Tricks with Transform-Based Registration

L29

Goal: To find out how rotations and scales can be discovered from transformation-based registration methods.

Rotations

We have seen that rotations are invariant under the Fourier transform. That is, rotating an image by θ also rotates its FT by θ.

What if an image is rotated AND translated?

Consider the FT of $f(Rx+b)$, where $x, b \in \mathbb{R}^2$. (or 3, or D.)

$$\tilde{F}(\omega) = \int f(Rx+b)\, e^{-2\pi i (\omega \cdot x)}\, dx \qquad \omega^T x$$

Let $y = Rx+b \Rightarrow dy = \det\left(\frac{\partial y}{\partial x}\right) dx = \det(R)\, dx = dx$

$$\Rightarrow x = R^{-1}(y-b)$$

$$\tilde{F}(\omega) = \int f(y)\, e^{-2\pi i \omega^T (R^{-1}(y-b))}\, dy$$

$$= \int f(y)\, e^{-2\pi i [\omega^T R^{-1} y - \omega^T R^{-1} b]}\, dy$$

These are scalar, so we can transpose them.
Note that R is orthogonal, so $R^{-1} = R^T$.

$$= \int f(y)\, e^{-2\pi i [y^T R \omega - b^T R \omega]}\, dy$$

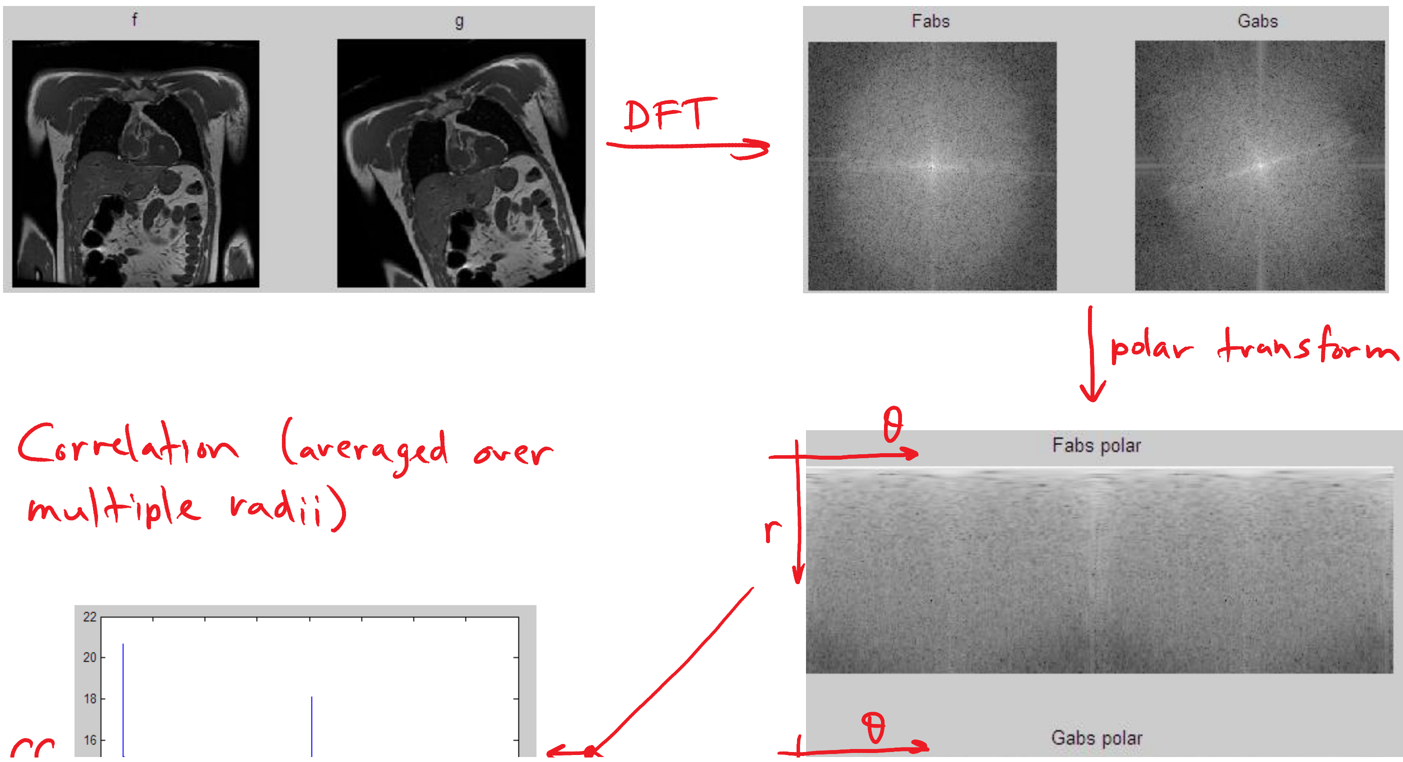
$$= e^{2\pi i b^T R \omega} \int f(y)\, e^{-2\pi i y^T R \omega}\, dy$$

