MODULARITY OF TRIANGULINE GALOIS REPRESENTATIONS

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ABSTRACT. We use the theory of trianguline (φ, Γ)-modules over pseudorigid spaces to prove a modularity lifting theorem for certain Galois representations which are trianguline at p, including those with characteristic p coefficients. The use of pseudorigid spaces lets us construct integral models of the trianguline varieties of [BHS17], [Che13], and we carry out a Taylor–Wiles patching argument for families of overconvergent modular forms. This permits us to construct a patched quaternionic eigenvariety and deduce our modularity results.

1. INTRODUCTION

The Fontaine–Mazur conjecture predicts that representations of Galois groups of number fields which are sufficiently nice should come from geometry. In practice, the way one proves this is by proving so-called automorphy lifting theorems, relating the Galois representations of interest to Galois representations already known to have the desired properties.

In this context, if $\rho : \mathrm{Gal}_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ is the representation, "sufficiently nice" includes a condition on the local Galois group at p called being *geometric*. In the present paper, motivated by a question of Andreatta–Iovita–Pilloni [AIP18], we consider a characteristic p analogue of this conjecture. There is no definition of "geometric" for a Galois representation with positive characteristic coefficients, but we replace it with the condition *trianguline*:

Theorem. *Assume $p \geq 5$, and let L be a finite extension of $\mathbf{F}_p((u))$. Let $\rho : \mathrm{Gal}_{\mathbf{Q}} \to \mathrm{GL}_2(\mathscr{O}_L)$ be an odd continuous Galois representation unramified away from p such that the (φ, Γ)-module $D_{\mathrm{rig}}(\rho|_{\mathrm{Gal}_{Q_p}})$ is trianguline with regular parameters. Assume moreover that the reduction $\overline{\rho}$ is modular and satisfies certain additional technical hypotheses. Then ρ corresponds to a point on an eigencurve $\mathscr{X}_{\mathrm{GL}_2}$ with tame level $\Gamma_1(N)$ for any $N \geq 5$ prime to p.*

The eigencurve was originally constructed by Coleman–Mazur, and it is a rigid analytic space whose points correspond to *overconvergent modular forms*. Each point of the eigencurve has a Galois representation attached, but Kisin [Kis03] showed that the Galois representations at non-classical points are not geometric at p. Instead, they are trianguline (though he did not use this terminology; it was introduced subsequently by Colmez). A converse was proved by Emerton [Eme11, Theorem 1.2.4] when the coefficients are p-adic.

Given a p-adic Galois representation ρ, there is an associated object $D_{\mathrm{rig}}(\rho)$ called a (φ, Γ)-module; at the expense of making the coefficients more complicated, the

Galois representation can be captured as the action of a semi-linear operator φ together with the action of a 1-dimensional p-adic Lie group Γ. Then even if ρ is irreducible, it is possible for $D_{\mathrm{rig}}(\rho)$ to be reducible. Kisin showed that this happens in small neighborhoods of classical points on the eigencurve; if ρ_x is the Galois representation attached to a point x, there is an exact sequence

$$0 \to D_1 \to D_{\mathrm{rig}}(\rho_x) \to D_2 \to 0$$

where D_1 and D_2 are rank-1 (φ, Γ)-modules. There is a basis element $\mathbf{e}_1$ of D_1 such that φ acts on $\mathbf{e}_1$ by the U_p-eigenvalue at x and Γ acts on $\mathbf{e}_1$ trivially. This construction was extended over (a normalization of) the eigencurve in separate work of [KPX14] and [Liu15].

More recently, the construction of the eigencurve has been extended to mixed characteristic by Andreatta–Pilloni–Iovita [AIP18], [AIP16] and Johansson–Newton [JN16], using Huber's theory of adic spaces instead of Tate's theory of rigid analytic spaces. These authors construct *pseudorigid spaces* containing characteristic 0 eigenvarieties as open subspaces, with non-empty characteristic p loci.

In previous work, we generalized the construction of (φ, Γ)-module to families of Galois representations with pseudorigid coefficients [Bel20] and showed that the triangulation of the eigencurve extends to the boundary characteristic p points [Bel21]. In the present paper, we use that machinery to prove a modularity result for Galois representations trianguline at p.

The proof rests on the Taylor–Wiles patching method, as reformulated in [Sch18]. However, there are a number of technical complications. For example, to carry out some preliminary reductions, we first prove a version of the Jacquet–Langlands correspondence on eigenvarieties extending the construction of [Bir19], and we characterize the image of the cyclic base change morphism $\mathscr{X}_{\mathrm{GL}_2/\mathbf{Q}} \to \mathscr{X}_{\mathrm{GL}_2/F}$ of [JN19a]. The latter uses the construction of an auxiliary "$\mathrm{Gal}(F/\mathbf{Q})$-fixed" eigenvariety, which may be of independent interest. This permits us to transfer the problem to overconvergent quaternionic modular forms over an extension of $\mathbf{Q}$.

The modules of quaternionic automorphic forms we patch are those constructed in [JN16]. We construct trianguline deformation rings which act on them, and we patch by introducing ramification at additional primes. But the construction of trianguline deformation rings is delicate, because in general triangulations of (φ, Γ)-modules do not interact well with integral structures on the corresponding Galois representation. Thus, we crucially use the pseudorigid theory of triangulations (and not just the rigid analytic theory) to ensure that we can construct an integral quotient of a Galois deformation ring whose analytic points are trianguline.

This leads to a further difficulty, which is that it is difficult to study the components of the trianguline deformation ring directly. Instead, we patch families of overconvergent automorphic forms, which lets us compare the Galois representation we are interested in with "nearby" representations which are known to be automorphic. Along the way, we construct local pieces of a patched quaternionic eigenvariety $\mathscr{X}_{\underline{D}^\times}^\infty$, together with a morphism to a trianguline variety and a patched module of overconvergent modular forms.

We have not attempted to work in maximum generality. In particular, it should be possible to relax the ramification condition and prove an overconvergent modularity lifting theorem for certain totally real fields. However, this would require constructing and studying a cyclic base change morphism for more general extensions of number fields. More seriously, in the course of the proof we pass to an extension $F/\mathbf{Q}$, and for technical reasons we need to ensure that the class group of F has order prime to p; this is possible in our setting but will be much more difficult in general.

We have also not optimized the tame level in the conclusion of Theorem 1. We expect ρ to correspond to a point on the eigencurve of tame level 1, and it should be possible to prove this by being more careful in the construction of the cyclic base change morphism and the Jacquet–Langlands morphism.

The work of Breuil–Hellmann–Schraen [BHS17] constructs a similar patched eigenvariety for unitary groups, using completed cohomology rather than overconvergent cohomology. It would be extremely interesting to relate these two constructions.

We now describe the structure of this paper. We begin by recalling the theory of trianguline (φ, Γ)-modules and their deformations; this permits us to construct and study pseudorigid trianguline varieties (generalizing those of [Che13] and [BHS17]). We compute the dimension of these pseudorigid trianguline varieties with fixed determinant and weight, and we show that they have an integral model.

We then turn to the automorphic theory we will need. We prove that so-called *twist classical* points are very Zariski dense in the eigenvariety $\mathscr{X}_{\underline{D}^\times}$, which permits us to interpolate the Jacquet–Langlands correspondence to extended eigenvarieties and permits us to conclude that $\mathscr{X}_{\underline{D}^\times}$ is reduced (extending the results of [Bir19] and [Che05]). We also study the cyclic base change morphism $\mathscr{X}_{\mathrm{GL}_2/\mathbf{Q}} \to \mathscr{X}_{\mathrm{GL}_2/F}$ of [JN19a]; when F is totally real and $[F : \mathbf{Q}]$ is prime to p, we show that $x \in \mathscr{X}_{\mathrm{GL}_2/F}$ is in the image if and only if it is fixed by $\mathrm{Gal}(F/\mathbf{Q})$. To do this, we construct a "$\mathrm{Gal}(F/\mathbf{Q})$-fixed eigenvariety" and show that classical points are dense in it.

Finally, we turn to the patching argument. We show that our modules of integral overconvergent automorphic forms are projective, and we show that we can add certain kinds of level structure. Then using the standard Taylor–Wiles patching construction, we construct a patched module with the support we expect.

2. Trianguline varieties and Galois deformation rings

2.1. **Galois deformation rings.** Let $\mathbf{F}$ be a finite field of characteristic p, and let G be a profinite group satisfying Mazur's condition Φ_p. The two cases we will be most interested in are $G = \mathrm{Gal}_K$ and $G = \mathrm{Gal}_{F,S}$, where K is a finite extension of $\mathbf{Q}_p$, and F is a number field, and S is a set of places of F.

Suppose we have a continuous homomorphism $\overline{\rho} : G \to \mathrm{GL}_d(\mathbf{F})$. Then we may construct the univeral framed deformation ring $R_{\overline{\rho}}^{\square}$, which pro-represents the functor

$$A \rightsquigarrow \{\rho : G \to \mathrm{GL}_d(A)\}/\sim$$

of lifts of $\overline{\rho}$, together with a basis. If $\mathrm{End}_G(\overline{\rho}) = \mathbf{F}$ (for example, if $\overline{\rho}$ is absolutely irreducible), we additionally have the universal (unframed) deformation ring $R_{\overline{\rho}}$ parametrizing deformations of ρ.

If R is a complete local noetherian $\mathbf{Z}_p$-algebra with maximal ideal $\mathfrak{m}_R$ and finite residue field, and $\psi : \mathrm{Gal}_K \to R^{\times}$ is a continuous character such that $\det \overline{\rho} = \psi$ mod $\mathfrak{m}_R$, there is a quotient $R \,\widehat{\otimes}\, R_{\overline{\rho}}^{\square} \twoheadrightarrow R_{\overline{\rho}}^{\square,\psi}$ parametrizing lifts of $\overline{\rho}$ with determinant ψ. Indeed, there is a homomorphism $R_{\det \overline{\rho}} \to R_{\overline{\rho}}^{\square}$ given by the determinant map, and the choice of ψ defines a homomorphism $R_{\det \overline{\rho}} \to R$; then $R_{\overline{\rho}}^{\square,\psi} = R \,\widehat{\otimes}_{R_{\det \overline{\rho}}}\, R_{\overline{\rho}}^{\square}$. If $\mathrm{End}_G(\overline{\rho}) = \mathbf{F}$, there is similarly a quotient $R \,\widehat{\otimes}\, R_{\overline{\rho}} \twoheadrightarrow R_{\overline{\rho}}^{\psi}$ parametrizing deformations of $\overline{\rho}$ with determinant ψ.

Now we specialize to the arithmetic situation of interest. Let F be a number field and let $\Sigma_p := \{v \mid p\}$. If $\rho : \mathrm{Gal}_F \to \mathrm{GL}_d(\mathbf{F})$ is a continuous representation and v is a place of F, we let ρ_v denote $\rho|_{\mathrm{Gal}_{F_v}}$. Suppose that $\overline{\rho}$ is absolutely irreducible, and let S be a finite set of places of F containing Σ_p and the infinite places such that $\overline{\rho}$ is unramified outside S. Then we let $R_{\overline{\rho},S}$ denote the universal deformation ring parametrizing deformations unramified outside of S, and we let $R_{\overline{\rho},S}^{\square}$ denote the universal deformation ring whose A-points are deformations ρ_A of $\overline{\rho}$ unramified outside of S, together with bases for $\rho_A|_{\mathrm{Gal}_{F_v}}$ for each $v \in \Sigma_p$. We also let $R_{\overline{\rho},\mathrm{loc}}^{\square} := \otimes_{v \in \Sigma_p} R_{\overline{\rho}_v}^{\square}$.

If $\psi : \mathrm{Gal}_F \to R^{\times}$ is a continuous character as above, we let

$$R_{\overline{\rho},S}^{\psi} := R \underset{R_{\det \overline{\rho},S}}{\widehat{\otimes}} R_{\overline{\rho},S}$$

$$R_{\overline{\rho},S}^{\square,\psi} := R \underset{R_{\det \overline{\rho},S}}{\widehat{\otimes}} R_{\overline{\rho},S}^{\square}$$

$$R_{\overline{\rho},\mathrm{loc}}^{\square,\psi} := R \underset{R_{\det \overline{\rho},\mathrm{loc}}}{\widehat{\otimes}} R_{\overline{\rho},\mathrm{loc}}^{\square}$$

For any place $v \in \Sigma_p$, restriction from $\mathrm{Gal}_{F,S}$ to Gal_{F_v} defines a homomorphism $R_{\overline{\rho}_v}^{\square} \to R_{\overline{\rho},S}^{\square}$, and so we obtain homomorphisms

$$R_{\overline{\rho},\mathrm{loc}}^{\square} \to R_{\overline{\rho},S}^{\square}$$

and

$$R_{\overline{\rho},\mathrm{loc}}^{\square,\psi} \to R_{\overline{\rho},S}^{\square,\psi}$$

We can relate our local and global deformation rings more precisely:

Lemma 2.1.1. *Suppose that $p \nmid d$. Let h^1 denote the dimension (as an $\mathbf{F}$-vector space) of*

$$\ker\left(H^1(\mathrm{Gal}_{F,S}, \mathrm{ad}^0(\overline{\rho})) \to \prod_{v \in \Sigma_p} H^1(\mathrm{Gal}_{F_v}, \mathrm{ad}^0(\overline{\rho}_v))\right)$$

let $\delta_F := \dim_{\mathbf{F}} H^0(\mathrm{Gal}_{F,S}, \mathrm{ad}\,\overline{\rho})$, and for $v \in \Sigma_p$ let $\delta_v := \dim_{\mathbf{F}} H^0(\mathrm{Gal}_{F_v}, \mathrm{ad}\,\overline{\rho}_v)$. Then $R_{\overline{\rho},S}^{\square,\psi}$ can be topologically generated over $R_{\overline{\rho},\mathrm{loc}}^{\square,\psi}$ by $g := h^1 + \sum_{v \in \Sigma_p} \delta_v - \delta_F$ elements.

Proof. Let $\mathfrak{m}_{\mathrm{loc}}$ denote the maximal ideal of $R_{\overline{\rho},\mathrm{loc}}^{\square,\psi}$ and let $\mathfrak{m}_S$ denote the maximal ideal of $R_{\overline{\rho},S}^{\square,\psi}$. We need to compute the relative tangent space $\left(\mathfrak{m}_S/(\mathfrak{m}_S^2,\mathfrak{m}_{\mathrm{loc}})\right)^*$ of $R_{\overline{\rho},S}^{\square,\psi}/\mathfrak{m}_{\mathrm{loc}}$. But the maximal ideal of R is contained in $\mathfrak{m}_{\mathrm{loc}}$, so we may assume that ψ is constant, and the result follows from [Kis09b, Lemma 3.2.2]. $\qquad\square$

2.2. **Deformations of trianguline (φ,Γ)-modules.** Trianguline (φ,Γ)-modules are those which are extensions of (φ,Γ)-modules of character type. More precisely,

Definition 2.2.1. Let X be a pseudorigid space over $\mathscr{O}_E$ for some finite extension $E/\mathbf{Q}_p$, let $K/\mathbf{Q}_p$ be a finite extension, and let $\underline{\delta} = (\delta_1,\ldots,\delta_d) : (K^\times)^d \to \Gamma(X,\mathscr{O}_X^\times)$ be a d-tuple of continuous characters. A (φ,Γ_K)-module D is *trianguline with parameter $\underline{\delta}$* if (possibly after enlarging E) there is an increasing filtration $\mathrm{Fil}^\bullet D$ by (φ,Γ_K)-modules and a set of line bundles $\mathscr{L}_1,\ldots,\mathscr{L}_d$ such that $\mathrm{gr}^i D \cong \Lambda_{X,\mathrm{rig},K}(\delta_i) \otimes \mathscr{L}_i$ for all i.

If $X = \mathrm{Spa}\,R$ where R is a field, we say that D is *strictly trianguline with parameter $\underline{\delta}$* if for each i, $\mathrm{Fil}^{i+1} D$ is the unique sub-(φ,Γ_K)-module of D containing $\mathrm{Fil}^i D$ such that $\mathrm{gr}^{i+1} D \cong \Lambda_{R,\mathrm{rig},K}(\delta_{i+1})$.

As in the characteristic 0 situation treated in [BC09], we may define and study deformations of trianguline (φ,Γ)-modules:

Definition 2.2.2. Let R be a finite extension of $\mathbf{F}_p((u))$ and let D be a fixed (φ,Γ_K)-module of rank d over $\Lambda_{R,\mathrm{rig},K}$ equipped with a triangulation $\mathrm{Fil}^\bullet D$ with parameter $\underline{\delta}$. Let $\mathcal{C}_R$ denote the category of artin local $\mathbf{Z}_p$-algebras R' equipped with an isomorphism $R'/\mathfrak{m}_{R'} \xrightarrow{\sim} R$. The trianguline deformation functor $\mathrm{Def}_{D,\mathrm{Fil}^\bullet} : \mathcal{C}_R \to \underline{\mathrm{Set}}$ is defined to be the set of isomorphism classes

$$\mathrm{Def}_{D,\mathrm{Fil}^\bullet}(R') := \{(D_{R'},\mathrm{Fil}^\bullet D_{R'},\iota)\}/\sim$$

where $D_{R'}$ is a (φ,Γ_K)-module over $\Lambda_{R',\mathrm{rig},K}$, $\mathrm{Fil}^\bullet D_{R'}$ is a triangulation, and $\iota : R \otimes_{R'} D_{R'} \xrightarrow{\sim} D$ is an isomorphism which also defines isomorphisms $R \otimes_{R'} \mathrm{Fil}^i D_{R'} \xrightarrow{\sim} \mathrm{Fil}^i D$.

One of the consequences of the proof of [Bel20, Proposition 5.1.1] is that when $d = 1$, $\mathrm{Def}_{D,\mathrm{Fil}^\bullet}$ is formally smooth. As in the characteristic 0 situation, the same is true for general d, so long as the parameter satisfies a certain regularity condition.

Proposition 2.2.3. *Suppose the parameter $\underline{\delta}$ of $\mathrm{Fil}^\bullet D$ satisfies the property that $\delta_i\delta_j^{-1} \neq \chi_{\mathrm{cyc}}$ for any $i < j$. Then $\mathrm{Def}_{D,\mathrm{Fil}^\bullet}$ is formally smooth.*

Proof. The proof is essentially identical to that of [BC09, Proposition 2.3.10], but we sketch it here for the convenience of the reader. We proceed by induction on d; the case $d = 1$ follows from the proof of [Bel20, Proposition 5.1.1], so we assume the result for trianguline deformations of (φ,Γ)-modules of rank $d - 1$. Let $I \subset R'$ be a square-zero ideal. We need to prove that $\mathrm{Def}_{D,\mathrm{Fil}^\bullet}(R') \to \mathrm{Def}_{D,\mathrm{Fil}^\bullet}(R'/I)$ is surjective, so we may factor $R' \twoheadrightarrow R'/I$ into a series of small extensions and assume that I is principal and $I\mathfrak{m}_{R'} = 0$. By the inductive hypothesis, we may find a trianguline deformation D' of $\mathrm{Fil}^{d-1} D$ over $\Lambda_{R',\mathrm{rig},L}$. By twisting, we may assume that δ_d is trivial. Then we need to show that the natural map $H^1_{\varphi,\Gamma}(D') \to H^1_{\varphi,\Gamma}(\mathrm{Fil}^{d-1})$ is surjective. But the cokernel of this map is $H^2_{\varphi,\Gamma}(I \otimes_{R'/\mathfrak{m}_{R'}} \mathrm{Fil}^{d-1} D(\delta_d^{-1})) =$

$I \otimes_{R'/\mathfrak{m}_{R'}} H^2_{\varphi,\Gamma}(\mathrm{Fil}^{d-1} D(\delta_d^{-1}))$, which is 0 by assumption and [Bel20, Corollary 4.2.3]. □

This motivates the following definition.

Definition 2.2.4. Let $\mathcal{T} := \widehat{K^\times}$ be the pseudorigid space representing the functor on pseudorigid spaces whose X-points are given by $\mathcal{T}(X) := \mathrm{Hom}_{\mathrm{cts}}(K^\times, \mathscr{O}(X)^\times)$. We say that a continuous character $\kappa : K^\times \to \mathscr{O}(X)^\times$ is *regular* if for all maximal points $x \in X$, the residual character $\kappa_x : K^\times \to k(x)^\times$ is not of the form $\alpha \mapsto \alpha^{-\mathbf{i}}$ or $\alpha \mapsto \alpha^{\mathbf{i+1}}|\alpha|$ for $\mathbf{i} \in \mathbf{Z}_{\geq 0}^{\mathrm{Hom}(K,k(x))}$ (if x is a characteristic 0 point), or trivial or $\chi_{\mathrm{cyc}} \circ \mathrm{Nm}_{K/\mathbf{Q}_p}$ (if x is a characteristic p point).

The space of *regular parameters* $\mathcal{T}^d_{\mathrm{reg}} \subset \mathcal{T}^d$ is the Zariski-open subspace whose X-points are given by parameters $\underline{\delta} : (K^\times)^d \to \mathscr{O}(X)^\times$ such that $\delta_i \delta_j^{-1} : K^\times \to \mathscr{O}(X)^\times$ is regular for all $j > i$.

