Abstract

This report explores the design and characteristics of P, PD, PID and phase-lead controllers for a type I, III order industrial plant process in MATLAB®. Controllers were designed to meet one of two performance specifications. After comparison it was concluded that PD controllers achieved the best performance.

Nomenclature

$e(\infty)$	Steady-state error	[%]
p	Pole in Laplace domain	[rad s^{-1}]
s	Laplace domain variable	[rad s^{-1}]
t	Time domain variable	[s]
z	Zero in Laplace domain	[rad s^{-1}]
ζ	Damping factor	
ω_n	Natural frequency	[rad s^{-1}]
G	Plant process transfer function	
G_c	Controller transfer function	
K	Plant process gain factor	
K_c	Controller gain factor	
K_d	Derivative gain factor	
K_i	Integral gain factor	
K_p	Proportional gain factor	
M_0	Peak overshoot	[%]
T_d	Derivative time constant	[s]
T_i	Integral time constant	[s]
T_r	Rise time (10 % to 90 %)	[s]
T_s	Settling time (2 %)	[s]

1 Introduction

Control systems find applications across engineering, economics and the environment.

Robotics are the obvious application where servo and stepper-motors are used in control systems to actuate precise armature movement.

Robots mimic control systems found in nature, such as the feedback between motor and sensory neurons in animals.

Many markets trade automatically, employing algorithms originally developed for electronic control theory.

2 Background

Figure 1 illustrates a closed loop system. The controller is designed to drive the output $y(s)$ of the plant process to the desired output $r(s)$ by minimising the error $e(s)$.
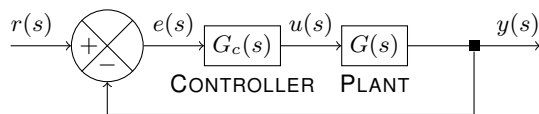

FIGURE 1: Block diagram of the closed loop system.

The controller gain factor K_c is a key design parameter and its effect on the system poles can be represented by the root locus. Each point on the root locus satisfies the angle criterion, given by Dorf et al. [1, p. 447] as:

$$\sum_i \angle (s - p_i) - \sum_j \angle (s - z_j) \equiv \pi \mod 2\pi \quad (1)$$

This allows the root locus to be manipulated deterministically by adding poles and/or zeros. The controller gain at a point on the root locus can be determined using the magnitude criterion:

$$K_c \cdot K \equiv \frac{\prod_i |s - p_i|}{\prod_j |s - z_j|} \quad (2)$$

where K is the gain factor of the plant process transfer function.

2.1 Performance

For low frequency applications the controller performance can be characterised by analysing the step response, which can be predicted using the following equations laid out in by Ogata et al. [2, pp. 231–233].

Overshoot M_0 is the the peak response in proportion to the final output value and is soley dependent on the damping factor ζ for second order systems:

$$M_0 = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) * \quad (3)$$

The settling time T_s of the controller is the time for the output oscillations to decay to within some margin of the final value. Settling time is also dependent on the natural frequency ω_n:

$$T_s = -\frac{\ln(2\%)}{\zeta\omega_n} \approx \frac{4}{\zeta\omega_n} * \quad (4)$$

The rise time T_r of the controller is the time for the output to transition from 10 % to 90 % of the final value:

$$T_r \leq \frac{\ln(90\%/10\%)}{\zeta\omega_n} \approx \frac{2.2}{\zeta\omega_n} * \quad (5)$$

These three parameters are typically specified as upper bounds and constitute the dynamic specification.

The steady state error $e(\infty)$ for a step input is defined and given in [2, p. 290] as follows:

$$e(\infty) := \lim_{t \to \infty} e(t) \quad (6)$$

$$= \begin{cases} \frac{1}{1 + \lim\limits_{s \to +0} G_c(s)G(s)}, & \text{type 0} \\ 0, & \text{otherwise} \end{cases} \quad (7)$$

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*Equality holds for second order systems only, but may be used to approximate higher order characteristics.

where the type of the open loop transfer function $G_c(s)G(s)$ is the number of poles at the origin.

In controller design there is a tradeoff between the dynamic specification and the steady state error. The type of controller has a large impact on how the tradeoff is done.