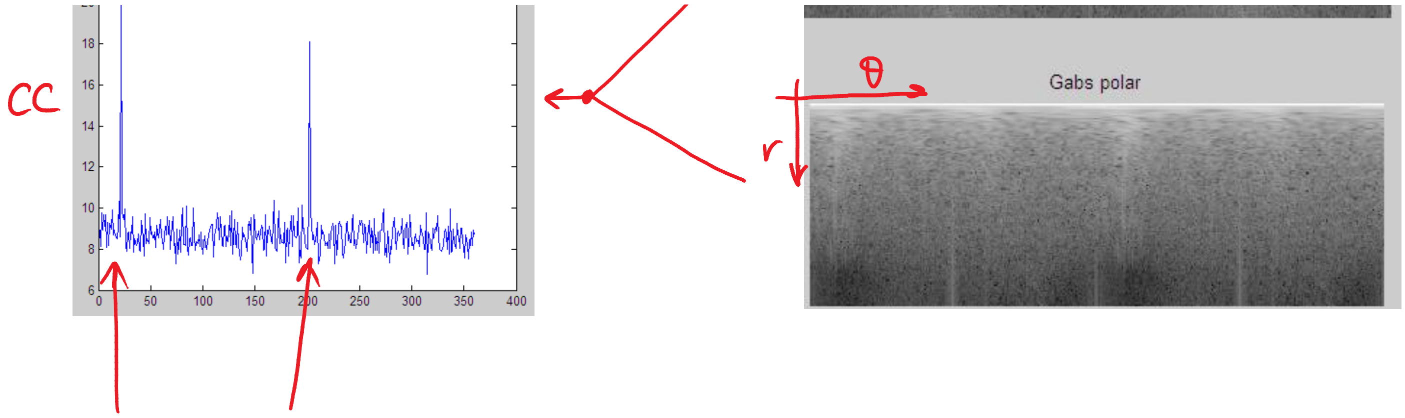
$$= e^{2\pi i b^T R\omega} F(R\omega)$$

This is just a phase-ramped, rotated version of $F(\omega)$, but the direction of the ramp is rotated also.

Since the translation only affects the phase of the Fourier coefs, we can look at the **modulus** of the coefs to try to estimate rotation. Once the rotation is accounted for, we can look for the phase differeneces to estimate translation. In this way, rotation and translation are **decoupled** in the frequency domain.

The reason that rotation and translation are decoupled in the frequency domain is that the centre of rotation is known... the origin. For 2D rotation, this turns the problem of finding the optimal rotation into a 1D shift problem; we convert our FTs into polar coordinates and look for the best shift along the θ-axis.





Notice there are two peaks. Why?

Scaling

Consider the function $f(x)$, and a scaled version $f(sx)$.

Define $g(\log x) = f(x)$.

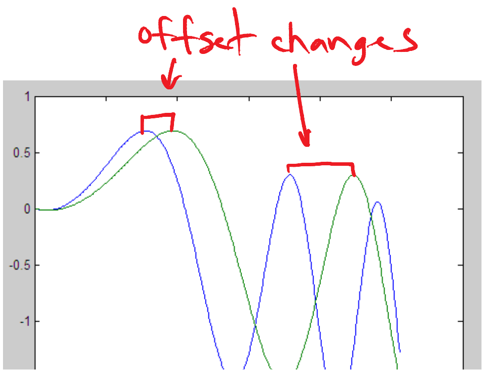
What is $f(\frac{x}{2})$ in terms of g?

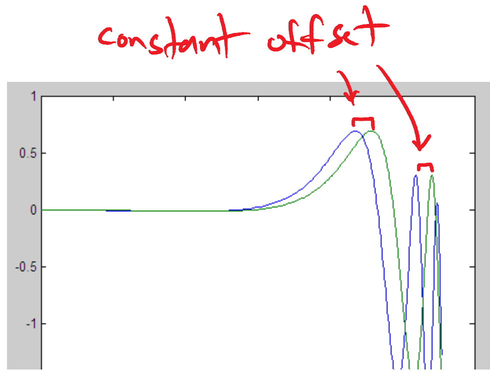
$$f\left(\frac{x}{2}\right) = g\left(\log \frac{x}{2}\right) = g(\log x - \log 2)$$

This is a shifted version of g.

