Variational Principles

Lecture Notes

Rashid M. Talha

School of Natural Sciences, NUST
January 18, 2023
Email: rmathsphys@gmail.com

Contents

Abstract

Lecture notes for an introductory course on variational principles.
Currently a work-in-progess.

Chapter 1

Introduction

Starting the topic with a standard functional, motivating the topic with many examples, aim of this text, roadmap of the things to come, things that are left out, recap of standard calculus (1D and ND), the fundamental theorem of calculus of variations.

$$I[f] = \int_{x_1}^{x_2} f(x, y(x), y'(x))\,\mathrm{d}x. \tag{1}$$

This is a functional.

1.1 Functionals & Variations

Lorem ipsum dolor sit amet, consectetur adipisicing elit, sed do eiusmod tempor incididunt ut labore et dolore magna aliqua. Ut enim ad minim veniam, quis nostrud exercitation ullamco laboris nisi ut aliquip ex ea commodo consequat. Duis aute irure dolor in reprehenderit in voluptate velit esse cillum dolore eu fugiat nulla pariatur. Excepteur sint occaecat cupidatat non proident, sunt in culpa qui officia deserunt mollit anim id est laborum.

1.2 Examples of Variational Problems

Lorem ipsum dolor sit amet, consectetur adipisicing elit, sed do eiusmod tempor incididunt ut labore et dolore magna aliqua. Ut enim ad minim veniam, quis nostrud exercitation ullamco laboris nisi ut aliquip ex ea commodo consequat. Duis aute irure dolor in reprehenderit in voluptate velit esse cillum dolore eu fugiat nulla pariatur. Excepteur sint occaecat cupidatat non proident, sunt in culpa qui officia deserunt mollit anim id est laborum.

1.3 Overview of these notes

Lorem ipsum dolor sit amet, consectetur adipisicing elit, sed do eiusmod tempor incididunt ut labore et dolore magna aliqua. Ut enim ad minim veniam, quis nostrud exercitation ullamco laboris nisi ut aliquip ex ea commodo consequat. Duis aute irure dolor in reprehenderit in

voluptate velit esse cillum dolore eu fugiat nulla pariatur. Excepteur sint occaecat cupidatat non proident, sunt in culpa qui officia deserunt mollit anim id est laborum.

- Introduce the calculus of variations and proof some standard results.
- Derive and use the Euler-Lagrange equations for various different types of Lagrangians.
- Explore the concept of symmetries, and study its relation to conservation laws; Noether's theorem.
- Establish a duality between differential equations and certain optimisation problems.
- Discuss constrained optimisation problems, along with several types of contraints.
- Discuss how modern theoretical physics is predominantly studied in terms of Lagrangians and Hamiltonians.
- Introduce classical mechanics in the framework of lagrangian mechanics; introduce Hamiltonian mechanics, and the concept of phase space, but not explore it is any detail.
- Apply variational techniques to domains other than physics; e.g. entropy minimisation in statistics.

Chapter 2

Calculus of real valued functions

2.1 Functions of a single variable

Lorem ipsum dolor sit amet, consectetur adipisicing elit, sed do eiusmod tempor incididunt ut labore et dolore magna aliqua. Ut enim ad minim veniam, quis nostrud exercitation ullamco laboris nisi ut aliquip ex ea commodo consequat. Duis aute irure dolor in reprehenderit in voluptate velit esse cillum dolore eu fugiat nulla pariatur. Excepteur sint occaecat cupidatat non proident, sunt in culpa qui officia deserunt mollit anim id est laborum.

2.2 Functions of a several variables

Lorem ipsum dolor sit amet, consectetur adipisicing elit, sed do eiusmod tempor incididunt ut labore et dolore magna aliqua. Ut enim ad minim veniam, quis nostrud exercitation ullamco laboris nisi ut aliquip ex ea commodo consequat. Duis aute irure dolor in reprehenderit in voluptate velit esse cillum dolore eu fugiat nulla pariatur. Excepteur sint occaecat cupidatat non proident, sunt in culpa qui officia deserunt mollit anim id est laborum.

2.3 Other

Lorem ipsum dolor sit amet, consectetur adipisicing elit, sed do eiusmod tempor incididunt ut labore et dolore magna aliqua. Ut enim ad minim veniam, quis nostrud exercitation ullamco laboris nisi ut aliquip ex ea commodo consequat. Duis aute irure dolor in reprehenderit in voluptate velit esse cillum dolore eu fugiat nulla pariatur. Excepteur sint occaecat cupidatat non proident, sunt in culpa qui officia deserunt mollit anim id est laborum.

Chapter 3

Euler-Lagrange equations

$\mathcal{L}(x, y, y')$, $\mathcal{L}(x, y, y', ..., yn)$, first integrals, and the definition of $\mathcal{H}$, $\mathcal{L}(x, yi, yi')$, $\mathcal{L}(x1, x2, y, y')$.

3.1 This type

Let $y = y(x)$ be a C^2 function of a single variable x, and consider the function $f(x, y, y'')$.

