VECTOR DIFFERENTIATION

1. If $R(u) = x(u)\hat{i} + y(u)\hat{j} + z(u)\hat{k}$, where x, y, z are differentiable functions of a scalar u, proved that $\frac{dR}{du} = \frac{dx}{du}\hat{i} + \frac{dy}{du}\hat{j} + \frac{dz}{du}\hat{k}$

Solu^n: $\frac{dR}{du} = \lim_{\Delta u \to 0} \frac{R(u+\Delta u) - R(u)}{\Delta u}$

$$= \lim_{\Delta u \to 0} \frac{\left[x(u+\Delta u)\hat{i} + y(u+\Delta u)\hat{j} + z(u+\Delta u)\hat{k}\right] - \left[x(u)\hat{i} + y(u)\hat{j} + z(u)\hat{k}\right]}{\Delta u}$$

$$= \lim_{\Delta u \to 0} \left[\frac{x(u+\Delta u) - x(u)}{\Delta u}\hat{i} + \frac{y(u+\Delta u) - y(u)}{\Delta u}\hat{j} + \frac{z(u+\Delta u) - z(u)}{\Delta u}\hat{k}\right]$$

$$= \lim_{\Delta u \to 0} x(\Delta u) = \frac{dx}{du}\hat{i} + \frac{dy}{du}\hat{j} + \frac{dz}{du}\hat{k}.$$

Proved

N.B → ① $\frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0}\left[\frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}\right]$

② $\frac{d\vec{r}}{dt}$ হল,
- direction হবে $\vec{R}$ বরাবর /
- tangent বরাবর।

2. Given, $R = \sin t\,\hat{i} + \cos t\,\hat{j} + t\hat{k}$ find, (a) $\frac{dR}{dt}$ (b) $\frac{d^2R}{dt^2}$ (c) $\left|\frac{dR}{dt}\right|$ (d) $\left|\frac{d^2R}{dt^2}\right|$

Solu^n: (a) $\frac{dR}{dt} = \frac{d}{dt}(\sin t)\hat{i} + \frac{d}{dt}(\cos t)\hat{j} + \frac{d}{dt}(t)\hat{k}$

$$= \cos t\,\hat{i} - \sin t\,\hat{j} + \hat{k}.$$

(c) $\left|\frac{dR}{dt}\right| = \sqrt{(\cos t)^2 + (-\sin t)^2 + (1)^2} = \sqrt{2}.$

(b) $\frac{d^2R}{dt^2} = \frac{d}{dt}(\cos t)\hat{i} + \frac{d}{dt}(\sin)\hat{j} + \frac{d}{dt}(1)\hat{k}$

$= -\sin t\hat{j} - \cos t\hat{j}$

(d) $\left|\frac{d^2R}{dt^2}\right| = \sqrt{(-\sin t)^2 + (-\cos t)^2} = 1$

Ans:

3. A particle moves along a curve whose parametric equations are $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$. where t is the time,

(a) determine its velocity and acceleration at any time.

(b) Find the magnitudes of the velocity and acc. at $t=0$

Solun:

(a) Position vector, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$= (e^{-t})\hat{i} + (2\cos 3t)\hat{j} + (2\sin 3t)\hat{k}$

velocity, $\vec{v} = \frac{d\vec{r}}{dt} = -e^{-t}\hat{i} - 6\sin 3t\hat{j} + 6\cos 3t\hat{k}$

acceleration, $\vec{a} = \frac{d\vec{v}}{dt} = e^{-t}\hat{i} - 18\cos 3t\hat{j} - 18\sin 3t\hat{k}$

Ans:

(b) At, $t=0$, $\vec{v} = -\hat{i} + 6\hat{k}$

$\vec{a} = \hat{i} - 18\hat{j}$

Magnitude, $|\vec{v}| = \sqrt{(-1)^2 + (6)^2} = \sqrt{37}$

$|\vec{a}| = \sqrt{(1)^2 + (-18)^2} = \sqrt{325}$

Ans:

Q. 4. A particle moves along the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$, where t is the time. Find the component of its velocity and acceleration at time $t = 1$ in the direction $\underline{i} - 3\underline{j} + 2\underline{k}$.

Solu^n: Position vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

$$= (2t^2)\hat{i} + (t^2 - 4t)\hat{j} + (3t - 5)\hat{k}$$

velocity, $\vec{v} = \frac{d\vec{r}}{dt} = 4t\,\hat{i} + (2t - 4)\hat{j} + 3\hat{k}$

acceleration, $\vec{a} = \frac{d\vec{v}}{dt} = 4\hat{i} + 2\hat{j}$

when, $t = 1$, then,

$$\vec{v} = 4\hat{i} + (2-4)\hat{j} + 3\hat{k} = 4\hat{i} - 2\hat{j} + 3\hat{k}$$

$$\vec{a} = 4\hat{i} + 2\hat{j}$$

The component of velocity in the direction of $(\underline{i} - 3\underline{j} + 2\underline{k})$ is

$$\vec{v} \cdot \hat{q} = (4\hat{i} - 2\hat{j} + 3\hat{k}) \left(\frac{i - 3j + 2k}{\sqrt{1^2 + (3)^2 + 2^2}} \right)$$

$$= (4\underline{i} - 2\underline{j} + 3\underline{k}) \left(\frac{\underline{i} - 3\underline{j} + 2\underline{k}}{\sqrt{14}} \right)$$

$$= \frac{4 + 6 + 6}{\sqrt{14}} = \frac{16}{\sqrt{14}}. \quad \text{Ans:}$$

The component of acceleration in the direction of $(\underline{i} - 3\underline{j} + 2\underline{k})$ is

$$\vec{a} \cdot \hat{q} = (4\hat{i} + 2\hat{j}) \left(\frac{\underline{i} - 3\underline{j} + 2\underline{k}}{\sqrt{14}} \right)$$

$$= \frac{4 - 6}{\sqrt{14}} = \frac{-2}{\sqrt{14}}. \quad \text{Ans:}$$

5. A curve C is defined by parametric equations $x = x(s)$, $y = y(s)$, $z = z(s)$, where, s is the arc length of C measured from a fixed point on C. If r is the position vector of any point on C, show that dr/ds is a unit vector tangent to C.

Solun: Position vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\frac{d\vec{r}}{ds} = \frac{d}{ds}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k}$$

$$\left|\frac{d\vec{r}}{ds}\right| = \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2}$$

$$= \sqrt{\frac{(dx)^2 + (dy)^2 + (dz)^2}{(ds)^2}}$$

$$= 1, \left[\text{since, } (ds)^2 = (dx)^2 + (dy)^2 + (dz)^2\right]$$

So, $\frac{d\vec{r}}{ds}$ is a unit vector tangent to C.

6. (a) Find the unit tangent vector to any point on the curve $x = t^2 + 1$, $y = 4t - 3$, $z = 2t^2 - 6t$.

(b) Determine the unit tangent at the point where $t = 2$.

Solun:

(a) Position vector, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$= (t^2+1)\hat{i} + (4t-3)\hat{j} + (2t^2 - 6t)\hat{k}$$

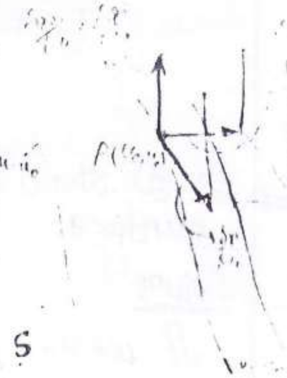
Tangent vector, $\frac{d\vec{r}}{dt} = 2t\hat{i} + 4\hat{j} + (4t-6)\hat{k}$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{(2t)^2 + (4)^2 + (4t-6)^2} = \sqrt{4t^2 + 16 + (4t-6)^2}$$

(b) Show that $\frac{\delta r}{\delta u} \times \frac{\delta r}{\delta v}$ ...
the surface.

Solu^n:

Consider point P having coordinates (u_0, v_0) on a surface S. The vector $\delta r/\delta u$ at P is obtained by differentiating r with respect to u, keeping $v = \text{const} = v_0$. From the theory of space curves, it follows that $\delta r/\delta u$ at P represents a vector tangent to the curve $v = v_0$ at P.



Similarly, $\delta r/\delta v$ at P, represents a vector tangent to the curve $u = \text{const.} = u_0$.

Since $\delta r/\delta u$ & $\delta r/\delta v$ represent vectors at P tangent to curves which lie on the surface S at P, it follows that these vectors are tangent to the surface at P, Hence, it follows that,

$\delta r/\delta u \times \delta r/\delta v$ reprents a vector normal to the surface.

26 Find the equ^n for the tangent plane to the surface $z = x^2+y^2$ at the point $(1,-1,2)$

Solu^n: Let, $x = u,\ y = v,\ z = x^2+y^2 = u^2+v^2$ be the parametric equation of the surface.

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$= u\hat{i} + v\hat{j} + (u^2+v^2)\hat{k}$$

$$\therefore \frac{d\vec{r}}{du} = \hat{i} + 2u\hat{k}$$
$$= \hat{i} + 2\hat{k} \text{ at } (1,-1,2)$$

$$\frac{d\vec{r}}{dv} = \hat{j} + 2v\hat{k}$$
$$= \hat{j} - 2\hat{k} \text{ at } (1,-1,2)$$

Normal of the surface, $\vec{n} = \dfrac{d\vec{r}}{du} \times \dfrac{d\vec{r}}{dv}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2 \\ 0 & 1 & -2 \end{vmatrix}$$

$$= -2\hat{i} + 2\hat{j} + \hat{k}$$

Now, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{r_0} = \hat{i} - \hat{j} + 2\hat{k}$ at $(1,-1,2)$

$(\vec{r}-\vec{r_0})$ is perpendicular to $\vec{n}$.

The required equ^n of tangent plane, $(\vec{r}-\vec{r_0})\cdot\vec{n} = 0$

$$\Rightarrow \left[(x-1)\hat{i} + (y+1)\hat{j} + (z-2)\hat{k}\right]\left(-2\hat{i} + 2\hat{j} + \hat{k}\right) = 0$$

$$\Rightarrow \quad 2x - 2y - z = 2 \quad \text{Ans:}$$

Q9. If A has constant magnitude show that A and $\frac{dA}{dt}$ are perpendicular provided $\left|\frac{dA}{dt}\right| \neq 0$.

Solun: Since A has constant magnitude;

$$A \cdot A = \text{constant}.$$

$$\Rightarrow \frac{d}{dt}(A \cdot A) = 0;$$

$$\Rightarrow A \cdot \frac{dA}{dt} + A \frac{dA}{dt} = 0$$

$$\Rightarrow 2A \cdot \frac{dA}{dt} = 0$$

$$\Rightarrow A \cdot \frac{dA}{dt} = 0$$

Thus, A is perpendicular to $\frac{dA}{dt}$ provided $\left|\frac{dA}{dt}\right| \neq 0$. Proved

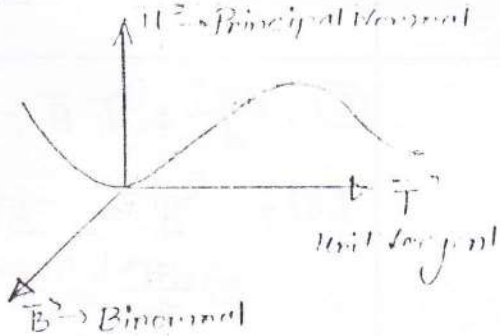
Frenet - Serret Formula:

ⓐ $\frac{d\vec{T}}{ds} = \kappa \vec{N}$; $\vec{N}$ = Principal Normal
κ = Curvature
$\rho = \frac{1}{\kappa}$ = radius of curvature.

ⓑ $\frac{d\vec{B}}{ds} = -\tau \vec{N}$; $\vec{B}$ = Binormal
τ = Torsion
$\sigma = \frac{1}{\tau}$ = Radius of Torsion

ⓒ $\frac{d\vec{N}}{ds} = \tau \vec{B} - \kappa \vec{T}$.

