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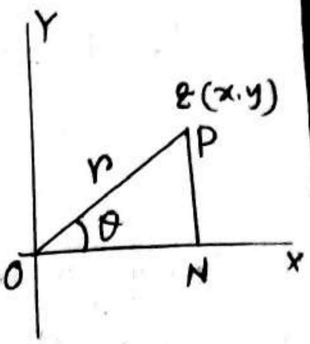
Complex variable: Any variable which contains complex number is called complex variable.

Ex: $z = a+ib$ Here z is a complex variable.

$z = OP = a+ib = re^{i\theta} = x+iy$.



$x = r\cos\theta$
$y = r\sin\theta$ $\Rightarrow z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$

θ is called amplitude or argument of z

i.e. $\arg z$

$|z| = r = \sqrt{x^2+y^2}$

equation

Modulus of the equation of complex variable is the $\boxed{\text{radius}}$ of a circle

$x^2+1 = 0 \Rightarrow x^2 = -1 \Rightarrow x^2 = i^2 \quad \therefore x = \sqrt{-1} = i$

Find $|e^z|$ if $z = x+iy$

Soln: We have $z = x+iy$

$|e^z| = |e^{x+iy}| = |e^x \cdot e^{iy}| = |e^x \cdot (\cos y + i\sin y)|$.

$= e^x \sqrt{\sin^2 y + \cos^2 y}$ $\quad [z = x+iy,\ |z| = \sqrt{x^2+y^2}]$

$= e^x \cdot 1 = e^x$ Ans.

If $z = x+iy$, then Find $|e^{iz}|$

Soln: We have $z = x+iy$

$|e^{iz}| = |e^{i(x+iy)}| = |e^{ix + i^2 y}| = |e^{ix - y}| = |e^{ix} \cdot e^{-y}|$

$= e^{-y}\sqrt{\cos^2 x + \sin^2 x} = e^{-y}$. Ans

If $z = 6e^{\pi i/3}$, evaluate $|e^{iz}|$

Soln: $e^{iz} = e^{i6e^{\pi i/3}} = e^{i6e^{i\pi/3}} = e^{i6(\cos\pi/3 + i\sin\pi/3)}$

$= e^{i6(\frac{1}{2} + i\sqrt{3}/2)} = e^{3i - 3\sqrt{3}} = e^{3i} \cdot e^{-3\sqrt{3}}$

$= e^{-3\sqrt{3}} \cdot (\cos 3 + i\sin 3)$

$\therefore |e^{iz}| = \sqrt{(e^{-3\sqrt{3}})^2 (\cos^2 3 + \sin^2 3)}$

$= e^{-3\sqrt{3}}$ Ans.

Complex conjugate:

A complex conjugate of a complex number $z = a+ib$ is $\bar{z} = a-ib$ while $|z\bar{z}| = |z|^2$

Prove that:

i) $\overline{z_1+z_2} = \bar{z_1} + \bar{z_2}$

ii) $\overline{z_1-z_2} = \bar{z_1} - \bar{z_2}$

iii) $\overline{z_1 z_2} = \bar{z_1}\,\bar{z_2}$

iv) $z\bar{z} = |z|^2$

Q) $\overline{z_1+z_2} = \bar{z_1} + \bar{z_2}$

Soln: Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

L.H.S $= \overline{(x_1+iy_1) + (x_2+iy_2)}$

$= \overline{(x_1+x_2) + i(y_1+y_2)}$

$= (x_1+x_2) - i(y_1+y_2)$ | (conjugate)

$= (x_1 - iy_1) + (x_2 - iy_2)$

$= \bar{z_1} + \bar{z_2}$ | (conjugate of z_1, z_2)

$=$ R.H.S (proved)

$\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

$z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

S = $\overline{z_1 - z_2}$

$= \overline{(x_1+iy_1)-(x_2+iy_2)}$

$= \overline{(x_1-x_2)+i(y_1-y_2)}$

$= (x_1-x_2) - i(y_1-y_2)$

$= (x_1 - iy_2) - (x_2 - iy_2)$

$= \overline{z_1} - \overline{z_2}$

= R.H.S (proved)

ii) $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

L.H.S = $\overline{z_1 z_2}$

$= \overline{(x_1+iy_1)(x_2+iy_2)}$

$= \overline{x_1x_2 + ix_1y_2 + iy_1x_2 - y_1y_2}$

$\cancel{\overline{x_2(x_1+iy_1)+iy_2(x_1+iy_1)}}$

$\cancel{=(x_1+iy_1)(x_2+iy_2)}$

$= (x_1x_2 - y_1y_2) - i(x_1y_2 + y_1x_2)$

$= x_1x_2 - y_1y_2 - ix_1y_2 - iy_1x_2$

$= x_2(x_1 - iy_1) - iy_2(x_1 - iy_1)$

$= (x_1 - iy_1)(x_2 - iy_2)$

$= \overline{z_1}\,\overline{z_2}$ = R.H.S

proved.

iv) $z\overline{z} = |z|^2$

Let $z = x + iy$.

$\overline{z} = x - iy$.

$z\overline{z} = (x+iy)(x-iy)$

$= x^2 - i^2y^2$

$= x^2 + y^2$

$= \sqrt{(x^2+y^2)}^2$

$= |z|^2$ (proved)

Prove that:

i) $|z_1 + z_2| \le |z_1| + |z_2|$

ii) $|z_1 - z_2| \geq |z_1| - |z_2|$

iii) $|z_1 z_2| = |z_1||z_2|$

iv) $|\overline{z}| = |z|$.

1. i) $|z_1 + z_2| \le |z_1| + |z_2|$

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$\therefore |z_1| = \sqrt{x_1^2 + y_1^2}$

and $|z_2| = \sqrt{x_2^2 + y_2^2}$

$\therefore z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$

$= (x_1 + x_2) + i(y_1 + y_2)$

$\therefore |z_1 + z_2| = \sqrt{(x_1+x_2)^2 + (y_1+y_2)^2}$

$\Rightarrow |z_1 + z_2|^2 = (x_1+x_2)^2 + (y_1+y_2)^2$

$= x_1^2 + x_2^2 + 2x_1x_2 + y_1^2 + y_2^2 + 2y_1y_2$

$= (x_1^2 + y_1^2) + (x_2^2 + y_2^2) + 2(x_1x_2 + y_1y_2)$

P.T.

$= |z_1|^2 + |z_2|^2 + 2\sqrt{(x_1x_2+y_1y_2)^2}$

$= |z_1|^2 + |z_2|^2 + 2\sqrt{x_1^2x_2^2+y_1^2y_2^2+2x_1x_2y_1y_2}$

$\Rightarrow |z_1+z_2|^2 \leqq |z_1|^2+|z_2|^2+2\sqrt{x_1^2x_2^2+y_1^2y_2^2+\cancel{2x_1x_2y_1y_2}\ x_1^2y_2^2+x_2^2y_1^2}$

$(a-b)^2$

$[\because (x_1y_2-x_2y_1)^2 \geqslant 0]$

$[\therefore x_1^2y_2^2+x_2^2y_1^2 \geqslant 2x_1x_2y_1y_2]$

$\Rightarrow |z_1+z_2|^2 \leqq |z_1|^2+|z_2|^2+2\sqrt{\{x_1^2(x_2^2+y_2^2)+y_1^2(y_2^2+x_2^2)\}}$

$\leq |z_1|^2+|z_2|^2+2|z_1||z_2|$

$\leq \{|z_1|+|z_2|\}^2$

$\therefore |z_1+z_2| \leq |z_1|+|z_2|$ (Proved)

Extra:

Let $z_1 = x_1+iy_1 \quad \therefore |z_1| = \sqrt{x_1^2+y_1^2}$

$z_2 = x_2+iy_2 \quad \therefore |z_2| = \sqrt{x_2^2+y_2^2}$

$z_3 = x_3+iy_3 \quad |z_3| = \sqrt{x_3^2+y_3^2}$

Now.

$|z_1+z_2+z_3| = |z_1+(z_2+z_3)| \leq |z_1|+|z_2+z_3|$

From Proof ①:

$|z_1+z_2+z_3| \leq |(z_1+z_2)+z_3|$

$\leq |z_1+z_2|+|z_3|$

$\leq |z_1|+|z_2|+|z_3|$.

ii) $|z_1-z_2| \geqslant |z_1|-|z_2|$

Soln: Let $z_1 = x_1+iy_1$, $|z_1| = \sqrt{x_1^2+y_1^2}$

and $z_2 = x_2+iy_2$, $|z_2| = \sqrt{x_2^2+y_2^2}$

$\therefore |z_1-z_2| = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

$|z_1-z_2|^2 = (x_1-x_2)^2+(y_1-y_2)^2$

$= x_1^2+x_2^2-2x_1x_2+y_1^2+y_2^2-2y_1y_2$

$= (x_1^2+y_1^2)+(x_2^2+y_2^2)-2(x_1x_2+y_1y_2)$

$\therefore |z_1-z_2|^2 = |z_1|^2+|z_2|^2-2\sqrt{(x_1x_2+y_1y_2)^2}$

$= |z_1|^2+|z_2|^2-2\sqrt{(x_1^2x_2^2+y_1^2y_2^2+2x_1x_2y_1y_2)}$

$\geqslant |z_1|^2+|z_2|^2-2\sqrt{(x_1^2x_2^2+y_1^2y_2^2+x_1^2y_2^2+x_2^2y_1^2)}$

$\geqslant |z_1|^2+|z_2|^2-2|z_1||z_2|$

$\geqslant \{|z_1|-|z_2|\}^2$

$\therefore |z_1-z_2| \geqslant |z_1|-|z_2|$

(Proved)

iv) $|\bar{z}| = |z|$

Let $z = x+iy$. $\bar{z} = x-iy$.

$|z| = \sqrt{x^2+y^2}$, $|\bar{z}| = \sqrt{x^2+y^2}$

$\therefore |\bar{z}| = |z|$ Proved

iii) $|z_1 z_2| = |z_1||z_2|$

Let $z_1 = x_1 + iy_1$, $|z_1| = \sqrt{x_1^2+y_1^2}$

$z_2 = x_2 + iy_2$, $|z_2| = \sqrt{x_2^2+y_2^2}$

$z_1 z_2 = (x_1+iy_1)(x_2+iy_2)$

$= x_1 x_2$

$|z_1 z_2| = x_1x_2 + iy_2x_1 + iy_1x_2 - y_1y_2$

$= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$

$\therefore |z_1 z_2| = \sqrt{(x_1x_2-y_1y_2)^2 + (x_1y_2+y_1x_2)^2}$

$= \sqrt{x_1^2x_2^2 + y_1^2y_2^2 \cancel{- 2x_1y_1x_2y_2} + x_1^2y_2^2 + y_1^2x_2^2 \cancel{+ 2x_1y_1x_2y_2}}$

$= \sqrt{x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + y_1^2x_2^2}$

$= \sqrt{x_2^2(x_1^2+y_1^2) + y_2^2(x_1^2+y_1^2)}$

$= \sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}$

$= \sqrt{x_1^2+y_1^2}\,\sqrt{x_2^2+y_2^2}$

$= |z_1| . |z_2|$.

$\therefore |z_1 z_2| = |z_1| . |z_2|$ (Proved)

Another proof of ①

$(|z_1| + |z_2|)^2 = (|z_1|)^2 + (|z_2|)^2 + 2|z_1||z_2|$

$\Rightarrow (|z_1| + |z_2|)^2 \geq z_1^2 + z_2^2 + 2z_1z_2$

$\Rightarrow (|z_1| + |z_2|)^2 \geq (z_1+z_2)^2$

$\Rightarrow (|z_1| + |z_2|)^2 \geq (|z_1+z_2|)^2$

$\therefore |z_1| + |z_2| \geq |z_1+z_2|$ (proved)

Prove that:

i) $\arg(z_1z_2) = \arg z_1 + \arg z_2$

ii) $\arg \bar{z} = -\arg z$

iii) $\arg(z_1/z_2) = \arg z_1 - \arg z_2$

i) $\arg(z_1z_2) = \arg z_1 + \arg z_2$

Soln:

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $\arg z_1 = \theta_1$

and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, $\arg z_2 = \theta_2$

$\therefore z_1z_2 = r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= r_1r_2(\cos\theta_1\cos\theta_2 + i\sin\theta_1\cos\theta_2 + i\sin\theta_2$
$+ i^2\sin\theta_1\sin\theta_2)$

$= r_1r_2\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}$

$\therefore \arg(z_1z_2) = \theta_1 + \theta_2$

$= \arg z_1 + \arg z_2$

$\therefore \arg(z_1z_2) = \arg z_1 + \arg z_2$

(Proved)

ii) $\arg \bar{z} = -\arg z$

Let $z = x + iy = re^{i\theta}$

$= r(\cos\theta + i\sin\theta)$, $\arg z = \theta$

$\bar{z} = r = \sqrt{x^2+y^2} =$

iii) $\arg(z_1/z_2) = \arg z_1 - \arg z_2$

Soln: Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$\therefore \arg z_1 = \theta_1$

and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\therefore \arg z_2 = \theta_2$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\{\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\}$$

$\therefore \arg(z_1/z_2) = \theta_1 - \theta_2$

$= \arg z_1 - \arg z_2$

(Proved)

State & prove De Moivere's Theorem.

Statement: De Moivere's theorem states that.

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

where n is any positive integer

Proof: Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Now. $z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2 \{(\cos\theta_1 + i\sin\theta_1) \times (\cos\theta_2 + i\sin\theta_2)\}$

$= r_1 r_2 \{(\cos\theta_1\cos\theta_2 + i\sin\theta_1\cos\theta_2 + i\sin\theta_2\cos\theta_1 + i^2\sin\theta_1\sin\theta_2)\}$

$= r_1 r_2 \{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)\}$

$= r_1 r_2 \{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}$

If $z_1, z_2, z_3 \cdots z_n = r_1, r_2, r_3 \cdots r_n$

$= \cos(\theta_1+\theta_2+\theta_3 \cdots \theta_n) + i\sin(\theta_1+\theta_2+\cdots$

$\therefore z^n = \cos n\theta + i\sin n\theta.$

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Roots of complex number: From De Moivre's theorem we can show that if ~~m~~ n is positive integer.

$$z^{1/n} = r^{\frac{1}{n}} \left\{ \cos\left(\frac{\theta + 2k\pi}{n}\right) + i \sin\left(\frac{\theta + 2k\pi}{n}\right) \right\}.$$

where, $k = 0, 1, 2, 3 \cdots n-1$.

Prove that, $e^{i\theta} = e^{i(\theta + 2k\pi)}$, where $k = 0, \pm 1, \pm 2 \ldots$

Soln: $e^{i(\theta+2k\pi)} = \cos(\theta + 2k\pi) + i \sin(\theta + 2k\pi)$.

$= \cos\theta + i \sin\theta = e^{i\theta}$ (Proved).

Find the roots of $z^5 + 32 = 0$ and locate the values of z^5 in the complex plane

Soln: $z^5 + 32 = 0 \Rightarrow z^5 = -32 = 32(\cos\pi + i\sin\pi) = 2^5(\cos\pi + i\sin\pi)$.

$\Rightarrow z^5 = 2^5 \left\{ \cos(\pi + 2k\pi) + i \sin(\pi + 2k\pi) \right\}$.

