Fourier Representation of continuous time signals

Properties of Fourier Transform[a]

- Translation Shifting a signal in time domain introduces linear phase in the frequency domain.

$$f(t) \longleftrightarrow F(\omega)$$

$$f(t - t_0) \longleftrightarrow e^{-j\omega t_0} F(\omega)$$

Proof:

[a] $\mathcal{F}$ *and* F^{-1} *correspond to the Forward and Inverse Fourier transforms*

$$F(\omega) = \int_{-\infty}^{+\infty} f(t - t_0) e^{-j\omega t}\, dt$$

Put $\tau = t - t_0$

$$\begin{aligned}
F(\omega) &= \int_{-\infty}^{+\infty} f(\tau) e^{-j\omega(\tau + t_0}\, dt \\
&= e^{-j\omega t_0} \int_{-\infty}^{+\infty} f(\tau) e^{-j\omega\tau}\, d\tau && (1) \\
&= F(\omega) e^{-j\omega t_0} && (2)
\end{aligned}$$

- Modulation A linear phase shift introduced in time domain signals results in a frequency domain.

$$f(t) \longleftrightarrow F(\omega)$$

$$e^{j\omega_0 t} f(t) \longleftrightarrow F(\omega - \omega_0)$$

Proof:

$$\begin{aligned}
F(\omega) &= \int_{-\infty}^{+\infty} f(t) e^{j\omega_0 t} e^{-j\omega t}\, dt \\
&= \int_{-\infty}^{+\infty} f(t) e^{-j(\omega-\omega_0)t}\, dt && (3) \\
&= F(\omega - \omega_0) && (4)
\end{aligned}$$

- Scaling Compression of a signal in the time domain results in an expansion in frequency domain and vice-versa.

$$\begin{aligned} f(t) &\longleftrightarrow F(\omega) \\ f(at) &\longleftrightarrow \frac{1}{|a|}F(\frac{\omega}{a}) \end{aligned}$$

Proof:

$$F(\omega) = \int_{-\infty}^{+\infty} f(at)e^{-j\omega t}\,dt$$

Put $\tau = at$

If $a > 0$

$$\begin{aligned} \mathcal{F}(f(at)) &= \int_{-\infty}^{+\infty} f(\tau)e^{-j\frac{\omega}{a}\tau}\,d\tau \\ &= \frac{1}{a}F(\frac{\omega}{a}) \end{aligned}$$

If $a < 0$

$$\begin{aligned}
\mathcal{F}(f(at)) &= - \int_{-\infty}^{+\infty} f(\tau) e^{-j\frac{\omega}{a}\tau}\, d\tau \\
&= \frac{1}{a} F(\frac{\omega}{a}) \\
Therefore \\
\mathcal{F}(f(at)) &= \frac{1}{|a|} F(\frac{\omega}{a})
\end{aligned}$$

- Duality

$$\begin{aligned}
f(t) &\longleftrightarrow F(\omega) \\
F(t) &\longleftrightarrow 2\pi f(-\omega)
\end{aligned}$$

Replace t with ω and ω with t in

$$
\begin{aligned}
F(\omega) &= \int_{-\infty}^{+\infty} f(t) e^{-j\omega t}\, dt \\
F(t) &= \int_{-\infty}^{+\infty} f(\omega) e^{-jt\omega}\, d\omega
\end{aligned}
$$

But the inverse Fourier transform of a given FT $f(\omega)$ is

$$
g(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\omega) e^{j\omega t}\, d\omega
$$

Therefore

$$
F(t) = 2\pi \mathcal{F}^{-1}(f(-\omega))
$$

or

$$F(t) \longleftrightarrow 2\pi f(-\omega)$$

Example:

$$\begin{aligned} \delta(t) &\longleftrightarrow 1 \\ 1 &\longleftrightarrow 2\pi\delta(-\omega) \\ & \quad = 2\pi\delta(\omega)^{\mathrm{b}} \end{aligned}$$

- Convolution Convolution of two signals in the time domain results in multiplication of their Fourier transforms.

$$f_1(t) * f_2(t) \longleftrightarrow F_1(\omega)F_2(\omega)$$

$$g(t) = f_1(t) * f_2(t) = \int -\infty^{+\infty} f_1(\tau) f_2(t-\tau) d\tau$$

Proof:

$$\begin{aligned}
\mathcal{F}(g(t)) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_1(\tau) f_2(t-\tau) d\tau \, e^{-j\omega t} dt \\
&= \int_{-\infty}^{+\infty} f_1(t) \int_{-\infty}^{+\infty} f_2(t) e^{-j\omega t} dt d\tau \\
&= \int_{-\infty}^{+\infty} f_1(\tau) F_2(\omega) e^{-j\omega\tau} d\tau \\
&= F_1(\omega) F_2(\omega)
\end{aligned}$$

- Multiplication Multiplication of two signas in the time domain results in convolution of their Fourier transforms

$$f_1(t)f_2(t) \longleftrightarrow \frac{1}{2\pi}F_1(\omega) * F_2(\omega)$$

This can be easily proved using the **Duality Property**

- Differentiation in time

$$\frac{d}{dt}f(t) \longleftrightarrow j\omega F(\omega)$$

Proof:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)e^{j\omega t}d\omega$$

Differentiating both sides w.r.t t yields the result.

- Differentiation in Frequency

$$(-jt)^n f(t) \longleftrightarrow \frac{d^n F(\omega)}{d\omega}$$

This follows from the duality property.

- Integration in time

$$\int_{-\infty}^{t} f(t)dt \longleftrightarrow \frac{1}{j\omega} F(\omega)$$