Fall, 2016 CIS 515

Fundamentals of Linear Algebra and Optimization
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Homework 4

October 27, 2016; Due November 8, 2016, beginning of class

Problem B1 (50 pts).

(1)Let's define matrix A as : $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ Then, we get: $a_{11} + a_{12} = c_1 = a_{21} + a_{22}$ and $a_{11} + a_{21} = c_2 = a_{12} + a_{22}$. We can rewrite the equations as:

$$a_{11} + a_{12} - a_{21} - a_{22} = 0 \tag{1}$$

$$a_{11} + a_{21} - a_{12} - a_{22} = 0 \tag{2}$$

From part d of the duality theorem (3.14), we know that

$$dim(U) + dim(U^0) = dim(E)$$

$dim(E)$ is n^2 or 4 and $dim(U)$ is $2(n-1)$ or 2 since there are two linearly indepedent equations. Then, $dim(U^0)$ is $4 - 2 = 2$.

(2) Prove that the dimension of the subspace of 2×2 matrices A, such that the sum of the entries of every row is the same (say c_1), the sum of entries of every column is the same (say c_2), and $c_1 = c_2$, is also 2. Prove that every such matrix is of the form

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix},$$

and give a basis for this subspace.

The equations corresponding to the three conditions listed (sum of entries in each row are the same, sum of entries in each column are the same, $c_1 = c_2$) are the following:

$$a_{11} + a_{12} - a_{21} - a_{22} = 0 \tag{3}$$

$$a_{11} + a_{21} - a_{12} - a_{22} = 0 \tag{4}$$

$$a_{12} - a_{21} = 0 \tag{5}$$

$$a_{11} - a_{22} = 0 \tag{6}$$

$$a_{22} - a_{11} = 0 \tag{7}$$

$$a_{21} - a_{12} = 0 \tag{8}$$

It is clear that only the first two equations are linearly independent so $dim(U) = 2$ and the dimension of the subspace of 2×2 matrices is still 2 as it is in section (1) of this problem.

From equations (5) and (6), we get that every such matrix is of the form $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ and a basis for the subspace consists of 4 matrices which are $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$

(3) Prove that the dimension of the subspace of 3×3 matrices A, such that the sum of the entries of every row is the same (say c_1), the sum of entries of every column is the same (say c_2), and $c_1 = c_2$, is 5. Begin by showing that the above constraints are given by the set of equations

$$\begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{21} \\ a_{22} \\ a_{23} \\ a_{31} \\ a_{32} \\ a_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Prove that every matrix satisfying the above constraints is of the form

$$\begin{pmatrix} a+b-c & -a+c+e & -b+c+d \\ -a-b+c+d+e & a & b \\ c & d & e \end{pmatrix},$$

with $a, b, c, d, e \in \mathbb{R}$. Find a basis for this subspace. (Use the method to find a basis for the kernel of a matrix).

Using the duality theorem again, we get: $n^2 - 2(n-1) = 9 - 2(3-1) = 5$ as the dimension of the subspace of the matrices A. The set of equations are given as

$$\begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{21} \\ a_{22} \\ a_{23} \\ a_{31} \\ a_{32} \\ a_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

For example, by looking at the equation corresponding to the first row, we have: $a_{11}+a_{12}+a_{13}=c_1$ and $a_{21}+a_{22}+a_{23}=c_1$ so $a_{11}+a_{12}+a_{13}-a_{21}-a_{22}-a_{23}=0$. The same reasoning applies for all 5 of the equations represented by the matrix.
In rref , the matrix becomes:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which shows 4 linearly independent equations corresponding to the $2(n-1)$ term in $n^2 - 2(n-1)$. We can find a basis for the kernel of this matrix by applying the algorithm from the notes. So then,

$$BK = \begin{pmatrix} 1 & 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 & 0 \\ -1 & -1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We arrive at 5 matrices that form the basis which are:

$$M_1 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} M_2 = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} M_3 = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} M_4 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} M_5 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Problem B2 (10 pts).
Because A is symmetric and positive definite then $\forall x \neq 0$, $x^\top(B^\top AB)x = (Bx)^\top A(Bx)$, since B is invertible then $Bx \neq 0$, thus it comes $(Bx)^\top A(Bx) > 0$, which means $B^\top AB$ is also positive definite.
Conversely, when $B^\top AB$ is positive definite, then it can be written as $B^\top AB = DD^\top \Rightarrow A = (B^\top)^{-1}D((B^\top)^{-1}D)^\top$. Since B is invertible then $(B^\top)^{-1}D$ is also invertible ,which means that A is positive definite.