$\Rightarrow z = 2 \left\{ \cos\left(\frac{\pi + 2k\pi}{5}\right) + i \sin\left(\frac{\pi + 2k\pi}{5}\right) \right\}$.

$\Rightarrow z = 2 \cdot e^{i(\pi + 2k\pi/5)}$.

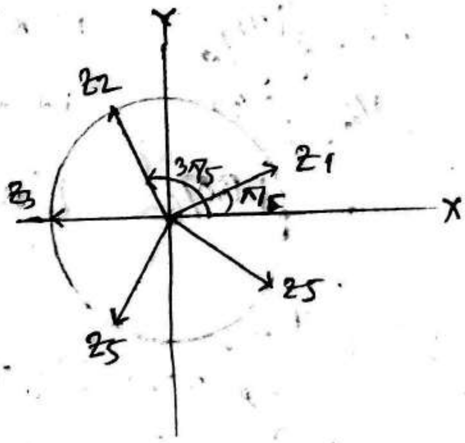
$\therefore z_1 = 2 \cdot e^{i\pi/5}$ if $k = 0$,

$z_2 = 2 \cdot e^{i3\pi/5}$ if $k = 1$.

$z_3 = 2 \cdot e^{i5\pi/5}$ if $k = 2$.

$z_4 = 2 \cdot e^{i7\pi/5}$ if $k = 3$.

$z_5 = 2 \cdot e^{i9\pi/5}$ if $k = 4$.



Angle গুলো Principle ভ্যালু থেকে নিবে.

Ex: $\frac{9\pi}{5} = 324°$:

Find the roots of the following equations.

i) $z^2 + a^2 = 0$.

ii) $z^6 + 2 = 0$.

iii) $z^4 + a^4 = 0$

Ext.

1. $z^2 + \pi^2 = 0$
2. $z^6 + 1 = 0$.
3. $z^2 + 1 = 0$
4. $z^4 + 16 = 0$
5. $z^4 + a^4 = 0$
6. $z^4 + 1 = 0$

S

$\therefore$ i). $z^2 + a^2 = 0 \Rightarrow z^2 = -a^2 = i^2 a^2 \Rightarrow z = \pm ai$.

ii) $z^6 + 2 = 0 \Rightarrow z^6 = -2 = 2(\cos\pi + i\sin\pi)$

$\Rightarrow z^6 = 2\{\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)\}$.

$\Rightarrow z = 2^{\frac{1}{6}}\left\{\cos\frac{(\pi+2k\pi)}{6} + i\sin\frac{(\pi+2k\pi)}{6}\right\}$.

$= 2^{\frac{1}{6}} \cdot e^{i\left(\frac{\pi+2k\pi}{6}\right)}$.

$z_1 = 2^{\frac{1}{6}} e^{i\pi/6}$ if $k=0$.

$z_2 = 2^{\frac{1}{6}} \cdot e^{i\pi/2}$ if $k=1$.

$z_3 = 2^{\frac{1}{6}} \cdot e^{i \cdot 5\pi/6}$ if $k=2$.

$z_4 = 2^{1/6} \cdot e^{i 7\pi/6}$ if $k=3$.

$z_5 = 2^{1/6} \cdot e^{i\frac{3\pi}{2}}$, if $k=4$

$z_6 = 2^{1/6} \cdot e^{i 11\pi/6}$ if $k=5$.

iii) $z^4 + a^4 = 0$

$\Rightarrow z^4 = -a^4 \Rightarrow z^4 = a^4(\cos\pi + i\sin\pi)$

$\Rightarrow z^4 = a^4\{\cos(\pi + 2k\pi) + i\sin(\pi+2k\pi)\}$

$\therefore z = a^{\frac{4}{4}}\left\{\cos\left(\frac{\pi+2k\pi}{4}\right) + i\sin\left(\frac{\pi+2k\pi}{4}\right)\right\}$.

$z = a\, e^{i\left(\frac{\pi+2k\pi}{4}\right)}$.

$\therefore z_1 = a \cdot e^{i\pi}$ if $k=0$.

$z_2 = a \cdot e^{i3\pi/4}$ if $k=1$

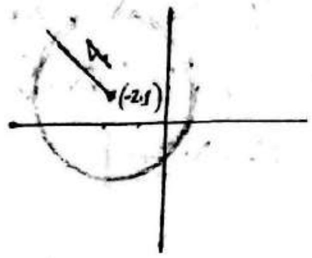
$z_3 = a \cdot e^{i5\pi/4}$ if $k=2$.

$z_4 = a \cdot e^{i7\pi/4}$ if $k=3$.

Find an equation for a circle of radius 4 with centre at (-2, 1) in complex plane.

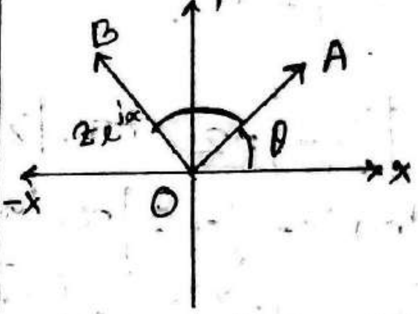
Soln: The centre can be represented by the complex number $(-2+i)$. If z is any point on the circle, then the distance from z to $(-2+i)$ is $|z-(-2+i)|=4$.

$\Rightarrow |z+2-i|=4$ is the required equation.



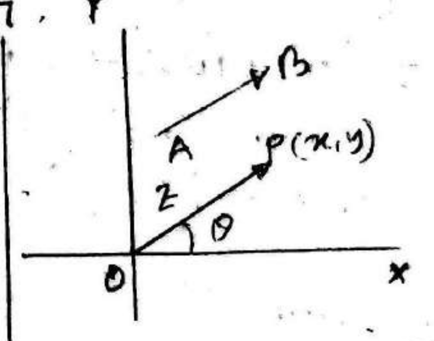
Given a complex number z, interpret geometrically $ze^{i\alpha}$ when α is real.

Soln: Let $z=re^{i\theta}$ be represented graphically by a vector OA in figure. Then $ze^{i\alpha}=re^{i\theta}\cdot e^{i\alpha}=re^{i(\theta+\alpha)}$ is the vector represented by OB. Hence multiplication of a vector z by $e^{i\alpha}$ amount to rotating z anticlockwise through angle α.



Vector interpretation of complex number.

$z=x+iy=\overline{OP}=\overline{AB}$



Dot & cross product of complex number.

$$z_1\cdot z_2=|z_1||z_2|\cos\theta=x_1x_2+y_1y_2=\underbrace{Re}_{Real}\{\bar{z}_1 z_2\}$$

$$=\frac{1}{2}\{\bar{z}_1 z_2+z_1\bar{z}_2\}$$

$$z_1\times z_2=|z_1||z_2|\sin\theta=x_1y_2-y_1x_2=\underbrace{Imagin}_{Imaginary}\{\bar{z}_1 z_2\}=\frac{1}{2i}\{\bar{z}_1 z_2-z_1\bar{z}_2\}.$$

when θ is the angle between z_1 & z_2.

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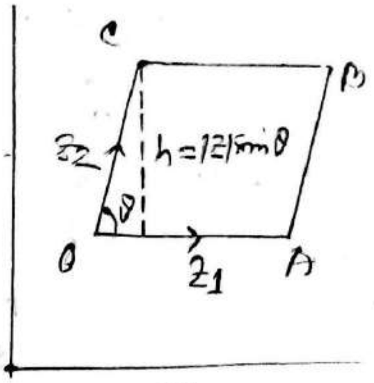
Prove that the area of a parallelogram having sides z_1 & z_2 is $|z_1 \times z_2|$.



Proof: Area of a parallelogram

$= (\text{base}) \times (\text{height})$

$= |z_1||z_2|.\sin\theta$

$= |z_1 \times z_2|$. (proved)

Represent graphically the set of values z in which $\left|\frac{z-3}{z+3}\right| = 2$.

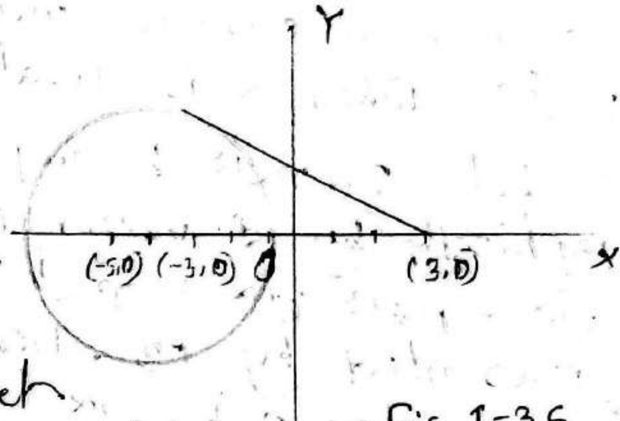


Fig 1-3.6

Soln: $\left|\frac{z-3}{z+3}\right| = 2$ or, $|z-3| = 2|z+3|$.

$\Rightarrow |x+iy-3| = 2|x+iy+3|$

$\Rightarrow \sqrt{(x-3)^2+y^2} = 2\sqrt{(x+3)^2+y^2}$

Equating both sides & simplification, we get

$x^2-6x+9+y^2 = 4(x^2+6x+9+y^2)$

$\Rightarrow x^2-4x^2-6x-24x+9-36+y^2-4y^2=0$

$\Rightarrow x^2+10x+9+y^2=0$

$\Rightarrow (x+5)^2+y^2=4^2$

$\Rightarrow |z+5|^2 = 4^2$.

Repost: If $z = 6e^{i\pi/3}$, evaluate $|e^{iz}|$.

Soln: $\therefore e^{iz} = e^{i6e^{i\pi/3}} = e^{i6}(\cos\pi/3 + i\sin\pi/3)$

$= e^{i6}\left(\frac{1}{2} + i\sqrt{3}/2\right) = e^{i3}.e^{-3\sqrt{3}}$.

$= e^{-3\sqrt{3}}.(\cos 3 + i\sin 3)$

$\therefore |e^{iz}| = \sqrt{(e^{-3\sqrt{3}})^2(\cos^2 3 + \sin^2 3)} = e^{-3\sqrt{3}}$ Ans

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Find two complex numbers whose sum is 4 and whose products is 8.

Explain the fallacy: $-1 = \sqrt{(-1)(-1)} = \sqrt{1} = 1$.

Hence. $-1 = 1$, it is not possible.

If $W = 3iz - z^2$ & $z = x+iy$ find. $|W|^2$ in terms of x & y.

1. Find the real & imaginary parts of the followings.

i). $3x + 2iy - ix + 5y = 7 + 5i$

ii) $2x - 3iy + 4ix - 2y - 5 - 10i = (x+y+2) - i(y-x+3)$.

iii) $f(z) = 2e^{iz}$

iv) $f(z) = \sqrt{z}\, e^{z}$

v) $z = \dfrac{-1+i\sqrt{3}}{2}$.

ii) Soln: ~~$2x - 3iy - ix + 5y = 7 + 5i$~~

$2x - 3iy + 4ix - 2y - 5 - 10i = (x+y+2) - i(y-x+3)$.

$\Rightarrow (2x - 2y - 5) - i(3y - 4x + 10) = (x+y+2) - i(y-x+3)$

Equating real & imaginary part.

$2x - 2y - 5 = x + y + 2 \Rightarrow x - 3y = 7 \rightarrow (1)$.

And. $3y - 4x + 10 = y - x + 3 \Rightarrow 2y - 3x = -7 \rightarrow (2)$.

$(1)\times 2 + 3(2) \Rightarrow -7x = -7 \quad \therefore x = 1$.

from (1). $1 - 3y = 7 \Rightarrow y = -2$.

iii) $f(z) = 2e^{iz} = re^{i\theta}.e^{iz} = re^{i(\theta+z)}$.

$= r.(\cos(\theta+z) + i\sin(\theta+z))$ $\quad \therefore x = r\cos(\theta+z)$

$y = r\sin(\theta+z)$

ix) $f(z) = \sqrt{z} = \sqrt{re^{i\theta}} = \sqrt{r} \cdot e^{i\theta/2}$

$= \sqrt{r} \left\{ \cos \theta/2 + i \sin \theta/2 \right\}$ $\therefore x = \sqrt{r} \cos \theta/2$

$y = \sqrt{r} \sin \theta/2$

x) $z = \frac{-1+\sqrt{3}i}{2} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$

$\therefore x = -1/2, \; y = \sqrt{3}/2$

1). $3x + 2iy - ix + 5y = 7 + 5i$

$\Rightarrow (3x + 5y) + i(2y - x) = 7 + 5i$

equating real & imaginary parts

$\therefore 3x + 5y = 7 \longrightarrow (1)$

$2y - x = 5 \longrightarrow (2)$

$\therefore x = 2y - 5 \quad \therefore (1) \Rightarrow 3(2y-5) + 5y = 7 \Rightarrow 6y - 15 + 5y = 7$

$\Rightarrow 11y = 22 \quad \therefore y = 2$. Ans

$\therefore x = 4 - 5 = -1$.

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Function, Limit & Continuity

Function: $w = f(z) = z^2$ is a function. Two types function.

i) Single valued function: If only one value of w corres-ponds to each value of z, we say that w is a single valued function of z.

Ex: $w = f(z) = z^2$ is a single valued function.

ii) Multiple valued function: If more than one value of w corresponds to each ~~other~~ value of z, we say that w is a multiple valued function of z.

Ex: $w = f(z) = \sqrt{z}$ is a multiple valued function.

Limit: Let $f(z)$ be defined and single valued, we say that the number l is the limit of $f(z)$ as z approaches z_0 and write $\lim_{z \to z_0} f(z) = l$ if for any positive number ε (however small) we can find another positive number δ (depends on ϵ). such that $|f(z) - l| < \varepsilon$ when ever. $0 < |z - z_0| < \delta$

Evaluate $\lim_{z \to 0} \frac{\bar{z}}{z}$. or. Prove that $\lim_{z \to 0} \frac{\bar{z}}{z}$ does not exists.

Soln: $\lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{(x,y) \to 0} \frac{x - iy}{x + iy} = \lim_{\substack{x \to 0 \\ y = 0}} \frac{x}{x} = 1$. if $y = 0$.

Similarly $\lim_{\substack{y \to 0 \\ x = 0}} \frac{-iy}{iy} = -1$. if $x = 0$.

Since the two approaches do not give same results. so the limit does not exists.

$\lim_{z\to i}\left\{(z-i)\dfrac{e^{imz}}{z^2+1}\right\}$

Soln: $\lim_{z\to i}\left\{\cancel{(z-i)}\dfrac{e^{imz}}{\cancel{(z-i)}(z+i)}\right\} = \lim_{z\to i}\dfrac{e^{imz}}{z+i} = \dfrac{e^{im.i}}{i+i} = \dfrac{e^{-m}}{2i} = \dfrac{1}{e^{m}2i}$

Evaluate: $\lim_{z\to ae^{i\pi/4}}\left\{(z-ae^{i\pi/4})\cdot\dfrac{1}{z^4+a^4}\right\}$

Soln: $\lim_{z\to ae^{i\pi/4}}\left\{(z-ae^{i\pi/4})\dfrac{1}{4z^3}\right\}$ [L'Hospital]

$= \dfrac{\cancel{(ae^{i\pi/4}-ae^{i\pi/4})}}{\cancel{4z^3}} = \dfrac{1}{4a^3}\cdot e^{-i\frac{3\pi}{4}}$ Ans.

