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Chapter-1
Complex numbers

Natural numbers:- 1, 2, 3, 4 ... are also called positive integers.

Negative integers:- Denoted -1, -2, -3, -4 and 0 respectively.

Rational numbers:- such as $3/4$, $-8/3$ arose to permit solutions of equations such as $bx = a$ for all integers a and b where $b \neq 0$.

Irrational numbers:- Such as $\sqrt{2} = 1.41423 \ldots$, $\pi = 3.14159 \ldots$ respectively.

Complex number:- We can consider a complex number as having the form $a+bi$ where a, b are real numbers i called the imaginary part, has the property $i^2 = -1$. If $z = a+bi$ then a is called the real part of z and b is called the Imaginary part of z and denoted by $Re\{z\}$ and $I_m(z)$ respectively. The symbol, z which can stand for any of a set of complex numbers, is called a complex variable.

Fundamental operations with complex numbers:-

1. Addition:- $(a+bi) + (c+di) = (a+c) + i(b+d)$
2. Subtraction:- $(a+bi) - (c+di) = (a-c) + i(b-d)$
3. Multiplication:- $(a+bi)(c+di) = ac + adi + bci + bdi^2$
$$= (ac - bd) + i(ad + bc)$$

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De MOIVRE's theorem:- If $z_1 = x_1 + iy_1 = r_1\cos\theta_1 + r_1 i\sin\theta_1$ $= r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ we can show that

$$z_1 z_2 = r_1 r_2 \{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\{\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\}$$

If $z_1 z_2 \cdots z_n = r_1 r_2 \cdots r_n\{\cos(\theta_1+\theta_2+\cdots\theta_n) + i\sin(\theta_1+\theta_2+\cdots\theta_n)\}$

then $z^n = \{r(\cos\theta + i\sin\theta)\}^n = r^n(\cos n\theta + i\sin n\theta)$ which is often called de-Moivre's theorem.

Roots of complex numbers:- A number w is called an nth root of a complex number z if $w^n = z$ and we write $w = z^{1/n}$. From De moivre's theorem we can show that if n is a positive integer,

$$z^{1/n} = \{r(\cos\theta + i\sin\theta)\}^{\frac{1}{n}}$$

$$= r^{1/n}\left\{\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right\} \quad k = 0,1,2\cdots n-1$$

Euler's formula:- By assuming the finite series expansion $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ of elementary calculus holds when $x = i\theta$,

$$e^{i\theta} = \cos\theta + i\sin\theta$$ which is called euler formula.

$$\therefore e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i\sin y).$$

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Q. Find real numbers x and y such that $3x + i2y - ix + 5y = 7 + 5i$

Soln:- Given that, $3x + 2iy - ix + 5y = 7 + 5i$

$$\Rightarrow 3x + 5y + i(2y - x) = 7 + 5i \quad --- (i)$$

Equating the real and imaginary part,

$3x + 5y = 7 \quad --- (ii)$

$2y - x = 5 \quad --- (iii)$

solving we get, $x = -1$, $y = 2$ (Ans:-)

Q. Absolute value:- The absolute value or modulus of a complex number $a+bi$ is defined as $|a+bi| = \sqrt{a^2+b^2}$.

$\therefore |-4+2i| = \sqrt{16+4} = \sqrt{20}$

Division:- $\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(a+di)(c-di)}$

$$= \frac{ac - adi + bic + bd}{d^2 + c^2}$$

$$= \frac{(ac+bd) + i(bc-ad)}{c^2+d^2}$$

$$= \frac{ac+bd}{c^2+d^2} + i\,\frac{bc-ad}{c^2+d^2}$$

Q. polar form of complex number:- If p is the point in the complex plane corresponding to the complex number (x, y)

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then $x = r\cos\theta$ $y = r\sin\theta$

where $r = \sqrt{x^2+y^2} = |x+iy|$ is called the absolute value of $z = x+iy$ and θ is called the amplitude or argument of $z = x+iy$.

It follows that, $z = x+iy = r(\cos\theta + i\sin\theta)$ which is called the polar form of the complex number and r and θ are called polar co-ordinate.

Ex. $\frac{5+5i}{3-4i} + \frac{20}{4+3i}$

$= \frac{5+5i}{3-4i} \cdot \frac{(3+4i)}{(3+4i)} + \frac{20}{4+3i} \cdot \frac{(4-3i)}{(4-3i)}$

$= \frac{15+20i+15i-20}{9+16} + \frac{80-60i}{16+9}$

$= \frac{-5+35i}{25} + \frac{80-60i}{25}$

$= -\frac{1}{5} + \frac{80}{25} + \frac{35i}{25} - \frac{60i}{25}$

$= \frac{-5+80}{25} + \frac{35i-60i}{25}$

$= 3-i$ (Ans:)

Ex. $\frac{3i^{30} - i^{19}}{2i-1} = \frac{3(i^2)^{15} - (i^2)^9 \cdot i}{2i-1} = \frac{-3+i}{2i-1}$

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$= \frac{(-3+i)}{(2i-1)} \cdot \frac{(2i+1)}{(2i+1)}$

$= \frac{-6i-3+2i^2+i}{4i^2-1}$

$= \frac{-6i-3-2+i}{-4-1}$

$= \frac{-5i-5}{-5}$

$= 1+i$ (Ans)

prove that, (a) $\overline{z_1+z_2} = \bar{z_1} + \bar{z_2}$

(b) $|z_1 z_2| = |z_1||z_2|$

sol^n(a): let, $z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$

$\therefore \overline{z_1+z_2} = \overline{x_1+iy_1+x_2+iy_2}$

$= \overline{(x_1+x_2)+i(y_1+y_2)}$

$= (x_1+x_2) - i(y_1+y_2)$

$= x_1 - iy_1 + x_2 - iy_2$

$= \overline{x_1+iy_1} + \overline{x_2+iy_2}$

$= \bar{z_1} + \bar{z_2}$

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Soln (b):- $|z_1 z_2| = |(x_1+iy_1)(x_2+iy_2)|$

$= |x_1x_2 + ix_1y_2 + ix_2y_1 - y_1y_2|$

$= |x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)|$

$= \{(x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2\}^{1/2}$

$= \{(x_1^2 + y_1^2)(x_2^2 + y_2^2)\}^{1/2}$

$= \sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}$

$= |z_1||z_2|$ (proved)

Q: prove that, $|z_1 + z_2| \leq |z_1| + |z_2|$ or prove that the modulus of the sum of two complex numbers does not exceed the sum of their modulus.

Analytically:- Let, $z_1 = x_1 + iy_1$ $z_2 = x_2 + iy_2$

Given that, $|z_1 + z_2| \leq |z_1| + |z_2|$

$\Rightarrow \sqrt{(x_1+x_2)^2 + (y_1+y_2)^2} \leq \sqrt{x_1^2+y_1^2} + \sqrt{x_2^2+y_2^2}$

squaring both sides, this will be true if,

$\Rightarrow (x_1+x_2)^2 + (y_1+y_2)^2 \leq x_1^2+y_1^2+2\sqrt{(x_1^2+y_1)^2(x_2^2+y_2^2)} + x_2^2+y_2^2$

$\Rightarrow 2x_1x_2 + 2y_1y_2 \leq 2\sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}$

$\Rightarrow (x_1x_2 + y_1y_2)^2 \leq \{\sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}\}^2$

$\Rightarrow x_1^2x_2^2 + 2x_1x_2y_1y_2 + y_1^2y_2^2 \leqq x_1^2x_2^2 + x_1^2y_2^2 + x_2^2y_1^2 + y_1^2y_2^2$

$\Rightarrow 2x_1x_2y_1y_2 \leqq x_1^2y_2^2 + x_2^2y_1^2$

$\Rightarrow 0 \leqq (x_1y_2)^2 - 2x_1x_2y_1y_2 + (y_1x_2)^2$

$\Rightarrow 0 \leqq (x_1y_2 - x_2y_1)^2$

$\therefore (x_1y_2 - x_2y_1)^2 \geqslant 0$ which is true.

Graphically :-

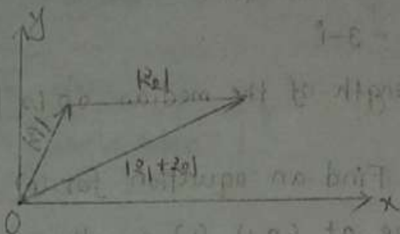

Fig:- $|z_1 + z_2| \leq |z_1| + |z_2|$

The result follows graphically from the fact that $|z_1|, |z_2|$, $|z_1 + z_2|$ represent the lengths of the sides of a triangle, and that the sum of the lengths of two sides of a triangle is greater than or equal to the length of third side.

Ex. Let $A(1,-2)$, $B(-3,4)$, $C(2,2)$ be the three vertices of triangle ABC. Find the length of the median from C to the side AB.

Solⁿ:- The position vectors of A,B,C are given by $z_1 = 1-2i$, $z_2 = -3$
$z_3 = 2+2i$ respectively.

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$AC = z_3 - z_1 = 2+2i-1+2i = 4i+1$

$BC = z_3 - z_2 = 2+2i+3-4i = 5-2i$

$AB = z_2 - z_1 = -3+4i-1+2i = 6i-4$

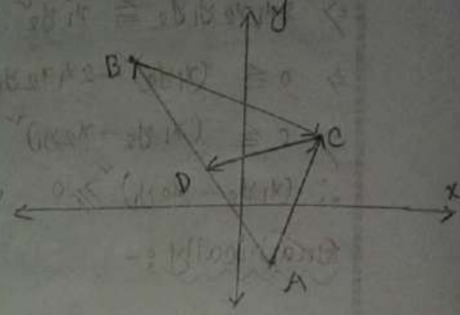
$AD = \frac{1}{2}AB = 3i-2$

Here $AC + CD = AD$

$\Rightarrow CD = AD - AC$

$= -2+3i-4i-1$

$= -3-i$



The length of the median CD is $|CD| = \sqrt{(-3)^2+(-1)^2} = \sqrt{10}$ (Ans:)

Ex. Find an equation for (a) a circle of radius 4 with centre at $(-2,1)$ (b) an ellipse with major axis of length 10 and foci at $(-3,0)$ and $(3,0)$

soln (a):- $(x+2)^2+(y-1)^2 = 4^2$

soln (b):- $|z+3| + |z-3| = 10$

$\Rightarrow |(x+3)+iy| + |(x-3)+iy| = 10$

$\Rightarrow \sqrt{(x+3)^2+y^2} + \sqrt{(x-3)^2+y^2} = 10$

$\Rightarrow (x+3)^2+\cancel{y^2} = 100 - 20\sqrt{(x-3)^2+y^2} + (x-3)^2+\cancel{y^2}$

$\Rightarrow \cancel{x^2}+6x+\cancel{9} = 100 - 20\sqrt{(x-3)^2+y^2} + \cancel{x^2}-6x+\cancel{9}$

$\Rightarrow 12x = 100 - 20\sqrt{(x-3)^2+y^2}$

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$\Rightarrow 3x = 25 - 5\sqrt{(x-3)^2+y^2}$

$\Rightarrow \{5\sqrt{(x-3)^2+y^2}\}^2 = (25-3x)^2$

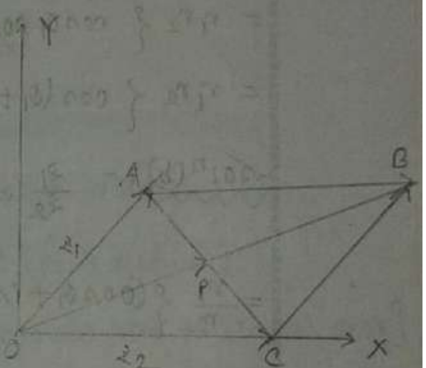
$\Rightarrow 25(x^2-6x+9+y^2) = 625 - 150x + 9x^2$

$\Rightarrow 16x^2 + 25y^2 = 400$

$\therefore \frac{x^2}{25} + \frac{y^2}{16} = 1$ (Ans:-)

Q. prove that the diagonals of a parallelogram bisect each other.

Soln:- Let, OABC the given parallelogram with diagonals inter secting at P.



Since $z_1 + AC = z_2$. $AC = z_2 - z_1$

$\therefore AP = m(z_2 - z_1)$ where $0 \leq m \leq 1$.

since, $OB = z_1 + z_2 = OP = n(z_1 + z_2)$

where $0 \leq n \leq 1$

But $OA + AP = OP$

$\Rightarrow z_1 + m(z_2 - z_1) = n(z_1 + z_2)$

$\Rightarrow (1-m-n)z_1 + (m-n)z_2 = 0$

Hence, $1-m-n=0$

$m-n=0$

$\therefore m = \frac{1}{2}, n = 1/2$ so P is the midpoint of both diagonals.

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Q. If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

prove that (a) $z_1 z_2 = r_1 r_2 \{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}$

(b) $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\{\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\}$

Solⁿ (a):- $z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1)\, r_2(\cos\theta_2 + i\sin\theta_2)$

$= (r_1\cos\theta_1 + ir_1\sin\theta_1)(r_2\cos\theta_2 + ir_2\sin\theta_2)$

$= r_1r_2\cos\theta_1\cos\theta_2 + ir_1r_2\cos\theta_1\sin\theta_2 + ir_1r_2\sin\theta_1\cos\theta_2 - r_1r_2\sin\theta_1\sin\theta_2$

$= r_1r_2\{\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\} + ir_1r_2\{\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2\}$

$= r_1r_2\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}$

Solⁿ (b):- $\dfrac{z_1}{z_2} = \dfrac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$

$= \dfrac{r_1}{r_2}\left\{\dfrac{(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}\right\}$

$= \dfrac{r_1}{r_2}\left\{\dfrac{\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2}{\cos^2\theta_2 + \sin^2\theta_2}\right\}$

$= \dfrac{r_1}{r_2}\left\{\dfrac{\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}\right\}$

$= \dfrac{r_1}{r_2}\{\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\}$

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Q. state and prove De ~~Moivre's~~ moivre's theorem:-

statement:- For any complex number (and in particular for a real number) x and integer n it holds that

$$(\cos x + i\sin x)^n = \cos(nx) + i\sin(nx)$$

proof:- we consider three cases,

(i) For $n>0$ we proceed by mathematical induction, when $n=1$ the result is clearly true.

The result is true for some integer $n=k$

$$(\cos x + i\sin x)^k = \cos(kx) + i\sin(kx)$$

Now considering $n=k+1$,

$$(\cos x + i\sin x)^{k+1} = (\cos x + i\sin x)^k (\cos x + i\sin x)$$

$$= \{\cos(kx) + i\sin(kx)\}\cdot(\cos x + i\sin x)$$

$$= \cos(kx)\cos x - \sin(kx)\sin(x) + i[\cos k(x)\sin x + \sin(kx)\cos x]$$

$$= \cos(kx+x) + i\sin(kx+x)$$

$$= \cos\{x(k+1)\} + i\sin\{(k+1)x\}$$

The result is true for $n=k+1$

Hence for every $n>1$ the result holds to be true.

(ii) when $n=0$ the formulae is true, $= \cos(0\cdot x) + i\sin(0\cdot x)$

$= 1$

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(iii) when $n<0$ we consider a integer $-m$ such that $n=-m$

$\therefore (\cos x+ i\sin x)^n = (\cos x + i\sin x)^{-m}$

$$= \frac{1}{\cos(mx) + i\sin(mx)}$$

$$= \frac{\cos(mx) - i\sin(mx)}{\cos^2(mx) + \sin^2(mx)}$$

$$= \cos(-mx) + i\sin(-mx)$$

$$= \cos(nx) + i\sin(nx)$$

Hence the theorem is true for any integer value of n.

Q: show that, (a) $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ (b) $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

proof: (a) $\frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{\cos\theta + i\cancel{\sin\theta} + \cos\theta - i\cancel{\sin\theta}}{2}$

$$= \frac{2\cos\theta}{2} = \cos\theta$$

(b) $\frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{\cancel{\cos\theta} + i\sin\theta - \cancel{\cos\theta} + i\sin\theta}{2i}$

$$= \frac{\cancel{2i}\sin\theta}{\cancel{2i}}$$

$$= \sin\theta$$

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Q. prove that (a) $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$

(b) $\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$

proof: (a) $\sin^3\theta = \left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^3$

$$= \frac{(e^{i\theta}-e^{-i\theta})^3}{8i^3} = \frac{-1}{8i}\left(e^{3i\theta} - 3e^{2i\theta}\cdot e^{-i\theta} + 3e^{i\theta}\cdot e^{-2i\theta} - e^{-3i\theta}\right)$$

$$= \frac{-1}{8i}\left(e^{3i\theta} - 3e^{i\theta} + 3e^{-i\theta} - e^{-3i\theta}\right)$$

$$= \frac{3}{4i}\left(\frac{e^{i\theta}-e^{-i\theta}}{2}\right) - \frac{1}{4}\left(\frac{e^{3i\theta}-e^{-3i\theta}}{2i}\right)$$

$$= \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta \quad \text{(proved)}$$

(b) $\cos^4\theta = \frac{(e^{i\theta}+e^{-i\theta})^4}{16}$

$$= \frac{1}{16}\left\{(e^{i\theta}+e^{-i\theta})^2\right\}^2$$

$$= \frac{1}{16}\left(e^{2i\theta} + 2e^{i\theta}\cdot e^{-i\theta} + e^{-2i\theta}\right)^2$$

$$= \frac{1}{16}\left(e^{4i\theta} + 4 + e^{-4i\theta} + 4e^{2i\theta} + 4e^{-2i\theta} + 2\right)$$

$$= \frac{1}{16}\left(e^{4i\theta} + e^{-4i\theta} + 6 + 4e^{2i\theta} + 4e^{-2i\theta}\right)$$

$$= \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8} \quad \text{(proved)}$$

Q. prove $e^{i\theta} = e^{i(\theta+2k\pi)}$, $k = 0, \pm 1, \pm 2 \cdots$

$\therefore e^{i(\theta+2k\pi)} = \cos(\theta+2k\pi) + i\sin(\theta+2k\pi)$

$$= \cos\theta + i\sin\theta = e^{i\theta} \text{ (proved)}$$

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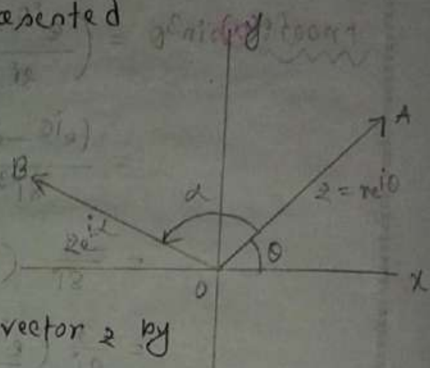
3. Given a complex number z, interpret geometrically $ze^{i\alpha}$ where α is real.

SOL^n:- Let, $z = re^{i\theta}$ be represented graphically by vector OA.



$ze^{i\alpha} = re^{i\theta} \cdot e^{i\alpha}$

$= re^{i(\theta+\alpha)}$ is the vector represented by OB.

Hence multiplication of a vector z by $e^{i\alpha}$ amounts to rotating z counter clockwise through α. We can consider $e^{i\alpha}$ as an operator which acts on z to produce this rotation.

5. Find the square roots of $-15-8i$

SOL^n:- $-15-8i = -15-8i$

$= 1-8i-16$

$= 1-8i+16i^2$

$= 1-2 \cdot 4i + (4i)^2$

$= (1-4i)^2$

$\therefore \sqrt{-15-8i} = \pm(1-4i)$

$= 1-4i, -1+4i$ (Ans:-)

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Q. solve this equation, $z^2+(2i-3)z+5-i=0$

Soln:- $z^2+(2i-3)z+5-i=0$

$\therefore \quad z = \dfrac{-(2i-3)\pm\sqrt{(2i-3)^2-4(1)(5-i)}}{2(1)}$

$= \dfrac{3-2i\pm\sqrt{-15-8i}}{2}$

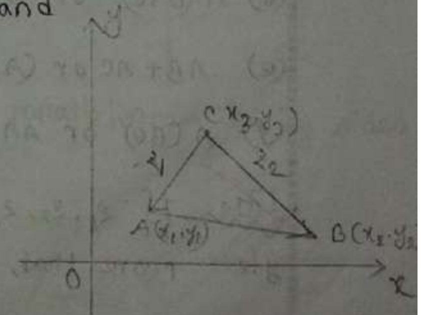
$= \dfrac{3-2i\pm\sqrt{-15-8i}}{2}$

$= \dfrac{3-2i\pm(1-4i)}{2}$

$= 2-3i$, or $1+i$ (Ans:)

Q. Find the area of the triangle with vertices at $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$

Soln:- The vector from C to A and B respectively.



