Heaven's light is our guide"

Rajshahi University of Engineering & Technology
Department of Computer Science & Engineering

Network Security
Course No. : 305
Chapter 7: Relations
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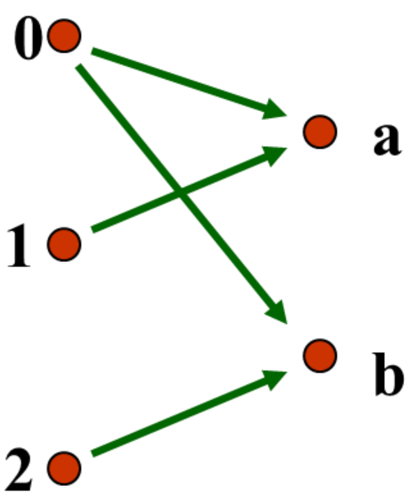
7.1 Relations and Their Properties

7.1 Relations and Their Properties

- **DEFINITION 1:**

  Let A and B be sets. A ***binary relation*** from A to B is a subset of A x B.
- ✓ In other words, a binary relation from A to B is a set R of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B.
- ✓ Notation: aRb $\Leftrightarrow$ (a, b) $\in$ R

  aRb $\Leftrightarrow$ (a, b) $\notin$ R
- ✓ When (a, b) belongs to R, a is said to be related to b by R.
- ✓ Binary relations represent relationships between the elements of two sets.



R	a	b
0	X	X
1	X	
2		X

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- **EXAMPLE 2:** Let A be the set of all cities, and let B be the set of the 50 states in the United States of America. Define the relation R by specifying that (a , b) belongs to R if city a is in state b. For instance, (Boulder, Colorado), (Bangor, Maine), (Ann Arbor, Michigan), (Middletown, New Jersey), (Middletown, New York), (Cupertino, California), and (Red Bank, New Jersey) are in R .
- **Functions as relations:**
  - ✓ The graph of a function f is the set of ordered pairs (a, b) such that b = f(a)
  - ✓ The graph of f is a subset of A * B → it is a relation from A to B
  - ✓ Conversely, if R is a relation from A to B such that every element in A is the first element of exactly one ordered pair of R, then a function can be defined with R as its graph
  - ✓ Relations are generalization of functions
- **Definition 2:**

  A relation on the set A is a relation from A to A.

❖ In other words, a relation on a set A i s a subset o f A x A .

7.1 Relations and Their Properties

- **EXAMPLE 4:** Let A be the set { l , 2, 3 , 4}. Which ordered pairs are in the relation R = {(a , b) | a divides b}?

  Solution: Because (a , b) is in R if and only if a and b are positive integers not exceeding 4 such that a divides b, we see that

  R = {(1 , 1) , (1 , 2), (1 , 3), (1 , 4), (2, 2), (2, 4), (3 , 3), (4, 4) } .
- **EXAMPLE 6:** How many relations are there on a set with n elements?

  Solution: A relation on a set A is a subset of A x A . Because A x A has n^2 elements when A has n elements, and a set with m elements has 2^m subsets, there are 2^{n^2} subsets of A x A . Thus, there are 2^{n^2} relations on a set with n elements. For example, there are $2^{3^2} = 2^9 = 512$ relations on the set {a , b, c } .

Properties of Relations

- **Definition 3:**

  A relation R on a set A is called ***reflexive*** if $(a, a) \in R$ for every element $a \in A$.

  $\forall a((a, a) \in R)$

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- **EXAMPLE 7:** Consider the following relations on { l , 2, 3 , 4} :

  $R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$
  $R_2 = \{(1,1), (1,2), (2,1)\}$
  $R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (3,4), (4,1), (4,4)\}$
  $R_4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$
  $R_5 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$
  $R_6 = \{(3,4)\}$

  Which of these relations are reflexive?

  Solution: R_3 and R_5: reflexive $\Leftarrow$ both contain all pairs of the form (a, a): (1,1), (2,2), (3,3) & (4,4).

  R_1, R_2, R_4 and R_6: not reflexive $\Leftarrow$ not contain all of these ordered pairs. (3,3) is not in any of these relations.