কাজ Diff:

① $\lim_{z\to\frac{-a+\sqrt{a^2-b^2}}{b}}\left\{\left(z-\dfrac{-a+\sqrt{a^2-b^2}}{b}\right)\dfrac{2}{bz^2+2aiz-b}\right\}$ (কাজ) | Ans: $\dfrac{1}{\sqrt{a^2-b^2}\,i}$

② Evaluate $\lim_{z\to\pi i}\dfrac{d}{dz}\left\{(z-\pi i)^2\dfrac{e^z}{(z^2+\pi^2)^2}\right\}$ কাজ. | Ans: $\dfrac{\pi+i}{4\pi^2}$.

Continuity: $f(z)$ be continuous at $z=z_0$ if

i) $\lim_{z\to z_0} f(z) = l$ must exist

ii) $f(z_0)$ must exist. i.e. $f(z)$ is defined at $z=z_0$.

iii) $l = f(z_0)$.

If. $f(z) = \begin{cases} z^2 ; & z\neq i \\ 0 ; & z=i\end{cases}$ is the function continuous at $z=i$? If not, redefine The function To be continuous

Soln: $f(z)$ be continuous at $z=i$, if

i) $\lim_{z\to z_0} f(z) = \lim_{z\to i} f(z) = \lim_{z\to i} z^2 = i^2 = -1$ exists.

ii) $f(z_0) = f(i) = 0$ exists

iii) $l \neq f(z_0)$ i.e. $-1\neq 0$, So. $f(z)$ is not continuous at $z=i$.

If we redefine $f(z)=z^2$ for all values of z.

It is continuous at $z=i$

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If $f(z) = \begin{cases} \frac{z^2+4}{z-2i} & ; z \neq 2i \\ 3+4i & ; z = 2i \end{cases}$ is the function continuous at $z=2i$. If not, redefine, the function to be continuous.

Soln: i) $\lim_{z \to 2i} \frac{z^2+4}{z-2i} = \frac{(z+2i)(z-2i)}{(z-2i)} = z+2i = 4i$.

ii) Here. $f(z_0) = f(2i) = 3+4i$.

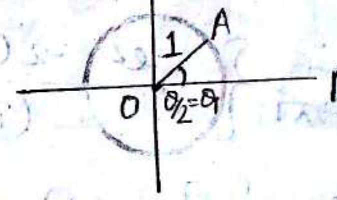
iii) Here, $l \neq f(z_0)$ i.e. $4i \neq 3+4i$

$\therefore$ The $f(z)$ is not continuous at $z=2i$

Redefine: If we define $f(z) = \frac{z^2+4}{z-2i}$ for all value of $z=i$

Branch Point & Branch Line:

$w = f(z) = \sqrt{z} = z^{1/2} = (re^{i\theta})^{1/2} = \sqrt{r}.\ e^{i\theta/2}$.



OB is called branch line

O is called branch point.

Prove that the zeros of ① $\sin z$ and ⓘⓘ $\cos z$ are all real and find them.

1). $\sin z = \frac{e^{iz} - e^{-iz}}{2i} = 0$ or. $e^{iz} = \frac{1}{e^{iz}} \Rightarrow e^{2iz} = 1$.

$\Rightarrow \sin z = e^{2k\pi i} = 1$

$\therefore z = k\pi$ i.e. $z = 0, \pm\pi, \pm 3\pi$

ii) $\cos z = \frac{e^{iz} + e^{-iz}}{2i} = 0 \Rightarrow e^{2iz} = -1 = e^{(2k+1)\pi i}$.

$\therefore z = (k + \frac{1}{2})\pi$ $\therefore$ i.e. $z = \pm\pi/2 ; \pm \frac{3\pi}{2} \cdots$

1). $\lim\limits_{z \to \frac{-a+\sqrt{a^2-b^2}}{b}} \left\{ \left(z - \frac{-a+\sqrt{a^2-b^2}}{b}\right) \frac{2}{bz^2+2aiz-b} \right\}$

$\lim\limits_{z \to \frac{-a+\sqrt{a^2-b^2}}{b}} \left\{ \frac{\cancel{2}}{\cancel{2}bz+\cancel{2}ai} \right\} = \frac{1}{\cancel{b} \times \frac{-a+\sqrt{a^2-b^2}}{\cancel{b}} + ai}$

$= \frac{1}{-a+\sqrt{a^2-b^2}+ai} = \frac{1}{\sqrt{a^2-b^2}+a(i-1)} =$

---

ii) $\lim\limits_{z\to\pi i} \frac{d}{dz} \left\{ (z-\pi i)^2 \frac{e^z}{(z^2+\pi^2)^2} \right\} = \lim\limits_{z\to\pi i} \frac{d}{dz} \left\{ (z-\pi i)^2 \frac{e^z}{(z-\pi i)^2(z+\pi i)^2} \right\}$

$= \lim\limits_{z\to\pi i} \frac{d}{dz} \left\{ \frac{e^z}{(z+\pi i)^2} \right\}$.

$= \lim\limits_{z\to\pi i} \left\{ \frac{e^z \cdot -2(z+\pi i) + (z+\pi i)^2 e^z}{(z+\pi i)^4} \right\}$

$= \lim\limits_{z\to\pi i} \left\{ e^z \cdot \times (-2) \cdot (z+\pi i)^{-3} + (z+\pi i)^{-2} e^z \right\}$.

$= e^{\pi i}(-2) \frac{1}{(\pi i+\pi i)^3} + \frac{e^{\pi i}}{(\pi i+\pi i)^2}$

$= \frac{e^{\pi i}}{(2\pi i)^2} \left\{ \frac{-\cancel{2}}{\cancel{2}\pi i} + 1 \right\}$

$= \frac{e^{\pi i}}{(2\pi i)^2} \left(\frac{-1+\pi i}{\pi i} \right)$

$= \frac{e^{\pi i}}{4\pi^2 i^2} \left(\frac{-1+\pi i}{\pi i} \right)$

$= \frac{1}{4\pi^2(-1)} (\cos\pi + i\sin\pi) \left\{ \frac{-1+\pi i}{\pi i} \right\}$.

$= \frac{\pi i - 1}{4\pi^3 i} = \frac{\pi i + i^2}{4\pi^3 i} = \frac{\pi + i}{4\pi^3}$ Ans.

Complex differentiation and Cauchy-Rieman Equation:

⇒ Definition of Derivatives: If $f(z)$ is single valued in some region R of the z-plane, the derivative of $f(z)$ is defined as $f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$, provided the limit exists, independent of the manner in which $\Delta z \to 0$.

08 ⇒ Analytic or Regular function: If the derivative $f'(z)$ exists at a point z of a region R, then $f(z)$ is said to be analytic in R.

Prove that, a (i) Necessary and (ii) sufficient condition that $w = f(z) = u(x,y) + iv(x,y)$ be analytic in a region R is that the Cauchy Rieman equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ are satisfied in } R$$

Proof: (i):

In order for $f(z)$ to be analytic the limit $f(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$\text{or, } f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)\} - \{u(x,y) + iv(x,y)\}}{\Delta x + i\Delta y} \rightarrow (1)$$

must exists independent of the manner in which $\Delta z \to 0$

Case 1: When $\Delta y = 0$, then $\Delta x \to 0$, in this case eqn (1) becomes,

$$f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y = 0}} \frac{\{u(x+\Delta x, y) + iv(x+\Delta x, y)\} - \{u(x,y) + iv(x,y)\}}{\Delta x}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y = 0}} \left[\left\{ \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} \right\} + i \left\{ \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} \right\} \right]$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \text{ provided the partial derivative exists}$$

Case II: When, $\Delta x=0$, $\Delta y\to 0$, in this case eqn (1) becomes,

$$f'(z)=\lim_{\substack{\Delta y\to 0\\ \Delta x=0}}\frac{\{u(x,y+\Delta y)+iv(x,y+\Delta y)\}-\{u(x,y)+iv(x,y)\}}{i\Delta y}$$

$$=\lim_{\substack{\Delta y\to 0\\ \Delta x=0}}\left[\left\{\frac{u(x,y+\Delta y)-u(x,y)}{i\Delta y}\right\}+i\left\{\frac{v(x,y+\Delta y)-v(x,y)}{i\Delta y}\right\}\right]$$

$$=\frac{1}{i}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}$$

$$=-i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}$$ provided the partial derivative exists

Now, $f(z)$ can not possibly be analytic unless these two cases i.e. there two ~~limits~~ are identified. Thus a necessary condition that $f(z)$ be analytic is case I = case II.

i.e, $-i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$.

So, we have, $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ (Proved)

Proof (II):

Since $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are supposed continuous, we have

$$\Delta u=u(x+\Delta x,y+\Delta y)-u(x,y)$$

$$=\{u(x+\Delta x,y+\Delta y)-u(x,y+\Delta y)\}+\{u(x,y+\Delta y)-u(x,y)\}$$

$$=\left(\frac{\partial u}{\partial x}+\epsilon_1\right)\Delta x+\left(\frac{\partial u}{\partial y}+\eta_1\right)\Delta y$$

$$=\frac{\partial u}{\partial x}\Delta x+\frac{\partial u}{\partial y}\Delta y+\epsilon_1\Delta x+\eta_1\Delta y.$$

Where: $\epsilon_1\to 0$, and $\eta_1\to 0$, as $\Delta x\to 0$ and $\Delta y\to 0$

Similarly, since $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are supposed continuous, we have, $\Delta v=\left(\frac{\partial v}{\partial x}+\epsilon_2\right)\Delta x+\left(\frac{\partial v}{\partial y}+\eta_2\right)\Delta y=\frac{\partial v}{\partial x}\Delta x+\frac{\partial v}{\partial y}\Delta y+\epsilon_2\Delta x+\eta_2\Delta y$

Where, $\epsilon_2\to 0$. $\eta_2\to 0$. as $\Delta x\to 0$ and $\Delta y\to 0$,

D.T.O

Then, $\Delta w = \Delta u + i\Delta v = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Delta y + \epsilon\Delta x + \eta\Delta y \rightarrow (2)$

Where, $\epsilon = \epsilon_1 + i\epsilon_2 \rightarrow 0$. and $\eta = \eta_1 + i\eta_2$ as. $\Delta x \rightarrow 0$ & $\Delta y \rightarrow 0$

By the cauchy - Rieman equations (2) can be written,

$$\Delta w = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\Delta y + \epsilon\Delta x + \eta\Delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\Delta x + i\Delta y) + \epsilon\Delta x + \eta\Delta y$$

Then on dividing by $\Delta z = \Delta x + i\Delta y$ and taking the limit as $\Delta z \rightarrow 0$, we see that.

$$\frac{dw}{dz} = f'(z) = \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

So, that the derivative exists and is unique i.e. $f(z)$ is analytic in R.

Test the analyticity of $f(z) = \bar{z}$

Sol: By definition

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$ must exist indpendent of the manner in which $\Delta z (\Delta x \text{ or } \Delta y) \rightarrow 0$.

Thus $$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z} = \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \frac{x - iy + \Delta x - i\Delta y - (x - iy)}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y = 0}} \frac{\Delta x}{\Delta x} = 1. \text{ if } \Delta y = 0.$$

Similarly, $$f'(z) = \lim_{\substack{\Delta y \rightarrow 0 \\ \Delta x = 0}} \frac{-i\Delta y}{i\Delta y} = -1. \text{ if, } \Delta x = 0$$

Since, the limits dependent on the manner in which $\Delta z \rightarrow 0$, the derivative does not exist. Thus the $f(z) = \bar{z}$ is not analytic function.

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Test the analyticity of the followings:

1) $f(z) = ize^{-z}$, ii) $f(z) = ze^{-z}$

1) Soln: $f(z) = u(x,y) + iv(x,y) = ize^{-z}$

$= i(x+iy)\, e^{-(x+iy)} = (ix-y)\, e^{-x} \cdot e^{-iy}$

$= (ix-y)\, e^{-x} \cdot (\cos y - i\sin y)$

$= e^{-x}(ix\cos y + x\sin y - y\cos y + iy\sin y)$

$= e^{-x}(x\sin y - y\cos y) + ie^{-x}(x\cos y + y\sin y)$.

Now, $\frac{\partial u}{\partial x} = e^{-x}\sin y - e^{-x}x\sin y + e^{-x}y\cos y$

$\frac{\partial v}{\partial y} = -e^{-x}x\sin y + e^{-x}\sin y + e^{-x}y\cos y$

$= -xe^{-x}\sin y + e^{-x}(y\cos y + \sin y)$

$= e^{-x}\sin y - e^{-x}x\sin y + e^{-x}y\cos y$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, which satisfies cauchy Rieman equation

hence, $f(z) = ize^{-z}$ is analytic

ii) $f(z) = ze^{-z} = u(x,y) + iv(x,y) = (x+iy)\, e^{-(x+iy)}$

$= (x+iy)\, e^{-x} \cdot e^{-iy} = (x+iy)\, e^{-x}(\cos y - i\sin y)$

$= e^{-x}(x\cos y - ix\sin y + iy\cos y + y\sin y)$

$= e^{-x}(x\cos y + y\sin y) + ie^{-x}(y\cos y - x\sin y)$

$\therefore \frac{\partial u}{\partial x} = e^{-x}\cos y - e^{-x}x\cos y - e^{-x}y\sin y$

$\therefore \frac{\partial v}{\partial y} = e^{-x}\cos y - ye^{-x}\sin y - xe^{-x}\cos y$

Here, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, which satisfies cauchy Rieman equation.

Hence, $f(z) = ze^{-z}$ is analytic.

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⊞ Laplacian: The operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is called the Laplacian.

⊞ Harmonic function: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ & $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ is called is called the harmonic function.

Prove that real and imaginary part of an analytic function $f(z)$ satisfy Laplacian's / Laplace's equation.

⊞ Proof: Since the function $f(z) = u(x,y) + iv(x,y)$ be an analytic function. So it satisfies the cauchy's Rieman. equation.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \longrightarrow ① \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \longrightarrow ②$$

Differentiating ① and, w.r.to x, we get, $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x . \partial y} \longrightarrow ③$

② w.r.to. y, we get. $\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x . \partial y} \longrightarrow ④$

By adding ③ & ④ we get Laplace's equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x . \partial y} = 0$$

Similarly diff eqn ① w.r.to y and diff eqn. ② w.r.to. x, we get the other Laplace's equation.

$$\frac{\partial^2 u}{\partial x . \partial y} = \frac{\partial^2 v}{\partial y^2} \longrightarrow ⑤ \quad \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2} \longrightarrow ⑥$$

By substracting ⑤ & ⑥ we have

$$\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial x . \partial y} = \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} = 0$$

$$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Hence, we can say that real & imaginary part of an analytic function $f(z)$ satisfy Laplace's equation.

Show that (a) $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic.
(b) Find v such that $f(z) = u + iv$ is analytic.
(c) Also find $f(z)$ in terms of z.

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Soln: (a). Given. $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$.

$\therefore \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x$

$\therefore \frac{\partial^2 u}{\partial x^2} = 6x + 6 \longrightarrow (1)$.