$z_1 = (x_1-x_3)+i(y_1-y_3)$

$z_2 = (x_2-x_3)+i(y_2-y_3)$

Area of the triangle,

$= \frac{1}{2}|z_1 \times z_2|$

$= \frac{1}{2}\left| Im\{[(x_1-x_3)-i(y_1-y_3)][(x_2-x_3)+i(y_2-y_3)]\}\right|$

$= \frac{1}{2}\left|(x_1-x_3-iy_1+iy_3)(x_2-x_3+iy_2-iy_3)\right|$

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$= \frac{1}{2} |(x_1 - x_3)(y_2 - y_3) - (y_1 - y_3)(x_2 - x_3)|$

$= \frac{1}{2} | x_1 y_2 - y_1 x_2 + x_2 y_3 - y_2 x_3 + x_3 y_1 - y_3 x_1 |$

$= \frac{1}{2} \left| \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right|$

Q. Given the point sets $A = \{3, -i, 2+i, 5\}$

$B = \{-i, 0, -1, 2+i\}$

$C = \{-\sqrt{2}i, 1/2, 3\}$

(a) $A+B$ or $A \cup B = \{3, -i, 2+i, 5, 0, -1\}$

(b) AB or $A \cap B = \{-i, 2+i\}$

(c) AC or $A \cap C = \{3\}$

(d) $A(B+C)$ or $A \cap (B \cup C) = \{3, -i, 2+i\}$

(e) $AB + AC$ or $(A \cap B) \cap (A \cap C) = \{-i, 2+i, 3\}$

(f) $A(BC)$ or $A \cap (B \cap C) = \phi$

Q. If z_1, z_2, z_3 represent vertices of an equilateral triangle prove that, $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$

Solⁿ:- $z_2 - z_1 = e^{\pi i/3}(z_3 - z_1)$ - - - - - - - (i)

$z_1 - z_3 = e^{\pi i/3}(z_2 - z_3)$ - - - - - (ii)

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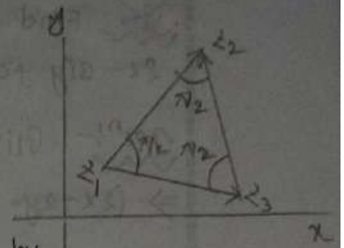
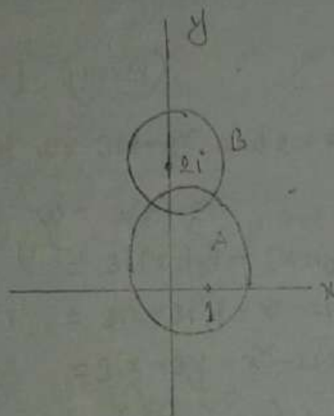
Then by division,

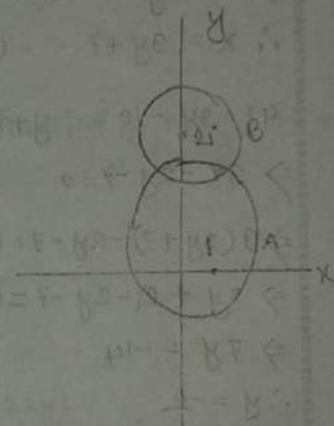
$$\frac{z_2-z_1}{z_1-z_3} = \frac{z_3-z_1}{z_2-z_3}$$

$\therefore z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$



Q. Given the set A and B represented by $|z-1|<3$ and $|z-2i|<2$. Represent geometrically (a) $A \cap B$ or AB (b) $A+B$ or $A \cup B$





Q. prove that the area of the parallelogram having sides z_1 and z_2 is $|z_1 \times z_2|$

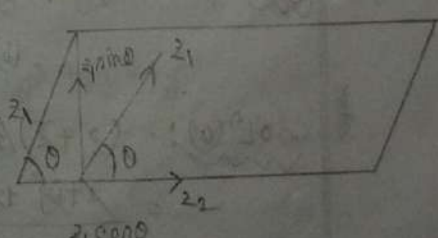
Area of the parallelogram

$= (\text{base})(\text{height})$

$= (|z_2|)(|z_1| \sin\theta)$

$= |z_1||z_2| \sin\theta$

$= |z_1 \times z_2|$ (Ans.)



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Find the number x and y such that

$2x - 3iy + 4ix - 2y - 5 - 10i = (x+y+2) - i(y-x+3)$

sol^n:- Given that, $2x - 3iy + 4ix - 2y - 5 - 10i = (x+y+2) - i(y-x+3)$

$\Rightarrow (2x-2y-5) + i(4x-3y-10) = (x+y+2) - i(y-x+3)$

Equating real and imaginary part,

$2x - 2y - 5 = x + y + 2$

$\Rightarrow x - 3y - 7 = 0$

$\therefore x = 3y + 7 \quad \text{--- (i)}$

$4x - 3y - 10 = -y + x - 3$

$\Rightarrow 3x - 2y - 7 = 0$

$\Rightarrow 3(3y+7) - 2y - 7 = 0$

$\Rightarrow 9y + 21 - 2y - 7 = 0$

$\Rightarrow 7y = -14$

$\therefore y = -2$

$\therefore x = 1$

$\therefore (x,y) = (1,-2)$ (Ans:)

Prove that (a) $Re\{z\} = (z+\bar{z})/2$

(b) $Im\{z\} = (z-\bar{z})/2i$

sol^n(a):- $(z+\bar{z})/2$

$= \dfrac{\cancel{x}+\cancel{iy}+\cancel{x}-\cancel{iy}}{2}$

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$= \frac{2x}{2}$

$= x = Re\{z\}$ (proved)

(b) $(z-\bar{z})/2i$

$= \frac{x+iy-(x-iy)}{2i}$

$= \frac{\cancel{x}+iy-\cancel{x}+iy}{2}$

$= iy$

$= Im\{z\}$ (proved)

Q. If $w = 3iz - z^2$ and $z = x+iy$ find $|w|^2$ in terms of x and y

sol^n:- $w = 3iz - z^2$

$= 3i(x+iy) - (x+iy)^2$

$= 3ix + 3i^2y - x^2 - 2ixy + y^2$

$= 3ix - 3y - x^2 - 2ixy + y^2$

$= (y^2 - x^2 - 3y) + i(3x - 2xy)$

$\Rightarrow |w| = \sqrt{(y^2-x^2-3y)^2 + (3x-2xy)^2}$

$\Rightarrow (|w|)^2 = y^4 + x^4 + 9y^2 - 2x^2y^2 + 6x^2y - 6y^3 + 9x^2 - 12x^2y + 4x^2y^2$

$\therefore (|w|)^2 = x^4 + y^4 + 2x^2y^2 - 6x^2y - 6y^3 + 9x^2 + 9y^2$

(Ans:-)

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Q. If $z = 6e^{\pi i/3}$ evaluate $|e^{iz}|$?

Solⁿ:- $|e^{iz}|$

$= |e^{i6e^{\pi i/3}}|$

$= |e^{i6(\cos\pi/3 + i\sin\pi/3)}|$

$= |e^{i6(\frac{1}{2} + \frac{\sqrt{3}}{2}i)}|$

$= |e^{3i - 3\sqrt{3}}|$

$= |e^{3i}|\, e^{-3\sqrt{3}}$

$= e^{-3\sqrt{3}}$ $\quad [\because |e^{i\theta}| = 1]$

Q. show that, $2+i = \sqrt{5}\, e^{i\tan^{-1}(1/2)}$

Solⁿ:- $r = \sqrt{4+1} = \sqrt{5}$

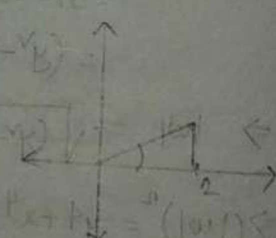
argument, $\theta = \tan^{-1}\frac{1}{2}$

$\therefore 2+i = r(\cos\theta + i\sin\theta)$

$= \sqrt{5}\,(\cos(\tan^{-1}1/2) + i\sin(\tan^{-1}1/2))$

$= \sqrt{5}\, e^{i\tan^{-1}(1/2)}$

Ans:



Q. An airplane travels 150 km southeast, 100 km due west, 225 km 30° north of east, and then 323 km northeast. Determine analytically and graphically how far and in what direction it is

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it's starting point.

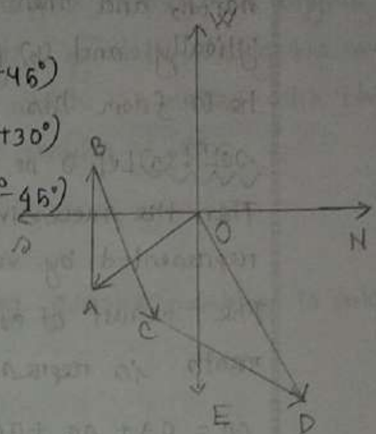
Sol^n:- $OA = 150\{\cos 45^\circ + i\sin 45^\circ\}$

$AB = 100\{\cos(180^\circ + 45^\circ) + i\sin(180^\circ + 45^\circ)$

$BC = 225\{\cos(180^\circ + 30^\circ) + i\sin(180^\circ + 30^\circ)$

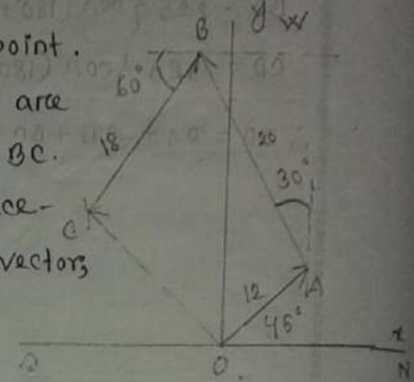
$CD = 323\{\cos(180^\circ - 45^\circ) + i\sin(180^\circ - 45^\circ)$

$\therefore OD = OA + AB + BC + CD$



Q. A man travels 12 miles northeast, 20 miles 30° west of north, and then 18 miles 60° south of west. Determine (a) Analytically and (b) graphically how far and in what direction he is from his starting point.

SOL^n: (a) Let O be the starting point. Then the successive displacements are represented by vectors OA, AB, BC. The result of all three displacements is represented by the vector,



$OC = OA + AB + BC$

$= 12(\cos 45^\circ + i\sin 45^\circ)$

$= 12e^{\pi i/4} + 20\{\cos(90^\circ+30^\circ) + i\sin(90^\circ+30^\circ)\} + 18\{\cos(180^\circ+60^\circ) + i\sin(180^\circ+60^\circ)\}$

$= 12e^{\pi i/4} + 20e^{2\pi i/3} + 18e^{4\pi i/3}$

Then, $OC = 12e^{\pi i/4} + 20e^{2\pi i/3} + 18e^{4\pi i/3}$

$= (12\cos 45^\circ + 20\cos 120^\circ + 18\cos 240^\circ) + i(12\sin 45^\circ + 20\sin 120^\circ + 18\sin 240^\circ)$

$= (6\sqrt{2} - 19) + (6\sqrt{2} + \sqrt{3})i$

$= \sqrt{(6\sqrt{2}-19)^2 + (6\sqrt{2}+\sqrt{3})^2}$

$= 14.7$ approximately

$\therefore \theta = \cos^{-1}\left(\frac{6\sqrt{2}-19}{14.7}\right) = 135^\circ 49'$

Thus the man is 14.7 miles from his starting point in a direction $135^\circ 49' - 90^\circ = 45^\circ 49'$ west of north.

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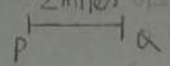
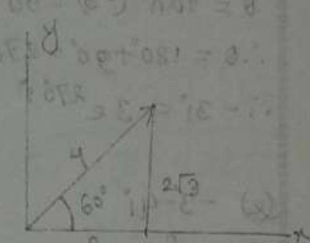
(b) Graphically:- Using a convenient unit of length such PQ which represents 2 miles, and a protractor to measure angles, construct vectors OA, AB and BC. Then by determining the number of units in OC and the angle which OC makes with the y axis, we obtain the approximate results of (a).



Ex: Express each of the following complex numbers in polar form:-

(a) $2+2\sqrt{3}i$



$\therefore$ Modulus, $r = \sqrt{4+12} = 4$

Argument, $\theta = \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right)$

$= 60^\circ = \pi/3$

$\therefore 2+2\sqrt{3}i = 4(\cos \pi/3 + i \sin \pi/3)$

$= 4e^{\pi i/3}$

(b) $-5+5i$

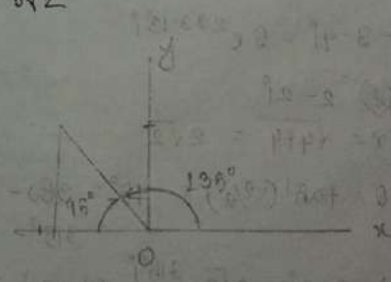
$\therefore$ Modulus, $r = \sqrt{5^2+5^2} = 5\sqrt{2}$

Argument, $\theta = \tan^{-1}\left(\frac{5}{-5}\right)$

$= -45^\circ$

$\theta = 180^\circ - 45^\circ = 135^\circ$

$\therefore -5+5i = 5\sqrt{2}\, e^{135^\circ i}$



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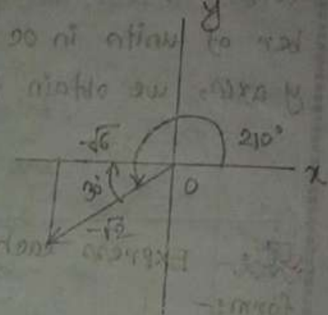
(c) $-\sqrt{6}+\sqrt{2}i$

$r = \sqrt{6+2} = 2\sqrt{2}$

$\theta = \tan^{-1}\left(\frac{-\sqrt{2}}{-\sqrt{6}}\right) = 30^\circ$

$\therefore \theta = 180^\circ + 30^\circ$
$= 210^\circ$

$-\sqrt{6} - \sqrt{2}i = 2\sqrt{2}e^{i210^\circ}$



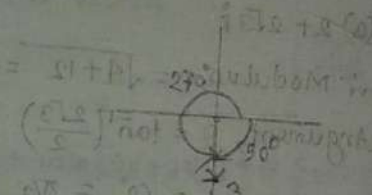
(d) $-3i$

$\therefore r = \sqrt{3^2} = 3$

$\theta = \tan^{-1}(-3) = 90^\circ$

$\therefore \theta = 180^\circ + 90^\circ = 270^\circ$

$\therefore -3i = 3e^{270^\circ i}$



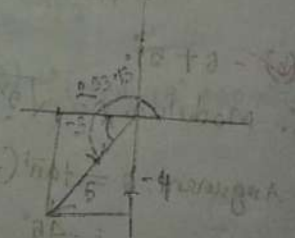
(e) $-3-4i$

$\therefore r = 5$

$\theta = \tan^{-1}\left(\frac{-4}{-3}\right) = 53.13^\circ$

$\therefore \theta = 180^\circ + 53.13^\circ = 233.13^\circ$

$-3-4i = 5e^{233.13^\circ i}$

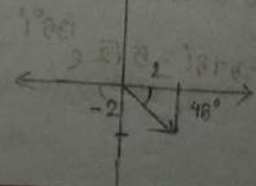


(f) $2-2i$

$r = \sqrt{4+4} = 2\sqrt{2}$

$\theta = \tan^{-1}(-2/2) = -45^\circ = 360^\circ - 45^\circ$
$= 315^\circ$

$\therefore 2-2i = 2\sqrt{2}e^{315^\circ i}$ (Ans)

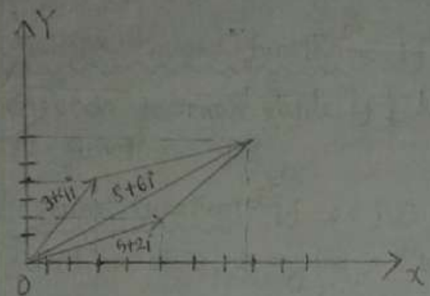


Q. Graphically representation of complex numbers:

(a) $(3+4i)+(5+2i)$

Analytically, $= 3+4i+5+2i = 8+6i$

Graphically $= (3+4i)+(5+2i)$

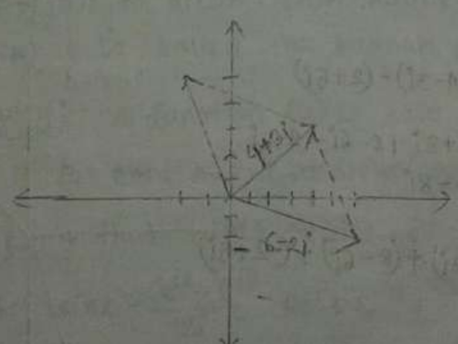


(b) $(6-2i)-(2-5i)$

Analytically, $= 6-2i-2+5i$

$= 4+3i$

graphically, $= (6-2i)+(-2+5i)$



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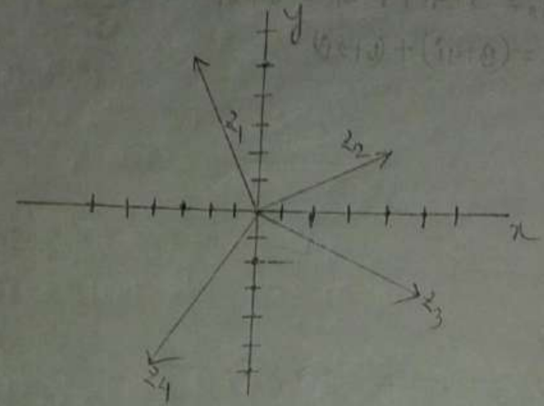
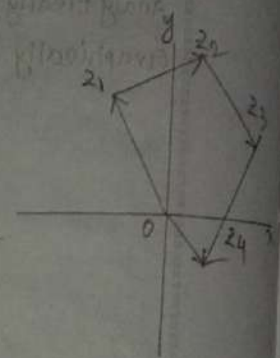
(c) $(-3+5i) + (4+2i) + (5-3i) + (-4-6i)$

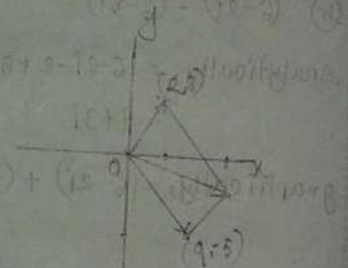
Analytically $= 2-2i$





(d) $(2+3i) + (4-5i)$

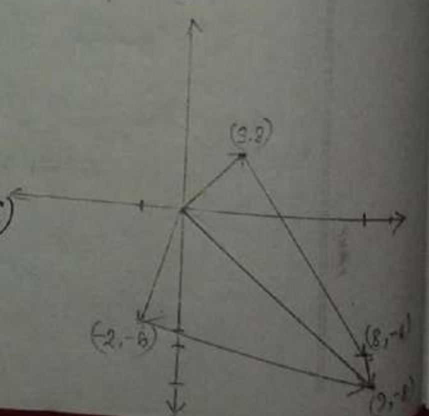
Analytically, $= 6-2i$



(e) $3(1+i) + 2(4-3i) - (2+5i)$

Analytically, $= 3+3i+8-6i-2-5i$
$= 9-8i$

Graphically, $= (3+3i) + (8-6i) + (-2+5i)$



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Chapter-2

Functions, Limits, continuity

Single valued function:- If only one value of w corresponds to each value of z we say that w is a single valued function of z that is $f(z)$ is single valued.

Multiple valued function:- If more than one value of w corresponds to each value of z we say that w is a multiple valued function of z.

Inverse function:- If $w = f(z)$ then we can also consider z as a function of w, $z = g(w) = f^{-1}(w)$. The function f^{-1} is often called called the Inverse function corresponding to f.

BranchLine:- It is clear that each branch of the function is single valued. In order to keep the function single valued we set-up an artificial barrier such as OB where B is at infinity [although any other line from O can be used] which we agree not to cross. This barrier is called the brachline or point O is called the branch point.

Continuity:- A function $f(z)$ is said to be continuous in a region if it is continuous at all points of the region.

Ex. prove that, $\sin^2 z + \cos^2 z = 1$

proof:- $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$ $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$

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$\sin^2 z + \cos^2 z$

$= \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2 + \left(\frac{e^{iz}+e^{-iz}}{2}\right)^2$

$= -\frac{e^{2iz}-2+e^{-2iz}}{4} + \frac{e^{2iz}+2+e^{-2iz}}{4}$

$= \frac{\cancel{-e^{2iz}}+4+\cancel{e^{2iz}}-\cancel{e^{-2iz}}+\cancel{e^{-2iz}}}{4}$

$= 1$ (proved)

Ex. prove that $e^{iz} = \cos z + i\sin z$
$e^{-iz} = \cos z - i\sin z$

proof:- $e^{iz} - e^{-iz} = 2i\sin z$ —— (i)

$e^{iz} + e^{-iz} = 2\cos z$ —— (ii)

By adding (i) and (ii)

$e^{iz} = \cos z + i\sin z$

By subtracting (i) and (ii)

$e^{-iz} = \cos z - i\sin z$

(proved)

Ex. prove that $\sin(z_1+z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$

proof:- $\sin(z_1+z_2) = \frac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{2i}$

$= \frac{e^{iz_1}\cdot e^{iz_2} - e^{-iz_1}\cdot e^{-iz_2}}{2i}$

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$= \frac{1}{2i}\{(\cos z_1 + i\sin z_1)(\cos z_2 + i\sin z_2) - (\cos z_1 - i\sin z_2)(\cos z_2 - i\sin z_2)$

$= \frac{1}{2i}\{\cos z_1 \cancel{\cos z_2} + i\sin z_2 \cos z_1 + i\sin z_1 \cos z_2 - \sin z_1 \cancel{\sin z_2} - \cos z_1 \cancel{\cos z_2} + i\sin z_2 \cos z_1 + i\sin z_1 \cos z_2 + \sin z_1 \cancel{\sin z_2}$

$= \frac{1}{2i}\{2i \sin z_2 \cos z_1 + 2i \sin z_1 \cos z_2\}$

$= \cos z_1 \sin z_2 + \sin z_1 \cos z_2$ (proved)

Q. prove that, (a) $\sin(-z) = -\sin z$ (b) $\cos(-z) = \cos z$
(c) $\tan(-z) = -\tan z$

soln (a):- $\sin(-z) = \frac{e^{i(-z)} - e^{-i(-z)}}{2i} = \frac{e^{-iz} - e^{iz}}{2i} = -\left(\frac{e^{iz} - e^{-iz}}{2i}\right)$
$= -\sin z$

(b) $\cos(-z) = \frac{e^{i(-z)} + e^{-i(-z)}}{2} = \frac{e^{-iz} + e^{iz}}{2} = \cos z$

(c) $\tan(-z) = \frac{\sin(-z)}{\cos(-z)} = \frac{-\sin z}{\cos z} = -\tan z$

Q. prove that (a) $1 - \tanh^2 z = \operatorname{sech}^2 z$
(b) $\sin iz = i\sinh z$
(c) $\cos iz = \cosh z$
(d) $\sin(x+iy) = \sin x \cosh y + i\cos x \sinh y$

proof:- (a) $\cosh^2 z - \sinh^2 z = \left(\frac{e^z + e^{-z}}{2}\right)^2 - \left(\frac{e^z - e^{-z}}{2}\right)^2$

$= \frac{e^{2z} + 2 + e^{-2z} - e^{2z} + 2 - e^{-2z}}{4} = 1$

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$\Rightarrow \dfrac{\cosh^2 z - \sinh^2 z}{\cosh^2 z} = \dfrac{1}{\cosh^2 z}$

$\Rightarrow 1 - \tanh^2 z = \operatorname{sech}^2 z$ (proved)

(b) $\sin iz = \dfrac{e^{i(iz)} - e^{-i(iz)}}{2i} = \dfrac{e^{-z} - e^{z}}{2i} = \dfrac{(e^{z} - e^{-z})\, i}{2} = i\sinh z$

(c) $\cos iz = \dfrac{e^{i(iz)} + e^{-i(iz)}}{2} = \dfrac{e^{-z} + e^{z}}{2} = \cosh z$

(d) $\sin(x+iy) = \sin x \cos iy + \cos x \sin iy$

$= \sin x \cosh y + i \cos x \sinh y$ (proved)

Ex:- prove that $\lim\limits_{z\to 0} \dfrac{\bar{z}}{z}$ does not exist.

proof:- let $z \to 0$ along the x-axis. Then $y = 0$, and $z = x+iy = x$ and $\bar{z} = x - iy = x$ so that the required limit is,

$\lim\limits_{z\to 0} \dfrac{\bar{z}}{z} = \lim\limits_{x\to 0} \dfrac{x}{x} = 1$

let $z \to 0$ along the y axis then $x = 0$. $z = x+iy = iy$, $\bar{z} = x - iy = -iy$ So that the required limit is,

$\lim\limits_{z\to 0} \dfrac{\bar{z}}{z} = \lim\limits_{y\to 0} \dfrac{-iy}{iy} = -1$

Since the two approaches do not give the same answer so that the limit does not exist.