Proof:



ⓐ $\dfrac{d\vec{T}}{ds} = k\vec{N}$

Proof:

$\vec{T}\cdot\vec{T} = T\cdot T\cos 0^\circ = 1$

$\Rightarrow \vec{T}\cdot\dfrac{d\vec{T}}{ds} + \vec{T}\cdot\dfrac{d\vec{T}}{ds} = 0$

$\Rightarrow \vec{T}\cdot\dfrac{d\vec{T}}{ds} = 0$

$\dfrac{d\vec{T}}{ds}$ is perpendicular to $\vec{T}$.

If $\vec{N}$ is a unit vector in the dirrection of $\dfrac{d\vec{T}}{ds}$,

Then,
$$\dfrac{d\vec{T}}{ds} = k\vec{N}$$
where, $\vec{N}$ = principal normal

k = curvature

and $\rho = \dfrac{1}{k}$ = radius of kurvature.

ⓑ $\dfrac{d\vec{B}}{ds} = -\tau\vec{N}$

Proof: Let, $\vec{T}\times\vec{N} = \vec{B}$

$$\Rightarrow \dfrac{d\vec{B}}{ds} = \vec{T}\times\dfrac{d\vec{N}}{ds} + \vec{N}\times\dfrac{d\vec{T}}{ds}$$
$$= \vec{T}\times\dfrac{d\vec{N}}{ds} + k\vec{N}\times\vec{N}$$
$$= \vec{T}\times\dfrac{d\vec{N}}{ds}$$
$$\Rightarrow \vec{T}\cdot\dfrac{d\vec{B}}{ds} = \vec{T}\cdot\left(\vec{T}\times\dfrac{d\vec{N}}{ds}\right)$$
$$\Rightarrow \vec{T}\cdot\dfrac{d\vec{B}}{ds} = 0 \quad \left[\because \vec{A}\cdot(\vec{A}\times\vec{C}) = 0\right]$$

so, $\dfrac{d\vec{B}}{ds}$ is perpendicular to $\vec{T}$.

But, $\vec{B}\cdot\vec{B} = 1$

$\Rightarrow \vec{B}\cdot\dfrac{d\vec{B}}{ds} = 0$

so, $\dfrac{d\vec{B}}{ds}$ is perpendicular to $\vec{B}$.

$\therefore \dfrac{d\vec{B}}{ds} = \cancel{-\vec{T}} - \tau\vec{N}$ where, τ → Torsion

$\sigma = \dfrac{1}{\tau}$ = Radius of torsion.

(c) $\frac{d\vec{N}}{ds} = \tau \vec{B} - k\vec{T}$

Let, $\vec{N} = \vec{B} \times \vec{T}$

$$\Rightarrow \frac{d\vec{N}}{ds} = \vec{B} \times \frac{d\vec{T}}{ds} + \vec{T} \times \frac{d\vec{B}}{ds}$$

$$= \vec{B} \times k\vec{N} - \tau \vec{N} \times \vec{T}$$

$$= k(-\vec{T}) - \tau(-\vec{B})$$

$$= \tau \vec{B} - k\vec{T}$$ Ans:

c # 19 The space $x = 3\cos t$; $y = 3\sin t$ and $z = 4t$; find;

(a) The unit tangent $\vec{T}$

(b) The principal normal $\vec{N}$, curvature k and radius of curvature ρ.

(c) The binormal B, torsion τ and radius of torsion

Solu^n:

(a) $\vec{r} = (3\cos t)\hat{i} + (3\sin t)\hat{j} + 4t\hat{k}$

$$\frac{d\vec{r}}{dt} = -3\sin t\,\hat{i} + 3\cos t\,\hat{j} + 4\hat{k}$$

Unit tangent $\vec{T} = \frac{d\vec{r}}{dt} \Big/ \left|\frac{d\vec{r}}{dt}\right|$

$$= \frac{-3\sin t\,\hat{i} + 3\cos t\,\hat{j} + 4\hat{k}}{\sqrt{9\sin^2 t + 9\cos^2 t + 16}}$$

$$= -\frac{3}{5}\sin t\,\hat{i} + \frac{3}{5}\cos t\,\hat{j} + \frac{4}{5}\hat{k}$$ Ans!

(b) $\frac{d\vec{T}}{dt} = -\frac{3}{5}\cos t\,\hat{i} + -\frac{3}{5}\sin t\,\hat{j}$

$\frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right| = \sqrt{9(\sin^2 t + \cos^2 t) + 16} = 5$

$\frac{d\vec{T}}{ds} = \frac{\left(\frac{d\vec{T}}{dt}\right)}{\left(\frac{ds}{dt}\right)} = -\frac{3}{25}\cos t\,\hat{i} \div \frac{3}{25}\sin t\,\hat{j}$

Now, $\frac{d\vec{T}}{ds} = \kappa\vec{N}$

$\left|\frac{d\vec{T}}{ds}\right| = |\kappa|\,|\vec{N}|$

$\Rightarrow \sqrt{\left(-\frac{3}{25}\cos t\right)^2 + \left(-\frac{3}{25}\sin t\right)^2} = |\kappa|\cdot 1$

$\Rightarrow \kappa = \frac{3}{25}$

$\therefore \rho = \frac{1}{\kappa} = \frac{25}{3}$

$\vec{N} = \frac{1}{\kappa}\frac{d\vec{T}}{ds} = \frac{25}{3}\left(-\frac{3}{25}\cos t\,\hat{i} - \frac{3}{25}\sin t\,\hat{j}\right)$

$= -\cos t\,\hat{i} - \sin t\,\hat{j}$ Ans:

(c) $\vec{B} = \vec{T}\times\vec{N}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{3}{5}\sin t & \frac{3}{5}\cos t & \frac{4}{5} \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

$= \hat{i}\left(0 + \frac{4}{5}\sin t\right) - \hat{j}\left(0 + \frac{4}{5}\cos t\right) + \hat{k}\left(\frac{3}{5}\sin^2 t + \frac{3}{5}\cos^2 t\right)$

$= \frac{4}{5}\sin t\,\hat{i} - \frac{4}{5}\cos t\,\hat{j} + \frac{3}{5}\hat{k}$

$\frac{d\vec{B}}{dt} = \frac{4}{5}\cos t\,\hat{i} + \frac{4}{5}\sin t\,\hat{j}$

$$\frac{d\vec{B}}{ds} = \frac{\left(\frac{d\vec{B}}{dt}\right)}{\left(\frac{ds}{dt}\right)} = \frac{4}{25}\cos t\,\hat{i} + \frac{4}{25}\sin t\,\hat{j}$$

We know,

$$\frac{d\vec{B}}{ds} = -\tau\vec{N}$$

$$\Rightarrow \left(\frac{4}{25}\cos t\,\hat{i} + \frac{4}{25}\sin t\,\hat{j}\right) = -\tau\left(-\cos t\,\hat{i} - \sin t\,\hat{j}\right)$$

$$\Rightarrow \tau = \frac{\frac{4}{25}\left(\cos t\,\hat{i} + \sin t\,\hat{j}\right)}{\left(\cos t\,\hat{i} + \sin t\,\hat{j}\right)}$$

$$= \frac{4}{25}$$

and, $\sigma = \frac{1}{\tau} = \frac{25}{4}$. Ans:

GRADIENT, DIVERGENCE &

CURL

Chapter - 4

$\vec{\nabla} = \frac{\delta}{\delta x}\underline{i} + \frac{\delta}{\delta y}\underline{j} + \frac{\delta}{\delta z}\underline{k} \Rightarrow$ vector operator / Del operator

$f(x,y,z)$; Gradient of f, $\vec{\nabla}f = \frac{\delta f}{\delta x}\hat{i} + \frac{\delta f}{\delta y}\hat{j} + \frac{\delta f}{\delta z}\hat{k}$.

$\vec{\nabla}f \rightarrow$ surface এর উপর লম্ব বরাবর।

$\vec{\nabla}f \cdot \hat{a} \rightarrow$ Directional Derivative.

C ⇒ 5 Show that $\vec{\nabla}\varphi$ is a vector perpendicular to the surface $\phi(x,y,z) = c$, where c is a constant.

Soluⁿ: Let, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector to any point $P(x,y,z)$ on the surface.

$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$ lies in the tangent plane to the surface at P.

But $d\varphi = 0$

$\rightarrow \frac{\delta\varphi}{\delta x}dx + \frac{\delta\varphi}{\delta y}dy + \frac{\delta\varphi}{\delta z}dz = 0$

$\Rightarrow \left(\frac{\delta\varphi}{\delta x}\hat{i} + \frac{\delta\varphi}{\delta y}\hat{j} + \frac{\delta\varphi}{\delta z}\hat{k}\right)\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right) = 0$

$\Rightarrow \quad \vec{\nabla}\varphi \; d\vec{r} = 0.$

so that, $\nabla\varphi$ is perpendicular to dr and therefore to the surface.

(*)(*) unit normal to the surface $= \dfrac{\vec{\nabla} f}{|\vec{\nabla} f|}$

(*)(*) Dirrectional derivative $= \vec{\nabla}\varphi \cdot \hat{a}$

C ⇒ 6 Find a unit normal to the surface $x^2y+2xz=4$ at the point $(2,-2,3)$.

Solun: $\vec{\nabla} f = \vec{\nabla}(x^2y+2xz) = \frac{\partial}{\partial x}(x^2y+2xz)\hat{i} + \frac{\partial}{\partial y}(x^2y+2xz)\hat{j}$

$+ \frac{\partial}{\partial z}(x^2y+2xz)\hat{k}$

$= (2xy+2z)\hat{i} + x^2\hat{j} + 2x\hat{k}$

$= -2\hat{i} + 4\hat{j} + 4\hat{k}$ at point $(2,-2,3)$

Then a unit normal to the surface is

$$= \frac{-2\hat{i}+4\hat{j}+4\hat{k}}{\sqrt{(-2)^2+4^2+4^2}}$$

$$= -\frac{2}{6}\hat{i} + \frac{4}{6}\hat{j} + \frac{4}{6}\hat{k}$$

$$= -\frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$$

C ⇒ 10 Find the dirrectional derivative of $\varphi = x^2yz + 4xz^2$ at $(1,-2,-1)$ in the dirrection $2\underline{i} - \underline{j} - 2\underline{k}$

Solun: $\varphi = x^2yz + 4xz^2$

$\vec{\nabla}\varphi = \vec{\nabla}(x^2yz+4xz^2) = (2xyz+4z^2)\hat{i} + (x^2z)\hat{j} + (x^2y+8x$

$= 8\hat{i} - \hat{j} - 10\hat{k}$; at $(1,-2,-1)$

unit vector in the dirrection of $2\hat{i} - \hat{j} - 2\hat{k}$ is,

$$\hat{a} = \frac{2\hat{i}-\hat{j}-2\hat{k}}{\sqrt{4+1+4}} = \frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}$$

$\therefore$ Required dirrectional derivative $= \vec{\nabla}\varphi \cdot \hat{a}$

$$= \left(8\hat{i} - \hat{j} - 10\hat{k}\right)\left(\tfrac{2}{3}\hat{i} - \tfrac{1}{3}\hat{j} - \tfrac{2}{3}\hat{k}\right)$$

$$= \frac{16}{3} + \frac{1}{3} + \frac{20}{3}$$

$$= \frac{37}{3}.$$

Since this is positive, φ is increasing in this dirrection.

Ans:

$*$ Find the dirrectional derivative of $\varphi = xyz^2$ at $(1,0,3)$ in the dirrection $\hat{i} - \hat{j} + \hat{k}$. Calculate the greatest rate of change of φ and dirrection of maximum rate of change.