$\frac{\partial u}{\partial y} = -6xy - 6y$

$\frac{\partial^2 u}{\partial y^2} = -6x - 6 \longrightarrow (2)$

$(1) + (2) \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, hence u is harmonic

(b) From Cauchy Riemen equation, we have

$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x \longrightarrow (3)$

$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -(-6xy - 6y) = 6xy + 6y \longrightarrow (4)$.

Integrating (3) w.r.to y and keeping x be constant, we get

$\int dv = \int (3x^2 - 3y^2 + 6x)\, dy \Rightarrow v = 3x^2y - y^3 + 6xy + F(x) \longrightarrow (5)$

where. $F(x)$ is an arbitrary real function of x. Now,

substracting (5) into (4). we have.

$6xy + 6y + F'(x) = 6xy + 6y$

$\Rightarrow F'(x) = 0, \Rightarrow F(x) = c$, [by integrating]

so, (5) becomes. $v = 3x^2y - y^3 + 6xy + c$.

(c) $f(z) = f(x+iy) = (x^3 - 3xy^2 + 3x^2 - 3y^2 + 1) + i(3x^2y - y^3 + 6xy + c)$.

Putting. $y=0$, $f(z) = u(x,0) + iv(x,0)$ (Replacing x by z).

$f(z) = u(z,0) + iv(z,0) \longrightarrow (6)$.

Now ~~(x,0)~~ $u(x,0) = x^3 + 3x^2 + 1$ $\quad \therefore u(z,0) = z^3 + 3z^2 + 1$

$v(x,0) = c$ $\quad \therefore v(z,0) = c$

Hence from (6). $f(z) = z^3 + 3z^2 + 1 + ic$.

Show that (i) $u = 2x(1-y)$ is harmonic (ii) Find v such that $f(z) = u + iv$ is analytic (iii) Find also $f(z)$ in terms of z.

Soln: (i) Given. $u = 2x - 2xy$

$\therefore \frac{\partial u}{\partial x} = 2 - 2y$, $\frac{\partial^2 u}{\partial x^2} = 0 \rightarrow ①$

$\frac{\partial u}{\partial y} = -2x$, $\frac{\partial u^2}{\partial y^2} = 0 \rightarrow ②$

$① + ② \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial u^2}{\partial y^2} = 0$, hence, it is harmonic.

(ii) From Cauchy's Riemann equation we have.

$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2 - 2y \rightarrow ③$.

$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -(-2x) = 2x \rightarrow ④$

Integrating equation ③ w.r.to y and kepping x constant.

$\int dv = \int (2-2y)\, dy \Rightarrow v = 2y - y^2 + F(x) \rightarrow ⑤$

Where, $F(x)$ is an arbitrary real function of x. Now substi-tuting ⑤ in eqn ④ we have

$\frac{\partial}{\partial x}\{2y - y^2 + F(x)\} = 2x$

$\Rightarrow F'(x) = 2x$, $\Rightarrow F(x) = x^2 + c$ [integrating].

$\therefore v = 2y - y^2 + x^2 + c$

(iii) $f(z) = f(x+iy) = u + iv = (2x - 2xy) + i(x^2 - y^2 + 2y + c)$

Putting. $y = 0$.

$\therefore f(x) = u(x,0) + iv(x,0)$.

Replacing x by z

$f(z) = u(z,0) + iv(z,0) \rightarrow ⑥$

Now. $u(x,0) = 2x$.

$v(x,0) = x^2 + c$

$\therefore u(z,0) = 2z$.

$v(z,0) = z^2 + c$

$\therefore f(z) = 2z + i(z^2 + c)$

Complex Integration.

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+idy)$$
$$= \int_C (udx - vdy) + i\int_C vdx + udy$$

Evaluate, $\int \bar{z}\,dz$, from $z=0$, to $z=4+2i$ along the curve C given by $z=t^2+it$.

Soln: The limit point if $z=0$ then $t=0$
and when $z=4+2i$, " $t=2$

$\therefore$ The limit integral $\int_C \bar{z}\,dz = \int_{t=0}^{2} \overline{(t^2+it)}\,d(t^2+it)$

$$= \int_{t=0}^{2} (t^2-it)(2t+i)\,dt = \int_0^2 (2t^3 - it^2 + t)\,dt$$

$$= \left[2\frac{t^4}{4} - i\frac{t^3}{3} + \frac{t^2}{2}\right]_0^2$$

$$= \left[\left(\frac{16}{2} - i\frac{8}{3} + \frac{4}{2}\right) - 0\right] = \left[10 - i\,8/3\right] \text{ Ans;}$$

Evaluate: $\int_{(0,3)}^{(2,4)} (2y+x^2)dx + (3x-y)dy$ along

i) the parabola $x=2t$, $y=t^2+3$
ii) the straight line from (0,3) to (2,4).

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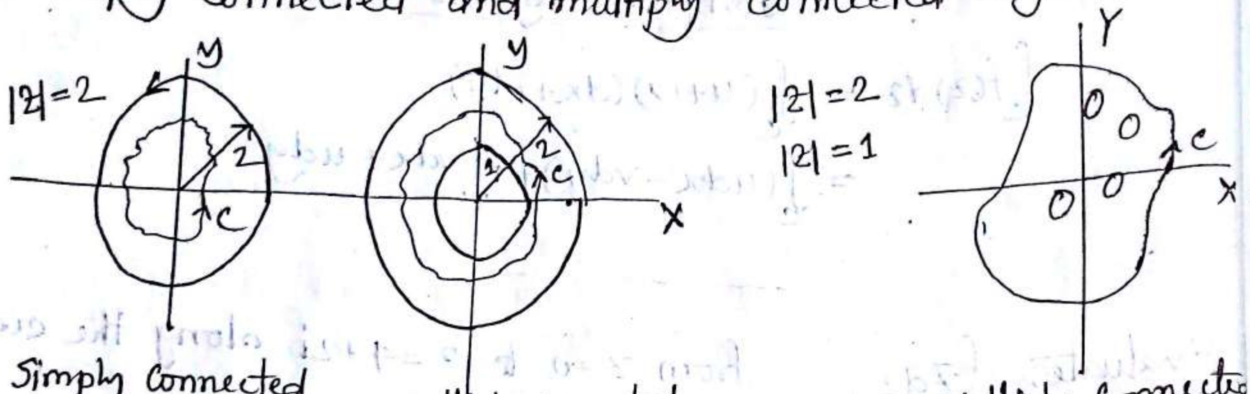
Simply Connected and multiply connected region:

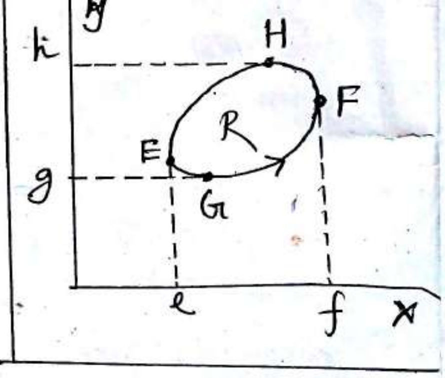


Simply Connected   Multiply connected   Multiply Connected

Green's theorem in the Plane:

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

State & prove Green's theorem:



Let, the equations of the curves EGF & EHF be $y = Y_1(x)$ and $y = Y_2(x)$ respectively. If R is the region bounded by C. we have

$$\iint_R \frac{\partial P}{\partial y}\, dx\,dy = \int_{x=e}^{f} \left[\int_{y=Y_1(x)}^{Y_2(x)} \frac{\partial P}{\partial y}\, dy\right] dx$$

$=$

ok/

Evaluate $\int_{(0,3)}^{(2,4)} (2y+x^2)dx + (3x-y)\,dy$ along

a) the parabola $x=2t$; $y=t^2+3$

b) Straight lines from $(0,3)$ to $(2,3)$ and then from $(2,3)$ to $(2$

c) a straight line from $(0,3)$ to $(2,4)$.

a) The points $(0,3)$ and $(2,4)$ on the parabola correspond to t
and $t=1$ respectively. Then the given integral equals

$$\int_{t=0}^{1} \{2(t^2+3)+(2t)^2\}\,2dt + \{3(2t)-(t^2+3)\}\,2t\,dt.$$

$$= \int_0^1 (24t^2+12-2t^3-6t)\,dt = 33/2.$$

b) Along the straight line from $(0,3)$ to $(2,3)$, $y=3$, $dy=0$. and the
line integral equals.

$$\int_{x=0}^{2} (6+x^2)dx + (3x-3)0 = \int_{x=0}^{2} (6+x^2)dx = 44/3.$$

Along the straight line from $(2,3)$ to $(2,4)$, $x=2$, $dx=0$ and t
integral equals

$$\int_{y=3}^{4} (2y+4)0 + (6-y)\,dy = 5/2$$

$$\frac{x-0}{0-2} = \frac{y-3}{3-4}$$

c) An equation for the line joining $(0,3)$ and $(2,4)$ is $2y-x=6$. Solv
for x. we have $x=2y-6$. then the line integral equals.

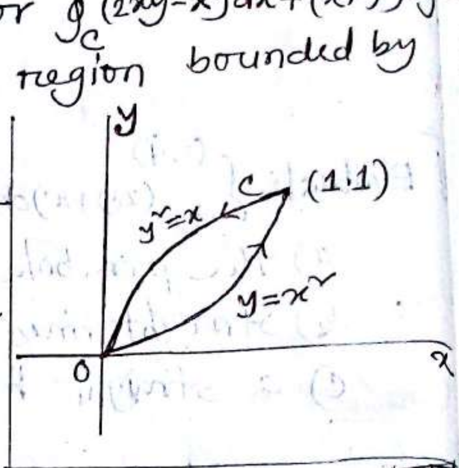
$$\int_{y=3}^{4} \{2y+(2y-6)^2\}2dy + \{3(2y-6)-y\}dy = \int_3^4 (8y^2-39y+54)\,dy =$$

The result can also be obtained by using $y=\frac{1}{2}(x+6$

4

Verify Green's theorem in the plane for $\oint_C (2xy-x^2)dx + (x+y^2)dy$ Where C is the closed curve of the region bounded by $y=x^2$ & $y^2=x$.

The plane curves $y=x^2$ and $y^2=x$ intersect at $(0,0)$ and $(1,1)$. The positive direction in traversing C is as shown in the figure



Along, $y=x^2$, the line integral equals

$$\int_{x=0}^{1} \{(2x)(x^2) - x^2\}dx + \{x+(x^2)^2\}d(x^2) \quad 2x\,dx = (2x^2+2x^5)dx$$

$$= \int_0^1 (2x^3 + x^2 + 2x^5)dx = 7/6.$$

Along, $y^2=x$, the line integral equals $\quad 2y\,dy = (4y^4 - 2y^5)dy.$

$$\int_{y=1}^{0} \{2(y^2)y - (y^2)^2\} dy^2 + (y^2+y^2)dy = \int_1^0 (4y^4 - 2y^5 + 2y^2)dy = -17/15$$

Then the required integral $= 7/6 - 17/15 = 1/30$

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy = \iint_R \left\{\frac{\partial}{\partial x}(x+y^2) - \frac{\partial}{\partial y}(2xy - x^2)\right\} dx\,dy$$

$$= \iint_R (1-2x)\,dx\,dy = \int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} (1-2x)\,dx\,dy = \int_{x=0}^{1} [y - 2xy]_{x^2}^{\sqrt{x}}\,dx$$

$$= \int_0^1 (x^{\frac{1}{2}} - 2x^{\frac{3}{2}} - x^2 + 2x^3)dx = \left[\frac{2}{3}x^{\frac{3}{2}} - 2\cdot\frac{2}{5}x^{\frac{5}{2}} - \frac{x^3}{3} + 2\frac{x^4}{4}\right]_0^1$$

$$= \frac{2}{3} - \frac{4}{5} - \frac{1}{3} + \frac{1}{2} = \frac{20-24-10+15}{30} = 1/30$$

Cauchy's Integral theorem: If $f(z)$ be analytic inside and on a simple closed curve region C. Then $\oint_C f(z)dz = 0$

Proof: Since $f(z)$ be analytic in the closed curve C, so cauchy, Rieman equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \rightarrow (1)$ are satisfied.

Now by definition. $\oint_C f(z)dz = \oint_C (u+iv)(dx+idy)$

$$= \oint_C (udx - vdy) + i\oint_C (vdx + udy) \rightarrow (2)$$

Apply Green's theorem in (2) we get.

$$\therefore \oint_C f(z)dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dxdy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dxdy$$

$$= \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right)dxdy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right)dxdy$$

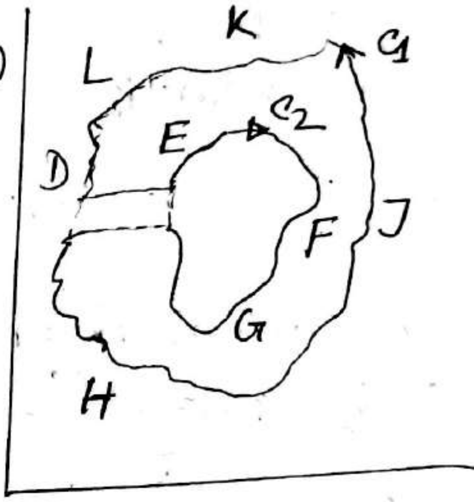
$$= 0 + 0 = 0.$$

Prove that (i) $\oint_C dz = 0$. (ii) $\oint_C z dz = 0$. where C is any simple closed curve.

Since two integration are zero. it must follows Cauchy's integral theorem. then $f(z) = 1$ as $f(z) = z$ must be analytic in the simple closed curve.

$$\oint_C (Pdx + Qdy) = \iint_R \left(\frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}\right) dx\, dy.$$

Let, $f(z)$ be analytic in a region R bounded by two simple closed curves C_1 and C_2 and also on C_1 & C_2, prove that $\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$ where C_1 & C_2 are both transversed in anticlockwise relative to the interiors.



Proof: Construct crosscut DE. Then since $f(z)$ is analytic in the region R. We have by cauchy's integral theorem:

$$\oint_{DEFGHJKLD} f(z)dz = 0$$

or. $$\int_{DE} f(z)dz + \int_{EFGE} f(z)dz + \int_{ED} f(z)dz + \int_{DHJKLD} f(z)dz = 0$$

Since. $\int_{DE} f(z)dz = -\int_{ED} f(z)dz$ are equal & opposite

$$\therefore \int_{DHJKLD} f(z)dz = -\int_{EFGH} f(z)dz = \int_{EGFE} f(z)dz$$

or. $$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz.$$

Evaluate $\oint_C \frac{dz}{z-a}$ where c is any simple closed curve and $z=a$ is i) outside c. (ii) inside c.

