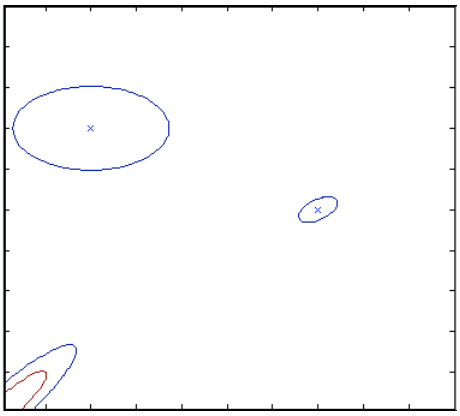
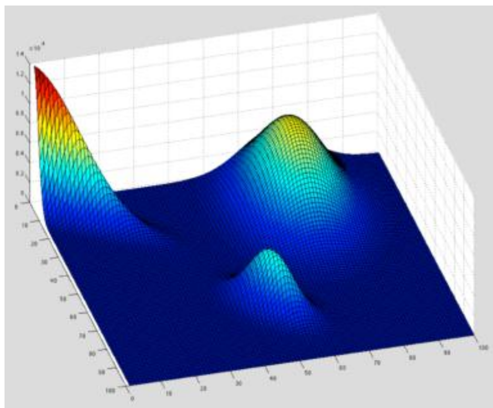
Gaussian Mixture Models (GMM)

$\boxed{L36}$

Goal: Learn how to do clustering registration without having to label points as assigned to one regressor.

Instead of representing clusters with regressors and computing cost as a sum of squared distances, we can model the pdf (probability density function) using **Gaussian kernels** and compute our cost as the **likelihood** of observing our data.





We denote the entire GMM as ϕ , representing each Gaussian component with μ and Σ.

$$N(I; \mu, \Sigma) = \frac{1}{\sqrt{(2\pi)^D |\Sigma|}} e^{-\frac{1}{2}(I-\mu)^T \Sigma^{-1} (I-\mu)}$$

I is our ensemble of images

Then, the probability of observing I from that pdf is $N(I; \mu, \Sigma)$.

The likelihood of observing I from the entire GMM is

$$p(I|\phi) = \sum_{k=1}^{K} \pi_k N(I; \mu_k, \Sigma_k)$$

weight of component k, where $\sum_k \pi_k = 1$.

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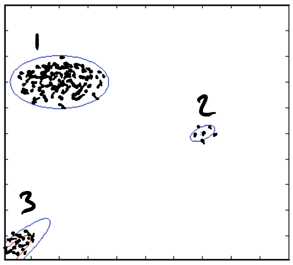
eg.



$\pi_1 = 0.8$

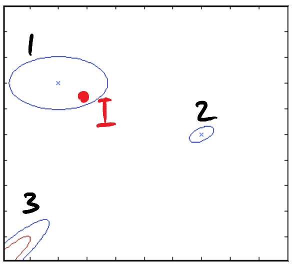
$\pi_2 = 0.15$

$\pi_3 = 0.05$

Generally speaking, π_k is the fraction of points that the k^{th} component explains.

In addition, each scatter point has a membership that is split among all the different Gaussian components.

eg.



"I" might have membership $(0.7, 0.1, 0.2)$

Then, the likelihood of observing ALL the pixels is the product of their individual probabilities,

$$L(I|\phi) = \prod_x p(I_x|\phi)$$

Note that image intensities are a function of the motion parameters θ . Thus, we wish to maximize

$$L(I|\theta,\phi) = \prod_x p(I_x^\theta|\phi)$$

It is actually more convenient to consider the log-likelihood,

$$\log L(I|\theta,\phi) = \sum_x \log p(I_x^\theta|\phi)$$

Density Estimation (finding ϕ)

Long story short… use estimation maximization (EM).

Iterate between these two steps...

Estimation step

$$\tau_{kx} = \frac{\pi_k \mathcal{N}(I_x^\theta|\phi_k)}{\sum_k \pi_k \mathcal{N}(I_x^\theta|\phi_k)}$$

Membership of pixel x to component k

Maximization step

$$\mu'_k = \frac{\sum_x \tau_{kx} I_x^\theta}{\sum_x \tau_{kx}},$$

Centroid of comp. k

$$\Sigma'_k = \frac{\sum_x \tau_{kx}(I_x^\theta - \mu'_k)(I_x^\theta - \mu'_k)^{\mathrm{T}}}{\sum_x \tau_{kx}}.$$

Covariance

$$\pi'_k = \frac{\sum_x \tau_{kx}}{\sum_k \sum_x \tau_{kx}}$$

Weight of comp. k

Motion Adjustment (finding θ)

Just like in calculus...

$$\frac{\partial}{\partial\theta} \log L(I|\theta,\phi) = 0$$

... crunch... crunch ... crunch ...

$$\left(\sum_x \frac{1}{p(I_x^\theta|\phi)} \sum_{k=1}^{K} \pi_k \mathcal{N}_k\left(I_x^\theta\right) \frac{\partial I_x^\theta}{\partial\theta} \Sigma_k^{-1} \frac{\partial I_x^\theta}{\partial\theta}^{\mathrm{T}}\right) \tilde{\theta} = \left(\sum_x \frac{1}{p(I_x^\theta|\phi)} \sum_{k=1}^{K} \pi_k \mathcal{N}_k\left(I_x^\theta\right) \frac{\partial I_x^\theta}{\partial\theta} \Sigma_k^{-1} \left(I_x^\theta - \mu_k\right)\right)$$

or simply

$$A\tilde{\theta} = b$$