabla} \cdot \vec{F}\,dx\,dy\,dz$$

$$= \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} \left\{ \frac{\partial}{\partial x}(4xz) - \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(yz) \right\} dx\,dy\,dz$$

$$= \int_0^1\int_0^1\int_0^1 (4z - 2y + y)\, dx\,dy\,dz$$

$$= \int_0^1\int_0^1\int_0^1 (4z - y)\, dz\,dy\,dx$$

$$= \int_0^1\int_0^1 \left[2z^2 - yz\right]_0^1 dy\,dx$$

$$= \int_0^1\int_0^1 (2-y)\, dy\,dx$$

$$= \int_0^1 \left[2y - \frac{y^2}{2}\right]_0^1 dx$$

$$= \int_0^1 \left(2 - \frac{1}{2}\right) dx$$

$$= \int_0^1 \frac{3}{2}\, dx = \left[\frac{3x}{2}\right]_0^1 = \frac{3}{2}$$

05-05-2018 : 10E : Saturday

→ Stoke's theorem

Let, S be a surface which is such that its projections on the xy and xz planes are regions bounded by simple closed curves, as indicated in the adjoining figure. Assume S to have representation $z=f(x,y)$ or $x=g(y,z)$ or $y=h(x,z)$, where f, g, h are single valued, continuous and differentiable functions.

$$\iint_S (\vec{\nabla}\times\vec{A})\,\hat{n}\,ds = \oint_C \vec{A}\cdot d\vec{r}$$

$$\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$$

$$L.H.S. = \iint_S \left\{\vec{\nabla}\times\left(A_1\hat{i}+A_2\hat{j}+A_3\hat{k}\right)\right\}\hat{n}\,ds$$

$$= \iint(\vec{\nabla}\times A_1\hat{i})\hat{n}\,ds + \iint(\vec{\nabla}\times A_2\hat{j})\,\hat{n}\,ds$$

$$+ \iint(\vec{\nabla}\times A_3\hat{k})\,\hat{n}\,ds$$

$$\vec{\nabla} \times A_1 \hat{i} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & 0 & 0 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}\left(0 - \frac{\partial A_1}{\partial z}\right) + \hat{k}\left(0 - \frac{\partial A_1}{\partial y}\right)$$

$$= \frac{\partial A_1}{\partial z}\hat{j} - \frac{\partial A_1}{\partial y}\hat{k}$$

$$(\vec{\nabla} \times A_1 \hat{i})\hat{n} = \frac{\partial A_1}{\partial z}\hat{j}\cdot\hat{n} - \frac{\partial A_1}{\partial y}\hat{k}\cdot\hat{n}$$

$$A(x,y,z) = A_1(x,y,z) + A_2(x,y,z) + A_3(x,y,z)$$

$$A_1(x,y,z) = A_1(x, y, f(x,y))$$

$$= F(x,y)$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = x\hat{i} + y\hat{j} + f(x,y)\hat{k}$$

$$\frac{\partial \vec{r}}{\partial y} = 0 + \hat{j} + \frac{\partial f}{\partial y}\hat{k}$$

$$\Rightarrow \hat{n}\cdot\frac{\partial \vec{r}}{\partial y} = \hat{n}\cdot\hat{j} + \frac{\partial f}{\partial y}\hat{n}\cdot\hat{k}$$

$$\Rightarrow 0 = \hat{n}\cdot\hat{j} + \frac{\partial f}{\partial y}\hat{n}\cdot\hat{k}$$

$$\Rightarrow \hat{n}\cdot\hat{j} = -\frac{\partial f}{\partial y}\hat{n}\cdot\hat{k}$$

$$(\vec{\nabla} \times A_1 \hat{i})\hat{n} = -\frac{\partial f}{\partial y}(\hat{n}\cdot\hat{k})\frac{\partial A_1}{\partial z} - \hat{n}\cdot\hat{k}\frac{\partial A_1}{\partial y}$$

$$= -\hat{n}\cdot\hat{k}\left(\frac{\partial z}{\partial y}\cdot\frac{\partial A_1}{\partial z} + \frac{\partial A_1}{\partial y}\right) \quad [\because z = f(x,y)]$$

