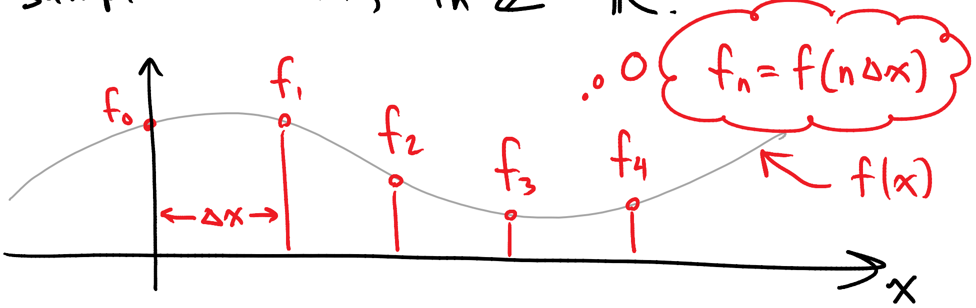
Sampling Theory

L07

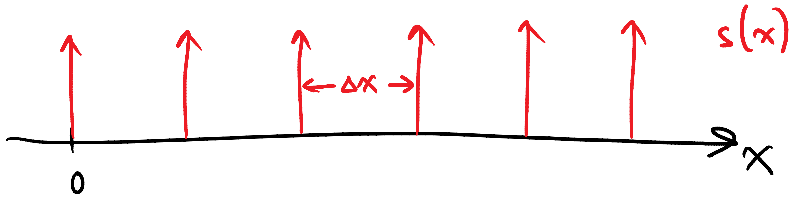
Goal: Images can be thought of as sampled versions of 2D functions. We want to develop a framework for understanding the relationship between a function and its samples.

Consider a continuous-domain function $f: \mathbb{R} \to \mathbb{R}$, and samples of it, $f_n: \mathbb{Z} \to \mathbb{R}$.



To sample f, we multiply by the Shah function. (or "comb")

$$s(x) = \sum_{n \in \mathbb{Z}} \delta(x - n\Delta x)$$



Then, the sampled version of f can be written

$$\bar{f}(x) = f(x)\, s(x) = f(x) \sum_{n \in \mathbb{Z}} \delta(x - n\Delta x)$$

Consider the FT of $\bar{f}(x)$.

$$\mathcal{F}\{\bar{f}(x)\}(\omega) = \mathcal{F}\{s(x)f(x)\}(\omega) = (S*F)(\omega)$$

What is $S(\omega)$?

$$S(\omega) = \mathcal{F}\{s(x)\}(\omega)$$

$$= \int_{-\infty}^{\infty} s(x)\, e^{-2\pi i \omega x}\, dx$$

$$= \int_{-\infty}^{\infty} \sum_n \delta(x - n\Delta x)\, e^{-2\pi i \omega x}\, dx$$

$$= \sum_n \underbrace{\int_{-\infty}^{\infty} \delta(x - n\Delta x)\, e^{-2\pi i \omega x}\, dx}$$ (after swapping Σ and $\int$)

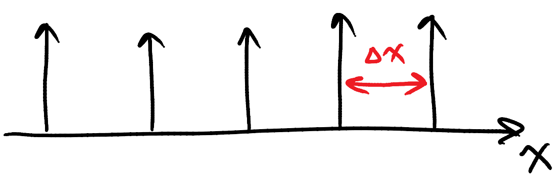
$$= \sum_{n \in \mathbb{Z}} \underbrace{e^{-2\pi i \omega (n\Delta x)}}$$

$= 1$ when $\omega \Delta x = k \in \mathbb{Z} \Rightarrow \omega = \frac{k}{\Delta x},\ k \in \mathbb{Z}$

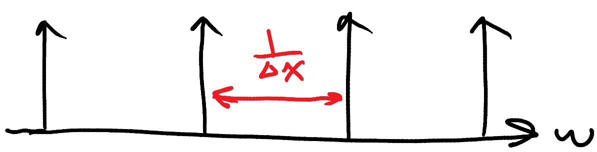
POOF $$= \sum_{k \in \mathbb{Z}} \delta\left(\omega - \frac{k}{\Delta x}\right)$$

Thus, the FT of the Shah function is also a Shah function, but with different spacing.

$s(x)$

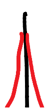


$S(\omega)$

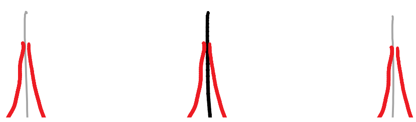


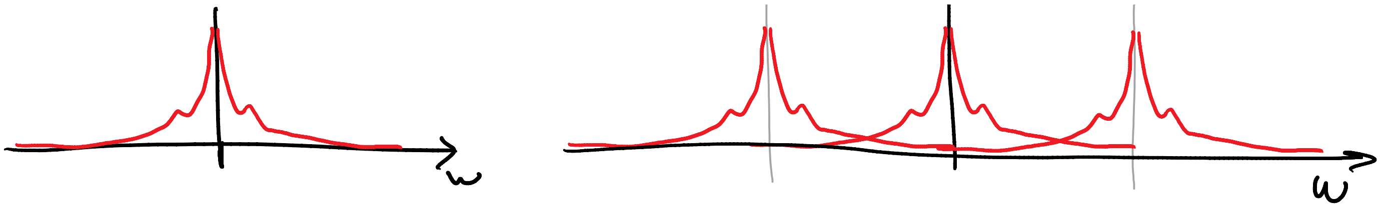
Ok, back to $\mathcal{F}\{f(x)s(x)\}(\omega) = (S*F)(\omega)$

$F(\omega) = \mathcal{F}\{f(x)\}(\omega)$



$(S*F)(\omega)$





So, sampling f gives us a periodic FT.

Likewise, a similar derivation can be used to show that a periodic f yields a discrete FT.

$f(x)$	$F(\omega)$
sampled	periodic
periodic	sampled
periodic & sampled	periodic & sampled

$\boxed{\text{TASK: Do the short quiz in D2L.}}$