Solu^n:

$$\vec{\nabla}\varphi = yz^2\hat{i} + xz^2\hat{j} + 2xyz\hat{k}$$

$$= 9\hat{j} \text{ at point } (1,0,3).$$

Projection/ dirrectional derivative $= \vec{\nabla}\varphi \cdot \hat{a}$

$$= 9\hat{j} \cdot \frac{\hat{i} - \hat{j} + \hat{k}}{\sqrt{3}}$$

$$= \frac{-9}{\sqrt{3}} = -3\sqrt{3}$$

$\therefore$ The greatest rate of change $= |\vec{\nabla}\varphi| = \sqrt{9^2} = 9$

The dirrection of the max. rate of change is y axis. because here exist only j.

C 2 Show that, the greatest rate of change of φ i.e. the maximum directional derivative, takes place in the direction of, and has the magnitude of, the vector $\nabla\varphi$.

Soluⁿ: $$\frac{d\varphi}{ds} = \frac{\delta\varphi}{\delta x}\cdot\frac{dx}{ds} + \frac{\delta\varphi}{\delta y}\cdot\frac{dy}{ds} + \frac{\delta\varphi}{\delta z}\cdot\frac{dz}{ds}$$
$$= \left(\frac{\delta\varphi}{\delta x}\hat{i} + \frac{\delta\varphi}{\delta y}\hat{j} + \frac{\partial\varphi}{\delta z}\hat{k}\right)\left(\frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k}\right)$$
$$= \vec{\nabla}\varphi \cdot \frac{d\vec{r}}{ds}$$

Now, $\frac{d\varphi}{ds}$ is the projection of $\vec{\nabla}\varphi$ in the direction $\frac{d\vec{r}}{ds}$. This projection will be maximum when $\vec{\nabla}\varphi$ and $\frac{d\vec{r}}{ds}$ have the same direction.

Then the maximum value of $\frac{d\varphi}{ds}$ takes place in the direction of $\vec{\nabla}\varphi$ and its magnitude is $|\vec{\nabla}\varphi|$.

7. Find an equⁿ for the tangent plane to the surface $2xz^2 - 3xy - 4x = 7$, at the point $(1, -1, 2)$.

Soluⁿ: $$\vec{\nabla}\varphi = \vec{\nabla}\left(2xz^2 - 3xy - 4x\right)$$
$$= \left(2z^2 - 3y - 4\right)\hat{i} + \left(-3x\right)\hat{j} + 4xz\hat{k}$$
$$= 7\hat{i} - 3\hat{j} + 8\hat{k} \quad \text{at point}(1,-1,2)$$

at, (x,y,z), $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

at, $(1,-1,2)$, $\vec{r_0} = \hat{i} - \hat{j} + 2\hat{k}$

$\vec{r} - \vec{r_0} = (x-2)\hat{i} + (y-2)\hat{j} + (z-1)\hat{k}$ $(x-1)\hat{i} + (y+1)\hat{j} + (z-2)\hat{k}$

For tangent plane,

$$(\vec{r} - \vec{r_0}) \cdot \vec{\nabla}\varphi = 0$$

$$\Rightarrow \left[(x-1)\hat{i} + (y+1)\hat{j} + (z-2)\hat{k}\right]\left[7\hat{i} - 3\hat{j} + 8\hat{k}\right] = 0$$

$$\Rightarrow (x-1)7 + (y+1)(-3) + (z-2)\cdot 8 = 0$$

$$\Rightarrow 7x - 7 - 3y - 3 + 8z - 16 = 0$$

$$\Rightarrow 7x - 3y + 8z = 26.$$ Ans:

(#) Find the equ^n of the tangent plane & normal plane line to the surface, $xyz^2 = 4$ at $(2, 2, 1)$.

Solu^n: $\vec{\nabla}\varphi = \vec{\nabla}(xyz^2)$

$$= yz^2\,\hat{i} + xz^2\,\hat{j} + 2xyz\,\hat{k}$$

$$= 2\hat{i} + 2\hat{j} + 8\hat{k}$$

at, (x, y, z), $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

at $(2, 2, 1)$, $\vec{r_0} = 2\hat{i} + 2\hat{j} + \hat{k}$

$$\vec{r} - \vec{r_0} = (x-2)\hat{i} + (y-2)\hat{j} + (z-1)\hat{k}$$

For, tangent plane, $(\vec{r} - \vec{r_0}) \cdot \vec{\nabla}\varphi = 0$

$$\Rightarrow (x-2).2 + (y-2)\cdot 2 + (z-1)\cdot 8 = 0$$

$$\Rightarrow 2x - 4 + 2y - 4 + 8z - 8 = 0$$

$$\Rightarrow 2x + 2y + 8z = 16$$

$$\therefore x + y + 4z = 8.$$ Ans:

For normal plane,

$$(\vec{r}-\vec{r_0}) \times \vec{\nabla}\varphi = 0$$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 8 \\ (x-2) & (y-2) & (z-1) \end{vmatrix} = 0$$

$$\Rightarrow \hat{i}(2z-2-8y+16) - \hat{j}(2z-2-8x+16) + \hat{k}(2y-4-2x+4) = 0$$

$$\Rightarrow \hat{i}(2z-8y+14) - \hat{j}(2z-8x+14) + \hat{k}(2y-2x) = 0$$

$$\Rightarrow \hat{i}(z-4y+7) - \hat{j}(z-4x+7) + \hat{k}(y-x) = 0$$

Ans:.

11 (a) In what direction from the point $(2,1,-1)$ is the directional derivative of $\varphi = x^2yz^3$ a maximum?

(b) What is the magnitude of this maximum?

Solution:

$$\vec{\nabla}\varphi = \vec{\nabla}(x^2yz^3)$$
$$= 2xyz^3\,\hat{i} + x^2z^3\,\hat{j} + 3x^2yz^2\,\hat{k}$$
$$= -4\hat{i} - 4\hat{j} + 12\hat{k} \quad \text{at } (2,1,-1)$$

(a) The directional derivative of φ a maimum in the direction, $\vec{\nabla}\varphi = -4\hat{i} - 4\hat{j} + 12\hat{k}$

(b) The magnitude of the maximum, $|\vec{\nabla}\varphi| = \sqrt{(-4)^2+(-4)^2+(12)^2}$
$$= \sqrt{176}$$
$$= 4\sqrt{11}.$$

Ans:.

Divergence:

$$\vec{\nabla}\cdot\vec{v} = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)\cdot\left(v_1\hat{i} + v_2\hat{j} + v_3\hat{k}\right)$$

$$= \frac{\delta}{\delta x}v_1 + \frac{\delta}{\delta y}v_2 + \frac{\delta}{\delta z}v_3.$$

C-P22 Determine the constant a so that the vector $v = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+az)\hat{k}$ is solenoidal.

Solun: A vector v is solenoidal, so,

$$\vec{\nabla}\cdot\vec{v} = 0$$

$$\Rightarrow \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)\left[(x+3y)\hat{i} + (y-2z)\hat{j} + (x+az)\hat{k}\right] = 0$$

$$\Rightarrow \frac{\delta}{\delta x}(x+3y) + \frac{\delta}{\delta y}(y-2z) + \frac{\delta}{\delta z}(x+az) = 0$$

$$\Rightarrow 1 + 1 + a = 0$$

$$\therefore a = -2.$$ Ans:

C-P 21. A fluid moves so that its velocity at any point is $v(x,y,z)$. Show that the gain of fluid per unit volum per unit time in a small parallelopide having centre at $P(x,y,z)$ and edges parallel to the co-ordinate axis and having magnitude $\Delta x, \Delta y, \Delta z$ respectively, is given approximately by $\text{div}\,\vec{v} = \vec{\nabla}\cdot\vec{v}$.

Curl:

$\vec{\nabla} \times \vec{A} = 0 \longrightarrow$ irrotational

$\vec{\nabla} \times \vec{A} \neq 0 \longrightarrow$ rotational

⊛ show $\vec{F} = y\hat{i} - x\hat{j}$ is Rotational or irrotational.

Solu^{n}:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \delta/\delta x & \delta/\delta y & \delta/\delta z \\ y & -x & 0 \end{vmatrix}$$

$$= \hat{i}(0+0) - \hat{j}(0-0) + \hat{k}(-1-1)$$

$$= -2\hat{k} \neq 0$$

$\therefore$ It is rotational. Ans:

29 Prove, $\nabla \times (\nabla \times A) = -\nabla^2 A + \nabla(\nabla \cdot A)$

Solu^{n}:

$$\nabla \times (\nabla \times A) = \nabla \times \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \delta/\delta x & \delta/\delta y & \delta/\delta z \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \nabla \times \left[\left(\frac{\delta A_3}{\delta y} - \frac{\delta A_2}{\delta z} \right)\hat{i} + \left(\frac{\delta A_1}{\delta z} - \frac{\partial A_3}{\delta x} \right)\hat{j} + \left(\frac{\delta A_2}{\delta x} - \frac{\delta A_1}{\delta y} \right)\hat{k} \right]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \delta/\delta x & \delta/\delta y & \delta/\delta z \\ \frac{\delta A_3}{\delta y} - \frac{\delta A_2}{\delta z} & \frac{\delta A_1}{\delta z} - \frac{\delta A_3}{\delta x} & \frac{\delta A_2}{\delta x} - \frac{\delta A_1}{\delta y} \end{vmatrix}$$

$$= \hat{i}\left[\frac{\delta}{\delta y}\left(\frac{\delta A_2}{\delta x} - \frac{\delta A_1}{\delta y}\right) - \frac{\delta}{\delta z}\left(\frac{\delta A_1}{\delta z} - \frac{\delta A_3}{\delta x}\right)\right] + \hat{j}\left[\frac{\delta}{\delta z}\left(\frac{\delta A_3}{\delta y} - \frac{\delta A_2}{\delta z}\right) - \frac{\delta}{\delta x}\left(\frac{\delta A_2}{\delta x} - \frac{\delta A_1}{\delta y}\right)\right]$$

$$+ \hat{k}\left[\frac{\delta}{\delta x}\left(\frac{\delta A_1}{\delta z} - \frac{\delta A_3}{\delta x}\right) - \frac{\delta}{\delta y}\left(\frac{\delta A_3}{\delta y} - \frac{\delta A_2}{\delta z}\right)\right].$$

$$= \left(-\frac{\delta^2 A_1}{\delta y^2} - \frac{\delta^2 A_1}{\delta z^2}\right)\hat{i} + \left(-\frac{\delta^2 A_2}{\delta z^2} - \frac{\delta^2 A_2}{\delta x^2}\right)\hat{j} + \left(-\frac{\delta^2 A_3}{\delta x^2} - \frac{\delta^2 A_3}{\delta y^2}\right)$$

$$+ \left(\frac{\delta^2 A_2}{\delta y\, \delta x} + \frac{\delta^2 A_3}{\delta z\, \delta x}\right)\hat{i} + \left(\frac{\delta^2 A_3}{\delta z\, \delta y} + \frac{\delta^2 A_1}{\delta x\, \delta y}\right)\hat{j} + \left(\frac{\delta^2 A_1}{\delta x\, \delta z} + \frac{\delta^2 A_2}{\delta y\, \delta z}\right)\hat{k}$$

$$= \left(-\frac{\delta^2 A_1}{\delta x^2} - \frac{\delta^2 A_1}{\delta y^2} - \frac{\delta^2 A_1}{\delta z^2}\right)\hat{i} + \left(-\frac{\delta^2 A_2}{\delta x^2} - \frac{\delta^2 A_2}{\delta y^2} - \frac{\delta^2 A_2}{\delta z^2}\right)\hat{j} + \left(-\frac{\delta^2 A_3}{\delta x^2} - \frac{\delta^2 A_3}{\delta y^2} - \frac{\delta^2 A_3}{\delta z^2}\right)\hat{k}$$

$$+ \left(\frac{\delta^2 A_1}{\delta x^2} + \frac{\delta^2 A_2}{\delta y\, \delta x} + \frac{\delta^2 A_3}{\delta z\, \delta x}\right)\hat{i} + \left(\frac{\delta^2 A_2}{\delta y^2} + \frac{\delta^2 A_3}{\delta z\, \delta y} + \frac{\delta^2 A_1}{\delta x\, \delta y}\right)\hat{j} + \left(\frac{\delta^2 A_3}{\delta z^2} + \frac{\delta^2 A_1}{\delta x\, \delta z} + \frac{\delta^2 A_2}{\delta y\, \delta z}\right)\hat{k}$$

$$= -\left(\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2}\right)\left(A_1\hat{i} + A_2\hat{j} + A_3\hat{k}\right)$$

$$+ \hat{i}\frac{\delta}{\delta x}\left[\frac{\delta A_1}{\delta x} + \frac{\delta A_2}{\delta y} + \frac{\delta A_3}{\delta z}\right] + \hat{j}\frac{\delta}{\delta y}\left[\frac{\delta A_1}{\delta x} + \frac{\delta A_2}{\delta y} + \frac{\delta A_3}{\delta z}\right] + \hat{k}\frac{\delta}{\delta z}\left[\frac{\delta A_1}{\delta x} + \frac{\delta A_2}{\delta y} + \frac{\delta A_3}{\delta z}\right]$$

$$= -\nabla^2 A + \nabla\left[\frac{\delta A_1}{\delta x} + \frac{\delta A_2}{\delta y} + \frac{\delta A_3}{\delta z}\right]$$

$$= -\nabla^2 A + \nabla\cdot(\nabla\cdot A)$$

Proved.

