07-02-2018 : 2C : Wednesday

Matrix: A system of any mn numbers arranged in a rectangular array of m rows and n columns is called a matrix of order $m \times n$

Column matrix: If there is only one column in a matrix, it is called a column matrix.

Example:

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

Row matrix: If in a matrix, there is only one row it is called a row matrix.

Example:

$$\begin{bmatrix} 1, & 2 & 3 \end{bmatrix}$$

Square matrix: If the number of rows and columns of a matrix are equal, then the matrix is of order $n \times n$ and is called a square matrix of order n.

Example:

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 5 & 2 \\ 7 & 6 & 9 \end{bmatrix}$$

Idempotent matrix: An idempotent matrix is a matrix which, when multiplied by itself, yields itself.

Example:

$$\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

Nilpotent matrix: A nilpotent matrix is a square matrix N such that $N^k = 0$ for some positive integer k. The smallest such k is sometimes called the index of N.

Example:

$$\begin{bmatrix} 0 & 2 & 1 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rotation matrix: A rotation matrix is a matrix that is used to perform a rotation in Euclidean space.

Example:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Transpose of a matrix: The transpose of a matrix is an operator which ~~filp~~ ~~filp~~ flips a matrix over its diagonal, that is it switches the row and ~~t~~ column indices of the matrix by producing another matrix.

Example:

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 & 6 \end{bmatrix} \xrightarrow{\text{Transpose}} \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ 4 & 6 \end{bmatrix}$$

Symmetric matrix: A symmetric matrix is a square matrix that is equal to its transpose.

Example:

$$\begin{bmatrix} 1 & 7 & 3 \\ 7 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

Skew-symmetric matrix: A skew-symmetric matrix is a square matrix whose transpose equals its negative.

Example:

$$\begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & -4 \\ 1 & 4 & 0 \end{bmatrix}$$

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Upper triangular matrix: A square matrix is called upper triangular matrix if all the entries below the main diagonal are zero.

Example:

$$A \begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & 9 \\ 0 & 0 & 6 \end{bmatrix}$$

~~Low~~ Lower triangular matrix: A square matrix is called lower triangular matrix if all the entries above the main diagonal are zero.

Example:

$$\begin{bmatrix} 2 & 0 & 0 \\ 4 & 6 & 0 \\ 5 & 9 & 3 \end{bmatrix}$$

14-02-2018: 3C: Wednesday

Rank of a matrix: A non-zero matrix A of order $m \times n$ is said to have rank r if at least one of its $r \times r$ square minor is different from zero while all the other minors (if any) of order $(r+1) \times (r+1)$ are zero.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}$$

Minor: Determinant of a square sub-matrix is called minor.

Example: Determinant of $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$ is a minor of A matrix.

There are three way of determine rank:

i) Minor test

ii) Normal form

iii) ~~Ekolon form~~ Echelon form

Ex:-1: Using minor test procedure find the rank of the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & -1 & 2 & 1 \\ 2 & 4 & 6 & 8 \end{bmatrix}$$

Solution:

Minors of order (3×3) are:

$$\begin{bmatrix} 2 & 3 & 4 \\ -1 & 2 & 1 \\ 4 & 6 & 8 \end{bmatrix} = 0 \qquad \begin{bmatrix} 1 & 3 & 4 \\ 1 & 2 & 1 \\ 2 & 6 & 8 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & 2 \\ 2 & 4 & 6 \end{bmatrix} = 0 \qquad \begin{bmatrix} 1 & 2 & 4 \\ 1 & -1 & 1 \\ 2 & 4 & 8 \end{bmatrix} = 0$$

Minors of order (2×2) are:

$$\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} = -1 - 2 = -3 \neq 0$$

Since there is a minor of order (2×2) which is different from zero and all the other minors of order (3×3) are zero. Therefore, the rank of the given matrix is ~~zero~~ two. $\rho(A) = 2$.

Ex:-2: Using minor test procedure find the rank of the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 3 & 6 & 3 \end{bmatrix}$$

Solution: Minors of order (3×3) are:

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 3 & 6 & 3 \end{bmatrix} = 0$$

Minors of order (2×2) are:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = 0, \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} = 0, \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} = 0, \begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = 0, \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} = 0$$

Since ~~there is no~~ all the minor of order (3×3) and (2×2) are zero.

Therefore, the rank of the given matrix, $\rho(A) = 1$.

Elementary row and column operators:

1. $R_{ij} \rightarrow$ Interchanging i-th and j-th rows.

2. $R_i(k) \rightarrow$ Multiplying each element of i-th row by a non-zero constant k.

3. $R_{ij}(k) \rightarrow$ Multiplying each element of j-th row by a non-zero constant k and then adding with the corresponding elements of i-th row.

* For column operation R is replaced by C.

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24-02-2018: 4C: Saturday

Normal form:

A non-zero matrix A of order $(m \times n)$ can be reduced by using elementary row and column operators to one of the four following forms:

i) $\begin{bmatrix} I_r & \bar{0} \\ \bar{0} & \bar{0} \end{bmatrix}$, ii) $\begin{bmatrix} I_r \\ \bar{0} \end{bmatrix}$

iii) $\begin{bmatrix} I_r & \bar{0} \end{bmatrix}$, iv) $\begin{bmatrix} I_r \end{bmatrix}$

These four forms are called normal form of A.

Let,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A \sim \begin{bmatrix} I_2 & \bar{0} \\ \bar{0} & \bar{0} \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B \sim \begin{bmatrix} I_3 & \bar{0} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C \sim \begin{bmatrix} I_3 \end{bmatrix}$$

In one non-zero matrix how many identity matrix is hidden is the work of normal form.

Rules for reducing a matrix to the normal form:

i) Make $A_{11}=1$ and other elements of column-1 and row-1 are zero.

ii) Make $A_{22}=1$ and other elements of column-2 and ~~under~~ below and right of A_{22} are zero.

iii) Make $A_{33}=1$ and other elements of ~~elo~~ column-3 below A_{33} are zero and such as A_{ii}.

Ex:-1: Raduce the following matrix to the normal form and hence find its rank:

$$A = \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$$

Solution: If any row or column is totally zero, then the row or column need to interchange with last row or column.

$$A = \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix} \underset{C_1\left(\frac{1}{2}\right)}{\sim} \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 1 & 3 & 7 & 5 \\ 1 & 5 & 11 & 6 \end{bmatrix}$$

$$\underset{\substack{C_{31}(-3) \\ C_{41}(-4)}}{\overset{C_{21}(1)}{\sim}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 4 & 1 \\ 1 & 4 & 4 & 1 \\ 1 & 6 & 8 & 2 \end{bmatrix} \underset{R_{41}(-1)}{\overset{R_{31}(-1)}{\sim}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 4 & 1 \\ 0 & 4 & 4 & 1 \\ 0 & 6 & 8 & 2 \end{bmatrix}$$

$$\overset{C_{24}}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 3 \\ 0 & 1 & 4 & 4 \\ 0 & 2 & 8 & 6 \end{bmatrix} \underset{C_{42}(-3)}{\overset{C_{32}(-4)}{\sim}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

$$\underset{R_{42}(-2)}{\overset{R_{32}(-1)}{\sim}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{C_{34}}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\sim \begin{bmatrix} I_3 & \bar{0} \\ \bar{0} & 0 \end{bmatrix}$ which is the required normal form of the given matrix A. The rank of the given matrix, $\rho(A) = 3$

Ex:- 2: Reduce the following matrix to the normal form and hence find its rank:

Solution:

$$A = \begin{bmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & -4 \\ 5 & 3 & 3 & 11 \end{bmatrix} \underset{C_{41}(-6)}{\overset{C_{21}(1)}{\underset{C_{31}(-3)}{\sim}}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 4 & -6 & -10 \\ 5 & 8 & -12 & -19 \end{bmatrix}$$

$$\underset{R_{21}(-1)}{\overset{R_{31}(-5)}{\sim}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & -6 & -10 \\ 0 & 8 & -12 & -19 \end{bmatrix} \overset{C_2(\frac{1}{4})}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -10 \\ 0 & 2 & -12 & -19 \end{bmatrix}$$

$$\underset{C_{42}(10)}{\overset{C_{32}(6)}{\sim}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \overset{R_{32}(-1)}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\overset{C_{34}}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} I_3 & \bar{O} \end{bmatrix}$$

which is the required normal form of the given matrix A. The rank of the given matrix, $\rho(A) = 3$

25-02-2018 : 4D : Sunday

Ex:-3 : Find the normal form of the following matrix and hence find its rank:

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}.$$

Solution:

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix} \overset{R_{12}}{\sim} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$$\underset{C_{41}(-1)}{\overset{C_{31}(-1)}{\sim}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & -3 & -1 \\ 1 & 1 & -3 & -1 \end{bmatrix} \underset{R_{41}(-1)}{\overset{R_{31}(-3)}{\sim}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

$$\underset{C_{42}(1)}{\overset{C_{32}(3)}{\sim}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \underset{R_{42}(-1)}{\overset{R_{32}(-1)}{\sim}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} I_2 & \bar{0} \\ \bar{0} & \bar{0} \end{bmatrix}$$

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This is the required normal form of the given matrix and hence rank of the matrix, $\rho(A) = 2$.

