$$\boxed{\textbf{Fall, 2016} \quad \textbf{CIS 515}}$$

Fundamentals of Linear Algebra and Optimization
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Homework 4

October 27, 2016; Due November 8, 2016, beginning of class

Problem B1 (50 pts).

(1)Let's define matrix A as : $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ Then, we get: $a_{11} + a_{12} = c_1 = a_{21} + a_{22}$ and $a_{11} + a_{21} = c_2 = a_{12} + a_{22}$. We can rewrite the equations as:

$$a_{11} + a_{12} - a_{21} - a_{22} = 0 \tag{1}$$

$$a_{11} + a_{21} - a_{12} - a_{22} = 0 \tag{2}$$

From part d of the duality theorem (3.14), we know that

$$dim(U) + dim(U^0) = dim(E)$$

$dim(E)$ is n^2 or 4 and $dim(U)$ is $2(n-1)$ or 2 since there are two linearly indepedent equations. Then, $dim(U^0)$ is $4 - 2 = 2$.

(2) The equations corresponding to the three conditions listed (sum of entries in each row are the same, sum of entries in each column are the same, $c_1 = c_2$) are the following:

$$a_{11} + a_{12} - a_{21} - a_{22} = 0 \tag{3}$$

$$a_{11} + a_{21} - a_{12} - a_{22} = 0 \tag{4}$$

$$a_{12} - a_{21} = 0 \tag{5}$$

$$a_{11} - a_{22} = 0 \tag{6}$$

$$a_{22} - a_{11} = 0 \tag{7}$$

$$a_{21} - a_{12} = 0 \tag{8}$$

It is clear that only the first two equations are linearly independent so $dim(U) = 2$ and the dimension of the subspace of 2×2 matrices is still 2 as it is in section (1) of this problem.

From equations (5) and (6), we get that every such matrix is of the form $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ and a basis for the subspace consists of 4 matrices which are $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$

(3) Using the duality theorem again, we get: $n^2 - 2(n-1) = 9 - 2(3-1) = 5$ as the dimension of the subspace of the matrices A. The set of equations are given as

$$\begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{21} \\ a_{22} \\ a_{23} \\ a_{31} \\ a_{32} \\ a_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

For example, by looking at the equation corresponding to the first row, we have: $a_{11} + a_{12} + a_{13} = c_1$ and $a_{21} + a_{22} + a_{23} = c_1$ so $a_{11} + a_{12} + a_{13} - a_{21} - a_{22} - a_{23} = 0$. The same reasoning applies for all 5 of the equations represented by the matrix.
In rref , the matrix becomes:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which shows 4 linearly independent equations corresponding to the $2(n-1)$ term in $n^2 - 2(n-1)$. We can find a basis for the kernel of this matrix by applying the algorithm from the notes. So then,

$$BK = \begin{pmatrix} 1 & 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 & 0 \\ -1 & -1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We arrive at 5 matrices that form the basis which are:

$$M_1 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} M_2 = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} M_3 = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} M_4 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} M_5 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Problem B2 (10 pts).
Because A is symmetric and positive definite then $\forall x \neq 0,\ x^\top (B^\top AB)x = (Bx)^\top A(Bx)$,

since B is invertible then $Bx \neq 0$, thus it comes $(Bx)^\top A(Bx) > 0$, which means $B^\top AB$ is also positive definite.
Conversely, when $B^\top AB$ is positive definite, then it can be written as $B^\top AB = DD^\top \Rightarrow A = (B^\top)^{-1}D((B^\top)^{-1}D)^\top$. Since B is invertible then $(B^\top)^{-1}D$ is also invertible ,which means that A is positive definite.

Problem B3 (100 pts).
(1)

$$SAA^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & ad-bc \end{pmatrix} A^{-1}$$

$$S = \begin{pmatrix} 1 & 0 \\ 0 & ad-bc \end{pmatrix} A^{-1}$$

To find A^{-1}, we use Gauss-Jordan elimination on $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We perform the following row operations:
1) $R_1/a \to R_1$
2) $-cR_1 + R_2 \to R_2$
3) $R_2/(d - \frac{cb}{a})$
4) $-\frac{b}{a}R_2 + R_1 \to R_1$
So $A^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$ and $S = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ -c & a \end{pmatrix}$
Indeed S is the product of at most 4 elementary matrices as we find that:

$$E_{12;\frac{-b}{d}} E_{22;(\frac{bc}{d}+a)} E_{21;c} E_{11;\frac{ad-bc}{d}} S = I$$