Consider the functor $\mathcal{S}^\square_d$ on pseudorigid spaces defined via

$$X \rightsquigarrow \{(D, \mathrm{Fil}^\bullet D, \underline{\delta}, \underline{\nu})\}/\sim$$

where D is a trianguline (φ, Γ_K)-module with filtration $\mathrm{Fil}^\bullet D$ and regular parameter $\underline{\delta} \in \mathcal{T}^d_{\mathrm{reg}}$, and $\underline{\nu}$ is a sequence of trivializations $\nu_i : \mathrm{gr}^i D \xrightarrow{\sim} \Lambda_{X,\mathrm{rig},K}$. There is a natural transformation $\mathcal{S}^\square_d \to \mathcal{T}^d_{\mathrm{reg}}$ given on X-points by

$$(D, \mathrm{Fil}^\bullet D, \underline{\delta}, \underline{\nu}) \rightsquigarrow \underline{\delta}$$

Exactly as in [Che13, Théorème 3.3] and [HS16, Theorem 2.4], we have the following:

Proposition 2.2.5. *The functor $\mathcal{S}^\square_d$ is representable by a pseudorigid space, which we also denote $\mathcal{S}^\square_d$, and the morphism $\mathcal{S}^\square_d \to \mathcal{T}^d_{\mathrm{reg}}$ is smooth of relative dimension $\frac{d(d-1)}{2}[K : \mathbf{Q}_p]$.*

One proves by induction on d that if D is a trianguline (φ, Γ_K)-module over X with parameter $\underline{\delta} \in (\mathcal{T}_{\mathrm{reg}})^d$, then $H^1_{\varphi,\Gamma_K}(D)$ is a vector bundle over X of rank $d[K : \mathbf{Q}_p]$ (the regularity assumption ensures that $H^0_{\varphi,\Gamma_K}(D) = H^2_{\varphi,\Gamma_K}(D) = 0$). Now $\mathcal{S}^\square_1 = \mathcal{T} = \mathcal{T}^1_{\mathrm{reg}}$, so $S^\square_1$ is representable and is smooth of the correct dimension over $\mathcal{T}^1_{\mathrm{reg}}$. Then one may proceed by induction on d again, and construct $\mathcal{S}^\square_d$ as the moduli space of extensions of the universal (φ, Γ_K)-module of character type $\Lambda_{\mathcal{T},\mathrm{rig},K}(\delta_{\mathrm{univ}}$ by the universal object $D_{d-1,\mathrm{univ}}$ over $\mathcal{S}^\square_{d-1}$. For a specified regular parameter $\underline{\delta} = (\delta_1, \ldots, \delta_d) \in \mathcal{T}^d_{\mathrm{reg}}(X)$, the fiber $\mathcal{S}^\square_d|_{\underline{\delta}}$ is equal to $\mathrm{Ext}^1(\Lambda_{X,\mathrm{rig},K}(\delta_d), D_{d-1,\mathrm{univ}}|_{(\delta_1,\ldots,\delta_{d-1})}) = H^1_{\varphi,\Gamma_K}(D_{d-1,\mathrm{univ}}|_{(\delta_1,\ldots,\delta_{d-1})}(\delta_d^{-1}))$. This is a rank-$(d-1)$ vector bundle over X, and the claim follows.

We also introduce a variant of $\mathcal{S}^\square_d$ with families of fixed determinant and weights. More precisely, suppose X is a pseudorigid space and we have a continuous character $\delta_{\mathrm{det}} : K^\times \to \mathscr{O}(X)^\times$ and a d-tuple of continuous characters $\underline{\kappa} := (\kappa_1, \ldots, \kappa_d) : \mathscr{O}_K^\times \to \mathscr{O}(X)^\times$. We say that δ_{det} and $\underline{\kappa}$ are *compatible* if $\delta_{\mathrm{det}}|_{\mathscr{O}_K^\times} = \kappa_1 \cdots \kappa_d$. If δ_{det} and $\underline{\kappa}$ are compatible, we consider the functor $\mathcal{S}^{\square,\delta_{\mathrm{det}},\underline{\kappa}}_d$ on pseudorigid spaces over X defined via

$$Y \rightsquigarrow \{(D, \mathrm{Fil}^\bullet D, \underline{\delta}, \underline{\nu}) \in \mathcal{S}^\square_d(Y) \mid \delta_i|_{\mathscr{O}_K^\times} = \kappa_i \text{ for all } i, \delta_1 \cdots \delta_d = \delta_{\mathrm{det}}\}/\sim$$

Proposition 2.2.6. *The functor $\mathcal{S}_d^{\square,\delta_{\det},\underline{\kappa}}$ is representable by a pseudorigid space over X, which we also denote $\mathcal{S}_d^{\square,\delta_{\det},\underline{\kappa}}$, and the morphism $\mathcal{S}_d^{\square,\delta_{\det},\underline{\kappa}} \to X$ is smooth of relative dimension $\frac{d(d-1)}{2}[K : \mathbf{Q}_p] + d - 1$.*

Proof. There is a morphism $\mathcal{T}^d \to \widehat{\mathscr{O}_K^\times}^d \times \mathbf{G}_m^{\mathrm{an}}$, given by $\underline{\delta} \mapsto \left(\delta_1|_{\mathscr{O}_K^\times}, \ldots, \delta_d|_{\mathscr{O}_K^\times}, \delta_1(\varpi_K) \cdots \delta_d(\varpi_K) \right)$, and it is smooth of relative dimension $d-1$. The choice of $\delta_{\det}$ and $\underline{\kappa}$ define a morphism $X \to \widehat{\mathscr{O}_K^\times}^d \times \mathbf{G}_m^{\mathrm{an}}$, and we have a pullback square

$$
\begin{array}{ccc}
\mathcal{S}_d^{\square,\delta_{\det},\underline{\kappa}} & \longrightarrow & \mathcal{S}_d^{\square} \\
\downarrow & & \downarrow \\
X & \longrightarrow & \widehat{\mathscr{O}_K^\times}^d \times \mathbf{G}_m^{\mathrm{an}}
\end{array}
$$

Then the result follows from Proposition 2.2.5. $\qquad\square$

2.3. **Structure of trianguline varieties.** Let $K/\mathbf{Q}_p$ be a finite extension, and let $\overline{\rho} : \mathrm{Gal}_K \to \mathrm{GL}_d(k)$ be a continuous representation, where k is a finite field containing the residue field of K. Let $X_{\mathrm{tri},\overline{\rho}}^{\square} \subset (\mathrm{Spa}\, R_{\overline{\rho}}^{\square})^{\mathrm{an}} \times \mathcal{T}^d$ be the Zariski closure of the set of maximal points $x = \{(\rho_x, \underline{\delta}_x)\}$, where ρ_x is a (framed) lift of $\overline{\rho}$ and $\underline{\delta}_x \in \mathcal{T}_{\mathrm{reg}}^d(L)$ is a regular parameter of $D_{\mathrm{rig}}(\rho_x)$.

Fix an d-tuple of characters $\underline{\kappa} := (\kappa_1, \ldots, \kappa_d)$, where $\kappa_i : \mathscr{O}_K^\times \to \mathscr{O}(X)^\times$ and $X := (\mathrm{Spa}\, R)^{\mathrm{an}}$. Over the pseudorigid space X, the character ψ corresponds to a rank-1 (φ, Γ)-module of the form $\mathscr{L} \otimes D_{\mathrm{rig}}(\delta_\psi)$, for some character $\delta_\psi : K^\times \to \mathscr{O}(X)^\times$. If δ_ψ and $\underline{\kappa}$ are compatible, we may define $X_{\mathrm{tri},\overline{\rho}}^{\square,\psi,\underline{\kappa}} \subset (\mathrm{Spa}\, R_{\overline{\rho}}^{\square,\psi})^{\mathrm{an}} \times \mathcal{T}^d$ to be the Zariski closure of the set of maximal points $x = \{(\rho_x, \underline{\delta}_x)\}$, where ρ_x is a framed lift of $\overline{\rho}$ with determinant ψ and $\underline{\delta}_x \in \mathcal{T}_{\mathrm{reg}}^d(L)$ is a regular parameter of $D_{\mathrm{rig}}(\rho_x)$ such that $\delta_i|_{\mathscr{O}_K^\times} = \kappa_i$.

In order to study the structure of $X_{\mathrm{tri},\overline{\rho}}^{\square}$ and $X_{\mathrm{tri},\overline{\rho}}^{\square,\psi,\underline{\kappa}}$, we will need to know something about the essential image of the functor from Galois representations to (φ, Γ)-modules:

Lemma 2.3.1. *The functor $M \rightsquigarrow D_{\mathrm{rig},K}(M)$ from Gal_K-representations to their associated (φ, Γ)-modules is formally smooth.*

Proof. We need to show that if D is a projective (φ, Γ_K)-module over a pseudoaffinoid algebra R, and $I \subset R$ is a square-zero ideal such that $(R/I) \otimes_R D$ arises from a family of Galois representations, then D also arises from a family of Galois representations. Indeed, we have a short exact sequence

$$
0 \to ID \to D \to (R/I) \otimes_R D \to 0
$$

By assumption, $D' := (R/I) \otimes_R D$ arises from a family of Gal_K representations M' over R/I, and since

$$
D'' := ID \cong I \otimes_R D \cong (R/\operatorname{ann}_R I) \otimes_{R/I} D
$$

it arises from a family of Gal_K representations M'' over $R/\operatorname{ann}_R I$. Since D has a model D_b over $\Lambda_{R,(0,b],K}$, we have a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{\Lambda}_{R,(0,b/p]} \otimes_R D'' & \longrightarrow & \widetilde{\Lambda}_{R,(0,b/p]} \otimes_R D & \longrightarrow & \widetilde{\Lambda}_{R,(0,b/p]} \otimes_R D' & \longrightarrow & 0 \\
& & \varphi-1 \big\uparrow & & \varphi-1 \big\uparrow & & \varphi-1 \big\uparrow & & \\
0 & \longrightarrow & \widetilde{\Lambda}_{R,(0,b]} \otimes_R D'' & \longrightarrow & \widetilde{\Lambda}_{R,(0,b]} \otimes_R D & \longrightarrow & \widetilde{\Lambda}_{R,(0,b]} \otimes_R D' & \longrightarrow & 0
\end{array}
$$

By construction, $\widetilde{\Lambda}_{R,(0,b]} \otimes_R D'' \cong \widetilde{\Lambda}_{R,(0,b]} \otimes \left(\widetilde{\Lambda}_{R_0,[0,b]} \otimes_{R_0} M_0'' \right)$ and $\widetilde{\Lambda}_{R,(0,b]} \otimes_R D' \cong \widetilde{\Lambda}_{R,(0,b]} \otimes \left(\widetilde{\Lambda}_{R_0,[0,b]} \otimes_{R_0} M_0' \right)$, for some integral models M_0'' and M_0' (perhaps after localizing on $\mathrm{Spa}\, R$ and shrinking b). Therefore, we have quasi-isomorphisms

$$
[M''] \xrightarrow{\sim} [\widetilde{\Lambda}_{R,[0,b]} \otimes_{R_0} M_0'' \xrightarrow{\varphi-1} \widetilde{\Lambda}_{R,[0,b/p]} \otimes_{R_0} M_0'']
$$
$$
\xrightarrow{\sim} [\widetilde{\Lambda}_{R,(0,b]} \otimes_R D'' \xrightarrow{\varphi-1} \widetilde{\Lambda}_{R,(0,b/p]} \otimes_R D'']
$$

and

$$
[M'] \xrightarrow{\sim} [\widetilde{\Lambda}_{R,[0,b]} \otimes_{R_0} M_0' \xrightarrow{\varphi-1} \widetilde{\Lambda}_{R,[0,b/p]} \otimes_{R_0} M_0']
$$
$$
\xrightarrow{\sim} [\widetilde{\Lambda}_{R,(0,b]} \otimes_R D' \xrightarrow{\varphi-1} \widetilde{\Lambda}_{R,(0,b/p]} \otimes_R D']
$$

Then the snake lemma implies that we have an exact sequence

$$
0 \to M'' \to \left(\widetilde{\Lambda}_{R,\mathrm{rig},K} \otimes D \right)^{\varphi=1} \to M' \to 0
$$

of R-modules equipped with continuous R-linear actions of Gal_K, with M' finite projective over R/I and $M'' \cong (R/\operatorname{ann}_R I) \otimes_{R/I} M'$. It follows that $M := \left(\widetilde{\Lambda}_{R,\mathrm{rig},K} \otimes D \right)^{\varphi=1}$ is a projective R-module of the same rank and $D_{\mathrm{rig},K}(M) = D$. $\qquad\square$

Proposition 2.3.2.   (1) *The space $X_{\mathrm{tri},\overline{\rho}}^{\square}$ (equipped with its underlying reduced structure) is equidimensional of dimension $d^2 + [K : \mathbf{Q}_p]\frac{d(d+1)}{2}$.*

(2) *The fibers of the morphism $X_{\mathrm{tri},\overline{\rho}}^{\square,\psi,\underline{\kappa}} \to (\mathrm{Spa}\, R)^{\mathrm{an}}$ are equidimensional of dimension $d^2 - 1 + [K : \mathbf{Q}_p]\frac{d(d-1)}{2}$.*

Proof. The proof of the first part is very similar to that of [BHS17, Théorème 2.6]. By construction, there is a universal framed deformation $\rho_{\mathrm{univ}} : \mathrm{Gal}_K \to \mathrm{GL}_d(R_{\overline{\rho}}^{\square})$ of $\overline{\rho}$, and we may pull it back to $X_{\mathrm{tri},\overline{\rho}}^{\square}$. Then there is a sequence of blow-ups and normalizations $f : \widetilde{X} \to X_{\mathrm{tri},\overline{\rho}}^{\square}$ and an open subspace $U \subset \widetilde{X}$ containing the characteristic p locus such that $f^*\rho_{\mathrm{univ}}|_U$ is trianguline with parameters $f^*\underline{\delta}$. Shrinking U if necessary, we may assume that $f^*\underline{\delta}$ is regular. Furthermore, there is a Zariski-dense and open subspace $V \subset X_{\mathrm{tri},\overline{\rho}}^{\square}$ such that $f^{-1}(V) \subset U$ and f defines an isomorphism $f^{-1}(V) \xrightarrow{\sim} V$.

Over U, the (φ, Γ_K)-module $D := D_{\mathrm{rig},K}(f^*\rho_{\mathrm{univ}})$ is equipped with an increasing filtration $\mathrm{Fil}^\bullet D$ such that $\mathrm{gr}^i D \cong \Lambda_{U,\mathrm{rig},K}(f^*\delta_i) \otimes \mathscr{L}_i$ for some line bundle $\mathscr{L}_i$ on U. We may therefore construct a $\left(\mathbf{G}_{\mathrm{m}}^{\mathrm{ad}} \right)^d$-torsor $U^{\square} \to U$ trivializing each of the $\mathscr{L}_i$; since $U^{\square}$ carries the data $(D, \mathrm{Fil}^\bullet D, f^*\underline{\delta}, \underline{\nu})$, where $\underline{\nu}$ is the set of trivializations $\nu_i : \mathrm{gr}^i D \xrightarrow{\sim} \Lambda_{U,\mathrm{rig},K}(f^*\delta_i)$, there is a morphism $U^{\square} \to \mathcal{S}_d^{\square}$.

Let $V^\square \subset U^\square$ denote the pullback of $U^\square \to U$ to V. We claim that $V^\square \to \mathcal{S}_d^\square$ is smooth of relative dimension d^2. To see this, suppose we have a pseudoaffinoid algebra R, a morphism $\mathrm{Spa}\, R \to \mathcal{S}_d^\square$, and a square-zero ideal $I \subset R$ such that the composition $\mathrm{Spa}\, R/I \hookrightarrow \mathrm{Spa}\, R \to \mathcal{S}_d^\square$ is in the image of $V^\square$. Then there is a ring of definition $R_0' \subset R/I$ such that the homomorphism $R_{\overline{\rho}}^\square \to R/I$ factors through R_0'; we let $M_0' \cong R_0'^{\oplus d}$ be the pullback of the universal framed deformation to R_0' and we let $M' := R/I \otimes_{R_0'} M_0'$.

By Lemma 2.3.1, there is a Gal_K-representation M over R such that $(R/I) \otimes_R M \xrightarrow{\sim} M'$. It follows that M_0' and its basis lift to a free module M_0 over some ring of definition $R_0 \subset R$, such that $R \otimes_{R_0} M_0 = M$. Moreover, M' is residually a lift of $\overline{\rho}$ at every maximal point of $\mathrm{Spa}\, R$, so M is as well. By [WE18, Theorem 3.8], M_0 corresponds to a $\mathrm{Spa}\, R_0$-point of $\mathrm{Spf}\, R_{\overline{\rho}}^\square$, and by construction M corresponds to a $\mathrm{Spa}\, R$-point of $\mathrm{Spa}\, R \to X_{\mathrm{tri}, \overline{\rho}}^\square$ deforming M'. Since M' corresponds to a $\mathrm{Spa}(R/I)$-point of the Zariski-open subspace $V \subset X_{\mathrm{tri}, \overline{\rho}}^\square$, the image of the morphism $\mathrm{Spa}\, R \to X_{\mathrm{tri}, \overline{\rho}}^\square$ also lands in V. Since D is trianguline with regular parameter and trivialized quotients, the morphism $\mathrm{Spa}\, R \to V$ lifts to a morphism $\mathrm{Spa}\, R \to V^\square$.

The claim that $V^\square \to \mathcal{S}_d^\square$ has relative dimension d^2 follows because "changing the framing" makes $V^\square$ a $(\mathrm{GL}_d)^{\mathrm{an}}$-torsor over its image in $\mathcal{S}_d^\square$.

Now by Proposition 2.2.5, we see that $V^\square$ is equidimensional of dimension $d^2 + \frac{d(d-1)}{2}[K : \mathbf{Q}_p] + d[K : \mathbf{Q}_p] + d$. Since $V^\square \to V$ is a $(\mathbf{G}_{\mathrm{m}}^{\mathrm{ad}})^d$-torsor, it follows that V is equidimensional of dimension $d^2 + \frac{d(d+1)}{2}[K : \mathbf{Q}_p]$. Finally, $V \subset X_{\mathrm{tri}, \overline{\rho}}^\square$ is Zariski-dense, so the first part follows.

For the second part, we define $V^{\square, \psi, \underline{\kappa}}$ via the pullback

$$
\begin{array}{ccc}
V^{\square, \psi, \underline{\kappa}} & \longrightarrow & V^\square \\
\downarrow & & \downarrow \\
\mathcal{S}_d^{\square, \delta_\psi, \underline{\kappa}} & \longrightarrow & \mathcal{S}_d^\square \\
\downarrow & & \downarrow \\
(\mathrm{Spa}\, R)^{\mathrm{an}} & \longrightarrow & \widehat{\mathscr{O}_K^\times}^d \times \mathbf{G}_m^{\mathrm{an}}
\end{array}
$$

where the morphism $(\mathrm{Spa}\, R)^{\mathrm{an}} \to \widehat{\mathscr{O}_K^\times}^d \times \mathbf{G}_m^{\mathrm{an}}$ is given by $\underline{\kappa}$ and δ_ψ.

We obtain a morphism

$$
V^{\square, \psi, \underline{\kappa}} \to (\mathrm{Spa}\, R)^{\mathrm{an}} \times V \subset (\mathrm{Spa}\, R)^{\mathrm{an}} \times \left(\mathrm{Spa}\, R_{\overline{\rho}}^\square\right)^{\mathrm{an}}
$$

and its image is $V^{\psi, \underline{\kappa}} := X_{\mathrm{tri}, \overline{\rho}}^{\square, \psi, \underline{\kappa}} \cap (\mathrm{Spa}\, R)^{\mathrm{an}} \times \left(\mathrm{Spa}\, R_{\overline{\rho}}^\square\right)^{\mathrm{an}}$, where the intersection is taken inside $(\mathrm{Spa}\, R)^{\mathrm{an}} \times X_{\mathrm{tri}, \overline{\rho}}^\square$.

Now we can compute dimensions: Since we have shown that $V^\square \to \mathcal{S}_d^\square$ is smooth of relative dimension d^2, the same is true for $V^{\square, \psi, \underline{\kappa}} \to \mathcal{S}_d^{\square, \psi, \underline{\kappa}}$, and it follows from Proposition 2.2.6 that $V^{\square, \psi, \underline{\kappa}} \to (\mathrm{Spa}\, R)^{\mathrm{an}}$ is smooth of relative dimension $\frac{d(d-1)}{2}[K : \mathbf{Q}_p] + d^2 + d - 1$. Since $V^{\square, \psi, \underline{\kappa}} \to V^{\psi, \underline{\kappa}}$ is a $(\mathbf{G}_{\mathrm{m}}^{\mathrm{ad}})^d$-torsor, the result follows. $\square$

Now we consider a global setup. Let F be a number field, and suppose that $\overline{\rho}$: $\mathrm{Gal}_F \to \mathrm{GL}_d(\mathbf{F})$ is an absolutely irreducible representation, unramified outside a finite set of primes S. Then the homomorphisms

$$R_{\overline{\rho}_v}^{\square} \to R_{\overline{\rho},S}^{\square}$$

for each $v \mid p$ induce a morphism

$$\left(\mathrm{Spa}\, R_{\overline{\rho},S}^{\square}\right)^{\mathrm{an}} \times \prod_{v|p} \mathcal{T}^d \to \prod_{v|p} \left(\left(\mathrm{Spa}\, R_{\overline{\rho}_v}^{\square}\right)^{\mathrm{an}} \times \mathcal{T}^d\right)$$

and we define $X_{\mathrm{tri},\overline{\rho},S}^{\square}$ to be the pre-image of $\prod_{v|p} X_{\mathrm{tri},\overline{\rho}_v}^{\square}$.

If R is a complete local noetherian $\mathbf{Z}_p$-algebra with maximal ideal $\mathfrak{m}_R$ and finite residue field, and $\psi : \mathrm{Gal}_F \to R^\times$ is a continuous character such that $\det \overline{\rho} = \psi$ mod $\mathfrak{m}_R$, the homomorphisms

$$R_{\overline{\rho}_v}^{\square,\psi_v} \to R_{\overline{\rho},S}^{\square,\psi}$$

and

$$R_{\overline{\rho},\mathrm{loc}}^{\square,\psi} \to R_{\overline{\rho},S}^{\square,\psi}$$

induce a sequence of morphisms

$$\left(\mathrm{Spa}\, R_{\overline{\rho},S}^{\square,\psi}\right)^{\mathrm{an}} \times \prod_{v|p} \mathcal{T}^d \longrightarrow \left(\mathrm{Spa}\, R_{\overline{\rho},\mathrm{loc}}^{\square,\psi}\right)^{\mathrm{an}} \times \prod_{v|p} \mathcal{T}^d \longrightarrow \prod_{v|p} \left(\left(\mathrm{Spa}\, R_{\overline{\rho}_v}^{\square,\psi_v}\right)^{\mathrm{an}} \times \mathcal{T}^d\right)$$

where $\psi_v := \psi|_{\mathrm{Gal}_{F_v}}$. We define $X_{\mathrm{tri},\overline{\rho},S}^{\square,\psi}$ and $X_{\mathrm{tri},\overline{\rho},\mathrm{loc}}^{\square,\psi}$ to be the pre-images of $\prod_{v|p} X_{\mathrm{tri},\overline{\rho}_v}^{\square,\psi_v}$ in $\left(\mathrm{Spa}\, R_{\overline{\rho},S}^{\square,\psi}\right)^{\mathrm{an}} \times \prod_{v|p} \mathcal{T}^d$ and $\left(\mathrm{Spa}\, R_{\overline{\rho},\mathrm{loc}}^{\square,\psi}\right)^{\mathrm{an}} \times \prod_{v|p} \mathcal{T}^d$, respectively.

