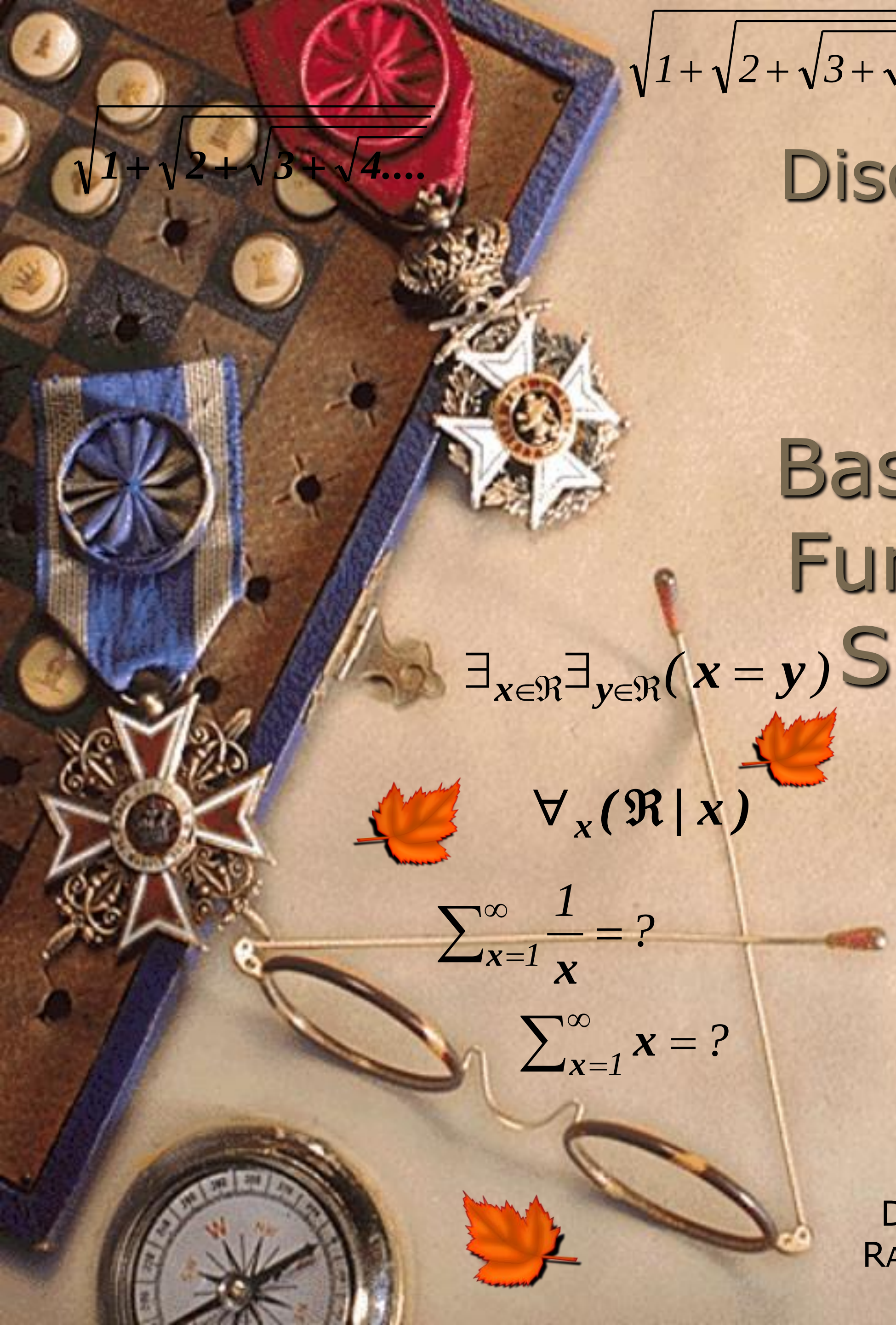

$$\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4....}}}}$$

$$\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4....}}}}$$

$$1-1+1-1+1.........=?$$

# Discrete mathematics

# Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

$$\exists_{x\in\Re}\exists_{y\in\Re}(x=y)$$

$$\forall_x(\Re \mid x)$$

$$\sum_{x=1}^{\infty}\frac{1}{x}=?$$

$$\sum_{x=1}^{\infty}x=?$$

## Chapter 2


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# Sequences and Summations

Section 2.4

# Section Summary

- Sequences.
  - Examples: Geometric Progression, Arithmetic Progression
- Recurrence Relations
  - Example: Fibonacci Sequence
- Summations
- Special Integer Sequences (*optional*)

# Introduction

- Sequences are ordered lists of elements.
  - 1, 2, 3, 5, 8
  - 1, 3, 9, 27, 81, .......
- Sequences arise throughout mathematics, computer science, and in many other disciplines, ranging from botany to music.
- We will introduce the terminology to represent sequences and sums of the terms in the sequences.

