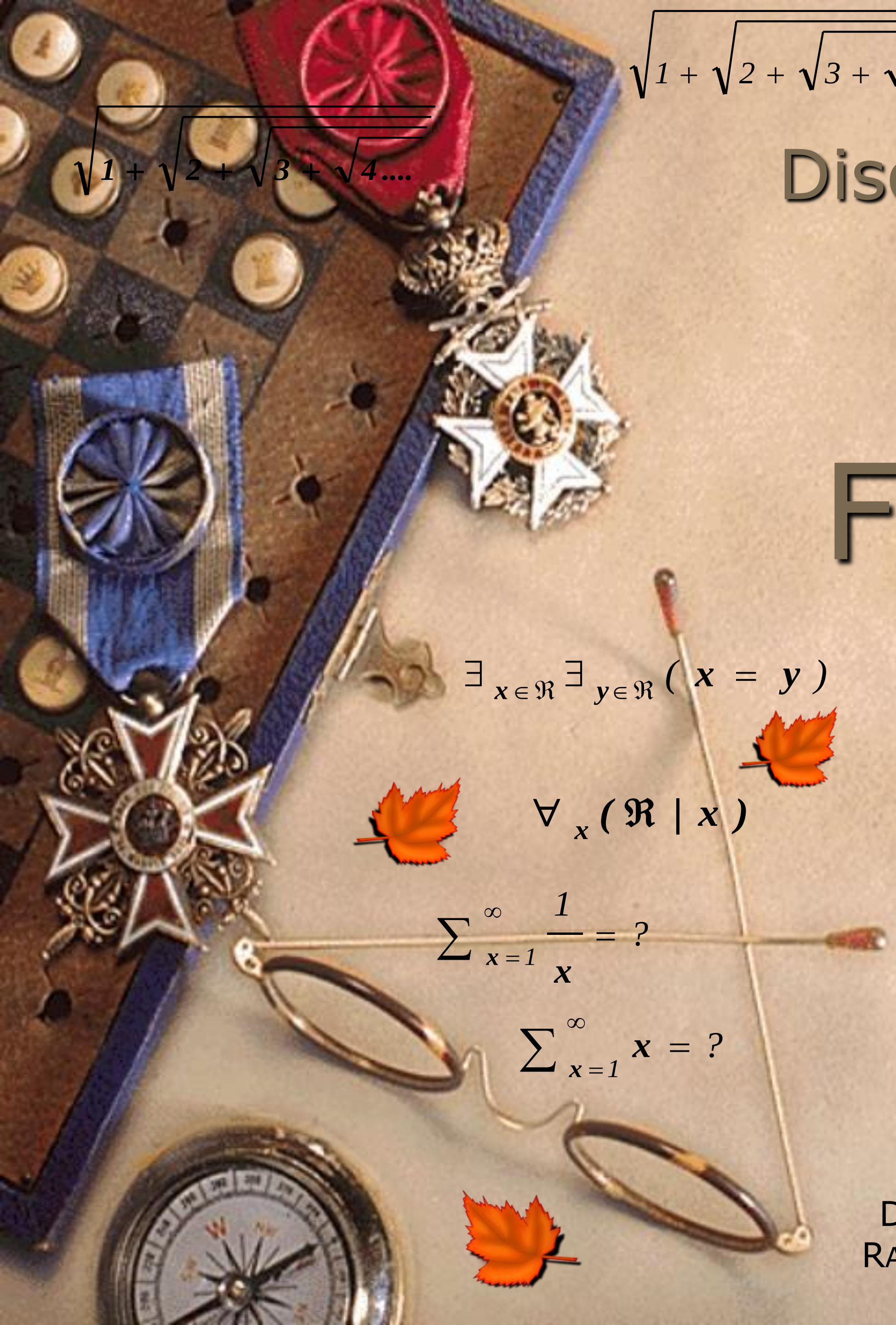

$$\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4\ldots}}}}$$

$$1 - 1 + 1 - 1 + 1 \ldots\ldots\ldots = ?$$

$$\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4\ldots}}}}$$

Discrete mathematics

The Foundations: Logic and Proofs

$$\exists_{x \in \Re} \exists_{y \in \Re} (x = y)$$

$$\forall_{x} (\Re \mid x)$$

$$\sum_{x=1}^{\infty} \frac{1}{x} = ?$$

$$\sum_{x=1}^{\infty} x = ?$$

Rizoan Toufiq

Assistant Professor

Department of Computer Science & Engineering

Rajshahi University of Engineering & Technology

Rules of Inference

Section 1.6

Section Summary

- Valid Arguments
- Inference Rules for Propositional Logic
- Using Rules of Inference to Build Arguments
  - Resolution
  - Fallacies
- Rules of Inference for Quantified Statements
- Combining Rules of Inference for Propositions and Quantified Statements

Valid Arguments in Propositional Logic

- We have the two premises:
  - “If you have a current password, then you can log onto the network.”
  - “You have a current password.”
- And the conclusion:
  - “You can log onto the network.”
- How do we get the conclusion from the premises?