Formula:

① $\vec{\nabla}\times(\vec{\nabla}\varphi) = 0$

② $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$

(32) (a) A vector V is called irrotational if ~~curl~~ curl $V = 0$. Find const. a, b, c so that $V = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$ is irrotational.

(b) show that V can be expressed as the gradient of a scalar function.

Solu^n: Since,

$$\text{curl } V = \nabla \times V = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \delta/\delta x & \delta/\delta y & \delta/\delta z \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

$$= \hat{i}(c+1) - \hat{j}(4-a) + \hat{k}(b-2)$$

Since $\vec{V}$ is irrotational, so,

$$\nabla \times V = 0$$

$$\Rightarrow \hat{i}(c+1) - \hat{j}(4-a) + \hat{k}(b-2) = 0$$

$$\therefore c = -1, \quad a = 4, \; b = 2.$$

We know $\nabla \times (\nabla\phi) = 0$

$$\therefore V = \nabla\phi.$$

$$\nabla\phi = V = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$$

$$\Rightarrow \frac{\delta\phi}{\delta x}\hat{i} + \frac{\delta\phi}{\delta y}\hat{j} + \frac{\delta\phi}{\delta z}\hat{k} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$$

So,
$$\frac{\delta\phi}{\delta x} = x+2y+az \quad \text{--- (i)}$$

$$\frac{\delta\phi}{\delta y} = bx-3y-z \quad \text{--- (ii)}$$

$$\frac{\delta\phi}{\delta z} = 4x+cy+2z \quad \text{--- (iii)}$$

(32) (a) A vector $\vec{v}$ is called irrotational if curl $v = 0$. Find const a, b, c so that $v = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$ is irrotational.

(b) Show that v can be expressed as the gradient of a scalar function.

Solu^n: Since,

$$\text{curl } v = \nabla \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \delta/\delta x & \delta/\delta y & \delta/\delta z \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

$$= \hat{i}(c+1) - \hat{j}(4-a) + \hat{k}(b-2)$$

Since $\vec{v}$ is irrotational, so

$$\nabla \times v = 0$$

$$\Rightarrow \hat{i}(c+1) - \hat{j}(4-a) + \hat{k}(b-2) = 0$$

$$\therefore c = -1, \quad a = 4, \quad b = 2.$$

We know $\nabla \times (\nabla \varphi) = 0$.

$$\therefore v = \nabla \varphi$$

$$\nabla \varphi = v = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$$

$$\Rightarrow \frac{\delta \varphi}{\delta x}\hat{i} + \frac{\delta \varphi}{\delta y}\hat{j} + \frac{\delta \varphi}{\delta z}\hat{k} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$$

So,

$$\frac{\delta \varphi}{\delta x} = x+2y+az \quad \text{—— (i)}$$

$$\frac{\delta \varphi}{\delta y} = bx-3y-z \quad \text{—— (ii)}$$

$$\frac{\delta \varphi}{\delta z} = 4x+cy+2z \quad \text{—— (iii)}$$

integrating equ^n ① with respect to $x \Rightarrow$

$$\varphi = \frac{x^2}{2} + 2yx + 4zx + f(y,z) \quad \text{——— (4)}$$

where $f(y,z)$ is a integrating const. independent of x.

differentiating ④ with. r.to. $y \rightarrow$

$$\frac{\partial \varphi}{\partial y} = 2x + \frac{\partial f}{\partial y} \quad \text{——— (5)}$$

comparing ② & ⑤ ;

$$\frac{\partial f}{\partial y} = -3y - z$$

$\Rightarrow f(y,z) = -\frac{3}{2}y^2 - zy + g(z)$ [integrating w.r.to y]

$\therefore g(z)$ is const. independent of x, y.

Now,

$$\varphi = \frac{x^2}{2} + 2yx + 4zx - \frac{3}{2}y^2 - zy + g(z) \quad \text{——— (6)}$$

diff. ⑥ w. r. to. z.

$$\frac{\partial \varphi}{\partial z} = 4x - y + g'(z) \quad \text{——— (7)}$$

comparing ③ & ⑦ $\Rightarrow$

$$g'(z) = 2z$$

$\therefore g(z) = z^2 + c$ [integ. w.r.to z]

c is arbitrary constant.

$$\therefore \varphi = \frac{x^2}{2} + 2yx + 4zx - \frac{3}{2}y^2 - zy + z^2 + c.$$

Ans.

Class Test

102. Show that, $\vec{A} = (6xy+z^3)\hat{i} + (3x^2-z)\hat{j} + (3xz^2-y)\hat{k}$ is irrotational. Find ϕ such that $A = \nabla\phi$.

Solu^n:
$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \delta/\delta x & \delta/\delta y & \delta/\delta z \\ 6xy+z^3 & 3x^2-z & 3z^2x-y \end{vmatrix}$$

$$= \hat{i}(-1+1) - \hat{j}(3z^2-3z^2) + \hat{k}(6x-6x)$$

$$= 0$$

$\therefore \vec{\nabla}\times\vec{A} = 0$; $\vec{A}$ is irrotational.

$$\vec{\nabla}\varphi = \vec{A} = (6xy+z^3)\hat{i} + (3x^2-z)\hat{j} + (3xz^2-y)\hat{k}$$

$$\Rightarrow \frac{\delta\varphi}{\delta x}\hat{i} + \frac{\delta\varphi}{\delta y}\hat{j} + \frac{\delta\varphi}{\delta z}\hat{k} = (6xy+z^3)\hat{i} + (3x^2-z)\hat{j} + (3xz^2-y)\hat{k}$$

$$\therefore \frac{\delta\varphi}{\delta x} = 6xy + z^3 \quad \text{——— (1)}$$

$$\frac{\delta\varphi}{\delta y} = 3x^2 - z \quad \text{——— (2)}$$

$$\frac{\delta\varphi}{\delta z} = 3xz^2 - y \quad \text{——— (3)}$$

Integrating (1) w. r. to. x $\Rightarrow$

$$\varphi = 3x^2y + z^3x + f(y,z) \quad \text{——— (4)}$$

where, $f(y,z)$ is a const. independent of x.

Differentiating (4) w. r. to. y,

$$\frac{\partial\varphi}{\delta y} = 3x^2 + \frac{\partial f}{\partial y} \quad \text{——— (5)}$$

comparing eq^n (2) & (5) $\Rightarrow$ $\frac{\partial f}{\delta y} = -z$

$$\Rightarrow f(y,z) = -zy + g(z) \text{ —— (6)}$$

integrating equ^n ① with respect to x ⇒

$$\varphi = \frac{x^2}{2} + 2yx + 4zx + f(y,z) \quad \text{—— (4)}$$

where $f(y,z)$ is a integrating const. independent of x.

differentiating ④ with r. to. y ⇒

$$\frac{\partial \varphi}{\partial y} = 2x + \frac{\delta f}{\delta y} \quad \text{—— (5)}$$

comparing ② & ⑤;

$$\frac{\delta f}{\delta y} = -3y - z$$

$$\Rightarrow f(y,z) = -\frac{3y^2}{2} - zy + g(z) \; [\text{integrating w.r. to } y]$$

∴ $g(z)$ is const. independent of x, y.

Now,

$$\varphi = \frac{x^2}{2} + 2yx + 4zx - \frac{3}{2}y^2 - zy + g(z) \quad \text{—— (6)}$$

diff. ⑥ w. r. to. z.

$$\frac{\delta \varphi}{\delta z} = 4x - y + g'(z) \quad \text{—— (7)}$$

comparing ③ & ⑦ ⇒

$$g'(z) = 2z$$

$$\therefore g(z) = z^2 + c \; [\text{integ. w.r. to } z]$$

c is arbitrary constant.

$$\therefore \varphi = \frac{x^2}{2} + 2yx + 4zx - \frac{3}{2}y^2 - zy + z^2 + c.$$

Ans.

[integrating w. r. to. y]

$\therefore \phi = 3x^2y + z^3x - yz + g(z)$ .... (6)

Diff. equn (6) w. r. to. z.

$\frac{\delta\phi}{\delta z} = 3z^2x - y + g'(z)$ .... (7)

Now comparing (3) & (7) $\Rightarrow$

$g'(z) = 0$ [Integrating ~~Diff.~~ w. r. to z]

$\therefore g(z) = 0 + c$

where c is arbitary constant.

So, $\phi = 3x^2y + z^3x - yz + c$. Ans:

C 103. Show that $E = r/r^2$ irrotational. Find ϕ such that $E = -\nabla\phi$ and such that $\phi(a) = 0$ where $a > 0$.

Soluno: Let, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$r^2 = x^2 + y^2 + z^2$

$$\vec{E} = \frac{\vec{r}}{r^2} = \frac{x}{x^2+y^2+z^2}\hat{i} + \frac{y}{x^2+y^2+z^2}\hat{j} + \frac{z}{x^2+y^2+z^2}\hat{k}$$

$$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \delta/\delta x & \delta/\delta y & \delta/\delta z \\ \frac{x}{x^2+y^2+z^2} & \frac{y}{x^2+y^2+z^2} & \frac{z}{x^2+y^2+z^2} \end{vmatrix}.$$

$$= \frac{1}{x^2+y^2+z^2} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \delta/\delta x & \delta/\delta y & \delta/\delta z \\ x & y & z \end{vmatrix}$$

$$= \frac{1}{x^2+y^2+z^2} \times 0$$

$$= 0$$

So, $\vec{E}$ is irrotational.