Problem B3 (100 pts). (1) Let A be any invertible 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Prove that there is an invertible matrix S such that

$$SA = \begin{pmatrix} 1 & 0 \\ 0 & ad-bc \end{pmatrix},$$

where S is the product of at most four elementary matrices of the form $E_{i,j;\beta}$.

Conclude that every matrix A in $\mathbf{SL}(2)$ (the group of invertible 2×2 matrices A with $\det(A) = +1$) is the product of at most four elementary matrices of the form $E_{i,j;\beta}$.

For any $a \neq 0, 1$, give an explicit factorization as above for

$$A = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}.$$

What is this decomposition for $a = -1$?

(2) Recall that a rotation matrix R (a member of the group $\mathbf{SO}(2)$) is a matrix of the form

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Prove that if $\theta \neq k\pi$ (with $k \in \mathbb{Z}$), any rotation matrix can be written as a product

$$R = ULU,$$

where U is upper triangular and L is lower triangular of the form

$$U = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}.$$

Therefore, every plane rotation (except a flip about the origin when $\theta = \pi$) can be written as the composition of three shear transformations!

(3)

(a) Write $A = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_i \\ \vdots \\ \alpha_j \\ \vdots \\ \alpha_n \end{pmatrix}$, $E = \begin{pmatrix} e_1 \\ \vdots \\ e_i \\ \vdots \\ e_j \\ \vdots \\ e_n \end{pmatrix}$, where e_i is the basis vector.

Choose $E_1(i,j) = \begin{pmatrix} e_1 \\ \vdots \\ e_i \\ \vdots \\ e_j - e_i \\ \vdots \\ e_n \end{pmatrix}$, then $A_1 = E_1(i,j)A = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_i \\ \vdots \\ \alpha_j - \alpha_i \\ \vdots \\ \alpha_n \end{pmatrix}$. Choose $E_2(i,j) =$

$\begin{pmatrix} e_1 \\ \vdots \\ e_i + e_j \\ \vdots \\ e_j \\ \vdots \\ e_n \end{pmatrix}$, then $A_2 = E_2(i,j)A_1 = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_j \\ \vdots \\ \alpha_j - \alpha_i \\ \vdots \\ \alpha_n \end{pmatrix}$. And $A_3 = E_1(i,j)A_2 = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_j \\ \vdots \\ -\alpha_i \\ \vdots \\ \alpha_n \end{pmatrix}$, $A_4 = E_{j,-1}A_3 = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_j \\ \vdots \\ \alpha_i \\ \vdots \\ \alpha_n \end{pmatrix}$, thus $E_{j,-1}E_1(i,j)E_2(i,j)E_1(i,j)A = P(i,j)A$.

Since $E_{j,-1}E_{j,-1} = I$, multiply $E_{j,-1}$ on both sides, we get $E_1(i,j)E_2(i,j)E_1(i,j)A = E_{j,-1}P(i,j)A$.

(b) Write $E_{n,d} = \begin{pmatrix} e_1 \\ \vdots \\ de_n \end{pmatrix}$, choose $E_3(i,d) = \begin{pmatrix} e_1 \\ \vdots \\ e_i - \frac{1-d}{d^2}e_n \\ \vdots \\ e_n \end{pmatrix}$, $E_4(i,d) = \begin{pmatrix} e_1 \\ \vdots \\ e_i \\ \vdots \\ e_n - de_i \end{pmatrix}$, $E_5(i,d) =$

$$\begin{pmatrix} e_1 \\ \vdots \\ e_i + \frac{1-d}{d} e_n \\ \vdots \\ e_n \end{pmatrix}, \ E_6(i,d) = \begin{pmatrix} e_1 \\ \vdots \\ e_i \\ \vdots \\ e_n + e_i \end{pmatrix}.$$ Then we can easily find that

$$E^{(1)}_{n,d} = E_3(i,d)E_{n,d} = \begin{pmatrix} e_1 \\ \vdots \\ e_i - \frac{1-d}{d} e_n \\ \vdots \\ de_n \end{pmatrix}$$

$$E^{(2)}_{n,d} = E_4(i,d)E^{(1)}_{n,d} = \begin{pmatrix} e_1 \\ \vdots \\ e_i - \frac{1-d}{d} e_n \\ \vdots \\ -de_i + e_n \end{pmatrix}$$

$$E^{(3)}_{n,d} = E_5(i,d)E^{(2)}_{n,d} = \begin{pmatrix} e_1 \\ \vdots \\ de_i \\ \vdots \\ -de_i + e_n \end{pmatrix}$$

$$E^{(4)}_{n,d} = E_6(i,d)E^{(3)}_{n,d} = \begin{pmatrix} e_1 \\ \vdots \\ de_i \\ \vdots \\ e_n \end{pmatrix}$$