Then

$$S = E^{-1}_{12;\frac{-b}{d}} E^{-1}_{22;(\frac{bc}{d}+a)} E^{-1}_{21;c} E^{-1}_{11;\frac{ad-bc}{d}}$$

For $A = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ the row operations to get the identity matrix are the following:
1. $R_1/a \to R_1$
2. $R_2/a^{-1} \to R_2$

so $E_{22;a^{-1}} E_{11;a} A = I$ and $A = E^{-1}_{11;a} E^{-1}_{22;a^{-1}}$.
When $a = -1$, the decomposition becomes:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = A$$

(2)

$$R = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} uv+1 & 2u+u^2v \\ v & uv+1 \end{pmatrix} = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$$

So then it is clear that $v = sin\theta$ and $u = \frac{cos\theta - 1}{sin\theta} = -tan\frac{\theta}{2}$

(3)

(a) Write $A = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_i \\ \vdots \\ \alpha_j \\ \vdots \\ \alpha_n \end{pmatrix}$, $E = \begin{pmatrix} e_1 \\ \vdots \\ e_i \\ \vdots \\ e_j \\ \vdots \\ e_n \end{pmatrix}$, where e_i is the basis vector.

Choose $E_1(i,j) = \begin{pmatrix} e_1 \\ \vdots \\ e_i \\ \vdots \\ e_j - e_i \\ \vdots \\ e_n \end{pmatrix}$, then $A_1 = E_1(i,j)A = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_i \\ \vdots \\ \alpha_j - \alpha_i \\ \vdots \\ \alpha_n \end{pmatrix}$. Choose $E_2(i,j) = \begin{pmatrix} e_1 \\ \vdots \\ e_i + e_j \\ \vdots \\ e_j \\ \vdots \\ e_n \end{pmatrix}$, then $A_2 = E_2(i,j)A_1 = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_j \\ \vdots \\ \alpha_j - \alpha_i \\ \vdots \\ \alpha_n \end{pmatrix}$. And $A_3 = E_1(i,j)A_2 = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_j \\ \vdots \\ -\alpha_i \\ \vdots \\ \alpha_n \end{pmatrix}$, $A_4 =$ $E_{j,-1}A_3 = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_j \\ \vdots \\ \alpha_i \\ \vdots \\ \alpha_n \end{pmatrix}$, thus $E_{j,-1}E_1(i,j)E_2(i,j)E_1(i,j)A = P(i,j)A$.

Since $E_{j,-1}E_{j,-1} = I$, multiply $E_{j,-1}$ on both sides, we get $E_1(i,j)E_2(i,j)E_1(i,j)A = E_{j,-1}P(i,j)A$.

(b) Write $E_{n,d} = \begin{pmatrix} e_1 \\ \vdots \\ de_n \end{pmatrix}$, choose $E_3(i,d) = \begin{pmatrix} e_1 \\ \vdots \\ e_i - \frac{1-d}{d^2} e_n \\ \vdots \\ e_n \end{pmatrix}$, $E_4(i,d) = \begin{pmatrix} e_1 \\ \vdots \\ e_i \\ \vdots \\ e_n - de_i \end{pmatrix}$, $E_5(i,d) = \begin{pmatrix} e_1 \\ \vdots \\ e_i + \frac{1-d}{d} e_n \\ \vdots \\ e_n \end{pmatrix}$, $E_6(i,d) = \begin{pmatrix} e_1 \\ \vdots \\ e_i \\ \vdots \\ e_n + e_i \end{pmatrix}$. Then we can easily find that

$$E^{(1)}_{n,d} = E_3(i,d)E_{n,d} = \begin{pmatrix} e_1 \\ \vdots \\ e_i - \frac{1-d}{d} e_n \\ \vdots \\ de_n \end{pmatrix}$$

$$E^{(2)}_{n,d} = E_4(i,d)E^{(1)}_{n,d} = \begin{pmatrix} e_1 \\ \vdots \\ e_i - \frac{1-d}{d} e_n \\ \vdots \\ -de_i + e_n \end{pmatrix}$$

$$E^{(3)}_{n,d} = E_5(i,d)E^{(2)}_{n,d} = \begin{pmatrix} e_1 \\ \vdots \\ de_i \\ \vdots \\ -de_i + e_n \end{pmatrix}$$

$$E^{(4)}_{n,d} = E_6(i,d)E^{(3)}_{n,d} = \begin{pmatrix} e_1 \\ \vdots \\ de_i \\ \vdots \\ e_n \end{pmatrix}$$