Echelon form : In every row the first non-zero element need to be 1. and below it every element of row will be zero. The number of non-zero row will be the rank of the matrix. Only row operation is used in it.

Ex:-1: Find the echelon form of the following matrix and hence find its rank:

$$A = \begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \overset{R_{12}}{\sim} \begin{bmatrix} 1 & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$\underset{R_{31}(-1)}{\overset{R_{21}(2)}{\sim}} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix} \overset{R_2\left(\frac{1}{3}\right)}{\sim} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$\underset{R_{32}(2)}{\overset{R_{42}(-1)}{\sim}} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the required echelon form of the given matrix and since there are two non-zero row, therefore the rank of the given matrix, $\rho(A) = 2$

Ex:-2: Find the echelon form of the following matrix and hence find its rank:

$$A = \begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & 1 & -2 & 3 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & 1 & -2 & 3 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix} \underset{\substack{R_{31}(-4) \\ R_{41}(-5)}}{\overset{R_{21}(-1)}{\sim}} \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 3 & -3 & 4 \\ 0 & 9 & -9 & 12 \\ 0 & 3 & -3 & 4 \end{bmatrix}$$

$$R_2\left(\frac{1}{3}\right)\begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & -1 & \frac{4}{3} \\ 0 & 9 & -9 & 12 \\ 0 & 3 & -3 & 4 \end{bmatrix} \underset{R_{42}(-3)}{\overset{R_{32}(-9)}{\sim}} \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & -1 & \frac{4}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the required echelon form of the given matrix and since there are two non-zero row, therefore the rank of the given matrix, $\rho(A) = 2$.

03-03-2018 : 5C : Saturday

Ex-3: Reduce the following matrix to the echelon form and hence find its rank:

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} \overset{R_{12}}{\sim} \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix},$$

$$\xrightarrow[R_{41}(-6)]{R_{21}(-2),\ R_{31}(-3)} \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} \xrightarrow{R_{23}(-1)} \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$$\xrightarrow[R_{42}(-9)]{R_{32}(-4)} \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix} \xrightarrow{R_3\left(\frac{1}{33}\right)} \begin{bmatrix} 1 & 1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 1 & \frac{2}{3} \\ 0 & 0 & 66 & 44 \end{bmatrix}$$

$$\xrightarrow{R_{43}(-66)} \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the required echelon form of the given matrix and since there are three non-zero row, therefore, the rank of the given matrix $\rho(A) = 3$.

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25-03-2018 : 7C : Sunday

System of Linear Equation

A system of linear equations is a collection of two or more linear equations involving the same set of variables.

Example:

$$3x + 2y - z = 1$$
$$2x - 2y + 4z = -2$$
$$-x + \frac{y}{2} - z = 0$$

is a system of three equations in the three variables x, y, z.

There are two types of linear system:

i) Homogeneous system

ii) Non-homogeneous system

i) Homogeneous system: A system of linear equation is homogeneous if all of the constant terms are zero.

Example:

$$x + 2y = 0$$
$$2x + 4y = 0$$

equivalent matrix equation $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow AX = \bar{O}$

It has unique solution (if rank and order of square matrix ~~are~~ are same) or infinity many solution (if rank is ~~greater~~ less than order of square matrix)

ii) Non-homogeneous system: A system of linear equations is non-homogeneous if any of the constant terms is not zero.

Example:

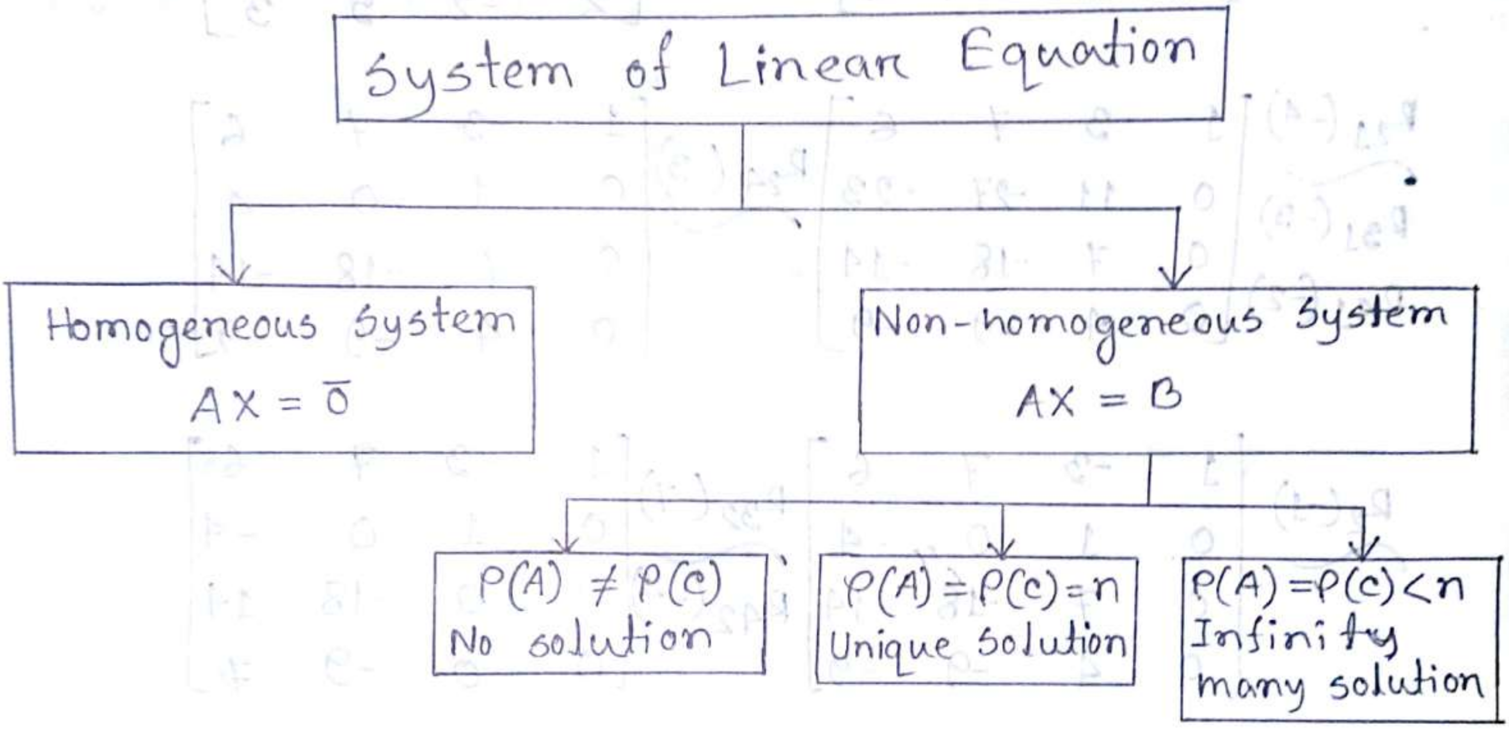
$$x - 2y + 3z = 2$$
$$2x - 3z = 0$$
$$x + y + z = 0$$

equivalent matrix equation

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & -3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow AX = B$

It has no solution (if rank of square matrix is equal to augmented matrix) or unique solution (if rank of square and augmented matrix is equal to the order of square matrix) or infinity many solution (if rank ~~a~~ of square and augmented ~~is~~ are equal but less than order of square matrix)



Here, C is the augmented matrix and n is the order of square matrix.

