Letting Your Past Define Your Taxes:
Optimal History-Dependent Income Taxation in General Equilibrium

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Introduction

- This paper studies optimal labor income taxation when taxes can depend on past income
- History dependence has large potential for reducing tax distortions
  - History dependent taxes can give incentives based on income history
  - Extract higher income from households with high earning potential
    - Give lower taxes, but only after investing in skills and earning high amounts
- In most countries, income taxes are based mostly on current income
  - But income history is recorded by governments and used in practice
    - e.g. public pensions/social security, income averaging in US (1964-1986)
- Question 1: How should labor income taxation vary with a household's income history?
- Question 2: What are the welfare implications of conditioning taxes on previous income?

What I Do

- Study history-dependent (HD) taxes in a model of life-cycle labor supply
  - Model ingredients
    1. Skill investment by households, e.g. decide to go to college or not
    2. Skill types are imperfectly substitutable in production, e.g. waiter vs. accountant
    3. Very general class of tax function
       - Average tax rate = *any* continuously differentiable function of entire income history
       - But taxes can only depend on income, not skills or hours worked
  - Record income history (36 years) + indv. state variables $\Longrightarrow$ up to 41 state variables

Background

- New feature: skill types are imperfectly substitutable in production
    - Widely held view of data: e.g. Katz and Murphy (1992), Acemoglu (2002)
        - Taxes have nontrivial effects on wages for labor
        - $\uparrow$ output for one worker $\implies$ $\uparrow$ marginal product (wage) of all workers
    - Creates role for history-dependence
        - Use HD to incentivize households with high earning potential to produce a lot
        - Can get large increases in wages for everyone else

- Existing literature on history-dependent taxation in dynamic models
    - Stantcheva (2017), Farhi and Werning (2013), Weinzierl (2011), and others
    - Common feature: skills are perfect substitutes in production
        - Wages received for labor are invariant to tax system (constant)
    - General finding: very small welfare benefits from HD taxes over just age-dependent

Methodology

- Imperfectly substitutable skills make standard solution methods intractable
  - Need to compute optimal taxes directly by specifying functional form for income taxes
- Need method to solve model with 41 state variables *and* compute welfare maximizing policy
  - Neural networks (NN) and new numerical libraries (e.g. Google's Tensorflow) make this possible
  - NN's designed to approximate high-dimensional functions quickly
    - Automatically allocate parameters to approximate functions with minimal parameters
- I use NN's to approximate optimal labor income tax function in steady state
  - Existing methods: use NN's to solve high-dimensional structural models
  - My method (Nested NN): use NN's to both solve the model *and* optimize income taxes
    - Flexible algorithm: can be applied to study optimal policies in a wide variety of models

How it Works

Findings

1. Optimal history-dependent (HD) tax function is:
   - For most people: tax rates (avg. & marginal) increase over life cycle
     - Wages are estimated to increase deterministically with age
     - Gov. increases taxes with age to smooth consumption
   - For high income history (top 5% of incomes): tax rates decrease over life cycle
     - As gov. learns which HH's have highest earning potential, rewards high earnings with lower taxes
       - Maintains high labor supply during working life
       - Incentivizes skill investment at beginning of life in anticipation of lower future taxes
     - High output $\implies$ Higher avg. wages $\implies$ Less redistribution/distortion under optimal tax
2. Welfare gain from HD tax large compared to optimal age-dependent tax (2% lifetime cons.)
   - Welfare gains are cut in half when wages for each skill are taken as given
   - Virtually no gains from HD taxation when wages *and* skills are taken as given
3. 90% of potential welfare gains achieved with parametric tax: $T($*average* past income, age$)$
   - Mimics full HD tax by making optimal taxes flatter in age with higher past income

Related Literature

Optimal taxation

- **Parametric (aka Quantitative Ramsey):** Heathcote, Storesletten and Violante (2017, 2020), Kapička (2020), Krueger and Ludwig (2016), Karabarbounis (2016), and others
- **History dependent (aka Dynamic Mirrlees):** Stantcheva (2017), Golosov, Troshkin and Tsyvinski (2016), Farhi and Werning (2013), Weinzierl (2011), and others
- **Static with differentiated labor:** Sachs, Tsyvinski and Werquin (2020), Scheuer and Werning (2016), Rothschild and Scheuer (2013), Stiglitz (1982, 1987), and others
- **Contribution:** General tax function that allows for history-dependence in model with differentiated labor

Neural networks to solve structural economic models

- Fernández-Villaverde, Nuño, Sorg-Langhans and Vogler (2020), Azinovic, Gaegauf and Scheidegger (2019), Maliar, Maliar and Winant (2019), and others
- **Contribution:** Application to optimal policy (nested neural network solution)

Plan

1. Model
   - Economic environment
   - Optimal taxation problem
2. Parameter Selection
3. Solution Method
4. Results
   - Optimal taxes
   - Welfare analysis
   - Simple implementation

Model

Agents

- Households (measure one, indexed by $i \in [0,1]$)
  - Work for A periods, age indexed by $a = 0, \ldots, A-1$
  - Choose one time, permanent investment in skills, s, before entering labor market
  - Each period choose consumption c_a, savings b_{a+1} and hours h_a
- Firm: produces consumption good using labor differentiated by skill s
- Government
  - Collects revenue from nonparametric history-dependent income taxes, $T(.)$
  - Uses tax revenue to fund expenditures, $G = g \times Y$ (constant % of output)
  - Taxes distort labor supply *and* skill investment: limit on optimal progressivity

Technology

- Output is a CES aggregate over continuum of skill types s

$$Y = \left(\int_0^\infty [N(s) f_s(s)]^{\frac{\omega-1}{\omega}} ds \right)^{\frac{\omega}{\omega-1}}$$

    - $N(s)$ is total labor supply and $f_s(s)$ is density for type s
    - ω is elasticity of substitution between skills
    - Note: higher skills not inherently more productive, but more valuable because rarer
        - Price for skills/skill premium: $p(s) = \left[\frac{Y}{N(s) f_s(s)} \right]^{\frac{1}{\omega}}$
        - Higher wage when Y larger $\implies$ spillovers from higher output

- Linear savings technology: households can transform one unit of consumption at a into $1+r$ units at $a+1$ risk-free

- Resource constraint

$$\sum_{a=0}^{A-1} \int (c_{ia} + b_{ia+1}) di + gY = (1+r) \sum_{a=0}^{A-1} \int b_{ia} di + Y$$