1). If $z=a$ is outside c, the function, ~~the function~~ $f(z) = \frac{1}{z-a}$ is analytic everywhere inside and one Here by Cauchy integral theorem: $\oint_C \frac{dz}{z-a} = 0$ [outside]

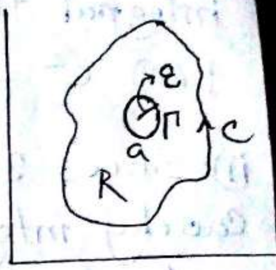
$$f(z) = \frac{1}{z-a} = \frac{1}{a-a} = \frac{1}{0} = \alpha = \text{undefined}.$$

Inside: If $z=a$ is inside c. we choose a circle of radius ε with centre at $z=a$, Then. on Γ $|z-a| = \varepsilon$ and $z-a = \varepsilon e^{i\theta}$ $[0 \le \alpha \le 2\pi]$

$\therefore dz = i\varepsilon e^{i\theta} d\theta$. thus $\oint_C \frac{dz}{z-a} = \int_0^{2\pi} i \frac{\varepsilon e^{i\theta} d\theta}{\varepsilon e^{i\theta}} = i\int_0^{2\pi} d\theta = 2\pi i$

Cauchy's integral formulas

If $f(z)$ is analytic inside and on its boundary C of simply connected region R except at the point a inside C, then



$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz$$

$$f'(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2}dz$$

$$f^n(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}dz$$

Proof: The function $\frac{f(z)}{z-a}$ is analytic and on C except at the point $z=a$ in figure we have $\oint_C \frac{f(z)}{z-a}dz = \oint_\Gamma \frac{f(z)}{z-a}dz \to ①$

Where, we can be choose a circle Γ of radius ε at the point $z=a$. Then on Γ, $|z-a| = \varepsilon$, $z-a = \varepsilon e^{i\theta}$ $0 \le \theta \le 2\pi$.

So, we can write; $\oint_C \frac{f(z)}{z-a}dz = \oint_\Gamma \frac{f(z)}{z-a}dz \to ①$

on the circle, $|z-a| = \varepsilon \Rightarrow z-a = \varepsilon e^{i\theta}$; $0 \le \theta \le 2\pi$

$\therefore z = a+\varepsilon e^{i\theta} \Rightarrow dz = i\varepsilon e^{i\theta}d\theta$

Now, $\oint_\Gamma \frac{f(z)}{z-a}dz = \oint_\Gamma \frac{f(z)}{z-a}dz = \int_0^{2\pi} \frac{f(a+\varepsilon e^{i\theta})}{\cancel{\varepsilon e^{i\theta}}} i\cancel{\varepsilon e^{i\theta}}d\theta$

$= i\int_0^{2\pi} f(a+\varepsilon e^{i\theta})d\theta \to ②$

Taking limit both side of ② as $\varepsilon \to 0$,

$\Rightarrow \oint_C \frac{f(z)}{z-a}dz = i\int_0^{2\pi} f(a)d\theta = 2\pi i f(a)$

$\Rightarrow f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz$ (proved)

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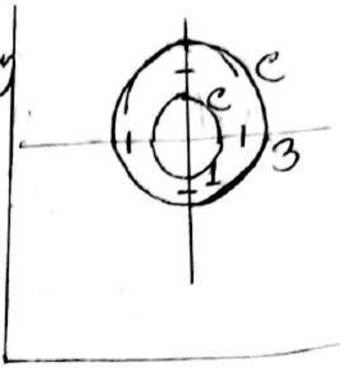
$\frac{1}{2\pi i}\oint_C \frac{e^z}{(z-2)}dz$, where c is the circle. i) $|z|=3$, ii) $|z|=1$

i) Since $z=2$ is inside c. so it follows Cauchy integral formula. Hence, $f(z)=e^z$, $a=2$ so



$$f(a)=e^2=\frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}dz \quad \text{Ans.}$$

ii) Since $z=2$ is outside c. so it follows Cauchy integral theorem. Hence

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz=0$$

Evaluate: $\oint_C \frac{e^{2z}}{(z+1)^4}dz$, where c is the circle $|z|=3$

A.

Hence $f(z)=e^{2z}$ and $a=-1$. it follows cauchy's integral formula.

$$f'(z)=2e^{2z}$$

$$f^3(z)=8e^{2z}$$

$$f^3(a)=8e^{-2}=\frac{3!}{2\pi i}\oint_C \frac{e^{2z}}{(z+1)}dz$$

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$$\text{or. } \oint_C \frac{e^{z}}{(z+1)^4}dz=\frac{2\pi i\times 8e^{-2}}{3\times 2\times 1}=\frac{8}{3}\pi i\, e^{-2} \quad \text{Ans.}$$

Soft 180.

Q. Prove that. i) $\oint_C dz=0$, ii) $\oint_C z\,dz=0$.

Soln: i). We know from Cauchy's Riemann integration theorem.

$\oint_C f(z)dz=0$, where. $f(z)$ is analytic

Hence. $f(z)=1=1+0.i$ $\therefore$ $u=1$, $v=0$

$$\frac{\partial u}{\partial x}=0, \quad \& \quad \frac{\partial v}{\partial y}=0$$

Since. $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$, so 1 is analytic inside and on a simple closed curve. from Cauchy Riemann theorem.

$$\oint_C 1\,dz=\oint_C dz=0$$

ii) Here $z = x+iy$

$\therefore u = x$ & $v = y$

$\frac{\partial u}{\partial x} = 1$. $\frac{\partial v}{\partial y} = 1$, Hence, z is analytic. So, from cauchy Riemann integration theorem $\oint_C z\,dz = 0$

Q Evaluate $\oint_C \frac{1}{z(z-2)^4} dz$. where c is the circle $|z| = 1$.

Soln: Let $f(z) = \frac{1}{(z-2)^4}$, So the integral theorem.

$$\oint_C \frac{1}{z(z-2)^4} dz = \oint_C \frac{f(z)}{(z-0)} dz$$

Here $f(z) = \frac{1}{(z-2)^4}$, $a = 0$, since $a = 0$ inside c. So it follows cauchy's integral formula.-

$$f(a) = f(0) = \frac{1}{(-2)^4} = \frac{1}{16} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-0)} dz = \frac{1}{2\pi i} \oint_C \frac{1}{z(z-2)^4} dz$$

$$\Rightarrow \oint_C \frac{1}{z(z-2)^4} dz = \frac{2\pi i}{16} = \frac{\pi i}{8} \quad \text{Ans.}$$

Cauchy's Residue theorem:

i) If $F(z)$ is analytic inside and on a simple closed curve c, except for a pole of order m at $z = a$ inside c. then.

$$\frac{1}{2\pi i} \oint_C F(z) dz = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left\{ (z-a)^m F(z) \right\}$$

ii) If there are two poles at $z = a_1$ and $z = a_2$ inside c of order m_1 and m_2 respectively, then.

$$\frac{1}{2\pi i} \oint_C F(z) dz = \lim_{z \to a_1} \frac{1}{(m_1-1)!} \frac{d^{m_1-1}}{dz^{m_1-1}} \left\{ (z-a_1)^{m_1} F(z) \right\} + \lim_{z \to a_2} \frac{1}{(m_2-1)!} \frac{d^{m_2-1}}{dz^{m_2-1}} \left\{ (z-a_2)^{m_2} F(z) \right\}.$$

iii) In general if $F(z)$ has a number of poles inside c with Residue $R_1, R_2, \ldots$ then $\oint_C F(z) dz = 2\pi i \{R_1 + R_2 + \cdots\} = 2\pi i \{\text{sum of residues}\}$

Evaluate: $\oint_C \frac{e^z}{(z^2+\pi^2)^2}dz$, where C in the circle $|z|=4$.

Sol^n: The poles of $\frac{e^z}{(z^2+\pi^2)^2} = \frac{e^z}{(z+\pi i)^2(z-\pi i)^2}$ are at $z=\pm\pi i$ inside C and both of order 2

Residue at $z=\pi i$ is $\lim_{z\to\pi i} \frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}\left\{(z-\pi i)^2 \frac{e^z}{(z-\pi i)^2(z+\pi i)^2}\right\}$

$$= \lim_{z\to\pi i}\frac{d}{dz}\left\{\frac{e^z}{(z+\pi i)^2}\right\}$$

$$= \frac{\pi+i}{4\pi^3}$$

Again. The residue at $z=-\pi i$ is $\lim_{z\to-\pi i}\frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}\left\{(z+\pi i)^2\frac{e^z}{(z-\pi i)^2(z+\pi i)^2}\right\}$

$$= \frac{\pi-i}{4\pi^3}.$$

Thus, $\oint_C \frac{e^z}{(z^2+\pi^2)^2}dz = 2\pi i\{\text{Sum of residue}\}$

$$= 2\pi i\left\{\frac{\pi+i}{4\pi^3}+\frac{\pi-i}{4\pi^3}\right\}.$$

$$= i/\pi \quad \text{Ans!}$$

Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{(z^2+1)^2}dz$ if $t>0$ and C is the circle $|z|=1$

Sol:

The poles of $\frac{e^{zt}}{(z^2+1)^2}$ are at $z=i$ inside both of order 2.

the residue at $z=i$

$$Lt_{z\to i}\frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}\left\{(z-i)^2\frac{e^{zt}}{(z^2+1)^2}\right\}$$

$$= Lt_{z\to i}\frac{d}{dz}\left\{\frac{e^{zt}}{(z+i)^2}\right\}.$$

$$= Lt_{z\to i}\left\{\frac{(z+i)^2te^{zt}-e^{zt}2(z+i)}{(z+i)^4}\right\}.$$

$$= \frac{te^{it}4(-1)-e^{zt}2.2i}{16}$$

$$= \frac{-4te^{it}-4ie^{it}}{16} = \frac{-4e^{it}(t+i)}{16}$$

$$= \frac{eit(t+i)}{-4} =$$

The residue at $z=-i$

$$Lt_{z\to -i}\frac{d}{dz}\left\{(z+i)^2\frac{e^{zt}}{(z^2+1)^2}\right\}$$

$$= Lt_{z\to -i}\frac{d}{dz}\left\{\frac{e^{zt}}{(z-i)^2}\right\}.$$

$$= Lt_{z\to -i}\left\{\frac{te^{zt}(z-i)^2-e^{zt}2(z-i)}{(z-i)^4}\right.$$

$$= \frac{te^{-it}(-2i)^2-e^{-it}2(-2i)}{(-2i)^4}$$

$$= \frac{-4te^{-it}+4ie^{-it}}{16} = \frac{te^{-it}-ie}{-4}$$

$\therefore \frac{1}{2\pi i}\oint_C\frac{e^{zt}}{(z^2+i)^2} = \{\text{Sum of resid}$

$$= \frac{-te^{it}-ie^{it}}{4} - \frac{te^{-it}-ie^{it}}{4}$$

$= -\frac{(e^{it} - ie^{it} + (e^{-it} + ie^{-it})}{4} + \frac{t}{2}\left\{ \frac{-i(e^{it}-e^{-it})}{2} - t\frac{(e^{it}+e^{-it})}{2} \right\}$

$= \frac{1}{2}\left\{ \frac{-i(e^{it}-e^{-it})}{-2i^2} - t\frac{(e^{it}+e^{-it})}{2} \right\} = \frac{1}{2}\left\{ \frac{e^{it}-e^{-it}}{2i} - t\frac{(e^{it}+e^{-it})}{2} \right\}.$

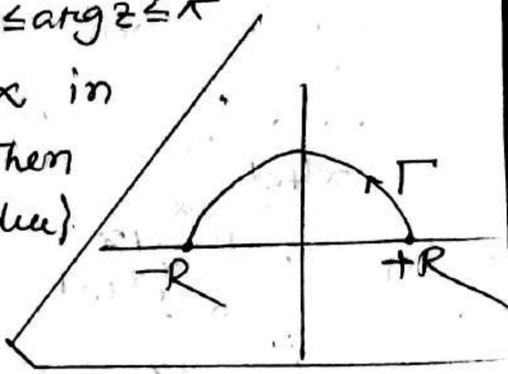
$= \frac{1}{2}(\sin t - t\cos t)$ Ans

Contour Integration.

Theorem: Consider the evaluation of integrals type $I = \int_{-\alpha}^{\alpha} F(x)dx$ where $F(z)$ is a function that satisfies the following conditions:

1) It is analytic in the upper half plane except at a finite number of poles.
2) It has no poles on the real axis.
3) $zF(z) \to 0$ uniformly as $|z| \to \alpha$ for $0 \le \arg z \le \pi$
4) When x is real, $xF(x) \to 0$ as $x \to \pm\alpha$ in such way that $\int_{-\alpha}^{\alpha} F(x)dx$ converges. Then

$I = \oint_C F(z)dz = \int_{-\alpha}^{\alpha} F(x)dx = 2\pi i\{\text{sum of Residue}\}$



Proof: $\oint_C F(z)dz = \int_{-R}^{R} F(x)dx + \int_{\Gamma} F(z)dz$

Taking limit $R \to \alpha$

$\therefore \oint_C F(z)dz = \int_{-\alpha}^{\alpha} F(x)dx + 0 = 2\pi i\{\text{Sum of residue}\}$ Ans.

Ex: Evaluate the contour $\int_0^{\alpha} \frac{dx}{x^4+a^4}$

Soln: Consider the integral $\oint_C \frac{dz}{z^4+a^4}$ where C is the closed contour of figure consisting of the line from $(-R)$ to $(+R)$ and the semicircle Γ traveled in anticlockwise.

The function $F(z) = \frac{1}{z^4+a^4}$ has simple poles $ae^{\pi i/4}, ae^{3\pi i/4}, ae^{5\pi i/4}, ae^{7\pi i/4}$ but only the first two poles $ae^{\pi i/4}$, and $ae^{3\pi i/4}$ lies in the upper half plane. The functions $F(z)$ clearly satisfies the all conditions of the theorem.

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Therefore. Residue at $z=ae^{\pi i/4}$ is $\lim_{z\to ae^{\pi i/4}}\left\{(z-ae^{\pi i/4})\frac{1}{z^4+a^4}\right.$

$$= \lim_{z\to ae^{\pi i/4}} \frac{1}{4z^3} = \frac{1}{4a^3}\cdot e^{-3\pi i/4}$$

again residue at. $z=ae^{3\pi i/4}$ is $\lim_{z\to ae^{\frac{3\pi i}{4}}}\left\{(z-ae^{3\pi i/4})\frac{1}{z^4+a^4}\right.$

$$= \lim_{z\to ae^{3\pi i/4}} \frac{1}{4z^3} = \frac{1}{4a^3}\cdot e^{-9\pi i/4}$$

Then. $\oint_C \frac{1}{z^4+a^4}\,dz = \int_{-R}^{R} \frac{1}{x^4+a^4}\,dx + \int \frac{1}{z^4+a^4}\,dz = 2\pi i\{\text{Sum of residue}\}$

$$= 2\pi i\left\{\frac{1}{4a^3}\,e^{-3\pi i/4} + \frac{1}{4a^3}\cdot e^{-9\pi i/4}\right\}$$

$$= 2\pi i \times \frac{1}{4a^3}\times -i\sqrt{2} = \pi/\sqrt{2}a^3 \quad \rightarrow (1)$$

Taking the limit of both side of (i) as $R\to\infty$

$$\int_{-\infty}^{\infty}\frac{dx}{x^4+a^4} + 0 = \pi/\sqrt{2}a^3$$

or. $2\int_0^{\infty}\frac{dx}{x^4+a^4} = \pi/\sqrt{2}a^3.$

$$\Rightarrow \int_0^{\infty}\frac{dx}{x^4+a^4} = \frac{\pi}{2\sqrt{2}a^3} \quad \text{Ans.}$$

Evaluate the followings:

i) $\int_0^{\infty}\frac{dx}{x^6+1}$, ii) $\int_0^{\infty}\frac{\cos mx}{x^2+1}\,dx.\ m>0.$ iii) $\int_0^{\infty}\frac{\cos mx}{(x^2+1)^2}\,dx.\ m>0.$

4). $\int_0^{\infty}\frac{\sin x}{x}\,dx.$

Solutions:

P.T.O

iii) Consider the integral $\oint_C \frac{e^{imz}}{(z^2+1)^2} dz$ where C is the closed contour of fig.1. Consisting of the line from $-R$ to $+R$ and the semicircle Γ traveled in anticlockwise. The integral has a poles at $z = \pm i$ of order 2 but only $z = +i$ lies inside C. clearly the function $F(z)$ satisfies all conditions.