Valid Arguments in Propositional Logic

- Use p to represent "You have a current password" and q to represent "You can log onto the network." Then, the argument has the form

$$\begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array}$$

TRUE

- Where ∴ is the symbol that denotes "therefore."

Q: The statement

$$((p \to q) \land p) \to q$$

is a tautology?

Valid Arguments in Propositional Logic

p	q	$p \rightarrow q$	$(p \rightarrow q) \land p$	$((p \rightarrow q) \land p) \rightarrow q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

Tautology

PROVED

We say this form of argument is **valid** because
Whenever all its **premises** are true,
The **conclusion** must also be true.

Valid Arguments in Propositional Logic

- p is true, but $p \to q$ is false

"If you have access to the network, then you can change your grade."
"You have access to the network."

∴ "You can change your grade."

- The argument we obtained is a valid argument, but because one of the premises, namely the first premise, is **false**, we cannot conclude that the conclusion is true.

Arguments in Propositional Logic

- A **argument** in propositional logic is a sequence of propositions.
- All but the final proposition are called **premises**.
- The last statement is the **conclusion**.
- The argument is **valid** if the premises imply the conclusion.
- An **argument form** is an argument that is valid no matter what propositions are substituted into its propositional variables.
- If the **premises** are $p_1, p_2, \ldots, p_n$ and the **conclusion** is q then $(p_1 \land p_2 \land \ldots \land p_n) \to q$ is a tautology.
- **Inference rules** are all simple argument forms that will be used to construct more complex argument forms.

Rules of Inference for Propositional Logic

Modus Ponens

$$\begin{array}{rl} & p \\ & p \to q \\ \hline \therefore & q \end{array}$$

Corresponding Tautology:

$$(p \wedge (p \to q)) \to q$$

Example:

Let p be "It is snowing."

Let q be "I will study discrete math."

"It is snowing."

"If it is snowing, then I will study discrete math."

Therefore , "I will study discrete math."

Rules of Inference for Propositional Logic

Modus Tollens

$$\begin{array}{rl} & \neg q \\ & p \to q \\ \hline \therefore & \neg p \end{array}$$

Corresponding Tautology:

$$(\neg q \wedge (p \to q)) \to \neg p$$

Example:

Let p be "it is snowing."

Let q be "I will study discrete math."

"I will not study discrete math."

"If it is snowing, then I will study discrete math."

Therefore , " it is not snowing."

Rules of Inference for Propositional Logic

Hypothetical Syllogism

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

Corresponding Tautology:
$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

Example:
Let *p* be "it snows."
Let *q* be "I will study discrete math."
Let *r* be "I will get an A."

"If it snows, then I will study discrete math."
"If I study discrete math, I will get an A."

"Therefore , If it snows, I will get an A."

Rules of Inference for Propositional Logic

Disjunctive Syllogism

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

Corresponding Tautology:
$(\neg p \wedge (p \vee q)) \rightarrow q$

Example:
Let p be "I will study discrete math."
Let q be "I will study English literature."

"I will study discrete math or I will study English literature."
"I will not study discrete math."

"Therefore , I will study English literature."

Rules of Inference for Propositional Logic

Addition

$$\frac{p}{\therefore p \vee q}$$

Corresponding Tautology:

$$p \rightarrow (p \vee q)$$

Example:

Let p be "I will study discrete math."
Let q be "I will visit Las Vegas."

"I will study discrete math."

"Therefore, I will study discrete math or I will visit Las Vegas."

Rules of Inference for Propositional Logic

Simplification

$$\begin{array}{c} p \wedge q \\ \hline \therefore q \end{array}$$

Corresponding Tautology:

$$(p \wedge q) \rightarrow p$$

Example:

Let p be "I will study discrete math."
Let q be "I will study English literature."

"I will study discrete math and English literature"

"Therefore, I will study discrete math."

Rules of Inference for Propositional Logic

Conjunction

$$\begin{array}{l} p \\ q \\ \hline \therefore p \land q \end{array}$$

Corresponding Tautology:
$((p) \land (q)) \rightarrow (p \land q)$

Example:
Let p be "I will study discrete math."
Let q be "I will study English literature."