# Sequences

**Definition**: A *sequence* is a function from a subset of the integers (usually either the set {0, 1, 2, 3, 4, …..} or {1, 2, 3, 4, ….} ) to a set $S$.

- The notation $a_n$ is used to denote the image of the integer $n$. We can think of $a_n$ as the equivalent of $f(n)$ where $f$ is a function from {0,1,2,…..} to $S$. We call $a_n$ a *term* of the sequence.

# Sequences

**Example**: Consider the sequence $\{a_n\}$ where

$$a_n = \frac{1}{n} \qquad \{a_n\} = \{a_1, a_2, a_3, \ldots\}$$

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots$$

# Geometric Progression

**Definition**: A *geometric progression* is a sequence of the form: a, ar, $ar^2$, ... ,$ar^n$ where the *initial term a* and the *common ratio r* are real numbers.

## Examples:

1. Let $a = 1$ and $r = -1$. Then:

$$\{b_n\} = \{b_0, b_1, b_2, b_3, b_4, \ldots\} = \{1, -1, 1, -1, 1, \ldots\}$$

2. Let $a = 2$ and $r = 5$. Then:

$$\{c_n\} = \{c_0, c_1, c_2, c_3, c_4, \ldots\} = \{2, 10, 50, 250, 1250, \ldots\}$$

3. Let $a = 6$ and $r = 1/3$. Then:

$$\{d_n\} = \{d_0, d_1, d_2, d_3, d_4, \ldots\} = \{6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots\}$$

# Arithmetic Progression

**Definition**: A *arithmetic progression* is a sequence of the form: a, a+d, a+2d, ... , a+nd,… where the *initial term a* and the *common difference d* are real numbers.

**Examples**:

1. Let $a = -1$ and $d = 4$:

$$\{s_n\} = \{s_0, s_1, s_2, s_3, s_4, \ldots\} = \{-1, 3, 7, 11, 15, \ldots\}$$

2. Let $a = 7$ and $d = -3$:

$$\{t_n\} = \{t_0, t_1, t_2, t_3, t_4, \ldots\} = \{7, 4, 1, -2, -5, \ldots\}$$

3. Let $a = 1$ and $d = 2$:

$$\{u_n\} = \{u_0, u_1, u_2, u_3, u_4, \ldots\} = \{1, 3, 5, 7, 9, \ldots\}$$

# Strings

**Definition**: A *string* is a finite sequence of characters from a finite set (an alphabet).

- Sequences of characters or bits are important in computer science.
- The *empty string* is represented by $\lambda$.
- The string *abcde* has *length* 5.

# Recurrence Relations

**Definition:** A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses $a_n$ in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers $n$ with $n \geq n_0$, where $n_0$ is a nonnegative integer.

- A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.
- The *initial conditions* for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

# Questions about Recurrence Relations

**Example 1**: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1,2,3,4,\ldots$ and suppose that $a_0 = 2$. What are $a_1$, $a_2$ and $a_3$?
[Here $a_0 = 2$ is the initial condition.]

**Solution**: We see from the recurrence relation that

$$a_1 = a_0 + 3 = 2 + 3 = 5$$
$$a_2 = 5 + 3 = 8$$
$$a_3 = 8 + 3 = 11$$

# Questions about Recurrence Relations

**Example** 2: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2,3,4,\ldots.$ and suppose that $a_0 = 3$ and $a_1 = 5$. What are $a_2$ and $a_3$?