Here, $\vec{E} = -\vec{\nabla}\varphi$

$$\Rightarrow \vec{\nabla}\varphi = -\vec{E}$$

$$\Rightarrow \frac{\delta\varphi}{\delta x}\hat{i} + \frac{\delta\varphi}{\delta y}\hat{j} + \frac{\delta\varphi}{\delta z}\hat{k} = -\frac{x}{x^2+y^2+z^2}\hat{i} - \frac{y}{x^2+y^2+z^2}\hat{j} - \frac{z}{x^2+y^2+z^2}\hat{k}$$

$$\therefore \frac{\delta\varphi}{\delta x} = -\frac{x}{x^2+y^2+z^2} \quad \text{--- (1)}$$

$$\frac{\delta\varphi}{\delta y} = -\frac{y}{x^2+y^2+z^2} \quad \text{--- (2)}$$

$$\frac{\delta\varphi}{\delta z} = -\frac{z}{x^2+y^2+z^2} \quad \text{--- (3)}$$

Integrating (1) w.r.to. x.

$$\varphi = -\frac{1}{2}\ln(x^2+y^2+z^2) + c$$

$$= -\ln(x^2+y^2+z^2)^{\frac{1}{2}} + c$$

$$= -\ln r + c \quad \text{--- (4)}$$

~~$\varphi(a) = -\ln r + c$~~

Now, $\varphi(a) = -\ln a + c$

$$\Rightarrow 0 = -\ln a + c$$

$$\therefore c = \ln a.$$

From equ^n (4) $\Rightarrow$.

$$\therefore \varphi = -\ln r + \ln a = \ln(a/r).$$

Ans:

~~61~~ **64** Find the equ^n for the tangent plane & normal line to the surface $z = x^2 + y^2$ at the point $(2,-1,5)$

<u>Solu^n:</u> $\varphi = x^2 + y^2 - z$

$$\vec{\nabla}\varphi = 2x\,\hat{i} + 2y\,\hat{j} - \hat{k}$$

$$= 4\hat{i} - 2\hat{j} - \hat{k} \quad \text{at point } (2,-1,5)$$

at, (x,y,z), $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

at, $(2,-1,5)$; $\vec{r_0} = 2\hat{i} - \hat{j} + 5\hat{k}$

$$\vec{r} - \vec{r_0} = (x-2)\,\hat{i} + (y+1)\,\hat{j} + (z-5)\,\hat{k}.$$

The equ^n for the tangent plane is,

$$(\vec{r} - \vec{r_0}) \cdot \vec{\nabla}\varphi = 0$$

$$\Rightarrow 4(x-2) + (y+1)(-2) + (z-5)(-1) = 0$$

$$\Rightarrow 4x - 8 - 2y - 2 - z + 5 = 0$$

$$\Rightarrow 4x - 2y - z = 5$$

Ans.

The equ^n for the normal line is,

$$\frac{x-2}{4} = \frac{y+1}{-2} = \frac{z-5}{-1}$$

Ans.

60 Find equ^{n}s for the tangent plane and normal line to the surface $xz^2+x^2y = z-1$ at the point $(1,-3,2)$

Solu^n:

$$\varphi = xz^2 + x^2y - z + 1$$

$$\vec{\nabla}\varphi = (z^2+2xy)\hat{i} + x^2\hat{j} + (2xz-1)\hat{k}$$

$$= -2\hat{i} + \hat{j} + 3\hat{k} \quad \text{at point } (1,-3,2)$$

Now, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$(1,-3,2)$, $\vec{r_0} = 1\hat{i} - 3\hat{j} + 2\hat{k}$

$$\vec{r} - \vec{r_0} = (x-1)\hat{i} + (y+3)\hat{j} + (z-2)\hat{k}$$

The tangent plane is —

$$(\vec{r} - \vec{r_0})\cdot\vec{\nabla}\varphi = 0$$

$$\Rightarrow (x-1)(-2) + (y+3)(1) + 3(z-2) = 0$$

$$\Rightarrow -2x+2+y+3+3z-6 = 0$$

$$\Rightarrow 2x - y - 3z + 1 = 0$$

Ans.

C. 65 Find the values of the constants a, b, c so that the directional derivative of $\varphi = axy^2 + byz + cz^2x^3$ at $(1,2,-1)$ has a maximum of magnitude 64 in a direction parallel to the z axis.

Solu^n:

$$\vec{\nabla}\varphi = \vec{\nabla}(axy^2 + byz + cz^2x^3)$$

$$= (ay^2 + 3cz^2x^2)\hat{i} + (2axy + bz)\hat{j} + (by + 2czx^3)\hat{k}$$

$$= (4a+3c)\hat{i} + (4a-b)\hat{j} + (2b-2c)\hat{k}$$

at point $(1,2,-1)$.

In the parallel to z axis,

$$\vec{\nabla}\varphi = (2b-2c)\hat{k}$$

Now, $|\vec{\nabla}\varphi| = 64$

$$\Rightarrow \sqrt{(2b-2c)^2} = 64$$

$$\Rightarrow b-c = 32$$

$$\Rightarrow b = c+32$$

Now,

$$4a+3c = 0$$

$$\underline{\overset{(-)}{}4a \underset{(+)}{-} b = 0}$$

$$3c+b = 0$$

$$\Rightarrow 4c+32 = 0$$

$$\therefore c = -8$$

$$\therefore b = 24$$

$$\therefore 4a+3(-8) = 0$$

$$\Rightarrow 4a-24 = 0$$

$$\Rightarrow 4a = 24$$

$$\therefore a = 6$$

$\therefore$ $a = 6$, $b = 24$ and $c = -8$. Ans:

C 67. Find the constants a and b so that the surface $ax^2 - byz = (a+2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at the point $(1,-1,2)$.

Solution:

$$\varphi_1 = ax^2 - byz - (a+2)x$$

$$\varphi_2 = 4x^2y + z^3 - 4$$

$$\vec{\nabla}\varphi_1 = (2ax - (a+2))\hat{i} + (-bz)\hat{j} + (-by)\hat{k}$$

$= (a-2)\hat{i} - 2b\hat{j} + b\hat{k}$ at point $(1,-1,2)$

=

$$\vec{\nabla}\varphi_2 = 8xy\,\hat{i} + 4x^2\hat{j} + 3z^2\hat{k}$$

$$= -8\hat{i} + 4\hat{j} + 12\hat{k}$$

As, they are orthogonal, so,

$$\vec{\nabla}\varphi_1 \cdot \vec{\nabla}\varphi_2 = 0$$

$$\Rightarrow \left[(a-2)\hat{i} - 2\hat{j} + b\hat{k}\right]\left[-8\hat{i} + 4\hat{j} + 12\hat{k}\right] = 0$$

$$\Rightarrow -8a + 16 - 8b + 12b = 0$$

$$\Rightarrow -8a + 4b + 16 = 0$$

$$\Rightarrow -2a + b + 4 = 0 \quad \text{———— (1)}$$

$(1, -1, 2)$ satisfy φ_1,

$$a(1)^2 - b(-1)(2) = (a+2)(1)$$

$$\Rightarrow a + 2b = a + 2$$

$$\Rightarrow 2b = 2$$

$$\Rightarrow b = 1.$$

In eq (1) putting $b = 1$,

$$-2a + 1 + 4 = 0$$

$$\Rightarrow a = \frac{5}{2}$$

so, $a = 5/2$, $b = 1$. Ans

12. Find the angle betw^n the surface $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at point $(2,-1,2)$.

Solu^n:

$$\varphi_1 = x^2+y^2+z^2-9$$

$$\Rightarrow \nabla\varphi_1 = 2x\hat{i}+2y\hat{j}+2z\hat{k}$$

$$= 4\hat{i}-2\hat{j}+4\hat{k} \text{ at point } (2,-1,2)$$

$$\varphi_2 = x^2+y^2-z-3$$

$$\Rightarrow \vec{\nabla}\varphi_2 = 2x\hat{i}+2y\hat{j}-\hat{k}$$

$$= 4\hat{i}-2\hat{j}-\hat{k}$$

$$\vec{\nabla}\varphi_1 \cdot \vec{\nabla}\varphi_2 = |\vec{\nabla}\varphi_1||\vec{\nabla}\varphi_2|\cos\theta$$

$$\Rightarrow \cos\theta = \frac{\vec{\nabla}\varphi_1 \cdot \vec{\nabla}\varphi_2}{|\vec{\nabla}\varphi_1||\vec{\nabla}\varphi_2|}$$

$$= \frac{(4\hat{i}-2\hat{j}+4\hat{k})(4\hat{i}-2\hat{j}-\hat{k})}{\sqrt{4^2+(-2)^2+4^2}\,\sqrt{4^2+(-2)^2+(-1)^2}}$$

$$= \frac{16+4-4}{6\sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{16}{6\sqrt{21}}\right)$$

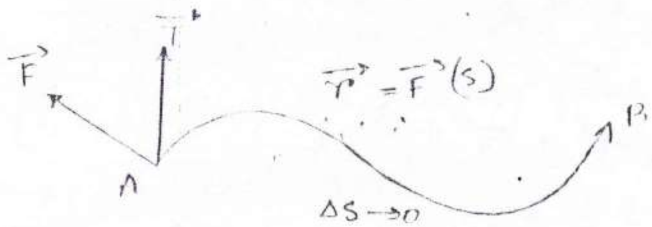
$$= 54^\circ 25'$$

Ans.

Chapter – 5

VECTOR INTEGRATION

LINE INTEGRATION



$$\vec{T} = \frac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|} = \frac{\frac{d\vec{r}}{dt}}{\frac{ds}{dt}} = \frac{d\vec{r}}{ds}$$

Now,

$$\Sigma \vec{F} \cdot \vec{T} \, \Delta s = \int_C \vec{F} \cdot \vec{T} \cdot ds$$

$$= \int_C \vec{F} \cdot d\vec{r} \quad \left[\because \vec{T} = \frac{d\vec{r}}{ds}\right]$$

Find the total work done in moving a particle in a force field given by $\vec{F} = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$ along the curve $x = t^2+1$, $y = 2t^2$, $z = t^3$ from $t=0$ to $t=1$.

Solu^n:

Total work done $= \int_C \vec{F} \cdot d\vec{r}$

$$= \int_C (3xy\hat{i} - 5z\hat{j} + 10x\hat{k})(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int_C (3xy \cdot dx - 5z \cdot dy + 10x \cdot dz)$$

$$= \int_0^1 \left[3(t^2+1)(2t^2) \cdot 2t - 5 \cdot (t^3)(4t) + 10(t^2+1) \cdot 3t^2\right] dt$$

$$= \int_0^1 \left[12t^3(t^2+1) - 20t^4 + 30t^2(t^2+1)\right] dt.$$

$$= \int_0^1 \left[12t^5 + 12t^3 - 20t^4 + 30t^4 + 30t^2\right] dt$$

$$= \int_0^1 \left(12t^3 + 12t^5 + 10t^4 + 30t^2\right) \cdot dt$$

$$= \left[12\frac{t^4}{4} + 12\frac{t^6}{6} + 10\frac{t^5}{5} + 30\frac{t^3}{3}\right]_0^1$$

$$= \left[3t^4 + 2t^6 + 2t^5 + 10t^3\right]_0^1$$

$$= 3 + 2 + 2 + 10$$

$$= 17.$$ Ans:

7. Find the total work done in moving a particle in a force field given by $F = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$ along the curve $x = t^2+1$, $y = 2t^2$, $z = t^3$ from $t = 1$ to $t = 2$.

Solun: Total work done $= \int_C \vec{F} \cdot d\vec{r}$

$$= \int_C \left(3xy\hat{i} - 5z\hat{j} + 10x\hat{k}\right)\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$$

$$= \int_C \left[3xy \cdot dx - 5z \cdot dy + 10x \cdot dz\right]$$

$$= \int_1^2 \left[3(t^2+1)(2t^2) \cdot 2t - 5t^3 \cdot 4t + 10(t^2+1) \cdot 3t^2\right] \cdot dt$$

$$= \int_1^2 \left(12t^3 + 12t^5 + 10t^4 + 30t^2\right) \cdot dt$$

$$= \left[3t^4 + 2t^6 + 2t^5 + 10t^3\right]_1^2$$

$$= 320 - 17$$

$$= 303.$$ Ans:

Conservative Force:

The force $\vec{F}$ is called conservative force if the amount of work done in displacing a unit particle/test body from one position to another position each independent of the path and depends on the end point only.

Imp. 6. If $A = (3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$; evalute $\int_C A\cdot dr$ from $(0,0,0)$ to $(1,1,1)$ along the following paths C.

(a) $x = t$, $y = t^2$, $z = t^3$

(b) The straight lines from $(0,0,0)$ to $(1,0,0)$ then to $(1,1,0)$ and then to $(1,1,1)$.

(c) The straight line joining $(0,0,0)$ and $(1,1,1)$.