Individual Wages and Income

- Hourly productivity:

$$\log \theta_{ia} = x(a) + z_{ia} + \varepsilon_{ia}$$

  - $x(a)$: deterministic age-productivity profile
  - $z_{ia} = z_{ia-1} + \eta_{ia}$, $\eta_{ia} \sim N(0, v_\eta)$: permanent shocks
  - $\varepsilon_{ia} \sim N(0, v_\varepsilon)$: transitory shocks

- Total labor income:

$$y_{ia} = \underbrace{p(s_i)}_{\text{skill price}} \times \underbrace{\theta_{ia}}_{\text{productivity}} \times \underbrace{h_{ia}}_{\text{hours worked}}$$

  - $p(s)$: skill price of type s labor = marginal product of type s

Preferences

- Preferences over consumption c, hours h, and skill-investment s for an individual i

$$U_i = -v_i(s_i) + E_0 \left[\left(\frac{1-\beta}{1-\beta^A} \right) \sum_{a=0}^{A-1} \beta^a u_i(c_{ia}, h_{ia}) \right]$$

where disutility from skill investment is

$$v_i(s_i) = \kappa_i^{-\frac{1}{\psi}} \frac{s_i^{1+\frac{1}{\psi}}}{1+\frac{1}{\psi}}, \ \kappa_i \sim Exp(1)$$

and utility from consumption and labor is

$$u_i(c_{ia}, h_{ia}) = \frac{\left[c_{ia}^{\phi_i} (1-h_{ia})^{1-\phi_i} \right]^{1-\gamma}}{1-\gamma}$$

where $\phi_i = \frac{1}{1+\exp \tilde{\phi}_i}$, $\tilde{\phi}_i \sim N(m_\phi, v_\phi)$

Household Problem

- Denote the vector of individual state variables as

$$S_{ia} \equiv (s_i, \phi_i, b_{ia}, z_{ia}, \varepsilon_{ia}, a, \{y_t\}_{t=0}^{a-1})$$

  - 41 individual state variables
    - 4 exogenous: $(\phi_i, z_{ia}, \varepsilon_{ia}, a)$
    - $2+a-1$ (37 when $a = A-1$) endogenous: $(s_i, b_{ia}, \{y_t\}_{t=0}^{a-1})$

- Individuals enter with zero savings $b_{i0} = 0$ and solve

$$v_{ia}(S_{ia}) = \max_{c_{ia}, h_{ia}, b_{ia+1}} u_i(c_{ia}, h_{ia}) + \beta E_a [v_{ia+1}(S_{ia+1})]$$

subject to

$$c_{ia} + b_{ia+1} = (1+r)b_{ia} + y_{ia} - T_a(y_{ia}; \{y_{ij}\}_{j=0}^{a-1})$$

and

$$c_{ia}, b_{ia+1} \geq 0, \;\; h_{ia} \in [0,1]$$

Equilibrium

- Stationary equilibrium is allocation functions $\left(s, \{c_a, h_a, b_{a+1}\}_{a=0}^{A-1}\right)$ and prices $p(s)$ such that
  - Households solve their problem
  - Skill price $p(s)$ is the marginal product of type s

$$p(s) = \left[\frac{Y}{N(s) f_s(s)}\right]^{\frac{1}{\omega}}$$

  - Densities for skills f_s and savings f_b are consistent with individual choices
  - Government budget is satisfied

$$gY \leq \sum_{a=0}^{A-1} \int T_a \left(y_{ia}; \{y_{it}\}_{t=0}^{a-1}\right) di$$

  - Markets clear

$$\sum_{a=0}^{A-1} \int c_{ia} di + gY = r \sum_{a=0}^{A-1} \int b_{ia} di + Y \text{ and } N(s) = \sum_{a=0}^{A-1} \int h_{ia}(s) \exp\{x(a) + z_{ia} + \varepsilon_{ia}\} di$$

Optimal Tax Problem

- Government's social welfare function is ex-ante expected utility of a household born into a stationary equilibrium

$$W = \int U_i di = \int \left\{ -v_i(s_i) + E_0 \left[\left(\frac{1-\beta}{1-\beta^A} \right) \sum_{a=0}^{A-1} \beta^a u_i(c_{ia}, h_{ia}) \right] \right\} di$$

- Government chooses the tax function $T(.)$ to maximize W subject to its budget

$$gY = \sum_{a=0}^{A-1} \int T_a \left(y_{ia}; \{y_{ij}\}_{j=0}^{a-1} \right) di$$

and that households solve their problem given the tax function

- Tax function $T(.)$ is of the form

$$T_a \left(y_a; \{y_j\}_{j=0}^{a-1} \right) = \tau_a \left(y_a; \{y_j\}_{j=0}^{a-1} \right) y_a$$

  - τ_a is a continuously differentiable function that depends on age and income history
  - Can prove this form has a unique mapping between tax rates and equilibrium allocations

Parameter Selection

Wage Estimation

- I estimate processes for wage shocks using data from the PSID
  - Regress log wages on a polynomial in age and demographic dummies

$$\log w_{ia} = x_0 + x_1 a + x_2 a^2 + D_i + \epsilon_{ia}$$

- Gives age profile $x(a) = x_0 + x_1 a + x_2 a^2$ and stochastic component of wages ϵ_{ia}
- Assume ϵ_{ia} is composed of permanent component z and transitory component ε

$$\epsilon_{ia} = z_{ia} + \varepsilon_{ia}$$

  where

$$z_{ia+1} = z_{ia} + \eta_{ia+1}, \ \eta_{ia} \sim N(0, v_\eta)$$
$$z_{i0} \sim N(0, v_z) \text{ and } \varepsilon_{ia} \sim N(0, v_\varepsilon)$$

- Estimate values of $(v_\eta, v_z, v_\varepsilon)$ to match $var(\epsilon_{ia}), var(\epsilon_{ia+2} - \epsilon_{ia})$ and $var(\epsilon_{ia+4} - \epsilon_{ia})$

Wage Parameters

Parameter	Description	Value
x_1	Linear component of life cycle profile	0.031
x_2	Quadratic component of life cycle profile	-0.0005
v_z	Variance of initial condition z_0	0.120
v_η	Variance of permanent shocks z	0.003
v_ε	Variance of transitory shocks ε	0.135

Table: Summary of Parameters for Wage Process

Fixed Parameters

Parameter	Description	Value	Source/Target
A	Years of working life	36	Heathcote et al. (2020)
ψ	Elasticity of skill investment to return	0.65	
ω	Elasticity of substitution across skills	3.124	
g	Government spending (% of output)	0.19	
β	Discount Factor	0.98	Golosov et al. (2016)
R	Return on savings	1/0.98	
m_ϕ	Mean of leisure disutilty	0.275	$H = 0.33$
v_ϕ	Variance of labor disutilty utility	0.026	$var(\log h_i) = 0.12$

Table: Summary of Fixed Parameters

Solution Method

Solving the Optimal Tax Problem

- Optimizing tax function directly requires keeping track of every previous level of income
  - With $A = 36$ periods of work, optimal tax has up to 36 arguments
  - Every argument of tax function becomes a state variable in HH problem
- Question: How do I approximate a nonlinear function with at least 36 arguments?