As a conclusion, $E_6(i,d)E_5(i,d)E_4(i,d)E_3(i,d)E_{n,d} = E_{i,d}$.
When $d = -1$, $E_3(i,d) = E_5(i,d)$ $E_4(i,d) = E_6(i,d)$.
(c) We can see that all the permutation matrix can be written as $P = \sum_{i,j} P(i,j)$, and from above we prove that $P(i,j)$ can be written as the product of elementary operation of the form $E_{k,\ell;\beta}$ and $E_{n,-1}$, such that $E_{j,-1}E_1(i,j)E_2(i,j)E_1(i,j) = P(i,j)$, so it is true for all permutation matrices.

(4) First, we can use the Gaussion elimination, such that a series of elementary matrices $(\prod_i E^i_{k,\ell;\beta})A = U$. Since A is invertible, we get $\prod_i u_{ii} \neq 0$, use $E_{i,u_{ii}^{-1}}$ to U, then we can get an upper-triangle matrix whose diagonal entries (except the last row) are 1, such that

$(\prod_i E_{i,u_{ii}^{-1}})(\prod_{(i)} E^i_{k,\ell;\beta})A = \begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 0 & 1 & \cdots & \vdots \\ \vdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$. Then apply a series of operations to it to make $a_{ij} = 0, \forall j > i, i \neq n$, such that $SA = (\prod_i E^{*(i)}_{k,\ell;\beta})(\prod_i E_{i,u_{ii}^{-1}})(\prod_{(i)} E^i_{k,\ell;\beta})A = \begin{pmatrix} I_{n-1} & 0 \\ 0 & d \end{pmatrix}$, since $\det(S)\det(A) = a_{nn} \Rightarrow d = \det(A)$

Prove that for every invertible $n \times n$ matrix A, there is a matrix S such that

$$SA = \begin{pmatrix} I_{n-1} & 0 \\ 0 & d \end{pmatrix} = E_{n,d},$$

with $d = \det(A)$, and where S is a product of elementary matrices of the form $E_{k,\ell;\beta}$.

In particular,every matrix in $\mathbf{SL}(n)$ (the group of invertible $n \times n$ matrices A with $\det(A) = +1$) can be written as a product of elementary matrices of the form $E_{k,\ell;\beta}$. Prove that at most $n(n+1) - 2$ such transformations are needed.

Extra Credit (20 points). Prove that every matrix in $\mathbf{SL}(n)$ can be written as a product of at most $(n-1)(\max\{n,3\}+1)$ elementary matrices of the form $E_{k,\ell;\beta}$.

Problem B4 (50 pts). A matrix, A, is called *strictly column diagonally dominant* iff

$$|a_{j\,j}| > \sum_{i=1,\, i\neq j}^{n} |a_{i\,j}|, \quad \text{for } j = 1, \ldots, n$$

Prove that if A is strictly column diagonally dominant, then Gaussian elimination with partial pivoting does not require pivoting, and A is invertible.

Problem B5 (40 pts). Let $(\alpha_1, \ldots, \alpha_{m+1})$ be a sequence of pairwise distinct scalars in $\mathbb{R}$ and let $(\beta_1, \ldots, \beta_{m+1})$ be any sequence of scalars in $\mathbb{R}$, not necessarily distinct.

(1) Prove that there is a unique polynomial P of degree at most m such that

$$P(\alpha_i) = \beta_i, \quad 1 \leq i \leq m+1.$$

Hint. Remember Vandermonde!