Chapter 4

Generalised coordinates

Lorem ipsum dolor sit amet, consectetur adipisicing elit, sed do eiusmod tempor incididunt ut labore et dolore magna aliqua. Ut enim ad minim veniam, quis nostrud exercitation ullamco laboris nisi ut aliquip ex ea commodo consequat. Duis aute irure dolor in reprehenderit in voluptate velit esse cillum dolore eu fugiat nulla pariatur. Excepteur sint occaecat cupidatat non proident, sunt in culpa qui officia deserunt mollit anim id est laborum.

Chapter 5

Hamilton's equations

Starting the topic with a standard functional, motivating the topic with many examples, aim of this text, roadmap of the things to come, things that are left out, recap of standard calculus (1D and ND), the fundamental theorem of calculus of variations.

Chapter 6

Conservation laws and symmetries

Lorem ipsum dolor sit amet, consectetur adipisicing elit, sed do eiusmod tempor incididunt ut labore et dolore magna aliqua. Ut enim ad minim veniam, quis nostrud exercitation ullamco laboris nisi ut aliquip ex ea commodo consequat. Duis aute irure dolor in reprehenderit in voluptate velit esse cillum dolore eu fugiat nulla pariatur. Excepteur sint occaecat cupidatat non proident, sunt in culpa qui officia deserunt mollit anim id est laborum.

Chapter 7

Constraints

Starting the topic with a standard functional, motivating the topic with many examples, aim of this text, roadmap of the things to come, things that are left out, recap of standard calculus (1D and ND), the fundamental theorem of calculus of variations.

Chapter 8

Second variation

Lorem ipsum dolor sit amet, consectetur adipisicing elit, sed do eiusmod tempor incididunt ut labore et dolore magna aliqua. Ut enim ad minim veniam, quis nostrud exercitation ullamco laboris nisi ut aliquip ex ea commodo consequat. Duis aute irure dolor in reprehenderit in voluptate velit esse cillum dolore eu fugiat nulla pariatur. Excepteur sint occaecat cupidatat non proident, sunt in culpa qui officia deserunt mollit anim id est laborum.

Chapter 9

Classical Mechanics

Lag and Ham mechanics, motivation of L=T-V, functional derivative, hamilton's principle of least action, central potentials

Chapter 10

Strum-Lioville Equation

Lorem ipsum dolor sit amet, consectetur adipisicing elit, sed do eiusmod tempor incididunt ut labore et dolore magna aliqua. Ut enim ad minim veniam, quis nostrud exercitation ullamco laboris nisi ut aliquip ex ea commodo consequat. Duis aute irure dolor in reprehenderit in voluptate velit esse cillum dolore eu fugiat nulla pariatur. Excepteur sint occaecat cupidatat non proident, sunt in culpa qui officia deserunt mollit anim id est laborum.

Appendix A

Convexity and Legendre Transforms

Lorem ipsum dolor sit amet, consectetur adipisicing elit, sed do eiusmod tempor incididunt ut labore et dolore magna aliqua. Ut enim ad minim veniam, quis nostrud exercitation ullamco laboris nisi ut aliquip ex ea commodo consequat. Duis aute irure dolor in reprehenderit in voluptate velit esse cillum dolore eu fugiat nulla pariatur. Excepteur sint occaecat cupidatat non proident, sunt in culpa qui officia deserunt mollit anim id est laborum.

1.1 Convex Functions

1.2 Legendre Transforms

Lorem ipsum dolor sit amet, consectetur adipisicing elit, sed do eiusmod tempor incididunt ut labore et dolore magna aliqua. Ut enim ad minim veniam, quis nostrud exercitation ullamco laboris nisi ut aliquip ex ea commodo consequat. Duis aute irure dolor in reprehenderit in voluptate velit esse cillum dolore eu fugiat nulla pariatur. Excepteur sint occaecat cupidatat non proident, sunt in culpa qui officia deserunt mollit anim id est laborum.

Appendix B

Lagrange Multipliers

Lorem ipsum dolor sit amet, consectetur adipisicing elit, sed do eiusmod tempor incididunt ut labore et dolore magna aliqua. Ut enim ad minim veniam, quis nostrud exercitation ullamco laboris nisi ut aliquip ex ea commodo consequat. Duis aute irure dolor in reprehenderit in voluptate velit esse cillum dolore eu fugiat nulla pariatur. Excepteur sint occaecat cupidatat non proident, sunt in culpa qui officia deserunt mollit anim id est laborum.

Appendix C

More Examples of Variational Problems

Shortest path on R2, shortest path on S2, Path of least time, caternary: least energy, fermat's principle in optics, specific systems from CM and theoretical engineering, least area with fixed perimeter. Geodesics.

Advanced theoretical physics (String theory: Nambu-Gato action, Maxwell's equations from the action, comment about the lagrangian for the standard model).

Bibliography

[1] R. Weinstock, Calculus of variations, with applications to physics and engineering, Dover Publications, 1974.

[2] F. Rindler, Calculus of variations, 1st Edition, Springer, 2018.

[3] I. M. Gelfand, S. V. Fomin, Calculus of variations, Prentice-Hall, 1963.

[4] L. D. Landau, E. M. Lifshitz, Mechanics, 2nd Edition, Pergamon Press, 1969.