Polynomial vs NN Approximation

- Suppose I tried to approximate the same tax function with polynomials and NN's
- With polynomials (e.g. Chebychev, splines)
  1. Additive: additional parameters improve approx. for only a single dimension
  2. Manual: I have to choose where to place additional parameters
  - With $A = 36$ and affine/linear functions in each dim, would need $2^{36} \approx 70$ billion parameters
    - Approximation is infeasible, even for crude approximation
- With neural networks
  1. Compositional: additional parameters improve approx. for all dimensions simultaneously
  2. Automatic: self-allocates parameters to find best low-dimensional representation of full data
  - Only have to approximate several thousand of parameters ($\approx 5{,}000$) instead of billions

Details

How it Works

- Sketch of optimal tax algorithm:
  1. Approximate tax function as a neural network
  2. Calculate change in social welfare from changes in tax function ($\partial W/\partial\tau$)
     - Calculate how individual choices and GE prices change under perturbed tax function ($\partial c/\partial\tau$)
  3. Update tax function to maximize welfare (gradient descent: $\tau_{new} = \tau_{old} + \frac{\partial W}{\partial \tau}$)
- Similar to variational method used in optimal tax literature to derive optimal nonlinear tax in static economy (e.g. Saez (2001))
  - Optimal tax function is such that no change produces a welfare gain ($\partial W/\partial\tau = 0$)

Technical Version | Check the Approximation

Results

Outline of Results

1. Optimal taxes
   - Baseline: CES Production ▸ Here
   - Counterfactual 1: Fixed prices ▸ Here
   - Counterfactual 2: Fixed prices and skills ▸ Here
2. Compare steady state welfare under HD optimum to different tax functions ▸ Here
   - Parametric and nonparametric on current income, $T(y)$, then current income and age, $T(y, a)$
3. Simple implementation ▸ Here
   - Parametric tax in avg. lifetime income gets most welfare gains from full HD policy

▸ Results with Separable Utility

Conclusion

- What I did
  - Developed nested NN method to compute optimal history-dependent taxes in an OLG economy
  - Method can be applied to wide variety of problems to compute optimal policies
- What I found
  - Welfare gain from history-dependence can be large
  - Elasticity of substitution between skill types is critical to optimal policy:
    - Key elasticity to be estimated in future work in optimal taxation
- Why it was useful
  - Demonstrates new possible source of welfare gains from income taxation
  - Full HD optimal tax provides a benchmark to compare with more implementable policies
    - Here, function in just age and avg past income gets close to full HD optimum

Thank You!

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Checking the Approximation

Back

- I can check if the NN is finding a maximum of the HH problem by manually computing derivatives around the final allocations

	Percentile of Distribution					
	10%	25%	50%	75%	90%	Average
First ($\partial V/\partial h$)	-1.4 $\times 10^{-2}$	-6.5 $\times 10^{-3}$	-1.6 $\times 10^{-4}$	6.4 $\times 10^{-3}$	1.5 $\times 10^{-2}$	7.9 $\times 10^{-4}$
Second ($\partial^2 V/\partial h^2$)	-0.228	-0.193	-0.134	-0.110	-0.086	-0.151

Table: Derivatives of Household Value Function at Approximation

- First derivatives are near zero and second derivatives are negative $\Longrightarrow$ NN is finding a maximum

Checking the Approximation

◂ Back

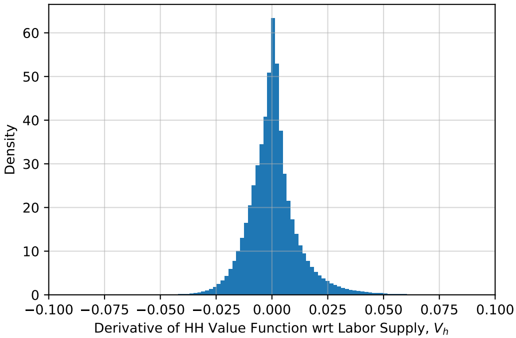
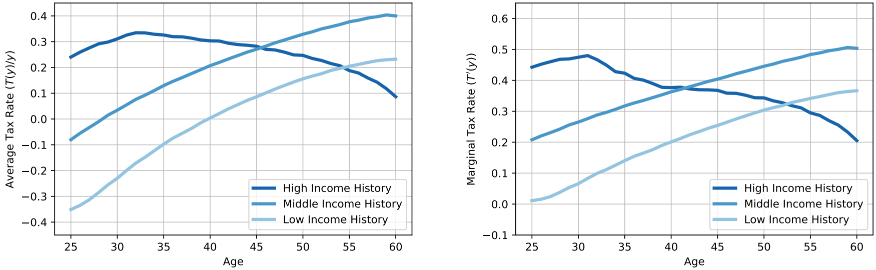


Figure: Histogram of Derivatives of HH Value Function ($\partial V/\partial h$)

Tax Rates Over the Life-cycle

Back



(a) Average Tax Rates (T_a/y_a) (b) Marginal Tax Rates ($\partial T_a/\partial y_a$)

Figure: Optimal History-Dependent Tax by Age

- Follow HH's that receive low (25th percentile), middle (75th) and high (99th) income each period
  - Middle income: taxes increase with deterministic life-cycle profile of wages
  - High income: taxes decrease as government learns their earning potential
    - Maintain high labor supply during life
    - Increase skill investment through expectation of lower future taxes