2.2 PID Control

One of the most common types of controller in industry is the PID controller, which can be easily implemented using microcontrollers. Figure 2 shows how a PID controller is composed in [1, p. 480].
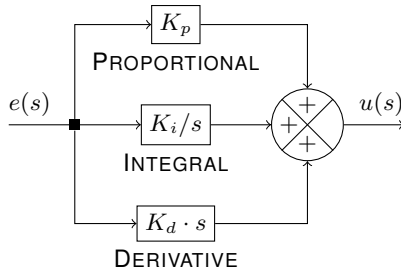
FIGURE 2: Block diagram of a PID controller

The relative contribution of the derivative factor K_d and intergral factor K_i can be expressed as time constants T_d and T_i as follows:

$$T_i = \frac{K_p}{K_i} \qquad\qquad T_d = \frac{K_d}{K_p} \qquad (8)$$

Consequently, an equivalent form of the controller in Figure 2 is as follows:

$$G_c(s) = K_p\left(1 + \frac{1}{T_i \cdot s} + T_d \cdot s\right) \qquad (9)$$

In any controller, the proportional gain K_p is used to set the tradeoff between overshoot and speed. Acording to [1, p. 483], a faster system has a slightly shorter rise and settling time but greater overshoot, achieved with a larger gain.

Increasing the contribution of the integrator term improves the steady-state performance at the expense of the dynamic performance. Generally, controllers with short T_i have greater overshoot, longer rise and settling time but smaller steady-state error.

Increasing the contribution of the differentiator term improves the dynamic performance at the expense of the steady-state performance. Generally, controllers with large T_d have smaller overshoot, shorter rise and settling time.

Some variations of the PID controller are named as follows:

- **P controller** where K_i and K_d are zero. Also known as gain compensation.

- **PI controller** where K_d is zero.

- **PD controller** where K_i is zero. Used in

2.3 Phase Compensation

Practical phase compensation involves adding real pole-zero pairs close to the origin to improve steady-state performance (phase-lag compensation) or left of the uncompensated root locus to improve dynamic peformance (phase-lead compensation) or both (lag-lead compenation). Figure 3 illustrates the block diagram of the compensator, similar to the designs specified in [2, pp. 421–423].
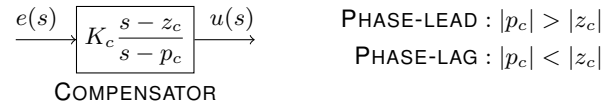
FIGURE 3: Block diagram of a phase-lead controller.

The compensator gain K_c is then used to tradeoff the overshoot and speed of the modified system, much like gain compensation.

3 Specification

The transfer function for the industrial plant process $G(s)$ was given as follows:

$$G(s) = \frac{s+1}{s(s^2 + 4s + 5)} \qquad (10)$$

The controller transfer functions $G_c(s)$ were chosen to meet the system performance specification detailed in Table 1.

TABLE 1: System perfomance specification.

SYSTEM	M_0 [%]	T_r [s]	T_s [s]	$e(\infty)$ [%]
1	5	1.0	4	—
2	20	0.5	—	5

4 Design

The transfer function (10) described a type I, 3rd order system.

The pole at the origin and equation (7) suggested the steady-state error would be zero. PI and phase-lag controllers seek to improve steady-state performance so these were not used.

The step response characteristics of gain compensators were compared against phase-lead compensators, to demonstrate how modifying the root locus affects performance.

The gain controllers were then compared against PD and PID controllers tuned in MATLAB® sisotool() using the 'robust response time' method with the same 'response time', varying the 'robust' parameter to achieve a similar degree of overshoot.

