CT Back Projection

L47

Goal: To see how the Fourier Projection Theorem can help us in CT reconstruction.

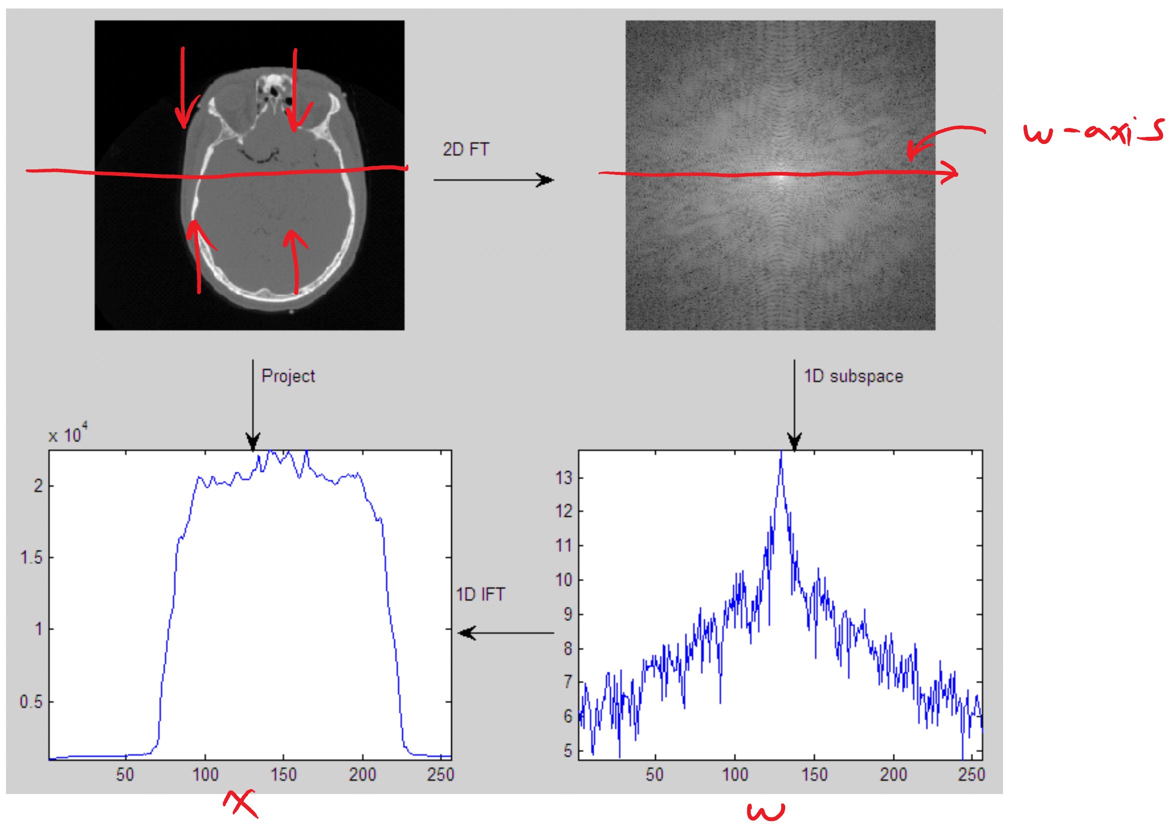
Recall the **Fourier Projection Theorem**
(see the end of L09)

$$\mathcal{F}\{\underbrace{p_x(f(\vec{x}))(x)}\}(\omega) = F(\omega, 0)$$

1D Fourier Transform

Radon proj. along y-axis is a function of x

1D subspace through 2D Fourier transform of f.



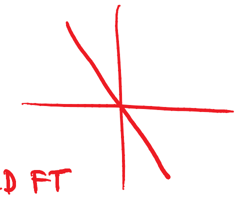
Theory of Back Projection

Since the Fourier transform is rotation invariant, we can apply the above in any direction.