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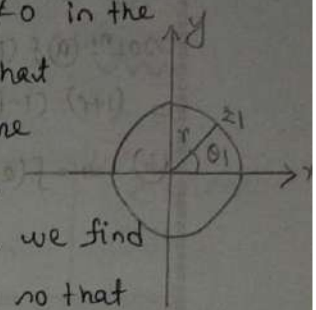
Q. prove that $f(z) = \operatorname{Ln} z$ has a branch point at $z = 0$.

proof:- we have $\ln z = \ln r + i\theta$

Suppose that we start at some point $z_1 \neq 0$ in the complex plane for which $r = r_1$, $\theta = \theta_1$ so that $\operatorname{Ln} z_1 = \ln r_1 + i\theta_1$. Then after making one complete circle about the origin in the positive or counterclockwise direction, we find on returning to z_1 that $r = r_1$, $\theta = \theta_1 + 2\pi$ so that $\operatorname{Ln} z_1 = \operatorname{Ln} r_1 + i(\theta_1 + 2\pi)$. Then we are on another branch on the function and so $z = 0$ is a branch point.



Q. $\frac{\sqrt{3}}{2} - \frac{3}{2} i$

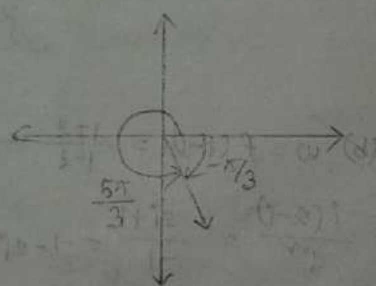
$\therefore r = \sqrt{\frac{3}{4} + \frac{9}{4}}$

$= \sqrt{3}$

$\theta = \tan^{-1}\left(\frac{-3/2}{\sqrt{3}/2}\right)$

$= \tan^{-1}(-\sqrt{3})$

$= \frac{-\pi}{3}$



$\therefore \theta = 2\pi - \pi/3$

$= \frac{6\pi - \pi}{3} = \frac{5\pi}{3}$

$\therefore z = r e^{5\pi i/3} = \sqrt{3} e^{5\pi/3}$

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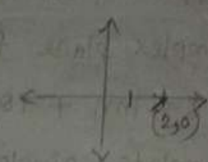
Q. Let $w = f(z) = z(2-z)$ Find the values of w corresponding to (a) $z = 1+i$ (b) $z = 2-2i$ and graph corresponding values in the w and z planes.

Sol^n: (a) $f(1+i) = (1+i)(2-1-i)$

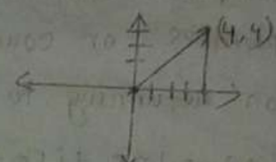
$= (1+i)(1-i) = 1-i^2 = 2$



(b) $w = f(2-2i) = z(2-z)$

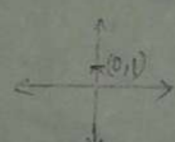
$= (2-2i)(2-2+2i)$

$= 4i - 4i^2 = 4+4i$



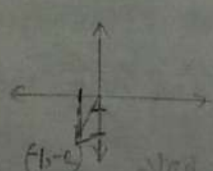
Q. If $w = f(z) = \dfrac{1+z}{1-z}$ find (a) $f(i)$ (b) $f(1-i)$ and represents it graphically.

Sol^n: (a) $f(i) = \dfrac{1+i}{1-i} = \dfrac{(1+i)(1+i)}{(1+i)(1-i)} = \dfrac{(1+i)^2}{1-i^2} = \dfrac{1+2i-1}{2} = i$



(b) $w = f(1-i) = \dfrac{1+z}{1-z} = \dfrac{1+1-i}{1-1+i} = \dfrac{2-i}{i} = \dfrac{2}{i} - 1 = \dfrac{2-i}{i}$

$= \dfrac{i(2-i)}{i^2} = \dfrac{2i+1}{-1} = -1-2i$



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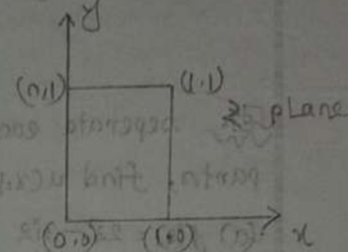
Q. A square S in the z plane has vertices at $(0,0), (1,0), (1,1)$ $(0,1)$. Determine the region in the w plane into which S is supplied under the transformation (a) $w = z^2$ (b) $w = \frac{1}{z+1}$

Sol^n: (a) we have $w = z^2$

$$= (x+iy)^2$$
$$= x^2 + 2xiy - y^2$$
$$= x^2 - y^2 + 2xiy$$



$\therefore u(x,y) = x^2 - y^2$

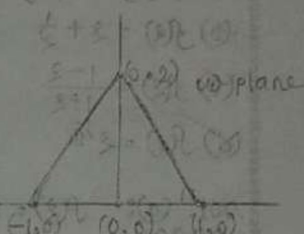
$v(x,y) = 2xy$

at $(0,0)$, $u(x,y) = 0$, $v(x,y) = 0$

at $(1,0)$, $u(x,y) = 1$, $v(x,y) = 0$

at $(1,1)$, $u(x,y) = 0$, $v(x,y) = 2$

at $(0,1)$, $u(x,y) = -1$, $v(x,y) = 0$



(b) $w = \frac{1}{z+1} = \frac{1}{x+iy+1} = \frac{1}{(x+1)+iy}$

$$= \frac{(x+1)-iy}{(x+1)^2 - i^2y^2}$$

$$= \frac{(x+1)-iy}{(x+1)^2 + y^2}$$

$\therefore w = u(x,y) + v(x,y) = \frac{x+1}{(x+1)^2+y^2} - i\frac{y}{(x+1)^2+y^2}$

Here, $u(x,y) = \frac{x+1}{(x+1)^2+y^2}$

$v(x,y) = -\frac{y}{(x+1)^2+y^2}$

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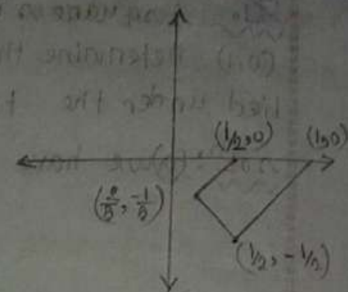
at $(0,0)$ $u(x,y) = 1, v(x,y) = 0$

at $(1,0)$ $u(x,y) = \frac{1}{2}, v(x,y) = 0$

at $(1,1)$ $u(x,y) = \frac{2}{5}, v(x,y) = -\frac{1}{5}$

at $(0,1)$ $u(x,y) = \frac{1}{2}, v(x,y) = -\frac{1}{2}$



Q. seperate each of the following into real and imaginary parts, find $u(x,y)$ and $v(x,y)$ such that $f(z) = u+iv$

(a) $f(z) = 2z^2 - 3iz$ (e) $f(z) = \sin 2z$

(b) $f(z) = z + \frac{1}{z}$

(c) $f(z) = \frac{1-z}{1+z}$

(d) $f(z) = z^{1/2}$

SOL^n (a):- $f(z) = 2z^2 - 3iz$

$$= 2(x+iy)^2 - 3i(x+iy)$$

$$= 2x^2 + 4ixy - 2y^2 - 3ix + 3y$$

$$= 2x^2 + 3y - 2y^2 + i(4xy - 3x)$$

$$= u(x,y) + iv(x,y)$$

$\therefore$ $u(x,y) = 2x^2 + 3y - 2y^2$

$v(x,y) = 4xy - 3x$

(b) $f(z) = z + \frac{1}{z}$

$$= \frac{z^2+1}{z} = \frac{x^2 + 2xiy - y^2 + 1}{x+iy}$$

$$= (x^2 + 2xiy - y^2 + 1)(x - iy) / x^2 + y^2$$

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$$= x^3 - xy^2 + x + 2xy^2 - x^2iy + iy^3 - iy + 2x^2y\,i \Big/ x^2+y^2$$

$$= \frac{x^3 + xy^2 + x}{x^2+y^2} - i\,\frac{(y - y^3 - x^2y)}{x^2+y^2} = u+iv$$

(c) $f(z) = \frac{1-z}{1+z}$

$$= \frac{1-(x+iy)}{1+(x+iy)}$$

$$= \frac{(1-x)-iy}{(1+x)+iy}$$

$$= \frac{\{(1-x)-iy\}\{(1+x)-iy\}}{\{(1+x)+iy\}\{(1+x)-iy\}}$$

$$= \frac{1-x^2-y^2-iy+xiy-iy-xiy}{(1+x)^2+y^2}$$

$$= \frac{1-x^2-y^2-2iy}{(1+x)^2+y^2}$$

$$= \frac{1-x^2-y^2}{(1+x)^2+y^2} - \frac{2iy}{(1+x)^2+y^2} = u+iv$$

(d) $f(z) = z^{1/2}$

$$= (re^{i\theta})^{1/2}$$

$$= r^{1/2} e^{i\theta/2}$$

$$= r^{1/2}(\cos\theta/2 + i\sin\theta/2)$$

$$= r^{\frac{1}{2}}\cos\theta/2 + ir^{1/2}\sin\theta/2$$

$$= u(x,y) + iv(x,y)$$

$\therefore u(x,y) = r^{1/2}\cos\theta/2 \quad v(x,y) = r^{\frac{1}{2}}\sin\theta/2$ (Ans:)

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(e) $f(z) = \sin 2z$

$= \sin 2(x+iy)$

$= \sin(2x+2iy)$

$= \sin 2x \cos 2iy + \cos 2x \sin 2iy$

$= \sin 2x \cosh 2y + i \cos 2x \sinh 2y$ (Ans:-)

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Chapter - 3

Analytic function:- If the derivative $f'(z)$ exists all points z of a region R, then $f(z)$ is said to be analytic in R and referred to as analytic function in R.

Cauchy - Riemann equations:- A necessary condition that $w = f(z) = u(x,y) + iv(x,y)$ be analytic in a region R is that, in R, u and v satisfy cauchy Riemann equations,

$$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}, \quad \frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$$

Harmonic functions:- The function $u(x,y)$ and $v(x,y)$ which satisfies the laplace equation in a region R are called harmonic functions.

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0, \quad \frac{\delta^2 v}{\delta x^2} + \frac{\delta^2 v}{\delta y^2} = 0$$

Singular points: A point at which $f(z)$ fails to be analytic is called a singular point or singularity of $f(z)$. Various types of singularity exist.

(i) **Isolated singularities:** The point $z = z_0$ is called an isolated singularities of $f(z)$ if we can find $\delta > 0$ such that the circle $|z - z_0| = \delta$ encloses no singular point other than z_0.

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(ii) **poles:** If we can find a positive integer n such that $\lim_{z\to z_0}(z-z_0)^n f(z) = A \neq 0$ then $z=z_0$ is a pole of order n. If $n=1$ then z_0 is called a simple pole.

$f(z) = \frac{1}{(z-2)^3}$ is pole of order 3 at $z=2$

(iii) **Branch points:-** Branch points of multiple valued functions are singular points.

$f(z) = (z-3)^{1/2}$ has a branch point $z=3$

$f(z) = \ln(z^2+z-2)$ has a branch point at $z=1,-2$

(iv) **Removable singularities:** The singular point z_0 is called a removable singularities of $f(z)$ if $\lim_{z\to z_0} f(z)$ exists.

The singular point $z=0$ is a removable singularity of $f(z)=\frac{\sin z}{z}$ since $\lim_{z\to 0}\frac{\sin z}{z}=1$

(v) **Essential singularities:** A singularity which is not a pole, branch point or removable singularity is called an essential singularity.

$f(z) = e^{1/z-2}$ has an essential singularity at $z=2$

(vi) **singularities at infinity:-** The type of singularity of $f(z)$ at $z=\infty$ is the same as that of $f(1/w)$ at $w=0$. The function $f(z)=z^3$ is a pole of order 3 at $z=\infty$ since $f(\frac{1}{w})=\frac{1}{w^3}$ has pole of order 3 at $w=0$.

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Q. (a) Necessary and (b) sufficient condition that $w = f(z) = u(x,y) + iv(x,y)$ be analytic in region R is that the cauchy Reimann equations $\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$, $\frac{\delta v}{\delta y} = -\frac{\delta u}{\delta x}$ are satisfied in R where it is supposed that these partial derivatives are continuous in R.

Necessity: (a) In order for $f(z)$ be analytic the limit,

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = f(z)$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)\} - \{u(x,y) + iv(x,y)\} / \Delta x + i\Delta y \quad \text{--- (i)}$$

Case-I:- $\Delta y = 0$, $\Delta x \to 0$

$$\lim_{\Delta x \to 0} \left\{ \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \left\{ \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} \right\} \right\}$$

$= \frac{\delta u}{\delta x} + i \frac{\delta v}{\delta x}$ provided the partial derivatives exist.

Case II:- $\Delta x = 0$, $\Delta y \to 0$

$$\lim_{\Delta y \to 0} \left\{ \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + \frac{v(x,y+\Delta y) - v(x,y)}{\Delta y} \right\}$$

$$= \frac{1}{i} \frac{\delta u}{\delta y} + \frac{\delta v}{\delta y} = \frac{i}{i^2} \frac{\delta u}{\delta y} + \frac{\delta v}{\delta y}$$

$$= -i \frac{\delta u}{\delta y} + \frac{\delta v}{\delta y}$$

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Now $f(z)$ can not be analytic unless these two Limits are identical. Thus a necessary condition that $f(z)$ analytic is,

$$\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x} = -i\frac{\delta u}{\delta y} + \frac{\delta v}{\delta y}$$

$$\therefore \frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}, \quad \frac{\delta v}{\delta x} = -\frac{\delta u}{\delta y}$$

(b) Sufficiency :- Since $\frac{\delta u}{\delta x}$ and $\frac{\delta u}{\delta y}$ are supposed continuous

$$\Delta u = u(x+\Delta x, y+\Delta y) - u(x,y)$$

$$= \{u(x+\Delta x, y+\Delta y) - u(x, y+\Delta y)\} + \{u(x, y+\Delta y) - u(x,y)\}$$

$$= \left(\frac{\delta u}{\delta x} + \epsilon_1\right)\Delta x + \left(\frac{\delta u}{\delta y} + \eta_1\right)\Delta y$$

$$= \frac{\delta u}{\delta x}\Delta x + \frac{\delta u}{\delta y}\Delta y + \epsilon_1 \Delta x + \eta_1 \Delta y$$

where, $\epsilon_1 \to 0$, $\eta_1 \to 0$ as $\Delta x \to 0 . \Delta y \to 0$

Similiarly $\frac{\delta v}{\delta x}$ and $\frac{\delta v}{\delta y}$ are supposed continuous we have,

$$\delta v = \left(\frac{\delta v}{\delta x} + \epsilon_2\right)\Delta x + \left(\frac{\delta v}{\delta y} + \eta_2\right)\Delta y$$

$$= \frac{\delta v}{\delta x}\Delta x + \frac{\delta v}{\delta y}\Delta y + \epsilon_2 \Delta x + \eta_2 \Delta y$$

where $\epsilon_2 \to 0$, $\eta_2 \to 0$, as $\Delta x \to 0$ and $\Delta y \to 0$ Then

$$\Delta w = \Delta u + i\Delta v$$

$$= \left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right)\Delta x + \left(\frac{\delta u}{\delta y} + i\frac{\delta v}{\delta y}\right)\Delta y + \epsilon\Delta x + \eta\Delta y \quad \text{--- (ii)}$$

where $\epsilon = \epsilon_1 + i\epsilon_2 \to 0$ and $\eta = \eta_1 + i\eta_2 \to 0$ $\Delta x \to 0$ $\Delta y \to 0$

$$\Delta w = \left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right)\Delta x + \left(-\frac{\delta v}{\delta x} + i\frac{\delta u}{\delta x}\right)\Delta y + \epsilon\Delta x + \eta\Delta y$$

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$= \left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right)(\Delta x + i\Delta y) + \epsilon\Delta x + \eta\Delta y$

Then on dividing by $\Delta z = \Delta x + i\Delta y$ and taking the limit as $\Delta z \to 0$

$\frac{dw}{dz} = f'(z) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}$

So the derivatives exist and is unique i.e. $f(z)$ is analytic in R.

Q: (a) prove that, $u = e^{-x}(x\sin y - y\cos y)$ is harmonic.

(b) Find v such that $f(z) = u + iv$ is analytic.

Sol^n (a):- Given that, $u = e^{-x}(x\sin y - y\cos y)$

$= e^{-x}x\sin y - e^{-x}y\cos y$

$\therefore \frac{\delta u}{\delta x} = e^{-x}\sin y - xe^{-x}\sin y + e^{-x}y\cos y$

$\frac{\delta^2 u}{\delta x^2} = -e^{-x}\sin y - \sin y\,(e^{-x} + x\cdot e^{-x}\cdot(-1)) - e^{-x}y\cos y$

$= -e^{-x}\sin y - e^{-x}\sin y + xe^{-x}\sin y - e^{-x}y\cos y$

$\therefore \frac{\delta^2 u}{\delta x^2} = -2e^{-x}\sin y + xe^{-x}\sin y - e^{-x}y\cos y \quad \text{---- (i)}$

$\frac{\delta u}{\delta y} = e^{-x}.x\cos y - e^{-x}\,y.-\sin y - e^{-x}.\cos y$

$= e^{-x}x\cos y + e^{-x}y\sin y - e^{-x}\cos y$

$\therefore \frac{\delta^2 u}{\delta y^2} = -e^{-x}x\sin y + e^{-x}y\cos y + e^{-x}\sin y + e^{-x}\sin y$

$= 2e^{-x}\sin y + e^{-x}y\cos y - e^{-x}x\sin y \quad \text{---- (ii)}$

(i) + (ii) we get, $\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0$ so it is harmonic.

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(b) Here, $f(z) = u+iv$ since $f(z)$ is analytic if its satisfies the cauchy riemann equations.

$$\frac{\delta v}{\delta y} = \frac{\delta u}{\delta x} = e^{-x}\sin y - xe^{-x}\sin y + e^{-x}y\cos y \quad \text{---(iii)}$$

$$\frac{\partial v}{\delta x} = -\frac{\delta u}{\delta y} = e^{-x}\cos y - xe^{-x}\cos y - e^{-x}y\sin y \quad \text{---(iv)}$$

Integrating (iii) with respect to y and get,

$$v = -e^{-x}\cos y + xe^{-x}\cos y + e^{-x}\{y\sin y + \cos y\} + c_1$$

$$= -e^{-x}\cos y + xe^{-x}\cos y + e^{-x}y\sin y + e^{-x}\cos y + c_1$$

Integrating (iv) with respect to x and get,

$$v = -e^{-x}\cos y - \cos y\,(-x\cdot e^{-x} - 1\cdot\int -e^{-x}dx) + e^{-x}y\sin y$$

$$= -e^{-x}\cos y - \cos y\,(-xe^{-x} - e^{-x}) + e^{-x}y\sin y + c_2$$

$$= -e^{-x}\cos y + xe^{-x}\cos y + e^{-x}\cos y + e^{-x}y\sin y + c_2$$

$$\therefore v = \cancel{-e^{-x}\cos y} + xe^{-x}\cos y + \cancel{e^{-x}\cos y} + e^{-x}y\sin y + c$$

$$= e^{-x}(x\cos y + y\sin y) + c \quad \text{(Ans:-)}$$

(a) $f(z) = u+iv$

$$= e^{-x}(x\sin y - y\cos y) + i\,(x\cos y + y\sin y)e^{-x}$$

$$= e^{-x}\left\{x\cdot\frac{e^{iy}-e^{-iy}}{2i} - y\cdot\frac{e^{iy}+e^{-iy}}{2}\right\} + ie^{-x}\left\{x\cdot\frac{e^{iy}+e^{-iy}}{2} + y\cdot\frac{e^{iy}-e^{-iy}}{2i}\right\}$$

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$= e^{-x}\left\{\frac{xe^{iy}-xe^{-iy}-iye^{iy}-iye^{-iy}}{2i} + ix\left\{\frac{ixe^{iy}+ixe^{-iy}+ye^{iy}-ye^{-iy}}{2i}\right\}\right.$

$= e^{-x}\left\{\frac{\cancel{xe^{iy}}-xe^{-iy}-\cancel{iye^{iy}}-iye^{-iy}-\cancel{xe^{iy}}-xe^{-iy}+\cancel{iye^{iy}}-iye^{-iy}}{2i}\right\}$

$= e^{-x}\left\{\frac{-2xe^{-iy}-2iye^{-iy}}{2i}\right\}$

$= \frac{-xe^{-(x+iy)}-iye^{-(x+iy)}}{i}$

$= i\,e^{-(x+iy)}(x+iy)$

$= ize^{-z}$ (Ans!)

4. Verify that the real and imaginary parts of the following functions satisfy the cauchy riemann equations and deduce their analycity of each function.

(a) $f(z) = z^2+5iz+3-i$

(b) $f(z) = ze^{-z}$

(c) $f(z) = \sin 2z$

solⁿ(a):- $f(z) = z^2+5iz+3-i$

$= (x+iy)^2+5i(x+iy)+3-i$

$= x^2+2xiy+i^2y^2+5ix+5i^2y+3-i$

$= x^2+2ixy-y^2+5ix-5y+3-i$

$= (x^2-y^2-5y+3)+i(2xy+5x-1)$

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$\therefore u = x^2 - y^2 - 5y + 3$

$v = 2xy + 5x - 1$

$\therefore \frac{\delta u}{\delta x} = 2x$, $\frac{\delta u}{\delta y} = -2y - 5 = -(2y+5)$

$\therefore \frac{\delta v}{\delta x} = 2y+5$, $\frac{\delta v}{\delta y} = 2x$

$\therefore \frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$ and $\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$ which satisfies the Cauchy Riemann equation and $f(z)$ is analytic.