$$\cancel{= \hat{n}\cdot\hat{k}\left(\frac{\partial A_1}{\partial y} + \frac{\partial A_1}{\partial y}\right)}$$

$$= -\hat{n}\cdot\hat{k}\left(\frac{\partial F}{\partial y}\right)$$

$$\therefore \iint_S (\vec{\nabla} \times A_1 \hat{i})\hat{n}\, ds = -\iint \hat{n}\cdot\hat{k}\,\frac{\partial F}{\partial y}\, ds$$

$$= -\iint \frac{\partial F}{\partial y}\, dx\, dy$$

$$= \int -\left\{\int \frac{\partial F}{\partial y}\, dy\right\} dx$$

$$= \int F\, dx$$

$$= \int A_1\, dx$$

Similarly,

$$\iint (\vec{\nabla} \times A_2 \hat{j})\hat{n}\, ds = \int A_2\, dy$$

$$\iint (\vec{\nabla} \times A_3 \hat{k})\hat{n}\, ds = \int A_3\, dz$$

$$L.H.S. = \iint \{ \vec{\nabla} \times (A_1\hat{i} + A_2\hat{j} + A_3\hat{k}) \hat{n}\, ds$$

$$= \int (A_1 dx + A_2 dy + A_3 dz) \cancel{(dx\hat{i} + dy\hat{j} + dz\hat{k})}$$

$$= \int \vec{A} \cdot d\vec{r}$$

$$= R.H.S.$$

$\therefore$ L.H.S. = R.H.S. (Proved)

23-06-2018: 11E: Saturday:

→ Verify the divergence theorem for $\vec{A} = 4x\hat{i} - 2xy^2\hat{j} + z^2\hat{k}$ taken over the region bounded by $x^2+y^2=4$, $z=0$ and $z=3$.

Solution:

Prove: $\iiint_V \vec{\nabla}\cdot\vec{A}\,dv = \iint_S \vec{A}\cdot\hat{n}\,ds$

$\vec{\nabla}\cdot\vec{A}$

$= \frac{d}{dx}(4x) - \frac{d}{dy}(2y^2) + \frac{d}{dz}(z^2)$

$= 4 - 4y + 2z$

In xy plane

$x^2+y^2=4$

$\Rightarrow y^2 = 4-x^2$

$\therefore y = \pm\sqrt{4-x^2}$

In x axis

$x^2 = 4$

$\therefore x = \pm 2$

$\therefore$ L.H.S. $= \iiint \vec{\nabla}\cdot\vec{A}\,dv$

$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{3} (4-4y+2z)\,dx\,dy\,dz$

$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4z - 4yz + z^2\right]_0^3 dx\,dy$

$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (12-12y+9)\,dx\,dy$

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$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21 - 12y)\, dx\, dy$$

$$= \int_{-2}^{2} \left[21y - 6y^2 \right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} \left[21\sqrt{4-x^2} - 6(4-x^2) + 21\sqrt{4-x^2} + 6(4-x^2) \right] dx$$

$$= 42 \int_{-2}^{2} \sqrt{4-x^2}\, dx$$

$$= 42 \left[\frac{x\sqrt{4-x^2}}{2} + 2 \sin^{-1}\frac{x}{2} \right]_{-2}^{2}$$

$$= 42 \left[0 + \pi - 0 + \pi \right]$$

$$= 84\pi$$

$$R.H.S = \iint_S \vec{A} \cdot \hat{n} \cdot ds$$

$$= \iint_{S_1} \vec{A} \cdot \hat{n}\, ds_1 + \iint_{S_2} \vec{A} \cdot \hat{n}\, ds_2 + \iint_{S_3} \vec{A} \cdot \hat{n}\, ds_3$$