$$\mathcal{F}_{1D}\left\{p_\theta(f(\vec{x}))(t)\right\}(\rho) = \left[\mathcal{F}_{2D}\left\{f(\vec{x})\right\}(\vec{\omega})\right]_\theta(\rho)$$

project onto angle θ

Extract subspace at angle θ



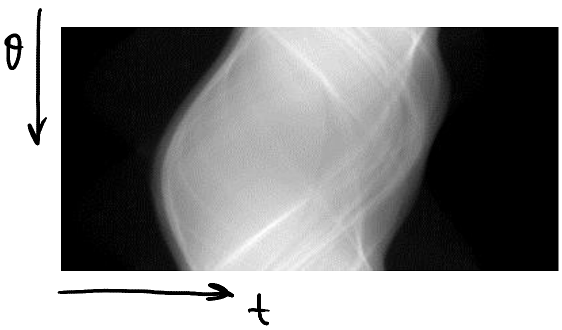
$$\underbrace{\mathcal{F}_{1D}\left\{p_\theta(f(\vec{x}))(t)\right\}(\rho)} = \underbrace{\left[\mathcal{F}_{2D}\left\{f(\vec{x})\right\}(\vec{\omega})\right]_\theta(\rho)}$$

Denote this as $P(\rho,\theta)$

The scanner gives us

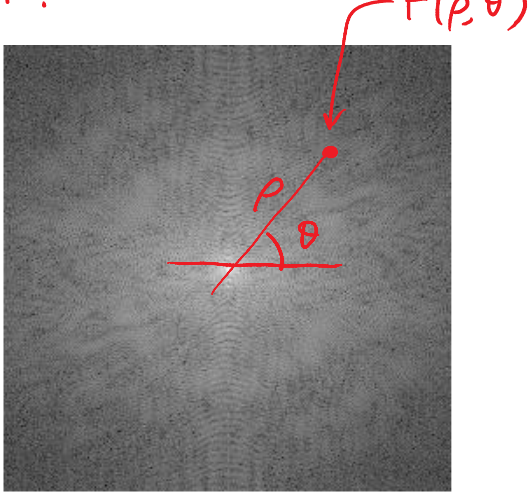
$p_\theta(t)$ or $p(t,\theta)$

The spatial-domain projection at angles θ.



We just take the 1D-FT over t to get $P(\rho,\theta)$

Notice that this is $F(\rho,\theta)$ simply a polar representation of F.



Hence, we can reconstruct f by taking the inverse FT, which itself is constructed using $P(\rho,\theta)$

We want

$$f(x,y) = \iint P(\rho(u,v), \theta(u,v))\, e^{2\pi i(ux+vy)}\, du\, dv$$

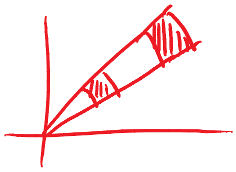
$$f(x,y) = \iint \underbrace{P(\rho(u,v), \theta(u,v))}_{\text{But this is in polar coords.}} e^{2\pi i (ux+vy)} \, du \, dv$$

Change of variables...

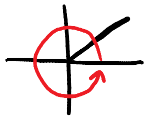
$$f(x,y) = \int_0^{\infty} \int_{-\pi}^{\pi} P(\rho,\theta) \, e^{2\pi i (x\rho\cos\theta + y\rho\sin\theta)} \, \rho \, d\theta \, d\rho$$

determinant of the Jacobian

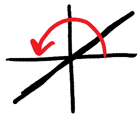
Area for $d\theta$ goes up linearly with ρ



Instead of $\rho \geq 0$ and $-\pi \leq \theta < \pi$



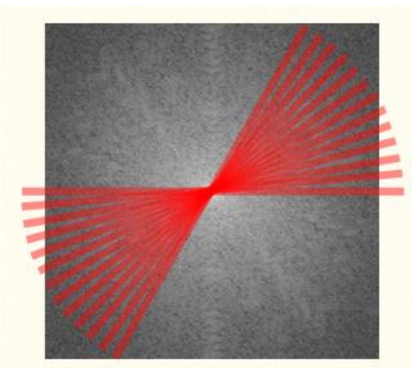
we can integrate $-\infty < \rho < \infty$ and $0 \leq \theta < \pi$



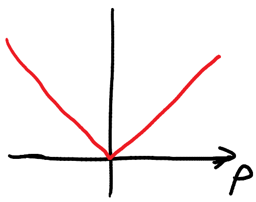
$$f(x,y) = \int_{-\infty}^{\infty} \int_0^{\pi} P(\rho,\theta) \, |\rho| \, e^{2\pi i \rho (x\cos\theta + y\sin\theta)} \, d\theta \, d\rho$$

This suggests that the way to reconstruct an image is to populate the frequency domain by adding each $P(\rho,\theta)$.

(astounding PowerPoint demo)



Then multiply by the cone filter.