5 Results

The step response of the gain compensator and the margins by which the system gain is allowed to increase or decrease were marked on Figure 4.
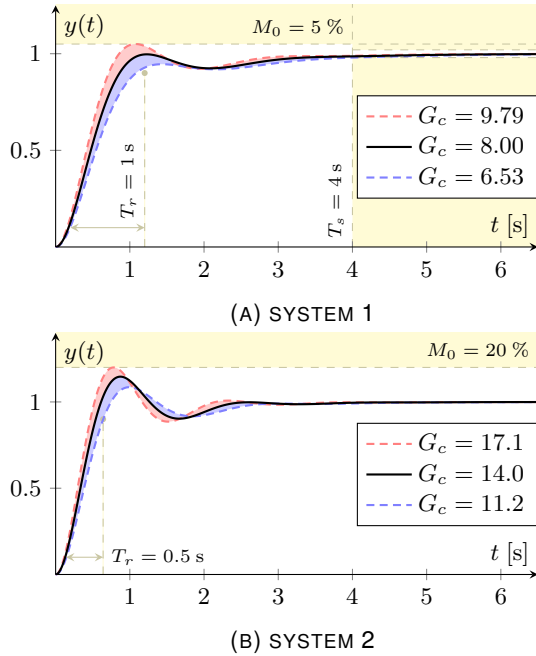

(A) SYSTEM 1

(B) SYSTEM 2

FIGURE 4: Step response of gain controller designed for each system. The gain limits which met the performance specification are also highlighted.

Figure 5 shows similar traces for the chosen phase-lead compensators.

The same gain compensators in Figure 4 are displayed alongside the designed PD and PID controllers in Figure 6.

6 Discussion

The gain margins indicated in Figure 4 show the tradeoff between overshoot and response time: as gain was increased, the system responded faster but exhibited more overshoot. Consequently there was a limit to the minimum rise time the controller could achieve before overshoot became unacceptable. Unsurprisingly SYSTEM 2, which had a more relaxed overshoot requirement, was able to rise and settle in less time. These values are detailed in Table 2. All performance requirments were satisfied.

The steady-state error was verified to be zero by taking the `dcgain()` of the closed loop system in MATLAB®.

Table 2 also lists the phase-lead compensators and their performance characteristics. By bending the root locus towards away from the orgin, the gain factor was able to be increased without the
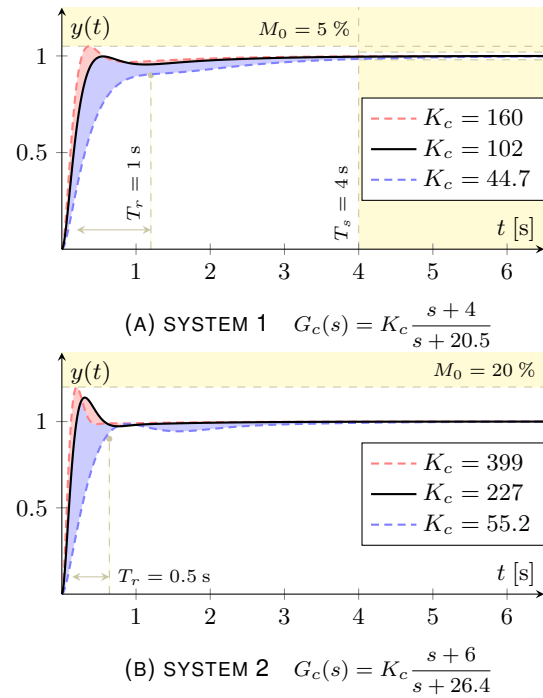

(A) SYSTEM 1 $G_c(s) = K_c \dfrac{s+4}{s+20.5}$

(B) SYSTEM 2 $G_c(s) = K_c \dfrac{s+6}{s+26.4}$

FIGURE 5: Step response of lead compensator designed for each system. The gain limits which met the performance specification are also highlighted.

(A) SYSTEM 1 $G_c(s) = \begin{cases} 8 & \text{P} \\ 41\left(1+0.26s\right) & \text{PD} \\ 33\left(1+\frac{1}{1.32s}+0.33s\right) & \text{PID} \end{cases}$