Solution:

$$\int_C \vec{A}\cdot d\vec{r} = \int_C \left[(3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}\right]\left[dx\hat{i} + dy\hat{j} + dz\hat{k}\right]$$

$$= \int_C \left[(3x^2+6y)\cdot dx - 14yz\cdot dy + 20xz^2\cdot dz\right]$$

(a) If $x = t$, $y = t^2$, $z = t^3$

$$\int_C \vec{A}\cdot d\vec{r} = \int_{t=0}^{t=1} \left[(3t^2+6t^2)\cdot dt - 14\cdot t^2\cdot t^3(2t) + 20\cdot t\cdot t^6\cdot 3t^2\right] dt$$

$$= \int_0^1 \left[9t^2 - 28t^6 + 60t^9\right]\cdot dt$$

$$= \left[\frac{9t^3}{3} - 28\frac{t^7}{7} + 60\frac{t^{10}}{10}\right]_0^1$$

$$= \left[3t^3 - 4t^7 + 6t^{10}\right]_0^1$$

$$= 3 - 4 + 6 = 5 \quad \text{Ans}$$

(b) Along the straight line $(0,0,0)$ to $(1,0,0)$.

$y=0,\ z=0,\ dy=0,\ dz=0,\ x$ varies from 0 to 1.

Then,
$$\int_C \vec{A}\cdot d\vec{r} = \int_{x=0}^{x=1}\left[(3x^2+0)\,dx - 14(0)(0)(0) + 20x(0)^2\cdot 0\right]$$

$$\int_0^1 3x^2\cdot dx.$$

$$= 3\left[\frac{x^3}{3}\right]_0^1$$

$$= \cancel{3}\left[x^3\right]_0^1 = 1 \quad \underline{\text{Ans.}} \qquad \text{(i)}$$

(c) Along straight line $(0,0,0)$ to $(1,1,1)$.

$x=t,\ y=t,\ z=t$ (Parametric form)

$$\int_C \vec{A}\cdot d\vec{r} = \int_0^t \left[(3t^2+6t) - 14t^2 + 20t\cdot t^2\right]dt$$

$$= \int_0^1 \left[3t^2+6t-14t^2+20t^3\right]dt$$

$$= \left[\cancel{3}\frac{t^3}{\cancel{3}} + \cancel{6}^{3}\frac{t^2}{\cancel{2}} - \cancel{14}^{4}\frac{t^3}{3} + \cancel{20}^{5}\frac{t^4}{\cancel{4}}\right]_0^1$$

$$\cancel{= \left[3t^2 - 4t^3 + 5t^4\right]_0^1} \qquad = \int_0^1 \left[6t - 11t^2 + 20t^3\right]dt$$

$$\cancel{= 3-4+5} \qquad = \left[\cancel{6}^{3}\frac{t^2}{\cancel{2}} - 11\frac{t^3}{3} + \cancel{20}^{5}\frac{t^4}{\cancel{4}}\right]_0^1$$

$$= \left[3t^2 - \frac{11}{3}t^3 + 5t^4\right]_0^1 = 3 - \frac{11}{3} + 5$$

$$8 - \frac{11}{3} = \frac{13}{3}$$

Ans.

(b) Along the straight line from $(1,0,0)$ to $(1,1,0)$.

$x = 1 \ ; \ z = 0$

$dx = 0 \ ; \ dz = 0$

while y varies from 0 to 1.

$$\int_{y=0}^{1} (3 \cdot 1^2 + 6y)\, 0 - 14 \cdot y \cdot 0 \cdot 0 + 20 \cdot 1 \cdot 0 \cdot 0$$

$$= \int_0^1 0 = 0 . \quad \text{——— (2)}$$

Along the straight line from $(1,1,0)$ to $(1,1,1)$.

$x = 1 \quad , \ y = 1$

$dx = 0 \quad \ dy = 0$

while z varies from 0 to 1.

$$\int_{z=0}^{z=1} (3 \cdot 1^2 + 6 \cdot 1) \cdot 0 - 14 \cdot 1 \cdot z \cdot 0 + 20 \cdot 1 \cdot z^2 \cdot dz$$

$$= 20 \int_0^1 z^2 \cdot dz$$

$$= 20 \left[\frac{z^3}{3} \right]_0^1$$

$$= \frac{20}{3} . \quad \text{——— (3)}$$

Adding (I) + (II) + (III) $\Rightarrow$

$$\int_C \vec{A} \cdot d\vec{r} = 1 + 0 + \frac{20}{3} = \frac{23}{3} . \quad \text{Ans.}$$

10(a) (a) If $\vec{F} = \vec{\nabla}\varphi$, where ϕ is single-valued and has continuous partial derivatives, show that the work done is moving a particle from one point $P_1 \equiv (x_1, y_1, z_1)$ in this field to another point $P_2 \equiv (x_2, y_2, z_2)$ is independent of the path joining the two points.

(b) Conversely, if $\int_C \vec{F}\cdot d\vec{r}$ is independent of the path C joining any two points, show that there exists a function ϕ such that $\vec{F} = \vec{\nabla}\varphi$.

Solun: (a) Work done $= \int_{P_1}^{P_2} \vec{F}\cdot d\vec{r}$

$$= \int_{P_1}^{P_2} \vec{\nabla}\varphi \cdot d\vec{r}$$

$$= \int_{P_1}^{P_2} \left(\frac{\delta\varphi}{\delta x}\hat{i} + \frac{\delta\varphi}{\delta y}\hat{j} + \frac{\delta\varphi}{\delta z}\hat{k}\right)\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$$

$$= \int_{P_1}^{P_2}\left(\frac{\delta\varphi}{\delta x}dx + \frac{\delta\varphi}{\delta y}dy + \frac{\delta\varphi}{\delta z}\cdot dz\right)$$

$$= \int_{P_1}^{P_2} d\varphi$$

$$= \varphi(P_2) - \varphi(P_1)$$

$$= \varphi(x_2, y_2, z_2) - \varphi(x_1, y_1, z_1)$$

It depends on point P_1 & P_2 but not on the path joining them.

(b) Let, $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$\int_C \vec{F}\cdot d\vec{r}$ is independent of path joining two points, (x_1, y_1, z_1) & (x_2, y_2, z_2) respectively.

$$\varphi(x, y, z) = \int_{(x_1, y_1, z_1)}^{(x, y, z)} \vec{F}\cdot d\vec{r}$$

$$\varphi(x+\Delta x, y, z) = \int_{(x_1, y_1, z_1)}^{(x+\Delta x, y, z)} \vec{F}\cdot d\vec{r}$$

$$\varphi(x+\Delta x, y, z) - \varphi(x, y, z) = \int_{(x_1, y_1, z_1)}^{(x+\Delta x, y, z)} \vec{F}\cdot d\vec{r} - \int_{(x_1, y_1, z_1)}^{(x, y, z)} \vec{F}\cdot d\vec{r}$$

$$= \int_{(x, y, z)}^{(x+\Delta x, y, z)} \vec{F}\cdot d\vec{r}$$

$$= \int_{(x, y, z)}^{(x+\Delta x, y, z)} (F_1\cdot dx + F_2\, dy + F_3\cdot dz)$$

Let, we can choose the paths to be straight line joining these points, so that, dy, dz are zero.

$$\varphi(x+\Delta x, y, z) - \varphi(x, y, z) = \int_{(x, y, z)}^{(x+\Delta x, y, z)} F_1\cdot dx.$$

$$\Rightarrow \lim_{\Delta x\to 0} \frac{\varphi(x+\Delta x, y, z) - \varphi(x, y, z)}{\Delta x} = \lim_{\Delta x\to 0} \frac{1}{\Delta x} \int_{(x, y, z)}^{\varphi(x+\Delta x, y, z)} \vec{F_1}\cdot dx.$$

$$\Rightarrow \frac{\partial\varphi}{\partial x} = F_1$$

Similarly, $\frac{\partial\varphi}{\partial y} = F_2$ and $\frac{\partial\varphi}{\partial z} = F_3$

$$\vec{F} = \frac{\delta\varphi}{\delta x}\hat{i} + \frac{\delta\varphi}{\delta y}\hat{j} + \frac{\delta\varphi}{\delta z}\hat{k}$$

$$\Rightarrow \vec{F} = \vec{\nabla}\varphi.$$

Proved

16. If $\varphi = 2xyz^2$, $F = xy\hat{i} - z\hat{j} + x^2\hat{k}$ and C is the curve $x = t^2, y = 2t, z = t^3$ from $t = 0$ to $t = 1$ evaluate the line integrals (a) $\int_C \phi\, dr$ (b) $\int_C F \times dr$

Solⁿ: (a) Along C, $\phi = 2xyz^2 = 2t^2 \cdot 2t \cdot (t^3)^2 = 4t^9$

$$r = dx\hat{i} + dy\, x\hat{i} + y\hat{j} + z\hat{k} = t^2\hat{i} + 2t\hat{j} + t^3\hat{k}$$

$$dr = (2t\hat{i} + 2\hat{j} + 3t^2\hat{k})\, dt \quad \text{then,}$$

$$\int_C \phi\, d\vec{r} = \int_{t=0}^{t=1} 4t^9 (2t\hat{i} + 2\hat{j} + 3t^2\hat{k}) \cdot dt$$

$$= \int_0^1 \left[8t^{10}\hat{i} + 8t^9\hat{j} + 12t^{11}\hat{k}\right] dt.$$

$$= \left[8\frac{t^{11}}{11}\hat{i} + \frac{8t^{10}}{10}\hat{j} + 12\frac{t^{12}}{12}\hat{k}\right]_0^1$$

$$= \frac{8}{11}\hat{i} + \frac{4}{5}\hat{j} + \hat{k} \quad \text{Ans:}$$

(b) Along C, $\vec{F} = xy\hat{i} - z\hat{j} + x^2\hat{k} = 2t^3\hat{i} - t^3\hat{j} + t^4\hat{k}$

$$\vec{F} \times d\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2t^3 & -t^3 & t^4 \\ 2t & 2 & 3t^2 \end{vmatrix}.$$

$$= t^3 \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & t \\ 2t & 2 & 3t^2 \end{vmatrix}$$

$$= t^3\left[\hat{i}(-3t^2 - 2t) + \hat{j}(2t^2 - 6t^2) + \hat{k}(4 + 2t)\right]$$

$$= (-3t^5 - 2t^4)\hat{i} - 4t^5\hat{j} + (4t^3 + 2t^4)\hat{k}$$

$$\int_C \vec{F} \times d\vec{r} = \hat{i}\int_0^1 (-3t^5 - 2t^4)\cdot dt + \hat{j}\int_0^1 (-4t^5)\,dt + \hat{k}\int_0^1 (4t^3 + 2t^4)\,dt$$

$$= \hat{i}\left[-3\frac{t^6}{6} - 2\frac{t^5}{5}\right]_0^1 - 4\hat{j}\left[\frac{t^6}{6}\right]_0^1 + \hat{k}\left[4\frac{t^4}{4} + 2\frac{t^5}{5}\right]_0^1$$

$$= \hat{i}\left(-\frac{1}{2} - \frac{2}{5}\right) - 4\hat{j}\cdot\frac{1}{6} + \hat{k}\left[1 + \frac{2}{5}\right]$$

$$= -\frac{9}{10}\hat{i} - \frac{2}{3}\hat{j} + \frac{7}{5}\hat{k} \quad \text{Ans.}$$

SURFACE INTEGRAL:

17. Give a definition of $\iint_S \vec{A}\cdot\hat{n}\cdot d\vec{S}$ over a surface S in terms of limit of a sum.

Soluⁿ: Subdivide the area S into M elements of area ΔS_p where $p = 1, 2, 3 \ldots M$. Choose any point P_p within ΔS_p whose co-ordinates are (x_p, y_p, z_p).