"I will study discrete math."
"I will study English literature."

"Therefore, I will study discrete math and I will study English literature."

Rules of Inference for Propositional Logic

TABLE 1 **Rules of Inference.**

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$(p \land (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \land (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \land (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \lor q$ $\neg p$ $\therefore q$	$((p \lor q) \land \neg p) \to q$	Disjunctive syllogism
p $\therefore p \lor q$	$p \to (p \lor q)$	Addition

Rules of Inference for Propositional Logic

TABLE 1 **Rules of Inference.**

Rule of Inference	*Tautology*	*Name*
$\begin{array}{rl} & p \wedge q \\ \hline \therefore & p \end{array}$	$(p \wedge q) \rightarrow p$	Simplification
$\begin{array}{rl} & p \\ & q \\ \hline \therefore & p \wedge q \end{array}$	$((p) \wedge (q)) \rightarrow (p \wedge q)$	Conjunction
$\begin{array}{rl} & p \vee q \\ & \neg p \vee r \\ \hline \therefore & q \vee r \end{array}$	$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$	Resolution

Using the Rules of Inference to Build Valid Arguments

- A **valid argument** is a sequence of statements.
- Each statement is either a **premise** or follows from **previous statements** by rules of inference.
- The last statement is called **conclusion**.
- A **valid argument** takes the following form:

$$
\begin{array}{l}
S_1 \\
S_2 \\
\vdots \\
S_n \\
\\
\therefore C
\end{array}
$$

Using the Rules of Inference to Build Valid Arguments

Example 1: From the single proposition

$$p \wedge (p \rightarrow q)$$

Show that q is a conclusion.

Solution:

Step	**Reason**
1. $p \wedge (p \rightarrow q)$	Premise
2. p	Simplification using (1)
3. $p \rightarrow q$	Simplification using (1)
4. q	Modus Ponens using (2) and (3)

Using the Rules of Inference to Build Valid Arguments

Example 2:

- With these **hypotheses/premises**:

  "It is not sunny this afternoon and it is colder than yesterday."

  "We will go swimming only if it is sunny."

  "If we do not go swimming, then we will take a canoe trip."

  "If we take a canoe trip, then we will be home by sunset."

- Using the inference rules, construct a valid argument for the **conclusion**:

  "We will be home by sunset."

Solution:

1. Choose propositional variables:

   p : "It is sunny this afternoon."

   q : "It is colder than yesterday."

   r : "We will go swimming."

   s : "We will take a canoe trip."

   t : "We will be home by sunset."

Using the Rules of Inference to Build Valid Arguments

Example 2:

- With these **hypotheses/premises**:
  "It is not sunny this afternoon and it is colder than yesterday."
  "We will go swimming only if it is sunny."
  "If we do not go swimming, then we will take a canoe trip."
  "If we take a canoe trip, then we will be home by sunset."
- Using the inference rules, construct a valid argument for the **conclusion**:
  "We will be home by sunset."

2. Translation into propositional logic:

$$\text{Hypotheses: } \neg p \wedge q,\ r \rightarrow p,\ \neg r \rightarrow s,\ s \rightarrow t$$
$$\text{Conclusion: } t$$

Using the Rules of Inference to Build Valid Arguments

Example 2:

- With these **hypotheses/premises**:
  "It is not sunny this afternoon and it is colder than yesterday."
  "We will go swimming only if it is sunny."
  "If we do not go swimming, then we will take a canoe trip."
  "If we take a canoe trip, then we will be home by sunset."
- Using the inference rules, construct a valid argument for the **conclusion**:
  "We will be home by sunset."

3. Construct the Valid Argument

Step	**Reason**
1. $\neg p \land q$	Premise
2. $\neg p$	Simplification using (1)
3. $r \rightarrow p$	Premise
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \rightarrow s$	Premise
6. s	Modus ponens using (4) and (5)
7. $s \rightarrow t$	Premise
8. t	Modus ponens using (6) and (7)

Resolution

Resolution

$$\begin{array}{l} \neg p \vee r \\ p \vee q \\ \hline \therefore q \vee r \end{array}$$

Corresponding Tautology:

$((\neg p \vee r) \wedge (p \vee q)) \rightarrow (q \vee r)$

*Resolution plays an important role in AI and is used in Prolog.

Example:

Let p be "I will study discrete math."
Let r be "I will study English literature."
Let q be "I will study databases."

"I will not study discrete math or I will study English literature."
"I will study discrete math or I will study databases."

"Therefore, I will study databases or I will study English literature."