(2) Let $L_i(X)$ be the polynomial of degree m given by

$$L_i(X) = \frac{(X-\alpha_1)\cdots(X-\alpha_{i-1})(X-\alpha_{i+1})\cdots(X-\alpha_{m+1})}{(\alpha_i-\alpha_1)\cdots(\alpha_i-\alpha_{i-1})(\alpha_i-\alpha_{i+1})\cdots(\alpha_i-\alpha_{m+1})}, \quad 1 \leq i \leq m+1.$$

The polynomials $L_i(X)$ are known as *Lagrange polynomial interpolants*. Prove that

$$L_i(\alpha_j) = \delta_{i\,j} \quad 1 \leq i, j \leq m+1.$$

Prove that

$$P(X) = \beta_1 L_1(X) + \cdots + \beta_{m+1} L_{m+1}(X)$$

is the unique polynomial of degree at most m such that

$$P(\alpha_i) = \beta_i, \quad 1 \leq i \leq m+1.$$

(3) Prove that $L_1(X), \ldots, L_{m+1}(X)$ are lineary independent, and that they form a basis of all polynomials of degree at most m.

How is 1 (the constant polynomial 1) expressed over the basis $(L_1(X), \ldots, L_{m+1}(X))$?

Give the expression of every polynomial $P(X)$ of degree at most m over the basis $(L_1(X), \ldots, L_{m+1}(X))$.

(4) Prove that the dual basis $(L_1^*, \ldots, L_{m+1}^*)$ of the basis $(L_1(X), \ldots, L_{m+1}(X))$ consists of the linear forms L_i^* given by

$$L_i^*(P) = P(\alpha_i),$$

for every polynomial P of degree at most m; this is simply *evaluation at* α_i.

Problem B6 (60 pts). (a) Find a lower triangular matrix E such that

$$E \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix}.$$

(b) What is the effect of the product (on the left) with

$$E_{4,3;-1} E_{3,2;-1} E_{4,3;-1} E_{2,1;-1} E_{3,2;-1} E_{4,3;-1}$$

on the matrix

$$Pa_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix}.$$

(c) Find the inverse of the matrix Pa_3.

(d) Consider the $(n+1) \times (n+1)$ Pascal matrix Pa_n whose ith row is given by the binomial coefficients

$$\binom{i-1}{j-1},$$

with $1 \leq i \leq n+1$, $1 \leq j \leq n+1$, and with the usual convention that

$$\binom{0}{0} = 1, \quad \binom{i}{j} = 0 \quad \text{if} \quad j > i.$$

The matrix Pa_3 is shown in question (c) and Pa_4 is shown below:

$$Pa_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix}.$$

Find n elementary matrices $E_{i_k,j_k;\beta_k}$ such that

$$E_{i_n,j_n;\beta_n} \cdots E_{i_1,j_1;\beta_1} Pa_n = \begin{pmatrix} 1 & 0 \\ 0 & Pa_{n-1} \end{pmatrix}.$$

Use the above to prove that the inverse of Pa_n is the lower triangular matrix whose ith row is given by the signed binomial coefficients

$$(-1)^{i+j-2}\binom{i-1}{j-1},$$

with $1 \leq i \leq n+1$, $1 \leq j \leq n+1$. For example,

$$Pa_4^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ -1 & 3 & -3 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix}.$$

Hint. Given any $n \times n$ matrix A, multiplying A by the elementary matrix $E_{i,j;\beta}$ *on the right* yields the matrix $AE_{i,j;\beta}$ in which β times the ith column is added to the jth column.

Problem B7 (30 pts). Given any two subspaces V_1, V_2 of a finite-dimensional vector space E, prove that

$$(V_1 + V_2)^0 = V_1^0 \cap V_2^0$$
$$(V_1 \cap V_2)^0 = V_1^0 + V_2^0.$$

Beware that in the second equation, V_1 and V_2 are subspaces of E, not E^*.

Hint. To prove the second equation, prove the inclusions $V_1^0 + V_2^0 \subseteq (V_1 \cap V_2)^0$ and $(V_1 \cap V_2)^0 \subseteq V_1^0 + V_2^0$. Proving the second inclusion is a little tricky. First, prove that we can pick a subspace W_1 of V_1 and a subspace W_2 of V_2 such that

1. V_1 is the direct sum $V_1 = (V_1 \cap V_2) \oplus W_1$.
2. V_2 is the direct sum $V_2 = (V_1 \cap V_2) \oplus W_2$.
3. $V_1 + V_2$ is the direct sum $V_1 + V_2 = (V_1 \cap V_2) \oplus W_1 \oplus W_2$.

TOTAL: 340 + 20 points.