(b) $f(z) = z e^{-z}$

$= (x+iy)\ e^{-(x+iy)}$

$= (x+iy)\ e^{-x} \cdot e^{-iy}$

$= e^{-x}(x+iy)\ (\cos y - i\sin y)$

$= e^{-x}(x\cos y - ix\sin y + iy\cos y + y\sin y)$

$= e^{-x}x\cos y - ixe^{-x}\sin y + iye^{-x}\cos y + e^{-x}y\sin y$

$= (e^{-x}x\cos y + e^{-x}y\sin y) + i(ye^{-x}\cos y - xe^{-x}\sin y)$

$\therefore u = e^{-x}x\cos y + e^{-x}y\sin y$

$v = ye^{-x}\cos y - xe^{-x}\sin y$

$\frac{\delta u}{\delta x} = \cos y\,(e^{-x} - xe^{-x}) - e^{-x}y\sin y$

$= \cos y\, e^{-x} - xe^{-x}\cos y - e^{-x}y\sin y$

$\frac{\delta u}{\delta y} = -e^{-x}x\sin y + e^{-x}(y\cdot\cos y + \sin y)$

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$= -xe^{-x}\sin y + ye^{-x}\cos y + e^{-x}\sin y$

$\frac{\delta v}{\delta x} = -e^{-x}y\cos y - \sin y(-xe^{-x} + e^{-x})$

$= -e^{-x}y\cos y + xe^{-x}\sin y - e^{-x}\sin y$

$\frac{\delta v}{\delta y} = e^{-x}(-y\sin y + \cos y) - xe^{-x}\cos y$

$= -e^{-x}y\sin y + e^{-x}\cos y - xe^{-x}\cos y$

$\therefore \frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$ and $\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$ which satisfies cauchy riemann equation and its analytic.

(c) $f(z) = \sin 2z$

$= \sin 2(x+iy)$

$= \sin(2x+2iy)$

$= \sin 2x \cos 2iy + \cos 2x \sin 2iy$

$= \sin 2x \cosh 2y + \cos 2x \, i\sinh 2y$

$= \sin 2x \cosh 2y + i\cos 2x \sinh 2y$

$\therefore u = \sin 2x \cosh 2y$

$v = \cos 2x \sinh 2y$

$\frac{\delta u}{\delta x} = 2\cos 2x \cosh 2y \qquad \frac{\delta v}{\delta x} = -2\sin 2x \sinh 2y$

$\frac{\delta u}{\delta y} = 2\sin 2x \sinh 2y \qquad \frac{\delta v}{\delta y} = 2\cos 2x \cosh 2y$

which satisfies Cauchy riemann equations and it is analytic.

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Q. (a) prove that the function $u = 2x(1-y)$ is harmonic

(b) Find a function v such that $f(z) = u + iv$

(c) Express $f(z)$ interms of z.

Soln:- (a) $u = 2x - 2xy$

$\frac{\delta u}{\delta x} = 2 - 2y \qquad \frac{\delta u}{\delta y} = -2x$

$\frac{\delta^2 u}{\delta x^2} = 0 \qquad \frac{\delta^2 u}{\delta y^2} = 0$

$\therefore \frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0$ so it is harmonic.

(b) Hence, $f(z) = u + iv$

so $\frac{\delta v}{\delta y} = \frac{\delta u}{\delta x} = 2 - 2y$ --- (i)

$\frac{\delta v}{\delta x} = -\frac{\delta u}{\delta y} = 2x$ --- (ii)

Integrating (i) w.r.to y, $v = 2y - y^2 + c_1$

" (ii) " " x, $v = x^2 + c_2$

$\therefore v = x^2 + 2y - y^2 + c$

(c) $f(z) = u + iv$

$= 2x - 2xy + i(x^2 + 2y - y^2)$

$= 2x - 2xy + ix^2 + 2iy - iy^2$

$= 2(x + iy) + ix^2 - 2xy - iy^2$

$= 2(x + iy) + i(x^2 + 2xyi + i^2 y^2)$

$= 2z + iz^2$

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Q. (a) prove that the function $u = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic.

(b) Find a function v such that $f(z) = u + iv$

(c) Express $f(z)$ in terms of z.

Sol^n (a):- $u = x^2 - y^2 - 2xy - 2x + 3y$

$\therefore \frac{\delta u}{\delta x} = 2x - 2y - 2 \qquad \frac{\delta u}{\delta y} = -2y - 2x + 3$

$\frac{\delta^2 u}{\delta x^2} = 2 \qquad \frac{\delta^2 u}{\delta y^2} = -2$

$\therefore \frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0$ so it is harmonic.

(b) Hence $f(z) = u + iv$

$\frac{\delta v}{\delta y} = \frac{\delta u}{\delta x} = 2x - 2y - 2$ --- (i)

$\therefore v = 2xy - y^2 - 2y + c_1$

$\frac{\delta v}{\delta x} = -\frac{\delta u}{\delta y} = 2y + 2x - 3$ --- (ii)

$\therefore v = 2xy + x^2 - 3x + c_2$

$\therefore v = x^2 - y^2 + 2xy - 3x - 2y + c$

(c) $f(z) = u + iv$

$= x^2 - y^2 - 2xy - 2x + 3y + i(x^2 - y^2 + 2xy - 3x - 2y)$

$= x^2 - y^2 - 2xy - 2x + 3y + ix^2 - iy^2 + 2ixy - 3ix - 2iy$

$= i(x^2 + 2ixy + i^2y^2) + (x^2 + 2ixy + i^2y^2) - 2(x + iy) - 3ix + 3y$

$= iz^2 + z^2 - 2z - 3(ix - y) = iz^2 + z^2 - 2z - \underline{3(ix + i^2y)} \rightarrow 3i(x + iy) = 3iz$

$= iz^2 + z^2 - 2z - 3iz$ (Ans)

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Ex. Determine which of the following functions u are harmonic. For each harmonic function find the conjugate harmonic function v and express u+iv as an analytic function of z.

(a) $3x^2y + 2x^2 - y^3 - 2y^2$

(b) $2xy + 3xy^2 - 2y^3$

(c) $xe^x \cos y - ye^x \sin y$

(d) $e^{-2xy} \sin(x^2 - y^2)$

Sol^n (a):- $u = 3x^2y + 2x^2 - y^3 - 2y^2$

$\frac{\delta u}{\delta x} = 6xy + 4x \qquad \frac{\delta^2 u}{\delta x^2} = 6y + 4$

$\frac{\delta u}{\delta y} = 3x^2 - 3y^2 - 4y \qquad \frac{\delta^2 u}{\delta y^2} = -6y - 4$

$\therefore \frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0$ so it is harmonic.

Now, $\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y} = 6xy + 4x$

$\therefore v = 3xy^2 + 4xy + c_1$

$\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x} = \cancel{6y+4} \; -3x^2 + 3y^2 + 4y$

$\therefore v = \cancel{6xy+4x} - x^3 + 3xy^2 + 4xy + c_2$

$\therefore v = 3xy^2 + 4xy - x^3 + c$ (Ans:)

$\therefore f(z) = u + iv$

$= 3x^2y + 2x^2 - y^3 - 2y^2 + i(3xy^2 + 4xy - x^3)$

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$= 3x^2y + 2x^2 - y^3 - 2y^2 + 4xyi + 3xy^2i - ix^3$

$= 2(x^2 - y^2 + 2ixy) - i(x^3 + 3x^2yi + 3xy^2i^2 + y^3)$

$= 2z^2 - iz^3$ (Ans:)

(b) $u = 2xy + 3xy^2 - 2y^3$

$\frac{\delta u}{\delta x} = 2y + 3y^2 \quad \frac{\delta u}{\delta y} = 2x + 6xy - 6y^2$

$\frac{\delta^2 u}{\delta x^2} = 0 \quad \therefore \frac{\delta^2 u}{\delta y^2} = 6x - 12y$

$\therefore \frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} \neq 0$ so it is not harmonic.

(c) $u = xe^x \cos y - ye^x \sin y$

$\frac{\delta u}{\delta x} = xe^x \cos y + e^x \cos y - e^x y \sin y$

$\frac{\delta^2 u}{\delta x^2} = xe^x \cos y + e^x \cos y + e^x \cos y - e^x y \sin y$

$= xe^x \cos y + 2e^x \cos y - e^x y \sin y$

$\frac{\delta u}{\delta y} = -xe^x \sin y - e^x \{y \cos y + \sin y\}$

$= -xe^x \sin y - e^x y \cos y - e^x \sin y$

$\frac{\delta^2 u}{\delta y^2} = -xe^x \cos y + e^x y \sin y - e^x \cos y - e^x \cos y$

$= -xe^x \cos y + e^x y \sin y - 2e^x \cos y$

$\therefore \frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0$ so it is harmonic.

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$\therefore \frac{\delta u}{\delta x} = \frac{\delta v}{\delta y} = xe^x \cos y + e^x \cos y - e^x y \sin y$

$\therefore v = xe^x \sin y + e^x \sin y - e^x \left[y \int \sin y \, dy - \int \left\{ \frac{dy}{dy} \int \sin y \, dy \right\} dy \right]$

$= xe^x \sin y + e^x \sin y + e^x y \cos y - e^x \int \cos y \, dy$

$= xe^x \sin y + \cancel{e^x \sin y} + e^x y \cos y - \cancel{e^x \sin y}$

$= xe^x \sin y + e^x y \cos y + c_1 \quad \text{--- (i)}$

$\therefore \frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x} = e^x x \sin y + e^x y \cos y + e^x \sin y$

$\therefore v = \sin y \left[x \int e^x dx - \int \left\{ \frac{dx}{dx} \int e^x dx \right\} dx \right] + e^x y \cos y + e^x \sin y$

$= \sin y \left\{ xe^x - e^x \right\} + e^x y \cos y + e^x \sin y + c_2$

$= xe^x \sin y - \cancel{e^x \sin y} + e^x y \cos y + \cancel{e^x \sin y} + c_2$

$= xe^x \sin y + e^x y \cos y + c_2$

$\therefore v = xe^x \sin y + e^x y \cos y + c$

$\therefore f(z) = u + iv$

$= xe^x \cos y - ye^x \sin y + i(xe^x \sin y + e^x y \cos y)$

$= xe^x \cos y - ye^x \sin y + ixe^x \sin y + ie^x y \cos y$

$= xe^x (\cos y + i \sin y) + ie^x y \cos y + i^2 ye^x \sin y$

$= xe^x e^{iy} + ie^x y . e^{iy}$

$= (x + iy) e^{x+iy}$

$= ze^z$ (Ans)

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Wrong!!!

(d) $u = e^{-2xy} \sin(x^2 - y^2)$

$\therefore \frac{\delta u}{\delta x} = e^{-2xy} \cdot \cos(x^2 - y^2) \cdot 2x - \sin(x^2 - y^2) \cdot e^{-2xy} \cdot 2y$

$\frac{\delta^2 u}{\delta x^2} = -e^{-2xy} \cdot 2x \sin(x^2 - y^2) \cdot 2x + \cos(x^2 - y^2) \{ e^{-2xy} \cdot 2 \cdot \cancel{xy} - 4x^2 e^{-2xy} \}$

$- e^{-2xy} \cdot 2y \cos(x^2 - y^2) \cdot 2x + \sin(x^2 - y^2) \cdot e^{-2xy} \cdot 4xy$

$= -4x^2 e^{-2xy} \sin(x^2 - y^2) + 2e^{-2xy} \cos(x^2 - y^2) - 4x^2 e^{-2xy} \cos(x^2 - y^2)$

$- e^{-2xy} 4xy \cos(x^2 - y^2) + 4xy\, e^{-2xy} \sin(x^2 - y^2)$

$\therefore \frac{\delta u}{\delta y} = e^{-2xy} \cos(x^2 - y^2) \cdot -2y - \sin(x^2 - y^2) e^{-2xy} 2x$

$= -2y\, e^{-2xy} \cos(x^2 - y^2) - 2x e^{-2xy} \sin(x^2 - y^2)$

$= -2y\, e^{-2xy} \times \sin(x^2 - y^2) \times -2y + \cos(x^2 - y^2) \{ -2y \cdot e^{-2xy} \cdot -2x$

$+ e^{-2xy} \cdot -2 \}$

$- [\{ 2x e^{-2xy} \cos(x^2 - y^2) \cdot -2y \} + \sin(x^2 - y^2) \{ 2x \cdot e^{-2xy} \cdot -2y \}$

$= -4y^2 e^{-2xy} \sin(x^2 - y^2) + 4xy\, e^{-2xy} \cos(x^2 - y^2) - \frac{2\cos(x^2 - y^2)}{e^{-2xy}}$

$+ 4xy\, e^{-2xy} \cos(x^2 - y^2) + 4xy\, e^{-2xy} \sin(x^2 - y^2)$

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chapter - 04

32:- Evaluate $\int_{(0,1)}^{(2,5)} (3x+y)\,dx + (2y-x)\,dy$ along (a) the curve $y = x^2+1$ (b) the straight Line joining $(0,1)$ and $(2,5)$ (c) the straight Lines from $(0,1)$ to $(0,5)$ and then from $(0,5)$ to $(2,5)$. and (d) the straight line from $(0,1)$ to $(2,1)$ and then from $(2,1)$ to $(2,5)$.

solution:- (a) we have $y = x^2+1$

$\therefore dy = 2x\,dx$

$\therefore \int_0^2 (3x + x^2+1)\,dx + (2x^2+2-x)\,dx \cdot 2x$

$= \int_0^2 (3x^2+x^2+1+4x^3+4x-2x^2)\,dx$

$= \int_0^2 (4x^3+7x-x^2+1)\,dx$

$= \left[\frac{x^4}{4}\times 4 + 7\cdot \frac{x^2}{2} - \frac{x^3}{3} + x\right]_0^2$

$= 2^4 + 14 - 8/3 + 2$

$= 32 - 8/3$

$= 88/3$ (Ans:-)

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(b) The equation of Line Joining (0,1) to (2,5)

$$\frac{x-0}{0-2} = \frac{y-1}{1-5}$$

$$\Rightarrow \frac{x}{-2} = \frac{y-1}{-4}$$

$$\Rightarrow y = 2x+1$$

$$\therefore dy = 2dx$$

$$\therefore \int_0^2 (3x+y)\,dx + (2y-x)\,dy$$

$$= \int_0^2 (3x+2x+1)\,dx + (4x+2-x)2dx$$

$$= \int_0^2 (5x+1 \ +8x+4-2x)\,dx$$

$$= \int_0^2 (11x+5)\,dx$$

$$= \left[11\,x^2/2 + 5x\right]_0^2$$

$= 32$ (Ans:-)

(c) Line from (0,1) to (0,5) where y varies from 1 to 5

$$\therefore x = 0$$

$$dx = 0$$

$$\therefore \int_1^5 2y\,dy$$

$$= [y^2]_1^5 = 25-1$$

$$= 24$$

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Line from (0.5) to (2.5) x varies from 0 to 2.

$y = 5$

$\therefore dy = 0$

$\therefore \int_0^2 (3x+y)\,dx$

$= \int_0^2 (3x+5)\,dx$

$= \left[\frac{3x^2}{2} + 5x\right]_0^2$

$= 6+10$

$= 16$ An

$\therefore \int_{(0,1)}^{(2,5)} (3x+y)\,dx + (2y-x)\,dy = 24+16$

$= 40$ (Ans:-)

(d) For the straight line from (0.1) to (2.1) here x varies 0 to 2. $y = 1$

$\therefore dy = 0$

$\therefore \int_0^2 (3x+1)\,dx$

$= \left[\frac{3x^2}{2} + x\right]_0^2$

$= 8$

For the straight line from (2.1) to (2.5), y varies from 1 to 5

$x = 2$

$\therefore dx = 0$

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$\therefore \int_1^5 (2y-2)\,dy$

$= \left[y^2 - 2y\right]_1^5$

$= [25 - 10 - 1 + 2]$

$= 16$

$\therefore \int_{(0,1)}^{(2,5)} (3x+y)\,dx + (2y-x)\,dy = 8 + 16$
$= 24$ (Ans:-)

33. (a) Evaluate $\oint_c (x+2y)\,dx + (y-2x)\,dy$ around the ellipse c defined by $x = 4\cos\theta$, $y = 3\sin\theta$. $0 \le \theta < 2\pi$ if c is described in a counterclockwise direction (b) what is the answer to (a) if c is described in a clockwise direction.

Soln:- (a) we have. $x = 4\cos\theta$

$dx = -4\sin\theta\,d\theta$

$y = 3\sin\theta$

$dy = 3\cos\theta\,d\theta$

$\therefore \oint_c (x+2y)\,dx + (y-2x)\,dy$

$= \int_0^{2\pi} (4\cos\theta + 6\sin\theta) - 4\sin\theta\,d\theta + (3\sin\theta - 8\cos\theta).\,3\cos\theta\,d\theta$

$= \int_0^{2\pi} -(16\sin\theta\cos\theta + 24\sin^2\theta)\,d\theta + (9\sin\theta\cos\theta - 24\cos^2\theta)\,d\theta$

$= \int_0^{2\pi} (-7\sin\theta\cos\theta - 24)\,d\theta$

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$$= -\frac{7}{2}\int_0^{2\pi} \sin 2\theta \, d\theta - 24\int_0^{2\pi} d\theta$$

$$= -\frac{7}{2}\left[\frac{-\cos 2\theta}{2}\right]_0^{2\pi} - 24\left[\theta\right]_0^{2\pi}$$

$$= -\frac{7}{2}\left(\cancel{-\tfrac{1}{2}} + \cancel{\tfrac{1}{2}}\right) - 24(2\pi - 0)$$

$$= -48\pi$$

(b) For clockwise direction, the range will be from 2π to 0. Hence the answer will be 48π.

34:- Evaluate $\oint (x^2 - iy^2)\,dz$ along (a) the parabala $y = 2x^2$ from (1,1) to (2,8) (b) the straight line from (1,1) to (1,8) and then from (1,8) to (2,8). (c) The straight line from (1,1) to (2,8).

(a):- we have $y = 2x^2$

$$\therefore dy = 4x\,dx$$

$$\therefore \oint_C (x^2 - iy^2)\,dz = \int_1^2 (dx + i\,dy)(x^2 - i4x^4)$$

$$= \int_1^2 (x^2 - i4x^4)(dx + i\,4x\,dx)$$

$$= \int_1^2 (x^2 - i4x^4 + i4x^3 + 16x^5)\,dx$$

$$= \left[\frac{x^3}{3} - i\cdot\frac{4x^5}{5} + i\cdot 4\cdot\frac{x^4}{4} + 8\cdot\frac{x^6}{3}\right]_1^2$$

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$$= \tfrac{8}{3} - i\cdot 4\cdot \tfrac{32}{5} + i\cdot 16 + 8\cdot\frac{64}{3} - \tfrac{1}{3} + i\,\frac{4}{5} - i - \tfrac{8}{3}$$

$$= \cancel{\tfrac{8}{3}} + \frac{512}{3} - \tfrac{1}{3} - \cancel{\tfrac{8}{3}} - i\left(\frac{128}{5}\right) + 16i - i + \frac{4}{5}i$$

$$= \frac{511}{3} - \tfrac{49}{5}\,i \quad \text{(Ans:-)}$$

(b):- For the straight line from $(1,1)$ to $(1,8)$ here y varies from 1 to 8. $x = 1$ $\therefore dx = 0$

$$\oint_c (x^2 - iy^2)\,dz = \oint_c (x^2 - iy^2)(dx + i\,dy)$$

$$= \int_1^8 (1 - iy^2)\cdot i\,dy$$

$$= \int_1^8 (i + y^2)\,dy$$

$$= \left[iy + y^3/3\right]_1^8$$

$$= \left[8i + \frac{8^3}{3} - i - \frac{1}{3}\right]$$

$$= 7i + \frac{511}{3}$$

For the line $(1,8)$ to $(2,8)$ $y = 8$ $\therefore dy = 0$

$$\oint_c (x^2 - iy^2)\,dz = \oint_c (x^2 - iy^2)(dx + i\,dy)$$

$$= \int_1^2 (x^2 - 64i)\,dx$$

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$= \left[\frac{x^3}{3} - 64xi\right]_1^2$

$= \frac{8}{3} - 64i \times 2 - \frac{1}{3} + 64i$

$= \frac{7}{3} - 64i$

$\therefore$ The answer is $= 7i + \frac{511}{3} + \frac{7}{3} - 64i$

$= \frac{518}{3} - 57i$

(a) The equation of straight line from (1,1) to (2,8)

$\frac{x-1}{1-2} = \frac{y-1}{1-8}$

$\Rightarrow 7x - 7 = y - 1$

$\Rightarrow y = 7x - 6$

$\therefore dy = 7dx$

$\therefore \oint_c (x^2 - iy^2)\,dz = \int_1^2 \{x^2 - i(7x-6)^2\}(dx + i\cdot 7dx)$

$= \int_1^2 (x^2 - i49x^2 + i84x - 36i)(dx + 7idx)$

$= \int_1^2 (x^2 - 49x^2 i + 84ix - 36i + 7ix^2 + 343x^2 - 588xi + 252)dx$

$= \int_1^2 (344x^2 - 42x^2 i - 588x + 84xi - 36i + 252)$

$= 344\cdot\left[\frac{x^3}{3}\right]_1^2 - 42i\cdot\left[\frac{x^3}{3}\right]_1^2 - 588\cdot\left[\frac{x^2}{2}\right]_1^2 + 84i\left[\frac{x^2}{2}\right]_1^2 -$

$36i[x]_1^2 + 252[x]_1^2$

$= \frac{518}{3} - 8i$ (Ans:)

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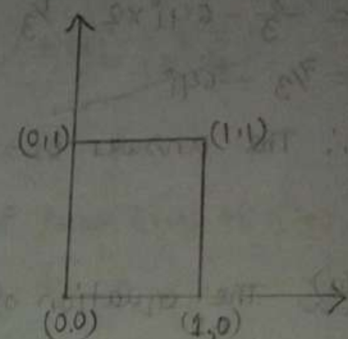
35:- Evaluate $\oint_C |z|^2 dz$ around the square with the vers. (0,0) (1,0) (1,1) (0,1).