→ solve by matrix method:

$$2x - 2y + 5z + 3w = 0$$
$$4x - y + z + w = 0$$
$$3x - 2y + 3z + 4w = 0$$
$$x - 3y + 7z + 6w = 0$$

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Solution:
The co-efficient matrix of the given system is

$$A = \begin{bmatrix} 2 & -2 & 5 & 3 \\ 4 & -1 & 1 & 1 \\ 3 & -2 & 3 & 4 \\ 1 & -3 & 7 & 6 \end{bmatrix} \overset{R_{14}}{\sim} \begin{bmatrix} 1 & -3 & 7 & 6 \\ 4 & -1 & 1 & 1 \\ 3 & -2 & 3 & 4 \\ 2 & -2 & 5 & 3 \end{bmatrix}$$

$$\overset{\substack{R_{21}(-4) \\ R_{31}(-3) \\ R_{41}(-2)}}{\sim} \begin{bmatrix} 1 & -3 & 7 & 6 \\ 0 & 11 & -27 & -23 \\ 0 & 7 & -18 & -14 \\ 0 & 4 & -9 & -9 \end{bmatrix} \overset{R_{24}(-3)}{\sim} \begin{bmatrix} 1 & -3 & 7 & 6 \\ 0 & -1 & 0 & 4 \\ 0 & 7 & -18 & -14 \\ 0 & 4 & -9 & -9 \end{bmatrix}$$

$$\overset{R_2(-1)}{\sim} \begin{bmatrix} 1 & -3 & 7 & 6 \\ 0 & 1 & 0 & -4 \\ 0 & 7 & -18 & -14 \\ 0 & 4 & -9 & -9 \end{bmatrix} \overset{\substack{R_{32}(-7) \\ R_{42}(-4)}}{\sim} \begin{bmatrix} 1 & -3 & 7 & 6 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & -18 & 14 \\ 0 & 0 & -9 & 7 \end{bmatrix}$$

$$\overset{R_3\left(-\frac{1}{18}\right)}{\sim} \begin{bmatrix} 1 & -3 & 7 & 6 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -\frac{7}{9} \\ 0 & 0 & -9 & 7 \end{bmatrix} \overset{R_{43}(9)}{\sim} \begin{bmatrix} 1 & -3 & 7 & 6 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -\frac{7}{9} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the echelon ~~of~~ form of the matrix A and hence rank of A, $\rho(A) = 3$

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since $\rho(A) < n$, therefore the given system has infinitely many solutions. In this case number of free variable equal to $n - \rho(A) = 4 - 3 = 1$.

Let, w be the free variable and $w = k$ where k is any real number.

Now, the equivalent system is

$$x - 3y + 7z + 6k = 0$$

$$y - 4k = 0 \Rightarrow y = 4k$$

$$z - \frac{7k}{9} = 0 \Rightarrow z = \frac{7k}{9}$$

$$w = k$$

$$x - (3 \times 4k) + \left(7 \times \frac{7k}{9}\right) + 6k = 0$$

$$\Rightarrow x - 12k + \frac{49k}{9} + 6k = 0$$

$$\Rightarrow x = 12k - \frac{49k}{9} - 6k$$

$$\Rightarrow x = \frac{108k - 49k - 54k}{9}$$

$$\therefore x = \frac{5k}{9}$$

Therefore the solution of the given system is $x = \frac{5k}{9}$, $y = 4k$, $z = \frac{7k}{9}$, $w = k$ where k is any real number.

31-03-2018: 8A: Saturday

Ex:-1: Solve the following system

$$x+y+4z=6$$
$$3x+2y-2z=9$$
$$5x+y+2z=13$$

Solution: The augmented matrix for the given system

$$C=\begin{bmatrix}1 & 1 & 4 & 6\\3 & 2 & -2 & 9\\5 & 1 & 2 & 13\end{bmatrix}\xrightarrow[R_{31}(-5)]{R_{21}(-3)}\begin{bmatrix}1 & 1 & 4 & 6\\0 & -1 & -14 & -9\\0 & -4 & -18 & -17\end{bmatrix}$$

$$\xrightarrow{R_2(-1)}\begin{bmatrix}1 & 1 & 4 & 6\\0 & 1 & 14 & 9\\0 & -4 & -18 & -17\end{bmatrix}\xrightarrow{R_{32}(4)}\begin{bmatrix}1 & 1 & 4 & 6\\0 & 1 & 14 & 9\\0 & 0 & 38 & 19\end{bmatrix}$$

$$\xrightarrow{R_3\left(\frac{1}{38}\right)}\begin{bmatrix}1 & 1 & 4 & 6\\0 & 1 & 14 & 9\\0 & 0 & 1 & \frac{1}{2}\end{bmatrix}$$

This is the echelon form of augmented matrix hence the rank of augmented and ~~m~~ square matrix $\rho(C)=\rho(A)=3=n$.

Since $\rho(A)=\rho(C)=n$, therefore the given system has unique solution.

Now the equivalent system is

$$x+y+4z=6$$

$$y+14z=9$$

$$z=\frac{1}{2}$$

$$\therefore y+\left(14\times\frac{1}{2}\right)=9$$

$$\Rightarrow y=9-7 \quad \therefore y=2$$

$$\therefore x+2+\left(4\times\frac{1}{2}\right)=6$$

$$\Rightarrow x=6-2-2$$

$$\therefore x=2$$

$\therefore$ Therefore the solution for the given system $x=2,\ y=2,\ z=\frac{1}{2}$.

Ex:-2: Solve the following system:

$$2x+4y-z=9$$

$$3x-y+5z=5$$

$$8x+2y+9z=19$$

Solution: The augmented matrix for the given system is

$$C = \begin{bmatrix} 2 & 4 & -1 & 9 \\ 3 & -1 & 5 & 5 \\ 8 & 2 & 9 & 19 \end{bmatrix} \overset{R_{12}(-1)}{\sim} \begin{bmatrix} -1 & 5 & -6 & 4 \\ 3 & -1 & 5 & 5 \\ 8 & 2 & 9 & 19 \end{bmatrix}$$

$$\overset{R_1(-1)}{\sim} \begin{bmatrix} 1 & -5 & 6 & -4 \\ 3 & -1 & 5 & 5 \\ 8 & 2 & 9 & 19 \end{bmatrix} \underset{R_{31}(-8)}{\overset{R_{21}(-3)}{\sim}} \begin{bmatrix} 1 & -5 & 6 & -4 \\ 0 & 14 & -13 & 17 \\ 0 & 42 & -39 & 51 \end{bmatrix}$$

$$\overset{R_2\left(\frac{1}{14}\right)}{\sim} \begin{bmatrix} 1 & -5 & 6 & -4 \\ 0 & 1 & -\frac{13}{14} & \frac{17}{14} \\ 0 & 42 & -39 & 51 \end{bmatrix} \overset{R_{32}(-42)}{\sim} \begin{bmatrix} 1 & -5 & 6 & -4 \\ 0 & 1 & -\frac{13}{14} & \frac{17}{14} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is echelon form of the given matrix hence the rank of the augmented matrix, $\rho(C) = \rho(A) = 2$

Since, $\rho(C) = \rho(A) < n$ therefore the given system has infinitely many solution. In this case number of free variable $= n - \rho(A) = 3 - 2 = 1$.

Let,

z be the free variable ~~also~~ and $z = k$ where k is any real number.

Now, the equivalent system is

$$x - 5y + 6k = -4$$

$$y - \frac{13k}{14} = \frac{17}{14}$$

$$\Rightarrow y = \frac{13k+17}{14}$$

$$\therefore x - \left(5 \times \frac{13k+17}{14}\right) + 6k = -4$$

$$\Rightarrow x = \frac{65k+85}{14} - 6k - 4$$

$$\Rightarrow x = \frac{65k+85-84k-56}{14}$$

$$\therefore x = \frac{29-19k}{14}$$

Therefore, the solution for the given system $x = \frac{29-19k}{14}$, $y = \frac{13k+17}{14}$, $z = k$ where k is any real number.

Ex:3: Solve the following system

$$x + y + z = 6$$

$$x + 2y + 3z = 14$$

$$x + 4y + 7z = 30$$

Solution: The augmented matrix for the given ~~matrix~~ system

$$C = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{bmatrix} \xrightarrow[R_{31}(-1)]{R_{21}(-1)} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 6 & 24 \end{bmatrix}$$

$$\xrightarrow{R_{32}(-3)} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the echelon form of the augmented matrix hence the rank of augmented matrix, $\rho(C) = \rho(A) = 2$

Since, $\rho(C) = \rho(A) < n$ therefor the given system has infinitely many solutions. In this case number of free variable $= n - \rho(A)$ $= 3-2 = 1$.

Let, z be the free variable and $z = k$ where k is any real number.

$$x + y + k = 6$$

$$y + 2k = 8$$

$$\Rightarrow y = 8 - 2k$$

$\therefore x+8-2k+k=6$

$\Rightarrow x = 6-8+k$

$\therefore x = k-2$

Therefore, the solution for the given system $x=k-2,\ y=8-2k,\ z=k$ where k is any real number.

07-04-2018 : 9A : Saturday

Characteristic roots and characteristic vectors

Eigen values and Eigen vectors

Let, A be a non-zero square matrix of order $n \times n$. Then

① The matrix $A-\lambda I$ is called the characteristic matrix of A.

② The determinant $|A-\lambda I|$ is called the ~~charaters~~ characteristic polynomial of A

③ The equation $|A-\lambda I|=0$ is called characteristic equation of A.

④ The values of λ for which the equation $|A-\lambda I|=0$ is satisfied are called the characteristic roots or eigen values of A.

⑤ Corresponding to each characterstic root λ there is a corresponding non-zero vector X which satisfies the equation $(A-\lambda I)X=\bar{0}$. These non-zero vector X are called the characterstics vector or eigen vector of A.

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

$$A-\lambda I = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 3-\lambda \end{bmatrix}$$

If λ is subcripted from diagonal values then it is characterstic matrix.

$$A-\lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 3-\lambda \end{bmatrix}$$

$$= (1-\lambda)(3-\lambda) - 4$$

$$= 3-\lambda-3\lambda+\lambda^2-4$$

$$= \lambda^2-4\lambda-1$$

It is the characterstic polynomial.