As a conclusion, $E_6(i,d)E_5(i,d)E_4(i,d)E_3(i,d)E_{n,d} = E_{i,d}$.
When $d = -1$, $E_3(i,d) = E_5(i,d)$ $E_4(i,d) = E_6(i,d)$.
(c) We can see that all the permutation matrix can be written as $P = \sum_{i,j} P(i,j)$, and from above we prove that $P(i,j)$ can be written as the product of elementary operation of the

form $E_{k,\ell;\beta}$ and $E_{n,-1}$, such that $E_{j,-1}E_1(i,j)E_2(i,j)E_1(i,j) = P(i,j)$, so it is true for all permutation matrices.

(4) First, we can use the Gaussion elimination, such that a series of elementary matrices $(\prod_i E^i_{k,\ell;\beta})A = U$. Since A is invertible, we get $\prod_i u_{ii} \neq 0$, use $E_{i,u_{ii}^{-1}}$ to U, then we can get an upper-triangle matrix whose diagonal entries (except the last row) are 1, such that $(\prod_i E_{i,u_{ii}^{-1}})(\prod_{(i)} E^i_{k,\ell;\beta})A = \begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 0 & 1 & \cdots & \vdots \\ \vdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$. Then apply a series of operations to it to make $a_{ij} = 0, \forall j > i, i \neq n$, such that $SA = (\prod_i E^{*(i)}_{k,\ell;\beta})(\prod_i E_{i,u_{ii}^{-1}})(\prod_{(i)} E^i_{k,\ell;\beta})A = \begin{pmatrix} I_{n-1} & 0 \\ 0 & d \end{pmatrix}$, since $\det(S)\det(A) = a_{nn} \Rightarrow d = \det(A)$.

Write $A = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ we can use induction,

when $n = 1$, we can find an entry in the first column such that $a_{i1} \neq 0$, otherwise $\det(A) = 0$, which is a contradiction. And then we use $E_{1,i,\frac{a_{11}-1}{a_{i1}}} = \begin{pmatrix} e_1 - \frac{a_{11}-1}{a_{i1}} e_i \\ \vdots \\ e_n \end{pmatrix}$, which makes the first entry $a_{11} = 1$, after a series operations such that $E_{1,j,a_{j1}} = \begin{pmatrix} e_1 \\ \vdots \\ e_j - a_{j1}e_1 \\ \vdots \\ e_n \end{pmatrix}$, we can make the other entries $a_{i1} = 0\ \forall i \neq 1$. These need n steps, however if a_{11} is the only non zero entry then we need another transportation matrix P to permute the first row to some row in the first column, thus we need $n+1$ matrices at most to get $A_1 = \begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \cdots & \cdots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{pmatrix}$.

It is the same when $n > 1$, while when we get the matrix $A_{n-1} = \begin{pmatrix} 1 & 0 & \cdots & a_{1n} \\ 0 & 1 & \cdots & a_{2n} \\ \vdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$, because A is invertible, comes the result that $a_{nn} = \det(A) = 1$, thus we only need $n-1$ matrices to get $A_n = I$.

As a conclusion, the total transformation is $(n+1)(n-1) + (n-1) = n(n+1) - 2$ at most.

Extra Credit (20 points). Prove that every matrix in $\mathbf{SL}(n)$ can be written as a product of at most $(n-1)(\max\{n,3\}+1)$ elementary matrices of the form $E_{k,\ell;\beta}$.