Fallacies

- Several common **fallacies** arise in **incorrect arguments**.
- The proposition $((p \rightarrow q) \land q) \rightarrow p$ is **not** a **tautology**, because it is **false** when p is **false** and q is **true**.
- **Example**: Is the following argument valid?
  - If you do every problem in this book, then you will learn discrete mathematics.
  - You learned discrete mathematics.
  - Therefore, you did every problem in this book.
- p: you do every problem in this book
- q: you will learn discrete mathematics
- r: you did every problem in this book.(**conclusion**)

$$\begin{array}{l} p \rightarrow q \\ q \\ \hline \therefore p \end{array}$$

- This type of incorrect reasoning is called **the fallacy of affirming the conclusion**.

Fallacies

- The proposition **$((p \rightarrow q) \land \neg p) \rightarrow \neg q$** is not a tautology, because it is **false** when **p is false and q is true**.
- Example: Is the following argument valid?
  - If you do every problem in this book, then you will learn discrete mathematics.
  - you did not do every problem in this book.
  - Therefore, You did not learned discrete mathematics.

$$\begin{array}{l} p \rightarrow q \\ \neg p \\ \hline \because \neg q \end{array}$$

- This type of incorrect reasoning is called **the fallacy of denying the hypothesis**.

Rules of Inference for Quantified Statements

TABLE 2 **Rules of Inference for Quantified Statements.**

Rule of Inference	*Name*
$\forall x P(x)$ $\therefore P(c)$	Universal instantiation
$P(c)$ for an arbitrary c $\therefore \forall x P(x)$	Universal generalization
$\exists x P(x)$ $\therefore P(c)$ for some element c	Existential instantiation
$P(c)$ for some element c $\therefore \exists x P(x)$	Existential generalization

Universal Instantiation (UI)

$$\frac{\forall x P(x)}{\therefore P(c)}$$

Example:

Our domain consists of all dogs and Fido is a dog.

"All dogs are cuddly."

"Therefore, Fido is cuddly."

Universal Generalization (UG)

$$\frac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$$

Used often implicitly in Mathematical Proofs.

Existential Instantiation (EI)

$$\frac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$$

Example:

"There is someone who got an A in the course."
"Let's call her *a* and say that *a* got an A"

Existential Generalization (EG)

$$\frac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$$

Example:

"Michelle got an A in the class."
"Therefore, someone got an A in the class."

Rules of Inference for Quantified Statements

Example 1: Given **premises**:

"Every man has two legs."

"John Smith is a man."

Using the rules of inference, construct a **valid argument** to show that

"John Smith has two legs"

Solution: Let

$M(x)$ denote "x is a man"

$L(x)$ " x has two legs"

John Smith be a member of the domain.

Valid Argument:

Step	**Reason**
1. $\forall x(M(x) \rightarrow L(x))$	Premise
2. $M(J) \rightarrow L(J)$	UI from (1)
3. $M(J)$	Premise
4. $L(J)$	Modus Ponens using (2) and (3)

Rules of Inference for Quantified Statements

Example 2: Use the rules of inference to construct a valid argument showing that the **conclusion**

"Someone who passed the first exam has not read the book."

follows from the **premises**

"A student in this class has not read the book."

"Everyone in this class passed the first exam."

Home Task

Universal Modus Ponens

Universal Modus Ponens combines **universal instantiation** and **modus ponens** into one rule.

$$\begin{array}{rl} & \forall x(P(x) \to Q(x)) \\ & P(a), \text{ where } a \text{ is a particular element in the domain} \\ \hline \therefore & Q(a) \end{array}$$

Universal Modus Tollens

Universal Modus Ponens combines **universal instantiation** and **modus tollens** into one rule.

$$
\begin{array}{ll}
& \forall x(P(x) \rightarrow Q(x)) \\
& \neg Q(a), \text{ where } a \text{ is a particular element in the domain} \\
\hline
\therefore & \neg P(a)
\end{array}
$$

Query???

$$\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4\ldots.}}}}$$

$$\exists_{x\in\Re}\exists_{y\in\Re}(\,x = y\,) = ?$$

$$\sum_{x=1}^{\infty} x = ?$$

$$\sum_{x=1}^{\infty}\frac{1}{x} = ?$$

$$\forall_{x}(\,\Re \mid x\,) = ?$$

$$\exists_{x\in\Re}\exists_{y\in\Re}(\,x = y\,) = ?$$

$$\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4\ldots.}}}} = ?$$

$$1-1+1-1+1\ldots\ldots\ldots = ?$$

$$\sum_{x=1}^{\infty}\frac{1}{x} = ?$$