$\lambda^2-4\lambda-1=0$ is a characterstic equation.

The power of λ = The ~~numbe~~ oreder ~~of~~ of identity matrix

⑥ ~~2~~

$\lambda^2 - 4\lambda - 1 = 0$

$\therefore \lambda = \dfrac{4 \pm \sqrt{16+4}}{2}$

$= \dfrac{4 \pm 2\sqrt{5}}{2}$

$= 2 \pm \sqrt{5}$ are eigen values

Ex:-1: Find all the eigen values and eigen vectors of the following matrix:

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

Solution: The characteristic equation for the given matrix is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 3 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 5-\lambda & 1 \\ 1 & 1 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (3-\lambda)\{(5-\lambda)(3-\lambda) - 1\} - 1(3-\lambda-1) + 1(1-5+\lambda) = 0$

$\Rightarrow (3-\lambda)(15 - 8\lambda + \lambda^2 - 1) + \lambda - 2 + \lambda - 4 = 0$

$\Rightarrow (3-\lambda)(\lambda^2 - 8\lambda + 14) + 2\lambda - 6 = 0$

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$$\Rightarrow 3\lambda^2 - 24\lambda + 42 - \lambda^3 + 8\lambda^2 - 14\lambda + 2\lambda - 6 = 0$$

$$\Rightarrow -\lambda^3 + 11\lambda^2 - 36\lambda + 36 = 0$$

$$\Rightarrow \lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

$$\Rightarrow (\lambda-6)(\lambda-3)(\lambda-2) = 0$$

$$\therefore \lambda = 2, 3, 6$$

These are the eigen values of A.

For $\lambda = 2$, the equation $(A - \lambda I)X = \bar{0}$ becomes

$$\left(\begin{bmatrix} 3 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 3 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 + x_2 + x_3 \\ x_1 + 3x_2 + x_3 \\ x_1 + x_2 + x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which implies $\quad x_1 + x_2 + x_3 = 0$

$$x_1 + 3x_2 + x_3 = 0$$

$$x_1 + x_2 + x_3 = 0$$

the co-efficient matrix for the above system is

$$A_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow[R_{31}(-1)]{R_{21}(-1)} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2(\frac{1}{2})} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This is the echelon form of the matrix hence the rank of the matrix, $\rho(A_1) = 2$

Since $\rho(A_1) < n$, therefore the above system has infinitely many solution. In this case, number of free variable $= n - \rho(A_1) = 3-2 = 1$

Let, x_3 be the free variable and $x_3 = k$ where k is any real number.

$$x_1 + x_2 + k = 0$$

$$x_2 = 0$$

$$\therefore x_1 + 0 + k = 0$$

$$\therefore x_1 = -k$$

The eigen vector corresponding to the eigen values $\lambda = 2$ is

$$X_{\lambda=2} = \begin{bmatrix} -k \\ 0 \\ k \end{bmatrix}$$

For $\lambda = 3$, the equation $(A - \lambda I)X = \bar{0}$ becomes

$$\left(\begin{bmatrix} 3 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 3 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_2 + x_3 \\ x_1 + 2x_2 + x_3 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which implies

$$x_2 + x_3 = 0$$
$$x_1 + 2x_2 + x_3 = 0$$
$$x_1 + x_2 = 0$$

the co-efficient matrix of the above system is.

$$A_2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \xrightarrow{R_{31}} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_{21}(-1)} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_{32}(-1)} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

This is the echelon form of the matrix hence the rank of the matrix, $\rho(A_2) = 2$.

Since $\rho(A_2) < n$, therefore the above system has infinitely many solution. In this case, number of free variable $= n - \rho(A_2) = 3-2 = 1$

Let, x_3 be the free variable and $x_3 = k$ where k is any real number.

$$x_1 + x_2 = 0 \qquad \therefore x_1 - k = 0$$
$$x_2 + k = 0 \qquad \therefore x_1 = k$$
$$\Rightarrow x_2 = -k$$

The eigen vector corresponding to the eigen values $\lambda = 3$ is

$$X_{\lambda=3} = \begin{bmatrix} k \\ -k \\ k \end{bmatrix}$$

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For $\lambda = 6$, the equation $(A - \lambda I)X = \bar{O}$ becomes

$$\left(\begin{bmatrix} 3 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 3 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -3 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -3x_1 + x_2 + x_3 \\ x_1 - x_2 + x_3 \\ x_1 + x_2 - 3x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which implies

$$-3x_1 + x_2 + x_3 = 0$$
$$x_1 - x_2 + x_3 = 0$$
$$x_1 + x_2 - 3x_3 = 0$$

the coefficient matrix of the above system is

$$A_3 = \begin{bmatrix} -3 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -3 \end{bmatrix} \overset{R_{21}}{\sim} \begin{bmatrix} 1 & -1 & 1 \\ -3 & 1 & 1 \\ 1 & 1 & -3 \end{bmatrix}$$

$$\underset{R_{31}(-1)}{\overset{R_{21}(3)}{\sim}} \begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 4 \\ 0 & 2 & -4 \end{bmatrix} \overset{R_2(-\frac{1}{2})}{\sim} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 2 & -4 \end{bmatrix}$$

$$R_{32}(-2)\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

This is the echelon form of the matrix hence the rank of the matrix, $\rho(A_3) = 2$

Since $\rho(A_3) < n$, therefore the above system has infinitely many solution. In this case, number of free variable $= n - \rho(A_3) = 3-2 = 1$

Let, x_3 be the free variable and $x_3 = k$ where k is any real number.

Now the equivalent system is

$$\cancel{x_1 + x_1 - x_2 + x_3}$$

$$x_1 - x_2 + k = 0$$

$$x_2 - 2k = 0$$

$$\therefore x_2 = 2k$$

$$\therefore x_1 - 2k + k = 0 \Rightarrow x_1 = k$$

The eigen ~~values~~ vectors corresponding to the eigen value $\lambda = 6$ is

$$X_{\lambda=6} = \begin{bmatrix} k \\ 2k \\ k \end{bmatrix}$$

16-04-2018 : 9C : Monday

→ Cayley-Hamilton theory :

- Every square matrix ~~satif~~ satisfies its characteristic equation.

It is used in determining inverse matrix.

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 3 \\ 4 & 5-\lambda \end{vmatrix} = 0$$

$\Rightarrow (2-\lambda)(5-\lambda) - 12 = 0$

$\Rightarrow 10 - 2\lambda - 5\lambda + \lambda^2 - 12 = 0$

$\Rightarrow \lambda^2 - 7\lambda - 2 = 0$

$\therefore A^2 - 7A - 2I = \bar{O}$ where A is a square matrix

$A^2 - 7A - 2I = \bar{O}$

$\Rightarrow (A^{-1} \times A^2) - 7(A^{-1} \times A) - 2(A^{-1} \times I) = A^{-1} \times \bar{O}$

$\Rightarrow A - 7I - 2A^{-1} = \bar{O}$

$\Rightarrow 2A^{-1} = A - 7I$

$$\therefore A^{-1} = \frac{A - 7I}{2}$$

Ex:-1: Verify Cayley-Hamilton theorem for the following matrix and hence find its inverse

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 2 & 2 & 1 \end{bmatrix}$$

Solution: The characteristic equation for the given matrix A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 2 & 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 2 & 2 \\ 1 & 3-\lambda & 1 \\ 2 & 2 & 1-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (1-\lambda)\{(3-\lambda)(1-\lambda)-2\} - 2(1-\lambda-2) + 2(2-6+2\lambda) = 0$$

$$\Rightarrow (1-\lambda)(3-3\lambda-\lambda+\lambda^2-2) + 2\lambda + 2 + 4\lambda - 8 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda + 1 - \lambda^3 + 4\lambda^2 - \lambda + 6\lambda - 6 = 0$$

$$\Rightarrow -\lambda^3 + 5\lambda^2 + \lambda - 5 = 0$$

$$\Rightarrow \lambda^3 - 5\lambda^2 - \lambda + 5 = 0$$

According to Cayley-Hamilton theory,
we have to show that $A^3-5A^2-A+5I=\bar{O}$

$$L.H.S. = A^3-5A^2-A+5I$$

$$= \begin{bmatrix} \cancel{31} & \cancel{62} & \cancel{32} \\ \cancel{31} & \cancel{63} & \cancel{31} \\ \cancel{32} & \cancel{62} & \cancel{31} \end{bmatrix}$$

$$= \begin{bmatrix} 31 & 62 & 32 \\ 31 & 63 & 31 \\ 32 & 62 & 31 \end{bmatrix} - 5\begin{bmatrix} 7 & 12 & 6 \\ 6 & 13 & 6 \\ 6 & 12 & 7 \end{bmatrix}$$

$$- \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 2 & 2 & 1 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 31-35-1+5 & 62-60-2 & 32-30-2 \\ 31-30-1 & 63-65-3+5 & 31-30-1 \\ 32-30-2 & 62-60-2 & 31-35-1+5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \bar{O} = R.H.S.$$