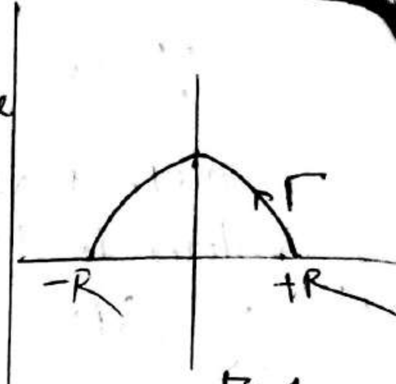

Fig.1

So the residue at $z = +i$ is $\lim_{z \to i} \frac{1}{(2-1)!} \frac{d^{2-1}}{dz^{2-1}} \left\{ (z-i)^2 \frac{e^{imz}}{(z^2+1)^2} \right\}$

$$= \lim_{z \to i} \frac{d}{dz} \left\{ \cancel{(z-i)^2} \frac{e^{imz}}{\cancel{(z-i)^2}(z+i)^2} \right\} = \lim_{z \to i} \frac{d}{dz} \left\{ \frac{e^{imz}}{(z+i)^2} \right\}.$$

$$= \frac{e^{-m}(1+m)}{4i}$$

Thus. $$\oint_C \frac{e^{imz}}{(z^2+1)^2} dz = \int_{-R}^{R} \frac{e^{imx}}{(x^2+1)^2} dx + \int_\Gamma \frac{e^{imz}}{(z^2+1)^2} dz = 2\pi i \{\text{sum of residues}\}$$

$$= 2\pi i \times \frac{e^{-m}(1+m)}{4i} = \pi \frac{e^{-m}(1+m)}{2} \longrightarrow ①.$$

Taking limit $R \to \infty$. both side ①.

$$\oint_C \frac{e^{imz}}{(z^2+1)^2} dz = \int_0^\infty \frac{e^{imx}}{(x^2+1)^2} dx + 0 = \frac{\pi e^{-m}(1+m)}{4} + i \times 0.$$

or, $$\int_0^\infty \frac{\cos mx}{(x^2+1)^2} dx + i \int_0^\infty \frac{\sin mx}{(x^2+1)^2} dx = \frac{\pi e^{-m}(1+m)}{4}$$

equating on both sides —

$$\int_0^\infty \frac{\cos mx}{(x^2+1)^2} dx = \frac{\pi e^{-m}(1+m)}{4}$$ Ans.

iv) $\int_0^\infty \frac{\sin x}{x}\,dx.$

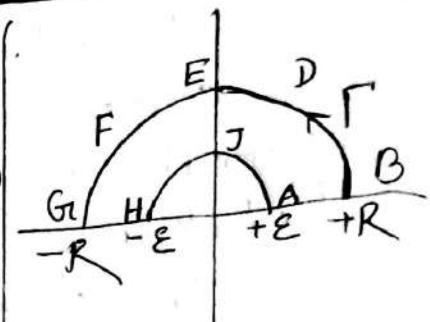

Fig: 2

The method of Cauchy's Residue theorem of Contour leads us to consider the integral of $\oint_C \frac{e^{iz}}{z}\,dz$ around the contour of fig:1 However, semicircle $z=0$. lies on this path of integration and since we can not integrate through a singularity at $z=0$ we modify that contour by intending the path at $z=0$ as shown in fig.2. Since $z=0$ is outside C' i.e ABDEFGHJA we have

$$\oint_{ABDEFGHJA} \frac{e^{iz}}{z}\,dz = 0$$

or. $$\int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\,dx + \int_{HJA} \frac{e^{iz}}{z}\,dz + \int_{\varepsilon}^{R} \frac{e^{ix}}{x}\,dx + \int_{BDEFG} \frac{e^{iz}}{z}\,dz = 0$$

compare the first & 3rd integral of left hand side we get

$$\Rightarrow \int_{\varepsilon}^{R} \frac{e^{ix} - e^{-ix}}{x}\,dx + \int_{HJA} \frac{e^{iz}}{z}\,dz + \int_{BDEFG} \frac{e^{iz}}{z}\,dz = 0$$

$$\Rightarrow 2i\int_{\varepsilon}^{R} \frac{\sin x}{x}\,dx = -\int_{HJA} \frac{e^{iz}}{z}\,dz - \int_{BDEFG} \frac{e^{iz}}{z}\,dz$$

Let. $\varepsilon \to 0$, and $R \to \infty$, the ~~first~~ 2nd integral on the right h side approaches zero. Let $z = \varepsilon e^{i\theta}$ in the first integ on the right hand side. we see that if-

$$-\int_{HJA} \frac{e^{iz}}{z}\,dz = -\lim_{\varepsilon\to 0} \int_{\pi}^{0} \frac{e^{i\varepsilon e^{i\theta}}}{\varepsilon e^{i\theta}}\, i\varepsilon e^{i\theta}\,d\theta = -\lim_{\varepsilon\to 0} \int_{\pi}^{0} i e^{i\varepsilon e^{i\theta}}$$

$$= i\int_0^{\pi} d\theta = \pi i$$

Thus. $2i\int_0^\infty \frac{\sin x}{x}\,dx = 0 + \pi i$ or. $\int_0^\infty \frac{\sin x}{x}\,dx = \pi/2$

Laplace

Laplace transform: Let $F(t)$ is a function of t where $t>0$. Then the laplace transform of $F(t)$ is denoted by $L\{F(t)\}$ or $f(s)$ is defined as

$$f(s) = L\{F(t)\} = \int_0^{\infty} e^{-st} F(t)\,dt$$

$$L\{e^{at}\} = \int_0^{\infty} e^{-st} e^{at}\,dt = \int_0^{\infty} e^{-t(s-a)}\,dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^{\infty}$$

$$= +\frac{1}{s-a}.$$

$F(t) = t^n$

$$L\{F(t)\} = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}} = f(s) = \{t^n\}.$$

$$L\{\sin at\} = \frac{a}{s^2+a^2}.$$

$$L\{\cos at\} = \frac{s}{s^2+a^2}.$$

$$L\{\sinh at\} = \frac{a}{s^2-a^2}.$$

$$L\{\cosh at\} = \frac{s}{s^2-a^2}$$

$$L\{\sin at + \cos 2t + e^{3t}\} = L\{\sin at\} + L\{\cos 2t\} + L\{e^{3t}\}.$$

$$= \frac{a}{s^2+a^2} + \frac{s}{s^2+2^2} + \frac{1}{s-3}$$

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Laws of Laplace transformation:-

1. $F(t)=1 \; ; \; L\{F(t)\} = f(s) = \frac{1}{s} \; ; \; s>0$

2. $F(t)=t \; ; \; L\{F(t)\} = f(s) = \frac{1}{s^2} \; ; \; s>0$

3. $F(t)=t^n \; ; \; L\{F(t)\} = f(s) = \frac{n!}{s^{n+1}} \; ; \; s>0$
   $n = 0, 1, 2, \ldots$

4. $F(t) = e^{at} \; ; \; L\{F(t)\} = f(s) = \frac{1}{s-a} \; ; \; s>a.$

5. $L\{e^{-at}\} = \frac{1}{s+a} \; ; \; s>a.$

6. $L\{\sin at\} = \frac{a}{s^2+a^2} \; ; \; s>0.$

7. $L\{\cos at\} = \frac{s}{s^2+a^2} \; ; \; s>0$

8. $L\{e^{at} t^n\} = \frac{n!}{(s-a)^{n+1}} \; ;$

9. $L\{\sinh at\} = \frac{a}{s^2-a^2} \; ;$

10. $L\{\cosh at\} = \frac{s}{s^2-a^2} \; ;$

11. $L\{e^{at} \sinh bt\} = \frac{b}{(s-a)^2-b^2} \; ;$

12. $L\{e^{at} \cosh bt\} = \frac{s-a}{(s-a)^2-b^2}$

13. $L\{e^{at} \sin bt\} = \frac{b}{(s-a)^2+b^2}$

14. $L\{e^{at} \cos bt\} = \frac{s-a}{(s-a)^2+b^2}.$

15. $L\{F(at)\} = \frac{1}{a} f(s/a) \; ;$

16. $L\{e^{at} F(t)\} = f(s-a) \; ;$

17. $L\{t^n e^{nt}\} = n/(s-n)^{n+1} \; ;$ 18. $L\{t^2 e^{2t}\} = \frac{2}{(s-2)^3} \; ;$

* $\sin at = \frac{e^{iat} - e^{-iat}}{2i}$
* $\cos at = \frac{e^{iat} + e^{-iat}}{2}$
* $\sinh at = \frac{e^{at} - e^{-at}}{2}$
* $\cosh at = \frac{e^{at} + e^{-at}}{2}$
* $e^{i\theta} = \cos\theta + i\sin\theta.$
* $e^{-i\theta} = \cos\theta - i\sin\theta.$
* $\int uv = u\int v\,dx - \int\left\{\frac{d}{dx}u \int v\,dx\right\}dx$

Q. Prove that: $L\{1\} = \frac{1}{s}$

solve: $L\{1\} = \int_0^\infty e^{-st}.1.dt = \int_0^\infty e^{-st} dt = -\frac{1}{s}\left[e^{-st}\right]_0^\infty = \frac{1}{s}$

Q. Prove that. $L\{t^n\} = \frac{\lfloor n}{s^{n+1}}$

solve: $L\{t^n\} = \int_0^\infty e^{-st}.t^n.dt = \int_0^\infty e^{-x} t^n . \frac{dx}{s}$ | Let. $st = x$, $t = x/s$, $dt = \frac{dx}{s}$.

$= \int_0^\infty e^{-x}.\left(\frac{x}{s}\right)^n . \frac{dx}{s} = \frac{1}{s^{n+1}} \int_0^\infty e^{-x}.x^n.dx$

$= \frac{\lfloor n}{s^{n+1}}$ $\left[\int_0^\infty e^{-x} x^n dx = \lfloor n\right]$.

Q. Prove that: $L\{e^{at}\} = \frac{1}{s-a}$; $s>a$. (আগের পৃষ্ঠায় আছে).

Q. Prove that: $L\{e^{-at}\} = \frac{1}{s+a}$; $s>a$.

solve: $L\{e^{-at}\} = \int_0^\infty e^{-st} e^{-at} dt = \int_0^\infty e^{-t(s+a)} dt$

$= -\frac{1}{s+a}\left[e^{-(s+a)t}\right]_0^\infty = -\frac{1}{s+a}.(-1) = \frac{1}{s+a}$

Q. Prove that, $L\{\sin at\} = a/s^2+a^2$

solve: we know, $\sin at = \frac{e^{iat} - e^{-iat}}{2i}$

$\therefore L\{\sin at\} = L\left\{\frac{e^{iat} - e^{-iat}}{2i}\right\} = \frac{1}{2i}\left[L\{e^{iat}\} - L\{e^{-iat}\}\right]$

$= \frac{1}{2i}\left[\int_0^\infty e^{-st}.e^{iat}.dt - \int_0^\infty e^{-st} e^{-iat} dt\right]$

$= \frac{1}{2i}\left[\int_0^\infty e^{-t(s-ia)} dt - \int_0^\infty e^{-t(s+ia)} dt\right]$

$= \frac{1}{2i}\left[\frac{-1}{(s-ia)}\left[e^{-(s-ia)t}\right]_0^\infty - \frac{-1}{(s+ia)}\left[e^{-t(s+ia)}\right]_0^\infty\right]$

$= \frac{1}{2i}\left[\frac{-1}{s-ia}(-1) - \frac{-1}{s+ia}(-1)\right] = \frac{1}{2i}\left[\frac{1}{s-ia} - \frac{1}{s+ia}\right] = \frac{1}{2i}\left[\frac{s+ia-s+ia}{s^2-i^2a^2}\right]$

$= \frac{a}{s^2+a^2}$ Ans

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Prove that. $L\{\cos at\} = \frac{s}{s^2+a^2}$

Solve: $L\{\cos at\} = L\left\{\frac{e^{iat}+e^{-iat}}{2}\right\}$, we know.

$= \frac{1}{2}\left[L\{e^{iat}\} + L\{e^{-iat}\}\right] = \frac{1}{2}\left[\int_0^\alpha e^{-st}e^{iat}dt + \int_0^\alpha e^{-st}e^{-iat}dt\right]$

$= \frac{1}{2}\left[\int_0^\alpha e^{-t(s-ia)}dt + \int_0^\alpha e^{-t(s+ia)}dt\right]$

$= \frac{1}{2}\left[\frac{-1}{(s-ia)}\left[e^{-(s-ia)t}\right]_0^\alpha + \frac{-1}{(s+ia)}\left[e^{-(s+ia)t}\right]_0^\alpha\right]$

$= \frac{1}{2}\left[\frac{1}{s-ia} + \frac{1}{s+ia}\right] = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}s}{s^2+a^2} = \frac{s}{s^2+a^2}.$

$L\{\cos at\} = s/s^2+a^2$. (Proved)

Prove that: $L\{\cosh at\} = s/s^2-a^2$.

Solve: $L\{\cosh at\} = L\left\{\frac{e^{at}+e^{-at}}{2}\right\} = \frac{1}{2}\left[L\{e^{at}\}+L\{e^{-at}\}\right]$

$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{1}{2}\,\frac{s+\cancel{a}+s-\cancel{a}}{s^2-a^2} = \frac{s}{s^2-a^2}$ (Proved)

Prove that: $L\{\sinh at\} = a/s^2-a^2$.

Solve: $L\{\sinh at\} = L\left\{\frac{e^{at}-e^{-at}}{2}\right\} = \frac{1}{2}\left[L\{e^{at}\} - L\{e^{-at}\}\right]$

$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+a-s+a}{s^2-a^2}\right] = \frac{a}{s^2-a^2}$ (Proved)

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CL First translation or shifting property:

If $L\{F(t)\} = f(s)$ then $L\{e^{at} F(t)\} = f(s-a)$ for periodic function.

proof: $L\{e^{at} F(t)\} = \int_0^\alpha e^{-st} e^{at} F(t)\, dt$

$= \int_0^\alpha e^{-t(s-a)} F(t)\,dt.$

$= f(s-a).$

$L\{F(t)\} = \int_0^\alpha e^{-st} F(t)\,dt$
$= f(s).$

$L\{\sin t\} = \frac{1}{s^2+1} = \frac{1}{s^2+1} = f(s)$

$L\{e^{2t} \sin t\} = \frac{1}{(s-2)^2+1} = f(s-2).$

#Sp: 2nd. Translation or shifting property.