On S_1 surface $z=0$ $\therefore \hat{n} = -\hat{k}$

$\therefore \vec{A}\cdot\hat{n} = -z^2 = 0$

$\therefore \iint_{S_1} \vec{A}\cdot\hat{n}\, ds_1 = 0$

On S_2 surface $z=3$ $\therefore \hat{n} = \hat{k}$

$\therefore \vec{A}\cdot\hat{n} = z^2 = 9$

$\therefore \iint_{S_2} \vec{A}\cdot\hat{n}\, ds_2$

$= 9 \iint_{S_2} ds_2$

$= 9 \times 4\pi$ $\left[\iint ds_2 \text{ is the area of the circle } x^2+y^2=4 \therefore \text{Area} = \pi r^2 = \pi \times 2^2 = 4\pi \right]$

$= 36\pi$

On S_3 surface,

Let, $\phi = x^2+y^2-4$

$$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{2(x\hat{i}+y\hat{j})}{2\sqrt{x^2+y^2}} = \frac{x\hat{i}+y\hat{j}}{\sqrt{4}} = \frac{x\hat{i}+y\hat{j}}{2}$$

$\therefore \vec{A}\cdot\hat{n} = 2x^2 - y^3$

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$$\therefore \iint_{s_3} \vec{A}\cdot\hat{n}\, ds_3 \qquad \left[x = r\cos\theta,\ y = r\sin\theta \quad ds = r\,d\theta\,dz\right]$$

Let,

$$x = 2\cos\theta$$

$$y = 2\sin\theta$$

$$ds_2 = 2\,d\theta\,dz$$

$$= \iint_{s_3} \left(2x^2 - y^3\right) ds_3$$

$$= \int_{0}^{2\pi}\int_{0}^{3} \left(8\cos^2\theta - 8\sin^3\theta\right) 2\,d\theta\,dz$$

$$= 16\int_{0}^{2\pi} \Big[(\cos^2\theta - \sin^3\theta)\,(z)\Big]_0^3\, d\theta$$

$$= \cancel{16}\ 48\int_{0}^{2\pi} \left(\cos^2\theta - \sin^3\theta\right) d\theta$$

$$= 48\int_{0}^{2\pi} \cos^2\theta\, d\theta - 48\int_{0}^{2\pi} \sin^3\theta\, d\theta$$

$$= 24\int_{0}^{2\pi} 2\cos^2\theta\, d\theta - 0 \qquad \left[\text{As } \sin^3\theta \text{ is odd function} \quad \int_{0}^{2\pi} \sin^3\theta\, d\theta = 0\right]$$

$$= 24\int_{0}^{2\pi} (1 + \cos 2\theta)\, d\theta$$

$$= 24\int_{0}^{2\pi} d\theta + 24\int_{0}^{2\pi} \cos 2\theta\, d\theta$$

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$$= 24\left[\theta\right]_0^{2\pi} + 24\left[\frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= 48\pi$$

$$\therefore R.H.S. = \iint_{S_1} \vec{A}\cdot\hat{n}\, ds_1 + \iint_{S_2} \vec{A}\cdot\hat{n}\, ds_2 + \iint_{S_3} \vec{A}\cdot\hat{n}\, ds_3$$

$$= 0 + 36\pi + 48\pi$$

$$= 84\pi$$

$\therefore L.H.S. = R.H.S.$ (verified)

30-06-2018 : 12E : Saturday

→ Verify Stoke's theorem for $\vec{A} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$ where s is the upper half surface of the sphere $x^2+y^2+z^2=1$ and C is its boundary.

Solution:

$$\oint \vec{A}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{A})\,\hat{n}\cdot ds$$

Let, $x = \cos t$, $y = \sin t$, $z = 0$ $\left[t = 0, 2\pi\right]$

$\Rightarrow dx = -\sin t\, dt$, $dy = \cos t\, dt$, $dz = 0$

$$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

$$\vec{A}\cdot d\vec{r} = (2x-y)\,dx - yz^2 dy - y^2z\, dz$$

$$= -(2\cos t - \sin t)\sin t\, dt$$

$$= (\sin^2 t - 2\sin t\cos t)\, dt$$

$$= (\sin^2 t - \sin 2t)\, dt$$

$$\therefore \text{L.H.S.} = \oint \vec{A} \cdot d\vec{r}$$

$$= \int_0^{2\pi} (\sin^2 t - \sin 2t)\, dt$$

$$= \frac{1}{2}\int_0^{2\pi} (1 - \cos 2t)\, dt - \int_0^{2\pi} \sin 2t\, dt$$

$$= \frac{1}{2}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi} + \left[\frac{\cos 2t}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2}[2\pi] + [1 - 1]$$