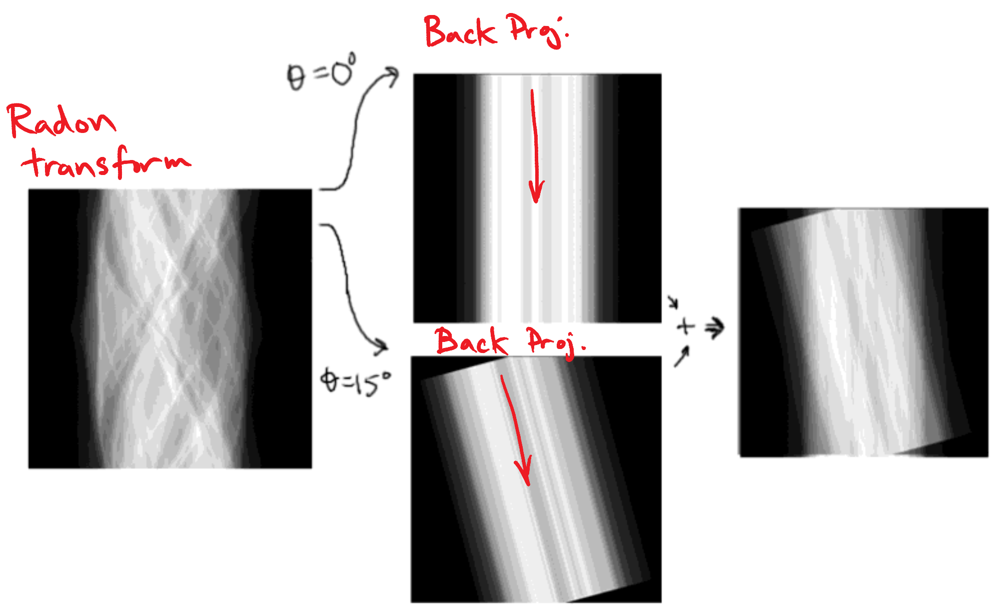


The cone filter compensates for the **higher density** of projections near the origin. It's essentially a **high-pass** filter.

Filtered Back Projection

The method described above is called "filtered back projection".
However, while this frequency-domain method works in theory, it requires **resampling** in the frequency domain, which can sometimes be **problematic.**

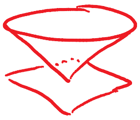
Instead, we can do the equivalent operations in the spatial domain.
We take each projection and **"back-project"** it. That is, we **smear** it back across the image in the **direction** it was acquired (ie. the gantry angle).



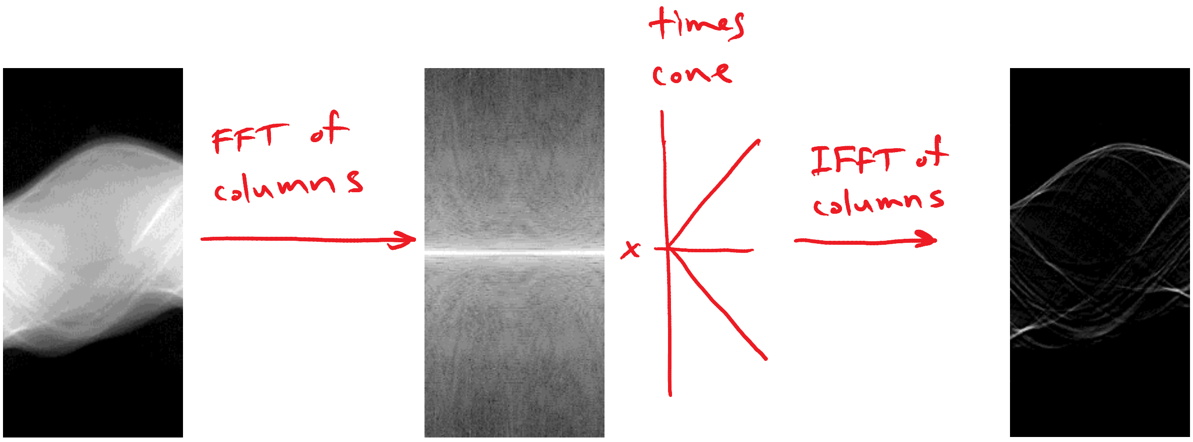
Then we simply add the backprojections together... one for each projection in our Radon Transform.

The resulting image will be blurry.

The resulting image will be blurry. That's because we haven't applied the **filter** yet $(|\rho|)$. We can do that by multiplying the 2D FFT of our image by a cone.



This filter can be applied in 1D to the projections themselves.



Then we do the back projection, and get a much crisper image.