we have. $\oint_C |z|^2 dz$



$= \oint_C (x^2+y^2)\,(dx+i\,dy)$

when $(0,0) \rightarrow (1,0)$

$y=0 \therefore dy=0$

$= \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$

$(1,0) \rightarrow (1,1)$ we get,

$x=1 \therefore dx=0$

$\int_0^1 (1+y^2)\,i\,dy$

$= i\left[y + y^3/3\right]_0^1$

$= i[1+1/3]$

$= \frac{4i}{3}$

when $(1,1) \rightarrow (0,1)$ we get,

$y=1$

$\therefore dy=0$

$\int_1^0 (x^2+1)\,dx = \left[\frac{x^3}{3}+x\right]_1^0 = -4/3$

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when $(0,1) \to (0,0)$ then we get,

$x = 0 \therefore dx = 0$

$\int_1^0 y^2 i\,dy$

$= i\left[\frac{y^3}{3}\right]_1^0$

$= -i/3$

$\therefore \oint_C |z|^2 dz = 1/3 + 4i/3 - 4/3 - i/3$

$= -1 + i$ (Ans:-)

36:- Evaluate $\oint_C (z^2+3z)dz$ along (a) the circle $|z| = 2$ from $(2,0)$ to $(0,2)$ in a counterclockwise direction (b) the straight line from $(2,0)$ to $(0,2)$ (c) the straight line from $(2,0)$ to $(2,2)$ and $(2,2)$ to $(0,2)$

Solution:- (a) we have $|z| = 2$

$\Rightarrow z = 2e^{i\theta} \quad 0 \le \theta \le 2\pi$

$\therefore dz = 2e^{i\theta} i\,d\theta$

Now $\oint_C (z^2+3z)\,dz = \int_0^{\pi/2} (4e^{2i\theta} + 6e^{i\theta})\, 2ie^{i\theta}\, d\theta$

$= 4i\int_0^{\pi/2} (2e^{3i\theta} + 3e^{2i\theta})\, d\theta$

$= 4i\left[\frac{2\cdot e^{3i\theta}}{3i} + \frac{3\cdot e^{2i\theta}}{2i}\right]_0^{\pi/2}$

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$$= 4i\left[\cancel{x}\, \tfrac{2}{3i}\left\{e^{3i\cdot\pi/2}\right\} + \tfrac{3}{2i}\left(e^{2i\cdot\pi/2}\right) - \left(\tfrac{2}{3i}e^{0} + \tfrac{3}{2i}\cdot e^{0}\right)\right]$$

$$= 4i\left[\frac{2}{3i}\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) + \frac{3}{2i}\left(\cos\pi + i\sin\pi\right) - \left(\frac{2}{3i} + \frac{3}{2i}\right)\right]$$

$$= 4i\left[\frac{2}{3i}(0-i) + \tfrac{3}{2i}(-1+0) - \frac{13\cancel{i}}{6i}\right]$$

$$= 4i\left[-\tfrac{2}{3} - \frac{3}{2i} - \frac{13}{6i}\right]$$

$$= \frac{-8i}{3} - \frac{88}{6}$$

$$= -\frac{44}{3} - \frac{8}{3}i \text{ (Ans:-)}$$

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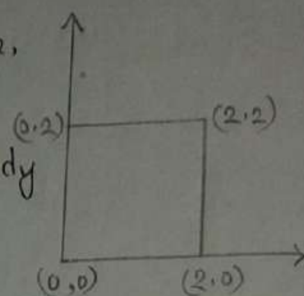
46:- Verify Green's theorem in the plane for $\oint_C (x^2-2xy)\,dx + (y^2 - x^3y)\,dy$ where c is a square with vertices at (0.0), (2.0) (2.2), (0.2)

From green's theorem we have,

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}\right) dx\,dy$$



Given that,

$$\oint_C (x^2-2xy)\,dx + (y^2-x^3y)\,dy$$

Here $(0,0) \longrightarrow (2,0)$

$y=0 \quad \therefore \; dy = 0$

$$\int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$$

$(2,0) \longrightarrow (2,2)$

$x=2, \quad dx = 0$

$$\int_0^2 (y^2-8y)\,dy = \left[y^3/3 - 4y^2\right]_0^2 = \frac{8}{3} - 16$$

$(2,2) \rightarrow (0,2) \quad \therefore \; y = 2 \;\therefore dy = 0$

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$\int_2^0 (x^2 - 2x2)\,dx = \int_2^0 (x^2 - 4x)\,dx$

$= \left[\frac{x^3}{3} - 2x^2\right]_2^0$

$= 8 - 8/3$

$(0.2) \rightarrow (0.0) \quad x = 0.\ dx = 0$

$\int_2^0 y^2\,dy = [y^3/3]_2^0 = -8/3$

$\therefore \oint_c (x^2 - 2xy)\,dx + (y^2 - x^3y)\,dy = \cancel{8/3} + \cancel{8/3} - 16 + 8 - \cancel{8/3} - \cancel{8/3}$

$= -8$

For R.H.S green's theorem. $P = x^2 - 2xy$

$Q = y^2 - x^3y$

$\iint_R \left(\frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}\right) dx\,dy = \int_0^2\int_0^2 -3x^2y - (-2x)\,dx\,dy$

$= \int_0^2 \int_0^2 (2x - 3x^2y)\,dx\,dy$

$= \int_0^2 \left[2xy - 3/2\, x^2y^2\right]_0^2 dx$

$= \int_0^2 (4x - 6x^2)\,dx$

$= \left[2x^2 - 2x^3\right]_0^2$

$= 8 - 16$

$= -8$ (proved)

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Q 38:- Evaluate $\int_{i}^{2-i} (3xy + iy^2)\, dz$ (a) along the straight line joining $z = i$, $z = 2-i$ (b) along the curve $x = 2t-2$, $y = 1+t-t^2$

(a) The straight Line from $(0,1)$ to $(2,-1)$ is,

$$\frac{y-1}{1+1} = \frac{x-0}{0-2}$$

$$\Rightarrow -y+1 = x$$

$$\Rightarrow x = 1-y$$

$$\therefore dx = -dy$$

$$\int_{i}^{2-i} (3xy + iy^2)\, dz = \int_{1}^{-1} \{3(1-y)y + iy^2\} (dx + idy)$$

$$= \int_{1}^{-1} (3y - 3y^2 + iy^2)(-dy + idy)$$

$$= \int_{1}^{-1} (3y - 3y^2 + iy^2 + 3iy - 3iy^2 - y^2)\, dy$$

$$= \int_{1}^{-1} (-4iy^2 + 3iy - 3y + 2y^2)\, dy$$

$$= \left[-4i \cdot y^3/3 + 3i \cdot y^2/2 - 3 \cdot y^2/2 + 2 \cdot y^3/3\right]_{1}^{-1}$$

$$= \left[-4i \cdot -1/3 + 3i/2 - 3 \cdot 1/2 - 2/3\right] - \left[-4i/3 + 3i/2 - 3/2 + 2/3\right]$$

$$= \frac{4i}{3} + \cancel{3i/2} - \cancel{3/2} - 2/3 + 4i/3 - \cancel{3i/2} + \cancel{3/2} - 2/3$$

$$= \frac{8i}{3} - \frac{4}{3} \quad \text{(Ans:-)}$$

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(b) at $x = 0,\ 2t - 2 = 0,\ t = 1$

$x = 2,\ 2t - 2 = 2 \therefore t = 2$

$$\int_1^2 \left\{ 3(2t-2)(1+t-t^2) + i(1+t-t^2)^2 \right\} \left\{ d(2t-2) + i\,d(1+t-t^2) \right.$$

$= \cdot\ -$

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39:- Evaluate $\oint_C \bar{z}^2 dz$ around the circles (a) $|z|=1$ (b) $|z-1|=1$

(a) $|z|=1$

$\Rightarrow z=e^{i\theta}$

$\therefore dz = ie^{i\theta}d\theta$

$\bar{z}=e^{-i\theta}$

$\oint_C \bar{z}^2 dz$

$= \int_0^{2\pi} (e^{-i\theta})^2 \cdot i \cdot e^{i\theta} d\theta$

$= \int_0^{2\pi} e^{-2i\theta} \cdot i \cdot e^{i\theta} d\theta$

$= i\int_0^{2\pi} e^{-i\theta} d\theta$

$= i\left[\frac{e^{-i\theta}}{-i}\right]_0^{2\pi}$

$= \left[e^{-i\theta}\right]_{2\pi}^{0}$

$= e^0 - e^{-2\pi i}$

$= 1-\{\cos(-2\pi) + \sin(-2\pi)\}$

$= 1-(1+0)$

$= 1-1$

$= 0$ (Ans:-)

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(b) $\oint_C \bar{z}^2 dz$ $\qquad |z-1| = 1$

$\quad$ i.e. $z - 1 = e^{i\theta}$

$= \int_0^{2\pi} (1+e^{-i\theta})^2 \cdot i \cdot e^{i\theta} d\theta$ $\quad \Rightarrow z = 1 + e^{i\theta}$

$\quad \bar{z} = 1 + e^{-i\theta}$

$= \int_0^{2\pi} (1 + e^{-2i\theta} + 2e^{-i\theta})$ $\quad \therefore dz = ie^{i\theta} d\theta$

$\quad i.e^{i\theta} d\theta$

$= \int_0^{2\pi} (ie^{i\theta} + ie^{-i\theta} + 2i) d\theta$

$= \left[\frac{i \cdot e^{i\theta}}{i} + i \cdot \frac{e^{-i\theta}}{-i} + 2i\theta \right]_0^{2\pi}$

$= \left[e^{i\theta} - e^{-i\theta} + 2i\theta \right]_0^{2\pi}$

$= e^{2\pi i} - e^{-2\pi i} + 2i \cdot 2\pi$

$= 4\pi i$ (Ans:-)

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chapter 4

Q. Evaluate $\int_{(0,3)}^{(2,4)} (2y+x^2)\,dx + (3x-y)\,dy$ along (a) the parabola $x=2t$, $y=t^2+3$ (b) the straight from (0,3) to (2,3) and then from (2,3) to (2,4) (c) a straight line from (0,3) to (2,4).

Soln:- (a) when, $x=2t$ $\quad y=t^2+3$

$\therefore t=0 \qquad \therefore t=0$

$\therefore 2=2t \qquad 4=t^2+3$

$\therefore t=1 \qquad \therefore t=1$

The given integrals, $\int_0^1 \{2(t^2+3)+4t^2\}2dt + \{3\times(2t) - (t^2+3)\}2t\,dt$

$= \int_0^1 (2t^2+6+4t^2)\,2dt + \int_0^1 (6t-t^2-3)\,2t\,dt$

$= \int_0^1 (4t^2+12+8t^2)\,dt + \int_0^1 (12t^2-2t^3-6t)\,dt$

$= \left[\frac{12t^3}{3} + 12t + \frac{12t^3}{3} - \frac{2t^4}{4} - 6t^2/2\right]_0^1$

$= [4+12+4-1/2-3]$

$= 8+24+8-1-6/2$

$= 33/2$

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(b) The straight from $(0,3)$ to $(2,3)$ so that, $y=3, dy=0$

$\therefore \int_0^2 (6+x^2)\,dx + 0$

$= \left[6x + \frac{x^3}{3}\right]_0^2$

$= 12 + 8/3$

$= \frac{44}{3}$

The straight Line from $(2,3)$ to $(2,4)$ $x=2$, $dx=0$

$\therefore \int_3^4 (2y+4)\times 0 + (6-y)\,dy$

$= \left[6y - \frac{y^2}{2}\right]_3^4$

$= 24 - 8 - 18 + 9/2$

$= 5/2$ $\therefore \frac{44}{3} + 5/2 = 103/6$

(c) A straight line for $(0,3)$ to $(2,4)$

$$\frac{y-3}{3-4} = \frac{x-0}{0-2}$$

$\Rightarrow -2y + 6 = -x$

$\Rightarrow 2y - x = 6$

$\Rightarrow x = 2y - 6$

$\therefore dx = 2dy$

$\therefore \int_3^4 \{2y + (2y-6)^2\}\, 2dy + \{3(2y-6) - y\}dy$

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$= \int_3^4 (8y^2 - 39y + 54)\, dy$

$= \left[\frac{8y^3}{3} - \frac{39y^2}{2} + 54y\right]_3^4$

$= 97/6$

Q. Evaluate $\oint_c \bar{z}\, dz$ from $z = 0$ to $z = 4+2i$ along the curv c given by (a) $z = t^2 + it$ (b) the line from $z = 0$ to $z = 2i$ an then the line from $z = 2i$ to $z = 4+2i$

soln:- The given Integrals.

$\oint_c (x - iy)(dx + idy) = \oint_c x dx + y dy + i\oint_c x dy - y dx$

The curve $z = t^2 + it$

$x = t^2, \; y = t$

Here, $0 = t^2 \therefore t = 0$, $0 = t \therefore t = 0$

$t^2 = 4 \therefore t = 2$ $\quad y = t \therefore t = 2$

$\therefore \int_0^2 t^2\, 2t dt + t dt + i\int_0^2 t^2 dt - t \times 2t dt$

$= \int_0^2 2t^3 dt + t dt + i\int_0^2 t^2 dt - 2t^2 dt$

$= \left[\frac{2t^4}{4} + t^2/2\right]_0^2 + i\left[-t^3/3\right]_0^2$

$= 10 - 8i/3$

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(b) The given integrals, $\oint_C (x-iy)(dx+idy)$

$= \oint_C xdx + ydy + i\oint_C xdy - ydx$

The line from $z=0$, to $z=2i$ as same as $(0,0)$ $(0,2)$

$x=0 \therefore dx=0$

$= \int_0^2 0\times 0 + ydy + i\int_0^2 0\times dy - y\times 0$

$= [y^2/2]_0^2$

$= 2$

The line from $z=2i$ to $z=4+2i$ is as same as $(0,2)(4,2)$

$y=2 \therefore dy=0$

$\int_0^4 xdx + 2\times 0 + i\int_0^4 x\cdot 0 - 2dx$

$= [x^2/2]_0^4 + i[-2x]_0^4$

$= 8-8i$

$\therefore$ The required value $= 2+8-8i$

$= 10-8i$

Q. Verify green's theorem for $\oint_C (2xy-x^2)dx + (x+y^2)dy$ where c is the closed curve of the region bounded by $y=x^2$ and $y^2=x$.

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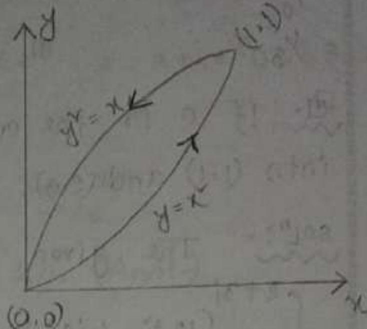
solⁿ:- when $y = x^2$ then

$$\int_0^1 \left[\{2x \cdot x^2 - x^2\}\, dx + \{x + x^4)\, 2x\,dx\right]$$



$$= \int_0^1 (2x^3 + x^2 + 2x^5)\, dx$$

$$= 7/6$$

Along $y^2 = x$ then,

$$\int_1^0 \left[(2y^3 - y^4)\, 2y\,dy + (y^2 + y^2)\, dy\right]$$

$$= \int_1^0 (4y^4 - 2y^5 + 2y^2)\, dy$$

$$= -17/15 \qquad \therefore \; 7/6 - 17/15 = 1/30$$

$$\therefore \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= \iint_R -\left\{\frac{\partial}{\partial x}(x + y^2) - \frac{\partial}{\partial y}(2xy - x^2)\right\} dx\,dy$$

$$= \iint_R (1 - 2x)\, dx\,dy$$

$$= \int_{x=0}^{1} \int_{x^2}^{\sqrt{x}} (1 - 2x)\, dx\,dy$$

$$= \int_0^1 (y - 2xy)\Big|_{x^2}^{\sqrt{x}}$$

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$= \int_0^1 (x^{1/2} - 2x^{3/2} - x^2 + 2x^3)\,dx$

$= 1/30$

Q. If c is the curve $y = x^3 - 3x^2 + 4x - 1$ joining points (1,1) and (2,3) Find the value of $\oint_C (12z^2 - 4iz)\,dz$

Soln:- The given integrals is,

$$\int_{1+i}^{2+3i} (12z^2 - 4iz)\,dz$$

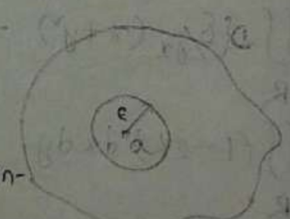
$$= \left[4z^3 - 2iz^2\right]_{1+i}^{2+3i}$$

$$= 4(2+3i)^3 - 2i(2+3i)^2 - (1+i)^3 + 2i(1+i)^2$$

$$= -156 + 38i$$

Q. Evaluate $\oint_C \frac{dz}{z-a}$ where c is any simple closed curve c and $z=a$ is (i) outside c (ii) inside c.

Soln:- (i) If a outside c



$f(z) = 1/z-a$ is analytic everywhere inside and on c. Hence by cauchy integral theorem,

$\oint dz/z-a = 0$

(ii) Suppose a is inside c and let T be a circle of radious r with centre $z=a$ so that T is inside c.

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$\oint_C \frac{dz}{z-a} = \oint_T dz/z-a$ - - (i)

So on T, $|z-a| = \epsilon$ or $z-a = \epsilon e^{i\theta}$ $\therefore z = a+\epsilon e^{i\theta}$ $0 \le \theta < 2\pi$

$\therefore dz = i\epsilon e^{i\theta} d\theta$

$\therefore \int_{\theta=0}^{2\pi} \frac{i\epsilon \cancel{e^{i\theta}} d\theta}{\epsilon \cancel{e^{i\theta}}} = i\int_0^{2\pi} d\theta = i[\theta]_0^{2\pi}$

$= 2\pi i$ (Ans:-)

$\therefore \oint_T \frac{f(z)}{z-a}$

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30:- Evaluate $\frac{1}{2\pi i}\oint_c \frac{e^z dz}{z-2}$ if c is (a) the circle $|z|=3$ (b) $|z|=1$

Solution:- Let $f(z)=e^z$

here $a=2$

Now from cauchy integral formula,

$$f(a)=\frac{1}{2\pi i}\oint_c \frac{f(z)\,dz}{z-a}$$

$$\therefore f(2)= \frac{1}{2\pi i}\oint \frac{e^z}{z-2}\,dz$$

(a) Since $z=2$ inside the circle $|z|=3$ hence by cauchy integral formula,

$$f(2)=\frac{1}{2\pi i}\oint_c \frac{e^z dz}{z-2}$$

Now, $f(z)=e^z$

$\therefore f(2)=e^2$ (Ans:-)

(b) Since $z=2$ outside the circle, $|z|=1$ hence by cauchy integral formula, $f(z)=0$

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33:- Evaluate $\frac{1}{2\pi i}\oint_C \frac{\cos\pi z}{z^2-1}dz$ around a rectangle vertices at (a) $2\pm i, -2\pm i$ (b) $-i, 2-i, 2+i, i$

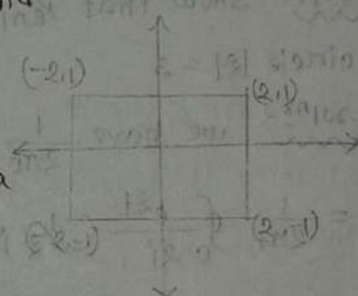
Soln:- We have, $\frac{1}{2\pi i}\oint_C \frac{\cos\pi z}{z^2-1}dz$

$$= \frac{1}{2\pi i}\oint_C \frac{\cos\pi z}{(z+1)(z-1)}dz$$

$$= \frac{1}{4\pi i}\left[\oint_C \frac{\cos\pi z}{z-1}dz - \oint_C \frac{\cos\pi z}{z+1}dz\right]$$

$$= \frac{1}{4\pi i}\oint \frac{\cos\pi z}{z-1}dz - \frac{1}{4\pi i}\oint \frac{\cos\pi z}{z+1}dz$$

(a) since the poles $z=1$ and $z=-1$ both are inside the rectangle. Hence by cauchy integral formula we get,



$$\frac{1}{2\pi i}\oint_C \frac{\cos\pi z}{z^2-1}dz = \frac{1}{2}\left[\frac{1}{2\pi i}\oint_C \frac{\cos\pi z}{z-1}dz - \frac{1}{2\pi i}\oint_C \frac{\cos\pi z}{z+1}dz\right]$$

$$= \frac{1}{2}\left[f(1) - f(-1)\right]$$

$$= \frac{1}{2}\left[\cos\pi(1) - \cos\pi(-1)\right]$$

$$= \frac{1}{2}(-1+1) = 0$$

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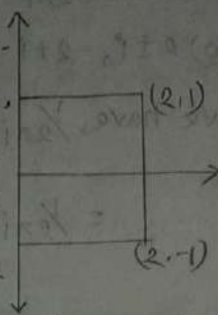
Since $z=1$ is inside the rectangle and $z=-1$ is outside the rectangle we get,



$$\frac{1}{2\pi i}\oint_C \frac{\cos\pi z\,dz}{z^2-1} = \frac{1}{2}\left[\frac{1}{2\pi i}\oint_C \frac{\cos\pi z\,dz}{z-1} - \frac{1}{2\pi i}\oint_C \frac{\cos\pi z\,dz}{z+1}\right.$$

$$= \frac{1}{2}\left[f(1) - 0\right]$$

$$= \frac{1}{2}\left[\cos\pi(1)\right]$$

$$= \frac{1}{2}\times(-1) = -\frac{1}{2}$$

34:- Show that $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2+1}\,dz = \sin t$ if $t>0$ and C is a circle $|z|=3$

Soln:- we have, $\frac{1}{2\pi i}\oint_C \frac{e^{zt}\,dz}{z^2+1} = \frac{1}{2\pi i}\oint_C \frac{e^{zt}\,dz}{z^2-i^2}$

$$= \frac{1}{2\pi i}\oint_C \frac{e^{zt}}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)dz$$

$$= \frac{1}{4\pi i^2}\left[\oint_C \frac{e^{zt}}{z-i}\,dz - \oint_C \frac{e^{zt}}{z+i}\,dz\right]$$

$$= \frac{1}{2i}\left[\frac{1}{2\pi i}\oint_C \frac{e^{zt}\,dz}{z-i} - \frac{1}{2\pi i}\oint_C \frac{e^{zt}\,dz}{z+i}\right]$$

Since both $z=i$ and $z=-i$ both are inside C by Cauchy integral formula,

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$$\frac{1}{2\pi i}\oint_C \frac{e^{zt}\,dz}{z^2+1} = \frac{1}{2i}\left[f(i) - f(-i)\right]$$

$$= \frac{1}{2i}\left(e^{it} - e^{-it}\right)$$

$$= \sin t \text{ [showed]}$$

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chapter - 5

Q. Evaluate $\oint_C \frac{\sin 3z}{(z+\pi/2)}$ if c is the circle $|z| = 5$

Soln:- We have cauchy integral formula,

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{z-a}$$

$$\Rightarrow \oint_C \frac{f(z)\,dz}{z-a} = 2\pi i f(a)$$

Here. $a = -\pi/2$

$f(z) = \sin 3z$

Since $z = -\pi/2$ is inside the circle $|z| = 5$

$\therefore f(-\pi/2) = \sin 3(-\pi/2)$

$= 1$

$$\therefore \oint_C \frac{\sin 3z}{z+\pi/2}\,dz = 2\pi i \cdot 1$$

$$= 2\pi i$$

Q. Evaluate $\oint_C \frac{e^{3z}}{(z-\pi i)}$ if c is (a) the circle $|z-1| = 4$ (b) the ellipse $|z-2| + |z+2| = 6$

SoLn:- We have cauchy integral formula,

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\,dz$$

Here, $f(z) = e^{3z}$

$\therefore a = \pi i$

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(a) Since $z = \pi i$ is inside the circle $|z-1| = 4$

$$f(\pi i) = e^{3\cdot\pi i} = \cos 3\pi + i\sin 3\pi$$
$$= -1$$

$$\therefore f(\pi i) = \frac{1}{2\pi i}\oint_C \frac{e^{3z}dz}{z-\pi i}$$

$$\Rightarrow \oint_C \frac{e^{3z}dz}{z-\pi i} = 2\pi i \times -1$$
$$= -2\pi i$$

(b) Since $z = \pi i$ is outside the region of ellipse $|z-2| + |z+2| = 6$

Hence by Cauchy's integral formula,

$$\oint_C e^{3z}/z-\pi i \; dz = 0$$

Q. Evaluate $\oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)} dz$ where C is the circle $|z| = 5$

Soln:- Since, $\frac{1}{(z-2)(z-1)} = \frac{1}{z-2} - \frac{1}{z-1}$

we have, $\oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)} dz$

$$= \oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{z-2} dz - \oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{z-1} dz$$

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By Cauchy's integral formula,

$$\oint_c \frac{\sin\pi z^2 + \cos\pi z^2}{z-2}\,dz = 2\pi i\,\{\sin\pi(2^2) + \cos\pi(2)^2\}$$
$$= 2\pi i$$

$$\oint_c \frac{\sin\pi z^2 + \cos\pi z^2}{z-1}\,dz = 2\pi i\,\{\sin\pi(1)^2 + \cos\pi(1)^2\}$$
$$= -2\pi i$$

Since $z=1$ and $z=2$ are inside the circle.

