M3/4/5P12 PROGRESS TEST #2

PLEASE WRITE YOUR NAME AND CID NUMBER ON EVERY SCRIPT THAT YOU HAND IN. FAILURE TO DO THIS MAY RESULT IN YOU NOT RECEIVING MARKS FOR QUESTIONS THAT YOU ANSWER.

Note: all representations are assumed to be on finite dimensional complex vector spaces.

Question 1.

(1) We define a representation $(\mathrm{Hom}(V, W), \rho_{\mathrm{Hom}(V,W)})$ by setting $\rho_{\mathrm{Hom}(V,W)}(g)(f) = \rho_W(g) \circ f \circ \rho_V(g^{-1})$.

(2) Suppose $f : V \to W$ is G-linear. Then for any $g \in G$,

$$\rho_{\mathrm{Hom}(V,W)}(g)(f) = \rho_W(g) \circ f \circ \rho_V(g^{-1}) = \rho_W(g) \circ (\rho_W(g^{-1}) \circ f) = f$$

so $f \in \mathrm{Hom}(V, W)^G$.

On the other hand, suppose $\rho_{\mathrm{Hom}(V,W)}(g)(f) = f$ for all $g \in G$. Then $f \circ \rho_V(g^{-1}) = \rho_W(g^{-1}) \circ f$ for every $g \in G$, and since $\{g^{-1}\}_{g \in G} = G$, this implies that f is G-linear.

(3) The formula is $\chi_{\mathrm{Hom}(V,W)} = \overline{\chi_V} \cdot \chi_W$.

Suppose that W is the trivial 1-dimensional representation, so that $\mathrm{Hom}(V, W) = V^*$ as a representation. If we fix $g \in G$, we may choose a basis of V so that $\rho_V(g)$ is diagonal. We choose the dual basis for V^*; if M is the matrix for $\rho_V(g)$, then the matrix for $\rho_{V^*}(g)$ with respect to the dual basis is $\overline{M}^t$. This is still a diagonal matrix, and

$$\chi_{V^*}(g) = \mathrm{Tr}((M^{-1})^t) = \mathrm{Tr}(M^{-1}) = \overline{\chi_V(g)}$$

(4) The conjugacy classes of D_{10} are $\{e\}$, $\{s, s^{-1}\}$, $\{s^2, s^{-2}\}$, and $\{t, st, s^2t, s^3t, s^4t\}$. There are two 1-dimensional representations of D_{10}, namely the trivial representation and (U, ρ_U) given by $\rho_U(s) = 1$, $\rho_U(t) = -1$. Thus, we have characters

	$\{e\}$	$\{s^{\pm 1}\}$	$\{s^{\pm 2}\}$	$\{s^i t\}$
size of conjugacy class	1	2	2	5
$\chi_{\mathrm{triv}}(g)$	1	1	1	1
$\chi_U(g)$	1	1	1	-1
$\chi_V(g)$	2	$\zeta + \zeta^{-1}$	$\zeta^2 + \zeta^{-2}$	0
$\chi_W(g)$	2	$\zeta^2 + \zeta^{-2}$	$\zeta + \zeta^{-1}$	0
$\chi_{\mathrm{Hom}(V,W)}(g)$	4	$\zeta + \zeta^2 + \zeta^3 + \zeta^4$	$\zeta + \zeta^2 + \zeta^3 + \zeta^4$	0

Since $\chi_{\mathrm{Hom}(V,W)} = \chi_V + \chi_W = \chi_{V \oplus W}$, it follows that $\mathrm{Hom}(V, W) \cong V \oplus W$ as isomorphisms of D_{10}.

Alternatively, we can compute the inner products of $\chi_{\mathrm{Hom}(V,W)}$ with the irreducible characters of D_{10}:

$$\begin{aligned}
\langle \chi_{\mathrm{triv}}, \chi_{\mathrm{Hom}(V,W)}(g)\rangle &= \frac{1}{10}\left(4 + 2\cdot(\zeta+\zeta^2+\zeta^3+\zeta^4) + 2\cdot(\zeta+\zeta^2+\zeta^3+\zeta^4) + 0\right)\\
&= \frac{4}{10}\left(1+\zeta+\zeta^2+\zeta^3+\zeta^4\right) = 0\\
\langle \chi_U, \chi_{\mathrm{Hom}(V,W)}(g)\rangle &= \frac{1}{10}\left(4 + 2\cdot(\zeta+\zeta^2+\zeta^3+\zeta^4) + 2\cdot(\zeta+\zeta^2+\zeta^3+\zeta^4) + 0\right) = 0\\
\langle \chi_V, \chi_{\mathrm{Hom}(V,W)}(g)\rangle &= \frac{1}{10}\left(8 + 2(\zeta+2\zeta^2+2\zeta^3+\zeta^4+2) + 2(2+2\zeta+\zeta^2+\zeta^3+2\zeta^4) + 0\right)\\
&= \frac{1}{10}\left(16+6\zeta+6\zeta^2+6\zeta^3+6\zeta^4\right) = 1\\
\langle \chi_W, \chi_{\mathrm{Hom}(V,W)}(g)\rangle &= \frac{1}{10}\left(8 + 2(2+2\zeta+\zeta^2+\zeta^3+2\zeta^4) + 2(\zeta+2\zeta^2+2\zeta^3+\zeta^4+2) + 0\right)\\
&= \frac{1}{10}\left(16+6\zeta+6\zeta^2+6\zeta^3+6\zeta^4\right) = 1
\end{aligned}$$

We have used the fact that $1+\zeta+\zeta^2+\zeta^3+\zeta^4 = 0$.

Thus, $\mathrm{Hom}(V,W) \cong V \oplus W$.

Question 2.

(1) Let $\chi_1, \ldots, \chi_r$ be the irreducible characters of G and let $g_1 = e, \ldots, g_r$ be representatives for the conjugacy classes $C_1, \ldots, C_r$ of G. Then the row relations state that

$$\langle \chi_i, \chi_j\rangle = \frac{1}{|G|}\sum_{g\in G}\chi_i(g)\overline{\chi_j(g)} = \begin{cases} 0 & \text{if } i \neq j\\ 1 & \text{if } i = j\end{cases}$$

The column relations state that

$$\sum_k \overline{\chi_k(g_i)}\chi_k(g_j) = \begin{cases} 0 & \text{if } i \neq j\\ \frac{|G|}{|C_i|} & \text{if } i = j\end{cases}$$

(2) Character tables are square, so there are three more irreducible representations. If (V_i, ρ_i) are the remaining irreducible representations, then $1^2 + 1^2 + \sum_i (\dim V_i)^2 = 20$, so $\sum_i(\dim V_i)^2 = 18$. The only way to write 18 as the sum of three squares is $18 = 1^2+1^2+4^2$, so there are two more 1-dimensional representations and one 4-dimensional irreducible representation.

(3) If χ_2 is the character of the 1-dimensional representations (V_2, ρ_2), then $(V_2\otimes V_2, \rho_{V_2\otimes V_2})$ is another 1-dimensional representation with character χ_2^2. Similarly, $(V_2^*, \rho_{V_2^*}) \cong (V_2\otimes V_2\otimes V_2, \rho_{V_2\otimes V_2\otimes V_2})$ is another 1-dimensional representation with character $\overline{\chi_2} = \chi_2^3$.

(4) The character table is now

	$g_1 = e$	g_2	g_3	g_4	g_5
size of conjugacy class	1	4	5	5	5
$\chi_1 = \chi_{\mathrm{triv}}$	1	1	1	1	1
χ_2	1	1	i	-1	$-i$
χ_3	1	1	-1	1	-1
χ_4	1	1	$-i$	-1	i
χ_5	4	?	?	?	?

The column relations imply that

$$\sum_k \overline{\chi_k(g_1)}\chi_k(g_2) = 4 + 4\chi_5(g_2) = 0$$

so $\chi_5(g_2) = -1$.

The column relations imply that

$$\sum_k \overline{\chi_k(g_1)}\chi_k(g_3) = (1+i-1-i) + 4\chi_5(g_3) = 4\chi_5(g_3) = 0$$

so $\chi_5(g_3) = 0$.

The column relations imply that

$$\sum_k \overline{\chi_k(g_1)}\chi_k(g_4) = (1-1+1-1) + 4\chi_5(g_4) = 4\chi_5(g_4) = 0$$

so $\chi_5(g_4) = 0$.

The column relations imply that

$$\sum_k \overline{\chi_k(g_1)}\chi_k(g_5) = (1-i-1+i) + 4\chi_5(g_5) = 4\chi_5(g_5) = 0$$

so $\chi_5(g_5) = 0$.