Derivatives and Fourier Theory

L23

Goal: To find out how image derivatives relate to the Fourier transform.

Recall the inverse Fourier transform,

$$f(x) = \int F(\omega)\, e^{2\pi i \omega x}\, d\omega$$

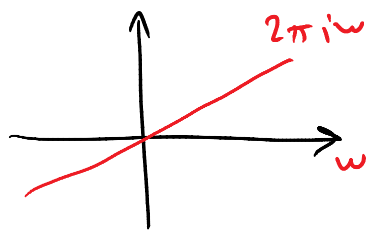
Now take its derivative.

$$\frac{df(x)}{dx} = \int F(\omega)\,(2\pi i \omega)\, e^{2\pi i \omega x}\, d\omega$$

$$= \int \big(2\pi i \omega F(\omega)\big)\, e^{2\pi i \omega x}\, d\omega$$

Hence, another way to compute the derivative is by manipulating the FT by

$$\frac{df(x)}{dx} = \mathcal{F}^{-1}\left\{ 2\pi i \omega F(\omega) \right\}$$



This can be approximated on discrete signals.

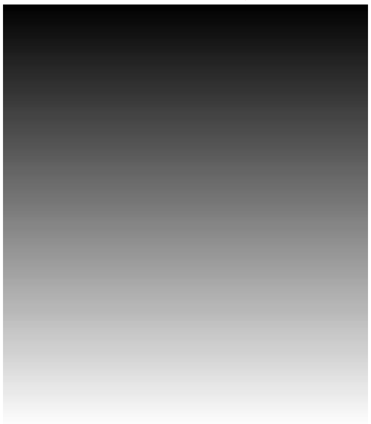
$$\frac{df(x_n)}{dx} \approx \mathrm{IDFT}\left\{ 2\pi i \overset{\omega}{\overset{\downarrow}{\frac{k}{N}}} F_k \right\}_n$$

Example: (See Matlab demo)

ω (omega)

$\frac{\partial f}{\partial r}$ (via FFT)

ω (omega) $\frac{\partial}{\partial r}$ (via FFT)



Now consider the discrete case: Central differencing

$$\frac{\partial f_n}{\partial x} = (f_{n+1} - f_{n-1})\frac{1}{2}$$

$$\mathcal{F}\left\{\frac{\partial f_n}{\partial x}\right\} = \left[\mathcal{F}\{f_{n+1}\} - \mathcal{F}\{f_{n-1}\}\right]\frac{1}{2}$$

Because the FT is linear

$$= \left[e^{\frac{2\pi i k}{N}} F_k - e^{\frac{-2\pi i k}{N}} F_k\right]\frac{1}{2}$$

$$= \left(e^{\frac{2\pi i k}{N}} - e^{\frac{-2\pi i k}{N}}\right)\frac{F_k}{2}$$

Let $\alpha = \frac{2\pi k}{N}$

$$= \left(\cancel{\cos\alpha} + i\sin\alpha - \cancel{\cos(-\alpha)} + i\sin\alpha\right) F_k \frac{1}{2}$$

$$= i\sin\frac{2\pi k}{N} F_k$$

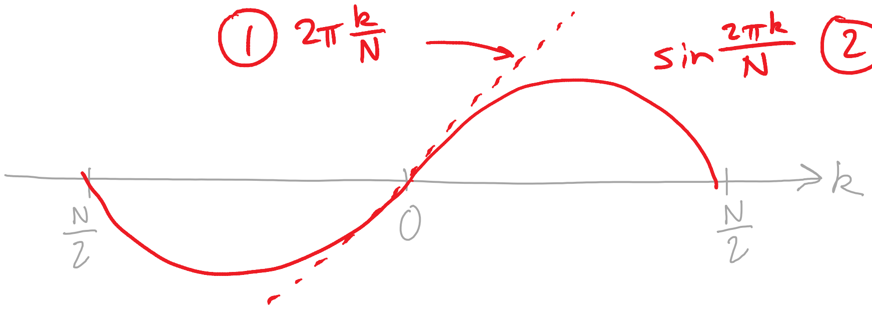
Summary of Derivatives & the Fourier Transform:

Continuous derivative → discrete approx.

$$\mathrm{DFT}\left\{\frac{df(x_n)}{dx}\right\}_k \simeq 2\pi i \frac{k}{N} F_k \qquad (1)$$

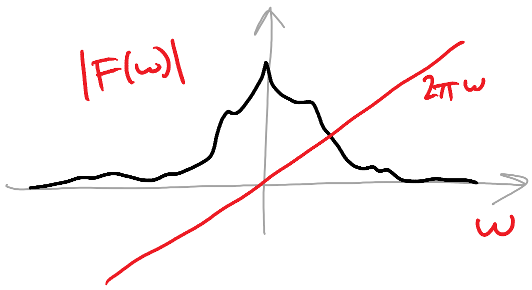
Discrete approx of derivative

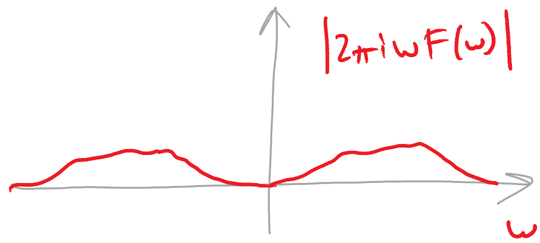
$$\mathrm{DFT}\left\{ \frac{df(x_n)}{dx} \right\}_k \approx i \sin \frac{2\pi k}{N} F_k \qquad (2)$$



Can you guess how noise will impact $\frac{df(x)}{dx}$?

Consider $F(\omega) = \mathcal{F}\{f(x)\}(\omega)$.
The derivative is like a high-pass filter, emphasizing the high frequencies, and de-emphasizing the low frequencies.

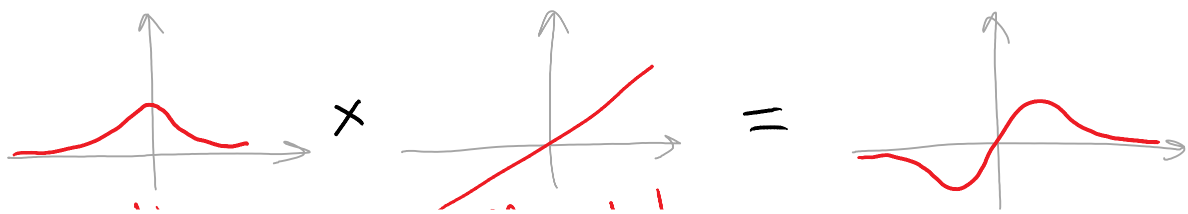


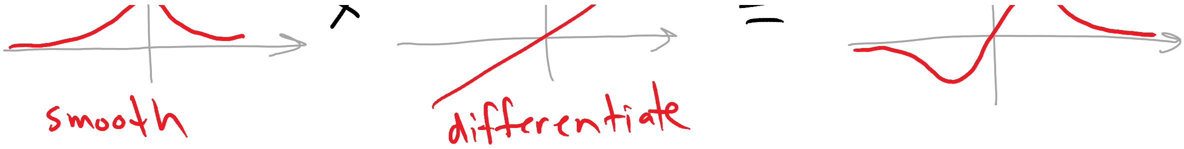


The noise tends to be amplified even though the overall signal is dampened.

For this reason, people often smooth the image before taking the derivative.

If we smooth with a Gaussian, then take the derivative...





Because both smoothing (by convolution) and differentiation are linear operations, the order they are applied does not matter.

ie. $\frac{d}{dx}(f*g) = \frac{df}{dx} * g = f * \frac{dg}{dx}$

(since the $\frac{d}{dx}$ operator is odd-symmetric...)

Both can be applied by convolving f with $\frac{dg}{dx}$.

(see Matlab demo)

Blurred df/dr