$= 2\pi i - (-2\pi i)$

$= 4\pi i$

Q. Evaluate $\oint_c \frac{e^z\,dz}{(z^2+\pi^2)^2}$ where c is the circle $|z|=4$

Soln:- $\oint_c \frac{e^z dz}{(z^2+\pi^2)^2} = \oint_c \frac{e^z dz}{(z-\pi i)^2(z+\pi i)^2}$

The poles are $z=\pm\pi i$ of order two and inside c.

Residue at $z=\pi i$

$$\lim_{z\to\pi i} \frac{1}{1!}\frac{d}{dz}\left\{(z-\pi i)^2 \frac{e^z}{(z-\pi i)^2(z+\pi i)^2}\right\}$$

$$= \lim_{z\to\pi i} \frac{d}{dz}\frac{e^z}{(z+\pi i)^2}$$

$$= \frac{\pi+i}{4\pi^3}$$

Residue at $z=-\pi i$, $\lim_{z\to-\pi i} \frac{1}{1!}\frac{d}{dz}\left\{(z+\pi i)^2 \frac{e^z}{(z-\pi i)^2(z+\pi

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$= \lim_{z \to -\pi i} \frac{d}{dz}\left\{ e^z/(z-\pi i)^2 \right\}$

$= \frac{\pi - i}{4\pi^3}$

$\therefore \oint_c \frac{e^z dz}{(z^2+\pi^2)^2} = 2\pi i$ (sum of residues)

$= 2\pi i \left(\frac{\pi + i}{4\pi^3} + \frac{\pi - i}{4\pi^3} \right)$

$= 2\pi i \times \frac{2\pi}{4\pi^3}$

$= \frac{i}{\pi}$

chapter-5

Evaluate $\oint_c \frac{e^{2z}}{(z+1)^4} dz$ c is the curve $|z| = 3$

Residue at $z = -1$

$$\lim_{z \to -1} \frac{1}{3!} \frac{d^3}{dz^3} \left\{ \cancel{(z+1)^4} \frac{e^{2z}}{\cancel{(z+1)^4}} \right\}$$

$$= \lim_{z \to -1} \frac{1}{3!} \frac{d^2}{dz^2} (2e^{2z})$$

$$= \lim_{z \to -1} \frac{1}{6} \frac{d}{dz} (4e^{2z})$$

$$= \lim_{z \to -1} \frac{1}{6} (8e^{2z})$$

$$= \frac{1}{6} 8e^{-2}$$

$$\therefore \oint_c \frac{e^{2z}}{(z+1)^4} dz = 2\pi i \times \frac{8}{6} e^{-2}$$

$$= \frac{8}{3} \pi i e^{-2}$$

Evaluate $\oint_c \frac{e^{iz}}{z^3} dz$ where c is the curve $|z| = 2$

Residue at $z = 0$ is $\lim_{z \to 0} \frac{1}{2!} \frac{d^2}{dz^2} \left\{ \frac{e^{iz}}{\cancel{z^3}} \times \cancel{z^3} \right\}$

$$= \lim_{z \to 0} \frac{1}{2!} (ie^{iz}) \cdot \frac{d}{dz}$$

$$= \lim_{z \to 0} \frac{1}{2!} (i^2 e^{iz})$$

$$= -\frac{1}{2}$$

$$\therefore \oint_c \frac{e^{iz}}{z^3} dz = -\pi i$$

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4. Find the value of (a) $\oint_C \frac{\sin^6 z}{z-\pi/6}\,dz$ (b) $\oint_C \frac{\sin^6 z}{(z-\pi/6)^3}$ if c is the circle $|z|=1$

(a) Here, $a = \pi/6$

$\therefore f(a) = \sin^6(\pi/6)$

$= \sin(\pi/6)^6$

$= \frac{1}{64}$

$\therefore \oint_C \frac{\sin^6 z}{z-\pi/6} = 2\pi i \times \frac{1}{64}$

$= \frac{\pi i}{32}$

(b) Residue at $z = \pi/6$

$\lim_{z\to\pi/6} \frac{1}{2!} \frac{d^2}{dz^2} \left\{ \cancel{(z-\pi/6)^3} \times \frac{\sin^6 z}{\cancel{(z-\pi/6)^3}} \right\}$

$= \lim_{z\to\pi/6} \frac{1}{2}\, d/dz\, (6\sin^5 z \cdot \cos z)$

$= \lim_{z\to\pi/6} \frac{1}{2} (6\sin^5 z \cdot -\sin z + \cos z \cdot 5\sin^4 z \cdot \cos z)$

$= \frac{1}{2}(6 \cdot \frac{1}{32} \times -1/2 + \sqrt{3}/2 \cdot 5 \cdot \frac{1}{16} \cdot \sqrt{3}/2)$

$= \frac{21}{32}$

$\therefore \oint_C \frac{\sin^6 z}{(z-\pi/6)^3} = 2\pi i \times \frac{21}{32}$

$= \frac{21\pi i}{16}$

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Q. Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{(z^2+1)^2}\,dz$ if $t>0$ and c is the circle $|z|=3$.

Soln:- Residue at $z=-i$,

$$= \lim_{z\to -i} \frac{1}{1!} \frac{d}{dz}\left\{\frac{e^{zt}\times \cancel{(z+i)^2}}{(z-i)^2\cancel{(z+i)^2}}\right\}$$

$$= \lim_{z\to -i} \frac{1}{1!} \frac{(z-i)^2\cdot te^{zt}-e^{zt}\cdot 2(z-i)\cdot}{(z-i)^4}$$

$$= \frac{(-i-i)^2\cdot te^{-it}-e^{-it}2(-i-i)}{(-i-i)^4}$$

$$= \frac{4i^2te^{-it}+e^{-it}4i}{16}$$

$$= \frac{4ie^{-it}-4te^{-it}}{16}$$

$$= \frac{-e^{-it}(t-i)}{4}$$

Residue at $z=i$ $\lim_{z\to i} \frac{1}{1!} \frac{d}{dz}\left(\frac{e^{zt}}{(z+i)^2}\right)$

$$= \lim_{z\to i} \frac{1}{1!}\left\{\frac{(z+i)^2 e^{zt}\cdot t - e^{zt}\cdot 2(z+i)}{(z+i)^4}\right\}$$

$$= \lim_{z\to i} \frac{1}{1!}\left\{\frac{(z+i)^2e^{zt}\cdot t-e^{zt}\cdot 2(z+i)}{(z+i)^4}\right\}$$

$$= \frac{-4e^{it}(t+i)}{16}$$

$$= \frac{-e^{it}(t+i)}{4}$$

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$\therefore \frac{1}{2\pi i} \oint \frac{e^{zt}}{(z^2+1)^2} dz = \frac{-e^{it}(t+i)}{4} + \frac{-e^{-it}(t-i)}{4}$

$= \frac{-te^{-it} + ie^{-it} - te^{it} - ie^{it}}{4}$

$= \frac{-t(e^{-it}+e^{it}) - i(e^{it}-e^{-it})}{4}$

$= \frac{-t(e^{it}+e^{-it})}{4} + \frac{-i(e^{it}-e^{-it})}{4}$

$= -\frac{t}{4}(\cos t + \cancel{i\sin t} + \cos t - \cancel{i\sin t}) + \frac{i(e^{it}-e^{-it})}{4i^2}$

$= -t/2 \cos t + \frac{1}{4i}(\cancel{\cos t} + i\sin t - \cancel{\cos t} + i\sin t)$

$= -t/2 \cos t + 1/2 \sin t$

$= 1/2 (\sin t - t\cos t)$

Q. prove that cauchy's integral formula $f(a) = 1/2\pi i \oint_C \frac{f(z)dz}{z-a}$

solⁿ:- The function $f(z)/z-a$ is analytic inside and on c at centre $z=a$

$\oint_C \frac{f(z)\,dz}{z-a} = \oint_T \frac{f(z)\,dz}{z-a}$ — (i)

we choose T is a circle of radious ϵ.

$|z-a| = \epsilon$

$\Rightarrow z-a = \epsilon e^{i\theta}$

$\Rightarrow z = a+\epsilon e^{i\theta} \quad \therefore dz = i\epsilon e^{i\theta}$

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$$\therefore \oint_{\gamma} \frac{f(z)\,dz}{z-a} = \int_0^{2\pi} \frac{f(a+i\epsilon e^{i\theta})}{\cancel{\epsilon e^{i\theta}}}\, i\cancel{\epsilon e^{i\theta}}\,d\theta$$

$$\therefore \oint_c \frac{f(z)\,dz}{z-a} = i\int_0^{2\pi} f(a+i\epsilon e^{i\theta})$$

$$\Rightarrow \oint_c \frac{f(z)\,dz}{z-a} = \lim_{\epsilon\to 0} i\int_0^{2\pi} f(a+i\epsilon e^{i\theta})\,d\theta$$

$$= i\int_0^{2\pi} f(a)\,d\theta$$

$$= 2\pi i f(a)$$

$$\therefore f(a) = \frac{1}{2\pi i}\oint_c \frac{f(z)\,dz}{z-a}$$

(proved)

☐. Green's theorem :-

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Chapter-6

Q1:- Expand $f(z) = 1/_{z-3}$ in a Laurent series for (a) $|z|<3$ (b) $|z|>3$

Soln:- (a) If $|z|<3$

$$\therefore 1/_{z-3} = \frac{1}{-3(1-z/3)}$$

$$= -1/3\,(1+z/3 + z^2/9 + z^3/27 + - -)$$

(b) if $|z|>3$

$$\therefore \frac{1}{z-3} = \frac{1}{z(1-3/z)} = \frac{1}{z}(1+3/z + 9/z^2 + - -)$$

$$= \frac{1}{z} + \frac{3}{z^2} + \frac{9}{z^3} + - -$$

Q2:- Expand $f(z) = \frac{z}{(z-1)(2-z)}$ in a Laurent series valid for (a) $|z|<1$ (b) $1<|z|<2$ (c) $|z|>2$ (d) $|z-1|>1$

Solution:- $f(z) = \frac{z}{(z-1)(2-z)} = \frac{A}{z-1} + B/_{2-z}$ - - (i)

$$\therefore z = A(2-z) + B(z-1) \quad - - - \text{(ii)}$$

if $z=2$, $2 = 0 + B$. $B = 2$

if $z=1$, $1 = A + 0$. $A = 1$

$$\therefore \frac{z}{(z-1)(2-z)} = \frac{1}{z-1} + \frac{2}{2-z}$$

Date : ....

(a) if $|z|<1$

$$\frac{1}{z-1} = \frac{1}{-1(1-z)} = -1(1+z+z^2+z^3+\cdots)$$
$$= (-1-z-z^2-z^3-\cdots)$$

if $|z|<2$

$$\frac{z}{z-z} = \frac{z}{z(1-z/2)} = 1+z/2+z^2/4+z^3/8+\cdots$$

For valid $|z|<1$, $=(1-1)+(z/2-z)+(z^2/4-z^2)+(z^3/8-z^3)+\cdots$

$$= -z/2 - 3z^2/4 - \frac{7z^3}{8} \cdots$$

(b) $1<|z|<2$

when $|z|>1$,

$$\frac{1}{z-1} = \frac{1}{z(1-1/z)}$$
$$= \frac{1}{z}(1-1/z)^{-1}$$
$$= 1/z\,(1+1/z+\frac{1}{z^2}+\frac{1}{z^3}+\cdots)$$
$$= \frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\cdots$$

when $|z|<2$

$$\frac{z}{z-z} = \frac{z}{z(1-z/2)} = (1-z/2)^{-1}$$
$$= 1+z/2+z^2/4+z^3/8+\cdots$$

valid for $1<|z|<2$ $= (1/z+\frac{1}{z^2}+\frac{1}{z^3})+\cdots+(1+z/2+z^2/4+z^3/8)$

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(c) For $|z|>2$ $\therefore |z|>2$ and $|z|>1$ also,

when $|z|>1$ $\frac{1}{z-1} = \frac{1}{z(1-1/z)}$

$= \frac{1}{z}(1-1/z)^{-1}$

$= 1/z(1+1/z+\frac{1}{z^2}+\frac{1}{z^3}+\cdots)$

$= \frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\cdots$

when, $|z|>2$, $\frac{2}{2-z} = \frac{2}{-z(1-2/z)}$

$= -2/z(1-2/z)^{-1}$

$= -\frac{2}{z}(1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\cdots)$

$= -\frac{2}{z}-\frac{4}{z^2}-\frac{8}{z^3}-\frac{16}{z^4}-\cdots$

$\therefore$ For $|z|>2$, $= (1/z+1/z^2+1/z^3+\cdots)+(-2/z-4/z^2-8/z^3-\cdots)$

$= -1/z-3/z^2-7/z^3-15/z^4-\cdots$

(d) For $|z-1|>1$

Let $z-1=u$

$\Rightarrow z=u+1$

Then $|u|>1$

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$\therefore f(z) = 2/2-z + \frac{1}{z-1}$

$= \frac{2}{2-u-1} + \frac{1}{u+1-1}$

$= \frac{2}{1-u} + \frac{1}{u}$

$= \frac{2}{-u(1-1/u)} + \frac{1}{u}$

$= \frac{1}{u} - 2/u\,(1-1/u)^{-1}$

$= \frac{1}{u} - 2/u\,(1+1/u + 1/u^2 + 1/u^3 + \cdots)$

$= \frac{-1}{u} - \frac{2}{u^2} - \frac{2}{u^3} - \cdots$

$= -(z-1)^{-1} - 2(z-1)^{-2} - 2(z-1)^{-3} - \cdots$

Q. Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in a Laurent series valid for

(a) $1 < |z| < 3$ (b) $|z| > 3$ (c) $0 < |z+1| < 2$ (d) $|z| < 1$

soln:- we have. $f(z) = \frac{1}{(z+1)(z+3)}$

$= \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z+3}\right]$

$= \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$

(a) if $|z| > 1$ $\quad \frac{1}{2(z+1)} = \frac{1}{2z(1+1/z)}$

$= \frac{1}{2z}(1 + 1/z + \frac{1}{z^2} + \frac{1}{z^3} + \cdots)$

$= 1/2z - 1/2z^2 + 1/2z^3 - 1/2z^4 + \cdots$

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if $|z|<3$, $\frac{1}{z(z+3)} = \frac{1}{z\cdot 3(1+z/3)} = \frac{1}{6(1+z/3)}$

$= \frac{1}{6}(1+z/3)^{-1}$

$= \frac{1}{6}(1 - z/3 + z^2/9 - z^3/27 + \cdots)$

$= \frac{1}{6} - z/18 + z^2/54 - z^3/162 + \cdots$

For valid, $1<|z|<3$

$-1/2z^4 + 1/2z^3 - 1/2z^2 + 1/2z - 1/6 + z/18 - z^2/54 + z^3/162 - \cdots$

(b):- For $|z|>3$ and $|z|>1$ also

$\frac{1}{z(z+1)} = 1/z^2 - 1/z^3 + \frac{1}{z^4} - \frac{1}{z^5} + \cdots$

For $|z|>3$

$\frac{1}{z(z+3)} = \frac{1}{z^2(1+3/z)}$

$= \frac{1}{z^2}(1+3/z)^{-1}$

$= 1/z^2 (1 - 3/z + 3^2/z^2 - 27/z^3 + \cdots)$

For $|z|>3$

$= (\frac{1}{z^2} - 1/z^3 + 1/z^4 - \frac{1}{z^5} + \cdots) - (1/z^2 - 3/z^3 + 9/z^4$

$= 1/z^2 - 4/z^3 + 13/z^4 - 40/z^5 + \cdots - 27/z^5 + 81/z^6 - \cdots)$

Date :

(c) For $0<|z+1|<2$ let $z+1=u$ $\therefore 0<|u|<2$

Then, $\dfrac{1}{(z+1)(z+3)} = \dfrac{1}{u(u+2)}$

if $|u|<2$ $\dfrac{1}{u(u+2)} = \dfrac{1}{2u(1+2/u)}$

$$= \frac{1}{2u(1+u/2)}$$

$$= \frac{1}{2u}(1+u/2)^{-1}$$

$$= \frac{1}{2u}\left(1 - u/2 + u^2/4 - \frac{u^3}{8} + \cdots\right)$$

$$= \frac{1}{2u} - \frac{1}{4} + u/8 - u^2/16 + u^3/32 - \cdots$$

For $0<|z+1|<2$

$$= \frac{1}{2(z+1)} - 1/4 + \frac{z+1}{8} - \frac{(z+1)^2}{16} + (z+1)^3/32 - \cdots$$

(d) For $|z|<1$ $\therefore |z|<3$ also

$$\frac{1}{2(z+1)} = 1/2\,(z+1)^{-1}$$

$$= 1/2\left(1 - z/2 + z^2/4 - z^3/9 + \cdots\right)$$

also $|z|<3$ $\dfrac{1}{2(z+3)} = \dfrac{1}{2\cdot 3(1+z/3)}$

$$= \frac{1}{6}(1+z/3)^{-1}$$

$$= 1/6\left(1 - z/3 + z^2/9 - z^3/27 + \cdots\right)$$

$$= 1/6 - z/18 + z^2/54 - z^3/162 + \cdots$$

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For $|z|<1$ we get,

$(1/2 - z/2 + z^2/2 - z^3/2 + z^4/2 - \cdots) - (1/6 - z/18 + \frac{z^2}{54} - \frac{z^3}{162}$

$= \frac{1}{3} - \frac{4}{9}z + \frac{13z^2}{27} - \frac{40z^3}{81} + \cdots$

Q. Expand $f(z) = \frac{1}{z(z-2)}$ in a laurent series valid for (a) $0<|z|<2$ (b) $|z|>2$

sol^n:- For $|z|<2$,

(a) $\frac{1}{z(z-2)} = \frac{1}{2z(z/2-1)}$

$= -\frac{1}{2z}(1-z/2)^{-1}$

$= -\frac{1}{2z}(1 + z/2 + z^2/4 + z^3/8 + \cdots)$

$= \frac{-1}{2z} - \frac{1}{4} - z/8 - z^2/16 - $

(b) For $|z|>2$

$\frac{1}{z(z-2)} = \frac{1}{z \cdot z(1-2/z)}$

$= \frac{1}{z^2}(1-2/z)^{-1}$

$= \frac{1}{z^2}(1+2/z + 4/z^2 + 9/z^3 + \cdots)$

$= \frac{1}{z^2} + \frac{2}{z^3} + 4/z^4 + 9/z^5 + \cdots$

Date :......................

chapter-6

Q. Find laurent series about the indicated singularity for each of the following functions:-

(a) $\frac{e^{2z}}{(z-1)^3}$ $\therefore z-1=u$

$\therefore z=1+u$

$= \frac{e^{2+2u}}{u^3}$

$= \frac{e^2 \cdot e^{2u}}{u^3}$

$= e^2/u^3 \left(1+2u+\frac{(2u)^2}{2!}+(2u)^3/3!+\cdots\right)$

$= e^2/(z-1)^3 + \frac{2e^2}{(z-1)^2} + \frac{2e^2}{z-1} + \cdots$

$z=1$ is a pole of order 3 or tripple pole

The series of converges for all values of $z \neq 1$.