If $L\{F(t)\} = f(s)$ and $G(t) = \begin{cases} F(t-a) & t>a. \\ 0 & t<a. \end{cases}$ then. $L\{G(t)\} = e^{-as} f(s)$

Proof: $L\{G(t)\} = \int_0^\alpha e^{-st} G(t)\,dt = \int_0^a e^{-st} G(t)\,dt + \int_a^\alpha e^{-st} G(t)\,dt$

$= 0 + \int_a^\alpha e^{-st} F(t-a)\,dt$

$= \int_0^\alpha e^{-s(u+a)} F(u)\,du.$

$= e^{-sa} \int_0^\alpha e^{-su} F(u)\,du.$

$= e^{-sa} f(u).$

Let, $t-a=u$
$dt = du.$
$t\to a.\ u=0.$
$t\to \alpha\ u=\alpha.$

II $L\{F'(t)\} = s f(s) - F(0).$

$L\{\sin at\} = \frac{a}{s^2+a^2}.$

$L\{a\cos at\} = s.f(s) - F(0) = s \cdot \frac{a}{s^2+a^2} - 0$

$= \frac{sa}{s^2+a^2}.$

$F(0) = \sin 0.t = 0.$

Alternative: $L\{a\cos at\} = a \cdot \frac{s}{s^2+a^2} = \frac{as}{s^2+a^2}.$

If $L\{F(t)\} = f(s)$ then prove $L\{F''(t)\} = s^2 f(s) - sF(0) - F'(0)$

Proof: Let, $G(t) = F'(t)$

$G'(t) = F''(t)$

$G(0) = F'(0)$.

we know. $L\{G'(t)\} = s g(s) - G(0)$ SP

$= s L\{G(t)\} - G(0)$.

$\Rightarrow L\{F''(t)\} = s L\{F'(t)\} - F'(0)$

$= s[s f(s) - F(0)] - F'(0)$.

$= s^2 f(s) - sF(0) - F'(0)$. (Proved)

Find $L\{F(t)\}$ where $F(t) = \begin{cases} \cos(t - 2\pi/3) & t > 2\pi/3 \\ 0 & t < 2\pi/3. \end{cases}$

Soln: Let, $G(t) = \cos(t)$.

$L\{G(t)\} = \frac{s}{s^2+1} = g(s)$.

$L\{F(t)\} = e^{-as} g(s) = e^{-2\pi/3 s} \frac{s}{s^2+1}$ (Proved)

If $L\{F(t)\} = f(s)$ then. $L\{t^n F(t)\} = (-1)^n f^n(s)$

$f(s) = \int_0^\infty e^{-st} F(t)\, dt$

$\Rightarrow f'(s) = \int_0^\infty \frac{\partial}{\partial s}(e^{-st} F(t))dt = \int_0^\infty -t e^{-st} F(t) dt = \int_0^\infty e^{-st} [-t \cdot F(t)] dt$

$= L\{-t F(t)\} = -L\{t F(t)\}$

$\therefore L\{t F(t)\} = (-1)^1 f'(s)$. (Proved)

$L\{t \sin 3t\}$.

Let, $F(t) = \sin 3t$.

$L\{F(t)\} = \frac{3}{s^2+9} = f(s)$.

$\therefore L\{t \sin 3t\} = (-1) f'(s) = -\frac{d}{ds}\left(\frac{3}{s^2+9}\right)$

$= \frac{6s}{(s^2+9)^2}$ (Proved)

If $L\{F(t)\} = f(s)$ then. $L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)\,du$.

Soln: Let. $\frac{F(t)}{t} = G(t) \Rightarrow tG(t) = F(t)$

Taking Laplace transform on both sides.

$$L\{tG(t)\} = L\{F(t)\}$$

$$-\frac{d}{ds} L\{G(t)\} = f(s)$$

$$L\{G(t)\} = -\int_\infty^s f(u)\,du = \int_s^\infty f(u)\,du$$

$$L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)\,du.$$

Find. $L\{\sin t/t\}$.

Soln: We have. $L\{\sin t\} = \frac{1}{s^2+1} = f(s)$

$$\therefore L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{u^2+1}\,du = \tan^{-1} u \Big|_s^\infty = \frac{\pi}{2} - \tan^{-1} s = \tan^{-1} 1/s$$

Laplace transform of integral.

If $L\{F(t)\} = f(s)$, then. $L\left\{\int_0^t F(u)\,du\right\} = f(s)/s$.

Proof: Let. $G(t) = \int_0^t F(u)\,du$

$$G(0) = 0$$
$$G'(t) = F(t).$$

Taking Laplace on both sides.

$$L\{G'(t)\} = L\{F(t)\}.$$

$$\Rightarrow sL\{G(t)\} - G(0) = f(s).$$

$$\Rightarrow sL\left\{\int_0^t F(u)\,du\right\} = f(s).$$

$$\therefore L\left\{\int_0^t F(u)\,du\right\} = f(s)/s \quad \text{(Proved)}.$$

$$\int_a^x \sin x = [-\cos x + c]_a^x = (-\cos x + c) - (-\cos a + c)$$
$$= -\cos x + \cos a,$$

if $a = \pi/2 \therefore = -\cos x$

Find $L\left\{\int_0^t \frac{\sin t}{t} dt\right\}$.

Soln: Let, $F(t) = \sin t / t$.

$\therefore L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\frac{1}{s} = f(s)$

$\therefore L\left\{\int_0^t \frac{\sin t}{t} dt\right\} = \frac{f(s)}{s} = \frac{1}{s}\tan^{-1}\frac{1}{s}$

Evaluate: $\int_0^\infty e^{-2t}\, t \cos t\, dt$.

Soln: $L\{t\cos t\} = (-1)^1 \frac{d}{ds}\frac{s}{s^2+1} = -\left[\frac{(s^2+1) - s.2s}{(s^2+1)^2}\right]$

$= -\left[\frac{s^2+1-2s^2}{(s^2+1)^2}\right] = -\frac{1-s^2}{(s^2+1)^2}$.

Again. $L\{t\cos t\} = \int_0^\infty e^{-st}\, t\cos t\, dt$.

or. $\int_0^\infty e^{-st}\, t \cos t\, dt = -\frac{1-s^2}{(s^2+1)^2}$

or. $\int_0^\infty e^{-2t}\, t\cos t\, dt = -\frac{1-4}{5^2} = +\frac{3}{25}$ Ans

Evaluate: $\int_0^\infty \frac{e^{-t} - e^{-3t}}{t} dt$

Soln: Let. $F(t) = e^{-t} - e^{-3t}$

$L\{F(t)\} = \frac{1}{s+1} - \frac{1}{s+3} = f(s)$

$L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)\, du = \int_s^\infty\left(\frac{1}{u+1} - \frac{1}{u+3}\right)du = \ln(u+1) - \ln(u+3)$

$= \ln \frac{u+1}{u+3}\Big|_s^\infty = -\ln\frac{s+1}{s+3} = \ln\frac{s+3}{s+1}$.

$\therefore L\left\{\frac{F(t)}{t}\right\} = \ln\frac{s+3}{s+1}$

or. $\int_0^\infty e^{-st}\frac{F(t)}{t} dt = \ln\frac{s+3}{s+1} \Rightarrow \int_0^\infty e^{-st}\frac{e^{-t} - e^{-3t}}{t} dt = \ln\frac{s+3}{s+1}$.

putting. $s = 0$, $\int_0^\infty \frac{e^{-t} - e^{-3t}}{t} dt = \ln 3$. Ans

Find $L\{\sin\sqrt{t}\}$.

$$\sin\sqrt{t} = \sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \left(\frac{\sqrt{t}}{5!}\right)^5 - \cdots + \cdots$$

$$= t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \cdots$$

$$L\{\sin\sqrt{t}\} = L\left\{ t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} \cdots \right\}.$$

$\Gamma(n+1) = n\Gamma(n)$
$= n(n-1)\Gamma(n-1)$
$\Gamma(1/2) = \sqrt{\pi}$

$$= \frac{\Gamma(3/2)}{s^{3/2}} - \frac{\Gamma(5/2)}{3!\, s^{5/2}} + \frac{\Gamma(7/2)}{5!\, s^{7/2}}$$

$$= \frac{\frac{1}{2}\Gamma(1/2)}{s^{3/2}} - \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma(1/2)}{3!\, s^{5/2}} + \frac{5/2\cdot 3/2\cdot 1/2\cdot\Gamma(1/2)}{5!\, s^{7/2}}$$

$$= \frac{\sqrt{\pi}}{2\cdot s^{3/2}}\left(1 - \frac{1}{4s} + \frac{1}{2!}\left(\frac{1}{4s}\right)^2 - \cdots\right).$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\cdot e^{-1/4s} \quad \text{Ans.}$$

Find. $L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}$.

Sol^n: Let. $F(t) = \sin\sqrt{t}$

$F(0) = 0$. we have
$L\{F(t)\} = \frac{\sqrt{\pi}}{2s^{3/2}}\cdot e^{-1/4s}$.

$\Rightarrow F'(t) = \frac{1}{2}\cdot\frac{\cos\sqrt{t}}{\sqrt{t}}$

$\Rightarrow L\{F'(t)\} = sf(s) - F(0)$.

$\Rightarrow \frac{1}{\cancel{2}} L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = s\cdot\frac{\sqrt{\pi}}{\cancel{2}\, s^{3/2}}\cdot e^{-1/4s} - 0$.

$\therefore L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \frac{\sqrt{\pi}}{s^{1/2}}\cdot e^{-1/4s}$ Ans.

H.W 31 Page: 58 → a, b, e.
80,

$L\{e^{at}\} = \frac{1}{s-a}$

$L^{-1}\{\frac{1}{s-a}\} = e^{at}$

$L\{1\} = \frac{1}{s}$

$L^{-1}\{\frac{1}{s}\} = 1.$

$L\{t\} = \frac{1}{s^2}$

$L^{-1}\{\frac{1}{s^2}\} = t.$

$F(t) = \begin{cases} G(t-a) & t>a. \\ 0 & t<a. \end{cases}$

$L\{F(t)\} = e^{-as} g(s)$

$\overline{L^{-1}}\left\{\frac{e^{-\pi/3 s}}{s^2+1}\right\} = \begin{cases} \sin(t-\pi/3) & t> \\ 0 & t< \end{cases}$

$\overline{L}\{t\sin t\} = \frac{2s}{s^2+1}.$

$L^{-1}\left\{\frac{2s}{s^2+1}\right\} = t\sin t.$

y''

Soln: To

$\Rightarrow$

$\Rightarrow$

$\Rightarrow$

79. $L^{-1}\left\{\frac{6s-4}{s^2-4s+20}\right\} = L^{-1}\left\{\frac{6s-4}{(s-2)^2+4^2}\right\} =$

$\Rightarrow L^{-1}\left\{\frac{6(s-2)}{(s-2)^2+4^2}\right\} + L^{-1}\left\{\frac{2.4}{(s-2)^2+4^2}\right\}$

$= 6\,L^{-1}\left\{\frac{(s-2)}{(s-2)^2+4^2}\right\} + 2.\,L^{-1}\left\{\frac{4}{(s-2)^2+4^2}\right\}.$

$= 6\,e^{2t}\cos 4t + 2.\,e^{2t}\sin 4t$

বাকী গুলো H.W. (49 Page).

H.W = 13. 25. 26. 27. 55. 56.

24. $L^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\} = L^{-1}\left\{\frac{3s+7}{s^2-3s+s-3}\right\} = L^{-1}\left\{\frac{3s+7}{(s-3)(s+1)}\right\}.$

$= L^{-1}\left\{\frac{4}{s-3} - \frac{1}{s+1}\right\} = 4\,e^{3t} - e^{-t}.$

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$y'' + y = t$. $\quad y(0) = 1$. $y'(0) = -2$.

Soln: Taking Laplace transform. we get.

$$s^2 \bar{y} - s y(0) - y'(0) + \bar{y} = \frac{1}{s^2}$$

$$\Rightarrow s^2 \bar{y} - s + 2 + \bar{y} = \frac{1}{s^2}$$

$$\Rightarrow \bar{y}(s^2+1) = \frac{1}{s^2} + s - 2$$

$$\Rightarrow \bar{y} = \frac{1}{s^2(s^2+1)} + \frac{s}{s^2+1} - \frac{2}{s^2+1}.$$

$$= \frac{1}{s^2} - \frac{1}{s^2+1} + \frac{s}{s^2+1} - \frac{2}{s^2+1}.$$

$$= \frac{1}{s^2} + \frac{s}{s^2+1} - \frac{3}{s^2+1}$$

Taking inverse Laplace transform we get

$$y = t + \cos t - 3\sin t.$$

Fourier Series:

Periodic Function: If $f(x+T) = f(x)$, then $f(x)$ is called periodic function and T is called its period.

Ex: $\tan(\pi + x) = \tan x$ and its period is π

Where, $\tan x$ is a periodic function

Sectionally continuous function: A function $f(x)$ is said to be sectionally continuous function on an interval $a \le x \le b$ If the interval can be subdivided into a finite number of intervals in each of which the function is continuous and has finite right & left hand limits. The requirement is that a function be sectionally continuous on some interval $[a, b]$ is equivalent to the requirement that it meets dirichlet conditions on the interval.

Odd function: A function $f(x)$ is said to be odd if $f(-x) = -f(x)$ for all x.

Ex. Let $f(-x) = \sin(-x) = -\sin x$, So, this is an odd function

Even function: A function $f(x)$ is said to be even if $f(-x) = f(x)$ for all x.

Ex: $f(x) = \cos x \Rightarrow f(-x) = \cos(-x) = \cos x$, so this is an even function

Fourier Series: Let $f(x)$ be a periodic function with period $2l$ and defined in the interval $c < x < c+2l$ then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{2} \right) \rightarrow (1)$$

where, $a_0 = \frac{1}{l} \int_c^{c+2l} f(x)\,dx$

$$a_n = \frac{1}{l} \int_c^{c+2l} f(x) \cos \frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l} \int_c^{c+2l} f(x) \sin \frac{n\pi x}{l}\,dx$$

This series (1) called fourier series and a_0, a_n, b_n are called fourier co-efficients

MZA 2π: If $f(x)$ is defined in the interval $(-\pi,\pi)$ and its period is 2π then the fourier series of $f(x)$ is defined as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$.

