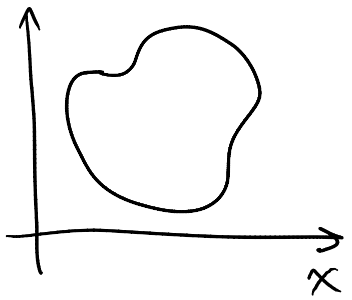
Introduction to Level Sets

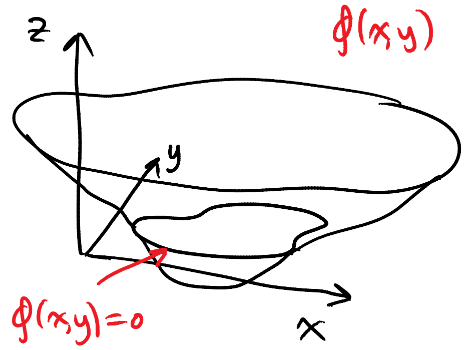
L41

Goal: An overview of the levelset method for image segmentation.

The level set method is a different way to formulate the active contour idea. Instead of **explicitly** modelling the curve, the curve is **implicitly** modelled as a **zero level set** of a higher-dimensional function. To model a curve in $\mathbb{R}^n$, you use an embedding function in $\mathbb{R}^n \to \mathbb{R}$.

eg.



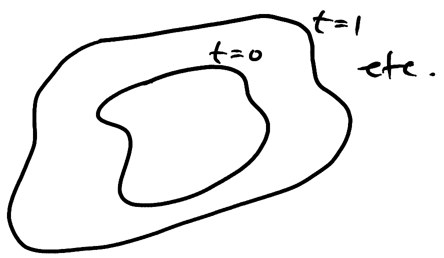


Note: Unless otherwise specified, in this course we will always assume that ϕ opens upward. (WLOG)

Since $\phi: \mathbb{R}^n \to \mathbb{R}$, the curve X is given by the inverse of ϕ, $\phi^{-1}: \mathbb{R} \to \mathbb{R}^n$. That is,

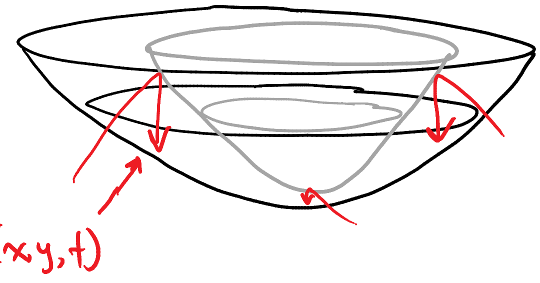
$$X = \phi^{-1}(0) \equiv \{\text{the set of all } (x,y) \text{ pts. s.t. } \phi(x,y)=0\}$$

Now consider an evolving level set, $X(s,t) = (x(s,t), y(s,t))$.



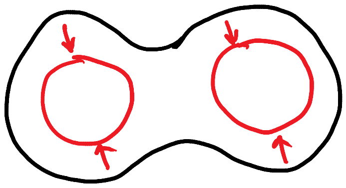
We can still model this with level sets: $\phi(x,y,t)$

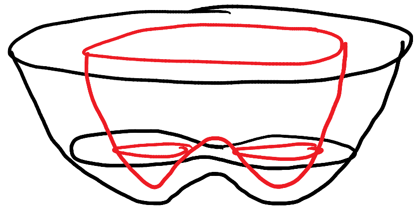
As time progresses, the embedding function ϕ can move and change so that the zero level set takes on the desired curve.



Since the curve is defined implicitly, it is easy to address topological changes.

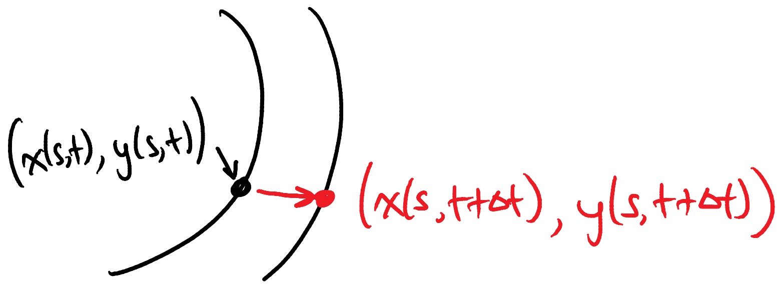
e.g.





Level Set Evolution

Consider the point $X(s,t) = (x(s,t), y(s,t))$ as the curve evolves.

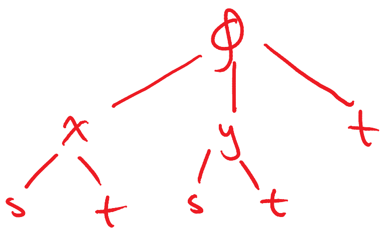


Hence, $\phi(x(s,t), y(s,t), t) = 0 = \phi(x(s,t+\Delta t), y(s,t+\Delta t), t+\Delta t)$

Using a Taylor expansion,

$$\phi(x(s,t+\Delta t), y(s,t+\Delta t), t+\Delta t)$$

$$= \phi(x(s,t), y(s,t), t) + \underbrace{\frac{d}{dt}\phi(x(s,t), y(s,t), t)}_{= \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial t}} \Delta t + \overset{\text{ignore}}{\cancel{O(\Delta t^2)}}$$



$$= \cancel{\phi}^{\,0} + \left[\frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial t}\right]\Delta t \stackrel{\sim}{=} 0$$

$$\Rightarrow \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial t} = 0$$

$$\frac{\partial \phi}{\partial t} = -\nabla\phi \cdot \left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}\right)$$ ← velocity of the point $(x(s,t), y(s,t))$

The velocity of (x(s,t), y(s,t)) is the motion of the curve, and we want to manipulate it to achieve the curve we're after. We can represent our velocity as a speed V_N in the direction normal (orthogonal) to the curve.
(Note: speed V_N is a scalar, not a vector)
The **gradient** of a function is orthogonal to its level curves. Thus, our velocity is parallel to $\nabla\phi$.

So we can represent $\left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}\right)$ as $V_N \frac{\nabla\phi}{\|\nabla\phi\|}$

Thus, our embedding function evolves according to

$$\frac{\partial \phi}{\partial t} = -\nabla\phi \cdot \left(V_N \frac{\nabla\phi}{\|\nabla\phi\|}\right) = -V_N \frac{\|\nabla\phi\|^2}{\|\nabla\phi\|}$$

$$\Rightarrow \boxed{\frac{\partial \phi}{\partial t} = -V_N \|\nabla\phi\|} \quad (1)$$

—END