Problem B4 (50 pts).
We can use the induction to show that after k round Gaussian elimination, such that $A^* = \begin{pmatrix} U & V \\ \mathbf{0} & A^{(k)} \end{pmatrix}$, $A^{(k)}$ is strictly column diagonally dominant and it does not require pivoting.
Suppose $A = \begin{pmatrix} \alpha & u \\ v & B \end{pmatrix}$,
When $k=1$, there comes the Gaussian elimination which makes $A = \begin{pmatrix} 1 & 0 \\ \frac{v}{\alpha} & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & B - \frac{uv}{\alpha} \end{pmatrix}$.
Therefore $A^{(1)} = B - \frac{uv}{\alpha}$. We need to prove $A^{(1)}$ is also strictly column diagonally dominant:

$$|a^{(1)}_{j\,j}| > \sum_{i=1,\, i\neq j}^{n} |a^{(1)}_{i\,j}|, \quad \text{for } j = 1, \ldots, n$$

Let $a^{(1)}_{ij}$ be the i,j entry of $A^{(1)}$, then

$$\begin{aligned} a^{(1)}_{ij} &= b_{ij} - \frac{u_j v_i}{\alpha} \\ \sum_{i\geq 2, i\neq j} |a^{(1)}_{ij}| &= \sum_{i\geq 2, i\neq j} |b_{ij} - \frac{u_j v_i}{\alpha}| \\ &\leq \sum_{i\geq 2, i\neq j} |b_{ij}| + \frac{|u_j|}{\alpha} \sum_{i\geq 2, i\neq j} |v_i| \end{aligned}$$

Since A is strictly column diagonally dominant, it follows that

$$\begin{aligned} \sum_{i\geq 2, i\neq j} |a^{(1)}_{ij}| &< (|b_{jj}| - |u_j|) + \frac{|u_j|}{\alpha}(\alpha - |v_j|) \\ &= |b_{jj}| - \frac{|u_j|}{\alpha}|v_j| \\ &\leq |b_{jj} - \frac{u_j}{\alpha} v_j| \\ &= |a^{(1)}_{jj}| \end{aligned}$$

Thus $A^{(1)}$ is strictly column diagonally dominant, there is no need to pivot. We can easily use the same way to prove that when $k > 1$, every round of Gaussian elimination leads to a strictly column diagonally dominant $A^{(k)}$.
Because $|a_{jj}| > \sum_{i=1,\, i\neq j}^{n} |a^{(1)}_{i\,j}| \geq 0 \Rightarrow a_{jj} > 0$ and $\det(A) = \prod_i a_{ii}$, thus A is invertible.

Problem B5 (40 pts).
(1) Write the polynomial as $P(X) = a_0 + a_1 X + a_2 X^2 + \cdots a_m X^m$. Then from equations$P(\alpha_i) =$

$\beta_i, \quad 1 \le i \le m+1.$, we can get the matrix form such that

$$A = \begin{pmatrix} 1 & \alpha_1^1 & \alpha_1^2 & \cdots & \alpha_m^m \\ 1 & \alpha_2^1 & \alpha_2^2 & \cdots & \alpha_2^m \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ 1 & \alpha_{m+1}^1 & \alpha_{m+1}^2 & \cdots & \alpha_{m+1}^m \end{pmatrix}$$
$$X = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_m \end{pmatrix}$$
$$b = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{m+1} \end{pmatrix}$$
$$AX = b$$

Where we change (a_i) $0 \le i \le m$ as our variables. Then consider $\det(A)$, think about the Vandermonde Matrix, we can easily get $\det(A) = \prod_{1 \le i < j \le m}(a_j - a_i)$. And since $(\alpha_1, \ldots, \alpha_{m+1})$ is a sequence of pairwise distinct scalars, there comes $\det(A) \neq 0$, then we can get the unique solution of $X = A^{-1}b$, which means the coefficients of polynomial are unique, so P is unique.

(2)
(a)
If $i \neq j$, then $\prod_{k \neq i}(X - \alpha_k) = \prod_{k \neq i}(\alpha_j - \alpha_k) = 0 \Rightarrow L_i(\alpha_j) = 0$.
If $i = j$, then $\frac{\prod_{k \neq i}(X - \alpha_k)}{\prod_{k \neq i}(\alpha_i - \alpha_k)} = \frac{\prod_{k \neq i}(\alpha_i - \alpha_k)}{\prod_{k \neq i}(\alpha_i - \alpha_k)} = 1 \Rightarrow L_i(\alpha_j) = 1$.
Thus $L_i(\alpha_j) = \delta_{i\,j} \quad 1 \le i, j \le m+1$.
(b)
The degree of $L_i(X)$ is m and $P(\alpha_i) = \sum_j \beta_j L_j(\alpha_i) = \sum_j \beta_j \delta_{i\,j} = \beta_i$. So the only thing we need to do is to prove the uniqueness of $P(X)$.
Suppose there is another polynomial $Q(X)$ satisfying our equations $P(\alpha_i) = \beta_i, \quad 1 \le i \le m+1$, then it must lead to the conclusion that $H(X) = Q(X) - P(X)$ is zero at α_i, $1 \le i \le m+1$. However, according to Fundamental Theorem of Algebra, $H(X)$ is at most m degree and can has at most m zero points unless it is identically zero polynomial. Therefore, $H(X)$ is the identically zero polynomial and $P(X) \equiv Q(X)$, which means that $P(X)$ is unique.