Discussion

◂ Back

- Why does history-dependent (HD) taxation increase skill investment?
- Optimality condition for skill investment:

$$\underbrace{v'(s_i)}_{\text{Marginal Cost of Skills}} = \underbrace{\sum_{a=0}^{A-1} \beta^a \underbrace{\frac{\partial y_{ia}}{\partial s_i} \left[1 - \overbrace{\sum_{k=a}^{A-1} \frac{\partial T_k\left(\{y_{ij}\}_{j=0}^{k}\right)}{\partial y_{ia}}}^{\text{Marginal Effect on Future Taxes}} \right]}_{\text{After-tax Wage}} \frac{1}{c_{ia}}}_{\text{Marginal Benefit of Skills}}$$

  - HH's w/ high earning potential anticipate low taxes after many years of high earnings
    - High effective after-tax wage early in life via lower future taxes: low $\partial T_k / \partial y_{ia}$
  - But households can't borrow, so high after-tax wage $\not\Longrightarrow$ high consumption, c_{ia}
  - High after-tax wage + low consumption = high skill investment
- Result: high skill investment $\Longrightarrow$ high output Y $\Longrightarrow$ high $p(s) = \left[\frac{Y}{N(s)f_s(s)}\right]^{\frac{1}{\omega}}$ for all s

Optimal HD Tax: Consumption and Income Profiles

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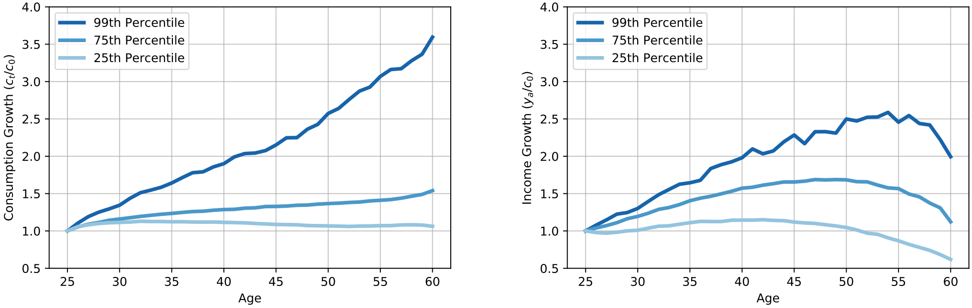


(a) Percentiles of Consumption by Age

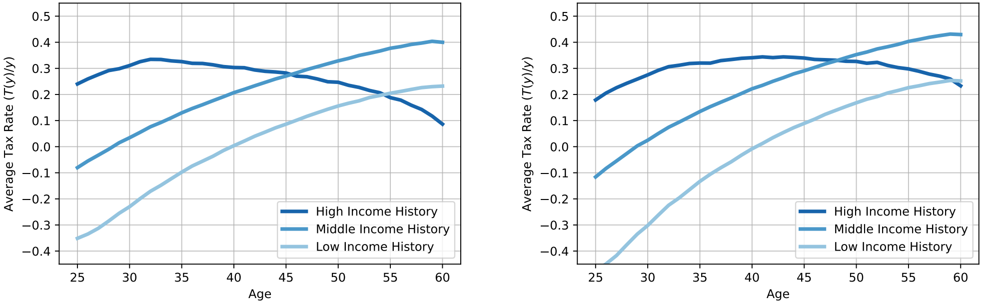
(b) Percentiles of Income by Age

Figure: Life-cycle Percentiles of Consumption and Income

- Consumption and income growth for 25th, 75th and 99th percentile of consumption and income
- Consumption is smooth for middle income HH's, but strongly increases for highest incomes

CF 1: Optimal HD Tax with Fixed Wages

Back



(a) Avg Tax Rates, Baseline Model (**Endogenous**)

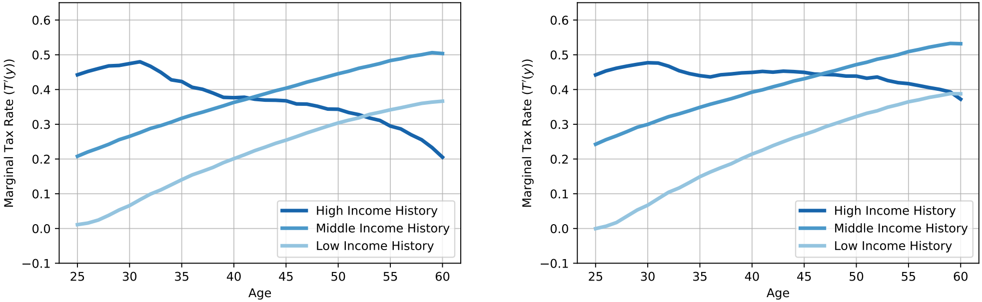
(b) Avg Tax Rates, **Exogenous** Wages

Figure: Optimal History-Dependent Tax by Age

- Fix prices (under US tax system): optimal taxes with endogenous skills, but fixed wage for each skill
- Taxes now about flat for high income history
  - History-dependence used less than before: less benefit from maintaining high output

CF 1: Optimal HD Tax with Fixed Wages

Back



(a) Marginal Tax Rates, Baseline Model (Endogenous) (b) Marginal Tax Rates, Exogenous Wages

Figure: Optimal History-Dependent Tax by Age

- Marginal taxes also flatter than with imperfect substitutability

CF 2: Optimal HD Tax with Fixed Wages and Skills

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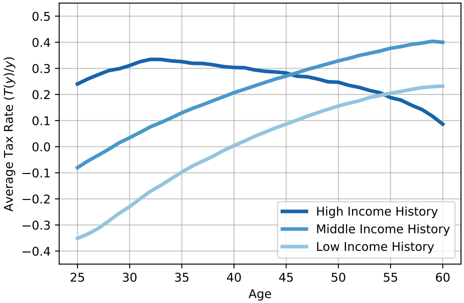
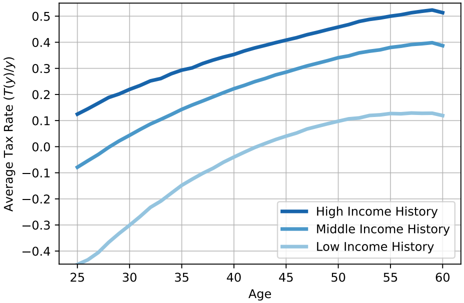


(a) Avg Tax Rates, Baseline Model (**Endogenous**)



(b) Avg Tax Rates, **Exogenous** Wages *and* Skills

Figure: Optimal History-Dependent Tax by Age

- Fix prices *and* skills (under US tax system)
- Taxes for high income history increase like others: virtually no benefit from increasing output
  - History-dependent optimum achievable with age-dependent tax