If we additionally have d-tuples of characters $\underline{\kappa}_v := (\kappa_{v,1}, \ldots, \kappa_{v,d})$, where $\kappa_{v,i} : \mathscr{O}_{F_v}^\times \to \mathscr{O}(X)^\times$ is a continuous character, and we set $X := (\mathrm{Spa}\, R)^{\mathrm{an}}$, we may form the spaces

$$
\begin{array}{ccccc}
X_{\mathrm{tri},\overline{\rho},S}^{\square,\psi,\underline{\kappa}} & \longrightarrow & X_{\mathrm{tri},\overline{\rho},\mathrm{loc}}^{\square,\psi,\underline{\kappa}} & \longleftarrow & \prod_{v|p} X_{\mathrm{tri},\overline{\rho}_v}^{\square,\psi_v,\underline{\kappa}_v} \\
\cap & & \cap & & \cap \\
\left(\mathrm{Spa}\, R_{\overline{\rho},S}^{\square,\psi}\right)^{\mathrm{an}} \times \prod_{v|p} \mathcal{T}^d & \longrightarrow & \left(\mathrm{Spa}\, R_{\overline{\rho},\mathrm{loc}}^{\square,\psi}\right)^{\mathrm{an}} \times \prod_{v|p} \mathcal{T}^d & \longrightarrow & \prod_{v|p} \left(\left(\mathrm{Spa}\, R_{\overline{\rho}_v}^{\square,\psi_v}\right)^{\mathrm{an}} \times \mathcal{T}^d\right)
\end{array}
$$

In particular, suppose we have fixed a neat level $K = K^p I$, as in sections 3 and 4, and consider the ring $R = \mathbf{Z}_p[\![T_0/\overline{Z(K)}]\!]$ corresponding to integral weight space. Since $T_0 = \prod_{v|p}(\mathrm{Res}_{\mathscr{O}_{F_v}/\mathbf{Z}_p} T_v)(\mathbf{Z}_p)$, we have homomorphisms $\mathbf{Z}_p[\![T_v(\mathscr{O}_{F_v})]\!] \to R$, and hence morphisms $\mathrm{Spa}\, R \to \mathrm{Spa}\, \mathbf{Z}_p[\![T_v(\mathscr{O}_{F_v})]\!]$. Suppose we have a determinant character $\psi : \mathrm{Gal}_F \to R^\times$ and a set of weights $\underline{\kappa}_v := (\kappa_{v,1}, \ldots, \kappa_{v,d}) : \mathscr{O}_{F_v}^\times \to \mathscr{O}(\mathscr{W}_F)^\times$ for each $v \mid p$, such that $\psi|_{\mathrm{Gal}_{F_v}}$ and $\underline{\kappa}_v$ are compatible for all v, and such that ψ_v and $\underline{\kappa}_v$ factor through $\mathbf{Z}_p[\![T_v(\mathscr{O}_{F_v})]\!] \to R$ for all v, i.e., they depend only on the projection to $\mathrm{Spa}\, \mathbf{Z}_p[\![T_v(\mathscr{O}_{F_v})]\!]$.

Proposition 2.3.3. *Under the above assumptions, the morphism $X_{\mathrm{tri},\overline{\rho},\mathrm{loc}}^{\square,\psi,\underline{\kappa}} \to \mathscr{W}_F$ has fibers that are equidimensional of dimension $|\Sigma_p|(d^2 - 1) + [F : \mathbf{Q}]\frac{d(d-1)}{2}$.*

Proof. Viewing ψ_v as a character $\mathrm{Gal}_{F_v} \to \mathbf{Z}_p[\![T_v(\mathscr{O}_{F_v})]\!]^\times$ and viewing $\underline{\kappa}_v = (\kappa_{v,1}, \ldots, \kappa_{v,d})$ as a d-tuple of characters $\mathscr{O}_{F_v}^\times \to \mathbf{Z}_p[\![T_v(\mathscr{O}_{F_v})]\!]^\times$, we have a pullback diagram

$$\begin{array}{ccc}
X^{\square,\psi,\underline{\kappa}}_{\mathrm{tri},\overline{\rho},\mathrm{loc}} & \longrightarrow & \prod_{v|p} X^{\square,\psi_v,\underline{\kappa}_v}_{\mathrm{tri},\overline{\rho}_v} \\
\downarrow & & \downarrow \\
\mathscr{W}_F & \longrightarrow & \prod_{v|p} (\mathrm{Spa}\,\mathbf{Z}_p[\![T_v(\mathscr{O}_{F_v})]\!])^{\mathrm{an}}
\end{array}$$

The right vertical morphism has relative dimension

$$\sum_{v|p} \left(d^2 - 1 + [F_v : \mathbf{Q}_p]\frac{d(d-1)}{2} \right) = |\Sigma_p|(d^2 - 1) + [F : \mathbf{Q}]\frac{d(d-1)}{2}$$

so the morphism $X^{\square,\psi,\underline{\kappa}}_{\mathrm{tri},\overline{\rho},\mathrm{loc}} \to \mathscr{W}_F$ does, as well. $\qquad\square$

The case we will be most interested in is the case where $F/\mathbf{Q}$ is cyclic and totally split at p, and $d = 2$. In that case, $X^{\square,\psi_v,\underline{\kappa}_v}_{\mathrm{tri},\overline{\rho}_v} \to (\mathrm{Spa}\,\mathbf{Z}_p[\![T_v(\mathscr{O}_{F_v})]\!])^{\mathrm{an}}$ has relative dimension 4 for each $v \mid p$, and hence $X^{\square,\psi,\underline{\kappa}}_{\mathrm{tri},\overline{\rho},\mathrm{loc}} \to \mathscr{W}_F$ has relative dimension $4[F : \mathbf{Q}]$.

2.4. Trianguline deformation rings.

We have constructed the trianguline varieties $X^{\square}_{\mathrm{tri},\overline{\rho}}$ and $X^{\square,\psi,\underline{\kappa}}_{\mathrm{tri},\overline{\rho}}$ as subspaces of the (non-quasicompact) pseudorigid space $(\mathrm{Spa}\,R^{\square}_{\overline{\rho}})^{\mathrm{an}} \times \mathcal{T}^d$. However, the advantage of working with general pseudorigid spaces is that we can construct integral models, so long as we bound the slope.

Proposition 2.4.1. *Let $\mathfrak{X}$ be a normal execellent formal scheme, which is nowhere discrete. If $Z \subset X := \mathfrak{X}^{\mathrm{an}}$ is a Zariski-closed adic subset, then there is a closed formal subscheme $\mathfrak{Z} \subset \mathfrak{X}$ such that $Z = \mathfrak{Z}^{\mathrm{an}}$.*

Proof. We may assume that $\mathfrak{X} = \mathrm{Spf}\,R$, where R is a normal excellent domain with ideal of definition $J = (f_1, \ldots, f_n)$. Then by the definitions of [JN19a, §2.1], there is a coherent sheaf $\mathcal{I} \subset \mathscr{O}_X$ of ideals such that $Z = \{x \in X \mid \mathcal{I}_x \neq \mathscr{O}_{X,x}\}$. We need to show that there is an ideal $I \subset R$ whose associated sheaf agrees with $\mathcal{I}$ on X.

We define $\mathcal{I}^+ := \mathcal{I} \cap \mathscr{O}_X^+$, and we set $I := \Gamma(X, \mathcal{I}^+)$; by [Lou17, Proposition 6.2], $R = \Gamma(X, \mathscr{O}_X^+)$, so we may view I as an ideal of R. It remains to show that for each affinoid open subspace $\mathrm{Spa}\,R' \subset X$, $R' \otimes_R I = \mathcal{I}(\mathrm{Spa}\,R')$. To see this, we observe that we have a finite cover $X = \cup_i \mathrm{Spa}\,R\left\langle \frac{J}{f_i} \right\rangle$, so it suffices to check this with $R' = \mathrm{Spa}\,R\left\langle \frac{J}{f_i} \right\rangle$.

Setting $R_i := R\left\langle \frac{J}{f_i} \right\rangle$ and $U_i = \mathrm{Spa}\,R\left\langle \frac{J}{f_i} \right\rangle$, we have an exact sequence of R-modules

$$0 \to I \to \prod_i \mathcal{I}^+(U_i) \rightrightarrows \prod_{i,j} \mathcal{I}^+(U_i \cap U_j)$$

For any fixed index i_0, we may tensor with $R_{i_0}^\circ$ and complete f_{i_0}-adically; as R is noetherian, our sequence

$$0 \to R_{i_0}^\circ \widehat{\underset{R}{\otimes}} I \to \prod_i \left(R_{i_0}^\circ \widehat{\underset{R}{\otimes}} \mathcal{I}^+(U_i) \right) \rightrightarrows \prod_{i,j} \left(R_{i_0}^\circ \widehat{\underset{R}{\otimes}} \mathcal{I}^+(U_i \cap U_j) \right)$$

remains exact. But $R_{i_0}^\circ \,\widehat{\otimes}_R\, \mathcal{I}^+(U_i)$ generates $\mathcal{I}(U_{i_0} \cap U_i)$ and $R_{i_0}^\circ \,\widehat{\otimes}_R\, \mathcal{I}^+(U_i \cap U_j)$ generates $\mathcal{I}(U_{i_0} \cap U_i \cap U_j)$ after inverting a pseudo-uniformizer u_{i_0} of R_{i_0} for all i, j, and $\{U_{i_0} \cap U_i\}_i$ is a cover of U_{i_0}, so in fact $R_{i_0} \,\widehat{\otimes}_R\, I = \mathcal{I}(U_{i_0})$, as desired. $\quad\square$

We will apply this to find formal models for our trianguline varieties. Recall that when K is a finite extension of $\mathbf{Q}_p$ and $\overline{\rho}$ is a representation of Gal_K, we defined $X_{\mathrm{tri},\overline{\rho}}^\square$ and $X_{\mathrm{tri},\overline{\rho}}^{\square,\psi,\kappa}$ as analytic subspaces of $\left(\mathrm{Spa}\, R_{\overline{\rho}}^\square\right)^{\mathrm{an}} \times \mathcal{T}^d$ and $\left(R \,\widehat{\otimes}\, R_{\overline{\rho}}^\square\right)^{\mathrm{an}} \times \mathcal{T}^d$, respectively. Here $\mathcal{T} = \widehat{\mathscr{O}_K^\times} \times \mathbf{G}_{\mathrm{m}}^{\mathrm{ad}}$; $\widehat{\mathscr{O}_K^\times} = \left(\mathbf{Z}_p\llbracket\mathscr{O}_K^\times\rrbracket\right)^{\mathrm{an}}$ so it has a natural formal model, but $\mathbf{G}_{\mathrm{m}}^{\mathrm{ad}}$ does not (in particular, it is not equal to the analytic locus of $\mathrm{Spf}\, \mathbf{Z}_p\left\langle T, T^{-1}\right\rangle$).

However, let $\mathbf{B}_{\leq h} := \mathrm{Spf}\, \mathbf{Z}_p\llbracket p^h T\rrbracket$ be the formal ball of radius h and let $\mathbf{B}_{\leq h}^{\mathrm{an}}$ be its analytic locus. Then $\cup_h \mathbf{B}_h^{\mathrm{an}}$ is a quasi-Stein space, and $\mathbf{G}_{\mathrm{m}}^{\mathrm{ad}} = (\cup_h \mathbf{B}_h^{\mathrm{an}}) \smallsetminus \{0\}$. If we define $\mathcal{T}_{\leq h} := \widehat{\mathscr{O}_K^\times} \times \mathbf{B}_{\leq h}^{\mathrm{an}}$, we may set

$$X_{\mathrm{tri},\overline{\rho},\leq h}^\square := X_{\mathrm{tri},\overline{\rho}}^\square \cap \left(\mathrm{Spa}\, R_{\overline{\rho}}^\square\right)^{\mathrm{an}} \times \mathcal{T}_{\leq h}^d$$

and

$$X_{\mathrm{tri},\overline{\rho},\leq h}^{\square,\psi,\kappa} := X_{\mathrm{tri},\overline{\rho}}^\square \cap \left(\mathrm{Spa}\, R \,\widehat{\otimes}\, R_{\overline{\rho}}^\square\right)^{\mathrm{an}} \times \mathcal{T}_{\leq h}^d$$

When F is a totally real field and $\overline{\rho}$ is a representation of Gal_F unramified outside a finite set of places S, we may similarly define bounded global trianguline varieties $X_{\mathrm{tri},\overline{\rho},S,\leq h}^\square$ and $X_{\mathrm{tri},\overline{\rho},S,\leq h}^{\square,\psi,\kappa}$ as subspaces of $\left(\mathrm{Spa}\, R_{\overline{\rho},S}^\square\right)^{\mathrm{an}} \times \prod_{v|p} \mathcal{T}_{\leq h}^d$ and $\left(\mathrm{Spa}\, R \,\widehat{\otimes}\, R_{\overline{\rho},S}^\square\right)^{\mathrm{an}} \times \prod_{v|p} \mathcal{T}_{\leq h}^d$, respectively.

Corollary 2.4.2. *Suppose that $\overline{\rho}$ is a representation of Gal_K, where K is a finite extension of $\mathbf{Q}_p$, or of Gal_F, where F is a totally real number field (in which case we assume ρ is unramified outside a finite set of places S). Then there are affine formal schemes $\mathfrak{X}_{\mathrm{tri},\overline{\rho},\leq h}^\square = \mathrm{Spf}\, R_{\mathrm{tri},\overline{\rho},\leq h}^\square$ (resp. $\mathfrak{X}_{\mathrm{tri},\overline{\rho},S,\leq h}^\square = \mathrm{Spf}\, R_{\mathrm{tri},\overline{\rho},S,\leq h}^\square$) and $\mathfrak{X}_{\mathrm{tri},\overline{\rho},\leq h}^{\square,\psi,\kappa}$ (resp. $\mathfrak{X}_{\mathrm{tri},\overline{\rho},S,\leq h}^{\square,\psi,\kappa}$) such that $\left(\mathfrak{X}_{\mathrm{tri},\overline{\rho},\leq h}^\square\right)^{\mathrm{an}} = X_{\mathrm{tri},\overline{\rho},\leq h}^\square$ (resp. $\left(\mathfrak{X}_{\mathrm{tri},\overline{\rho},S,\leq h}^\square\right)^{\mathrm{an}} = X_{\mathrm{tri},\overline{\rho},S\leq h}^\square$) and $\left(\mathfrak{X}_{\mathrm{tri},\overline{\rho},\leq h}^{\square,\psi,\kappa}\right)^{\mathrm{an}} = X_{\mathrm{tri},\overline{\rho},\leq h}^{\square,\psi,\kappa}$ (resp. $\left(\mathfrak{X}_{\mathrm{tri},\overline{\rho},S,\leq h}^{\square,\psi,\kappa}\right)^{\mathrm{an}} = X_{\mathrm{tri},\overline{\rho},S,\leq h}^{\square,\psi,\kappa}$).*

3. Extended eigenvarieties

3.1. Definitions. We briefly recall the construction of extended eigenvarieties in the two cases of interest to us. Fix a number field F and a reductive group H over F which is split at all places above p; then we define $\mathbf{G} := \mathrm{Res}_{F/\mathbf{Q}}\, \mathrm{H}$. If we choose split models $\mathrm{H}_{\mathscr{O}_{F_v}}$ over $\mathscr{O}_{F_v}$ for each place $v \mid p$, along with split maximal tori and Borel subgroups $\mathrm{T}_v \subset \mathbf{B}_v \subset \mathrm{H}_{\mathscr{O}_{F_v}}$, we obtain an integral model $\mathbf{G}_{\mathbf{Z}_p} := \prod_{v|p} \mathrm{H}_{\mathscr{O}_{F_v}}$ of $\mathbf{G}$, as well as closed subgroup schemes

$$\mathrm{T} := \prod_{v|p} \mathrm{Res}_{\mathscr{O}_{F_v}/\mathbf{Z}_p}\, \mathrm{T}_v \subset \mathbf{B} := \prod_{v|p} \mathrm{Res}_{\mathscr{O}_{F_v}/\mathbf{Z}_p}\, \mathbf{B}_v$$

Let $T_0 := \mathrm{T}(\mathbf{Z}_p)$, and let the Iwahori subgroup $I \subset \mathbf{G}_{\mathbf{Z}_p}(\mathbf{Z}_p)$ be the pre-image of $\mathbf{B}(\mathbf{F}_p)$ under the reduction map $\mathbf{G}_{\mathbf{Z}_p}(\mathbf{Z}_p) \to \mathbf{G}_{\mathbf{Z}_p}(\mathbf{F}_p)$.

We choose a tame level by choosing compact open subgroups $K_\ell \subset \mathbf{G}(\mathbf{Q}_\ell)$ for each prime $\ell \neq p$, such that $K_\ell = \mathcal{G}(\mathbf{Z}_\ell)$ for almost all primes ℓ (where $\mathcal{G}$ is some reductive model of $\mathbf{G}$ over $\mathbf{Z}[1/M]$ for some integer M). Then we put $K^p := \prod_{\ell \neq p} K_\ell$ and

$K := K^p I$; we assume throughout that K contains an open normal subgroup K' such that $[K : K']$ is prime to p and

(3.1.1) $\qquad\qquad x^{-1} D^\times x \cap K' \subset \mathscr{O}_F^{\times,+}$ for all $x \in (\mathbf{A}_{F,f} \otimes_F D)^\times$

which is the neatness hypothesis of [JN19b].[1] If $\mathbf{Z}$ denotes the center of $\mathbf{G}$, we let $Z(K) := \mathbf{Z}(\mathbf{Q}) \cap K$ and let $\overline{Z(K)} \subset T_0$ denote its p-adic closure. We also let $K_\infty \subset \mathbf{G}(\mathbf{R})$ be a maximal compact and connected subgroup at infinity, and let $Z_\infty^\circ \subset Z_\infty =: \mathbf{Z}(\mathbf{R})$ denote the identity component.

Finally, let $\Sigma \subset T_0$ be the kernel of some splitting of the inclusion $T_0 \subset \mathrm{T}(\mathbf{Q}_p)$; there are then certain submonoids $\Sigma^{\mathrm{cpt}} \subset \Sigma^+ \subset \Sigma$, and we fix some $t \in \Sigma^{\mathrm{cpt}}$.

In the cases of interest to us, F will be a totally real field, completely split at p, and H will be either GL_2 or the reductive group $\underline{D}^\times$ corresponding to the units of a totally definite quaternion algebra over F split at every place above p. Fixing isomorphisms $D_v \xrightarrow{\sim} \mathrm{Mat}_2(F_v)$ for each place v where D is split, we may define integral models of H_v via $\mathrm{H}_{\mathscr{O}_{F_v}}(R_0) := (R_0 \otimes \mathrm{Mat}_2(\mathscr{O}_{F_v}))^\times$ for all $\mathscr{O}_{F_v}$-algebras R_0 (whether $\mathrm{H} = \mathrm{GL}_2$ or $\underline{D}^\times$). In either case, we let $\mathbf{B}_v \subset \mathrm{H}_{\mathscr{O}_{F_v}}$ be the standard upper-triangular Borel and we let $\mathrm{T}_v \subset \mathbf{B}_v$ be the standard diagonal maximal torus.

For either choice of H, we define $\Sigma_v^+ := \left\{ \begin{pmatrix} p^{a_1} & 0 \\ 0 & p^{a_2} \end{pmatrix} \mid a_2 \geq a_1 \right\}$ and $\Delta_v := I_v \Sigma_v^+ I_v$. Similarly, we define $\Sigma^+ := \prod_{v \mid p} \Sigma_v^+$ and $\Delta_p := I \Sigma^+ I = \prod_{v \mid p} \Delta_v$. Then we fix $U_v := \left[I_v \begin{pmatrix} 1 \\ & p \end{pmatrix} I_v \right] \in I_v \backslash \mathrm{H}(F_v)/I_v$ and $U_p := \prod_{v \mid p} U_v$.

For each prime $\ell \neq p$, we fix a monoid $\Delta_\ell \subset \mathbf{G}(\mathbf{Q}_\ell)$ containing K_ℓ, which is equal to $\mathbf{G}(\mathbf{Q}_\ell)$ when $K_\ell = \mathcal{G}(\mathbf{Z}_\ell)$, such that (Δ_ℓ, K_ℓ) is a Hecke pair and the Hecke algebra $\mathbb{T}(\Delta_\ell, K_\ell)$ over $\mathbf{Z}_p$ is commutative. Then we define $\Delta^p := \prod_{\ell \neq p}' \Delta_\ell$ and $\Delta := \Delta^p \Delta_p$.

A *weight* is a continuous homomorphism $\kappa : T_0 \to R^\times$ which is trivial on $Z(K)$, where R is a pseudoaffinoid algebra over $\mathbf{Z}_p$. We define *weight space* $\mathscr{W}$ via

$$\mathscr{W}(R) := \{\kappa \in \mathrm{Hom}_{\mathrm{cts}}(T_0, R^\times) \mid \kappa|_{Z(K)} = 1\}$$

It can be written explicitly as the analytic locus of $\mathrm{Spa}\left(\mathbf{Z}_p[\![T_0/\overline{Z(K)}]\!], \mathbf{Z}_p[\![T_0/\overline{Z(K)}]\!]\right)$. Then $\mathscr{W}$ is equidimensional of dimension $1 + [F : \mathbf{Q}] + \mathfrak{d}$, where $\mathfrak{d}$ is the defect in Leopoldt's conjecture for F and p.

