Properties of the FT

L09

Goal: To survey some of the properties of the Fourier transform that are useful in image processing (beyond the convolution theorem).

But first, a quick look at what a 2D Fourier coefficient means.

$\boxed{\text{Demo: L09\_wavefront.m}}$

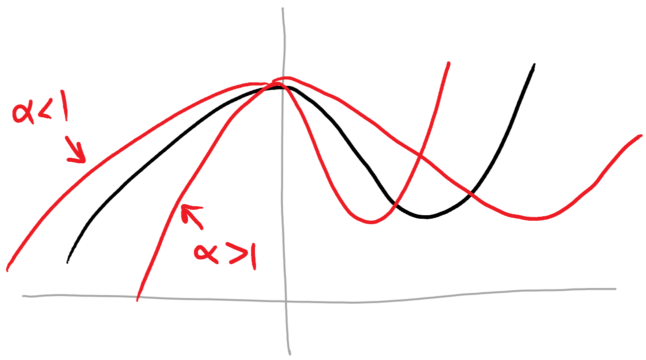
Scale Property

What happens to $F(\omega)$ if we scale f?

Let $\bar{f}(x) = f(\alpha x)$

$$\bar{F}(\omega) = \int_{-\infty}^{\infty} \bar{f}(x) e^{-2\pi i \omega x} dx$$

$$= \int_{-\infty}^{\infty} f(\alpha x) e^{-2\pi i \omega x} dx$$



Change of variables: $y = \alpha x \Rightarrow dy = \alpha dx$

$$\frac{dy}{\alpha} = dx$$

Note: Lots of Fourier proofs involve a change of variables.

$$\bar{F}(\omega) = \frac{1}{\cdot} \int^{\infty} f(u) e^{-2\pi i \frac{\omega y}{\alpha}} du = \frac{1}{\cdot} F\left(\frac{\omega}{\cdot}\right)$$

$$F(\omega) = \frac{1}{\alpha}\int_{-\infty}^{\infty} f(y)\, e^{-2\pi i \frac{\omega y}{\alpha}}\, dy = \frac{1}{\alpha} F\left(\frac{\omega}{\alpha}\right)$$

$$\boxed{\text{Thus, } \mathcal{F}\{f(\alpha x)\}(\omega) = \frac{1}{\alpha}\mathcal{F}\{f(x)\}\left(\frac{\omega}{\alpha}\right)}$$

So, as $f(x)$ stretches, $F(\omega)$ contracts & shrinks (and vice versa).

Example: Consider the Shah function with spacing Δx, $s(x)$. Recall the spacing of $S(\omega) = \mathcal{F}\{s(x)\}(\omega)$.

Exercise:

What is the spacing of $s(\alpha x)$, $\alpha \neq 0$?

What is the spacing of $S(\alpha x)$?

There is also a scale property for the DFT, though the intepretation is different. Suppose f_n and F_k are each an array of N numbers. What do those numbers mean? What points in the spatial/frequency domain do those values correspond to?

$$F_k = \sum_{n=0}^{N-1} f_n\, e^{-2\pi i \frac{nk}{N}}$$

One period... $0 \le \frac{nk}{N} < N$ (*)

Notice we can introduce an arbitrary spatial scale, $0 \le \frac{n}{N} < 1 \Rightarrow 0 \le \frac{nL}{N} < L$

$\Rightarrow 0 \le x < L$

We compensate by scaling the frequency variable,

(*) $\Rightarrow 0 \le \frac{nL}{N}\frac{k}{L} < N$

i.e. $x = \frac{nL}{N}$ for $n = 0, \ldots, N-1$

$w = \frac{k}{L}$ for $k = 0, \ldots, N-1$

Thus, if $x \in \left\{ \frac{0L}{N}, \frac{L}{N}, \frac{2L}{N}, \ldots, \frac{(N-1)L}{N} \right\}$ L "Field of view" (FOV) or period

then $w \in \left\{ \frac{0}{L}, \frac{1}{L}, \frac{2}{L}, \ldots, \frac{N-1}{L} \right\}$ $\frac{N}{L}$

$$\boxed{\text{In general, } (FOV_x)(\Delta w) = (FOV_w)(\Delta x) = 1}$$

$(FOV_x)(\Delta w) \to 1$ $\quad$ $(FOV_w)(\Delta x) \to 1$

$(L)\left(\frac{1}{L}\right) = 1$ $\quad$ $\left(\frac{N}{L}\right)\left(\frac{L}{N}\right) \to 1$