CF 2: Optimal HD Tax with Fixed Wages and Skills

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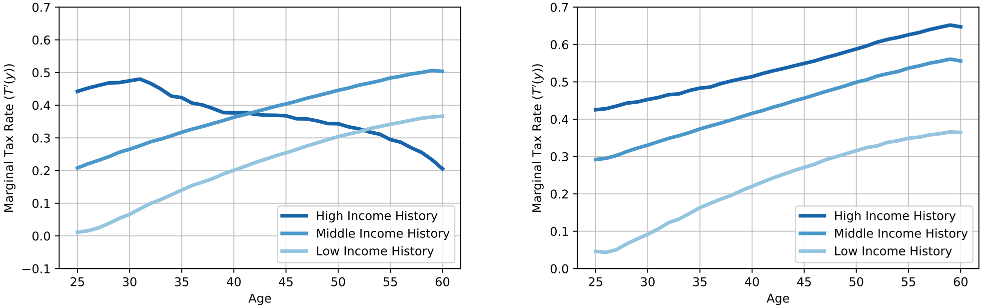

(a) Marginal Tax Rates, Baseline Model (Endogenous) (b) Marginal Tax Rates, Exogenous Wages and Skills

Figure: Optimal History-Dependent Tax by Age

- Marginal taxes also increasing in age for all income histories (not much need for HD taxes)

Consumption Equivalent Welfare Gains

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- To compare welfare impact of each policy, I use consumption equivalent welfare
- Consumption-equivalent welfare gain of moving to a new tax function T^* from T is the g such that

$$W\left(s^*, \{(c_a^*, h_a^*\}_{a=0}^{A-1}; T^*\right) = W\left(s, \{(1+g)c_a, h_a\}_{a=0}^{A-1}; T\right)$$

where W is social welfare

$$W\left(s, \{c_a, h_a\}_{a=0}^{A-1}; T\right) = \int \left\{ -v_i\left(s_i(T)\right) + E_0\left[\left(\frac{1-\beta}{1-\beta^A}\right)\sum_{a=0}^{A-1}\beta^a u_i\left(c_{ia}(T), h_{ia}(T)\right)\right]\right\} di$$

- g : percent gain of lifetime consumption necessary to deliver same welfare as T^*

Comparison with Restricted Tax Systems

- I compare the full history-dependent tax to more restricted systems
- Parametric tax functions that depend on (log-linear class used by Benabou (2000, 2002), Karabarbounis (2016) and Heathcote, Storesletten and Violante (2017, 2020))
  - Current income
$$T(y) = y - (1-\tau)y^{\rho}$$
  - Current income and age
$$T_a(y) = y - (1-\tau(a))y^{\rho(a)}$$
$$\tau(a) = \tau_0 + \tau_1 a + \tau_2 a^2 \text{ and } \rho(a) = \rho_0 + \rho_1 a + \rho_2 a^2$$
- Nonparametric tax functions that can depend on
  - Only current income
$$T(y) = \tau(y)y$$
  - Age and current income
$$T_a(y) = \tau(y,a)y$$
  where τ can be *any* differentiable function

Welfare Gains

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- Gains of moving *to* most complex policy *from* more restricted tax systems:

	Income History	Age and Current Income	Only Current Income
Nonparametric	0.0%	1.82%	5.43%
Log-linear		1.98%	6.31%

Table: Welfare Gain of Moving to Nonparametric, History Dependent Taxes

- Gain of HD tax over AD tax equivalent to 1.82 percent increase in lifetime consumption
- About 10-20 times bigger than existing studied of HD taxation ($< 0.2\%$)

Optimal Restricted Tax Functions | Summary of Tax Functions | Allocations | Present Value of Taxes Paid

Welfare Gains in Alternate Economies

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- Gains of moving *to* optimal HD tax system *from* more restricted tax systems:

Economy	Age-Dependent	Current Income Only
Endogenous Skills and Wages (Baseline)	1.82%	5.43%
Endogenous Skills, Exogenous Wages	0.81%	4.36%
Exogenous Skills and Wages	0.10%	4.20%

Table: Welfare Gain of Moving to History-Dependent Tax System from Optimal Restricted Tax

- Removing endogenous effects on wages eliminates most gains from HD over age-dependent
- Removing effects on *both* wages and skill investment virtually eliminates all gains from HD
  - Consistent with existing studies of HD taxation

What does this mean for policy? Back

- Non-parametric HD policy not easily implemented in reality and hard to interpret
  - What simple parametric policy can achieve similar levels of welfare?

- Full optimum gives you guidance on which simpler policies can achieve highest possible welfare
  - E.g., let log-linear tax vary with age *and* average lifetime income ($\bar{y} = \frac{1}{a-1}\sum_{t=0}^{a-1} y_t$)

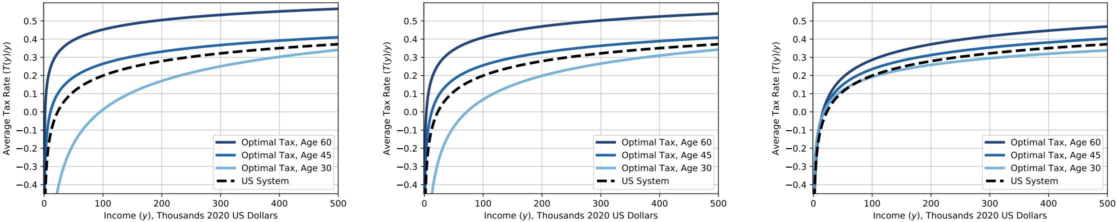
$$T(y; a, \bar{y}) = y - (1 - \tau(a, \bar{y}))y^{\rho(a,\bar{y})}$$

$$\tau(a, \bar{y}) = \tau_0 + \tau_1 a + \tau_2 a^2 + \tau_3 a\bar{y} + \tau_4 a^2 \bar{y} \text{ and } \rho(a, \bar{y}) = \rho_0 + \rho_1 a + \rho_2 a^2 + \rho_3 a\bar{y} + \rho_4 a^2 \bar{y}$$

  - Turns out to capture 90% of welfare gain of HD tax compared to just parametric AD tax
    - Only lose 0.2% of consumption compared to full non-parametric HD tax

History-Dependent Parametric Tax

Back



(a) Optimal Tax by Age, $\bar{y} = \$25,000$ (b) Optimal Tax by Age, $\bar{y} = \$100,000$ (c) Optimal Tax by Age, $\bar{y} = \$500,000$