The next step is to construct a sheaf of Hecke modules over weight space, such that U_p acts compactly and admits a Fredholm determinant. We will actually use two such sheaves. If $\kappa : T_0 \to R^\times$ is a weight, then [JN16] construct certain modules of analytic functions $\mathcal{A}_\kappa^r$ and distributions $\mathcal{D}_\kappa^r$. Here $r \in (r_\kappa, 1)$, where $r_\kappa \in [1/p, 1)$. When $r_\kappa \in (1/p, 1)$, they also constructed $\mathcal{A}_\kappa^{<r}$ and $\mathcal{D}_\kappa^{<r}$, so that $\mathcal{D}_\kappa^r$ is the dual of $\mathcal{A}_\kappa^{<r}$ and $\mathcal{A}_\kappa^r$ is the dual of $\mathcal{D}_\kappa^{<r}$. As in [HN17] we fix augmented Borel–Serre complexes $C_\bullet^{\mathrm{BM}}(K, -)$ and $C_c^\bullet(K, -)$ for Borel–Moore homology and compactly

---

[1]The authors assume throughout that the level is neat; to relax this assumption, one chooses an open normal subgroup $K' \subset K$ of index prime to p such that K' is neat, and considers the complexes $C_c^\bullet(K', -)^{K/K'}$ and $C_\bullet^{\mathrm{BM}}(K', -)_{K/K'}$. Since K/K' has order prime to p, the finite-slope subcomplexes $C_c^\bullet(K, \mathscr{D}_\kappa)_{\leq h}^{K/K'}$ and $C_\bullet^{\mathrm{BM}}(K', -)_{\leq h, K/K'}$ remain perfect.

supported cohomology, respectively, and we consider

$$C_\bullet^{\mathrm{BM}}(K, \mathcal{A}_\kappa^r)$$

as well as

$$C_c^\bullet(K, \mathcal{D}_\kappa^r) \quad \text{and} \quad C_c^\bullet(K, \mathcal{D}_\kappa^{<r})$$

Now $\mathcal{A}_\kappa^r$ and $\mathcal{D}_\kappa^r$ are potentially orthonormalizable, so $C_*^{\mathrm{BM}}(K, \mathcal{A}_\kappa^r) := \oplus_i C_i^{\mathrm{BM}}(K, \mathcal{A}_\kappa^r)$ and $C_c^*(K, \mathcal{D}_\kappa^r) := \oplus_i C_c^i(K, \mathcal{D}_\kappa^r)$ are, as well. Since U_p acts compactly on $\mathcal{A}_\kappa^r$ and $\mathcal{D}_\kappa^r$, this implies that there are Fredholm determinants $F_\kappa^{r,\prime}$ and F_κ^r for its action on $C_*^{\mathrm{BM}}(K, \mathcal{A}_\kappa^r)$ and $C_c^*(K, \mathcal{D}_\kappa^r)$, respectively.

It turns out that $F_\kappa^{r,\prime}$ and F_κ^r are independent of r, by [JN16, Proposition 4.1.2]; we set $\mathscr{D}_\kappa := \varprojlim_r \mathcal{D}_\kappa^r$ and $\mathscr{A}_\kappa := \varinjlim_r \mathcal{A}_\kappa^r$, and we write F_κ and F_κ' for the Fredholm determinants of U_p on $C_c^*(K, \mathscr{D}_\kappa)$ and $C_*^{\mathrm{BM}}(K, \mathscr{A}_\kappa)$, respectively. Then F_κ and F_κ' define spectral varieties $\mathscr{Z} \subset \mathbf{A}_{\mathscr{W}_F}^1$ and $\mathscr{Z}' \subset \mathbf{A}_{\mathscr{W}_F}^1$. We let $\pi : \mathscr{Z} \to \mathscr{W}_F$ and $\pi' : \mathscr{Z}' \to \mathscr{W}_F$ be the projection on the first factor.

By [JN16, Theorem 2.3.2], $\mathscr{Z}$ has a cover by open affinoid subspaces V such that $U := \pi(V)$ is an open affinoid subspace of $\mathscr{W}_F$ and $\pi|_V : V \to U$ is finite of constant degree. This implies that over such a V, F factors as $F_V = Q_V S_V$ where Q_V is a multiplicative polynomial, S_V is a Fredholm series, and Q_V and S_V are relatively prime. Equivalently, there is some rational number h such that (U, h) is a slope datum for $(\mathscr{W}_F, F_\kappa)$, and we write $\mathscr{Z}_{U,h} := V$.

If such a factorization exists, we may make $C_c^\bullet(K_F, \mathscr{D}_V)$ into a complex of $\mathscr{O}_{\mathscr{Z}}$-modules by letting T act via U_p^{-1}. Then the assignment $V \mapsto \ker Q_V^*(U_p) \subset C^\bullet(K_F, \mathscr{D}_V)$ defines a bounded complex $\mathscr{K}^\bullet$ of coherent $\mathscr{O}_{\mathscr{Z}}$-modules, where $Q_V^*(T) := T^{\deg Q_V} Q_V(1/T)$. Equivalently, $\ker Q_V^*(U_p)$ is the slope-$\leq h$ subcomplex of $C^\bullet(K_F, \mathscr{D}_V)$. We set

$$\mathscr{M}_c^* := \oplus_i H^i(\mathscr{K}) = \oplus_i H_c^i(K, \mathscr{D}_\kappa)_{\leq h}$$

which is a coherent sheaf on $\mathscr{Z}$.

Such factorizations exist locally, by an extension of a result of [AS]:

Proposition 3.1.1. *Let R be a pseudoaffinoid algebra, and let $x_0 \in \operatorname{Spa} R$ be a maximal point. Let $F(T) \in R\{\{T\}\}$ be a Fredholm power series and fix $h \in \mathbf{Q}$. Suppose $F_{x_0} \neq 0$, and let $F_{x_0} = Q_0 S_0$ be the slope $\leq h$-factorization of the specialization of F at x_0. Then there is an open affinoid subspace $U \subset \operatorname{Spa} R$ containing x_0 such that F_U has a slope $\leq h$-factorization $F_U = QS$ with Q specializing to Q_0 and S specializing to S_0 at x_0.*

Proof. The existence of the factorization of F_{x_0} follows from the version of the Weierstrass preparation theorem proved in [AS, Lemma 4.4.3]. Then the proof of the proposition is nearly identical to that of [AS, Theorem 4.5.1], up to replacing p with u and translating the numerical inequalities into rational localization conditions. $\square$

We further observe that we have inclusions $\mathcal{D}_\kappa^r \subset \mathcal{D}_\kappa^{<r} \subset \mathcal{D}_\kappa^s$ for any $r_\kappa \leq s < r$. Thus, the fact that $F_\kappa^r = F_\kappa^s$ implies that $\mathscr{M}_c^* = \oplus_i H_c^i(K, \mathscr{D}_\kappa^{<r})_{\leq h}$ for any $r > r_\kappa$.

We may carry out the same procedure for the action of U_p on $C_*^{\mathrm{BM}}(K, \mathscr{A}_\kappa)$, and obtain a coherent sheaf $\mathscr{M}_*^{\mathrm{BM}} = \oplus_i H_i^{\mathrm{BM}}(K, \mathscr{A}_\kappa)_{\leq h}$ on $\mathscr{Z}'$. Both $\mathscr{M}_c^*$ and $\mathscr{M}_*^{\mathrm{BM}}$ are Hecke modules, so we have constructed *eigenvariety data* $(\mathscr{Z}, \mathscr{M}_c^*, \mathbb{T}(\Delta, K), \psi)$ and

$(\mathscr{L}', \mathscr{M}_*^{\mathrm{BM}}, \mathbb{T}(\Delta, K), \psi')$ (where $\psi : \mathbb{T}(\Delta, K) \to \mathrm{End}_{\mathscr{O}_{\mathscr{L}}}(\mathscr{M}_c^*)$ and $\psi' : \mathbb{T}(\Delta, K) \to$ $\mathrm{End}_{\mathscr{O}_{\mathscr{L}'}}(\mathscr{M}_*^{\mathrm{BM}})$ give the Hecke-module structures).

Finally, we may construct eigenvarieties from the eigenvariety data. Let $\mathscr{T}$ and $\mathscr{T}'$ denote the sheaves of $\mathscr{O}_{\mathscr{L}}$-algebras generated by the images of ψ and ψ', respectively; in particular, if $\mathscr{L}_{U,h} \subset \mathscr{L}$ is an open affinoid corresponding to the slope datum (U, h), then

$$\mathscr{T}(\mathscr{L}_{U,h}) = \mathrm{im}\left(\mathscr{O}(\mathscr{L}_{U,h}) \otimes_{\mathbf{Z}_p} \mathbb{T}(\Delta, K_F) \to \mathrm{End}_{\mathscr{O}(\mathscr{L}_{U,h})}\left(H_c^*(K_F, \mathscr{D}_U)_{\leq h}\right)\right)$$

and

$$\mathscr{T}'(\mathscr{L}'_{U,h}) = \mathrm{im}\left(\mathscr{O}(\mathscr{L}'_{U,h}) \otimes_{\mathbf{Z}_p} \mathbb{T}(\Delta, K_F) \to \mathrm{End}_{\mathscr{O}(\mathscr{L}'_{U,h})}\left(H_*^{\mathrm{BM}}(K_F, \mathscr{A}_U)_{\leq h}\right)\right)$$

Then we set

$$\mathscr{X}_{\mathbf{G}}^{\mathbb{T}} := \underline{\mathrm{Spa}}\,\mathscr{T}$$

and

$$\mathscr{X}_{\mathbf{G}}^{\mathbb{T},'} := \underline{\mathrm{Spa}}\,\mathscr{T}'$$

and we have finite morphisms $q : \mathscr{X}_{\mathbf{G}} \to \mathscr{L}$ and $q' : \mathscr{X}'_{\mathbf{G}} \to \mathscr{L}'$, and $\mathbf{Z}_p$-algebra homomorphisms $\phi_{\mathscr{X}} : \mathbb{T} \to \mathscr{O}(\mathscr{X}_{\mathbf{G}}^{\mathbb{T}})$ and $\phi_{\mathscr{X}'} : \mathbb{T} \to \mathscr{O}(\mathscr{X}_{\mathbf{G}}^{\mathbb{T},'})$. If the choice of Hecke operators is clear from context, we will drop $\mathbb{T}$ from the notation.

We emphasize that, unlike [JN16], we have added the Hecke operators U_v at places $v \mid p$ to our Hecke algebras (and hence to the coordinate rings of our eigenvarieties), not just the controlling operator U_p.

3.2. **The middle-degree eigenvariety.** When $F = \mathbf{Q}$ and $\mathbf{G} = \mathrm{H} = \mathrm{GL}_2$, for any fixed slope h such that $C_c^\bullet(K, \mathscr{D}_\kappa)$ has a slope-$\leq h$ decomposition, the complex $C_c^\bullet(K, \mathscr{D}_\kappa)_{\leq h}$ has cohomology only in degree 1, and $H_c^1(K, \mathscr{D}_\kappa)_{\leq h}$ is projective. As a result, the eigencurve is reduced and equidimensional, and classical points are very Zariski-dense. For a general totally real field F, the situation is more complicated. The complex $C_c^\bullet(K, \mathscr{D}_\kappa)_{\leq h}$ lives in degrees $[0, 2d]$ and we are still primarily interested in the degree-d cohomology; indeed, the discussion of [Har87, §3.6] shows that cuspidal cohomological automorphic forms contribute only to middle degree cohomology. However, there is no reason to expect the other cohomology groups to vanish.

Instead, following [BH17] we will sketch the construction of an open subspace $\mathscr{X}_{\mathrm{GL}_2/F,\mathrm{mid}} \subset \mathscr{X}_{\mathrm{GL}_2/F}$ where $H_c^i(K, \mathscr{D}_\kappa)$ vanishes for $i \neq d$. The cohomology and base change result [JN16, Theorem 4.2.1] shows that the locus where $H_c^i(K, \mathscr{D}_\kappa) = 0$ for $i \geq d+1$ is open, but we need to use the homology complexes $C_\bullet^{\mathrm{BM}}(K, \mathcal{A}_\kappa)$ to control $H_c^i(K, \mathscr{D}_\kappa)$ for $i \leq d-1$.

As in [BH17], the key points are a base change result for Borel–Moore homology, and a universal coefficients theorem relating it to compactly supported cohomology:

Proposition 3.2.1. • *There is a third-quadrant spectral sequence*

$$E_2^{i,j} = \mathrm{Tor}_{-i}^R(H_{-j}^{\mathrm{BM}}(K, \mathscr{A}_\kappa)_{\leq h}, S) \Rightarrow H_{-i-j}^{\mathrm{BM}}(K, \mathscr{A}_{\kappa_S})_{\leq h}$$

• *There is a second-quadrant spectral sequence*

$$E_2^{i,j} = \mathrm{Ext}_R^i(H_j^{\mathrm{BM}}(K, \mathscr{A}_\kappa)_{\leq h}, R) \Rightarrow H_c^{i+j}(K, \mathscr{D}_\kappa)_{\leq h}$$

These are spectral sequences of $\mathbb{T}(\Delta^p, K^p)$-modules.

The proof uses both the fact that $\mathcal{D}_\kappa^{<r}$ is the continuous dual of $\mathcal{A}_\kappa^r$, and the fact that $H_c^i(K, \mathcal{D}_\kappa^{<r})_{\leq h} = H_c^i(K, \mathcal{D}_\kappa^r)_{\leq h}$ for all $r > r_\kappa$.

Proposition 3.2.2. *If (U, h) is a slope datum, then we have a natural commuting diagram*

$$
\begin{array}{ccc}
\mathcal{O}(U) \otimes \mathbb{T}(\Delta, K) & \longrightarrow & \mathbb{T}'_{\kappa, \leq h} \\
\downarrow & & \vdots \\
\mathbb{T}_{\kappa, \leq h} & \longrightarrow & \mathbb{T}^{\mathrm{red}}_{\kappa, \leq h}
\end{array}
$$

Thus, we have a morphism $\tau : \mathscr{X}^{\mathrm{red}}_{\mathrm{GL}_2/F} \to \mathscr{X}'_{\mathrm{GL}_2/F}$ and a closed immersion $i : \mathscr{X}^{\mathrm{red}}_{\mathrm{GL}_2/F} \hookrightarrow \mathscr{X}_{\mathrm{GL}_2/F}$.

Definition 3.2.3.

$$
\mathscr{X}_{\mathrm{GL}_2/F,\mathrm{mid}} := \mathscr{X}_{\mathrm{GL}_2/F} \smallsetminus \left[\left(\cup_{j=d+1}^{2d} \mathrm{supp}(\mathscr{M}_c^j) \right) \cup \left(\cup_{j=0}^{d-1} \mathrm{supp}(i_* \tau^* \mathscr{M}_j^{\mathrm{BM}}) \right) \right]
$$

By construction, a point $x \in \mathscr{X}_{\mathrm{GL}_2/F}$ of weight λ_x lies in the Zariski-open subspace $\mathscr{X}_{\mathrm{GL}_2/F,\mathrm{mid}} \subset \mathscr{X}_{\mathrm{GL}_2/F}$ if and only if $H_c^j(K, k_x \otimes \mathscr{D}_{\lambda_x}) = 0$ for all $j \neq d$.

Proposition 3.2.4.
- *The coherent sheaf $\mathscr{M}_c^d|_{\mathscr{X}_{\mathrm{GL}_2/F,\mathrm{mid}}}$ is flat over $\mathscr{W}$.*
- *$\mathscr{X}_{\mathrm{GL}_2/F,\mathrm{mid}}$ is covered by open affinoids W such that W is a connected component of $(\pi \circ q)^{-1}(U)$, where (U, h) is some slope datum, and $\mathscr{T}(W)$ acts faithfully on $\mathscr{M}_c^d(W) \cong e_W H_c^d(K, \mathscr{D}_\kappa)_{\leq h}$ (where e_W is the idemopotent projector restricting from $(\pi \circ q)^{-1}(U)$ to W).*

Proof. This follows from the base change spectral sequence, and the criterion for flatness. $\qquad\square$

3.3. Jacquet–Langlands.

The classical Jacquet–Langlands correspondence lets us transfer automorphic forms between GL_2 and quaternionic algebraic groups. Over $\mathbf{Q}$, this correspondence was interpolated in [Che05] to give a closed immersion of eigencurves $\mathscr{X}^{\mathrm{rig}}_{D^\times/\mathbf{Q}} \hookrightarrow \mathscr{X}^{\mathrm{rig}}_{\mathrm{GL}_2/\mathbf{Q}}$; this interpolation was given for general totally real fields in [Bir19]. We give the corresponding result for extended eigenvarieties. However, as we have elected to work with the eigenvariety for GL_2/F constructed in [JN16], we will never get an isomorphism of eigenvarieties, even when $[F : \mathbf{Q}]$ is even.

Let D be a totally definite quaternion algebra over F, split at every place above p, and let $\mathfrak{d}_D$ be its discriminant. For any ideal $\mathfrak{n} \subset \mathscr{O}_F$ with $(\mathfrak{d}_D, \mathfrak{n}) = 1$, we define the subgroup $K_1^{D^\times}(\mathfrak{n}) \subset (\mathscr{O}_D \otimes \widehat{\mathbf{Z}})^\times$

$$
K_1^{D^\times}(\mathfrak{n}) := \left\{ g \in (\mathscr{O}_D \otimes \widehat{\mathbf{Z}})^\times \mid g \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix} \right) \pmod{\mathfrak{n}} \right\}
$$

We may define a similar subgroup $K_1^{\mathrm{GL}_2/F}(\mathfrak{n}) \subset \mathrm{Res}_{\mathscr{O}_F/\mathbf{Z}_p} \mathrm{GL}_2(\widehat{\mathbf{Z}})$.

A *classical algebraic weight* is a tuple $(k_\sigma) \in \mathbf{Z}_{\geq 0}^{\Sigma_\infty}$ together with a tuple $(v_\sigma) \in \mathbf{Z}^{\Sigma_\infty}$ such that $(k_\sigma) + (v_\sigma) = (r, \ldots, r)$ for some $r \in \bar{\mathbf{Z}}$, where Σ_∞ is the set of embeddings

$F \hookrightarrow \mathbf{R}$. Set $e_1 := (\frac{r+k_\sigma}{2})$ and $e_2 := (\frac{r-k_\sigma}{2})$, and define characters $\kappa_i : F^\times \to \mathbf{R}^\times$ for $i = 1, 2$ via

$$\kappa_i(x) = \prod_{\sigma \in \Sigma_\infty} \sigma(x)^{e_{i,\sigma}}$$

Then (κ_1, κ_2) is a character on $\mathrm{T}(\mathbf{Z})$ which is trivial on a finite-index subgroup of the center $Z_G(\mathbf{Z}) = \mathscr{O}_F^\times$.

Then we have the classical Jacquet–Langlands correspondence:

Theorem 3.3.1. *Let κ be a classical weight, and let $\mathfrak{n} \subset \mathscr{O}_F$ be an ideal such that $(\mathfrak{n}, \mathfrak{d}_D) = 1$. There is a Hecke-equivariant isomorphism of spaces of cusp forms*

$$S_\kappa^{D^\times}(K_1^{D^\times}(\mathfrak{n})) \xrightarrow{\sim} S_\kappa^{\mathfrak{d}_D - \mathrm{new}}(K_1^{\mathrm{GL}_2 / F}(\mathfrak{n}\mathfrak{d}_D))$$

We will interpolate this correspondence to a closed immersion $\mathscr{X}_{D^\times} \hookrightarrow \mathscr{X}_{\mathrm{GL}_2 / F}$, where the source has tame level $K_1^{D^\times}(\mathfrak{n})$ and the target has tame level $K_1^{\mathrm{GL}_2 / F}(\mathfrak{n})$. We use the interpolation theorem of [JN19a]:

Theorem 3.3.2 ([JN19a, Theorem 3.2.1]). *Let $\mathfrak{D}_i = (\mathscr{X}_i, \mathscr{M}_i, \mathbb{T}_i, \psi_i)$ for $i = 1, 2$ be eigenvariety data, with corresponding eigenvarieties $\mathscr{X}_i$, and suppose we have the following:*

- *A morphism $j : \mathscr{X}_1 \to \mathscr{X}_2$*
- *A $\mathbf{Z}_p$-algebra homomorphism $\mathbb{T}_2 \to \mathbb{T}_1$*
- *A subset $\mathscr{X}^{\mathrm{cl}} \subset \mathscr{X}_1$ of maximal points such that the $\mathbb{T}_2$-eigensystem of x appears in $\mathscr{M}_2(j(\mathfrak{q}_1(x)))$ for all $x \in \mathscr{X}^{\mathrm{cl}}$.*

Let $\overline{\mathscr{X}} \subset \mathscr{X}_1$ denote the Zariski closure of $\mathscr{X}^{\mathrm{cl}}$ (with its underlying reduced structure). Then there is a canonical morphism $i : \overline{\mathscr{X}} \to \mathscr{X}_2$ lying over j, such that $\phi_{\overline{\mathscr{X}}} \circ \sigma = i^* \circ \phi_{\mathscr{X}_2}$. If j is a closed immersion and σ is a surjection, then i is a closed immersion.

We take $\mathscr{X}_1 = \mathscr{X}_2 = \mathscr{W}_F \times \mathbf{G}_\mathrm{m}$. In order to define $\mathbb{T} = \mathbb{T}_1 = \mathbb{T}_2$, we set

$$\Delta_v = \begin{cases} \mathrm{GL}_2(F_v) & \text{if } v \nmid p\mathfrak{d}_D\mathfrak{n} \\ K_1^{D^\times}(\mathfrak{n})_v & \text{if } v \mid \mathfrak{d}_D\mathfrak{n} \end{cases}$$

For $v \mid p$, we take Δ_v as in §3.1. In other words, $\mathbb{T}$ is the commutative $\mathbf{Z}_p$-algebra generated by $T_v := [K_v \left(\begin{smallmatrix} 1 & \\ & \varpi_v \end{smallmatrix} \right) K_v]$ and $S_v := [K_v \left(\begin{smallmatrix} \varpi_v & \\ & \varpi_v \end{smallmatrix} \right) K_v]$ for $v \nmid p\mathfrak{d}_D\mathfrak{n}$ and U_v for $v \mid p$.

However, we cannot immediately combine this interpolation theorem with the Jacquet–Langlands correspondence, because our choice of weight space means that classical weights may not be Zariski dense unless Leopoldt's conjecture is true. More precisely, given a classical algebraic weight, we constructed a character on $\mathrm{T}(\mathbf{Z})$ trivial on a finite-index subgroup of $\mathscr{O}_F^\times$, and conversely, characters on $\mathrm{T}(\mathbf{Z})$ trivial on a finite-index subgroup of $\mathscr{O}_F^\times$ yield classical algebraic weights. This equivalence relies on Dirichlet's unit theorem.

This means that there are two natural definitions of p-adic families of weights, $\mathscr{W}_F' = \mathrm{Spa}\, \mathbf{Z}_p[\![(\mathrm{Res}_{\mathscr{O}_F / \mathbf{Z}}\, \mathbf{G}_\mathrm{m}) \times \mathbf{Z}_p^\times]\!]^{\mathrm{an}}$ interpolating classical algebraic weights, and $\mathscr{W}_F$ interpolating characters on T_0, and the equivalence of those two definitions depends on Leopoldt's conjecture.