- **Definition 4:**

  A relation R on a set A is called ***symmetric*** if $(b,a)\in R$ whenever $(a,b)\in R$, for all $a,b\in A$.
  A relation R on a set A such that for all $a,b\in A$, if $(a,b)\in R$ and $(b,a)\in R$, then $a=b$ is called ***antisymmetric*.**

$$\forall a\forall b((a,b)\in R \rightarrow (b,a)\in R)$$
$$\forall a\forall b(((a,b)\in R \wedge (b,a)\in R) \rightarrow (a=b))$$

7.1 Relations and Their Properties

- **EXAMPLE 10:** Consider the following relations on { l , 2, 3 , 4} :

  R_1 = {(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)}
  R_2 = {(1,1), (1,2), (2,1)}
  R_3 = {(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (3,4), (4,1), (4,4)}
  R_4 = {(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)}
  R_5 = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)}
  R_6 = {(3,4)}

  Which of these relations are symmetric and which are anti symmetric?

  Solution:

  R_2 & R_3: symmetric $\Leftarrow$ each case (b, a) belongs to the relation whenever (a, b) does.

  For R_2: only thing to check that both (1,2) & (2,1) belong to the relation

  For R_3: it is necessary to check that both (1,2) & (2,1) belong to the relation.

  None of the other relations is symmetric: find a pair (a, b) so that it is in the relation but (b, a) is not.

  R_4, R_5 and R_6: antisymmetric $\Leftarrow$for each of these relations there is no pair of elements a and b with $a \neq b$ such that both (a, b) and (b, a) belong to the relation.

  None of the other relations is antisymmetric.: find a pair (a, b) with $a \neq b$ so that (a, b) and (b, a) are both in the relation.

7.1 Relations and Their Properties

- **EXAMPLE 12:** Is the "divides" relation on the set of positive integers symmetric? Is it anti symmetric?

  Solution: This relation is not symmetric because $1 \mid 2$, but $2 \nmid 1$. It is anti symmetric, for if a and b are positive integers with $a \mid b$ and $b \mid a$, then a = b.

- **Definition 5:**

  A relation R on a set A is called ***transitive*** if whenever $(a,b)\in R$ and $(b,c)\in R$, then $(a,c)\in R$, for all $a,b,c\in A$.

  $$\forall a\forall b\forall c(((a,b)\in R \land (b,c)\in R) \rightarrow (a,c)\in R)$$

- **EXAMPLE 15:** Is the "divides" relation on the set of positive integers transitive?

  Solution: Suppose that a divides b and b divides c. Then there are positive integers k and l such that b = ak and c = bl. Hence, c = a(kl), so a divides c. It follows that this relation is transitive.

7.1 Relations and Their Properties

- **EXAMPLE 13:** Consider the following relations on { l , 2, 3 , 4} :

  R_1 = {(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)}
  R_2 = {(1,1), (1,2), (2,1)}
  R_3 = {(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (3,4), (4,1), (4,4)}
  R_4 = {(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)}
  R_5 = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)}
  R_6 = {(3,4)}

  Which of these relations are transitive?

  Solution:

  - R_4 , R_5 & R_6 : transitive $\Leftarrow$ verify that if (a, b) and (b, c) belong to this relation then (a, c) belongs also to the relation

  R_4 transitive since (3,2) and (2,1), (4,2) and (2,1), (4,3) and (3,1), and (4,3) and (3,2) are the only such sets of pairs, and (3,1) , (4,1) and (4,2) belong to R_4. Same reasoning for R_5 and R_6.

  - R_1 : not transitive $\Leftarrow$ (3,4) and (4,1) belong to R_1, but (3,1) does not.
  - R_2 : not transitive $\Leftarrow$ (2,1) and (1,2) belong to R_2, but (2,2) does not.
  - R_3 : not transitive $\Leftarrow$ (4,1) and (1,2) belong to R_3, but (4,2) does not.

7.1 Relations and Their Properties

Combining Relations

- ✓ Two relations can be combined in any way two sets can be combined.
- **Example:** Let A = {1, 2, 3} and B = {1, 2, 3, 4, }. The relations R_1 = {(1,1), (2,2), (3,3)} and R_2 = {(1,1), (1,2), (1,3), (1,4)} can be combined to obtain:

$$R_1 \cup R_2 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (3,3)\}$$
$$R_1 \cap R_2 = \{(1,1)\}$$
$$R_1 - R_2 = \{(2,2), (3,3)\}$$
$$R_2 - R_1 = \{(1,2), (1,3), (1,4)\}$$