Figure: Optimal Parametric History-Dependent Tax

- Captures key feature of full HD policy: HH's are rewarded for high past output with smaller increase
  - Low income history $\implies$ taxes increase quickly with age
  - High income history $\implies$ taxes increase more slowly with age

Marginal tax Rates

How Neural Networks Work

Back

- **Neural Network**: *nonlinear transformation of weighted sums*

$$y(x;w) \approx \sum_{i=1}^{m} w_{2,i} f\left(\sum_{j=0}^{n} w_{1,j,i} x_j\right)$$

  - y is the function being approximated
  - x is the vector of n state variables
  - m is the degree of approximation (determines accuracy)
  - f is a nonlinear function, e.g. tanh
  - Number of parameters w to estimate $= m + m \times (n+1) \implies$ grows linearly in size of state x
- NN finds weighted sums of state variables as low dimension representation of full state
  - In many cases, some average can accurately represent key features of the full state
  - Especially useful with a large number of state variables that behave "similarly"
    - e.g. many locations, countries, sectors, wage shocks, previous income levels
- Universal Approximation Theorem: NN can approximate *any* continuous function

Deep Learning

Back

- More complex functions can be approximated by adding additional transformations of data (*layers*)
- With more layers, parameters still grow linearly with number of state variables (n)
- Two layers

$$y(x;w) \approx \sum_{i_2=1}^{m} w_{3,i_2} f\left(\sum_{i_1=1}^{m} w_{2,i_1,i_2} f\left(\sum_{j=1}^{n} w_{1,j,i_{i1}} x_j\right)\right)$$

$$\text{num. parameters} = n \times m + m^2 + m$$

- L layers (usually at most 5)

$$y(x;w) \approx \sum_{i_L=1}^{m} w_{L,i_L} f\left(\sum_{i_{L-1}=1}^{m} w_{L-1,i_{L-1},i_L} f\left(\cdots \sum_{i_2=1}^{m} w_{3,i_2,i_3} f\left(\sum_{i_1=1}^{m} w_{2,i_1,i_2} f\left(\sum_{j=1}^{n} w_{1,j,i_{i1}} x_j\right)\right)\dots\right)\right)$$

$$\text{num. parameters} = n \times m + (L-1)m^2 + m$$

Comparing Approximation Methods

Back

- **Standard Polynomial Approximation**: *weighted sum of nonlinear transformations*

$$T(x;w) \approx \sum_{i_a=1}^{n} \cdots \sum_{i_0=1}^{n} w_{i_0,\ldots,i_a} f_{i_0,\ldots,i_a}(x), \text{ where } x = (\{y_t\}_{t=0}^{a-1}, a)$$

  - Number of parameters w to estimate $= n^A \implies$ grows exponentially

- **Neural Network**: *nonlinear transformation of weighted sums*

$$T(x;w) \approx \sum_{i=1}^{m} w_{2,i} f\left(\sum_{j=0}^{a} w_{1,j,i} x_j\right), \text{ where } x = (\{y_t\}_{t=0}^{a-1}, a)$$

  - Number of parameters w to estimate $= m + m \times (a+1) \implies$ grows linearly

- NN finds weighted sum of state variables as low dimension representation of state
  - Usually some average can accurately represent the full state vector
  - Especially if state variables are similar (like with income history)

Solution Method for Optimal Taxes

Back

- Tax function, individual choices and prices are approximated as separate NN's

$$T\left(y_a; a, \{y_t\}_{t=0}^{a-1} \mid w_T\right),\ c(b, z, \varepsilon, s, \phi, a, \{y_t\}_{t=0}^{a-1} \mid w_c; w_p, w_T) \text{ and } p(s \mid w_p; w_c, w_T)$$

- Consider a perturbation to one weight of the tax function: $w_T + \Delta = (w_1, \ldots, w_i + \delta, \ldots, w_n)$
  - Update individual choices (and skill prices) under the perturbed tax function by gradient descent

$$\tilde{w}_c = w_c + \frac{\partial U(w_c; w_T + \Delta, \tilde{w}_p)}{\partial w_c}$$

  - Compute social welfare under the perturbed tax function: $W(\tilde{w}_c, \tilde{w}_p, w_T + \Delta)$

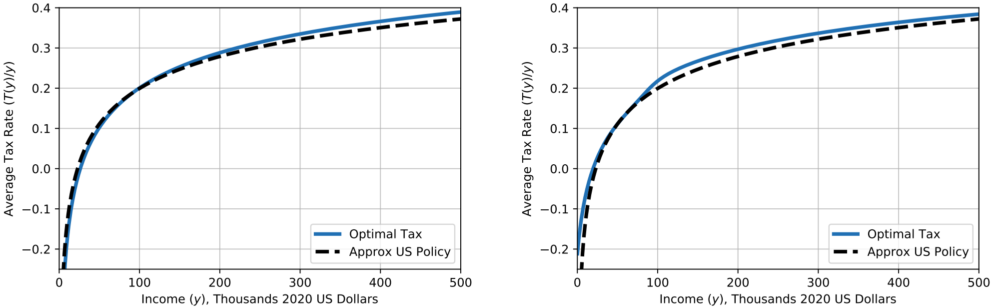
- Optimal tax system is the w_T such that a perturbation produces no welfare gain

$$\frac{\partial W(w_c, w_p, w_T + \Delta)}{\partial \delta} = 0$$

- All optimization done by computer: no taking foc's by hand

Optimal Tax on Current Income

Back



(a) Optimal Parametric Tax, $T(y) = y - (1 - \tau)y^{\rho}$ (b) Optimal Nonparametric Tax, $T(y)$

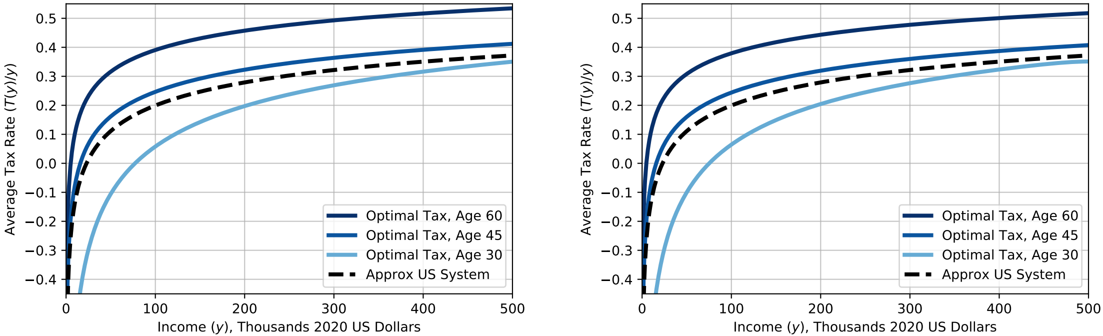
Figure: Optimal Tax on Current Income

- Optimal taxes have similar progressivity to US system
- Nonparametric tax has lower marginal tax rates for low and middle incomes ($<$ \$100K)