Fortunately, the gap between these weight spaces can be controlled: there is a closed embedding $\mathscr{W}_F' \hookrightarrow \mathscr{W}_F^{\mathrm{rig}}$, and the twisting action by characters on $\mathscr{O}_{F,p}^\times / \overline{\mathscr{O}_F^{\times,+}}$ defines a surjective map

$$\widehat{\mathscr{O}_{F,p}^\times / \overline{\mathscr{O}_F^{\times,+}}} \times \mathscr{W}_F' \to \mathscr{W}_F^{\mathrm{rig}}$$

We say that a weight $\lambda \in \mathscr{W}_F^{\mathrm{rig}}(\overline{\mathbf{Q}}_p)$ is *twist classical* if it is in the $\widehat{\mathscr{O}_{F,p}^\times / \overline{\mathscr{O}_F^{\times,+}}}(\overline{\mathbf{Q}}_p)$-orbit of a classical weight. Then twist classical weights are very Zariski dense in $\mathscr{W}_F$.

In addition, we may define a twisting action on Hecke modules, as in [BH17]. Let $\mathrm{Gal}_{F,p}$ denote the Galois group of the maximal abelian extension of F unramified away from p and ∞, and let $\eta : \mathrm{Gal}_{F,p} \to \overline{\mathbf{Q}}_p^\times$ be a continuous character. Global class field theory implies that $\mathrm{Gal}_{F,p}$ fits into an exact sequence

$$1 \to \mathscr{O}_{F,p}^\times / \overline{\mathscr{O}_F^{\times,+}} \to \mathrm{Gal}_{F,p} \to \mathrm{Cl}_F^+ \to 1$$

where Cl_F^+ is the narrow class group of F (and hence finite). Suppose M is an R-module equipped with an R-linear left Δ_p-action. Then we may define a new left Δ_p-module $M(\eta) := M \otimes \eta^{-1}|_{\mathscr{O}_{F,p}^\times}$, where the action of $g \in \Delta_p$ is given by

$$g \cdot m = \left(\eta^{-1}|_{\mathscr{O}_{F,p}^\times} \left(\det g \cdot p^{-\sum_{v|p} v(\det g)} \right) \right) \cdot (g \cdot m)$$

In particular, $\mathscr{D}_\kappa(\eta) \cong \mathscr{D}_{\eta^{-1} \cdot \kappa}$ by [BH17, Lemma 5.5.2], and there is an isomorphism

$$\mathrm{tw}_\eta : H_c^*(K, \mathscr{D}_\kappa) \xrightarrow{\sim} H_c^*(K, \mathscr{D}_{\eta^{-1} \cdot \kappa})$$

Suppose $x \in \mathscr{X}_{\underline{D}^\times}(\overline{\mathbf{Q}}_p)$ is a point with $\mathrm{wt}(x) =: \lambda$, corresponding to the system of Hecke eigenvalues $\psi_x : \mathbb{T} \to \overline{\mathbf{Q}}_p$. Then we define a new system of Hecke eigenvalues, via

$$\mathrm{tw}_\eta(\psi_x)(T) = \begin{cases} \eta(\varpi_v)\psi_x(T) & \text{if } v \nmid p\mathfrak{d}_D\mathfrak{n} \text{ and } T = T_v \\ \eta(\varpi_v)^2 \psi_x(T) & \text{if } v \nmid p\mathfrak{d}_D\mathfrak{n} \text{ and } T = S_v \\ \eta(\varpi_v)\psi_x(T) & \text{if } v \mid p \end{cases}$$

Then it follows from [BH17, Proposition 5.5.5] that $\mathrm{tw}_\eta(\psi_x)$ corresponds to a point $\mathrm{tw}_\eta(x) \in \mathscr{X}_{\underline{D}^\times}$ of weight $\eta^{-1}|_{\mathscr{O}_{F,p}^\times} \cdot \kappa$.

We say that a point $x \in \mathscr{X}_{\underline{D}^\times}(\overline{\mathbf{Q}}_p)$ is *twist classical* if it is in the $\widehat{\mathrm{Gal}_{F,p}}(\overline{\mathbf{Q}}_p)$-orbit of a point corresponding to a classical system of Hecke eigenvalues.

Proposition 3.3.3. *Twist classical points are very Zariski dense in $\mathscr{X}_{\underline{D}^\times}$.*

Proof. Recall that $\mathscr{X}_{\underline{D}^\times}$ admits a cover by affinoid pseudorigid spaces of the form $\mathrm{Spa}\,\mathscr{T}(\mathscr{Z}_{U,h})$, where $\pi : \mathscr{Z}_{U,h} \to U$ is finite of constant degree, and

$$\mathscr{T}(\mathscr{Z}_{U,h}) = \mathrm{im}\left(\mathscr{O}(\mathscr{Z}_{U,h}) \otimes_{\mathbf{Z}_p} \mathbb{T}^p \to \mathrm{End}_{\mathscr{O}(\mathscr{Z}_{U,h})}(H_c^*(K, \mathscr{D}_U)_{\leq h}) \right)$$

We write $U = \mathrm{Spa}\,R$ for some pseudo-affinoid algebra R over $\mathbf{Z}_p$. We will show that $\mathrm{Spec}\,\mathscr{T}(\mathscr{Z}_{U,h}) \to \mathrm{Spec}\,R$ carries irreducible components surjectively onto irreducible components, and we will construct a Zariski dense set of maximal points $W_{U,h}^{\mathrm{tw-cl}} \subset U$ such that the points of $\mathrm{wt}^{-1}(W_{U,h}^{\mathrm{tw-cl}})$ are twist classical. By [Che04, Lemme 6.2.8], this implies the desired result.

To see that irreducible components of $\operatorname{Spec}\mathscr{T}(\mathscr{Z}_{U,h})$ map surjectively onto irreducible components of $\operatorname{Spec}R$, we observe that D is totally definite, so the associated Shimura manifold is a finite set of points and $H_c^*(K,\mathscr{D}_U)$ vanishes outside degree 0. The base change spectral sequence of [JN16, Theorem 4.2.1] implies that the formation of $H^0(K,\mathscr{D}_U)_{\leq h}$ commutes with arbitrary base change on U, which implies that $H^0(K,\mathscr{D}_U)_{\leq h}$ is flat. Then [Che04, Lemme 6.2.10] implies that $\operatorname{Spec}\mathscr{T}(\mathscr{Z}_{U,h})\to\operatorname{Spec}R$ carries irreducible components surjectively onto irreducible components, as desired.

Thus, it remains to construct $W_{U,h}^{\mathrm{tw-cl}}$. Birkbeck proved a "small slope implies classical" result [Bir19, Theorem 4.3.7], and constructed a set $W_{U,h}^{\mathrm{cl}}$ Zariski dense in $U\cap\mathscr{W}_F'$ such that the points of $\operatorname{wt}^{-1}(W_{U,h}^{\mathrm{cl}})$ are classical (see the proof of [Bir19, Theorem 6.1.9]). Setting $W_{U,h}^{\mathrm{tw-cl}}$ to be the $\widehat{\mathscr{O}_{F,p}^\times/\overline{\mathscr{O}_F^{\times,+}}}(\overline{\mathbf{Q}}_p)$-orbit of $W_{U,h}^{\mathrm{cl}}$, [BH17, Lemma 6.3.1] implies that points of $\operatorname{wt}^{-1}(W_{U,h}^{\mathrm{tw-cl}})$ are twist classical, and we are done. $\qquad\square$

As a corollary, we deduce that $\mathscr{X}_{\underline{D}^\times}$ has no components supported entirely in characteristic p:

Corollary 3.3.4. $\mathscr{X}_{\underline{D}^\times}^{\mathrm{rig}}$ *is Zariski dense in* $\mathscr{X}_{\underline{D}^\times}$.

We may use similar arguments to show that $\mathscr{X}_{\underline{D}^\times}$ is reduced:

Proposition 3.3.5. *The eigenvariety* $\mathscr{X}_{\underline{D}^\times/F}$ *is reduced.*

Proof. We first show that $\mathscr{X}_{\underline{D}^\times}^{\mathrm{rig}}$ is reduced. By [JN16, Proposition 6.1.2] (which adapts [Che05, Proposition 3.9]), it is enough to find a Zariski dense set of twist classical weights $W_{U,h}^{\mathrm{ß}}\subset U\subset\mathscr{W}_F^{\mathrm{rig}}$ for each slope datum (U,h) such that $\mathscr{M}(\mathscr{Z}_{U,h})_\kappa$ is a semi-simple Hecke module for all $\kappa\in W_{U,h}^{\mathrm{ß}}$. Birkbeck [Bir19, Lemma 6.1.12] constructed sets $W_{U,h}'^{\mathrm{ß}}$ Zariski dense in $U\cap\mathscr{W}_F'^{,\mathrm{rig}}$ with this property, and we will again use twisting by p-adic characters to construct $W_{U,h}^{\mathrm{ß}}$.

If $\eta:\mathscr{O}_{F,p}^\times/\overline{\mathscr{O}_F^{\times,+}}\to\overline{\mathbf{Q}}_p^\times$ is a character, we have an isomorphism

$$\operatorname{tw}_\eta:H_c^*(K,\mathscr{D}_\kappa)\xrightarrow{\sim}H_c^*(K,\mathscr{D}_{\eta^{-1}\cdot\kappa}$$

By [BH17, Proposition 5.5.5], tw_η is Hecke-equivariant up to scalars, so $\mathscr{M}(\mathscr{Z}_{U,h})_\kappa$ is a semi-simple Hecke module if and only if $\mathscr{M}(\mathscr{Z}_{\eta^{-1}\cdot U,h})_{\eta^{-1}\cdot\kappa}$ is. Thus, we may take $W_{U,h}^{\mathrm{ß}}$ to be the $\widehat{\mathscr{O}_{F,p}^\times/\overline{\mathscr{O}_F^{\times,+}}}(\overline{\mathbf{Q}}_p)$-orbit of $\cup_{U'}W_{U',h}'^{,\mathrm{ß}}$, as (U',h) varies through slope data, and we see that $\mathscr{X}_{\underline{D}^\times}^{\mathrm{rig}}$ is reduced.

Now let (U,h) be a slope datum, and let $\{U_i\}$ be an open affinoid cover of $\mathscr{W}_F^{\mathrm{rig}}$. Since $\mathscr{W}_F$ has no components supported entirely in characteristic p, the natural map

$$\mathscr{O}(U)\to\prod_i\mathscr{O}(U_i)$$

is injective. Moreover, we saw in the proof of Proposition 3.3.3 that $H^0(K,\mathscr{D}_U)_{\leq h}$ a finite flat $\mathscr{O}(U)$-module, and [Che04, Lemme 6.2.10] implies that $\mathscr{T}(\mathscr{Z}_{U,h})$ is also

$\mathscr{O}(U)$-flat. Thus, the composition

$$\mathscr{T}(\mathscr{Z}_{U,h}) \to \mathscr{T}(\mathscr{Z}_{U,h}) \otimes_{\mathscr{O}(U)} \prod_i \mathscr{O}(U_i) \hookrightarrow \prod_i \mathscr{T}(\mathscr{Z}_{U_i,h})$$

remains injective, and since each $\mathscr{T}(\mathscr{Z}_{U_i,h})$ is reduced, so is $\mathscr{T}(\mathscr{Z}_{U,h})$. $\qquad\square$

Now the Jacquet–Langlands correspondence for eigenvarieties follows immediately:

Corollary 3.3.6. *There is a closed immersion $\mathscr{X}_{D^\times} \hookrightarrow \mathscr{X}_{\mathrm{GL}_2/F}$ interpolating the classical Jacquet–Langlands correspondence on (twist) classical points, where the source has tame level $K_1^{D^\times}(\mathfrak{n})$ and the target has tame level $K_1^{\mathrm{GL}_2/F}(\mathfrak{n})$.*

In particular, if $[F : \mathbf{Q}]$ is even, we can find D split at all finite places and ramified at all infinite places. Then we may take in particular $\mathfrak{n} = \mathscr{O}_F$ to obtain a morphism of eigenvarieties of tame level 1.

3.4. **Cyclic base change.** Fix an integer $N \in \mathbf{N}$, and let S be a finite set of primes containing every prime dividing pN. For any number field F, we again let $K_F^p \subset \mathrm{GL}_2(\mathbf{A}_F)$ be the compact open subgroup given by

$$K_F^p := \{g \in \mathrm{GL}_2(\mathbf{A}_F) \mid g \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \pmod{N}\}$$

and we let $K_F := K_F^p I$. We also define the Hecke algebra

$$\mathbb{T}_F^S := \mathbb{T}_{\mathrm{GL}_2/F}^S := \otimes_{v \notin S} \mathbb{T}(\mathrm{GL}_2(F_v), \mathrm{GL}_2(\mathscr{O}_{F_v}))$$

There is a homomorphism $\sigma_F^S : \mathbb{T}_F^S \to \mathbb{T}_{\mathbf{Q}}^S$ induced by unramified local Langlands and restriction of Weil representations from W_F to $W_{\mathbf{Q}}$.

Similarly, there is a morphism of weight spaces $\mathscr{W}_{\mathbf{Q},0} \hookrightarrow \mathscr{W}_{\mathbf{Q}} \to \mathscr{W}_F$ induced by the norm map $T_{F,0} \to T_{\mathbf{Q},0}$.

In the special case where $F/\mathbf{Q}$ is cyclic, [JN19a] interpolated the classical base change map:

Theorem 3.4.1 ([JN19a, Theorem 4.3.1]). [2] *There is a finite morphism*

$$\mathscr{X}_{\mathrm{GL}_2/\mathbf{Q},\mathrm{cusp},F-\mathrm{ncm}}^S \to \mathscr{X}_{\mathrm{GL}_2/F}^S$$

lying over $\mathscr{W}_{\mathbf{Q}} \to \mathscr{W}_F$ and compatible with the homomorphism σ_F^S.

Here the source includes only cuspidal components with a Zariski-dense set of forms without CM by an imaginary quadratic subfield of F.

We wish to characterize the image of this map when F is totally real and completely split at p (so that the "F - ncm" condition is vacuous). We further assume that $[F : \mathbf{Q}]$ is prime to p.

Remark 3.4.2. We expect that it is possible to construct a base change morphism and characterize its image for more general cyclic extensions of number fields F'/F; however, for simplicity (and compatibility with [JN19a]) we have chosen to restrict to this setting.

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[2]The authors only construct the morphism when $N \geq 5$, to maintain their running assumption that the level is actually neat (as opposed to containing an open neat subgroup with index prime to p). However, the argument is identical for small N.

Let $\mathrm{Gal}(F/\mathbf{Q}) = \langle \tau \rangle$. Then $\mathrm{Gal}(F/\mathbf{Q})$ acts on $\mathrm{GL}_{2/F}$, stabilizing $\mathrm{T} \subset \mathbf{B}$ and I, and also stabilizing the tame level K_F^p. We will construct a "$\mathrm{Gal}(F/\mathbf{Q})$-fixed $\mathrm{GL}_{2/F}$-eigenvariety" $\mathscr{X}_{\mathrm{GL}_2/F}^{S,\mathrm{Gal}(F/\mathbf{Q})}$ and show that it is the image of the cyclic base change map; Xiang [Xia18] used a similar idea to construct p-adic families of essentially self-dual automorphic representations.

We first observe that $\mathrm{Gal}(F/\mathbf{Q})$ acts on $\mathbb{T}_F^S$ via $(\tau \cdot T)(g) = T(\tau^{-1}(g))$ for all $T \in \mathbb{T}_F^S$ and $g \in \mathrm{GL}_2(\mathbf{A}_{F,f})$. Then for any $\delta \in \Delta$, $(\tau \cdot [K_F \delta K_F])(g) = [K_F \tau^{-1}(\delta) K_F](g)$, and in particular, $\tau \cdot U_v = U_{\tau(v)}$, and hence $\mathrm{Gal}(F/\mathbf{Q})$ fixes U_p. Similarly, we have an action of $\mathrm{Gal}(F/\mathbf{Q})$ on $\mathscr{W}_{\mathbf{Q}}$ given via $(\tau \cdot \lambda)(g) = \lambda(\tau^{-1}(g))$; the image of $\mathscr{W}_{\mathbf{Q}}$ in $\mathscr{W}_F$ is the diagonal locus, i.e., exactly the $\mathrm{Gal}(F/\mathbf{Q})$-fixed locus.

Since U_p is fixed by $\mathrm{Gal}(F/\mathbf{Q})$, we see that if κ is a weight fixed by $\mathrm{Gal}(F/\mathbf{Q})$, then the Fredholm determinant $F_\kappa(T)$ of the action of U_p on $C^\bullet(K_F, \mathscr{D}_\kappa)$ is fixed by $\mathrm{Gal}(F/\mathbf{Q})$. Thus, we have a spectral variety $\mathscr{Z}^{\mathrm{Gal}(F/\mathbf{Q})} \subset \mathscr{W}_F^{\mathrm{Gal}(F/\mathbf{Q})} \times \mathbf{A}^{1,\mathrm{an}}$ over $\mathscr{W}_F^{\mathrm{Gal}(F/\mathbf{Q})}$.

Lemma 3.4.3. *Let $\kappa : T_0 \to R^\times$ be a weight fixed by $\mathrm{Gal}(F/\mathbf{Q})$. There is an action of $\mathrm{Gal}(F/\mathbf{Q})$ on $C^\bullet(K_F, \mathscr{D}_\kappa)$ and if $\mathscr{D}_\kappa$ admits a slope-$\leq h$ decomposition, the action of $\mathrm{Gal}(F/\mathbf{Q})$ stabilizes $C^\bullet(K_F, \mathscr{D}_\kappa)_{\leq h}$.*

Proof. Referring to the definition of $\mathscr{D}_\kappa$ for an arbitrary weight κ, we have $\mathscr{D}_\kappa = \varprojlim \mathcal{D}_\kappa^r$, where $\mathcal{D}_\kappa^r$ is the completion of a module $\mathcal{D}_\kappa$ with respect to a norm $|\cdot|$. The module $\mathcal{D}_\kappa$ itself is the continuous dual of the space $\mathcal{A}_\kappa \subset \mathcal{C}(I, R)$ of continuous functions $f : I \to R$ such that $f(gb) = \kappa(b)f(g)$ for all $g \in I$ and $b \in B_0$. It follows that we have a map $\tau : \mathcal{A}_\kappa \to \mathcal{A}_{\tau(\kappa)}$ (since the action of $\mathrm{Gal}(F/\mathbf{Q})$ preserves both I and B_0). If κ is fixed by τ, we obtain a dual action of $\mathrm{Gal}(F/\mathbf{Q})$ on $\mathcal{D}_\kappa$, and hence $\mathcal{D}_\kappa^r$ and $\mathscr{D}_\kappa$.

Since K_F^p is also stable under the action of $\mathrm{Gal}(F/\mathbf{Q})$ and the actions of K_F^p and $\mathrm{Gal}(F/\mathbf{Q})$ on $\mathscr{D}_\kappa$ commute, by functoriality we obtain an action of $\mathrm{Gal}(F/\mathbf{Q})$ on $C^\bullet(K_F, \mathscr{D}_\kappa)$. Moreover, the action of $\mathrm{Gal}(F/\mathbf{Q})$ fixes the Hecke operator U_p, so [JN16, Proposition 2.2.11] implies that the action of $\mathrm{Gal}(F/\mathbf{Q})$ also preserves $C^\bullet(K_F, \mathscr{D}_\kappa)_{\leq h}$.   □

Lemma 3.4.4. *Let $\kappa : T_0 \to R^\times$ be a weight fixed by $\mathrm{Gal}(F/\mathbf{Q})$. For any $T \in \mathbb{T}_F^S$, we have $\tau \cdot T = \tau \circ T \circ \tau^{-1}$ as operators on $C^\bullet(K_F, \mathscr{D}_\kappa)$.*

Proof. We may assume $T = [K_F \delta K_F]$ for some $\delta \in \Delta$. Then $\tau \cdot [K_F \delta K_F] = [K_F \tau(\delta) K_F]$, and the corresponding morphism

$$C^\bullet(K_F, \mathscr{D}_\kappa) \to C^\bullet(\tau(\delta) K_F \tau(\delta)^{-1}, \mathscr{D}_\kappa)$$

is induced by the conjugation map $\tau(\delta) K_F \tau(\delta)^{-1} \to K_F$ and the map $\mathscr{D}_\kappa \to \mathscr{D}_\kappa$ given by $d \mapsto \tau(\delta) \cdot d$. But $\tau(\delta) K_F \tau(\delta)^{-1} = \tau\left(\delta \tau^{-1}(K_F) \delta^{-1}\right)$, so we may factor the conjugation map as

$$\tau(\delta) K_F \tau(\delta)^{-1} \xrightarrow{\tau^{-1}} \delta \tau^{-1}(K_F) \delta^{-1} \to \tau^{-1}(K_F) \xrightarrow{\tau} K_F$$

Similarly, $d \mapsto \tau(\delta) \cdot d$ factors as $\tau \circ T \circ \tau^{-1}$, so our morphism of complexes also factors as desired.   □

We may restrict $\mathscr{M}_c^*$ to $\mathscr{Z}^{\mathrm{Gal}(F/Q)}$, and by abuse of notation, we again use $\mathscr{T}$ to denote the sheaf generated by the image of $\mathbb{T}_F^S$ in $\mathscr{E}nd_{\mathscr{Z}^{\mathrm{Gal}(F/\mathbf{Q})}}(\mathscr{M}_c^*|_{\mathscr{Z}^{\mathrm{Gal}(F/\mathbf{Q})}})$. Then the slice of the eigenvariety $\mathscr{X}_{\mathrm{GL}_2/F}^S$ over $\mathscr{W}_F^{\mathrm{Gal}(F/\mathbf{Q})}$ is, by definition, $\underline{\mathrm{Spa}}\,\mathscr{T}$.

Both $\mathbb{T}(\Delta^p, K_F^p)$ and $\mathrm{End}_{\mathscr{O}(V)}(\mathscr{M}_c^*|_{\mathscr{Z}^{\mathrm{Gal}(F/\mathbf{Q})}})$ have actions of $\mathrm{Gal}(F/\mathbf{Q})$, and Lemma 3.4.4 implies that they are compatible. Thus, $\mathscr{T}(V)$ and $\mathscr{X}_{\mathrm{GL}_2/F}^S|_{\mathscr{W}_F^{\mathrm{Gal}(F/\mathbf{Q})}}$ have actions of $\mathrm{Gal}(F/\mathbf{Q})$.