(B) SYSTEM 2 $G_c(s) = \begin{cases} 14 & \text{P} \\ 81\left(1+0.10s\right) & \text{PD} \\ 65\left(1+\frac{1}{0.70s}+0.16s\right) & \text{PID} \end{cases}$
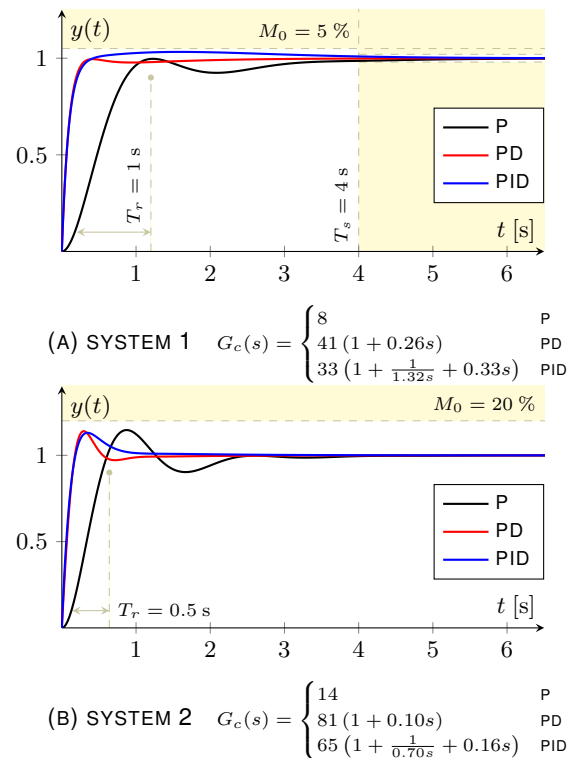
FIGURE 6: Step response of P, PD and PID controllers designed for each system.

damping factor becoming too small. Figure 7 illustrates how the compensated root loci extended away from the imaginary axis where the damping factor was small. The larger gain allowed faster rise

TABLE 2: Performance characteristics of gain and phase-lead compensators designed for each system.

	K_c	$\|z_c\|^*$	$\|p_c\|^*$	M_0 [%]	T_r [s]	T_s [s]	$e(\infty)$
SYSTEM 1	8	—	—	**0.0**	**0.69**	**3.35**	0
	102	4	20.5	**0.0**	**0.28**	**2.16**	0
SYSTEM 2	14	—	—	**14.6**	**0.41**	2.26	**0**
	227	6	26.4	**13.9**	**0.14**	0.98	**0**

* Units [rad s^{-1}]. Values in **bold** match the specification in Table 1.

TABLE 3: Performance characteristics of P, PD and PID controllers designed for each system.

	K_p	T_i [s]	T_d [s]	M_0 [%]	T_r [s]	T_s [s]	$e(\infty)$
SYSTEM 1	8	—	—	**0.0**	**0.69**	**3.35**	0
	41	—	0.26	**0.0**	**0.18**	**1.22**	0
	33	1.32	0.33	**3.3**	**0.20**	**2.93**	0
SYSTEM 2	14	—	—	**14.6**	**0.41**	2.26	**0**
	81	—	0.10	**14.0**	**0.13**	0.87	**0**
	65	0.70	0.16	**13.0**	**0.13**	0.87	**0**

Values in **bold** match the specification in Table 1.

and settling times.

(A) Uncompensated root locus

(B) Compensated for SYSTEM 1
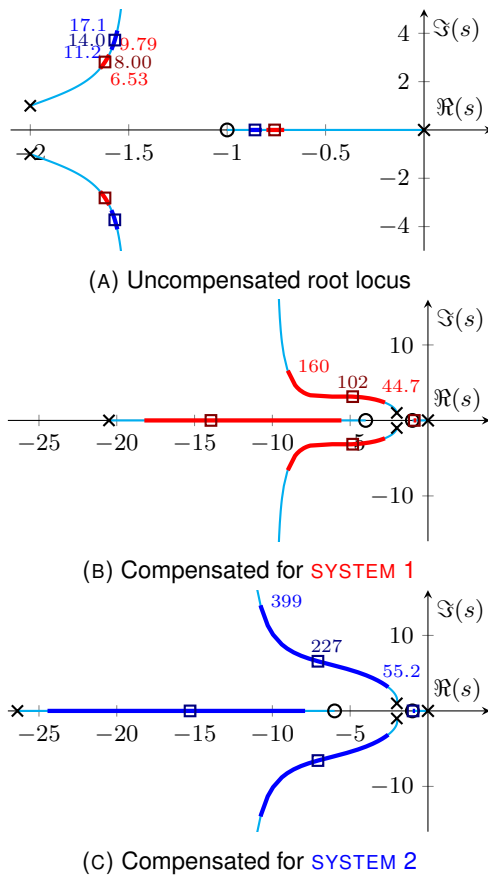

(C) Compensated for SYSTEM 2

FIGURE 7: Chosen closed loop pole locations, gains and acceptable limits for the gain and phase-lead compensator designed for both systems, marked on the original and compensated root loci.