(b)

$(z-3) \sin 1/z+2$ $\therefore z+2=u$

$= (u-5) \sin \frac{1}{u}$

$= (u-5) \left\{ 1/u - \frac{1}{3! u^3} + \frac{1}{5! u^5} - \cdots \right\}$

$= 1 - 5/u - \frac{1}{3! u^2} + \frac{5}{3! u^3} + \frac{1}{5! u^4} - \cdots$

$= 1 - 5/z+2 - \frac{1}{6(z+2)^2} + \cdots$

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$z=-2$ is an essential singularity

The series converges for all values of $z \neq -2$

(c) $\frac{z-\sin z}{z^3}$, $z=0$

$= \frac{1}{z^3}\left\{ z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \right) \right.$

$= \frac{1}{z^3}\left\{ \frac{z^3}{3!} - \frac{z^5}{5!} + \cdots \right\}$

$= \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \cdots$

$z=0$ is a removable singularity

The series converges for all values of z.

(d) $\frac{z}{(z+1)(z+2)}$ $\therefore z=-2$

$\therefore z+2=u$

$= \frac{u-2}{(u-1)u}$

$= \frac{2-u}{u} \cdot \frac{1}{1-u}$

$= \frac{2-u}{u} \cdot (1+u+u^2+u^3+\cdots)$

$= \frac{2}{u} + 1 + u + u^2 + \cdots$

$= \frac{2}{z+2} + 1 + (z+2) + (z+2)^2 + \cdots$

$z=-2$ is a pole of order 1 or simple pole.

The series converges for all values of z such that

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(e) $\dfrac{1}{z^2(z-3)^2}$ $\therefore z=3$

$z-3=u$

$$=\frac{1}{u^2(u+3)^2}$$

$$=\frac{1}{9u^2(1+u/3)^2}$$

$$=\frac{1}{9u^2}\left(1+(-2)(u/3)+\frac{(-2)(-3)}{2!}(u/3)^2+\frac{(-2)(-3)(-4)}{3!}(u/3)^3+\cdots\right)$$

$$=\frac{1}{9u^2}-\frac{2}{27u}+\frac{1}{27}-\cdots$$

$$=\frac{1}{9(z-3)^2}-\frac{2}{27(z-3)}+\frac{1}{27}-\cdots$$

$z=3$ is pole of order 2 or double pole.

The series of converges for all values $0<|z-3|<3$.

Chapter-7

Q. Evaluate $\int_0^{\alpha} \frac{dx}{x^6+1}$

Solⁿ:- Consider $\oint_C \frac{dz}{z^6+1}$, where c is the closed contour consisting of the line from $-R$ to R and the semicircle T, traversed in the counter clockwise sense.

Since $z^6+1=0$, when $z = e^{\pi i/6}, e^{3\pi i/6}, e^{5\pi i/6}, e^{7\pi i/6}$ ---- these are the simple poles of $\frac{1}{z^6+1}$. Only the poles $e^{\pi i/6}, e^{3\pi i/6}, e^{5\pi i/6}$ are lie within c. Using L-Hospital rule,

Residue at $e^{\pi i/6}$, $\lim_{z\to e^{\pi i/6}} \left\{(z-e^{\pi i/6})\frac{1}{z^6+1}\right\}$

$$= \lim_{z\to e^{\pi i/6}} \frac{1}{6z^5}$$

$$= \frac{1}{6e^{5\pi i/6}} = \frac{1}{6}e^{-5\pi i/6}$$

Residue at $e^{3\pi i/6}$, $\lim_{z\to e^{\frac{3\pi i}{6}}} \left\{(z-e^{3\pi i/6})\frac{1}{z^6+1}\right\}$

$$= \lim_{z\to e^{3\pi i/6}} \frac{1}{6z^5}$$

$$= \frac{1}{6}e^{-15\pi i/6}$$

Residue at, $e^{5\pi i/6}$, $\lim_{z\to e^{5\pi i/6}} \left\{(z-e^{5\pi i/6})\frac{1}{z^6+1}\right\}$

$$= \lim_{z\to e^{\frac{5\pi i}{6}}} \frac{1}{6z^5} = \frac{1}{6}e^{-25\pi i/6}$$

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Thus, $\oint_C \frac{dz}{z^6+1} = 2\pi i\left(\frac{1}{6}e^{-5\pi i/6} + \frac{1}{6}e^{-15\pi i/6} + \frac{1}{6}e^{-25\pi i/6}\right)$

$= \frac{2\pi i}{6}\left(\cos\left(\frac{-5\pi i}{6}\right) + i\sin\left(\frac{-5\pi i}{6}\right) + \cos\left(-\frac{15\pi i}{6}\right) + i\sin\left(-\right.\right.$

$+ \cos\left(-\frac{25\pi i}{6}\right) + i\sin\left(-\frac{25\pi i}{6}\right)\Big)$

$= \frac{2\pi i}{6}\left[-\cancel{\sqrt{3}/2} - i/2 + 0 - i + \cancel{\sqrt{3}/2} - i/2\right]$

$= \frac{2\pi i}{6}(-2i)$

$= \frac{-4\pi i^2}{6}$

$= \frac{2\pi}{3}$

$\therefore \int_{-R}^{R}\frac{dx}{x^6+1} + \int_T \frac{dz}{z^6+1} = \frac{2\pi}{3}$ $[\because T = \Gamma]$

$\Rightarrow \int_{-\alpha}^{\alpha}\frac{dx}{x^6+1} + 0 = \frac{2\pi}{3}$ $[\because$ Taking $R \to \alpha]$

$\Rightarrow 2\int_0^{\alpha}\frac{dx}{x^6+1} = \frac{2\pi}{3}$ $\left[\because \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx\right]$

$\therefore \int_0^{\alpha}\frac{dx}{x^6+1} = \frac{\pi}{3}$ (Ans:-)

$\therefore [\because \oint_C \frac{dz}{z^6+1}$ becomes infinitive when $z^6+1=0$ $[\because \oint_C \frac{dz}{0}$

when $z^6 = -1$ it is possible only when $z = \cos\pi + i\sin$

$= -1 + i\times 0 = -1$ or $e^{\pi i}$, Here power 6 so residue

$e^{3\pi i}, e^{5\pi i}$. Less than 6.$]$

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Ex 11:- Evaluate $\int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta}$

Soln:- let, $z = e^{i\theta}$

$dz = ie^{i\theta}d\theta$

$\therefore dz = iz\,d\theta$

Here, $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+z^{-1}}{2}$ $\qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-z^{-1}}{2i}$

$$\therefore \int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta} = \oint_c \frac{dz/iz}{3-2(z+z^{-1})/2+(z-z^{-1})/2i}$$

$$= \oint_c \frac{2dz}{6iz-(2iz^2+2i)+z^2-1} \quad \text{[multiplied by } 2iz]$$

$$= \oint_c \frac{2dz}{6iz-2iz^2-2i+z^2-1}$$

$$= \oint_c \frac{2dz}{(1-2i)z^2+6iz-1-2i}$$

where c is the circle of unit radius with centre at the origin. poles of $\frac{2}{(1-2i)z^2+6iz-1-2i}$

Date :

$$z = \frac{-6i \pm \{(-6i)^2 - 4\times(1-2i)(-1-2i)\}^{1/2}}{2\times(1-2i)}$$

$$= \frac{-6i \pm (-36+4(1-2i)(1+2i))^{1/2}}{2(1-2i)}$$

$$= \frac{-6i \pm \{-36+4(1+\cancel{2i}-\cancel{2i}-4i^2)\}^{1/2}}{2(1-2i)}$$

$$= \frac{-6i \pm \{-36+4(1+4)\}^{1/2}}{2(1-2i)}$$

$$= \frac{-6i \pm \sqrt{16i^2}}{2(1-2i)}$$

$$= \frac{-6i \pm 4i}{2(1-2i)}$$

$$= 2-i, \frac{(2-i)}{5}$$

Only $\frac{(2-i)}{5}$ inside the circle C.

Residue at $\left(\frac{2-i}{5}\right)$, $\lim\limits_{z\to\left(\frac{2-i}{5}\right)} \left\{z - \frac{(2-i)}{5}\right\} \left\{\frac{2}{(1-2i)z^2+6iz-1-2i}\right\}$

$$= \lim_{z\to\left(\frac{2-i}{5}\right)} \frac{2}{2(1-2i)z+6i}$$

$$= \frac{1}{2i}$$

Date :

$$\therefore \oint_C \frac{2dz}{(1-2i)z^2+6iz-1-2i} = 2\pi i\left(\frac{1}{2i}\right) = \pi \text{ (Ans:)}$$

12:- show that, $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$ if $a>|b|$

solⁿ:- let, $z = e^{i\theta}$

$$dz = ie^{i\theta}d\theta$$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

$$= \frac{z-z^{-1}}{2i}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \oint_C \frac{dz/iz}{a+b(z-z^{-1})/2i}$$

$$= \oint_C \frac{2dz}{bz^2+2aiz-b}$$

where c is the circle of unit radious with centre at origin.

The poles of $\frac{2dz}{bz^2+2aiz-b}$

$$\therefore z = \frac{-2ai \pm \sqrt{-4a^2+4b^2}}{2b}$$

$$= \frac{-ai \pm \sqrt{a^2-b^2}\, i}{b}$$

$$= \frac{-a+\sqrt{a^2-b^2}}{b}i, \quad \frac{-a-\sqrt{a^2-b^2}}{b}i$$

Date :

Only $\frac{-a+\sqrt{a^2-b^2}}{b}i$ lies inside of c since,

$$\left|\frac{-a+\sqrt{a^2-b^2}}{b}i\right| = \left|\frac{\sqrt{a^2-b^2}-a}{b} \cdot \frac{\sqrt{a^2-b^2}+a}{\sqrt{a^2-b^2}+a}\right| = \left|\frac{b}{\sqrt{a^2-b^2}+a}\right| < 1$$

[if $a > b > 0$]

Residue, at $z_1 = \frac{-a+\sqrt{a^2-b^2}}{b}i$

$$\lim_{z \to z_1} (z-z_1)\frac{2}{bz^2+2aiz-b}$$

$$= \lim_{z \to z_1} \frac{2}{2bz+2ai}$$

$$= \frac{2}{2\cancel{b}\left\{\frac{-a+\sqrt{a^2-b^2}}{\cancel{b}}i\right\}+2ai}$$

$$= \frac{2}{2(-a+\sqrt{a^2-b^2})i+2ai}$$

$$= \frac{1}{\cancel{-ai}+\sqrt{a^2-b^2}i+\cancel{ai}}$$

$$= \frac{1}{\sqrt{a^2-b^2}} \, {}^1/_i$$

$$\therefore \oint_c \frac{2dz}{bz^2+2aiz-b} = 2\pi \cancel{i}\left(\frac{1}{\sqrt{a^2-b^2}\,\cancel{i}}\right)$$

$$= \frac{2\pi}{\sqrt{a^2-b^2}} \quad \text{(proved)}$$

Date :

Q 13:- show that, $\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\, d\theta = \frac{\pi}{12}$

Solnᵗ:- let, $z = e^{i\theta}$

$dz = i e^{i\theta} d\theta$

$\therefore dz = iz\, d\theta$

$\cos\theta = \frac{z+z^{-1}}{2}$

$\therefore \cos 3\theta = \frac{z^3+z^{-3}}{2}$

$\therefore \int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\, d\theta = \oint_C \frac{(z^3+z^{-3})/2}{5-4(z+z^{-1})/2} \cdot \frac{dz}{iz}$

$= \oint_C \frac{z^6+1}{z^3(10-4z-4z^{-1})} \cdot \frac{dz}{iz}$

$= \frac{1}{2i} \oint_C \frac{z^6+1}{z^3(5-2z-2z^{-1})}\, dz/z$

$= \frac{1}{2i} \oint_C \frac{z^6+1}{z^3(5z-2z^2-2)}\, dz$

$= \frac{-1}{2i} \oint_C \frac{z^6+1}{z^3(2z^2-5z+2)}\, dz$

$= \frac{-1}{2i} \oint_C \frac{z^6+1}{z^3(2z^2-4z-z+2)}\, dz$

$= \frac{-1}{2i} \oint_C \frac{z^6+1}{z^3(z-2)(2z-1)}\, dz$

The integrand has a pole of order 3 at $z=0$ simple pole at $z = 1/2$ and 2 but only 0 and $1/2$ lies insides circle

Date :

Because 0 and $\frac{1}{2}$ less than 1 and 2 is greater than 1,
Here c is circle of unit radious.

Residue at $z=0$ $\lim_{z\to 0} \frac{1}{2!} \frac{d^2}{dz^2} \left\{ (z-0)^3 \cdot \frac{z^6+1}{z^3 \cdot (z-2)(2z-1)} \right\}$

$= \lim_{z\to 0} \frac{1}{2} \frac{d^2}{dz^2} \left\{ \frac{z^6+1}{(z-2)(2z-1)} \right\}$

$= \lim_{z\to 0} \frac{1}{2} \frac{d}{dz} \left\{ \frac{(2z^2-5z+2)\, 6z^5 - (z^6+1)(4z-5)}{(2z^2-5z+2)^2} \right\}$

$= \frac{1}{2} \lim_{z\to 0} \frac{d}{dz} \left\{ \frac{12z^7 - 30z^6 + 12z^5 - 4z^7 + 5z^6 - 4z + 5}{(2z^2-5z+2)^2} \right\}$

$= \frac{1}{2} \lim_{z\to 0} \frac{d}{dz} \left\{ \frac{8z^7 - 25z^6 + 12z^5 - 4z + 5}{(2z^2-5z+2)^2} \right\}$

$= \frac{1}{2} \lim_{z\to 0} \left\{ \frac{(2z^2-5z+2)^2 \cdot (56z^6 - 150z^5 + 60z^4 - 4\cancel{z}) - (8z^7 - 25z^6 + 12z^5 - 4z + 5) \cdot 2(2z^2-5z+2)(4z-5)}{(2z^2-5z+2)^4} \right\}$

$= \frac{1}{2} \times \frac{4\times(-4) - 5\times 2\times 2\times(-5)}{2^4} = \frac{-16+100}{32} = \frac{21}{8}$

Residue, $z = \frac{1}{2}$, $\lim_{z\to \frac{1}{2}} \left\{ (z-\tfrac{1}{2}) \cdot \frac{z^6+1}{z^3 \cdot (2z-1)(z-2)} \right\}$

$= \lim_{z\to \frac{1}{2}} \left\{ \left(\frac{\cancel{2z-1}}{2} \right) \cdot \frac{z^6+1}{z^3 \cdot (\cancel{2z-1})(z-2)} \right\}$

$= \frac{1}{2} \times \frac{(\frac{1}{2})^6+1}{(\frac{1}{2})^3 \cdot (\frac{1}{2}-2)}$

Date :

$$= \frac{1}{2} \times \frac{\frac{1}{64}+1}{\frac{1}{8} \times -\frac{3}{2}}$$

$$= \frac{-65}{24}$$

$$\therefore \oint_c \frac{-1}{2i} \frac{z^6+1}{z^3(z-2)(2z-1)} dz = \frac{-1}{2i} \times 2\pi i \left(\frac{21}{8} - \frac{65}{24}\right)$$

$$= -\pi\left(\frac{63-65}{24}\right)$$

$$= \frac{\pi}{12} \text{ (proved)}$$

Q.14:- Show that, $\int_0^{2\pi} \frac{d\theta}{(5-3\sin\theta)^2} = \frac{5\pi}{32}$

Solⁿ:- let, $z = e^{i\theta}$

$$dz = ie^{i\theta} d\theta$$

$$\therefore dz = iz\, d\theta$$

$$\sin z = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$= \frac{z - z^{-1}}{2i}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{(5-3\sin\theta)^2} = \oint_c \frac{dz/iz}{\left\{5 - 3\left(\frac{z-z^{-1}}{2i}\right)\right\}^2}$$

$$= \oint_c \frac{z\,dz/i}{\left\{5z - \frac{3(z^2-1)}{2i}\right\}^2}$$

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$$= \oint_c \frac{z\,dz/i}{\frac{(10iz - 3z^2 + 3)^2}{(2i)^2}} = -\frac{4}{i}\oint_c \frac{z\,dz}{(3z^2 - 10iz - 3)^2}$$

The integrand has a pole of order at 2,

$$z = \frac{-(-10i) \pm \{(-10i)^2 - 4\times 3(-3)\}^{1/2}}{2\times 3}$$

$$= \frac{10i \pm \sqrt{-100+36}}{6}$$

$$= \frac{10i \pm 8i}{6} = 3i, \; i/3$$

Only the pole $i/3$ is inside c and $3i$ is greater than 2.

Residue at $z = i/3$ is, $\lim_{z\to i/3} \frac{1}{1!}\frac{d}{dz}\left\{(z - i/3)^2 \frac{z}{(3z^2 - 10iz - 3)^2}\right\}$

$$= \lim_{z\to i/3} \frac{d}{dz}\left\{\left(\frac{3z-i}{3}\right)^2 \times \frac{z}{(3z^2 - 9iz - iz + i^2 3)^2}\right\}$$

$$= \lim_{z\to i/3} \frac{d}{dz}\left\{\frac{(3z-i)^2}{9} \times \frac{z}{\{3z(z-3i) - i(z-3i)\}^2}\right\}$$

$$= \lim_{z\to i/3} \frac{d}{dz}\left\{\frac{\cancel{(3z-i)^2}}{9} \times \frac{z}{(z-3i)^2\cancel{(3z-i)^2}}\right\}$$

$$= \frac{1}{9}\lim_{z\to i/3}\left\{\frac{(z-3i)^2\cdot 1 - z\cdot 2(z-3i)\cdot 1}{(z-3i)^4}\right\}$$

$$= \frac{1}{9}\left\{(i/3 - 3i) - i/3 \cdot 2\,(i/3 - 3i) \,/\, (i/3 - 3i)^4\right\}$$

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$$= \frac{1}{9} \times \frac{\frac{1}{9}(i-9i)^2 - 2\times \frac{i}{3} \times \frac{i-9i}{3}}{\left(\frac{i-9i}{3}\right)^4}$$

$$= \frac{1}{9} \times \frac{\frac{1}{9}\times(-8i)^2 - 2\times\frac{i}{3}\times\frac{-8i}{3}}{\left(\frac{-8i}{3}\right)^4}$$

$$= \frac{1}{9} \times \frac{\frac{1}{9}\times -64 - \frac{16}{9}}{\frac{4096}{81}}$$

$$= \frac{1}{9} \times \frac{\frac{-80}{9}}{\frac{4096}{81}} = \frac{1}{\cancel{9}} \times \frac{-80}{\cancel{9}} \times \frac{\cancel{81}}{4096} = \frac{-5}{256}$$

$$\therefore \frac{-4}{i}\oint_c \frac{z\,dz}{(3z^2-10iz-3)^2} = \frac{\cancel{-}4}{\cancel{i}} \times 2\pi\cancel{i}\left(\frac{\cancel{-}5}{256}\right)$$

$$= \frac{5\pi}{32} \quad \text{(proved)}$$

16:- show that, $\int_0^\infty \frac{\cos mx}{x^2+1}\,dx = \frac{\pi}{2} e^{-m} \quad m>0$

Sol^n:- consider $\oint_c \frac{e^{imz}}{z^2+1}\,dz$ where c is the contour.
The integrand has a simple pole at $z=\pm i$, but $z=i$ is inside the c. Residue at $z=i$ is $\lim_{z\to i} \frac{1}{\cancel{1!}} \cancel{\frac{d}{dz}}\left\{(z-i)\frac{e^{imz}}{(z-i)(z+i)}\right\}$

$$= \lim_{z\to i} \frac{e^{im\cdot i}}{(i+i)} = \frac{e^{-m}}{2i}$$

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Then, $\oint_C \frac{e^{imz}}{z^2+1}\, dz = 2\pi i \times \frac{e^{-m}}{2i} = \pi e^{-m}$

$\Rightarrow \int_{-R}^{R} \frac{e^{imx}}{x^2+1}\, dx + \int_T \frac{e^{imz}}{z^2+1}\, dz = \pi e^{-m}$

$\Rightarrow \int_{-R}^{R} \frac{\cos(mx) + i\sin(mx)}{x^2+1}\, dx + \int_T \frac{e^{imz}}{z^2+1}\, dz = \pi e^{-m}$

$\Rightarrow \int_{-R}^{R} \frac{\cos(mx)}{x^2+1}\, dx + i\int_{-R}^{R} \frac{\sin mx}{x^2+1}\, dx + \int_T \frac{e^{imz}}{z^2+1}\, dz = \pi e^{-m}$

$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos(mx)}{x^2+1}\, dx + 0 = \pi e^{-m}$ $[R \to \infty]$ $\left[\because \text{it is not } T, \text{ it is } \Gamma\right]$

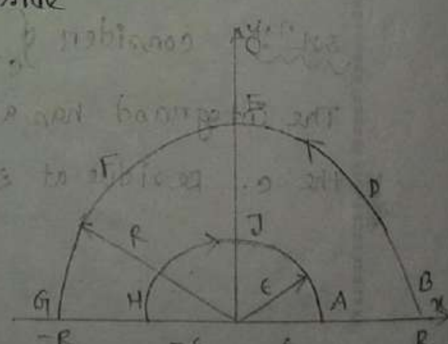
$\Rightarrow 2\int_0^{\infty} \frac{\cos mx}{x^2+1}\, dx = \pi e^{-m}$

$\therefore \int_0^{\infty} \frac{\cos mx}{x^2+1}\, dx = \frac{\pi}{2} e^{-m}$

Q.:- prove that, $\int_0^{\infty} \frac{\sin x}{x}\, dx = \pi/2$

Soln:- since $z=0$ is outside circle c' we have,

$\oint_{C'} \frac{e^{iz}}{z}\, dz = 0$



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$$\Rightarrow \int_{-R}^{-\epsilon} \frac{e^{ix}}{x}dx + \int_{BDEFG} \frac{e^{iz}}{z}dz + \int_{+\epsilon}^{R} \frac{e^{ix}}{x}dx + \int_{HJA} \frac{e^{iz}}{z}dz = 0$$

Replacing x by $-x$ in the ~~third~~ integral and combining with the third integral,

$$\int_{\epsilon}^{R} \frac{e^{ix} - e^{-ix}}{x}dx + \int_{HJA} \frac{e^{iz}}{z}dz + \int_{BDEFG} \frac{e^{iz}}{z}dz = 0$$

$$\Rightarrow 2i\int_{\epsilon}^{R} \frac{\sin x}{x}dx = -\int_{HJA} \frac{e^{iz}}{z}dz - \int_{BDEFG} \frac{e^{iz}}{z}dz$$

$\Big[\because$ Replacing x by $-x$, $\displaystyle\int_{-R}^{-\epsilon} \frac{e^{ix}}{x}dx$

$$= \int_{R}^{\epsilon} \frac{e^{-ix}}{-x}d(-x)$$

$$= -\int_{\epsilon}^{R} \frac{e^{-ix}}{x}dx \Big]$$

Let $\epsilon \to 0$ and $R \to \alpha$ and the second integral on the right approaches to zero.