(3) Prove that $L_1(X), \ldots, L_{m+1}(X)$ are lineary independent, and that they form a basis of all polynomials of degree at most m.

How is 1 (the constant polynomial 1) expressed over the basis $(L_1(X), \ldots, L_{m+1}(X))$?

Give the expression of every polynomial $P(X)$ of degree at most m over the basis $(L_1(X), \ldots, L_{m+1}(X))$.

(4) Prove that the dual basis $(L_1^*, \ldots, L_{m+1}^*)$ of the basis $(L_1(X), \ldots, L_{m+1}(X))$ consists of the linear forms L_i^* given by

$$L_i^*(P) = P(\alpha_i),$$

for every polynomial P of degree at most m; this is simply *evaluation at* α_i.

Problem B6 (60 pts). (a) Find a lower triangular matrix E such that

$$E \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix}.$$

(b) What is the effect of the product (on the left) with

$$E_{4,3;-1}E_{3,2;-1}E_{4,3;-1}E_{2,1;-1}E_{3,2;-1}E_{4,3;-1}$$

on the matrix

$$Pa_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix}.$$

(c) Find the inverse of the matrix Pa_3.

(d) Consider the $(n+1) \times (n+1)$ Pascal matrix Pa_n whose ith row is given by the binomial coefficients

$$\binom{i-1}{j-1},$$

with $1 \leq i \leq n+1$, $1 \leq j \leq n+1$, and with the usual convention that

$$\binom{0}{0} = 1, \quad \binom{i}{j} = 0 \quad \text{if} \quad j > i.$$

The matrix Pa_3 is shown in question (c) and Pa_4 is shown below:

$$Pa_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix}.$$

Find n elementary matrices $E_{i_k,j_k;\beta_k}$ such that

$$E_{i_n,j_n;\beta_n} \cdots E_{i_1,j_1;\beta_1} Pa_n = \begin{pmatrix} 1 & 0 \\ 0 & Pa_{n-1} \end{pmatrix}.$$

Use the above to prove that the inverse of Pa_n is the lower triangular matrix whose ith row is given by the signed binomial coefficients

$$(-1)^{i+j-2}\binom{i-1}{j-1},$$

with $1 \leq i \leq n+1$, $1 \leq j \leq n+1$. For example,

$$Pa_4^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ -1 & 3 & -3 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix}.$$

Hint. Given any $n \times n$ matrix A, multiplying A by the elementary matrix $E_{i,j;\beta}$ *on the right* yields the matrix $AE_{i,j;\beta}$ in which β times the ith column is added to the jth column.

Problem B7 (30 pts). Given any two subspaces V_1, V_2 of a finite-dimensional vector space E, prove that

$$\begin{aligned}(V_1 + V_2)^0 &= V_1^0 \cap V_2^0 \\ (V_1 \cap V_2)^0 &= V_1^0 + V_2^0.\end{aligned}$$

Beware that in the second equation, V_1 and V_2 are subspaces of E, not E^*.

Hint. To prove the second equation, prove the inclusions $V_1^0 + V_2^0 \subseteq (V_1 \cap V_2)^0$ and $(V_1 \cap V_2)^0 \subseteq V_1^0 + V_2^0$. Proving the second inclusion is a little tricky. First, prove that we can pick a subspace W_1 of V_1 and a subspace W_2 of V_2 such that

1. V_1 is the direct sum $V_1 = (V_1 \cap V_2) \oplus W_1$.
2. V_2 is the direct sum $V_2 = (V_1 \cap V_2) \oplus W_2$.
3. $V_1 + V_2$ is the direct sum $V_1 + V_2 = (V_1 \cap V_2) \oplus W_1 \oplus W_2$.

TOTAL: 340 + 20 points.