- **Definition 6:**

  Let R be relation from A to B, S be relation from B to C. The ***composite*** of R & S is the relation consisting of ordered pairs (a,c), where $a\in A$, $c\in C$, and there exists an element $b\in B$ such that $(a,b)\in R$ and $(b,c)\in S$. (denoted by $S\circ R$)
- **Example 20:** What is the composite of relations R & S, where R is the relation from {1, 2, 3} to {1, 2, 3, 4} with R={(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)} and S is the relation from {l,2,3,4} to {0,1, 2} with S={(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)}?
- **Solution:** $S\circ R$ is constructed using all ordered pairs in R and ordered pairs in S, where the second element of the ordered pair in R agrees with the first element of the ordered pair in S. For example, the ordered pairs (2, 3) in R and (3 , 1) in S

S produce the ordered pair (2, 1) in S○R. Computing all the ordered pairs in the composite, we find

S o R = {(1 , 0), (1 , 1) , (2, 1) , (2, 2), (3 , 0), (3 , 1) } .

- **Definition 7:**

  Let R be a relation on the set A. The *powers* R^n, n=1,2,3,…, are defined recursively by $R^1=R$, and $R^{n+1}=R^n \circ R$.

  The definition shows that $R^2 = R \circ R$, $R^3 = R^2 \circ R = (R \circ R) \circ R$, *and so on.*

- **Example 22:** Let R = {(1, 1), (2, 1), (3, 2), (4, 3)}. Find the powers R^n, n = 2, 3 4,. .

  Solution: Because R^2 = R o R, we find that R^2 = {(1, 1), (2, 1), (3, 1), (4, 2)} . Furthermore, because $R^3 = R^2$ oR , R^3 = {(1, 1), (2, 1), (3, 1), (4, 1)}. Additional computation shows that R^4 is the same as R^3 , so R^4 = (1, 1), (2, 1), (3, 1), (4, 1)}. It also follows that $R^n = R^3$ for n = 5 , 6, 7, ... The reader should verify this.

- **Theorem 1:** The relation R on a set A is transitive if and only if $R^n \subseteq R$ for n=1,2,3,…

8.2 n-ary Relations and Their Applications

8.2 n-ary Relations and Their Applications

- ✓ Relationship among elements of more than 2 sets often arise: n-ary relations.
- ✓ Used to represent computer databases.
- **Definition 1:**

  Let $A_1, A_2, \ldots, A_n$ be sets. An *n-ary relations* on these sets is a subset of $A_1 \times A_2 \times \ldots \times A_n$. The sets $A_1, A_2, \ldots, A_n$ are called the *domains* of the relation, and n is called its *degree*.
- **Example 1:** Let R be the relation on N * N * N consisting of triples (a, b, c) where a, b, and c are integers with a<b<c. Then (1,2,3) $\in$ R, but (2,4,3) $\notin$ R. The degree of this relation is 3. Its domains are equal to the set of integers.
- **Databases and Relations:**
  - ✓ ***Relational database model*** has been developed for information processing
  - ✓ A database consists of ***records***, which are n-tuples made up of ***fields***
  - ✓ The fields contains information such as:

    Name , Student # ,Major ,Grade point average of the student
  - ✓ The relational database model represents a database of records or n-ary relation
  - ✓ The relation is R(Student-Name, Id-number, Major, GPA)
  - ✓ Relations used to represent databases are called ***tables***, each column of the table corresponds to an attribute of the database.

8.2 n-ary Relations and Their Applications

- ✓ A domain of an n -ary relation is called a ***primary key*** when the value of the n -tuple from this domain determines the n -tuple. That is, a domain is a primary key when no two n -tuples in the relation have the same value from this domain.
- ✓ Records are often added to or deleted from databases. Because of this, the property that a domain is a primary key is time-dependent.
- ✓ Consequently, a primary key should be chosen that remains one whenever the database is changed.
- ✓ The current collection of n -tuples in a relation is called the ***extension*** of the relation.
- ✓ The more permanent part of a database, including the name and attributes of the database, is called its ***intension***.
- ✓ Combinations of domains can also uniquely identify n -tuples in an n-ary relation.
- ✓ When the values of a set of domains determine an n-tuple in a relation, the Cartesian product of these domains is called a ***composite key***.