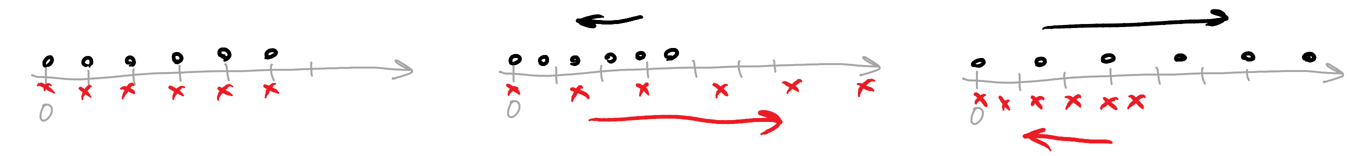
Equivalently, we could use

$$x \in \left\{ \frac{0}{L}, \frac{1}{L}, \frac{2}{L}, \ldots, \frac{N-1}{L} \right\}$$

$$w \in \left\{ \frac{0L}{N}, \frac{L}{N}, \frac{2L}{N}, \ldots, \frac{(N-1)L}{N} \right\}$$

Exercise: Verify that this sampling obeys the discrete scaling rule.

Visual Examples: For 6-vectors ($N=6$)...

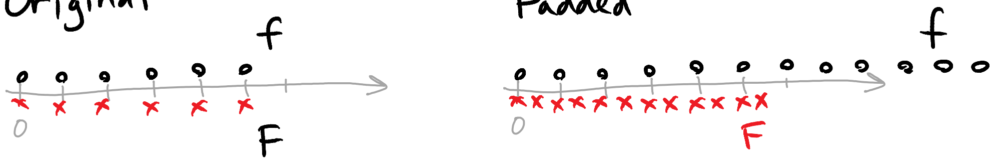


Side-Effect: Scaling using padding & cropping, & the DFT

Consider this situation...

- Pad f (add samples), then apply DFT.
  Where will the samples be in the freq. domain?

Original   Padded



In other words, padding in one domain can be thought of as subsampling in the other.

$\boxed{\text{Demo: Image scaling using the DFT.}}$

Shift Property

Consider f_n and its DFT F_k, $n,k=0,\ldots,N-1$.

Let $g_n = f_{n-d}$ (a shifted version of f_n).

$$G_k = \sum_{n=0}^{N-1} g_n e^{-2\pi i \frac{nk}{N}} \qquad k=0,\ldots,N-1$$

$$= \sum^{N-1} f_{n-d}\, e^{-2\pi i \frac{nk}{N}} \qquad \text{Change of vars: Let } m=n-d$$

$$= \sum_{n=0}^{N-1} f_{n-d}\, e^{-2\pi i \frac{nk}{N}}$$

Change of vars: Let $m = n-d$

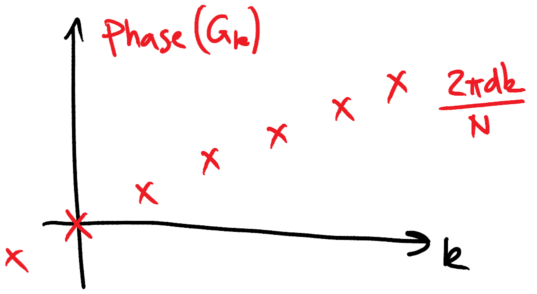
$n = m+d$

$$= \sum_{m=-d}^{N-1-d} f_m\, e^{-2\pi i \frac{(m+d)k}{N}}$$

Since f & e are N-periodic $= \sum_{m=0}^{N-1} f_m\, e^{-2\pi i \frac{mk}{N}}\, e^{-2\pi i \frac{dk}{N}}$

$$= e^{-2\pi i \frac{dk}{N}}\, F_k$$

That is, shifting f by d samples is equivalent to multiplying F by a "phase ramp".



Slope is dictated by d.

d larger ⇒ steeper

"Phase" because it only influences the phase of the Fourier coefs.

This property is also used quite often in MRI.

$\boxed{\text{Demo: Fourier Shift}}$

Rotational Invariance

If R is a rotation matrix, then $\bar{f}(\vec{x}) \equiv f(R\vec{x})$ is a rotated version of f. What does $\mathcal{F}\{\bar{f}(\vec{x})\}(\omega)$ look like?

$\mathcal{F}\{\tilde{f}(\vec{x})\}(\omega)$ look like?

As it turns out

$$\mathcal{F}\{f(R\vec{x})\}(\omega) = F(R\vec{\omega})$$

We won't prove this here, but it's quite simple. Thus, rotating f is equivalent to rotating F by the same amount.

This works for the FT, and also for the DFT with a few minor caveats.