Marginal Tax Rates

Optimal Age-Dependent Tax

Back



(a) Optimal Parametric Tax, $T(y,a) = y - (1-\tau(a))y^{\rho(a)}$

(b) Optimal Nonparametric Tax, $T(y,a)$

Figure: Optimal Age-Dependent Tax on Current Income

- Optimal taxes increase and become less progressive with age
- Mostly similar except nonparametric has low taxes on young with very high income

Marginal Tax Rates Present Values

Current Income Marginal Tax Rates

Back

	Current Income, Thousands of Dollars					
	10	25	50	100	200	500
US System	5%	17%	25%	32%	39%	47%
Parametric	4%	22%	25%	34%	41%	49%
Nonparametric	3%	15%	24%	40%	40%	47%

Table: Marginal Tax Rates, Taxes on Current Income

Age-Dependent Marginal Tax Rates

Back

	Current Income, Thousands of Dollars					
Age	10	25	50	100	200	500
30	-21%	1%	15%	28%	38%	50%
45	10%	22%	29%	36%	43%	50%
60	26%	36%	43%	49%	55%	61%

Table: Marginal Tax Rates, Parametric Age-Dependent Tax

	Current Income, Thousands of Dollars					
Age	10	25	50	100	200	500
30	-20%	0%	16%	28%	39%	19%
45	10%	21%	29%	36%	42%	50%
60	26%	35%	42%	48%	53%	59%

Table: Marginal Tax Rates, Nonparametric Age-Dependent Tax

Allocations with HD Policy

Back

- Compare steady state allocations under HD tax to allocations under AD tax
- Average gain of switching from AD tax to HD tax by present value of income:

$$PV_i(y) = \left(\frac{1-R^{-1}}{1-R^{-A}}\right)\sum_{a=0}^{A-1} R^{-a} y_{ia}$$

	Quartile, Present Value of Income				
	0-25%	25-50%	50-75%	75-100%	Total
Income ($PV(y)$)	-0.82%	-0.56%	-0.17%	5.20%	2.12%
Consumption ($PV(c)$)	0.35%	-0.10%	-0.23%	7.08%	2.85%
Leisure ($PV(1-h)$)	0.38%	0.40%	0.38%	-1.15%	0.01%
Skills (s)	-2.26%	-0.66%	0.65%	1.64%	0.51%
Skill Price ($p(s)$)	-0.39%	0.19%	0.75%	1.72%	0.70%

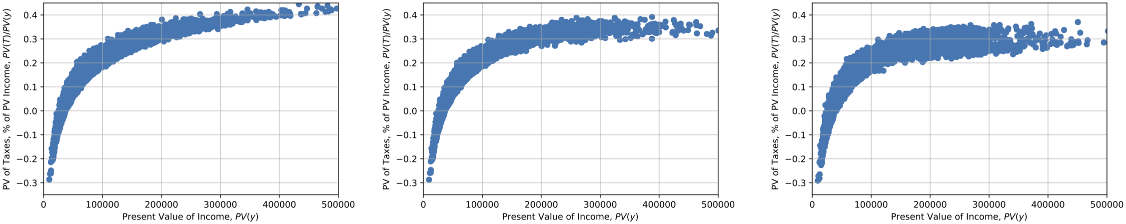
Table: Percent Gain in Average Allocations by Quartile of Present Value of Income, AD to HD Tax

- Note: $\phi = 0.275$, so leisure is valued about $3.6\times$ consumption
- Higher $Y \implies$ higher $p(s) = (Y/[N(s)f_s(s)])^{\frac{1}{\omega}} \implies$ similar c with lower h and s

Levels

Present Value of Taxes Paid

Back



(a) Parametric Age-Dependent Taxes (b) Nonparametric Age-Dependent Taxes (c) History-Dependent Taxes

Figure: Present Value of Taxes Paid by Present Value of Income

- Compare present value of taxes paid in initial period: $PV_i(T) = \left(\frac{1-R^{-1}}{1-R^{-A}}\right)\sum_{a=0}^{A-1} R^{-a} T_a(y_{ia})$
  - PV of taxes paid are similar in all three for incomes below \$100K
  - Nonparametric taxes flatter for incomes over \$200K
  - HD taxes more dispersed for incomes over \$100K

Allocations with HD Policy

Back

	Present Value of Income Quartile				
	0-25%	25-50%	50-75%	75-100%	Total
Income ($PV(y)$)	37,991	58,902	82,549	149,060	82,124
Consumption ($PV(c)$)	36,466	51,568	67,279	106,729	65,510
Taxes ($PV(T)$)	1,525	7,333	15,267	42,315	16,610
Avg. Tax Rates ($PV(T/y)$)	-4.4%	3.9%	9.4%	17.6%	6.6%
Leisure ($PV(1-h)$)	0.648	0.643	0.639	0.634	0.641
Skills (s)	0.221	0.333	0.460	0.823	0.459
Skill Price ($p(s)$)	0.179	0.197	0.218	0.279	0.218

Table: Average Allocations by Quartile of PV Income Distribution, AD Tax

	Quartile, Present Value of Income				
	0-25%	25-50%	50-75%	75-100%	Total
Income ($PV(y)$)	37,679	58,573	82,407	156,819	83,868
Consumption ($PV(c)$)	36,588	51,518	67,124	114,281	67,377
Taxes ($PV(T)$)	1,164	7,117	15,288	42,588	16,539
Avg. Tax Rates ($PV(T/y)$)	-5.6%	3.1%	9.0%	17.4%	6.0%
Leisure ($PV(l)$)	0.650	0.646	0.642	0.626	0.641
Skills (s)	0.216	0.331	0.463	0.837	0.462
Skill Price ($p(s)$)	0.178	0.198	0.219	0.284	0.220