The subspace of $\mathscr{X}_{\mathrm{GL}_2/F}$ fixed by $\mathrm{Gal}(F/\mathbf{Q})$ corresponds to the sheaf $V \mapsto \mathscr{T}(V)_{\mathrm{Gal}(F/\mathbf{Q})}$ of co-invariants of $\mathscr{T}$; by definition, $\mathscr{T}(V)_{\mathrm{Gal}(F/\mathbf{Q})}$ acts on $(\mathscr{H}^*)^{\mathrm{Gal}(F/\mathbf{Q})}$, and the map $\mathscr{T}(V)_{\mathrm{Gal}(F/\mathbf{Q})} \to \mathscr{E}nd_{\mathscr{Z}^{\mathrm{Gal}(F/\mathbf{Q})}}\left((\mathscr{H}^*)^{\mathrm{Gal}(F/\mathbf{Q})}\right)$ is injective.

The above discussion gives us an eigenvariety datum

$$(\mathscr{Z}^{\mathrm{Gal}(F/\mathbf{Q})}, (\mathscr{M}_c^*)^{\mathrm{Gal}(F/\mathbf{Q})}, (\mathbb{T}_F^S)_{\mathrm{Gal}(F/\mathbf{Q})}, \psi)$$

and we let $\mathscr{X}_{\mathrm{GL}_2/F}^{S,\mathrm{Gal}(F/\mathbf{Q})}$ denote the corresponding pseudorigid space.

Proposition 3.4.5. *There is a closed immersion $\mathscr{X}_{\mathrm{GL}_2/F}^{S,\mathrm{Gal}(F/\mathbf{Q})} \hookrightarrow \mathscr{X}_{\mathrm{GL}_2/F}^S$, and the morphism $\mathscr{X}_{\mathrm{GL}_2/\mathbf{Q},\mathrm{cusp}} \to \mathscr{X}_{\mathrm{GL}_2/F}$ constructed in [JN19a, §4.3] factors through $\mathscr{X}_{\mathrm{GL}_2}^{\mathrm{Gal}(F/\mathbf{Q})}$.*

Proof. The first assertion follows from [JN19a, Theorem 3.2.1], since $\mathbb{T}(\Delta^p, K^p) \twoheadrightarrow \mathbb{T}(\Delta^p, K^p)_{\mathrm{Gal}(F/\mathbf{Q})}$ is a surjection. The second assertion is simply the assertion that every system of Hecke eigenvalues in the image of the cyclic base change map is fixed by $\mathrm{Gal}(F/\mathbf{Q})$. But this follows from the corresponding fact for the classical cyclic base change map, plus the fact that classical points are very Zariski-dense in $\mathscr{X}_{\mathrm{GL}_2/\mathbf{Q}}$. $\qquad\square$

We let

$$\mathscr{X}_{\mathrm{GL}_2/F}^{S,\mathrm{Gal}(F/\mathbf{Q}),\circ} := \mathscr{X}_{\mathrm{GL}_2/F}^{S,\mathrm{Gal}(F/\mathbf{Q})} \cap \mathscr{X}_{\mathrm{GL}_2/F,\mathrm{mid}}^S$$

and we let $\overline{\mathscr{X}_{\mathrm{GL}_2/F}^{S,\mathrm{Gal}(F/\mathbf{Q}),\circ}}$ denote its Zariski closure in $\mathscr{X}_{\mathrm{GL}_2/F}^S$.

Lemma 3.4.6. *Classical points are very Zariski dense in $\mathscr{X}_{\mathrm{GL}_2/F}^{S,\mathrm{Gal}(F/\mathbf{Q}),\circ}$.*

Proof. If (U, h) is a slope datum and $W \subset \mathscr{X}_{\mathrm{GL}_2/F}^S$ is a connected affinoid subspace of the pre-image of U, then $\mathscr{T}(W) = e_W \mathscr{T}(U)$ and $\mathscr{M}_c^*(W) \cong e_W H_c^*(K, \mathscr{D}_U)_{\leq h}$, where e_W is the idempotent projector to W. If $W \subset \mathscr{X}_{\mathrm{GL}_2/F,\mathrm{mid}}^S$, then $\mathscr{M}_c^* \cong e_W H_c^d(K, \mathscr{D}_U)_{\leq h}$ and $H_c^d(K, \mathscr{D}_U)_{\leq h}$ is a projective $\mathscr{O}_{\mathscr{W}}(U)$-module. It follows that the restriction of $\mathscr{M}_c^*$ to $\mathscr{X}_{\mathrm{GL}_2/F}^{S,\mathrm{Gal}(\overline{F}/\mathbf{Q}),\circ}$ is a vector bundle, and since $|\mathrm{Gal}(F/\mathbf{Q})|$ is prime to p, its $\mathrm{Gal}(F/\mathbf{Q})$-invariants remain projective.

Now we may apply [Che04, Lemme 6.2.10] to conclude that $\mathscr{T}(W)$ is equidimensional of dimension $\dim \mathscr{O}_{\mathscr{W}_F^{\mathrm{Gal}(F/\mathbf{Q})}}(U)$, and every irreducible component of $\mathrm{Spec}\,\mathscr{T}(W)$ surjects onto an irreducible component of $\mathrm{Spec}\,\mathscr{O}_{\mathscr{W}_F^{\mathrm{Gal}(F/\mathbf{Q})}}(U)$. If $x \in W$ has a classical weight that is sufficiently large (where "sufficiently large" depends on h), then x corresponds to a classical Hilbert modular form. But sufficiently large classical weights are Zariski dense in U, so [Che04, Lemme 6.2.8] implies that classical points are dense in W. $\qquad\square$

Remark 3.4.7. The proof of Lemma 3.4.6 is the only time we use our assumption that $|\mathrm{Gal}(F/\mathbf{Q})|$ is prime to p.

Corollary 3.4.8. *The image of the cyclic base change morphism is exactly* $\overline{\mathscr{X}_{\mathrm{GL}_2/F}^{S,\mathrm{Gal}(F/\mathbf{Q}),\circ}}$.

Proof. Since the morphism $\mathscr{X}_{\mathrm{GL}_2/F}^{S,\mathrm{Gal}(F/\mathbf{Q}),\circ} \to \mathscr{X}_{\mathrm{GL}_2/F}$ is finite, it has closed image. Moreover, cyclic base change carries any classical point of $\mathscr{X}_{\mathrm{GL}_2/\mathbf{Q},\mathrm{cusp}}$ to a point of $\mathscr{X}_{\mathrm{GL}_2/F}^{S,\mathrm{Gal}(F/\mathbf{Q}),\circ}$. On the other hand, every classical point of $\mathscr{X}_{\mathrm{GL}_2/F}^{S,\mathrm{Gal}(F/\mathbf{Q}),\circ}$ is in the image of cyclic base change, by the classical theorem, so Lemma 3.4.6 implies the desired result. □

4. Overconvergent quaternionic modular forms

4.1. **Definitions.** We will use overconvergent cohomology to define and study spaces of overconvergent quaternionic modular forms. Let F be a totally real number field totally split at p, and let D be a totally definite quaternion algebra over F, split at all places above p, with fixed maximal order $\mathcal{D}$. Let H be the reductive group over F defined by $\mathrm{H}(R) := (R \otimes D)^{\times}$ for F-algebras R, and let $\mathbf{G} := \mathrm{Res}_{F/\mathbf{Q}}\,\mathrm{H}$. Fix a tame level $K^p \subset \mathbf{G}(\mathbf{A}_{\mathbf{Q},f}^p)$, and set $K := K^p I$. The adelic subgroup $K(N) \subset (\mathbf{A}_{F,f} \otimes D)^{\times}$ of full level N is neat for $N \geq 3$ such that N is prime to the finite places where D ramifies, so we may take K^p arbitrary.

The coefficients for our families of overconvergent modular forms will be a pseudoaffinoid algebra R over $\mathbf{Z}_p$. We also fix a pseudo-uniformizer $u \in R$.

In order to construct an eigenvariety for $\mathbf{G}$, we fixed a Borel–Serre complex $C^\bullet(K, -)$ and we considered the cohomology $C^\bullet(K, \mathscr{D}_\kappa)$. However, because we assumed D is totally definite, the associated Shimura manifold is a finite set of points, and so the cohomology of $C^\bullet(K, -)$ vanishes outside of degree 0.

Thus, we can give an extremely concrete description of the automorphic forms of interest to us and of the Hecke operators acting on them. Suppose Δ is a monoid satisfying $K \subset \Delta \subset (\mathbf{A}_{F,f} \otimes_F D)^{\times}$ and M is a left $R[\Delta]$-module. Then if $f : D^{\times}\backslash(\mathbf{A}_{F,f} \otimes_F D)^{\times} \to M$ is a function and $\gamma \in \Delta$, we define $_\gamma|f$ via $_\gamma|f(g) = \gamma \cdot f(g\gamma)$. Then

$$H^0(K, M) = \left\{ f : D^{\times}\backslash(\mathbf{A}_{F,f} \otimes_F D)^{\times} \to M \mid {}_\gamma|f = f \text{ for all } \gamma \in K \right\}$$

If $\psi : \mathbf{A}_{F,f}^{\times}/F^{\times} \to R_0^{\times}$ is a continuous character such that $\psi|_{K_v \cap \mathscr{O}_{F_v}^{\times}}$ agrees with the action of $K_v \cap \mathscr{O}_{F_v}^{\times}$ on M for all finite places v of F, we may extend the action of Δ on M to an action of $\Delta \cdot \mathbf{A}_{F,f}^{\times}$, by letting $\mathbf{A}_{F,f}^{\times}$ act by ψ. Then we define

$$H^0(K, M) := \{ f \in H^0(K, M) \mid {}_z|f = f \text{ for all } z \in \mathbf{A}_{F,f}^{\times} \}$$

If we write $D^{\times}\backslash(\mathbf{A}_{F,f} \otimes_F D)^{\times}/K = \coprod_{i \in I} D^{\times} g_i K$ for some finite set of elements $g_i \in (\mathbf{A}_{F,f} \otimes_F D)^{\times}$, the natural map

$$H^0(K, M)[\psi] \to \oplus_{i \in I} M^{(K\mathbf{A}_{F,f}^{\times} \cap g_i^{-1} D^{\times} g_i)/F^{\times}}$$
$$f \mapsto (f(g_i))$$

is an isomorphism.

To define Hecke operators attached to elements of Δ, let $K' \subset K$ be another compact open subgroup and $g \in \Delta$ and decompose the double coset $[KgK'] = \coprod_i g_i K'$ as a finite disjoint union (for some $g_i \in \Delta$). Then we may define the Hecke operator $[KgK'] : H^0(K, M) \to H^0(K, M)$ via

$$[KgK']f = \sum_i g_i | f$$

Then the Hecke algebra $\mathbb{T}(\Delta, K)$ is the $\mathbf{Z}_p$-algebra generated by the double cosets $[KgK]$ for $g \in \Delta$, with multiplication given by convolution. We will only consider monoids Δ such that convolution is commutative.

The coefficient modules of interest to us are the modules of distributions constructed in [JN16]. If $\kappa : T_0/\overline{Z(K)} \to R^\times$ is a weight, we choose a norm $|\cdot|$ on R so that $|\cdot|$ is adapted to κ and multiplicative with respect to u, and $\log_p|\cdot|$ is discrete (which we may do, by Lemma 4.1.1 below). Then the unit ball $R_0 \subset R$ is a ring of definition containing u. We let $\mathcal{D}_\kappa^{r,\circ} \subset \mathcal{D}_\kappa^r$ denote the unit ball, and we also consider larger modules of distributions $\mathcal{D}_\kappa^{<r} \supset \mathcal{D}_\kappa^r$, with unit ball $\mathcal{D}_\kappa^{<r,\circ} \subset \mathcal{D}_\kappa^{<r}$.

The construction of this norm is a variant of [JN16, Lemma 3.3.1], and we refer to that paper for the terminology:

Lemma 4.1.1. *If R is a pseudoaffinoid algebra over $\mathbf{Z}_p$ and $\kappa : T_0/\overline{Z(K)}$ is a weight, there is a norm $|\cdot|$ on R such that $|\cdot|$ is adapted to κ and multiplicative with respect to u, the unit ball R_0 is noetherian, and $\log_p|\cdot|$ is discrete.*

Proof. Choose a noetherian ring of definition $R_0 \subset R$ formally of finite type over $\mathbf{Z}_p$. As in the proof of [JN16, Lemma 3.3.1], $\kappa(T_0) \subset R^\circ$ and $\kappa(T_\epsilon) \subset 1 + R^{\circ\circ}$; since both groups are topologically finitely generated, we may replace R_0 with a finite integral extension and assume that $\kappa(T_0) \subset R_0$, and we may find some integer $m \geq 1$ so that $\kappa(T_\epsilon)^m \subset 1 + uR_0$.

Let $R' := R[u^{1/m}]$, let $R_0' := R_0[u^{1/m}]$, and let $u' := u^{1/m}$. Then R' is a finite R-module, so it has a canonical topology, and the subspace topology it induces on R agrees with the original topology on R. Now for any $a \in \mathbf{R}_{>1}$ we may define a norm $|\cdot|'$ on R' via

$$|r'|' = \inf\{a^s \mid u'^s r' \in R_0'\}$$

The restriction of $|\cdot|'$ to R has the desired properties. $\square$

If $v \mid p$, we define $\Sigma_v^+ := \left\{ \begin{pmatrix} \varpi_v^{a_1} & 0 \\ 0 & \varpi_v^{a_2} \end{pmatrix} \mid a_2 \geq a_1 \right\}$ and $\Delta_v := I_v \Sigma_v^+ I_v$. Similarly, we define $\Sigma^+ := \prod_{v\mid p} \Sigma_v^+$ and $\Delta_p := I\Sigma^+ I = \prod_{v\mid p} \Delta_v$. Then we fix $U_v := \left[I_v \left(\begin{smallmatrix} 1 & \\ & \varpi_v \end{smallmatrix} \right) I_v \right] \in I_v \backslash \mathrm{H}(F_v)/I_v$ and $U_p := \prod_{v\mid p} U_v$.

Recall that the operator U_p acts compactly on $\mathcal{D}_\kappa^r$, which is a potentially ON-able R-module. We therefore have a Fredholm power series $F_\kappa := \det(1 - TU_p \mid H^0(K, \mathcal{D}_\kappa^r))$, and it is independent of $r \geq r_\kappa$, by [JN16, Proposition 4.1.2]. If F_κ has a slope $\leq h$-factorization, then the formalism of slope decompositions implies that we have a slope decomposition $H^0(K, \mathcal{D}_\kappa^r) = H^0(K, \mathcal{D}_\kappa^r)_{\leq h} \oplus H^0(K, \mathcal{D}_\kappa^r)_{>h}$ for all $r \geq r_\kappa$. By [JN16, Proposition 4.1.3], if $s > r \geq r_\kappa$, the inclusion $\mathcal{D}_\kappa^s \subset \mathcal{D}_\kappa^r$ induces an isomorphism $\mathcal{D}_{\kappa,\leq h}^s \xrightarrow{\sim} \mathcal{D}_{\kappa,\leq h}^r$. Since $\mathscr{D}_\kappa = \varprojlim_r \mathcal{D}_\kappa^r$ and F_κ is independent of $r \geq r_\kappa$, we obtain a slope-$\leq h$ submodule $H^0(K, \mathscr{D}_\kappa)_{\leq h} \subset H^0(K, \mathscr{D}_\kappa)$.

Moreover, if $r' \in [r_\kappa, r)$, the inclusions

$$\mathcal{D}_\kappa^r \subset \mathcal{D}_\kappa^{<r} \subset \mathcal{D}_\kappa^{r'}$$

induce an isomorphism $\mathcal{D}_{\kappa, \leq h}^r \xrightarrow{\sim} \mathcal{D}_{\kappa, \leq h}^{r'}$. We may therefore define

$$\mathcal{D}_{\kappa, \leq h}^{<r} := \mathrm{im}\left(\mathcal{D}_{\kappa, \leq h}^r \to \mathcal{D}_\kappa^{<r}\right) = \mathcal{D}_{\kappa, \leq h}^{r'} \cap \mathcal{D}_\kappa^{<r}$$

and

$$H^0(K, \mathcal{D}_\kappa^{<r})_{\leq h} := \mathrm{im}\left(H^0(K, \mathcal{D}_\kappa^r)_{\leq h} \to H^0(K, \mathcal{D}_\kappa^{<r})\right)$$

We make the additional definitions

$$\mathcal{D}_{\kappa, \leq h}^{<r, \circ} := \mathrm{im}\left(\mathcal{D}_\kappa^{<r, \circ} \to \mathcal{D}_{\kappa, \leq h}^{<r}\right)$$

and

$$H^0(K, \mathcal{D}_\kappa^{<r, \circ})_{\leq h} := \mathrm{im}\left(H^0(K, \mathcal{D}_\kappa^{<r, \circ}) \to H^0(K, \mathcal{D}_\kappa^{<r}) \to H^0(K, \mathcal{D}_\kappa^{<r})_{\leq h}\right)$$

We are now in a position to define spaces of overconvergent quaternionic modular forms, together with an integral structure.

Definition 4.1.2. We define the modular forms of weight κ and slope $\leq h$ to be the module $S_\kappa(K)_{\leq h} := H^0(K, \mathscr{D}_\kappa)_{\leq h}$. It is a module over the Hecke algebra $\mathbb{T}_{\kappa, \leq h} := \mathrm{im}(R \otimes_{\mathbf{Z}_p} \mathbb{T}(\Delta, K) \to \mathrm{End}_R(S_\kappa(K)_{\leq h}))$.

We define integral modular forms of weight κ and slope $\leq h$ to be the R_0-submodule $S_\kappa^{<r, \circ}(K)_{\leq h} := H^0(K, \mathcal{D}_\kappa^{<r, \circ})_{\leq h}$, and it is a module over the integral Hecke algebra $\mathbb{T}_{\kappa, \leq h}^{<r, \circ} := \mathrm{im}\left(R_0 \otimes_{\mathbf{Z}_p} \mathbb{T}(\Delta, K) \to \mathrm{End}_{R_0}(S_\kappa^\circ(K)_{\leq h})\right)$.

If $\psi : \mathbf{A}_{F,f}^\times / F^\times \to R_0^\times$ is a continuous character as above, so that $\psi|_{K^p \cap \mathbf{A}_{F,f}^{p,\times}}$ is trivial and $\psi|_{I_v \cap \mathscr{O}_{F_v}^\times}$ is equal to the action of $I_v \cap \mathscr{O}_{F_v}^\times$ on $\mathcal{D}_\kappa$ for $v \mid p$, we define the modular forms with central character ψ to be

$$S_{\kappa, \psi}(K)_{\leq h} := \{f \in S_\kappa(K)_{\leq h} \mid {}_z|f = \psi(z)f \text{ for all } z \in F^\times \backslash \mathbf{Z}(\mathbf{A}_{F,f}) = F^\times \backslash \mathbf{A}_{F,f}^\times\}$$

and similarly for integral modular forms with central character ψ.

We now fix a choice of Hecke algebra. Let S denote the set of places of F such that $v \mid p$, D is ramified at v, or $K_v \neq \mathcal{O}_{D,v}^\times$. For $v \notin S$, we define

$$S_v := [K \left({}^{\varpi_v} \quad {}_{\varpi_v} \right) K], T_v := \left[K \left({}^1 \quad {}_{\varpi_v} \right) K\right] \in K \backslash (\mathbf{A}_{F,f} \otimes D)^\times / K$$

for some fixed uniformizer ϖ_v of $\mathcal{O}_{F_v}$.

We define the Hecke algebra $\mathbb{T}$ to be the free commutative $\mathbf{Z}_p$-algebra generated by $\{U_v\}_{v \mid p}$ and $\{S_w, T_w\}_{w \notin S}$. Since Δ_p acts on the modules of distributions $\mathcal{D}_\kappa^{<r, \circ}$ and Hecke operators away from p preserve the slope decomposition, we may view $S_\kappa^{<r, \circ}(K)_{\leq h}$ as a $\mathbb{T}$-module. We additionally set

$$\mathbb{T}_{K, \leq h}^{<r, \circ} := \mathrm{im}\left(\mathbb{T} \to \mathrm{End}_{R_0}(S_\kappa^{<r, \circ}(K)_{\leq h})\right)$$

We also describe the so-called diamond operators, after modifying the tame level K^p. Suppose we have a finite set Q of places of F such that for each $v \in Q$, $v \nmid p$, $\mathrm{Nm}\, v \equiv 1 \pmod{p}$, and D is split at v. For each $v \in Q$, we again let $K_0(v) \subset \mathrm{H}(F_v)$ denote the subgroup $\{\left({}^*_0 {}^*_* \right) \mod v\}$, and we consider the homomorphism

$$K_0(v) \to k(v)^\times \to \Delta_v$$

given by composing

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto ad^{-1}$$

with the projection to the p-power subgroup $\Delta_v \subset k(v)^\times$. Let $K^-(v)$ denote the group

$$K^-(v) \left\{ \begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \mid ad^{-1} \mapsto 1 \text{ in } \Delta_v \right\}$$

for each $v \in Q$, and let

$$K_0(Q) := \prod_{v \in Q} K_0(v) \cdot \prod_{v \notin Q} K_v$$

and

$$K^-(Q) := \prod_{v \in Q} K^-(v) \cdot \prod_{v \notin Q} K_v$$

Then $K_0(v)/K^-(v) \cong \Delta_v$, and $\Delta_Q := \prod_{v \in Q} \Delta_v$ acts naturally on $S_\kappa^{<r,\circ}(K^-(Q))$. Each $h \in \Delta_v$ yields a Hecke operator which we write $\langle h \rangle := \left[K^-(Q) \widetilde{h} K^-(Q) \right]$.

We let $\mathbb{T}_Q^-$ be the free commutative $\mathbf{Z}_p$-algebra generated by $\{U_v\}_{v|p}$, $\{S_v, T_v\}_{v \notin S}$, and $\{U_{\varpi_v}\}_Q$, where $U_{\varpi_v} := [K^-(v) \begin{pmatrix} 1 & 0 \\ 0 & \varpi_v \end{pmatrix} K^-(v)]$; it acts naturally on $S_\kappa^{<r,\circ}(K^-(Q))_{\leq h}$, and we let $\mathbb{T}_{K^-(Q),\leq h}^{<r,\circ}$ denote the R_0-algebra its image generates in $\mathrm{End}_{R_0}(S_\kappa^{<r,\circ}(K^-(Q))_{\leq h})$. Similarly, we let $\mathbb{T}_{0,Q}$ be the free commutative $\mathbf{Z}_p$-algebra generated by $\{U_v\}_{v|p}$, $\{S_v, T_v\}_{v \notin S}$, and $\{U_{\varpi_v}\}_Q$, where $U_{\varpi_v} := [K_0(v) \begin{pmatrix} 1 & 0 \\ 0 & \varpi_v \end{pmatrix} K_0(v)]$.