$$= \pi$$

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix}$$

$$= \hat{i}(-2yz + 2yz) - \hat{j}(0 - 0) + \hat{k}(0 + 1)$$

$$= \hat{k}$$

$$ds = \frac{dx\,dy}{\hat{n} \cdot \hat{k}}$$

$$\text{R.H.S.} = \iint_S (\vec{\nabla} \times \vec{A})\, \hat{n}\, ds$$

$$= \iint \hat{k} \cdot \hat{n}\, \frac{dx\,dy}{\hat{n} \cdot \hat{k}}$$

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$$= \iint dx\,dy$$

In xy plane,

$$x^2+y^2=1 \quad \left[\text{From } x^2+y^2+z^2=1\right]$$

$$\Rightarrow y = \pm\sqrt{1-x^2}$$

In x axis, $x^2=1 \quad \therefore x = \pm 1$

$$\therefore R.H.S = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx\,dy$$

$$= \int_{-1}^{1} \left[y\right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$$

$$= \int_{-1}^{1} \left(\sqrt{1-x^2}+\sqrt{1-x^2}\right) dx$$

$$= \int_{-1}^{1} 2\sqrt{1-x^2}\,dx$$

$$= 2\left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}x\right]_{-1}^{1}$$

$$= 2\left[0 + \cancel{0}\,\frac{1}{2}\cdot\frac{\pi}{2} - 0 - \left(-\frac{1}{2}\cdot\frac{\pi}{2}\right)\right]$$

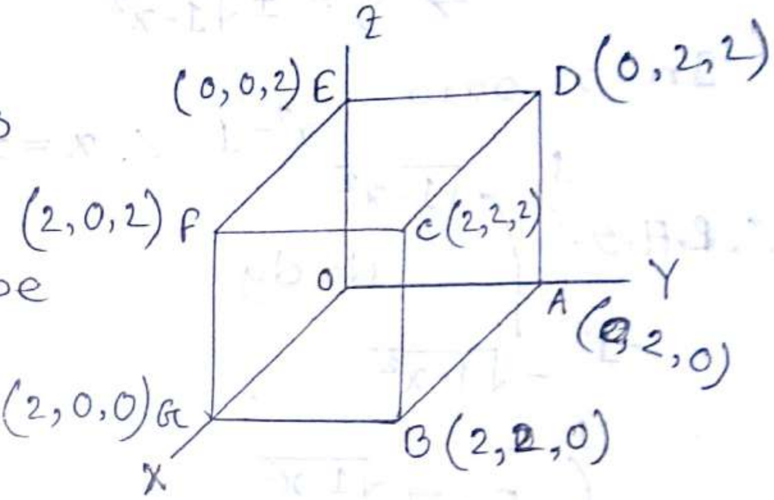
$$= 2\left[\frac{\pi}{4}+\frac{\pi}{4}\right] = 2\cdot\frac{\pi}{2} = \pi$$

$\therefore L.H.S = R.H.S$ (verified)

→ Verify stoke's theorem for $\vec{A} = (y - z + 2)\hat{i} + (yz + 4)\hat{j} - xz\hat{k}$ where s is the surface of the cube $x=0,\ y=0,\ z=0,\ x=2,\ y=2,\ z=2$ above the xy plane.

Solution:

$$\oint \vec{A}\cdot d\vec{r} = \iint (\vec{\nabla}\times\vec{A})\,\hat{n}\,ds$$



surface of the cube is CDEF.

$$\therefore L.H.S. = \oint \vec{A}\cdot d\vec{r}$$

$$= \int_{CD} \vec{A}\cdot d\vec{r} + \int_{DE} \vec{A}\cdot d\vec{r} + \int_{EF} \vec{A}\cdot d\vec{r} + \int_{FC} \vec{A}\cdot d\vec{r}$$

$$\vec{A}\cdot d\vec{r} = (y - z + 2)\,dx + (yz + 4)\,dy - xz\,dz$$

For CD: $y = 2,\ z = 2$, only x varies from 2 to 0.