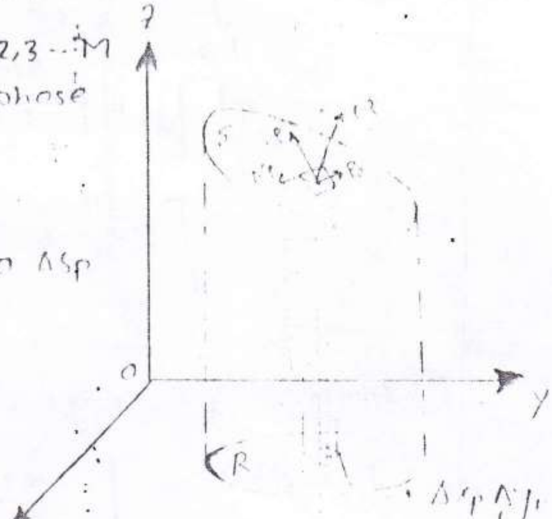
Define $A(x_p, y_p, z_p) = \vec{A}_p$.

Let, $\hat{n}_p$ be the positive unit normal to ΔS_p at P.

From the sum,

$$\sum_{p=1}^{M} \vec{A}_p \cdot \hat{n}_p\, \Delta S_p$$

where $\vec{A}_p\cdot\hat{n}_p$ is the normal component of A_p at P_p.



Now, if $M \to \infty$, then the largest dimention of each ΔS_p approches zero. This limit, if it exists, is called the surface integral of the normal component of $\vec{A}$ over $\vec{S}$ and is denoted by $\iint_S \vec{A}\cdot\hat{n}\cdot d\vec{S}$.

$$\vec{F} = \frac{\delta\varphi}{\delta x}\hat{i} + \frac{\delta\varphi}{\delta y}\hat{j} + \frac{\delta\varphi}{\delta z}\hat{k}$$

$$\Rightarrow \vec{F} = \vec{\nabla}\varphi.$$

Proved.

16. If $\varphi = 2xyz^2$, $F = xy\hat{i} - z\hat{j} + x^2\hat{k}$ and c is the curve $x = t^2, y = 2t, z = t^3$ from $t = 0$ to $t = 1$ evaluate the line integrals (a) $\int_c \phi\, dr$ (b) $\int_c F \times dr$

Soluⁿ: (a) Along c, $\phi = 2xyz^2 = 2 t^2 \cdot 2t \cdot (t^3)^2 = 4t^9$

$$r = \cancel{dx\hat{i} + dy}\ x\hat{i} + y\hat{j} + z\hat{k} = t^2\hat{i} + 2t\hat{j} + t^3\hat{k}$$

$$dr = (2t\hat{i} + 2\hat{j} + 3t^2\hat{k})\, dt \quad \text{then,}$$

$$\int_c \phi\, d\vec{r} = \int_{t=0}^{t=1} 4t^9 (2t\hat{i} + 2\hat{j} + 3t^2\hat{k}) \cdot dt$$

$$= \int_0^1 \left[8t^{10}\hat{i} + 8t^9\hat{j} + 12t^{11}\hat{k}\right] dt$$

$$= \left[8\frac{t^{11}}{11}\hat{i} + \overset{4}{\cancel{8}}\frac{t^{10}}{\underset{5}{\cancel{10}}}\hat{j} + \cancel{12}\frac{t^{12}}{\cancel{12}}\hat{k}\right]_0^1$$

$$= \frac{8}{11}\hat{i} + \frac{4}{5}\hat{j} + \hat{k} \quad \text{Ans:}$$

(b) Along c, $\vec{F} = xy\hat{i} - z\hat{j} + x^2\hat{k} = 2t^3\hat{i} - t^3\hat{j} + t^4\hat{k}$

$$\vec{F} \times d\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2t^3 & -t^3 & t^4 \\ 2t & 2 & 3t^2 \end{vmatrix}$$

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$$= t^3\left[\hat{i}(-3t^2 - 2t) + \hat{j}(2t^2 - 6t^2) + \hat{k}(4 + 2t)\right.$$

$$= (-3t^5 - 2t^4)\hat{i} + -4t^5\hat{j} + (4t^3 + 2t^4)\hat{k}$$

$$\int_C \vec{F} \times d\vec{r} = \hat{i}\int_0^1 (-3t^5 - 2t^4)\cdot dt + \hat{j}\int_0^1 (-4t^5)\,dt + \hat{k}\int_0^1 (4t^3 + 2t^4)\,dt$$

$$= \hat{i}\left[-3\frac{t^6}{6} - 2\frac{t^5}{5}\right]_0^1 - 4\hat{j}\left[\frac{t^6}{6}\right]_0^1 + \hat{k}\left[4\frac{t^4}{4} + 2\frac{t^5}{5}\right]_0^1$$

$$= \hat{i}\left(-\frac{1}{2} - \frac{2}{5}\right) - 4\hat{j}\cdot\frac{1}{6} + \hat{k}\left[1 + \frac{2}{5}\right]$$

$$= -\frac{9}{10}\hat{i} - \frac{2}{3}\hat{j} + \frac{7}{5}\hat{k} \quad \text{Ans.}$$

SURFACE INTEGRAL:

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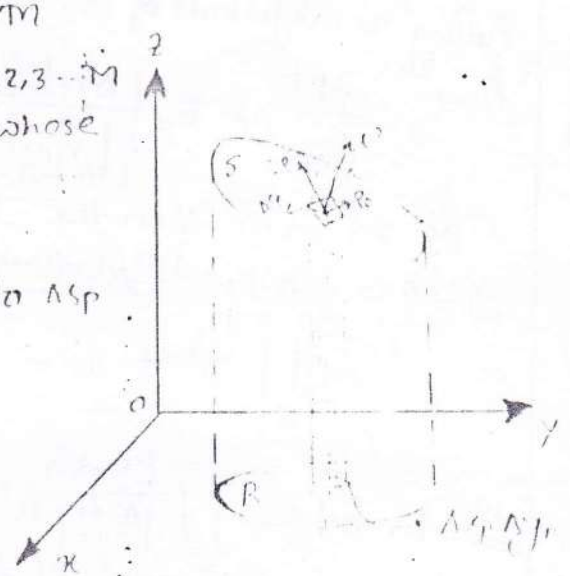
Let, $\hat{n}_p$ be the positive unit normal to ΔS_p at P.



From the sum,

$$\sum_{p=1}^{M} \vec{A}_p \cdot \hat{n}_p\, \Delta \vec{S}_p$$

where $\vec{A}_p\cdot\hat{n}_p$ is the normal component of A_p at P_p.

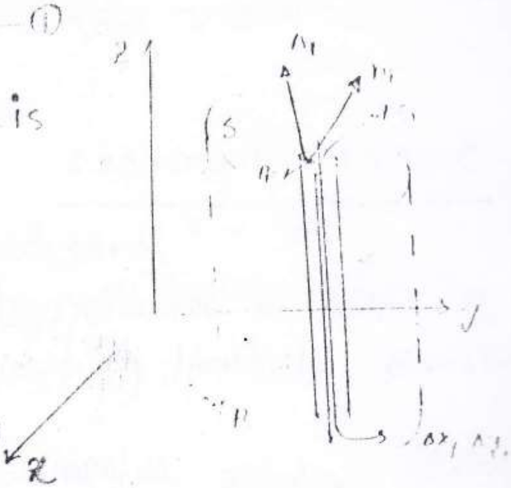
Now, if $M \to \infty$, then the largest dimention of each ΔS_p approches zero. This limit, if it exists, is called the surface integral of the normal component of $\vec{A}$ over $\vec{S}$ and is denoted by $\iint_S \vec{A}\cdot\hat{n}\cdot d\vec{S}$.

18. Suppose that the surface S has projection R on the xy plane. Show that
$$\iint_S \vec{A}\cdot\hat{n}\cdot d\vec{S} = \iint_R \hat{A}\cdot\hat{n}\cdot\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

Soln:

The surface integral is the limit of the sum,

$$\iint_S \vec{A}\cdot\hat{n}\cdot ds = \sum_{p=1}^{M} \vec{A}_p\,\hat{n}_p\cdot\Delta S_p \quad \text{--- (1)}$$



The projection of ΔS_p on the xy plane is $|(\hat{n}_p\Delta S_p)\cdot k|$ or $|\hat{n}_p\cdot k|\,\Delta S_p$. So,

$$|n_p\cdot k|\,\Delta S_p = \Delta x_p\,\Delta y_p$$
$$\therefore \Delta S_p = \frac{\Delta x_p\cdot\Delta y_p}{|\hat{n}_p\cdot\hat{k}|}$$

Putting the magnitude of ΔS_p into eqn

$$(1) \Rightarrow \sum_{p=1}^{M} \vec{A}_p\,\hat{n}_p\,\frac{\Delta x_p\,\Delta y_p}{|\hat{n}_p\cdot\hat{k}|} = \iint_S \vec{A}\cdot\hat{n}\cdot ds$$

If $M\to\infty$ then the largest Δx_p and Δy_p approach zero. So the equation would be—

$$\iint_S \vec{A}\cdot\hat{n}\cdot ds = \iint_R \vec{A}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} \quad \text{(Showed)}$$

19. Evalute $\iint_S \vec{A}\cdot\hat{n}\cdot ds$ where $A = 12z\hat{i} - 12\hat{j} + 3y\hat{k}$ and S is that part of the plane $2x+3y+6z=12$ which is located in the first octant.

Solu^n:

$$A = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$$

plane, $2x + 3y + 6z = 12$

$$\iint_S \vec{A}\cdot\hat{n}\, ds = \iint_R \vec{A}\cdot\hat{n}\, \frac{dx\, dy}{|\hat{n}\cdot\hat{k}|}$$

$$\vec{\nabla}(2x+3y+6z-12) = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)(2x+3y+6z-12)$$

$$= 2\hat{i} + 3\hat{j} + 6\hat{k}$$

$$\hat{n} = \frac{2\hat{i}+3\hat{j}+6\hat{k}}{\sqrt{2^2+3^2+6^2}} = \frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$$

$$\hat{n}\cdot\hat{k} = \left(\frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right)\cdot\hat{k} = \frac{6}{7}$$

$$\vec{A}\cdot\hat{n} = (18z\hat{i} - 12\hat{j} + 3y\hat{k})\left(\frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right)$$

$$= \frac{36z - 36 + 18y}{7} \qquad \left[z = \frac{12-2x-3y}{6}\right]$$

$$= \frac{6(12-2x-3y) - 36 + 18y}{7}$$

$$= \frac{72 - 12x - \cancel{18y} - 36 + \cancel{18y}}{7}$$

$$= \frac{36 - 12x}{7}$$

$$\iint_S \vec{A}\cdot\hat{n}\, ds = \iint_R \vec{A}\cdot\hat{n}\, \frac{dx\, dy}{|\hat{n}\cdot\hat{k}|}$$