Now pre-multiplying both side by A^{-1} we get

$$(A^{-1}\times A^{3}) - 5\,(A^{-1}\times A^{2}) - (A^{-1}\times A) + 5\,(A^{-1}\times I) = \bar{O}$$

$$\Rightarrow A^{2} - 5A - I + 5A^{-1} = \bar{O}$$

$$\Rightarrow 5A^{-1} = 5A + I - A^{2}$$

$$\therefore A^{-1} = A + \left(\frac{I - A^{2}}{5}\right)$$

$$= \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 2 & 2 & 1 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 7 & 12 & 6 \\ 6 & 13 & 6 \\ 6 & 12 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + \frac{1}{5} - \frac{7}{5} & 2 - \frac{12}{5} & 2 - \frac{6}{5} \\ 1 - \frac{6}{5} & 3 + \frac{1}{5} - \frac{13}{5} & 1 - \frac{6}{5} \\ 2 - \frac{6}{5} & 2 - \frac{12}{5} & 1 + \frac{1}{5} - \frac{7}{5} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{5} & -\frac{2}{5} & \frac{4}{5} \\ -\frac{1}{5} & \frac{3}{5} & \cancel{1}-\frac{1}{5} \\ \frac{4}{5} & -\frac{2}{5} & -\frac{1}{5} \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} -1 & -2 & 4 \\ -1 & 3 & -1 \\ 4 & -2 & -1 \end{bmatrix}$$

21-04-2018 : 10A : Saturday

Field

- $x, y \in \mathbb{R} \therefore x \pm y \in \mathbb{R}$ [$\mathbb{R}$ is closed under addition]
- $0 \in \mathbb{R}$, $x + 0 = 0 + x = x$ [Additive identity]
- $x \in \mathbb{R}$, $-x = \mathbb{R}$ [Additive inverse]
- $x, y \in \mathbb{R}$, $x + y = y + x$
- $x, y \in \mathbb{R}$, $x \cdot y \in \mathbb{R}$ [$\mathbb{R}$ is closed under multiplication]
- $1 \in \mathbb{R}$, $x \cdot 1 = 1 \cdot x = x$ [Multiplicative identity]
- $x, y \in \mathbb{R}$, $x \cdot y = y \cdot x$ [Multiplicatively commoditive]
- $x \in \mathbb{R}$, $x^{-1} \in \mathbb{R}$ [Multiplicative inverse]
- $x(y + z) = xy + yz$
  $(x+y)z = xz + yz$ [Dristributive property]

Field :

Vector Space

Let, V be a non-empty set and K be any scalar field. Now we define two operations on V as:

i) Vector addition: For any $u, v \in V$, $u+v \in V$

ii) Scalar Multiplication: For any $u \in V$ and $k \in K$, $ku \in V$

Then V is called a vector space over the scalar field K if the following axioms hold for any $u, v, w \in V$:

Addition Rules

A_1) $u + (v+w) = (u+v) + w$

A_2) There exist a $0 \in V$ such that $u+0 = 0+u = u$

A_3) For every $u \in V$ there exists an $-u \in V$ such that $u + (-u) = (-u) + u = 0$

A_4) $u+v = v+u$

Multiplication Rule:

m_1) $k(u+v) = ku + kv$ for all $k \in K$

m_2) $(a+b)u = au + bu$ for any $a, b \in K$

M_3) $(ab)u = a(bu)$ for any $a, b \in K$

M_4) $u.1 = 1.u = u$ for the unit scalar $1 \in K$

Example-1: $\mathbb{R}^2$ is a vector space over the scalar field $\mathbb{R}$.

Solution:

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(a,b) : a, b \in \mathbb{R}\}$$

$(a_1, b_1), (a_2, b_2) \in \mathbb{R}$

$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$

$k(a_1, b_1) = (ka_1, kb_1)$

$(0,0) \in \mathbb{R}^2$, $(a,b) \in \mathbb{R}^2$, $(-a,-b) \in \mathbb{R}^2$

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06-05-2018 : 11A : Sunday

→ Linear Combination :

Let, V be a vector space over the scalar field K. The vector $v \in V$ is called a linear combination of vectors $u_1, u_2, \ldots, u_m$ in V if there exist some scalars $a_1, a_2, \ldots, a_m$ in K such that

$$v = a_1 u_1 + a_2 u_2 + \cdots + a_m u_m$$

$$\{(1,0), (0,1)\} \in \mathbb{R}^2$$

$$(2,3) = \underbrace{2}_{a_1}\underbrace{(1,0)}_{u_1} + \underbrace{3}_{a_2}\underbrace{(0,1)}_{u_2}$$

(2,3) is linear combination of (1,0) and (0,1)

Ex-1: Express $v = (1,-2,5)$ in $\mathbb{R}^3$ as a linear combination of $u_1 = (1,1,1)$, $u_2 = (1,2,1)$, $u_3 = (2,-1,1)$

Solution:

Let, $(1,-2,5) = x(1,1,1) + y(1,2,3) + z(2,-1,1)$

$$\Rightarrow (1,-2,5) = (x+y+2z,\ x+2y-z,\ x+3y+z)$$

which implies that

$$x+y+2z = 1$$
$$x+2y-z = -2$$
$$x+3y+z = 5$$

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The augmented matrix of the above system is

$$C = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & 3 & 1 & 5 \end{bmatrix} \underset{R_{31}(-1)}{\overset{R_{21}(-1)}{\sim}} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 2 & -1 & 4 \end{bmatrix}$$

$$\overset{R_{32}(-2)}{\sim} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 5 & 10 \end{bmatrix} \overset{R_3(\frac{1}{5})}{\sim} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

This is the echelon form of the augmented matrix hence the rank of the matrix,

$\rho(C) = \rho(A) = 3$

since, $\rho(A) = \rho(C) \leq n$ therefore the above system has unique solution.

Now, the equivalent system is

$$x + y + 2z = 1$$
$$y - 3z = -3$$
$$z = 2$$

$\therefore y - (3 \times 2) = -3$

$\Rightarrow y - 6 = -3$

$\therefore y = 3$

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$x + 3 + (2\times 2) = 1$

$\Rightarrow x + 3 + 4 = 1$

$\therefore x = -6$

Thus linear combination is

$$(1,-2,5) = -6(1,1,1) + 3(1,2,3) + 2(2,-1,-1)$$

Ex:-2: Express $v = (3,7,-4)$ in $\mathbb{R}^3$ as a linear combination of the vectors $u_1 = (1,2,3)$, $u_2 = (2,3,7)$, $u_3 = (3,5,6)$

Solution:

Let, $(3,7,-4) = x(1,2,3) + y(2,3,7) + z(3,5,6)$

$\Rightarrow (3,7,-4) = (x+2y+3z,\ 2x+3y+5z,\ 3x+7y+6z)$

which implies

$$x + 2y + 3z = 3$$
$$2x + 3y + 5z = 7$$
$$3x + 7y + 6z = -4$$

The augmented matrix of the above system is

$$C = \begin{bmatrix} 1 & 2 & 3 & 3 \\ 2 & 3 & 5 & 7 \\ 3 & 7 & 6 & -4 \end{bmatrix} \xrightarrow[R_{31}(-3)]{R_{21}(-2)} \begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & -3 & -13 \end{bmatrix}$$

$$\xrightarrow{R_2(-1)} \begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & -3 & -13 \end{bmatrix} \xrightarrow{R_{32}(-1)} \begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -4 & -12 \end{bmatrix}$$

$$\xrightarrow{R_3(-\frac{1}{4})} \begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

This is the echelon form of the matrix hence rank of the matrix, $\rho(C) = \rho(A) = 3$

since $\rho(C) = \rho(A) = n$ therefore the system has unique solution.