$dy = 0,\ dz = 0$

$$\int_{CD} \vec{A}\cdot d\vec{r}$$

$$= \int_{CD}^{0} (y - z + 2)\,dx$$

$$= \int_{2}^{0} (2-2+2)\, dx$$

$$= \left[2x\right]_{2}^{0}$$

$$= -4$$

For DE : $x = 0,\ z = 2$, only y varies from 2 to 0.

$dx = 0,\ dz = 0$

$$\int_{DE} \vec{A} \cdot d\vec{r}$$

$$= \int_{DE} (yz + 4)\, dy$$

$$= \int_{2}^{0} (2y+4)\, dy$$

$$= \cancel{\left[2y+4\right]^{0}} \left[y^2 + 4y\right]_{2}^{0}$$

$$= -4 - 8$$

$$= -12$$

For EF : $y = 0,\ z = 2$, only x varies from 0 to 2

$dy = 0,\ dz = 0$

$$\int_{EF} \vec{A} \cdot d\vec{r}$$

$$= \int_{EF} (y-z+2)\, dx = \int_{0}^{2} (0-2+2)\, dx = 0.$$

For FC: $x = 2$, $z = 2$, only y varies to 0 to 2.

$dx = 0$, $dz = 0$

$$\int_{FC} \vec{A} \cdot d\vec{r}$$

$$= \int_{FC} (yz + 4)\, dy$$

$$= \int_0^2 (2y + 4)\, dy$$

$$= \left[y^2 + 4y \right]_0^2 = 4 + 8 = 12$$

$$\therefore L.H.S. = \int_{CD} \vec{A} \cdot d\vec{r} + \int_{DE} \vec{A} \cdot d\vec{r} + \int_{EF} \vec{A} \cdot d\vec{r} + \int_{FC} \vec{A} \cdot d\vec{r}$$

$$= -4 - 12 + 0 + 12$$

$$= -4$$

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y - z + 2 & yz + 4 & -xz \end{vmatrix}$$

$$= \hat{i}(0 - y) - \hat{j}(-z + 1) + \hat{k}(0 - 1)$$

$$= -y\hat{i} + (z - 1)\hat{j} - \hat{k}$$

On CDEF surface $\hat{n} = \hat{k}$, $z = 2$

$$\therefore R.H.S. = \iint_{CDEF} (\vec{\nabla} \times \vec{A})\, \hat{n}\, ds$$

$$= \int_0^2\int_0^2 \{-y\hat{i} + (2-1)\hat{j} - \hat{k}\} \cdot \hat{k}\, dx\,dy$$

$$= -\int_0^2\int_0^2 dx\,dy$$

$$= -\int_0^2 [y]_0^2\, dx$$

$$= -\int_0^2 2\,dx$$

$$= -2\,[x]_0^2$$

$$= -4$$

$\therefore$ L.H.S. = R.H.S. (verified)

07-07-2018 : 13E : Saturday

→ Verify divergence theorem for $\vec{A} = 2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k}$ taken over the region on the first octant bounded by $y^2+z^2=9$ and $x=2$.

Solution:

$$\iiint \vec{\nabla}\cdot\vec{A}\, dv = \iint \vec{A}\cdot\hat{n}\, ds$$

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial}{\partial x}(2x^2y) - \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(4xz^2)$$

$$= 4xy - 2y + 8xz$$

$\therefore$ L.H.S. $= \iiint \vec{\nabla}\cdot\vec{A}\, dv$

$$= \int_{x=0}^{2}\int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}} (4xy - 2y + 8xz)\, dx\, dy\, dz$$

$$= \int_0^2\int_0^3 \left[4xyz - 2yz + 4xz^2\right]_0^{\sqrt{9-y^2}} dx\, dy$$

$$= \int_0^2\int_0^3 4xy\sqrt{9-y^2} - 2y\sqrt{9-y^2} + 36x - 4xy^2\Big] dx\, dy$$

$$= \int_0^2 \left[\frac{4xy^3}{3} - \frac{2y^3}{3} + 36xy - \frac{4xy^3}{3}\right]_0^3 dx$$

$$= \int_0^2 (36x - 18 + 108x - 36x)\, dx$$

$$= \int_0^2 (108x - 18)\, dx$$

$$= \left[54x^2 - 18x\right]_0^2$$

$$= 216 - 36$$

$$= 180$$

Scanned by CamScanner

Volume of the first octant has three surface.