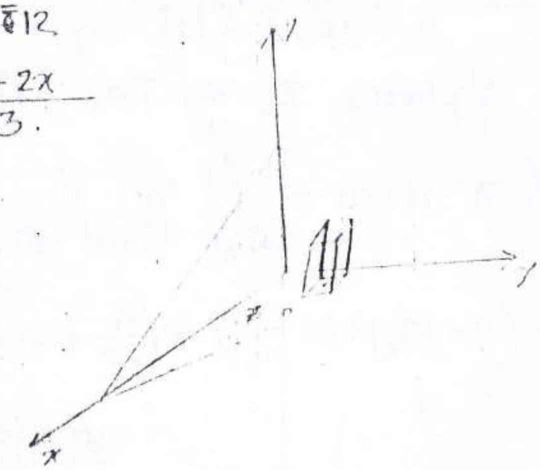
$$= \iint_R \frac{36-12x}{7}\cdot\frac{7\cdot dx\, dy}{6}$$

$$= \iint_R (6-2x)\, dx\cdot dy$$

=

At xy plane $2x+3y=12$

$$\Rightarrow y = \frac{12-2x}{3}.$$



$$= \int_0^6 \int_0^{\frac{12-2x}{3}} (6-2x)\, dx \cdot dy$$

$$= \int_0^6 \left[6y - 2xy\right]_0^{\frac{12-2x}{3}} dx$$

$$= \int_0^6 \left[6\left(\frac{12-2x}{3}\right) - 2x\left(\frac{12-2x}{3}\right)\right] dx$$

$$= \int_0^6 \left[24 - 4x - \frac{24x-4x^2}{3}\right] dx$$

$$= \int_0^6 \left[24 - 4x - 8x + \frac{4}{3}x^2\right] dx$$

$$= \int_0^6 \left[24 - 12x + \frac{4}{3}x^2\right] \cdot dx$$

$$= \left[24x - \cancel{12}^{\,6}\frac{x^2}{\cancel{2}} + \frac{4}{3}\frac{x^3}{3}\right]_0^6$$

$$= \left[24x - 6x^2 + \frac{4}{9}x^3\right]_0^6$$

$$= 144 - 216 + \frac{4}{\cancel{9}} \times \cancel{216}^{\,24}$$

$$= 144 - 216 + 96$$

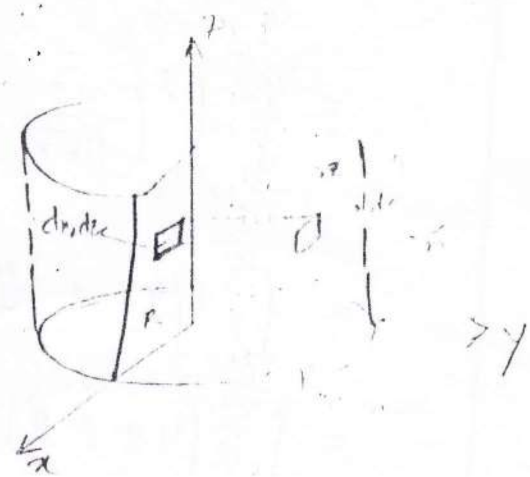
$$= 240 - \cancel{219}\ 216 = 24$$ Ans.

20 Evalute $\iint_S \vec{A}\cdot\hat{n}\cdot ds$ where, $\vec{A} = z\hat{i} + xj - 3y^2 zk$ and S is the surface of the cylinder $x^2+y^2=16$ included in the first octant betwn $z=0$ and $z=5$.

$$\vec{A} = z\hat{i} + x\hat{j} - 3y^2 z\hat{k}$$

$$x^2+y^2=16$$

$$z=0,\ z=5$$



$$\nabla(x^2+y^2-16) = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)(x^2+y^2-16)$$

$$\hat{n} = \frac{2x\hat{i}+2y\hat{j}}{\sqrt{(2x)^2+(2y)^2}} = 2x\hat{i}+2y\hat{j}$$

$$= \frac{2x\hat{i}+2y\hat{j}}{\sqrt{4(x^2+y^2)}}$$

$$= \frac{2x\hat{i}+2y\hat{j}}{\sqrt{4 \times 16}}$$

$$= \frac{x\hat{i}+y\hat{j}}{4}$$

$$\hat{n}\cdot\hat{j} = \left(\frac{x\hat{i}+y\hat{j}}{4}\right)\hat{j} = \frac{y}{4}$$

$$\vec{A}\cdot\hat{n} = \left(z\hat{i}+x\hat{j}-3y^2z\hat{k}\right)\left(\frac{1}{4}x\hat{i}+\frac{1}{4}y\hat{j}\right)$$

$$= \frac{1}{4}xz + \frac{1}{4}xy = \frac{1}{4}(xz+xy).$$

$$\iint_S \vec{A}\cdot\hat{n}\cdot ds = \iint_R \frac{1}{4}(xz+xy)\cdot\frac{dx\cdot dz}{|\hat{n}\cdot\hat{j}|}$$

$$= \iint_R \frac{1}{4}(zx+xy)\cdot\frac{4}{y}\,dx\cdot dz$$

$$= \int_{x=0}^{x=4}\int_{z=0}^{z=5} \left(\frac{xz + xy}{y}\right) \cdot \frac{dz}{dy} \cdot dy \cdot dx \quad \left[\begin{array}{l} \text{in } xz \text{ plane, } y=0 \\ x^2 = 16 \\ \Rightarrow x = 4 \end{array}\right]$$

$$= \int_0^4 \int_0^5 \left[\frac{xz}{\sqrt{16-x^2}} + x\right] \cdot dz \cdot dx.$$

$$= \int_{z=0}^{z=5}\int_{x=0}^{x=4} \left(\frac{xz}{\sqrt{16-x^2}} + x\right) \cdot dz \cdot dx$$

$$= \int_0^5 \left\{4z + \left[\frac{x^2}{2}\right]_0^4\right\} \cdot dz$$

$$= \int_0^5 \left[4z + 8\right] dz$$

$$= \left[\frac{\overset{2}{\cancel{4}}z^2}{\cancel{2}} + 8z\right]_0^5$$

$$= 2\times 5^2 + 8\times 5$$

$$= 50 + 40$$

$$= 90. \quad \underline{\text{Ans:}}$$

$$\int_0^4 \frac{x \cdot dx}{\sqrt{16-x^2}} \qquad x = 4\sin\theta \Rightarrow dx = 4\cos\theta \cdot d\theta$$

$$= \int_0^{\pi/2} \frac{4\sin\theta}{4\cos\theta} \cdot 4\cos\theta \cdot d\theta$$

$$= \int_0^{\pi/2} 4\sin\theta \cdot d\theta.$$

$$= -4\left[\cos\theta\right]_0^{\pi/2}$$

$$= -4\left[0 - 1\right] = 4$$

VOLUME INTEGRALS :

25 Let $\varphi = 45x^2y$ and let v denote the closed region bounded by the planes $4x+2y+z=8$, $x=0, y=0, z=0$.

(a) Express $\iiint_V \phi \, dv$ as the limit of a sum.

(b) Evalute the integral in (a).

Solu^n:

(b)

$4x+2y+z=8$

$\Rightarrow z = 8-4x-2y$

$4x+2y=8$

$\Rightarrow y = \frac{8-4x}{2} = 4-2x$

$$\iiint_v \phi \, dv = \int_{x=0}^{x=2} \int_{y=0}^{y=4-2x} \int_{z=0}^{z=8-4x-2y} (45x^2y) \, dx \cdot dy \cdot dz$$

$$= 45 \int_{x=0}^{2} \int_{y=0}^{4-2x} (x^2yz) \cdot dy \cdot dx$$

$$= 45 \int_{x=0}^{2} \int_{y=0}^{4-2x} \left[x^2y(8-4x-2y)\right] \cdot dy \cdot dx$$

$$= 45 \int_{x=0}^{2} \int_{y=0}^{4-2x} (8x^2y - 4x^3y - 2x^2y^2) \cdot dy \cdot dx$$

$$= 45 \int_{x=0}^{2} \left[8x^2\frac{y^2}{2} - 4x^3\frac{y^2}{2} - 2x^2\frac{y^3}{3} \right]^{4-2x} dx$$

$$= 45\int_{x=0}^{2} 4x^2(4-2x)^2 - 2x^3(4-2x)^2 - \frac{2}{3}x^2(4-2x)^3 \cdot dx$$

$$= 45\int_0^2 (4-2x)^2(4x^2-2x^3) - \frac{2}{3}x^2(4-2x)^3 \cdot dx$$

$$= 45\int_0^2 x^2(4-2x)^3 - \frac{2}{3}x^2(4-2x)^3 \cdot dx$$

$$= 45\int_0^2 x^2(4-2x)^3\left(1-\frac{2}{3}\right) \cdot dx$$

$$= \frac{45}{3}\int_0^2 x^2(4-2x)^3 \cdot dx$$

$$= \frac{45}{3}\int_0^2 x^2(64-96x+48x^2-8x^3) \cdot dx$$

$$= 15\int_0^2 (64x^2-96x^3+48x^4-8x^5) \cdot dx$$

$$= 15\left[64\frac{x^3}{3} - 96\frac{x^4}{4} + 48\frac{x^5}{5} - 8\frac{x^6}{6}\right]_0^2$$

$$= 15\left[\frac{512}{3} - 384 + \frac{1536}{5} - \frac{256}{3}\right]$$

$$= 128.$$ Ans:

26: Let, $f = 2xz\hat{i} - x\hat{j} + y^2\hat{k}$. Evalute $\iiint f \cdot dv$ when v is the region bounded by the surface $x=0$, $y=0$, $y=6$, $z=x^2$, $z=4$.

Solu^n:

$$\iiint_V \vec{F} \cdot dv = \int_{x=0}^{x=2}\int_{y=0}^{y=6}\int_{z=x^2}^{z=4} \left(2xz\hat{i} - x\hat{j} + y^2\hat{k}\right) dx \cdot dy \cdot dz$$

$$= \hat{i}\int_0^2\int_0^6\int_{x^2}^4 2xz \cdot dx \cdot dy\, dz - \hat{j}\int_0^2\int_0^6\int_{x^2}^4 x \cdot dx\, dy \cdot dz + \hat{k}\int_0^2\int_0^6\int_{x^2}^4 y^2 \cdot dx \cdot dy \cdot dz$$

$$= \hat{i}\int_0^2\int_0^6 \left[xz^2\right]_{x^2}^4 dy \cdot dx - \hat{j}\int_0^2\int_0^6 \left[xz\right]_{x^2}^4 dy \cdot dx + \hat{k}\int_0^2\int_0^6 \left[y^2 z\right]_{x^2}^4 dy \cdot dx$$

$$= \hat{i}\int_0^2\int_0^6 \left[16x - x^5\right] dy \cdot dx - \hat{j}\int_0^2\int_0^6 \left[4x - x^3\right] dy \cdot dx + \hat{k}\int_0^2\int_0^6 \left[4y^2 - x^2y^2\right] dy \cdot dx$$

$$= \hat{i}\int_0^2 \left[16xy - x^5y\right]_0^6 \cdot dx - \hat{j}\int_0^2 \left[4xy - x^3y\right]_0^6 dx + \hat{k}\int_0^2 \left[\frac{4y^3}{3} - \frac{x^2y^3}{3}\right]_0^6 \cdot dx$$

$$= \hat{i}\int_0^2 \left[96x - 6x^5\right] dx - \hat{j}\int_0^2 \left[24x - 6x^3\right] \cdot dx + \hat{k}\int_0^2 \left[288 - 72x^2\right] \cdot dx$$

$$= \hat{i}\left[96\frac{x^2}{2} - 6\frac{x^6}{6}\right]_0^2 - \hat{j}\left[24\frac{x^2}{2} - 6\frac{x^4}{4}\right]_0^2 + \hat{k}\left[288x - 72\frac{x^3}{3}\right]_0^2$$

$$= \hat{i}\,(192 - 64) - \hat{j}\,(48 - 24) + \hat{k}\,(576 - 192)$$

$$= 128\hat{i} - 24\hat{j} - 384\hat{k}$$

Ans.

27. Find the volume of the region common to the intersecting cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$.

Soluⁿ:

∴ Required volume = 8 times volume of region shown in

$$= 8\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\int_{z=0}^{\sqrt{a^2-x^2}} dz.dy.dx$$

$$= 8\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\left[(z)\right]_0^{\sqrt{a^2-x^2}}.dy.dx$$

$$= 8\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\sqrt{a^2-x^2}\,dy.dx$$

$$= 8\int_{x=0}^{a}\left[\sqrt{a^2-x^2}\,.\,y\right]_0^{\sqrt{a^2-x^2}}.dx$$

$$= 8\int_{x=0}^{a}(a^2-x^2).dx$$

$$= 8\int_{x=0}^{a}(a^2-x^2).dx$$

$$= 8.\left[a^2x-\frac{x^3}{3}\right]_0^a$$

$$= 8\left[a^3-\frac{a^3}{3}\right] = 8.\frac{2a^3}{3}$$

$$= \frac{16a^3}{3}$$ Ans.