Now, the equivalent system is,

$$x + 2y + 3z = 3$$

$$y + z = -1$$

$$z = 3$$

$\therefore y + 3 = -1$

$\Rightarrow y = -4$

$\therefore x + \{2 \times (-4)\} + (3 \times 3) = 3$

$\Rightarrow x - 8 + 9 = 3$

$\Rightarrow x = 2$

Thus, linear combination is

$$(3,7,-4) = 2(1,2,3) - 4(2,3,7) + 3(3,5,6)$$

Ex:-3: Express M as a linear combination of the matrices A, B, C where

$$M = \begin{bmatrix} 4 & 7 \\ 7 & 9 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$$

Solution:

Let,

$$\begin{bmatrix} 4 & 7 \\ 7 & 9 \end{bmatrix} = x\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + y\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + z\begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & 7 \\ 7 & 9 \end{bmatrix} = \begin{bmatrix} x+y+z & x+2y+z \\ x+3y+4z & x+4y+5z \end{bmatrix}$$

which implies that.

$$x+y+z = 4$$
$$x+2y+z = 7$$
$$x+3y+4z = 7$$
$$x+4y+5z = 9$$

The augmented matrix of the above system is

$$C = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & 2 & 1 & 7 \\ 1 & 3 & 4 & 7 \\ 1 & 4 & 5 & 9 \end{bmatrix} \begin{matrix} R_{21}(-1) \\ R_{31}(-1) \\ R_{41}(-1) \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 2 & 3 & 3 \\ 0 & 3 & 4 & 5 \end{bmatrix}$$

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$$\xrightarrow[R_{42}(-3)]{R_{32}(-2)} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 4 & -4 \end{bmatrix} \xrightarrow{R_3(\frac{1}{3})} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 4 & -4 \end{bmatrix}$$

$$\xrightarrow{R_{43}(-4)} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the echelon form of the matrix hence the rank of the matrix, $\rho(C) = \rho(A) = 3$

Since, $\rho(A) = \rho(C) = n$, therefore the above system has unique solution.

Now, the equivalend system is

$$x + y + z = 4$$

$$y = 3$$

$$z = -1$$

$$\therefore x + 3 - 1 = 4$$

$$\Rightarrow x = 2$$

Thus, linear combination is

$$\begin{bmatrix} 4 & 7 \\ 7 & 9 \end{bmatrix} = 2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + 3 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - 1 \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$$

Ex:-4: Express $V = 3t^2+5t-5$ as a linear combination of $P_1 = t^2+2t+1$, $P_2 = 2t^2+5t+4$ and $P_3 = t^2+3t+6$

Solution:

Let,

$$(3t^2+5t-5) = x(t^2+2t+1) + y(2t^2+5t+4) + z(t^2+3t+6)$$

$$\Rightarrow 3t^2+5t-5 = (x+2y+z)t^2 + (2x+5y+3z)t + (x+4y+6z)$$

which implies that

$$x+2y+z = 3$$
$$2x+5y+3z = 5$$
$$x+4y+6z = -5$$

The augmented matrix of the above system is

$$C = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 5 & 3 & 5 \\ 1 & 4 & 6 & -5 \end{bmatrix} \underset{R_{31}(-1)}{\overset{R_{21}(-2)}{\sim}} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 3 & -6 \end{bmatrix}$$

$$\overset{R_3(\frac{1}{3})}{\sim} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

This is the echelon form of the matrix hence the rank of the matrix, $\rho(c) = \rho(A) = 3$

Since $\rho(c) = \rho(A) = n$, therefore the above system has unique solution.

Now, the equivalent system is

$$x + 2y + z = 3$$
$$y + z = -1$$
$$z = -2$$

$\therefore y - 2 = -1 \Rightarrow y = 1$

$x + (2 \times 1) + -2 = 3$

$\Rightarrow x + 2 - 2 = 3 \Rightarrow x = 3$

Thus, linear combination is

$$3t^2 + 5t - 5 = 3(t^2 + 2t + 1) + 1(2t^2 + 5t + 4) - 2(t^2 + 3t + 6)$$

08-05-2018 : 11C : Tuesday

Spanning set:

Let, V be a vector over the scale field K. Vectors $u_1, u_2, ----, u_m$ in V are said to form a spanning set of V if every $v \in V$ is a linear combination of the vectors $u_1, u_2, ----, u_m$ that is if there exist some scalars $a_1, a_2, ----, a_m$ in K such that,

$$v = a_1 u_1 + a_2 u_2 + ----- + a_m u_m$$

$(1,0), (0,1) \in \mathbb{R}^2$

$(2,3) = 2(1,0) + 3(0,1)$

$(5,7) = 5(1,0) + 7(0,1)$

spanning set $= \{(1,0), (0,1)\}$

Ex-1: Test whether the following vectors in $\mathbb{R}^3$ form a spanning set or not?

i) $u_1 = (1,1,1)$, $u_2 = (1,1,0)$, $u_3 = (1,0,0)$

ii) $u_1 = (1,2,3)$, $u_2 = (1,3,5)$, $u_3 = (1,5,9)$

Solution:

i) Let, (a,b,c) be any vector in $\mathbb{R}^3$ and

$$(a,b,c) = x(1,1,1) + y(1,1,0) + z(1,0,0)$$

$$\Rightarrow (a,b,c) = (x+y+z,\ x+y,\ x)$$

which implies that

$$x+y+z = a$$
$$x+y = b$$
$$x = c$$

$\therefore \cancel{x+c=}$

$\therefore c+y = b \Rightarrow y = b-c$

$\therefore c+b-c+z = a$

$\Rightarrow z = a-b$

~~Thus~~ Since the above system has unique solution, thus the given vectors form a spanning set of $\mathbb{R}^3$.

ii) Let, (a,b,c) be any vector in $\mathbb{R}^3$ and

$$(a,b,c) = x(1,2,3) + y(1,3,5) + z(1,5,9)$$

$$\Rightarrow (a,b,c) = (x+y+z,\ 2x+3y+5z,\ 3x+5y+9z)$$

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which implies that,

$$x+y+z=a$$
$$2x+3y+5z=b$$
$$3x+5y+9z=c$$

The augmented matrix of the above system is,

$$C=\begin{bmatrix}1 & 1 & 1 & a\\ 2 & 3 & 5 & b\\ 3 & 5 & 9 & c\end{bmatrix} \xrightarrow[R_{31}(-3)]{R_{21}(-2)} \begin{bmatrix}1 & 1 & 1 & a\\ 0 & 1 & 3 & b-2a\\ 0 & 2 & 6 & c-3a\end{bmatrix}$$

$$\xrightarrow{R_{32}(-2)} \begin{bmatrix}1 & 1 & 1 & a\\ 0 & 1 & 3 & b-2a\\ 0 & 0 & 0 & c-2b+a\end{bmatrix}$$

Since $\rho(A) \neq \rho(C)$, therefore the above system has no solution.

Thus the given vectors do not form a spanning set of $\mathbb{R}^3$.

→ Subspace:

Let, V be a vector space over the scalar field K and W be a subset of V. Then W is called a subspace of V if ω is itself a vector space with respect to the operations of vector addition and scalar multiplication on V.

Ex-1: $\mathbb{R}^3$ is a vector space over $\mathbb{R}$. Then the subset
$U = \{(a,b,c) : a=b=c\}$ is a subspace of $\mathbb{R}^3$.

~~24-06-2~~

24-06-2018 : 12A : Sunday

→ Linear Dependence and Independence:

Let, V be a vector space over the scalar field K. The vectors $u_1, u_2, \ldots, u_m$ in V are called linearly dependent if there exist some scalars $a_1, a_2, \ldots, a_m$ in K not all of them are equal to zero such that

$$a_1 u_1 + a_2 u_2 + \cdots + a_m u_m = 0$$

Otherwise the vectors are said to be linearly independent.

Ex: Test whether the following vectors are linearly dependent or not?

(i) $u = (1,1,2)$, $v = (2,3,1)$, $w = (4,5,5)$

(ii) $u = (1,2,5)$, $v = (2,5,1)$, $w = (1,5,2)$

(iii) $u = (1,2,3)$, $v = (2,5,7)$, $w = (1,3,5)$

Solution:

(i) Let, $x(1,1,2) + y(2,3,1) + z(4,5,5) = (0,0,0)$

$\Rightarrow (x+2y+4z,\ x+3y+5z,\ 2x+y+5z) = (0,0,0)$

which implies

$$x + 2y + 4z = 0$$

$$x + 3y + 5z = 0$$

$$2x + y + 5z = 0$$

The coefficient matrix of the above system is,

$$A = \begin{vmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 2 & 1 & 5 \end{vmatrix} \xrightarrow[R_{31}(-2)]{R_{21}(-1)} \begin{vmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & -3 & -3 \end{vmatrix}$$

$$\xrightarrow{R_{32}(3)} \begin{vmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix}$$

which implies that $\rho(A) = 2$. Since $\rho(A) < n$, therefore the above system has infinitely many solutions. Hence, the vectors are linearly dependent.

(ii) Let, $x(1,2,5)+y(2,5,1)+z(1,5,2)=(0,0,0)$

$\Rightarrow (x+2y+z,\ 2x+5y+5z,\ 5x+y+2z)=(0,0,0)$

which implies,

$$x+2y+z=0$$
$$2x+5y+5z=0$$
$$5x+y+2z=0$$

The coefficient matrix of the above system is,

$$A=\begin{vmatrix}1 & 2 & 1\\ 2 & 5 & 5\\ 5 & 1 & 2\end{vmatrix} \xrightarrow[R_{31}(-5)]{R_{21}(-2)} \begin{vmatrix}1 & 2 & 1\\ 0 & 1 & 3\\ 0 & -9 & -3\end{vmatrix}$$

$$\xrightarrow{R_{32}(9)} \begin{vmatrix}1 & 2 & 1\\ 0 & 1 & 3\\ 0 & 0 & 24\end{vmatrix}$$

which implies that $\rho(A)=3$. since $\rho(A)=n$ therefore the above system has a unique solution.