[Here the initial conditions are $a_0 = 3$ and $a_1 = 5$. ]

**Solution**: We see from the recurrence relation that

$$a_2 = a_1 - a_0 = 5 - 3 = 2$$

$$a_3 = a_2 - a_1 = 2 - 5 = -3$$

# Fibonacci Sequence

**Definition**: Define the *Fibonacci sequence*, $f_0, f_1, f_2, \ldots$, by:

- Initial Conditions: $f_0 = 0$, $f_1 = 1$
- Recurrence Relation: $f_n = f_{n-1} + f_{n-2}$

**Example**: Find $f_2, f_3, f_4, f_5$ and $f_6$ .

**Answer**:

$$f_2 = f_1 + f_0 = 1 + 0 = 1,$$
$$f_3 = f_2 + f_1 = 1 + 1 = 2,$$
$$f_4 = f_3 + f_2 = 2 + 1 = 3,$$
$$f_5 = f_4 + f_3 = 3 + 2 = 5,$$
$$f_6 = f_5 + f_4 = 5 + 3 = 8.$$

# Solving Recurrence Relations

- Finding a formula for the *n*th term of the sequence generated by a recurrence relation is called *solving the recurrence relation*.
- Such a formula is called a *closed formula*.
- Various methods for solving recurrence relations will be covered in Chapter 8 where recurrence relations will be studied in greater depth.
- Here we illustrate by example the method of iteration in which we need to guess the formula. The guess can be proved correct by the method of induction (Chapter 5).

# Iterative Solution Example

**Method 1**: Working upward, forward substitution

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 2,3,4,....$ and suppose that $a_1 = 2$.

$$a_2 = 2 + 3$$

$$a_3 = (2 + 3) + 3 = 2 + 3 \cdot 2$$

$$a_4 = (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3$$

$$.$$

$$.$$

$$.$$

$$a_n = a_{n-1} + 3 = (2 + 3 \cdot (n - 2)) + 3 = 2 + 3(n - 1)$$

# Iterative Solution Example

**Method 2**: Working downward, backward substitution

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 2,3,4,\ldots.$ and suppose that $a_1 = 2$.

$$
\begin{aligned}
a_n &= a_{n-1} + 3 \\
&= (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2 \\
&= (a_{n-3} + 3) + 3 \cdot 2 = a_{n-3} + 3 \cdot 3 \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
&= a_2 + 3(n-2) = (a_1 + 3) + 3(n-2) = 2 + 3(n-1)
\end{aligned}
$$

# Financial Application

**Example**: Suppose that a person deposits \$10,000.00 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?

Let $P_n$ denote the amount in the account after 30 years. $P_n$ satisfies the following recurrence relation:

$$P_n = P_{n-1} + 0.11 P_{n-1} = (1.11)\ P_{n-1}$$

with the initial condition $P_0 = 10{,}000$

*Continued on next slide* →

# Financial Application

$$P_n = P_{n-1} + 0.11 P_{n-1} = (1.11)\ P_{n-1}$$

with the initial condition $P_0 = 10{,}000$

**Solution**: Forward Substitution

$$P_1 = (1.11) P_0$$

$$P_2 = (1.11) P_1 = (1.11)^2 P_0$$

$$P_3 = (1.11) P_2 = (1.11)^3 P_0$$

$$\vdots$$

$$P_n = (1.11) P_{n-1} = (1.11)^n \mathrm{P}_0 \quad = \quad (1.11)^n\ 10{,}000$$

$P_n = (1.11)^n\ 10{,}000$ (Can prove by induction, covered in Chapter 5)

$$P_{30} = (1.11)^{30}\ 10{,}000 = \$228{,}992.97$$

# Useful Sequences

**TABLE 1** Some Useful Sequences.

| *nth Term* | *First 10 Terms* |
|---|---|
| $n^2$ | 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ... |
| $n^3$ | 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, ... |
| $n^4$ | 1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, ... |
| $2^n$ | 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, ... |
| $3^n$ | 3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, ... |
| $n!$ | 1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, ... |
| $f_n$ | 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ... |

# Guessing Sequences (*optional*)

**Example**: Conjecture a simple formula for $a_n$ if the first 10 terms of the sequence $\{a_n\}$ are 1, 7, 25, 79, 241, 727, 2185, 6559, 19681, 59047.

**Solution**: Note the ratio of each term to the previous approximates 3. So now compare with the sequence $3^n$. We notice that the $n$th term is 2 less than the corresponding power of 3. So a good conjecture is that $a_n = 3^n - 2$.