On $S_1 (x=0)$, $\cancel{\hat{n} = -\hat{k}}$, $x=0$

$\hat{n} = -\hat{i}$

$$\iint_{S_1} \vec{A}\cdot\hat{n}\, dS_1 = \iint_{S_1} (-y^2\hat{j})\cdot(-\hat{i})\, dS_1 = 0$$

On $S_2 (x=2)$, $\hat{n} = \hat{i}$, $x=2$

$$\iint_{S_2} \vec{A}\cdot\hat{n}\, dS_2 = \iint_{S_2} (8y\hat{i} - y^2\hat{j} + 8z^2\hat{k})\,\hat{i}\cdot dy\,dz$$

$$= \int_0^3 \int_0^{\sqrt{9-y^2}} 8y\, dy\, dz$$

$$= 8\int_0^3 [yz]_0^{\sqrt{9-y^2}}\, dy$$

$$= 8\int_0^3 y\sqrt{9-y^2}\, dy$$

$$= 8\left[\frac{y^3}{3}\right]_0^3$$

$$= 8\times 9$$

$$= 72$$

On S_3 $(y^2+z^2=9)$,

Let, $\phi = y^2+z^2-9$

$\vec{\nabla}\phi = 2y\hat{j}+2z\hat{k}$

$|\vec{\nabla}\phi| = \sqrt{4y^2+4z^2}$

$= 2\sqrt{y^2+z^2} = 2\times 3 = 6$

$\therefore \hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{2y\hat{j}+2z\hat{k}}{6} = \frac{y\hat{j}+z\hat{k}}{3}$

$\hat{n}\cdot\hat{j} = \frac{y}{3}$ $\qquad ds_3 = \frac{dx\,dz}{\hat{n}\cdot\hat{j}}$

$= \frac{3\,dx\,dz}{y}$

$\vec{A}\cdot\hat{n}$

$= (2x^2y\,\hat{i} - y^2\hat{j} + 4xz^2\hat{k})\left(\frac{y\hat{j}+z\hat{k}}{3}\right)$

$= \frac{1}{3}(-y^3+4xz^3)$

$$\iint_{S_3} \vec{A}\cdot\hat{n}\,ds_3$$

$$= \int_{x=0}^{2}\int_{z=0}^{3}\left(\frac{-y^3+4xz^3}{3}\right)\frac{3\,dx\,dz}{y}$$

$$= \int_{x=0}^{2}\int_{z=0}^{3}\left(-y^2+\frac{4xz^3}{y}\right)dx\,dz$$

$$= \int_0^2 \int_0^3 \left[-(9-z^2) + \frac{4xz^3}{\sqrt{9-z^2}} \right] dx\,dz \quad [y^2 = 9-z^2]$$

$$= \int_0^2 \left[-9z + \frac{z^3}{3} + 4x\left(9z - \frac{z^3}{3}\right) \right]_0^3 dx$$

$$= \int_0^2 \left[-27 + 9 + 4x(27-9) \right] dx$$

$$= \int_0^2 (72x - 18)\, dx$$

$$= \left[36x^2 - 18x \right]_0^2 = 144 - 36 = 108$$

$$\therefore \text{R.H.S.} = \iint_S \vec{A} \cdot \hat{n}\, ds$$

$$= \iint_{S_1} \vec{A} \cdot \hat{n}\, ds_1 + \iint_{S_2} \vec{A} \cdot \hat{n} \cdot ds_2 + \iint_{S_3} \vec{A} \cdot \hat{n}\, ds_3$$

$$= 0 + 72 + 108$$

$$= \cancel{180}\ 108$$

$\therefore$ L.H.S. = R.H.S. (verified)