Heaven's light is our guide"

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Network Security

Course No. : 305

Chapter 6: Advance Counting Technique

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6.4 Generating Functions

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- **Use of Generating Function:**
  - ✓ To represent sequences efficiently by coding the terms of a sequence as coefficients of powers of a variable x in a formal power series.
  - ✓ Used to solve many types of counting problems
  - ✓ Used to solve recurrence relations
  - ✓ Used to prove combinatorial identities
  - ✓ A helpful tool for studying many properties of sequences
- **Definition 1:**

  The generating function for the sequence $a_0, a_1, \ldots, a_n, \ldots$ of real numbers is the infinite series:

  $$G(x) = a_0 + a_1 x + \ldots + a_k x^k + \ldots = \sum_{k=0}^{\infty} a_k x^k.$$

- ❖ **Remark:** The generating function for $\{a_k\}$ given in Definition 1 is sometimes called the ordinary generating function of $\{a_k\}$ to distinguish it from other types of generating functions for this sequence.
- **Example 1:** The generating functions for the sequences $\{a_k\}$ with $a_k = 3$, $a_k = k + 1$, and $a_k = 2^k$ are $\sum_{k=0}^{\infty} 3x^k$, $\sum_{k=0}^{\infty} (k+1)x^k$ and $\sum_{k=0}^{\infty} 2^k x^k$

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- **Example 2:** What is the generating function for the sequence 1, 1, 1, 1, 1, 1?

  Solution: The generating function of 1, 1, 1, 1, 1, 1 is

$$G(x) = 1 + x + x^2 + x^3 + x^4 + x^5.$$

$$= (x^6 -1) / (x -1)$$

<u>Useful Facts about Power Series</u>

- ✓ When generating functions are used to solve counting problems, they are usually considered to be ***formal power series***.
- ✓ Convergence Problems are ignored.
- ✓ A function has a unique power series around $x = 0$ will be important.
- **Example 4:** The function $f(x) = 1/(1-x)$ is the generating function of the series: 1,1,1,… because

  $1/(1-x) = 1 + x + x^2 + \dots$ for $|x| < 1$.
- **Example 5:** The function $f(x) = 1/(1-ax)$ is the generation function of the series: 1, a, a^2, a^3,… because

  $1/(1-ax) = 1 + ax + a^2x^2 + \dots$

  when lax l < 1 , or equivalently, for $|x| < 1 / |a|$ for $a \neq 0$.

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- **Theorem 1:** Let $f(x)=\sum_{k=0}^{\infty} a_k x^k$ and $g(x)=\sum_{k=0}^{\infty} b_k x^k$. Then
  $f(x) + g(x) = \sum_{k=0}^{\infty} (a_k+b_k)x^k$ and $f(x)g(x) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} a_j b_{k-j}\right)x^k$ and.
  It's valid only for power series that converge in an interval.
- **EXAMPLE 6:** Let $f(x) = 1/(1-x)^2$. Use Example 4 to find the coefficients a_0, a_1, a_2… . .. in the expansion $f(x) = \sum_{k=0}^{\infty} a_k x^k$
  Solution: From Example 4 we see that
  $1/(1-x) = 1 + x + x^2+\dots$
  Hence, from Theorem 1 , we have
  $1/(1-x)^2 = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} 1\right)x^k = \sum_{k=0}^{\infty} (k+1)x^k$
- **Definition 2:**
  Let u be a real number and k a nonnegative integer. Then the extended binomial coefficient (u,k) is defined by (u,k)=u(u-1)…(u-k+1)/k! if k>0, or 1 if k=0.
- **EXAMPLE 7:** Find the values of the extended binomial coefficients (-2, 3) and (1/2, 3) .
  Solution: Taking u = -2 & k = 3 in Definition 2 gives us (-2, 3) = $\frac{(-2)(-3)(-4)}{3!}$ = -4.
  Similarly, taking u = 1/2 and k = 3 gives us (1/2, 3) = $\frac{(1/2)(1/2-1)(1/2-2)}{3!} = \frac{1}{16}$

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- **Theorem 2: (The Extended Binomial Theorem)**

  Let x be a real number with |x| < 1 and let u be a real number. Then

  $$(1+x)^u = \Sigma_{k=0..\infty}(u,k)\, x^k.$$