Hence the vectors are linearly independent

(iii) Let,

$$x(1,2,3)+y(2,5,7)+z(1,3,5)=(0,0,0)$$

$$\Rightarrow (x+2y+z,\ 2x+5y+3z,\ 3x+7y+5z)=(0,0,0)$$

which implies,

$$x+2y+z=0$$
$$2x+5y+3z=0$$
$$3x+7y+5z=0$$

The coefficient matrix of the above system is,

$$A=\begin{vmatrix}1&2&1\\2&5&3\\3&7&5\end{vmatrix}\ \xrightarrow[R_{31}(-3)]{R_{21}(-2)}\ \begin{vmatrix}1&2&1\\0&1&1\\0&1&2\end{vmatrix}$$

$$\xrightarrow{R_{32}(-1)}\ \begin{vmatrix}1&2&1\\0&1&1\\0&0&1\end{vmatrix}$$

which implies that $\rho(A)=3$. since $\rho(A)=n$ therefore the above system has ~~infinitely many~~ an unique solution.

Hence the vectors are linearly independent.

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→ Basis of a vector space:

Let, V be a vector space over the scalar ~~field k~~ field K. A set $S = \{v_1, v_2, \ldots, v_n\}$ is a basis of V if it has the following two properties:

① S is linearly independent

② S spans V.

27-06-2018 : 12C : Tuesday :

Ex: Determine whether or not each of the following form a basis of $\mathbb{R}^3$:

① $\{(1,1,1), (1,2,3), (2,-1,1)\}$

② $\{(1,1,2), (1,2,5), (5,3,4)\}$

Solution:

① Let, $x(1,1,1) + y(1,2,3) + z(2,-1,1) = (0,0,0)$

$\Rightarrow (x+y+2z, x+2y-z, x+3y+z) = (0,0,0)$

which implies,

$$x+y+2z=0$$
$$x+2y-z=0$$
$$x+3y+z=0$$

The coefficient matrix of the above system is

$$A=\begin{vmatrix}1&1&2\\1&2&-1\\1&3&1\end{vmatrix}\xrightarrow[R_{31}(-1)]{R_{21}(-1)}\begin{vmatrix}1&1&2\\0&1&-3\\0&2&-1\end{vmatrix}$$

$$\xrightarrow{R_{32}(-2)}\begin{vmatrix}1&1&2\\0&1&-3\\0&0&5\end{vmatrix}$$

which implies that $\rho(A)=3$. since $\rho(A)=n$ thercefore the above system has unique solution and which is $x=0$, $y=0$, $z=0$

Hence, we can say that the vectors are linearly independent.

Let, (a, b, c) be any vector in $\mathbb{R}^3$. Also let

$$x(1,1,1) + y(1,2,3) + z(2,-1,1) = (a,b,c)$$

$$\Rightarrow (x+y+2z,\ x+2y-z,\ x+3y+z) = (a,b,c)$$

which implies,

$$x+y+2z = a$$
$$x+2y-z = b$$
$$x+3y+z = c$$

The augmented matrix for the above system is.

$$C = \left|\begin{array}{cccc} 1 & 1 & 2 & a \\ 1 & 2 & -1 & b \\ 1 & 3 & 1 & c \end{array}\right| \xrightarrow[R_{31}(-1)]{R_{21}(-1)} \left|\begin{array}{cccc} 1 & 1 & 2 & a \\ 0 & 1 & -3 & b-a \\ 0 & 2 & -1 & c-a \end{array}\right|$$

$$\xrightarrow{R_{32}(-2)} \left|\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & 1 & -3 & b-a \\ 0 & 0 & 5 & a-2b+c \end{array}\right|$$

which implies that $\rho(A) = \rho(C) = 3$.
Therefore the above system has a unique solution. Thus, we can say that the vectors do form a spanning set of $\mathbb{R}^3$. Hence, the given vectors do form a basis of $\mathbb{R}^3$.

(ii) Let,

$$x(1,1,2) + y(1,2,5) + z(5,3,4) = (0,0,0)$$

$$\Rightarrow (x+y+5z,\ x+2y+3z,\ 2x+5y+4z) = (0,0,0)$$

which implies

$$x+y+5z=0$$
$$x+2y+3z=0$$
$$2x+5y+4z=0$$

the coefficiend matrix of the above system is.

$$A = \begin{vmatrix} 1 & 1 & 5 \\ 1 & 2 & 3 \\ 2 & 5 & 4 \end{vmatrix} \xrightarrow[R_{31}(-2)]{R_{21}(-1)} \begin{vmatrix} 1 & 1 & 5 \\ 0 & 1 & -2 \\ 0 & 3 & -6 \end{vmatrix}$$

$$R_{32}(-3)\begin{vmatrix} 1 & 1 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{vmatrix}$$

which implies that $\rho(A) = 2$. since $\rho(A) < n$ therefore the above system has infinitely many solution.

Hence, we can say the vectors are linearly dependent.

So, the given vectors do not form a basis of $\mathbb{R}^3$.

→ Dimension of a vector space:

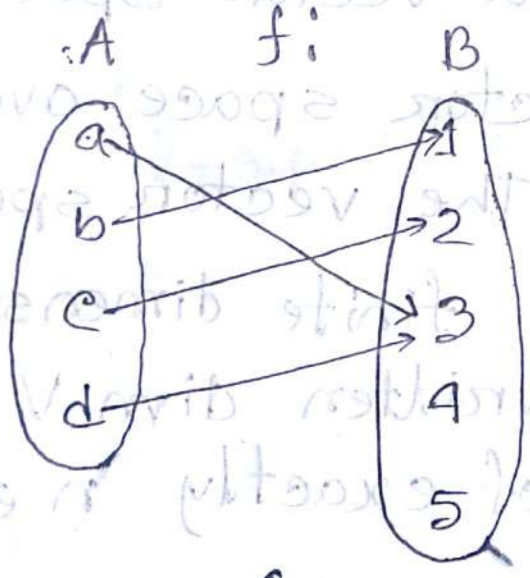
Let, V be a vector space over the scalar field K. The vector space V is said to be of finite dimensional or n-dimensional, written dim V = n if V has a basis of exactly n elements.

→ For example, the vector space $\mathbb{R}^2$ is of two-dimensional because it has a basis $\{(1,0),(0,1)\}$ of exactly two elements.

→ For example, the vector space $\mathbb{R}^3$ is of three dimensional because it has a basis $\{(1,0,0),(0,1,0),(0,0,1)\}$ of exactly three elements.

01-07-2018: 13A: Sunday:

Linear Mapping / Linear Transformation



$f: A \to B$ (f is mapping from A to B) condition for function:

i) Every domain (element of A) has an image.

ii) A domain can not have more than one image (co-domain).

→ $\begin{matrix} a=b \\ f(a)=f(b) \end{matrix}$ or $\left.\begin{matrix} a\neq b \\ f(a)\neq f(b) \end{matrix}\right\}$ if every domain has different image then it is a one-to-one function

→ If no co-domain element is free hence, if every co-domain is an image of any domain then it is onto function.

Let, V and U be two vector spaces over the same scalar field K. Then the mapping $f: V \to U$ is said to be linear. if ~~has~~ it has the following two properties:

i) for any $v, w \in V$, $F(v+w) = F(v) + F(w)$

ii) for any $v \in V, k \in K$, $F(kv) = kF(v)$

Ex: A mapping $F: \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $F(x,y) = (x+y, x)$, Prove that F is linear.

Solution:

Let, $(a_1, b_1), (a_2, b_2) \in \mathbb{R}^2$, then

$$F((a_1, b_1) + (a_2, b_2)) = F(a_1 + a_2,\ b_1 + b_2)$$
$$= (a_1 + a_2 + b_1 + b_2,\ a_1 + a_2)$$
$$= ((a_1 + b_1, a_1) + (a_2 + b_2, a_2))$$
$$= F(a_1, b_1) + F(a_2, b_2)$$

Again for any $k \in K$,

$$F(k(a_1, b_1)) = F(ka_1, kb_1)$$
$$= (ka_1 + kb_1,\ ka_1)$$
$$= k(a_1 + b_1,\ a_1)$$
$$= k\, F(a_1, b_1)$$

Hence, $F: \mathbb{R}^2 \to \mathbb{R}^2$ is linear.

Ex: A mapping $F: \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $F(x,y) = (x, x+y)$. Test the linearity of F.

