Fourier Series

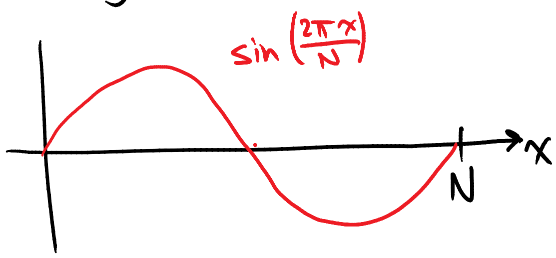
Goal: To introduce the Fourier transform & review complex numbers. The Fourier transform is fundamental not only to image processing, but it also plays a special role in medical imaging.

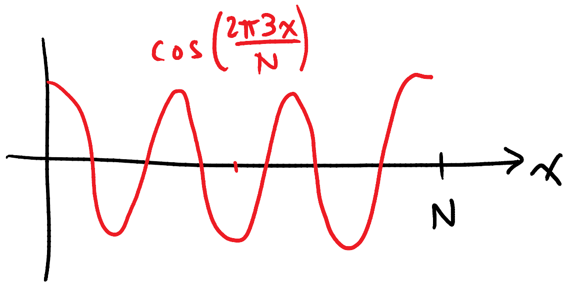
Consider the trigonometric functions of n,

$$\sin\left(\frac{2\pi k x}{N}\right) \quad \text{and} \quad \cos\left(\frac{2\pi k x}{N}\right) \qquad k \in \mathbb{Z}$$

They repeat when x increases by $\frac{N}{k}$.

Or, they repeat k times in $0 \leq \ \leq N$.





Theorem: Suppose f is a "nice" N-periodic function. There exist coefficients a_k & b_k such that

$$f(x) = a_0 + \sum_{k=1}^{\infty}\left[a_k \cos\left(\frac{2\pi k x}{N}\right) + b_k \sin\left(\frac{2\pi k x}{N}\right)\right]$$

This is known as a Fourier Series.

In practice, we approximate f with a truncated Fourier Series,

$$f(x) = a_0 + \sum_{k=1}^{m}\left[a_k \cos\left(\frac{2\pi k x}{N}\right) + b_k \sin\left(\frac{2\pi k x}{N}\right)\right] \quad (*)$$

Instead of treating the a's and b's separately, we can use the more sophisticated and compact complex notation,

$$f(x) = \sum_{k=-m}^{m} C_k\left(\cos\left(\frac{2\pi kx}{N}\right) + i\sin\left(\frac{2\pi kx}{N}\right)\right)$$

Notice the sum is now from $-m$ to m. Here's why. If $f(x) \in \mathbb{R}$, then we need to make sure all the imaginary parts cancel out.

$$f(x) = a_0 + \sum_{k=1}^{m}\left[C_k\cos\frac{2\pi kx}{N} + iC_k\sin\frac{2\pi kx}{N} + C_{-k}\cos\frac{-2\pi kx}{N} + iC_{-k}\sin\frac{-2\pi kx}{N}\right]$$

$$= a_0 + \sum_{k=1}^{m}\left[(C_k + C_{-k})\cos\frac{2\pi kx}{N} + i(C_k - C_{-k})\sin\frac{2\pi kx}{N}\right]$$

Comparing to ⊛ above...

$$\begin{cases} a_k = C_k + C_{-k} & ① \\ b_k = i(C_k - C_{-k}) & ② \end{cases}$$

$$① + i② \Rightarrow a_k + ib_k = 2C_{-k} \Rightarrow C_{-k} = \frac{a_k + ib_k}{2}$$

$$① - i② \Rightarrow a_k - ib_k = 2C_k \Rightarrow C_k = \frac{a_k - ib_k}{2}$$

Notice, then, that $C_k = \overline{C_{-k}}$ (complex conjugates)

$\boxed{\text{Task: Do the associated quiz in D2L.}}$