We conclude by recording some results on the existence of slope decompositions as we vary the tame level:

Lemma 4.1.3. *Let $K'^p \subset K^p$ be a finite-index subgroup, and let $K' := K'^p I$. If $H^0(K', \mathscr{D}_\kappa)$ has a slope-$\leq h$ decomposition, then so does $H^0(K, \mathscr{D}_\kappa)$. Similarly, if $\psi : \mathbf{A}_{F,f}^\times/F^\times \to R_0^\times$ is a character as above and $H^0(K', \mathscr{D}_\kappa)[\psi]$ has a slope-$\leq h$ decomposition, then so does $H^0(K, \mathscr{D}_\kappa)[\psi]$.*

Proof. We only prove the first statement; the second follows similarly. We may write $K = \coprod_{j \in J} h_j K'$ for a finite index set J and $h_j \in (\mathbf{A}_{F,f}^p \otimes D)^\times$. Then we have

$$H^0(K, \mathscr{D}_\kappa) = \ker \left(H^0(K', \mathscr{D}_\kappa) \xrightarrow{\oplus_j(\mathrm{id} - h_j|)} \oplus_{j \in J} H^0(K', \mathscr{D}_\kappa) \right)$$

The controlling operator U_p commutes with each h_j, so $\oplus_{j \in J} H^0(K', \mathscr{D}_\kappa)$ has a slope-$\leq h$ decomposition and $\oplus_j(\mathrm{id} - h_j|)$ also commutes with U_p. Then [JN16, Proposition 2.2.11] implies that $H^0(K, \mathscr{D}_\kappa)$ has a slope-$\leq h$ decomposition. $\square$

Lemma 4.1.4. *Suppose that $(K\mathbf{A}_{F,f}^\times \cap x^{-1}D^\times x)/F^\times$ is trivial for all $x \in (\mathbf{A}_{F,f} \otimes D)^\times$, and let $\psi : \mathbf{A}_{F,f}^\times/F^\times \to R_0^\times$ be a character as above. Let $K'^p \subset K^p$ be a finite-index subgroup, and let $K' := K'^p I$. If $H^0(K, \mathscr{D}_\kappa)[\psi]$ has a slope-$\leq h$ decomposition, then so does $H^0(K', \mathscr{D}_\kappa)[\psi]$.*

Proof. We write $D^\times \backslash (\mathbf{A}_{F,f} \otimes_F D)^\times/K = \coprod_{i \in I} D^\times g_i K$ and $K = \coprod_{j \in J} h_j K'$ for finite index sets I and J; since $K'_v = K_v$ for all $v \mid p$, we may take $h_j \in (\mathbf{A}_{F,f}^p \otimes D)^\times$. Then $D^\times \backslash (\mathbf{A}_{F,f} \otimes_F D)^\times/K = \coprod_{i,j} D^\times g_i h_j K'$ and we have an isomorphism

$$H^0(K', \mathscr{D}_\kappa)[\psi] \xrightarrow{\sim} \oplus_{j \in J} \oplus_{i \in I} \mathscr{D}_\kappa^{(K'\mathbf{A}_{F,f}^\times \cap (g_i h_j)^{-1} D^\times (g_i h_j))/F^\times} = \oplus_{j \in J} \oplus_{i \in I} \mathscr{D}_\kappa$$

This map sends $f \in H^0(K, \mathscr{D}_\kappa)$ to $\oplus_{i,j} f(g_i h_j)$, and we may group the summands to obtain an isomorphism

$$H^0(K', \mathscr{D}_\kappa)[\psi] \xrightarrow{\sim} \oplus_j h_j^{-1} \cdot {}_{h_j}|H^0(K, \mathscr{D}_\kappa)[\psi]$$

The action of U_p commutes with the action of h_j and ${}_{h_j}|$, so it preserves the decomposition on the right, and the result follows. $\qquad\square$

4.2. **Integral overconvergent quaternionic modular forms.** We need to make a closer study of the structure of the integral modules of distributions and their finite-slope subspaces.

Lemma 4.2.1. *If $\kappa : T_0/\overline{Z(K)} \to R^\times$ is a weight and F_κ has a slope $\leq h$-factorization, then $S_\kappa(K)_{\leq h}$ is a finite projective R-module and is compatible with arbitrary base change on R.*

Proof. The base change spectral sequence of [JN16, Theorem 4.2.1] and the vanishing of overconvergent cohomology in degrees greater than 0 imply that the formation of $H^0(K, \mathscr{D}_\kappa)_{\leq h}$ commutes with arbitrary base change on R. But this implies that it is flat. Since $S_\kappa(K)_{\leq h}$ is a finite R-module (by [JN16, Corollary 4.1.8]) and R is noetherian, this implies it is projective. $\qquad\square$

Corollary 4.2.2. *If $\kappa : T_0/\overline{Z(K)} \to R^\times$ is a weight and F_κ has a slope $\leq h$-factorization, then $\mathcal{D}_{\kappa, \leq h}^{<r, \circ}$ and $S_\kappa^{<r, \circ}(K)_{\leq h}$ are finite R_0-modules.*

Proof. This follows from the equalities $\mathcal{D}_{\kappa, \leq h}^r = \mathcal{D}_{\kappa, \leq h}^{<r, \circ}$ and $H^0(K, \mathcal{D}_\kappa^r)_{\leq h} = H^0(K, \mathcal{D}_\kappa^{<r})_{\leq h}$, and the fact that $\mathcal{D}_\kappa^{<r, \circ}$ is bounded in $\mathcal{D}_\kappa^{<r}$. $\qquad\square$

Now we consider the behavior of $\mathcal{D}_\kappa^{<r, \circ}$ and $H^0(K, \mathcal{D}_{\kappa, \leq h}^{<r, \circ})$ under change of coefficients. Let $\kappa_R : T_0/\overline{Z(K)} \to R^\times$ be a weight. If $f : R \to R'$ is a homomorphism of pseudoaffinoid algebras, we let $\kappa_{R'}$ denote the composition $T_0/\overline{Z(K)} \xrightarrow{\kappa_R} R^\times \xrightarrow{f} R'^\times$. By [JN16, Corollary A.14], f is topologically of finite type, so we have a surjection $R \langle X_1, \ldots, X_n \rangle \twoheadrightarrow R'$. If $R_0 \subset R$ is a ring of definition and $u \in R_0$ is a pseudo-uniformizer, we define $R_0' := R_0 \langle X_1, \ldots, X_n \rangle$ and $u' := f(u)$.

We extend $|\cdot|_R$ to $R \langle X_1, \ldots, X_n \rangle$ via

$$\left| \sum_\alpha r_\alpha \underline{X}^\alpha \right|_R := \sup_\alpha |r_\alpha|_R$$

where α is a multi-index, and we equip R' with the quotient norm $|\cdot|_{R'}$. It follows from the discreteness of $\log_p |\cdot|_R$ that the unit ball of R' is R_0'.

Lemma 4.2.3. *With notation as above, the natural map $R_0' \widehat{\otimes}_{R_0} \mathcal{D}_{\kappa_R}^{<r, \circ} \to \mathcal{D}_{\kappa_{R'}}^{<r, \circ}$ is a topological isomorphism (with respect to the u'-adic topology), where the completed tensor product is taken with respect to the u-adic topology on $\mathcal{D}_{\kappa_R}^{<r, \circ}$ and the u'-adic topology on R_0'.*

Proof. Since f factors as $R \hookrightarrow R \langle X_1, \ldots, X_n \rangle \twoheadrightarrow R'$, and the result is clear for $R_0 \to R_0 \langle X_1, \ldots, X_n \rangle$, we may assume that f is surjective.

We first check that the morphism $R_0' \widehat{\otimes}_{R_0} \mathcal{D}_{\kappa_R}^{<r,\circ} \to \mathcal{D}_{\kappa_{R'}}^{<r,\circ}$ is an isomorphism of R_0'-modules. The discussion after [JN16, Proposition 3.2.7] shows that

$$\mathcal{D}_{\kappa_R}^{<r,\circ} \cong \prod_\alpha R_0 \cdot u^{-n_R(r,u,\alpha)} \mathbf{n}^\alpha$$

where $n_R(r,u,\alpha) := \left\lfloor \frac{|\alpha| \log_p r}{\log_p |u|_R} \right\rfloor$, $\mathbf{n}$ is a certain (non-canonical but explicit) finite set (depending only on the group-theoretic data we fixed at the beginning of §3), and α is a multi-index (and similarly for $\mathcal{D}_{\kappa_{R'}}^{<r,\circ}$). Now R_0' is a finitely presented R_0-module, and for any finitely presented R_0-module M, the natural morphism $M \otimes_{R_0} \prod_\alpha R_0 \cdot u^{-n_R(r,u,\alpha)} \mathbf{b}^\alpha \to \prod_\alpha M \cdot u^{-n_R(r,u,\alpha)} \mathbf{b}^\alpha$ is an isomorphism. By construction, $n_R(r,u,\alpha) = n_{R'}(r,u',\alpha)$ for all α, so the claim follows.

Finally, the morphism $R_0' \widehat{\otimes}_{R_0} \mathcal{D}_{\kappa_R}^{<r,\circ} \to \mathcal{D}_{\kappa_{R'}}^{<r,\circ}$ is clearly continuous, so the open mapping theorem implies that it is a topological isomorphism. $\qquad\square$

Corollary 4.2.4. *Let* $\kappa : T_0/\overline{Z(K)} \to R^\times$ *be a weight. If* F_κ *has a slope-$\leq h$ factorization, then* $H^0(K, \mathcal{D}_\kappa^{<r,\circ})_{\leq h}$ *is a finite projective R_0-module.*

Proof. We may assume $\operatorname{Spa} R$ is connected, so that $H^0(K, \mathcal{D}_\kappa^{<r})_{\leq h}$ is a finite projective R-module of some constant rank d. We define $H^0(K, \mathcal{D}_\kappa^{<r,\circ})_{>h} := \operatorname{im}\left(H^0(K, \mathcal{D}_\kappa^{<r,\circ}) \to H^0(K, \mathcal{D}_\kappa^r)_{>h}\right)$, so that $H^0(K, \mathcal{D}_\kappa^{<r,\circ}) \cong H^0(K, \mathcal{D}_\kappa^{<r,\circ})_{\leq h} \oplus H^0(K, \mathcal{D}_\kappa^{<r,\circ})_{>h}$.

Writing $D^\times \backslash (\mathbf{A}_{F,f} \otimes_F D)^\times / K = \coprod_{i \in I} D^\times g_i K$ for some finite set of elements $g_i \in (\mathbf{A}_{F,f} \otimes_F D)^\times$, we have an isomorphism $H^0(K, \mathcal{D}_\kappa^{<r,\circ}) \cong \oplus_{i \in I} (\mathcal{D}_\kappa^{<r,\circ})^{(K\mathbf{A}_{F,f}^\times \cap g_i^{-1} D^\times g_i)/F^\times}$. For every maximal point $x : R \to L$ of $\operatorname{Spa} R$, Lemma 4.2.3 implies that the base change map

$$\mathcal{O}_L \otimes_{R_0} \oplus_i \mathcal{D}_\kappa^{<r,\circ} \to \oplus_i \mathcal{D}_{\kappa_x}^{<r,\circ}$$

is an isomorphism. Moreover, the calculations of [Tay06, Lemma 1.1] show that the order of $(K\mathbf{A}_{F,f}^\times \cap g_i^{-1} D^\times g_i)/F^\times$ is prime to p for all i, so the specialization map $\mathcal{O}_L \otimes_{R_0} H^0(K, \mathcal{D}_\kappa^{<r,\circ}) \to H^0(K, \mathcal{D}_{\kappa_x}^{<r,\circ})$ is an isomorphism.

On the other hand, the specialization map carries $H^0(K, \mathcal{D}_\kappa^{<r,\circ})_{\leq h}$ to $H^0(K, \mathcal{D}_{\kappa_x}^{<r,\circ})_{\leq h}$ and $H^0(K, \mathcal{D}_\kappa^{<r,\circ})_{>h}$ to $H^0(K, \mathcal{D}_{\kappa_x}^{<r,\circ})_{>h}$. Since $\mathcal{O}_L \otimes_{R_0} H^0(K, \mathcal{D}_\kappa^{<r,\circ}) \to H^0(K, \mathcal{D}_{\kappa_x}^{<r,\circ})$ is an isomorphism, this implies that $\mathcal{O}_L \otimes_{R_0} H^0(K, \mathcal{D}_\kappa^{<r,\circ})_{\leq h} \to H^0(K, \mathcal{D}_{\kappa_x}^{<r,\circ})_{\leq h}$ is also an isomorphism.

Since R_0 has no u-torsion and the specialization map $\operatorname{Max}(\operatorname{Spa} R) \to \operatorname{MaxSpec} R_0/u$ from maximal points is surjective, it suffices to check that $H^0(K, \mathcal{D}_{\kappa_x}^{<r,\circ})_{\leq h}$ is free of rank d for all x. But this follows from the projectivity and rank of $H^0(K, \mathcal{D}_\kappa^{<r})_{\leq h}$. $\qquad\square$

In fact, we can prove something stronger:

Proposition 4.2.5. *For any finite set of primes Q as in section 4.1, the module $S_\kappa^{<r,\circ}(K^-(Q))_{\leq h}$ is finite projective over $R_0[\Delta_Q]$ and the natural map*

$$\sum_{h \in \Delta_Q} \langle h \rangle : \left(S_\kappa^{<r,\circ}(K^-(Q))_{\leq h}\right)_{\Delta_Q} \to S_\kappa^{<r,\circ}(K_0(Q))_{\leq h}$$

is an isomorphism.

Proof. We first assume that K is neat. Then we can write $\mathbf{A}_{F,f} \otimes_F D^\times = \coprod_i D^\times g_i K$, where the disjoint union is finite, and $K_Q(0) = \coprod_{j \in \Delta_Q} k_j K^-(Q)$. We claim that Δ_Q acts freely on $D^\times \backslash (\mathbf{A}_{F,f} \otimes_F D)^\times / K^-(Q)$, from which the result follows. But if $D^\times g_i k_j K^-(Q) = D^\times g_{i'} k_{j'} K^-(Q)$, then the neatness hypothesis 3.1.1 implies that $i = i'$ and $j = j'$, and the result follows.

If $K' \lhd K$ with K' neat and $[K : K']$ prime to p, then $S_\kappa^{<r,\circ}(K^-(Q))_{\leq h} = \left(S_\kappa^{<r,\circ}(K'_Q)_{\leq h}\right)^{K/K'}$. Since $S_\kappa^{<r,\circ}(K'_Q)_{\leq h}$ is projective, its invariants by a prime-to-p group are also projective. $\qquad\square$

4.3. **Galois representations.** In [JN16, §5.4], the authors construct families of Galois determinants (in the sense of [Che14]) over the eigenvarieties $\mathscr{X}_{\mathbf{G}}$ when $\mathbf{G} = \mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_n$ and F is totally real or CM, and prove that they satisfy local-global compatibility at places away from p and the level. Then the Jacquet–Langlands correspondence lets us deduce the following:

Theorem 4.3.1. *Let D be a quaternion algebra over a totally real field F, such that F is totally split at p and D is split at all places above p. Let $K = K^p I \subset (\mathbf{A}_{F,f} \otimes D)^\times$ be the level, and let S be the set of finite places v of F for which D is ramified or $K_v \neq \mathrm{GL}_2(\mathscr{O}_{F_v})$. Then there is a continuous 2-dimensional Galois determinant $D : \mathrm{Gal}_{F,S} \to \mathscr{O}(\mathscr{X}_{\underline{D}^\times})^+$ such that*

$$D(1 - X \cdot \mathrm{Frob}_v) = P_v(X)$$

for all $v \notin S$, where $P_v(X)$ is the standard Hecke polynomial.

Moreover, if $v \mid p$ then for every maximal point $x \in \mathscr{X}_{\underline{D}^\times}$ of weight $\kappa_x = (\kappa_{x,1}, \kappa_{x,2})$, we let $\psi : \mathscr{O}(\mathscr{X}_{\underline{D}^\times})^+ \to k(x)^+$ denote the corresponding specialization map. Then the Galois representation corresponding to $D_x|_{\mathrm{Gal}_{F,v}}$ is triauguline with parameters $\delta_1, \delta_2 : F_v^\times \rightrightarrows k(v)^\times$, where

$$\delta_1|_{\mathscr{O}_{F_v}^\times} = \kappa_{x,2}^{-1}|_{T_v} \ and \ \delta_1(\varpi_v) = \psi(U_v)$$

and

$$\delta_2|_{\mathscr{O}_{F_v}^\times} = (\kappa_{x,1}\chi_{\mathrm{cyc}})^{-1} \ and \ \delta_2(\varpi_v) = \psi(I_v \left(\begin{smallmatrix} \varpi_v & \\ & 1 \end{smallmatrix}\right) I_v)$$

Proof. It only remains to check local-global compatibility at places above p. But this is true for classical points by work of Saito, Blasius–Rogawski, and Skinner, and it is true for twist classical points by the definition of twisting. Then the result follows from [Bel21, Theorem 6.3.1]. $\qquad\square$

5. Patching and modularity

5.1. **Set-up.** Let us recall our goal. Fix a non-archimedean local field L with ring of integers $\mathscr{O}_L$, residue field $\mathbf{F}$, and uniformizer u. Fix a continuous odd representation $\bar{\rho} : \mathrm{Gal}_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{F})$, such that:

- $\bar{\rho}$ is modular
- $\bar{\rho}|_{\mathrm{Gal}_{\mathbf{Q}(\sqrt{2},\sqrt{3},\zeta_p)}}$ is absolutely irreducible
- $\bar{\rho}$ is unramified at all places away from p
- $\bar{\rho} \not\sim \chi \otimes \left(\begin{smallmatrix} \bar{\chi}_{\mathrm{cyc}} & * \\ & 1 \end{smallmatrix}\right)$ for any character $\chi : \mathrm{Gal}_{\mathbf{Q}} \to \mathbf{F}^\times$.

We wish to prove the following modularity theorem:

Theorem 5.1.1. *Suppose $\rho : \mathrm{Gal}_{\mathbf{Q}} \to \mathrm{GL}_2(\mathscr{O}_L)$ is a continuous odd representation unramified away from p and trianguline at p, whose reduction modulo u is as above. Then ρ arises from an overconvergent modular form.*

The predicted weight κ can be read off from the parameters of the triangulation, as can the predicted slope h.

More precisely, we will show that ρ corresponds to a class in $S_\kappa(K)_{\leq h}$, where $K = I \cdot K_1(N)^p = I \cdot \prod_{\ell \neq p, \ell \nmid N} \mathrm{GL}_2(\mathbf{Q}_\ell) \cdot \prod_{\ell \mid N} K_1(\ell)$ for some $N \geq 5$ prime to p. To do this, we will consider an open weight $\kappa : T_0 \to \mathscr{O}(U)^\times$, where $U \subset \mathscr{W}$ contains a point corresponding to κ and (U, h) is a slope datum, and we will study the spaces $S_\kappa(K^-(Q))_{\leq h}$ for varying sets of primes Q.

5.2. Patched eigenvarieties.

In this section, we construct patched quaternionic eigenvarieties, using the language of ultrafilters of [Sch18, §9]. We fix a totally real field F split at all places above p and a totally definite quaternion algebra D over F, which is ramified at all infinite places and split at all finite places. We also fix the tame level $K^p := \mathrm{GL}_2(\mathbf{A}_{F,f}^p)$. We further assume that $F/\mathbf{Q}$ is abelian, so that Leopoldt's conjecture is known to hold, and we assume the class group of F has order prime to p. Unlike [Sch18], we do not assume that F has a unique prime above p; we let $\Sigma_p := \{v \mid p\}$. We expect these hypotheses can be relaxed considerably, but this is not necessary for our applications.

Recall that there are Galois deformation rings $R_{\overline{\rho}, \Sigma_p}^\square$ and $R_{\overline{\rho}, \Sigma_p}$, parametrizing deformations of $\overline{\rho}$ unramified outside of Σ_p, where $R_{\overline{\rho}}^\square$ additionally parametrizes framings of the deformations at places of Σ_p. There is also a local framed deformation ring $R_{\overline{\rho}, \mathrm{loc}}^\square := \widehat{\otimes}_{v \in \Sigma_p} R_{\overline{\rho}_v}^\square$, where $R_{\overline{\rho}_v}^\square$ parametrizes framed deformations of $\overline{\rho}|_{\mathrm{Gal}_{F_v}}$, and there is a natural map $R_{\overline{\rho}, \mathrm{loc}}^\square \to R_{\overline{\rho}}^\square$.

We define a distinguished family of characters $\chi_{\mathrm{cyc,univ}} : \mathrm{Gal}_F \to \mathbf{Z}_p[\![T_0/\overline{Z(K)}]\!]^\times$ over integral weight space which deforms the inverse of the mod p cyclotomic character. We have a universal weight $\underline{\lambda} = (\lambda_1, \lambda_2)$, where each λ_i is a character $\prod_{v \in \Sigma_p} \mathscr{O}_{F_v}^\times \to \mathbf{Z}_p[\![T_0/\overline{Z(K)}]\!]^\times$, and we define $\kappa_v : \mathscr{O}_{F_v}^\times \cong \mathbf{Z}_p^\times \to \mathbf{Z}_p[\![T_0/\overline{Z(K)}]\!]^\times$ via $\kappa_v(x) := \left(\lambda_1|_{\mathscr{O}_{F_v}^\times}(x)\lambda_2|_{\mathscr{O}_{F_v}^\times}(x)x\right)^{-1}$. Then because we have assumed that Leopoldt's conjecture holds for F, we see that κ_v is independent of $v \in \Sigma$; global class field theory gives us a corresponding character $\mathrm{Gal}_{\mathbf{Q}} \to \mathbf{Z}_p[\![T_0/\overline{Z(K)}]\!]^\times$, which we restrict to Gal_F to obtain $\chi_{\mathrm{cyc,univ}}$.