Solution:

Let, $(a_1, b_1), (a_2, b_2) \in \mathbb{R}^2$. Then

$$F((a_1, b_1) + (a_2, b_2)) = F(a_1+a_2, b_1+b_2)$$
$$= (a_1+a_2, a_1+a_2+b_1+b_2)$$
$$= (a_1, a_1+b_1) + (a_2, a_2+b_2)$$
$$= F(a_1, b_1) + F(a_2, b_2)$$

Again for any $k \in K$

$$F(k(a_1, b_1)) = F(ka_1, kb_1)$$
$$= (ka_1, ka_1+kb_1)$$
$$= k(a_1, a_1+b_1)$$
$$= k\,F(a_1, b_1)$$

Hence, $F: \mathbb{R}^2 \to \mathbb{R}^2$ is ~~lien~~ linear.

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03-07-2018 : 13C6 Tuesday :

Diagonalization

Ex : Consider the matrix $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$

a) Find all eigen values and eigen vectors of A.

b) find a non-singular matrix P such that $D = P^{-1}AP$ is diagonal.

c) Find the positive square root of A that is a matrix B such that $B^2 = A$

d) Compute A^{50} using diagonal factorization

Solution :

a) The characterstic equation of the given matrix is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

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$\Rightarrow (2-\lambda)(3-\lambda) - 2 = 0$

$\Rightarrow 6 - 2\lambda - 3\lambda + \lambda^2 - 2 = 0$

$\Rightarrow \lambda^2 - 5\lambda + 4 = 0$

$\Rightarrow (\lambda - 4)(\lambda - 1) = 0$

$\therefore \lambda = 4, 1$

For $\lambda = 4$, the equation is

$(A - 4I)X = \bar{0}$

$\Rightarrow \begin{bmatrix} 2-4 & 2 \\ 1 & 3-4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \bar{0}$

$\Rightarrow \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -2x + 2y \\ x - y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

which implies

$-2x + 2y = 0$

$x - y = 0$

the coeffient matrix for the above system is,

$A_1 = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \overset{R_2(-\frac{1}{2})}{\sim} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

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$R_{21}(-1)$ $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$\rho(A_4) = 1$ since $\rho(A_2) < n$ therefore the above system has infinitely many solutions. and number of free variable $= n - \rho(A_4)$
$= 2 - 1$
$= 1$

Let, y be the free variable and $y = k$ where k is ~~an~~ any integer.

The equivalent system,

$$x - y = 0 \quad \therefore x = k$$
$$y = k.$$

Hence, $X_{\lambda=4} = \begin{bmatrix} k \\ k \end{bmatrix}$

For a particular case we can take

$$u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda = 1$, the characterstic equation is

$$(A - \lambda I) X = \bar{0}$$

$$\Rightarrow \begin{bmatrix} 2-1 & 2 \\ 1 & 3-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \bar{0}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x+2y \\ x+2y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which implies,

$$x+2y = 0$$
$$x+2y = 0$$

The coeffient matrix of the above system is,

$$A_2 = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \xrightarrow{R_{21}(-1)} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$\rho(A_2) = 1$ since $\rho(A_2) < n$, therefore the above system has infinitely many solutions ~~let, y~~ and the number of free variable $n - \rho(A_2) = 2 - 1 = 1$

Let, y be the free variable and $y=k$ where k is any integer.

The equivalent system.

$$x+2y=0 \therefore x=-2k$$
$$y=k$$

Hence,
$$X_{\lambda=1}=\begin{bmatrix}-2k\\ k\end{bmatrix}$$

For a particular case we can take,
$$u_2=\begin{bmatrix}-2\\ 1\end{bmatrix}$$

b) Now, we can form the matrix

P as

$$P=\begin{bmatrix}u_2 & u_1\end{bmatrix}$$

$$=\begin{bmatrix}-2 & 1\\ 1 & 1\end{bmatrix}$$

Here, $|P|=-2-1=-3$

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Again, $P^{-1} = -\frac{1}{3}\begin{bmatrix} 1 & -1 \\ -1 & -2 \end{bmatrix}$

Now,

$$D = P^{-1}AP$$

$$= -\frac{1}{3}\begin{bmatrix} 1 & -1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= -\frac{1}{3}\begin{bmatrix} 1 & -1 \\ -4 & -8 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= -\frac{1}{3}\begin{bmatrix} -3 & 0 \\ 0 & -12 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

08-07-2018 : 14A : Sunday :

c) we have,

$$D = P^{-1}AP$$

$$\Rightarrow PD = PP^{-1}AP$$

$$\Rightarrow PD = AP$$

$$\Rightarrow PDP^{-1} = APP^{-1}$$

$\Rightarrow PDP^{-1} = A$

$\therefore A = PDP^{-1}$

Hence, $B = \sqrt{A} = P\sqrt{D}\,P^{-1}$

$$= -\frac{1}{3}\begin{bmatrix}-2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 4\end{bmatrix}^{\frac{1}{2}}\begin{bmatrix}1 & -1\\ -1 & -2\end{bmatrix}$$

$$= -\frac{1}{3}\begin{bmatrix}-2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}1 & -1\\ -1 & -2\end{bmatrix}$$

$$= -\frac{1}{3}\begin{bmatrix}-2 & 2\\ 1 & 2\end{bmatrix}\begin{bmatrix}1 & -1\\ -1 & -2\end{bmatrix}$$

$$= -\frac{1}{3}\begin{bmatrix}-4 & -2\\ -1 & -5\end{bmatrix}$$

d) we have,

$A = PDP^{-1}$

Then, $A^{50} = PD^{50}P^{-1}$

$$= -\frac{1}{3}\begin{bmatrix}-2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 4\end{bmatrix}^{50}\begin{bmatrix}1 & -1\\ -1 & -2\end{bmatrix}$$

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$$= -\frac{1}{3}\begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4^{50} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & -2 \end{bmatrix}$$

$$= -\frac{1}{3}\begin{bmatrix} -2 & 4^{50} \\ 1 & 4^{50} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & -2 \end{bmatrix}$$

$$= -\frac{1}{3}\begin{bmatrix} -2-4^{50} & 2-2(4^{50}) \\ 1-4^{50} & -1-2(4^{50}) \end{bmatrix}$$

Matrix Representation of a linear operator

Let, T be a linear operator from a vector space V into itself and suppose that $S = \{u_1, u_2, \text{---}, u_n\}$ in a basis of V. Now, $T(u_1), T(u_2), \text{-----}, T(u_n)$ are vectors in V and each ~~is~~ is a linear combination of basis vectors in the set S.

Say,

$$T(u_1) = a_{11} u_1 + a_{12} u_2 + \cdots + a_{1n} u_n$$

$$T(u_2) = a_{21} u_1 + a_{22} u_2 + \cdots + a_{2n} u_n$$

$$\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots$$

$$T(u_n) = a_{n1} u_1 + a_{n2} u_2 + \cdots + a_{nn} u_n$$

The transpose of the above matrix of co-efficient, denoted by $m_s(T)$ or $[T]_s$ is called the matrix representation of T relative to the basis S.

Ex 6: Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator defined by $F(x,y) = (2x+3y,\ 4x-5y)$. Find the matrix representation of F relative to the basis $S = \{(1,2), (2,5)\}$.

Solution:

Here,

$$T(1,2) = (8,-6)$$

Let, $(8,-6) = x(1,2) + y(2,5)$

$$\Rightarrow (8,-6) = (x+2y,\ 2x+5y)$$

which implies,

$$x+2y=8$$

$$2x+5y=-6$$

The augmented matrix of the above system is,

$$A_1 = \begin{bmatrix} 1 & 2 & 8 \\ 2 & 5 & -6 \end{bmatrix} \xrightarrow{R_{21}(-2)} \begin{bmatrix} 1 & 2 & 8 \\ 0 & 1 & -22 \end{bmatrix}$$

Hence, the equivalent system,

$$x+2y=8$$

$$y=-22$$

$$\therefore x=52$$

Thus, $T(1,2) = 52(1,2) - 22(2,5)$ - - - (i)

Again, $T(2,9) = (19,-17)$

Let, $(19,-17) = x(1,2) + y(2,5)$

$$\Rightarrow (19,-17) = (x+2y, 2x+5y)$$

which implies,

$$x+2y=19$$

$$2x+5y=-17$$

the augmented matrix of the above system. is,

$$A_2 = \begin{bmatrix} 1 & 2 & 19 \\ 2 & 5 & -17 \end{bmatrix} \overset{R_{21}(-2)}{\sim} \begin{bmatrix} 1 & 2 & 19 \\ 0 & 1 & -55 \end{bmatrix}$$

the equivalent system,

$$x+2y = 19$$
$$y = -55$$
$$\therefore x = 129$$

Thus, $T(2,5) = 129(1,2) - 55(2,5)$ ----(ii)

From (i) an (ii) we have

$$T(1,2) = 52(1,2) - 22(2,5)$$
$$T(2,5) = 129(1,2) - 55(2,5)$$

Here the coeffiient matrix is

$$\begin{bmatrix} 52 & -22 \\ 129 & -55 \end{bmatrix}$$

Therefore, $[F]_S = \begin{bmatrix} 52 & 129 \\ -22 & -55 \end{bmatrix}$

This is the matrix representation of F relative to the basis S.