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx.$$

$(0,\pi)$ If $f(x)$ is defined in the interval $(0,\pi)$ and its period is π then the fourier series of $f(x)$ is defined as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos 2nx + b_n \sin 2nx).$$

Where. $a_0 = \frac{1}{\pi/2} \int_0^{\pi} f(x)\,dx = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos 2nx\,dx$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin 2nx\,dx$$

(a,b) If $f(x)$ is defined in the interval $(0,\pi)$ and its period is $(b-a)$ then the fourier series of $f(x)$ is defined as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2n\pi x}{b-a} + b_n \sin \frac{2n\pi x}{b-a}\right).$$

Where, $a_0 = \frac{2}{b-a} \int_a^b f(x)\,dx$

$$a_n = \frac{2}{b-a} \int_a^b f(x) \cos\frac{2n\pi x}{b-a}\,dx.$$

$$b_n = \frac{2}{b-a} \int_a^b f(x) \sin\frac{2n\pi x}{b-a}\,dx$$

$(0, 2l) \Rightarrow$ If $f(x)$ is defined in the interval $(0, 2l)$ and its period is $2L$ then the fourier series of $f(x)$ defined as.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right).$$

Where $a_0 = \frac{1}{L}\int_0^{2l} f(x)\,dx.$

$$a_n = \frac{1}{L}\int_0^{2l} f(x)\cos\frac{n\pi x}{L}\,dx.$$

$$b_n = \frac{1}{L}\int_0^{2l} f(x)\sin\frac{n\pi x}{L}\,dx.$$

Find the fourier co-efficient:

Sol^n: If $f(x)$ is defined in the integral $(-\pi, \pi)$ then fourier series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \to (1)$

integrating (1) w.r.to x in the interval $(-\pi, \pi)$ we get

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} \frac{a_0}{2}\,dx + \sum_{n=1}^{\infty}\left[a_n\int_{-\pi}^{\pi}\cos nx\,dx + b_n\int_{-\pi}^{\pi}\sin nx\,dx\right]$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\,dx = \frac{a_0}{2}\cdot[x]_{-\pi}^{\pi} = \frac{a_0}{2}\cdot(\pi+\pi) = a_0\pi$$

$$\therefore \quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

multiplying (1) by $\cos nx$ and integrating w.r.to x in the interval $(-\pi, \pi)$, we get

$$\Rightarrow \int_{-\pi}^{\pi}\cos nx\, f(x)\,dx = \int_{-\pi}^{\pi}\frac{a_0}{2}\cos nx\,dx + \sum_{n=1}^{\infty}\left[a_n\int_{-\pi}^{\pi}\cos nx\cos nx\right. + b_n\int_{-\pi}^{\pi}\sin nx\cos nx$$

$$\Rightarrow \int_{-\pi}^{\pi}\cos nx\, f(x)\,dx = \sum_{n=1}^{\infty} a_n\pi = a_n\pi$$

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$$\therefore a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos mx\, f(x)\,dx.$$

$$\text{or. } a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx\, f(x)\,dx \quad \text{Ans.}$$

multiplying ① by $\sin mx$ and integrating w.r.to x in the interval $(-\pi, \pi)$, we get

$$\int_{-\pi}^{\pi} \sin mx\, f(x)\,dx = \int_{-\pi}^{\pi} \sin mx \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\int_{-\pi}^{\pi}\cos nx \sin mx\,dx + b_n\int_{-\pi}^{\pi}\sin mx \sin nx\,dx\right]$$

$$\Rightarrow \int_{-\pi}^{\pi} \sin mx\, f(x)\,dx = \sum_{n=1}^{\infty} b_m \pi$$

$$\therefore b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin mx\, f(x)\,dx$$

$$\therefore b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx\, f(x)\,dx.$$

Parsevals identity for fourier series. If $f(x)$ is defined in the interval $(-l, l)$ and $f(x)$ is converges in the period $2l$ then $\frac{1}{l}\int_{-l}^{l}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$.

Proof: we have the fourier series.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \rightarrow ①.$$

Where. $a_0 = \frac{1}{L}\int_{-l}^{l} f(x)\,dx$, $a_n = \frac{1}{L}\int_{-l}^{l} f(x) \cos\frac{n\pi x}{L}\,dx$.

$$b_n = \frac{1}{L}\int_{-l}^{l} f(x) \sin\frac{n\pi x}{L}\,dx$$

Multiplying ① $\boxed{\text{w.r.to}}$ by $f(x)$ and then integrating in the in interval $(-L, L)$ we get

$$\Rightarrow \int_{-L}^{L}[f(x)]^2\,dx = \frac{a_0}{2}\int_{-L}^{L} f(x)\,dx + \sum_{n=1}^{\infty}\left[a_n\int_{-l}^{l} f(x)\cos\frac{n\pi x}{L}\,dx + b_n\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx\right]$$

$$\Rightarrow \int_{-L}^{L}[f(x)]^2\,dx = \frac{a_0}{2}\cdot a_0 L + \sum_{n=1}^{\infty}(a_n \cdot L a_n + b_n \cdot L b_n).$$

$$\therefore \frac{1}{L}\int_{-L}^{L}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

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Fourier series of even function is defined in the interval $(-\pi, \pi)$ is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

Where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$, $a_n = \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

Proof; We have fourier series.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

[$f(x)$ is even]

Then. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx.$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx.$$

[$\because f(x)\sin nx$ is odd]

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

[~~an - bn~~.].

Expand in fourier series of $f(x) = x$ in the interval $(-\pi, \pi)$ and $f(x) = x^2$.

Soln: for $f(x) = x$ is an odd function. So the fourier series is $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ where. $b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx$

$$\therefore b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} x\sin nx\,dx$$

$$= \frac{2}{\pi}\left[-x\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\pi\cdot\frac{(-1)^n}{n}\right] = -2\cdot\frac{(-1)^n}{n}.$$

$$\therefore f(x) = \sum_{n=1}^{\infty} -\frac{2}{\pi}(-1)^n \sin nx$$

$$\therefore x = 2\sin x - \sin 2x + \frac{2}{3}\sin 3x$$

for. $f(x) = x^2$

$\therefore f(x) = x^2$ is an even function.

$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

Where. $a_0 = \frac{2}{\pi} \cdot \int_0^{\pi} f(x)\,dx$

$= \frac{2}{\pi} \int_0^{\pi} x^2\,dx = \frac{2\pi^2}{3}$.

$a_n = \frac{2}{\pi} \cdot \int_0^{\pi} f(x) \cos nx\,dx = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx\,dx$

$= \frac{2}{\pi} \left[\frac{x^2 \sin nx}{n} + 2x \frac{\cos nx}{n^2} - 2 \frac{\sin nx}{n^3} \right]_0^{\pi}$

$= \frac{2}{\pi} \cdot \frac{2\pi \cdot \cos n\pi}{n^2} = \frac{4}{n^2} (-1)^n$.

$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$\therefore x^2 = \frac{2\pi^2}{3 \cdot 2} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cos nx$.

$\Rightarrow x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cdot (-1)^n \cos nx = \frac{\pi^2}{3} - 4\cos x + \cos 2x - \frac{4}{9} \cos 3x + \cdots$

When $x = \pi/2$

$\therefore \frac{\pi^2}{4} = \frac{\pi^2}{3} - 1$

If. $f(x) = |x|$. $-\pi < x < \pi$. then show that

$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots \right]$.

Soln) Here. $f(-x) = |-x| = |x| = f(x)$. So, $f(x)$ is an even function in the interval $(-\pi, \pi)$ is $f(x)$

Then the fourier series

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

Here. $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx = \frac{2}{\pi} \int_0^{\pi} |x|\,dx = \frac{2}{\pi} \int_0^{\pi} x\,dx = \pi$.

and, $a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\,dx = \frac{2}{\pi} \int_0^{\pi} x \cos nx\,dx = \frac{2}{\pi} \left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2} \right]_0^{\pi}$

$= 2/\pi \cdot \left[\frac{(-1)^n}{n^2} - \frac{1}{n^2} \right] = \frac{2}{\pi n^2} \{(-1)^n - 1\} \rightarrow (3)$.

From (1): $f(x) = \pi/2 + \sum_{n=1}^{\infty} \frac{2}{\pi n^2} \{(-1)^n - 1\} \cos nx = \frac{\pi}{2} - \frac{4}{\pi} \left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} \cdots \right]$

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Expand in fourier series of $f(x) = x + x^2$ in $(-\pi, \pi)$.

$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$.

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\,dx = 0 + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]$

$= \frac{1}{\pi}\left[\frac{\pi^3}{3} + \frac{\pi^3}{3}\right] = \frac{2\pi^2}{3}$ Ans

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos nx\,dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx = 0 + \frac{4}{n^2}(-1)^n$.

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx\,dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx + 0 = \frac{2}{\pi}\left[-x\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$

$= -\frac{2}{\pi}(-1)^n \quad \therefore f(x) = \frac{2\pi^2}{3} - 4\left(\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right)$
$+ 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right)$

$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \pi^2/6$.

$\therefore f(x) = \frac{2\pi^2}{3} - 4\left(\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right) + 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3}\right)$

$\Rightarrow (x+x^2) = \frac{2\pi^2}{3} - 4\left(-1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2}\right)$.

$\Rightarrow \frac{2\pi^2}{3\times 4} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \qquad \therefore \frac{\pi^2}{6} = \frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} - -$

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If $f(x)=|x|$, $-\pi<x<\pi$, then show that

$$f(x)=\frac{\pi}{2}-\frac{4}{\pi}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\cdots\right].$$

& $\pi^2 = 1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$

Soln: Here, $f(-x)=|-x|=|x|=f(x)$. So, $f(x)$ is an even function

then, the fourier series is $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx \to (1)$

Where. $a_0=\frac{2}{\pi}\int_0^{\pi} f(x)dx=\frac{2}{\pi}\int_0^{\pi}|x|dx=\pi \to (2)$.

$a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx=\frac{2}{\pi}\left[\frac{x\sin nx}{n}+\frac{\cos nx}{n^2}\right]_0^{\pi}$.

$=\frac{2}{\pi n^2}\left\{(-1)^n-1\right\} \to (3)$.

from (1). $f(x)=\frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{2}{\pi n^2}\left\{(-1)^n-1\right\}\cos nx$

$=\frac{\pi}{2}-\frac{4}{\pi}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\cdots\right]$. (Proved)

from which. $f(0)=0$.

$\therefore f(0)=\frac{\pi}{2}-\frac{4}{\pi}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right]$

$\therefore \pi^2=1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$

Show that in the interval $(-\pi,\pi)$, $f(x)=x\sin x$ can be expand in the series $x\sin x=1-\frac{1}{2}\cos x-\frac{2}{3}\cos 2x+\frac{1}{4}\cos 3x\cdots$

and then show that $\frac{1}{2}+\frac{1}{1\cdot 3}-\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}-\cdots=\pi/4$.

Soln: Here, $f(-x)=x\sin x=f(x)$. So. $f(x)$ is an even function.

then the fourier series is $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx \to (1)$.

Where $a_0=\frac{2}{\pi}\int_0^{\pi} x\sin x\,dx=\frac{2}{\pi}\left[-x\cos x+\sin x\right]_0^{\pi}=\frac{2}{\pi}\left[-\pi\cos\pi\right]=2 \to (2)$

$a_n=\frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\,dx=\frac{2}{\pi}\cdot\frac{1}{2}\int_0^{\pi} 2x\sin x\cos nx\,dx$

$=\frac{1}{\pi}\int_0^{\pi} x\left\{\sin(n+1)x-\sin(n-1)x\right\}dx$

$=\frac{1}{\pi}\left\{\left[-x\frac{\cos(n+1)x}{n+1}+\frac{\sin(n+1)x}{(n+1)^2}+x\frac{\cos(n-1)x}{n-1}-\frac{\sin(n-1)x}{(n-1)^2}\right]_0^{\pi}\right.$, $n\neq 1$

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$$a_n = \frac{1}{\pi}\left\{-\pi\frac{\cos(n+1)\pi}{(n+1)} + \pi\frac{\cos(n-1)\pi}{(n-1)}\right\}, \quad n \neq 1$$

$$\longrightarrow = \left[\frac{\cos nx}{n+1}\right]_0^{\pi} - \left[\frac{\cos nx}{n-1}\right]_0^{\pi}$$

$$= \frac{-2(-1)^n}{n^2-1}$$

$$= \frac{2(-1)^{n-1}}{n^2-1} \longrightarrow (3).$$

If, $n = 1$. then $a_1 = \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx = \frac{1}{\pi}\left[-\frac{x\cos 2x}{2} + \frac{\sin 2x}{4}\right]_0^{\pi}$

$$= -\tfrac{1}{2} \longrightarrow (4).$$

$\therefore f(x) = \frac{a_0}{2} + a_1\cos x + \sum_{n=2}^{\infty} a_n\cos nx = \frac{2}{2} - \frac{1}{2}\cos x + \sum_{n=2}^{\infty}\frac{2(-1)^{n-1}}{n^2-1}\cos$

or. $x\sin x = 1 - \frac{1}{2}\cos x - \frac{2}{3}\cos 2x + \frac{1}{4}\cos 3x - \frac{2}{15}\cos 4x + \frac{1}{12}\cos$

$$-\frac{2}{35}\cos 6x + \cdots \longrightarrow (5).$$

if. $x = \pi/2$. then.

$\frac{\pi}{2}\sin\frac{\pi}{2} = 1 + \frac{2}{3} - \frac{2}{15} + \frac{2}{35} - \cdots$

$\frac{\pi}{2} = 1 + \frac{2}{3} - \frac{2}{15} + \frac{2}{35} - \cdots$

$\therefore \frac{\pi}{4} = \frac{1}{2} + \frac{1}{3} - \frac{1}{15} + \frac{1}{35} = \frac{1}{2} + \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots$ A

If $f(x) = \begin{cases} \pi + x, & -\pi < x < 0 \\ \pi - x & 0 < x < \pi. \end{cases}$

then show that fourier series in the interval $(-\pi$

is $f(x) = \frac{\pi}{2} + \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right)$

Soln; fourier series in the interval $(-\pi, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) \longrightarrow (1).$$

Here. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\,dx + \int_0^{\pi} f(x)\,dx\right\}.$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0}(\pi + x)\,dx + \int_0^{\pi}(\pi - x)\,dx\right\}$$

$$a_0 = \frac{1}{\pi}\left\{\left[\pi x + \frac{x^2}{2}\right]_{-\pi}^{0} + \left[\pi x - \frac{x^2}{2}\right]_0^{\pi}\right\}.$$

$$= \frac{1}{\pi}\left[\left(\pi^2 - \frac{\pi^2}{2}\right) + \left(\pi^2 - \frac{\pi^2}{2}\right)\right] = \pi \longrightarrow (2).$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx.$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_0^{\pi} f(x)\cos nx\,dx\right\}.$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0}(\pi+x)\cos nx\,dx + \int_0^{\pi}(\pi-x)\cos nx\,dx\right\}.$$

$$= \frac{1}{\pi}\left\{\left[(\pi+x)\frac{\sin nx}{n} - \left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{0} + \left[(\pi-x)\frac{\sin nx}{n} - (-1) - \frac{\cos nx}{n^2}\right]_0^{\pi}\right\}$$

$$= \frac{1}{\pi n^2}\left\{(1 - \cos n\pi) + (-\cos n\pi + 1)\right\}.$$

$$= \frac{2}{\pi n^2}\left\{1 - (-1)^n\right\} \longrightarrow (3).$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0}(\pi+x)\sin nx\,dx + \int_0^{\pi}(\pi-x)\sin nx\,dx\right\}.$$

$$= \frac{1}{\pi}\left\{\left[(\pi+x)\cdot -\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_{-\pi}^{0} + \left[(\pi-x) - \frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right]_0^{\pi}\right.$$

$$= \frac{1}{\pi}\left(-\frac{\pi}{n} + \pi/n\right) = 0 \longrightarrow (4).$$

$$\therefore f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi n^2}\left\{1-(-1)^n\right\}\cos nx$$

$$= \pi/2 + 4/\pi\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right)$$