Table 3 lists the performance characteristics of the P, PD and PID controllers illustrated in Figure 6. All performance requirements were satisfied.

The overshoot of each controller was similar by design, but the corresponding rise and settling times were significantly better for the PD controller than the baseline P controller. The derivative term controlled the output by predicting future error, avoiding overshoot, making the transient more robust.

The integral term in PID controllers had little addi-

tional effect, as the plant process already contained an integrator. The integral term slowed the transient response but had no effect on the already perfect steady-state error. This behaviour was consistent with other type I systems, such as the servo-motors described in [1, pp. 235–236], where the velocity of the motor is varied in order to set the servo position. For such applications, PD controllers are common as their decreased complexity results in comparable or better performance.

For similar reasons PD controllers outperformed phase-lead compensators. Practical phase-lead compensators have a real pole, which behaves as a low pass filter, much like an integrator. A PD controller is an ideal phase-lead compensator as the real pole is omitted.

7   Conclusion

Two P, PD, PID and phase-lead controllers were designed to successfully meet all of the performance requirements for two different systems using the same plant process.

The second system had a more relaxed overshoot requirement, so controllers could be designed with faster rise and settling times.

Steady-state requirement was not an issue as the plant-process was of type I, with an integrator at the origin.

The PD controllers outperformed the P controller, as the derivative term limited overshoot by predicting future error; the PID controller, which had similar performance but increased complexity because the integral term was redundant; and the practical phase-lead compensator, as the PD controller is also an ideal phase-lead compensator.

References

[1] R. C. Dorf and R. H. Bishop, *Modern Control Systems*, 12th. Pearson, 2010, pp. 251–483.

[2] K. Ogata and Y. Yang, *Modern control engineering*, 4th ed. Prentice hall India, 2002, pp. 231–293.

A   Controller Derivations

A.1   P Controller/Gain Compensator

The P controller was found by increasing the controller gain K_c or K_p in steps of 0.1 until the rise and settling time specifications were satisfied. The gain continued to be increased until the overshoot specifications were violated. The median gain was chosen for the controller, allowing for a margin of safety.

A.2   Phase-lead Compensator

Figure 8 illustrates how the added poles and zeroes were chosen when designing the phase-lead controllers.
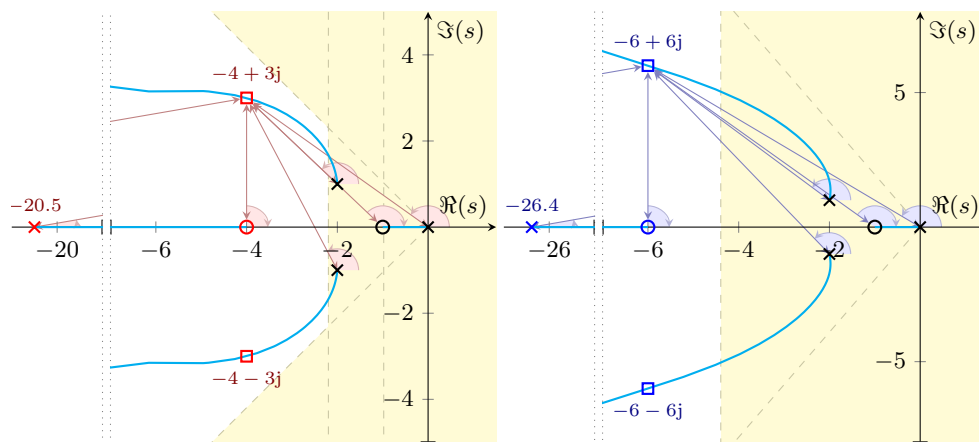

FIGURE 8: Ideal phase-lead compensated root locus for SYSTEM 1 and SYSTEM 2, designed using the angle and magnitude criteria.