$z = \epsilon e^{i\theta}$

$dz = \epsilon i e^{i\theta} d\theta$

$$\therefore 2i\int_{0}^{\alpha} \frac{\sin x}{x}dx = \lim_{\epsilon\to 0}\int_{\pi}^{0} \frac{e^{i\epsilon e^{i\theta}}}{\cancel{\epsilon e^{i\theta}}} i\cancel{\epsilon e^{i\theta}}d\theta$$

$$= \lim_{\epsilon\to 0}\int_{0}^{\pi} ie^{\epsilon e^{i\theta}}d\theta$$

$= \int_0^\pi i\,d\theta$

$= \pi i$

$\therefore \int_0^\alpha \frac{\sin x}{x}\,dx = \pi i \times \frac{1}{2i}$

$= \frac{\pi}{2}$ (proved)

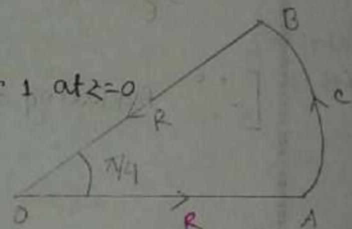
Q 19:- prove that, $\int_0^\alpha \sin x^2\,dx = \int_0^\alpha \cos x^2\,dx = \frac{1}{2}\sqrt{\pi/2}$

Sol^n:- consider $\oint_C e^{iz^2}\,dz = 0$

The integrand has a pole of order 1 at $z=0$

Residue, $z=0$, $\lim_{z\to 0} (z \times e^{iz^2})$

$= 0$



$\therefore \oint_C e^{iz^2}\,dz = 0$

$\Rightarrow \int_{OA} e^{iz^2}\,dz + \int_{AB} e^{iz^2}\,dz + \int_{BO} e^{iz^2}\,dz = 0$

$\Rightarrow \int_0^R e^{ix^2}\,dx + \int_0^{\pi/4} e^{iR^2 e^{2i\theta}} \cdot iRe^{i\theta}\,d\theta + \int_R^0 e^{ir^2 e^{\pi i/2}} e^{\pi i/4}\,dr = 0$

$\Rightarrow \int_0^R (\cos x^2 + i\sin x^2)\,dx = -\int_R^0 e^{ir^2 \times i} \cdot e^{\pi i/4}\,dr - \int_0^{\pi/4} e^{iR^2(\cos 2\theta + i\sin 2\theta)}\, iRe^{i\theta}\,d\theta$

$\Rightarrow \int_0^R (\cos x^2 + i\sin x^2)\,dx = e^{\pi i/4}\int_0^R e^{-r^2}\,dr - \int_0^{\pi/4} e^{iR^2\cos 2\theta - R^2 \sin 2\theta} \cdot iRe^{i\theta}\,d\theta$

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Hence, $z = Re^{i\theta}$ $\qquad z = re^{\pi i/4}$

$dz = iRe^{i\theta}d\theta \qquad dz = e^{\pi i/4}.dr$

$\therefore z^2 = r^2 e^{2\times\pi i/4}$

Taking $R\to\alpha$ first integral on the right,

$$e^{\pi i/4}\int_0^{\alpha} e^{-r^2}dr = \frac{\sqrt{\pi}}{\cancel{2}}(e^{\pi i/4})$$

$$= \frac{\sqrt{\pi}}{\cancel{2}}(\cos\pi/4 + i\sin\pi/4)$$

$$= \sqrt{\pi/2}\,(1/2 + i/2)$$

$$= \frac{1}{2}\sqrt{\pi/2} + i/2\sqrt{\pi/2}$$

The absolute value of the second integral, on the right,

$$\left|\int_0^{\pi/4} e^{iR^2\cos2\theta - R^2\sin2\theta}.\, iRe^{i\theta}d\theta\right| \le$$

$$\int_0^{\pi/4}\left|e^{iR^2\cos2\theta}.\, e^{-R^2\sin2\theta}.\, iRe^{i\theta}\right|d\theta$$

$$\le \int_0^{\pi/4} e^{-R^2\sin2\theta}\,\underbrace{\left|e^{iR^2\cos2\theta}\right|}_{1}\,\underbrace{\left|ie^{i\theta}\right|}_{1}\,Rd\theta$$

$$\le R\int_0^{\pi/4} e^{-R^2\sin2\theta}\,d\theta$$

$$= R\int_0^{1/2\times\pi/2} e^{-R^2\sin2\theta}\,d\theta \qquad \left[\begin{array}{l}\because 2\theta=\varphi\\ \sin\theta \ge \frac{2\varphi}{\pi}\end{array}\right]$$

$$= \frac{R}{2}\int_0^{\pi/2} e^{-R^2\cdot\frac{2\varphi}{\pi}}.\,d\varphi = R/2\left[\frac{e^{-R^2 2\varphi/\pi}}{\frac{-2R^2}{\pi}}\right]_0^{\pi/2}$$

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$= R/2 \times \frac{-\pi}{2R^2}\left[e^{-2R^2\varphi/\pi}\right]_0^{\pi/2}$

$= \frac{-\pi}{4R}\left(e^{-2R^2 \times \pi/2 \times \frac{1}{\pi}} - e^0\right)$

$= \frac{-\pi}{4R}\left(e^{-R^2} - 1\right)$

$= \frac{\pi}{4R}\left(1 - e^{-R^2}\right)$

Taking limit at $R \to \alpha$ this above integral zero,

$\int_0^\alpha (\cos x^2 + i\sin x^2)\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}} + \frac{i}{2}\sqrt{\frac{\pi}{2}}$

Equating the real and imaginary part,

$\int_0^\alpha \cos x^2\, dx = \frac{1}{2}\sqrt{\pi/2}$

$\int_0^\alpha \sin x^2\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$ (proved)

Q.20:- Show that $\int_0^\alpha \frac{x^{p-1}}{1+x}\, dx = \frac{\pi}{\sin p\pi}$, $0 < p < 1$

SOL^n:- consider $\oint_C \frac{z^{p-1}}{z+1}\, dz$

The integrand has a simple pole at $z = -1$ inside C, Residue at $z = -1, = e^{\pi i}$ is $\lim_{z \to -1}\left\{(z+1)\frac{z^{p-1}}{(1+z)}\right\}$

$= \left(e^{\pi i}\right)^{p-1}$

$= \left(e^{p-1}\right)^{\pi i}$

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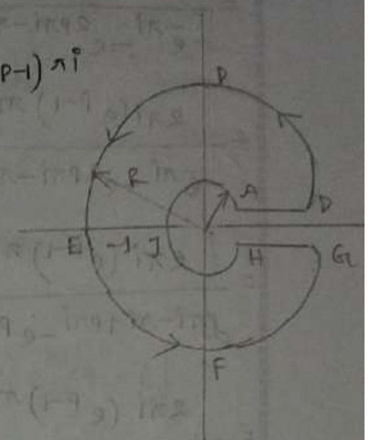
Then $\oint_C \frac{z^{p-1}}{1+z}\,dz = 2\pi i \times (e^{(p-1)\pi i})$

$$\Rightarrow \int_{AB} + \int_{BDEFG} + \int_{GH} + \int_{HJA} = 2\pi i\,(e^{(p-1)\pi i})$$



$$\Rightarrow \int_{\epsilon}^{R} \frac{x^{p-1}}{1+x}\,dx + \int_0^{2\pi} \frac{(Re^{i\theta})^{p-1}\cdot iRe^{i\theta}\,d\theta}{1+Re^{i\theta}}$$

$$+ \int_R^{\epsilon} \frac{(xe^{2\pi i})^{p-1}}{1+xe^{2\pi i}}\,dx$$

$$+ \int_{2\pi}^{0} \frac{(\epsilon e^{i\theta})^{p-1}\cdot i\epsilon e^{i\theta}\,d\theta}{1+\epsilon e^{i\theta}} = 2\pi i\, e^{(p-1)\pi i} \quad \left[\because e^{2\pi i} = 1\right]$$

Taking Limit, $\epsilon \to 0$ and $R \to \alpha$

$$\int_0^{\alpha} \frac{x^{p-1}}{1+x}\,dx + \int_{\alpha}^{0} \frac{e^{2\pi i(p-1)}\cdot x^{p-1}}{1+x}\,dx = 2\pi i\, e^{(p-1)\pi i}$$

$$\Rightarrow \int_0^{\alpha} \frac{x^{p-1}}{1+x}\,dx - \int_0^{\alpha} \frac{e^{2\pi i(p-1)}}{1+x}\cdot x^{p-1}\,dx = 2\pi i\, e^{(p-1)\pi i}$$

$$\Rightarrow \{1 - e^{2\pi i(p-1)}\} \int_0^{\alpha} \frac{x^{p-1}}{1+x}\,dx = 2\pi i\, e^{(p-1)\pi i}$$

$$\Rightarrow \int_0^{\alpha} \frac{x^{p-1}}{1+x}\,dx = \frac{2\pi i\, e^{(p-1)\pi i}}{1 - e^{2\pi i(p-1)}}$$

$$= \frac{2\pi i\, e^{(p-1)\pi i}}{-e^{\pi i} - e^{2p\pi i - 2\pi i}} \quad \left[\because e^{-\pi i} = -1\right]$$

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$$= \frac{2\pi i\, e^{(P-1)\pi i}}{-e^{-\pi i} - e^{2P\pi i - \pi i} \cdot e^{-\pi i}}$$

$$= \frac{2\pi i (e^{P-1})\pi i}{-e^{-\pi i} - e^{2P\pi i - \pi i} \cdot (-1)}$$

$$= \frac{2\pi i (e^{P-1})\pi i}{e^{P\pi i - \pi i + P\pi i} - e^{P\pi i - \pi i + P\pi i}}$$

$$= \frac{2\pi i (e^{P-1})\pi i}{e^{(P-1)\pi i} \cdot e^{P\pi i} - e^{(P-1)\pi i} \cdot e^{-P\pi i}}$$

$$= \frac{2\pi i \cancel{(e^{P-1})\pi i}}{\cancel{(e^{P-1})\pi i}\{e^{P\pi i} - e^{-P\pi i}\}}$$

$$= \frac{2\pi i}{\cancel{\cos p\pi} + i\sin p\pi - \cancel{\cos p\pi} + i\sin p\pi}$$

$$= \frac{\cancel{2\pi i}}{\cancel{2i}\sin p\pi}$$

$$= \frac{\pi}{\sin p\pi} \quad \text{(proved)}$$

No. 22 prove that $\int_0^{\alpha} \frac{\ln(x^2+1)}{x^2+1}\, dx = \pi \ln 2$

Soln :- Consider $\oint_C \frac{\ln(z+i)}{z^2+1}\, dz$

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The integral has a pole $z=\pm i$ but only i inside the circle c.

Residue $z=i$, $\lim_{z\to i}\left\{ \cancel{(z-i)} \dfrac{\ln(z+i)}{\cancel{(z-i)}(z+i)} \right\}$

$$= \frac{\ln(2i)}{2i}$$

$$\therefore \oint_c \frac{\ln(z+i)}{z^2+1}\,dz = 2\pi i \times \frac{\ln(2i)}{2i} = \pi \ln 2i$$

$$= \pi\ln 2 + \pi \ln i$$

$$= \pi \ln 2 + \pi \ln e^{\pi i/2} \quad \left[\because e^{\pi i/2} = i\right]$$

$$= \pi\ln 2 + \tfrac{1}{2}\pi^2 i$$

$$\Rightarrow \int_{-R}^{R} \frac{\ln(x+i)}{x^2+1}\,dx + \int_{\Gamma} \frac{\ln(z+i)}{z^2+1}\,dz = \pi\ln 2 + \frac{1}{2}\pi^2 i$$

$$\Rightarrow \int_{-R}^{0} \frac{\ln(x+i)}{x^2+1}\,dx + \int_{0}^{R} \frac{\ln(x+i)}{x^2+1}\,dx + \int_{\Gamma} \frac{\ln(z+i)}{z^2+1}\,dz = \pi\ln 2 + \tfrac{1}{2}\pi^2 i$$

Replacing x by $-x$ in the first integral,

$$\Rightarrow \int_{R}^{0} \frac{\ln(-x+i)\,d(-x)}{(-x)^2+1} + \int_{0}^{R} \frac{\ln(x+i)}{x^2+1}\,dx + \int_{\Gamma} \frac{\ln(z+i)}{z^2+1}\,dz = \pi\ln 2 + \frac{1}{2}\pi^2 i$$

$$\Rightarrow \int_{0}^{R} \frac{\ln(i-x)}{x^2+1}\,dx + \int_{0}^{R} \frac{\ln(x+i)}{x^2+1}\,dx + \int_{\Gamma} \frac{\ln(z+i)}{z^2+1}\,dz = \pi\ln 2 + \frac{1}{2}\pi^2 i$$

$$\Rightarrow [\ln(i+x) + \ln(i-x)] \int_{0}^{R} \frac{dx}{x^2+1} + \int_{\Gamma} \frac{\ln(z+i)}{z^2+1}\,dz = \pi\ln 2 + \frac{1}{2}\pi^2 i$$

$$\Rightarrow \ln(i^2 - x^2) \int_{0}^{R} \frac{dx}{x^2+1} + \int_{\Gamma} \frac{\ln(z+i)}{z^2+1}\,dz$$

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$\Rightarrow \operatorname{Ln}(-1-x^2)\int_0^R \frac{dx}{x^2+1} + \int_T \frac{\operatorname{Ln}(z+i)}{z^2+1}dz = \pi \operatorname{Ln}2 + \frac{\pi^2}{2}i$

$\Rightarrow \operatorname{Ln}\{-1(x^2+1)\}\int_0^R \frac{dx}{x^2+1} + \int_T \frac{\operatorname{Ln}(z+i)}{z^2+1}dz = \pi \ln 2 + \frac{\pi^2}{2}i$

$\Rightarrow [\operatorname{Ln}(x^2+1) + \ln(-1)]\int_0^R \frac{dx}{x^2+1} + \int_T \frac{\operatorname{Ln}(z+i)}{z^2+1}dz = \pi \operatorname{Ln}2 + \frac{\pi^2}{2}i$

$\Rightarrow [\ln(x^2+1) + \ln(e^{\pi i})]\int_0^R \frac{dx}{x^2+1} + \int_T \frac{\operatorname{Ln}(z+i)}{z^2+1}dz = \pi \operatorname{Ln}2 + \frac{1}{2}\pi^2 i$

$\Rightarrow [\ln(x^2+1) + \pi i]\int_0^R \frac{dx}{x^2+1} + \int_T \frac{\operatorname{Ln}(z+i)}{z^2+1}dz = \pi \ln 2 + \frac{1}{2}\pi i^2$

$\Rightarrow \int_0^R \frac{\operatorname{Ln}(x^2+1)}{x^2+1}dx + i\int_0^R \frac{\pi}{x^2+1}dx + \int_T \frac{\ln(z+i)}{z^2+1}dz = \pi \ln 2 + \frac{1}{2}\pi i^2$

putting $R \to \alpha$ and equating real parts of both sides,

$\int_0^\alpha \frac{\operatorname{Ln}(x^2+1)}{x^2+1}dx = \pi \ln 2$ (proved)

$\boxed{Q}$ 23:- prove that, $\int_0^{\pi/2} \ln \sin x\, dx = \int_0^{\pi/2} \ln \cos x\, dx = -\frac{1}{2}\pi \ln 2$

Soln:- We know that, $\int_0^\alpha \frac{\operatorname{Ln}(x^2+1)}{x^2+1}dx = \pi \operatorname{Ln}2$

put, $x = \tan\theta$

$dx = \sec^2\theta\, d\theta$ $\quad \int_0^{\pi/2} \frac{\ln(\tan^2\theta+1)}{\tan^2\theta+1}\cdot \sec^2\theta\, d\theta$

$x \to 0,\ \theta \to 0$

$x \to \alpha,\ \theta \to \pi/2$ $\quad = \int_0^{\pi/2} \operatorname{Ln}(\sec^2\theta)\, d\theta$

$= \int_0^{\pi/2} \operatorname{Ln}(\sec\theta)^2 d\theta$

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$= -2\int_0^{\pi/2} \ln(\cos\theta)\, d\theta$

$= -\cancel{2} \times -\pi/\cancel{2} \ln 2$

$= \pi \ln 2$

$\therefore -2\int_0^{\pi/2} \ln\cos\theta\, d\theta = \pi \ln 2$

$\therefore \int_0^{\pi/2} \ln\cos\theta\, d\theta = \frac{-\pi}{2} \ln 2$ ---- (i)

let $\theta = \pi/2 - \varphi$ in (i) we get,

$\int_0^{\pi/2} \ln \sin\varphi\, d\varphi = -\pi/2 \ln 2$ (proved)

Q.37 :- prove that, $\int_0^{\infty} \frac{(\ln u)^2}{u^2+1} du = \frac{\pi^3}{8}$

SOL^n:- consider $\oint_C \frac{(\ln z)^2}{z^2+1} dz$

The integrand has a simple pole, $z = \pm i$ and $z = i$ inside

Residue at $z = i$, $\lim_{z\to i} \left\{ \cancel{(z-i)} \frac{(\ln z)^2}{(z+i)\cancel{(z-i)}} \right\}$

$= \frac{(\ln i)^2}{2i} = \frac{(\ln e^{\frac{\pi i}{2}})^2}{2i}$

$= \frac{(\frac{\pi i}{2})^2}{2i} = \frac{-\pi^2}{8i}$

$\therefore \oint_C \frac{(\ln z)^2}{z^2+1} dz = 2\pi i \times \left(-\frac{\pi^2}{8i}\right) = \frac{-\pi^3}{4}$

$\Rightarrow \int_{\epsilon}^{R} \frac{(\ln z)^2}{z^2+1} dz + \int_{\Gamma_2} \frac{(\ln z)^2}{z^2+1} dz + \int_{-R}^{-\epsilon} \frac{(\ln z)^2}{z^2+1} + \int_{\Gamma_1} \frac{(\ln z)^2}{z^2+1} =$

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Taking $z = u$ on the first integral on the left

$dz = du \quad \ln z = \ln u$

Taking $z = -u$ in the 3rd integral.

$dz = -du \quad \ln z = \ln(-u)$

$= \ln u + \ln(-1)$

$= \ln u + \ln(e^{\pi i})$

$= \ln u + \pi i$

Taking $\epsilon \to 0$ and $R \to \alpha$

$$\int_0^{\alpha} \frac{(\ln u)^2}{u^2+1} du + \int_0^{\alpha} \frac{(\ln u + \pi i)^2}{u^2+1} du = \frac{-\pi^3}{4}$$

$$\Rightarrow \int_0^{\alpha} \frac{(\ln u)^2}{u^2+1} du + \int_0^{\alpha} \frac{(\ln u)^2 + 2\pi i \ln(u) + (\pi i)^2}{u^2+1} du = -\frac{\pi^3}{4}$$

$$\Rightarrow 2\int_0^{\alpha} \frac{(\ln u)^2}{u^2+1} du + 2\pi i \int_0^{\alpha} \frac{\ln u}{u^2+1} du - \pi^2 \int_0^{\alpha} \frac{du}{u^2+1} = -\frac{\pi^3}{4}$$

$$\Rightarrow 2\int_0^{\alpha} \frac{(\ln u)^2}{u^2+1} du + 2\pi i \int_0^{\alpha} \frac{\ln u}{u^2+1} du - \pi^2 \times \frac{\pi}{2} = -\frac{\pi^3}{4}$$

$$\Rightarrow 2\int_0^{\alpha} \frac{(\ln u)^2}{u^2+1} du + 2\pi i \int_0^{\alpha} \frac{\ln u}{u^2+1} du = \frac{\pi^3}{2} - \frac{\pi^3}{4} = \frac{\pi^3}{4}$$

$\therefore$ Equating the real and imaginary part,

$$\int_0^{\alpha} \frac{(\ln u)^2}{u^2+1} du = \frac{\pi^3}{8} \quad \text{(proved)}$$

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Q. prove that if $m>0$ $\int_0^\alpha \frac{\cos mx}{(x^2+1)^2}dx = \frac{\pi e^{-m}(1+m)}{4}$

Soln:- consider that $\oint_c \frac{e^{imz}}{(z^2+1)^2}dz$ where c is the contour.

The integrand has poles $z=\pm i$ and $z=i$ is inside c. Residue at $z=i$ is, $\lim_{z\to i} \frac{d}{dz}\left\{(z-i)^2 \frac{e^{imz}}{(z-i)^2(z+i)^2}\right\}$

$$= \lim_{z\to i} \frac{(z+i)^2 mi e^{imz} - e^{imz}.2(z+i)}{(z+i)^4}$$

$$= \frac{im(2i)^2 e^{i^2 m} - e^{i^2 m} 2.2i}{(i+i)^4}$$

$$= \frac{-4ime^{-m} - 4ie^{-m}}{16}$$

$$= \frac{-ie^{-m}(1+m)}{4} \times \frac{4}{1}$$

$$= \frac{-ie^{-m}(1+m)}{4}$$

$$\therefore \oint_c \frac{e^{imz}}{(z^2+1)^2}dz = 2\pi i \times \frac{-ie^{-m}(1+m)}{4}$$

$$= \frac{\pi}{2} e^{-m}(1+m)$$

$$\Rightarrow \int_{-R}^{R} \frac{e^{imx}}{(x^2+1)^2}dx + \int_\Gamma \frac{e^{imz}}{(z^2+1)^2}dz = \frac{\pi}{2}e^{-m}(1+m)$$

Taking $R\to\alpha$ and we get,

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$$\int_{-\alpha}^{\alpha} \frac{\cos mx + i\sin mx}{(x^2+1)^2}\, dx = \pi/2\, e^{-m}(1+m)$$

$$\Rightarrow 2\int_{0}^{\alpha} \frac{\cos mx + i\sin mx}{(x^2+1)^2}\, dx = \frac{\pi}{2}\, e^{-m}(1+m)$$

$$\therefore \int_{0}^{\alpha} \frac{\cos mx}{(x^2+1)^2}\, dx = \frac{\pi}{4}\, e^{-m}(1+m) \text{ (proved)}$$

Residue theorem:-