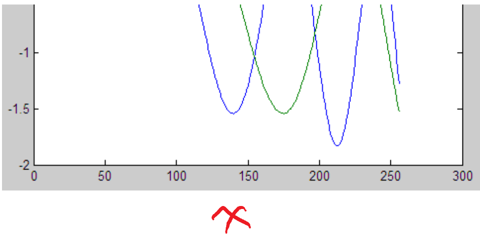
Hence, applying a spatial log mapping to an image turns scaling into **translation**. However, just like in rotations, the centre of scaling has to be the origin of the log mapping.

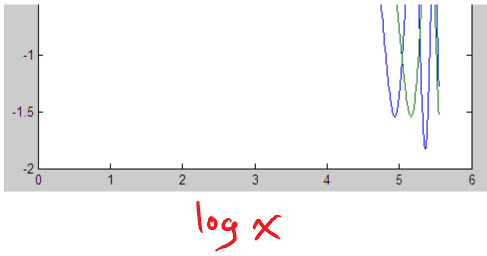
```
x = 1:256;
f = sin((x/64).^2) - x/256;
g = sin((x*0.8/64).^2) ...
      - x*0.8/256;

figure(1); plot(x,f,x,g);
```



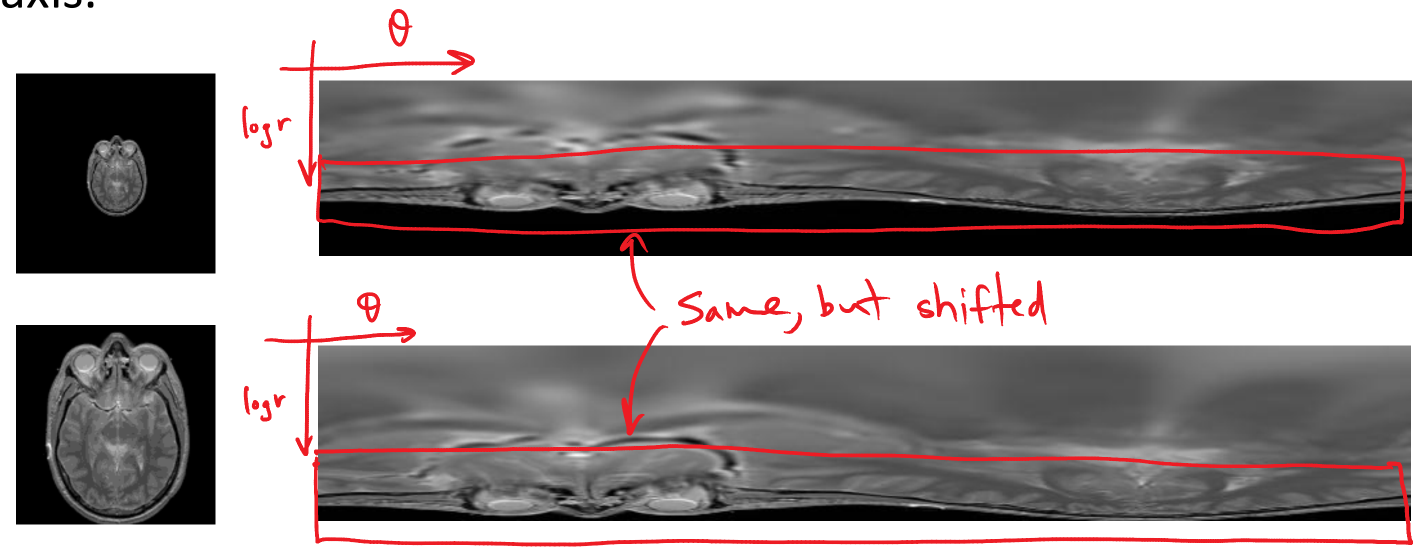


```
figure(1); plot(x,f,x,g);

figure(2); plot(log(x),f,lx,g);
```





In 2D, we can use the log-polar transform. A scale difference between the images turns into a shift along the log-radius axis.



Just like for rotation, we need to know the **centre of expansion**; in this way, scaling and translation are **coupled**.

However, it is easy to show that the Fourier transform is invariant under scaling, so one can do the same trick as for rotation. You can use the **modulus** of the Fourier coefficients to estimate the scale because the translation only affects the **phase**. Cool, non?!