The yellow region marks the closed loop pole locations which did not satisfy the performance specification. Values from Table 1 were substituted into (3), (4) and (5) to define the boundaries of these regions. The locus of $\zeta\omega_n$ manifests as a vertical line on the s-plane whilst the locus of ζ is a straight line through the origin in the second quadrant, reflected in the real axis.

Desired closed loop poles labelled in Figure 8 were chosen close to these boundaries to avoid large compensator parameters and ensure a practical design. The added zero was placed on the real axis in line with the labelled poles and the location of the real pole was determined by rearranging the angle criterion given in (1). This was done programmatically in MATLAB®:

```matlab
function [zc, pc, Kc] = find_lead_compensator(G, s)
  % Place real zero 'zc' below desired pole 's'
  zc = real(s);
  % Find real pole 'pc' using angle criterion
  th = sum(angle(s-pole(G))) - sum(angle(s-zero(G))) - angle(s-zc);
  th = mod(th-pi, 2*pi);
  pc = zc + imag(s)/tan(th);
  % Find gain 'Kc' using magnitude criterion
  K  = prod(abs(s-pole(G)))/prod(abs(s-zero(G))) * abs(s-pc)/abs(s-zc);
  Kc = K;
end % function
```

The magnitude criterion given in (2) was used to find the controller gain corresponding to this desired pole, which can be seen on lines 9 and 10, but this was not the DC gain used in the final controller. The DC gain was varied in MATLAB® `sisotool()` () to find the limiting gains which met the performance specification for each system. The median gains were chosen for the optimum controllers.

A.3   PID Tuning

Figure 9 captures how precision is the only limitation to the performance of the PD controller for a type I system, where steady-state error is not a concern.
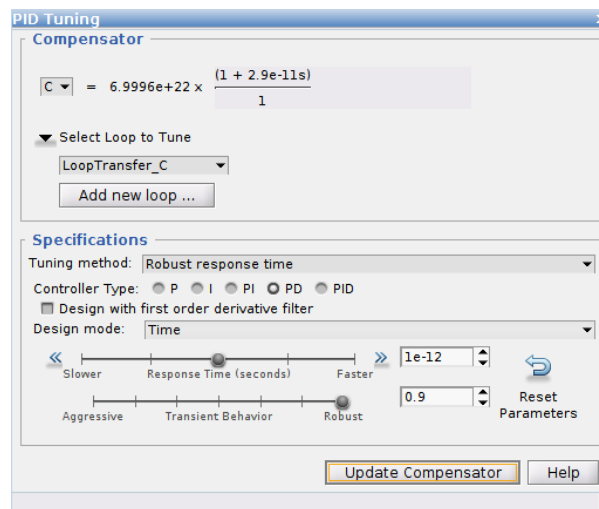

FIGURE 9: Demonstration of the arbitrary precision choice when tuing PID controllers using MATLAB® `sisotool()`.

So P, PD and PID controllers could be compared, the default 'response time' parameter was used—0.1836 s. Transient behaviour was varied until the overshoot was similar to that achieved by the gain and phase-lead compensators. This process can be seen in Figure 10
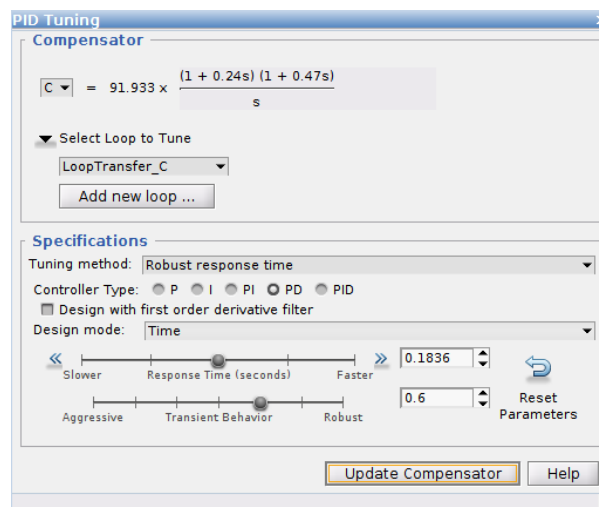

FIGURE 10: Tuning PID controllers with fixed 'response time' using MATLAB® `sisotool()`.