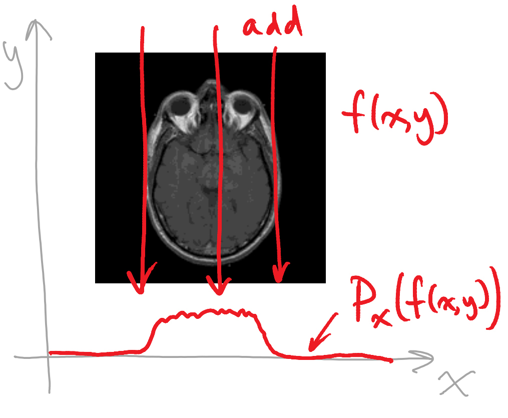
$\boxed{\text{Demo: Fourier rotation}}$

Fourier Projection Theorem

Suppose you have $f: \mathbb{C}^2 \to \mathbb{C}$, and $F(\omega, \lambda) = \mathcal{F}\{f(x,y)\}(\omega,\lambda)$.

Consider the Radon projection of f onto the x-axis,

$$\mathcal{P}_x\{f(x,y)\}(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$



For no apparent reason, let's take the 1D FT of $\mathcal{P}_x$.

$$\int_{-\infty}^{\infty} \mathcal{P}_x\{f(x,y)\}(x)\, e^{-2\pi i \omega x}\, dx = \int\int_{-\infty}^{\infty} f(x,y)\, e^{-2\pi i \omega x}\, dy\, dx$$

$$\int_{-\infty} \mathcal{P}_x\{f(x,y)\}(x)\, e^{\cdots}\, dx = \int_{-\infty}\int_{-\infty} f(x,y)\, e^{\cdots}\, dy\, dx$$

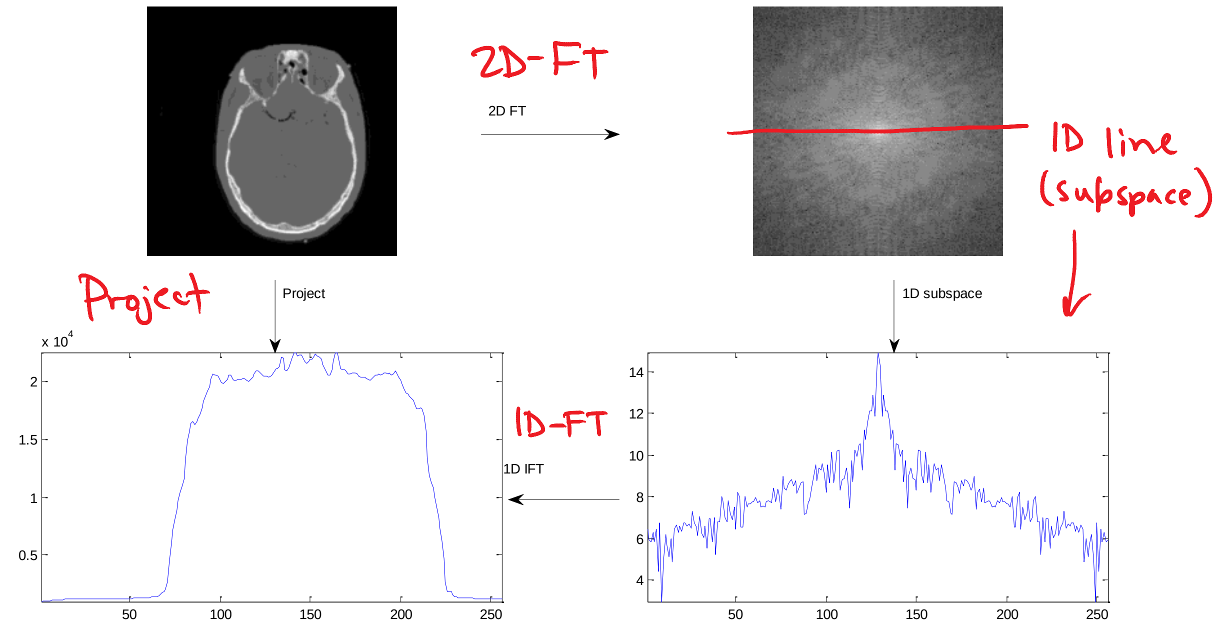
but $\lambda = 0$.

$$= \iint f(x,y)\, e^{-2\pi i(\omega x + \lambda y)}\, dy\, dx$$

$$= F(\omega, 0)$$

Thus, the 1D-FT of the projection is the same as the 1D line of Fourier coefs taken from the 2D-FT.

In a picture...



This works for a projection along any direction, as long as the subspace in the freq. domain has the same orientation.