Table: Average Allocations by Quartile of PV Income Distribution, HD Tax

Parametric History-Dependent Marginal Tax Rates

Back

	Current Income, Thousands of Dollars					
Age	10	25	50	100	200	500
30	-25%	-1%	14%	27%	38%	50%
45	13%	23%	30%	36%	42%	49%
60	33%	41%	47%	52%	57%	62%

Table: Marginal Tax Rates, Parametric History-Dependent Tax, $\bar{y} = \$50,000$

	Current Income, Thousands of Dollars					
Age	10	25	50	100	200	500
30	7%	16%	23%	29%	35%	42%
45	9%	20%	28%	35%	42%	49%
60	12%	25%	34%	42%	49%	57%

Table: Marginal Tax Rates, Parametric History-Dependent Tax, $\bar{y} = \$500,000$

Separable Utility

◂ Back

- Now consider the same exercise, but with a separable utility function

$$u_i(c_{ia}, h_{ia}) = \log c_{ia} - \exp \varphi_i \frac{h_{ia}^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}, \ \varphi \sim N(m_\varphi, v_\varphi)$$

  - Labor supply elasticity now constant for all households and equals ν

- Set $\nu = 0.5$ (standard value), re-calibrate (m_φ, v_φ) and compute welfare gains as before

	Income History	Age and Current Income	Only Current Income
Nonparametric	0.0%	1.55%	5.20%
Log-linear		1.65%	5.33%

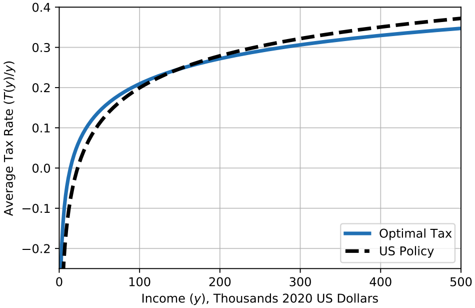
Table: Welfare Gain of Moving to Nonparametric, History Dependent Taxes

▸ Optimal Tax Functions

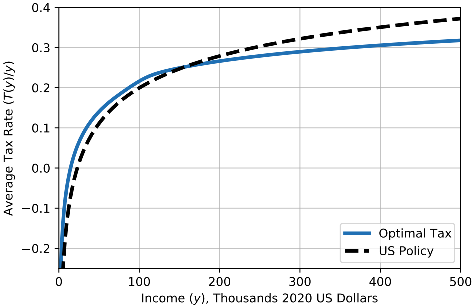
▸ Marginal Tax Rates

Separable Utility: Tax on Current Income

◂ Back



(a) Optimal Parametric Tax, $T(y) = y - (1-\tau)y^{\rho}$



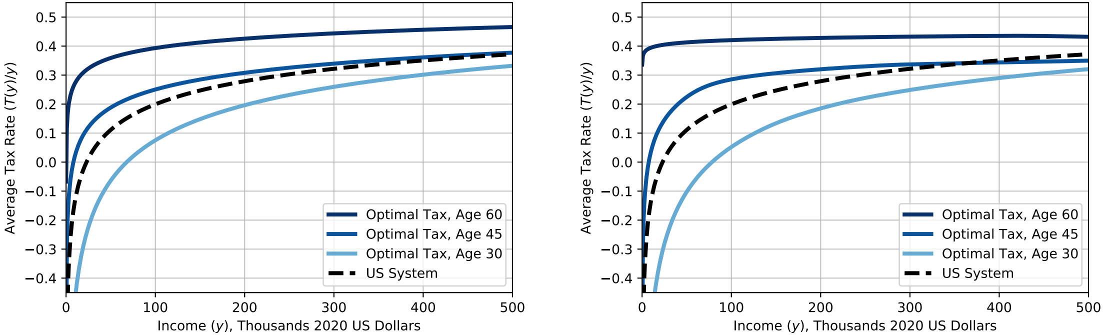
(b) Optimal Nonparametric Tax, $T(y)$

Figure: Optimal Tax on Current Income, Separable Utility

- Dotted line is parametric approximation of US income tax system as $T_{US}(y) = y - (1-\tau_{US})y^{\rho_{US}}$

Separable Utility: Age-Dependent Tax

Back



(a) Optimal Parametric Tax, $T(y,a) = y - (1-\tau(a))y^{\rho(a)}$

(b) Optimal Nonparametric Tax, $T(y,a)$

Figure: Optimal Age-Dependent Tax on Current Income, Separable Utility

Separable Utility: History-Dependent Tax

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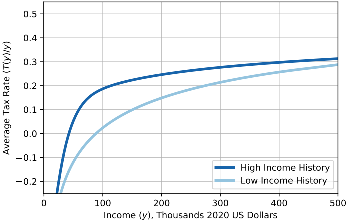
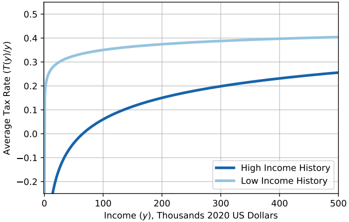
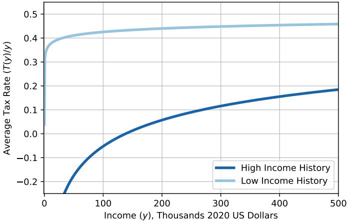


(a) Optimal Tax After 10 Years



(b) Optimal Tax After 20 Years



(c) Optimal Tax After 30 Years

Figure: Optimal Non-parametric History-Dependent Tax, Nonseparable Utility

- Tax rates of two households who each earn \$50K and \$400K for first 10, 20 and 30 years of working life

Separable Utility: History-Dependent Marginal Tax Rates

◂ Back

	Age			
	30	40	50	60
Low Income History ($50K)	-25%	30%	35%	49%
Middle Income History ($100K)	-5%	21%	38%	49%
High Income History ($400K)	29%	33%	20%	9%

Table: Average Tax Rates, History-Dependent Tax, Separable Utility

	Age			
	30	40	50	60
Low Income History ($50K)	6%	36%	39%	49%
Middle Income History ($100K)	20%	32%	41%	49%
High Income History ($400K)	37%	40%	32%	26%

Table: Marginal Tax Rates, History-Dependent Tax, Separable Utility

Summary of Tax Functions

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	History of Income	Age and Current Income	Only Current Income
Nonparametric	$T(y;\{y_t\}_{t=0}^{a-1},a)$	$T(y;a)$	$T(y)$
Log-linear		$y-(1-\tau(a))y^{\rho(a)}$	$y-(1-\tau)y^{\rho}$

Table: Summary of Tax Functions