# Summations

- Sum of the terms $a_m, a_{m+1}, \ldots, a_n$ from the sequence $\{a_n\}$
- The notation:

$$\sum_{j=m}^{n} a_j \quad \textstyle\sum_{j=m}^{n} a_j \quad \sum_{m \leq j \leq n} a_j$$

represents

$$a_m + a_{m+1} + \cdots + a_n$$

- The variable *j* is called the *index of summation*. It runs through all the integers starting with its *lower limit m* and ending with its *upper limit n*.

# Summations

- More generally for a set $S$:

$$\sum_{j \in S} a_j$$

- **Examples**:

$$r^0 + r^1 + r^2 + r^3 + \cdots + r^n = \sum_{0}^{n} r^j$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_{1}^{\infty} \frac{1}{i}$$

$$\text{If } S = \{2, 5, 7, 10\} \text{ then } \sum_{j \in S} a_j = a_2 + a_5 + a_7 + a_{10}$$

# Product Notation (*optional*)

- Product of the terms $a_m, a_{m+1}, \ldots, a_n$ from the sequence $\{a_n\}$
- The notation:

$$\prod_{j=m}^{n} a_j \qquad \textstyle\prod_{j=m}^{n} a_j \qquad \prod_{m \leq j \leq n} a_j$$

represents

$$a_m \times a_{m+1} \times \cdots \times a_n$$

# Geometric Series

**Sums of terms of geometric progressions**

$$\sum_{j=0}^{n} ar^j = \begin{cases} \frac{ar^{n+1}-a}{r-1} & r \neq 1 \\ (n+1)a & r = 1 \end{cases}$$

**Proof:** Let $S_n = \sum_{j=0}^{n} ar^j$

To compute $S_n$ , first multiply both sides of the equality by r and then manipulate the resulting sum as follows:

$$rS_n = r\sum_{j=0}^{n} ar^j$$

$$= \sum_{j=0}^{n} ar^{j+1}$$

*Continued on next slide* →

# Geometric Series

$$= \sum_{j=0}^{n} ar^{j+1}$$ From previous slide.

$$= \sum_{k=1}^{n+1} ar^{k}$$ Shifting the index of summation with $k = j + 1$.

$$= \left( \sum_{k=0}^{n} ar^{k} \right) + (ar^{n+1} - a)$$ Removing $k = n + 1$ term and adding $k = 0$ term.

$$= S_n + (ar^{n+1} - a)$$ Substituting $S$ for summation formula

$$\therefore \quad rS_n = S_n + (ar^{n+1} - a)$$

$$S_n = \frac{ar^{n+1} - a}{r - 1}$$ if $r \neq 1$

$$S_n = \sum_{j=0}^{n} ar^{j} = \sum_{j=0}^{n} a = (n+1)a$$ if $r = 1$

# Some Useful Summation Formulae

**TABLE 2** Some Useful Summation Formulae.

| *Sum* | *Closed Form* |
|---|---|
| $\sum_{k=0}^{n} ar^k \ (r \neq 0)$ | $\frac{ar^{n+1} - a}{r - 1}, r \neq 1$ |
| $\sum_{k=1}^{n} k$ | $\frac{n(n+1)}{2}$ |
| $\sum_{k=1}^{n} k^2$ | $\frac{n(n+1)(2n+1)}{6}$ |
| $\sum_{k=1}^{n} k^3$ | $\frac{n^2(n+1)^2}{4}$ |
| $\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$ | $\frac{1}{1-x}$ |
| $\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$ | $\frac{1}{(1-x)^2}$ |

Geometric Series: We just proved this.

Later we will prove some of these by **induction**.

Proof in text (requires calculus)

# Query???

$$\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4....}}}}$$

$$\exists_{x\in\Re}\exists_{y\in\Re}(x=y)=?$$

$$\sum_{x=1}^{\infty} x = ?$$

$$\sum_{x=1}^{\infty}\frac{1}{x}=?$$

$$\forall_x(\Re\,|\,x)=?$$

$$\exists_{x\in\Re}\exists_{y\in\Re}(x=y)=?$$

$$\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4....}}}}=?$$

$$1-1+1-1+1........=?$$

$$\sum_{x=1}^{\infty}\frac{1}{x}=?$$