M3/4/5P12 PROGRESS TEST #2

PLEASE WRITE YOUR NAME AND CID NUMBER ON EVERY SCRIPT THAT YOU HAND IN. FAILURE TO DO THIS MAY RESULT IN YOU NOT RECEIVING MARKS FOR QUESTIONS THAT YOU ANSWER.

Note: all representations are assumed to be on finite dimensional complex vector spaces.

Question 1. Let G be a finite group and let (V, ρ_V) and (W, ρ_W) be representations of G.

(1) Define the representation with underlying vector space $\mathrm{Hom}(V, W)$.
(2) Prove that $f : V \to W$ is G-linear if and only if $f \in \mathrm{Hom}(V, W)^G \subset \mathrm{Hom}(V, W)$, where $\mathrm{Hom}(V, W)^G$ denotes the invariant subspace.
(3) What is the formula for the character of the representation $\mathrm{Hom}(V, W)$ in terms of the characters χ_V and χ_W of (V, ρ_V) and (W, ρ_W)? Justify your answer if (W, ρ_W) is the trivial 1-dimensional representation.
(4) Let $G = D_{10} = \langle s, t : s^5 = t^2 = e, tst = s^{-1}\rangle$. There are two irreducible 2-dimensional representations of G, which we call (V, ρ_V) and (W, ρ_W). If we choose suitable bases of V and W, we have

$$\rho_V(s) = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \qquad \text{and} \qquad \rho_V(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$\rho_W(s) = \begin{pmatrix} \zeta^2 & 0 \\ 0 & \zeta^{-2} \end{pmatrix} \qquad \text{and} \qquad \rho_W(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $\zeta := e^{2\pi i/5}$.

Compute the decomposition of $\mathrm{Hom}(V, W)$ into irreducible representations of D_{10}. Justify your answer.

Question 2.

(1) Let G be a finite group. State the row and column orthogonality relations for a character table.
(2) For the rest of this question, G will be a group of order 20 with 5 conjugacy classes. Here are the first two lines of its character table:

	$g_1 = e$	g_2	g_3	g_4	g_5
size of conjugacy class	1	4	5	5	5
$\chi_1 = \chi_{\mathrm{triv}}$	1	1	1	1	1
χ_2	1	1	i	-1	$-i$

What are the dimensions of the remaining representations of G? Justify your answer.
(3) Find another 1-dimensional representation of G.
(4) Find the remaining entries of the character table, and justify your answer.