8.2 n-ary Relations and Their Applications

TABLE A: Students

Students Names	ID #	Major	GPA
Smith	3214	Mathematics	3.9
Stevens	1412	Computer Science	4.0
Rao	6633	Physics	3.5
Adams	1320	Biology	3.0
Lee	1030	Computer Science	3.7

8.2 n-ary Relations and Their Applications

Operations on n-ary Relations

- There are varieties of operations that are applied on n-ary relations in order to create new relations that answer eventual queries of a database**.**
- **Definition 2:**

  Let R be an n-ary relation and C a condition that elements in R may satisfy. Then the selection operator SC maps the n-ary relation R to the n-ary relation of all n-tuples from R that satisfy the condition C.
- **Example 7:** To find the records of computer science majors in the n-ary relation R shown in Table 1 , we use the operator S_{C_1} where C_1 is the condition Major = "Computer Science." The result is the two 4-tuples (Ackermann, 231455, Computer Science, 3.88) and (Chou, 102147, Computer Science, 3.49). Similarly, to find the records of students who have a grade point average above 3.5 in this database, we use the operator S_{C_2}, where C_2 is the condition GPA >3.5 . The result is the two 4-tuples (Ackermann, 231455, Computer Science, 3.88) and (Rao, 678543, Mathematics, 3.90). Finally, to find the records of computer science majors who have a GPA above 3.5, we use the operator S_{C_3} where C_3 is the condition (Major = "Computer Science" /\ GPA > 3.5). The result consists of the single 4-tuple (Ackermann, 231455, Computer Science, 3.88).

8.2 n-ary Relations and Their Applications

- **Definition 3:**
  The *projection* $P_{i_1 \, i_2 \dots . i_m}$ where $i_1 < i_2 < \dots . < i_m$,maps the n-tuple ($a_1$,$a_2$,…,$a_n$) to the m-tuple ($a_{i_1}$, a_{i_2},…, a_{i_m}), where m<=n.
- In other words, the projection deletes n – m of the components of n-tuple, leaving the i_1th, i_2th, …, and i_mth components.
- **EXAMPLE 8:** What results when the projection $P_{1,3}$ is applied to the 4-tuples (2, 3, 0, 4), (Jane Doe, 23411101, Geography, 3.14), and (a_1,a_2, a_3 , a_4) ?
  Solution: The projection $P_{1,3}$ sends these 4-tuples to (2, 0), (Jane Doe, Geography), and (a_1, a_3), respectively.
- **Example 9:** What relation results when the projection $P_{1,4}$ is applied to the relation in Table A?
  Solution: When the projection $P_{1,4}$ is used, the second and third columns of the table are deleted, and pairs representing student names and GPA are obtained. Table B displays the results of this projection.

Students Names	GPA
Smith	3.9
Stevens	4.0
Rao	3.5
Adams	3.0
Lee	3.7

TABLE B: GPAs

8.2 n-ary Relations and Their Applications

- **Definition 4:**

  Let R be a relation of degree m and S a relation of degree n. The *join* $J_p(R,S)$, where $p \le m$ and $p \le n$, is a relation of degree $m+n-p$ that consists of all $(m+n-p)$-tuples $(a_1,a_2,\ldots,a_{m-p},c_1,c_2,\ldots,c_p,b_1,b_2,\ldots,b_{n-p})$, where the m-tuple $(a_1,a_2,\ldots,a_{m-p}, c_1,c_2,\ldots,c_p)$ belongs to R and the n-tuple $(c_1,c_2,\ldots,c_p,b_1,b_2,\ldots,b_{n-p})$ belongs to S.
- **EXAMPLE 11:** What relation results when the operator J_2 is used to combine the relation displayed in tables C and D?

Professor	Dpt	Course #
Cruz	Zoology	335
Cruz	Zoology	412
Farber	Psychology	501
Farber	Psychology	617
Grammer	Physics	544
Grammer	Physics	551
Rosen	Computer Science	518
Rosen	Mathematics	575

TABLE C: Teaching Assignments

Dpt	Course #	Room	Time
Computer Science	518	N521	2:00 PM
Mathematics	575	N502	3:00 PM
Mathematics	611	N521	4:00 PM
Physics	544	B505	4:00 PM
Psychology	501	A100	3:00 PM
Psychology	617	A110	11:00 AM
Zoology	335	A100	9:00 AM
Zoology	412	A100	8:00 AM

TABLE D: Class Schedule

8.2 n-ary Relations and Their Applications

Solution: The join J_2 produces the relation shown in Table E

Professor	Dpt	Course #	Room	Time
Cruz	Zoology	335	A100	9:00 AM
Cruz	Zoology	412	A100	8:00 AM
Farber	Psychology	501	A100	3:00 PM
Farber	Psychology	617	A110	11:00 AM
Grammer	Physics	544	B505	4:00 PM
Rosen	Computer Science	518	N521	2:00 PM
Rosen	Mathematics	575	N502	3:00 PM

Table E: Teaching Schedule