We fix an unramified continuous character $\psi_0 : \mathrm{Gal}_F \to \mathbf{Z}_p[\![T_0/\overline{Z(K)}]\!]^\times$ such that the reduction $\overline{\psi}_0$ modulo the maximal ideal satisfies $\det \overline{\rho} = \overline{\psi}_0 \cdot \overline{\chi}_{\mathrm{cyc}}^{-1}$. Then we constructed quotients

$$\mathbf{Z}_p[\![T_0/\overline{Z(K)}]\!] \widehat{\otimes} R_{\overline{\rho}, \Sigma_p}^\square \twoheadrightarrow R_{\overline{\rho}, \Sigma_p}^{\square, \psi}$$

$$\mathbf{Z}_p[\![T_0/\overline{Z(K)}]\!] \widehat{\otimes} R_{\overline{\rho}, \mathrm{loc}}^\square \twoheadrightarrow R_{\overline{\rho}, \mathrm{loc}}^{\square, \psi}$$

$$\mathbf{Z}_p[\![T_0/\overline{Z(K)}]\!] \widehat{\otimes} R_{\overline{\rho}, \Sigma_p} \twoheadrightarrow R_{\overline{\rho}, \Sigma_p}^{\psi}$$

parametrizing families of deformations with fixed determinants.

We also define families of weights $\underline{\kappa}_v$ over $\mathscr{W}_F$ via

$$\underline{\kappa}_v = (\kappa_{v_1}, \kappa_{v,2}) = \left(\lambda_2|_{\mathscr{O}_{F_v}^\times}^{-1}, \lambda_1|_{\mathscr{O}_{F_v}^\times}^{-1} \right)$$

In order to find sets of Taylor–Wiles primes, we impose the following standard hypotheses:

(1) $p \geq 5$
(2) $\overline{\rho}|_{F(\zeta_p)}$ is absolutely irreducible
(3) If $p = 5$ and $\overline{\rho}$ has projective image $\mathrm{PGL}_2(\mathbf{F}_5)$, the kernel of $\overline{\rho}$ does not fix $F(\zeta_5)$

Then we have the following relative version of [Kis09a, Proposition 2.2.4] (since we assumed p splits completely in F, $[F : \mathbf{Q}] = |\Sigma_p|$):

Proposition 5.2.1. *Let* $g := \dim_{\mathbf{F}_q} H^1(\mathrm{Gal}_{F,\Sigma_p}, \mathrm{ad}^0 \overline{\rho}(1)) - 1$. *Then for each positive integer* n, *there exists a finite set* Q_n *of places of* F, *disjoint from* Σ_p, *of cardinality* $g + 1$, *such that*

(1) *for all* $v \in Q_n$, $\mathrm{Nm}(v) \equiv 1 \pmod{p^n}$, *and* $\overline{\rho}(\mathrm{Frob}_v)$ *has distinct eigenvalues*
(2) *the global relative Galois deformation ring* $R_{\overline{\rho},\Sigma_p \cup Q_n}^{\square,\psi}$ *parametrizing families of deformations with determinant* ψ *unramified outside* $\Sigma_p \cup Q_n$ *can be topologically generated as an* $R_{\overline{\rho},\mathrm{loc}}^{\square,\psi}$-*algebra by* g *elements.*

Proof. This follows from Lemma 2.1.1, as in [Kis09b, Proposition 3.2.5]. $\qquad\square$

We fix such a set Q_n for each $n \geq 1$, as well as a non-principal ultrafilter $\mathfrak{F}$ on $\{n \geq 1\}$ (more precisely, on its power set, ordered by inclusion). For notational convenience, we set $Q_0 := \emptyset$, and we let $Q'_n := Q_n \cup \Sigma_p$. For each n, we again let $K^-(Q_n) \subset K_0(Q_n) \subset G(\mathbf{A}_{F,f}^p) \cong \mathrm{GL}_2(\mathbf{A}_{F,f}^p)$ be the compact open subgroups

$$K^-(Q_n) := \prod_{v \notin Q_n} \mathrm{GL}_2(\mathscr{O}_{F_v}) \times \prod_{v \in Q_n} K_v(1) \subset \prod_{v \notin Q_n} \mathrm{GL}_2(\mathscr{O}_{F_v}) \times \prod_{v \in Q_n} K_v(0)$$

For each $v \in Q_n$, we fix a root α_v of the characteristic polynomial $X^2 - T_v X + \mathrm{Nm}(v)S_v$ of $\overline{\rho}(\mathrm{Frob}_v)$ (increasing $\mathbf{F}_q$ if necessary). Then we let $\mathbb{T}_{K_{Q_n}(i),\kappa,\leq h}^{<r,\circ} \subset \mathrm{End}(S_\kappa(K^-(Q_n))_{\leq h})$ denote the Hecke algebra generated by T_v and S_v for all $v \notin Q_n \cup \Sigma_p$, and $U_v := [K_v(i) \left(\begin{smallmatrix} 1 & 0 \\ 0 & \pi_v \end{smallmatrix}\right) K_v(i)]$ for $v \in Q_n$.

Thus, we have a collection of diagrams

$$\begin{array}{ccc} \mathscr{X}_{\underline{D}^\times/F}^{K^-(Q_n)} & \longrightarrow & \coprod_{\overline{\rho}} \mathrm{Spa}\, R_{\overline{\rho}, Q_n \cup \Sigma_p} \\ \downarrow{\scriptstyle\mathrm{wt}} & & \\ \mathscr{W}_F & & \end{array}$$

Now we fix a weight $\kappa : T_0 \to \mathscr{O}(U)^\times$, $U = \mathrm{Spa}\, R$ with with U either a maximal point or an open affinoid subspace of $\mathscr{W}_F$, such that (U, h) is a slope datum for $S_{\kappa,\psi_0}(K_0(Q_n))$ and $S_{\kappa,\psi_0}(K^-(Q_n))$ for all n (we use Lemmas 4.1.3 and 4.1.4 to find a uniform slope datum). We equip R with a norm adapted to κ, using Lemma 4.1.1, and we let $R_0 \subset R$ be the unit ball. Then we fix some $r > r_\kappa$. Letting $R_{\overline{\rho}, Q'_n}|_U$

denote the ring of definition of $U \times \operatorname{Spa} R_{\overline{\rho}, Q'_n}$ (with respect to R_0), we see that $\mathbb{T}^{<r,\circ}_{K^-(Q_n),\kappa,\leq h}$ is a $R_{\overline{\rho},Q'_n}|_U$-algebra.

The modularity of the residual representation $\overline{\rho}$ means that $\overline{\rho}$ corresponds to a maximal ideal $\mathfrak{m} \subset \mathbb{T}$. At the expense of possibly replacing $\mathbf{F} = \mathbb{T}/\mathfrak{m}$ with a quadratic extension, we may assume that the polynomials $X^2 - T_v X + \operatorname{Nm}(v)S_v$ have distinct roots $\{\alpha_v, \beta_v\}$ in $\mathbf{F}$. Since $\mathbf{Z}_p[\![T_0/\overline{Z(K)}]\!] \widehat{\otimes} \mathbb{T}$ is henselian, there are lifts A_v, B_v of α, β in $\left(\mathbf{Z}_p[\![T_0/\overline{Z(K)}]\!] \widehat{\otimes} \mathbb{T}\right)_{\mathfrak{m}}$. Let $\mathfrak{p} \subset \mathbb{T}^{<r,\circ}_{K,\kappa,\leq h}$ denote the ideal generated by $\mathfrak{m}$, and let $\mathfrak{p}^-_{Q_n} \subset \mathbb{T}^{<r,\circ}_{K^-(Q_n),\kappa,\leq h}$ (resp. $\mathfrak{p}_{0,Q_n} \subset \mathbb{T}^{<r,\circ}_{K_0(Q_n),\kappa,\leq h}$) be the ideal generated by $\mathfrak{m} \cap \mathbb{T}^-_{Q_n}$ (resp. $\mathfrak{m} \cap \mathbb{T}_{0,Q_n}$) and $U_v - \alpha_v$ for all $v \in Q_n$.

We consider the localizations $\left(S^{<r,\circ}_{\kappa,\psi_0}(K)_{\leq h}\right)_{\mathfrak{p}}$ and $\left(S^{<r,\circ}_{\kappa,\psi_0}(K_0(Q_n))_{\leq h}\right)_{\mathfrak{p}_{0,Q_n}}$. This amounts to restricting to the connected components of the eigenvariety where the Hecke eigenvalues lift those given by the maps $\mathbb{T} \to \mathbb{T}/\mathfrak{m}$ and $\mathbb{T}_{Q_n} \to \mathbb{T}_{Q_n}/\mathfrak{m} \cap \mathbb{T}_{Q_n}$. Then we have the following version of [Kis09a, Lemma 2.1.7]:

Proposition 5.2.2. *The map*
$$\prod_{v \in Q_n} (U_{\varpi_v} - B_v) : \left(S^{<r,\circ}_{\kappa,\psi_0}(K)_{\leq h}\right)_{\mathfrak{p}} \to \left(S^{<r,\circ}_{\kappa,\psi_0}(K_0(Q_n))_{\leq h}\right)_{\mathfrak{p}_{0,Q_n}}$$
is an isomorphism.

Proof. We may assume $Q_n = \{v\}$, by induction on the size of Q_n. Then the source and the target are finite projective R_0-modules, and by [Kis09a, Lemma 2.1.7] the map is an isomorphism when specialized to any classical weight. It follows that $\left(S^{<r,\circ}_{\kappa,\psi_0}(K)_{\leq h}\right)_{\mathfrak{p}}$ and $\left(S^{<r,\circ}_{\kappa,\psi_0}(K_0(Q_n))_{\leq h}\right)_{\mathfrak{p}_{0,Q}}$ have the same rank, so it suffices to check that $U_{\varpi_v} - B_v$ is surjective after specializing at every maximal ideal of R_0.

Thus, we need to check that
$$U_{\varpi_v} - B_v : \mathbf{F}' \otimes_{R_0} \left(S^{<r,\circ}_{\kappa,\psi}(K)_{\leq h}\right)_{\mathfrak{p}} \to \mathbf{F}' \otimes_{R_0} \left(S^{<r,\circ}_{\kappa,\psi_0}(K_0(Q_n))_{\leq h}\right)_{\mathfrak{p}_{0,Q}}$$
is surjective for any specialization $R_0 \to \mathbf{F}'$ at a maximal ideal. But this is a map of vector spaces of the same dimension, so it is enough to prove injectivity.

The module $\mathbf{F}' \otimes_{R_0} \left(S^{<r,\circ}_{\kappa,\psi_0}(K)_{\leq h}\right)_{\mathfrak{p}}$ is a finite module over the artin local ring $\mathbb{T}_{\mathfrak{m}}/p$, so if the kernel of $U_{\varpi_v} - B_v$ is non-trivial, it contains $f \neq 0$ which is $\mathfrak{m}$-torsion. In particular, $T_v(f) = (\alpha_v + \beta_v)x$ and $U_{\varpi_v}(f) = \beta_v$.

Since
$$[K_0(v) \left(\begin{smallmatrix} 1 & \\ & \varpi_v \end{smallmatrix}\right) K_v(v)] = \coprod_{\alpha \in k_v} \left(\begin{smallmatrix} 1 & \\ \widetilde{\alpha}\varpi_v & \varpi_v \end{smallmatrix}\right) K_0(v)$$

and
$$[\operatorname{GL}_2(\mathscr{O}_{F_v}) \left(\begin{smallmatrix} 1 & \\ & \varpi_v \end{smallmatrix}\right) \operatorname{GL}_2(\mathscr{O}_{F_v})] = \left(\begin{smallmatrix} \varpi_v & \\ & 1 \end{smallmatrix}\right) \operatorname{GL}_2(\mathscr{O}_{F_v}) \bigsqcup \coprod_{\alpha \in k_v} \left(\begin{smallmatrix} 1 & \\ \widetilde{\alpha}\varpi_v & \varpi_v \end{smallmatrix}\right) \operatorname{GL}_2(\mathscr{O}_{F_v})$$

where $\widetilde{\alpha}$ denotes a lift of α, we see that
$$\left(\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix}\right) f = (T_v - U_{\varpi_v})(f) = \alpha_v f$$

Then
$$\left(\begin{smallmatrix} 1 & \\ & \varpi \end{smallmatrix}\right) f = \left(\left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)\left(\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)\right) f = \left(\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix}\right) f = \alpha_v f$$

(since f is fixed by the action of $\mathrm{GL}_2(\mathscr{O}_{F_v})$). It follows that

$$U_{\varpi_v}(f) = \sum_{\alpha \in k_v} \left(\begin{smallmatrix} 1 & 0 \\ \tilde{\alpha}\varpi_v & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & \\ & \varpi \end{smallmatrix}\right) f = |k_v|\alpha_v f = \alpha_v f$$

which contradicts the assumption that $\alpha_v \neq \beta_v$. $\qquad \square$

Moreover, by Lemma 4.2.5 $S_{\kappa,\psi_0}(K^-(Q_n))_{\leq h,\mathfrak{p}_{Q_n}^-}$ is a projective $R_0[\Delta_{Q_n}]$-module, with $R_0 \otimes_{R_0[\Delta_{Q_n}]} S_{\kappa,\psi_0}(K^-(Q_n))_{\leq h,\mathfrak{p}_{Q_n}^-} \cong S_{\kappa,\psi_0}(K_0(Q_n))_{\leq h,\mathfrak{p}_{0,Q_n}}$.

Set $j = 4|\Sigma_p| - 1$ and $h = |Q_n| = g + 1$. By the existence of Galois representations, $S_{\kappa,\psi_0}(K^-(Q_n))_{\leq h,\mathfrak{p}_{Q_n}^-}^\circ$ is a module over $R_{\overline{\rho},Q_n'}^\psi$. Using local-global compatibility at places in Q_n, there is a homomorphism $R_0 \widehat{\otimes} \mathbf{Z}_p[\![y_1,\ldots,y_h]\!] \to R_{\overline{\rho},Q_n'}^\psi|_U$ such that the action of $R_0 \widehat{\otimes} \mathbf{Z}_p[\![y_1,\ldots,y_h]\!]$ on $S_{\kappa,\psi_0}(K^-(Q_n))_{\leq h,\mathfrak{p}_{Q_n}^-}^\circ$ is compatible with the action of $R_0[\Delta_{Q_n}]$ via a fixed surjection $R_0 \widehat{\otimes} \mathbf{Z}_p[\![y_1,\ldots,y_h]\!] \to R_0[\Delta_{Q_n}]$. Moreover, we may use local-global compatibility at places in Σ_p to make $S_{\kappa,\psi_0}(K^-(Q_n))_{\leq h,\mathfrak{p}_{Q_n}^-}^\circ$ into a module over $R_{\mathrm{tri},\overline{\rho},Q_n',\leq h}^{\psi,\underline{\kappa}}|_U$, where the coordinates of $\mathbf{G}_{\mathrm{m}}^{\Sigma_p}$ act as U_v^{-1}.

Since $R_{\overline{\rho},Q_n'}^\psi \to R_{\overline{\rho},Q_n'}^{\square,\psi}$ is formally smooth of dimension j, we may construct a homomorphism

$$R_0 \widehat{\otimes} \mathbf{Z}_p[\![y_1,\ldots,y_h,y_{h+1},\ldots,y_{h+j}]\!] \to R_{\mathrm{tri},\overline{\rho},Q_n',\leq h}^{\square,\psi,\underline{\kappa}}|_U$$

compatible with

$$R_0 \widehat{\otimes} \mathbf{Z}_p[\![y_1,\ldots,y_h]\!] \to R_{\overline{\rho},Q_n'}^\psi|_U$$

such that $y_{h+1},\ldots,y_{h+j}$ are the framing variables. Finally, we fix a surjection $R_{\overline{\rho},\mathrm{loc}}^{\square,\psi}[\![x_1,\ldots,x_g]\!] \twoheadrightarrow R_{\overline{\rho},Q_n'}^{\square,\psi}$ and a map $R_0 \widehat{\otimes} \mathbf{Z}_p[\![y_1,\ldots,y_{h+j}]\!] \to R_{\overline{\rho},\mathrm{loc}}^{\square,\psi}[\![x_1,\ldots,x_g]\!]|_U$ such that the corresponding diagram

$$
\begin{array}{ccc}
R_0 \widehat{\otimes} \mathbf{Z}_p[\![y_1,\ldots,y_{h+j}]\!] & \longrightarrow & R_{\mathrm{tri},\overline{\rho},\mathrm{loc},\leq h}^{\square,\psi,\underline{\kappa}}[\![x_1,\ldots,x_g]\!] \\
& \searrow & \downarrow \\
& & R_{\mathrm{tri},\overline{\rho},Q_n',\leq h}^{\square,\psi,\underline{\kappa}}
\end{array}
$$

commutes.

Now we can patch. Set $M_n := R_{\mathrm{tri},\overline{\rho},Q_n',\leq h}^{\square,\psi,\underline{\kappa}} \otimes_{R_{\mathrm{tri},\overline{\rho},Q_n',\leq h}^{\psi,\underline{\kappa}}} S_{\kappa,\psi_0}(K_{Q_n}^-)_{\leq h,\mathfrak{p}_{Q_n}^-}^\circ$, so that $R_0 \otimes_{R_0 \widehat{\otimes} \mathbf{Z}_p[\![y_1,\ldots,y_h]\!]} M_n \cong S_{\kappa,\psi_0}(K)_{\leq h,\mathfrak{p}}^\circ$ for all $n \geq 1$.

For any open ideal $I \subset R_0[\![y_i]\!]$, this implies that $\{R_0/I \otimes_{R_0[\![y_i]\!]} M_n\}$ has rank bounded uniformly in n, so we may take an ultraproduct as in [Sch18, §8], and conclude that $\prod_{n \geq 1} M_n/I$ is finitely presented over $\prod_{n \geq 1} R_0 \widehat{\otimes} \mathbf{Z}_p[\![y_i]\!]/I$. For any such I, our choice of non-principal ultrafilter gives a localization map

$$\prod_{n \geq 1} R_0 \widehat{\otimes} \mathbf{Z}_p[\![y_i]\!]/I \to R_0 \widehat{\otimes} \mathbf{Z}_p[\![y_i]\!]/I$$

and hence

$$M_I := R_0 \widehat{\otimes} \mathbf{Z}_p[\![y_i]\!]/I \otimes_{\prod_{n \geq 1} R_0 \widehat{\otimes} \mathbf{Z}_p[\![y_i]\!]/I} \prod_n M_n/I$$

Passing to the inverse limit, we obtain

$$M_\infty := \varprojlim_I M_I$$

Set $R^{\square,\psi,\kappa}_{\mathrm{tri},\overline{\rho},\infty,\leq h}|_U := \varprojlim_n R^{\square,\psi,\kappa}_{\mathrm{tri},\overline{\rho},Q'_n,\leq h}|_U$ and $\mathbb{T}^{<r,\circ}_{\infty,\kappa,\leq h} := \varprojlim_n \mathbb{T}^{<r,\circ}_{K^-(Q_n),\kappa,\leq h}$. Then we have a sequence of homomorphisms

$$R_0 \mathbin{\widehat{\otimes}} \mathbf{Z}_p[\![y_i]\!] \to R^{\square,\psi,\kappa}_{\mathrm{tri},\overline{\rho},\mathrm{loc},\leq h}[\![x_i]\!]|_U \to R^{\square,\psi,\kappa}_{\mathrm{tri},\overline{\rho},\infty,\leq h}|_U \to \mathbb{T}^{<r,\circ}_{\infty,\kappa,\leq h}$$

compatible with their actions on M_∞.

Proposition 5.2.3. M_∞ *is a finite projective* $R_0 \mathbin{\widehat{\otimes}} \mathbf{Z}_p[\![y_1, \ldots, y_{h+j}]\!]$*-module.*

Now we pass to the analytic loci of the corresponding map

$$\operatorname{Spa} R^{\square,\psi,\kappa}_{\mathrm{tri},\overline{\rho},\mathrm{loc},\leq h}[\![x_i]\!]|_U \to \operatorname{Spa} R_0 \mathbin{\widehat{\otimes}} \mathbf{Z}_p[\![y_i]\!]$$

and we consider M_∞ as a coherent sheaf over $X^{\square,\psi,\kappa}_{\mathrm{tri},\overline{\rho},\mathrm{loc},\leq h}|_U \times \mathbf{B}^{\mathrm{ad},g}_1$.

The support of M^{an}_∞ over $X^{\square,\psi,\kappa}_{\mathrm{tri},\overline{\rho},\mathrm{loc},\leq h}|_U \times \mathbf{B}^{\mathrm{ad},g}_1$ is a Zariski-closed subspace; since M^{an}_∞ is a vector bundle over $\operatorname{Spa} R_0 \mathbin{\widehat{\otimes}} \mathbf{Z}_p[\![y_i]\!]^{\mathrm{an}}$, the dimension of its support must be equal to

$$\dim \operatorname{Spa} R_0 \mathbin{\widehat{\otimes}} \mathbf{Z}_p[\![y_i]\!]^{\mathrm{an}} = \dim U + (g+1) + (4|\Sigma_p| - 1) = \dim U + g + 4|\Sigma_p|$$

But the morphism $X^{\square,\psi,\kappa_n}_{\mathrm{tri},\overline{\rho},\mathrm{loc}}|_U \to U$ has relative dimension $4|\Sigma_p|$ by Proposition 2.3.3, so the support of M^{an}_∞ on $X^{\square,\psi,\kappa}_{\mathrm{tri},\overline{\rho},\mathrm{loc},\leq h}|_U \times \mathbf{B}^{\mathrm{ad},g}_1$ is the union of irreducible components.

Finally, since we have a closed embedding

$$X^{\square,\psi,\kappa}_{\mathrm{tri},\overline{\rho},\infty,\leq h}|_U \hookrightarrow X^{\square,\psi,\kappa}_{\mathrm{tri},\overline{\rho},\mathrm{loc},\leq h}|_U \times \mathbf{B}^{\mathrm{ad},g}_1$$

we conclude that the support of M_∞ on $X^{\square,\psi,\kappa}_{\mathrm{tri},\overline{\rho},\infty,\leq h}|_U$ is also a union of irreducible components.

We summarize this discussion:

Theorem 5.2.4. *There is a space* $\mathscr{X}^\infty_{\underline{D}^\times,\leq h}|_U$ *(which we call the patched eigenvariety) and a morphism*

$$\mathscr{X}^\infty_{\underline{D}^\times,\leq h}|_U \to \operatorname{Spa} R^{\square,\psi,\kappa}_{\mathrm{tri},\overline{\rho},\mathrm{loc},\leq h}[\![x_i]\!]|_U$$

whose image is the union of irreducible components.

Since this morphism factors through the global trianguline variety, we also deduce the following corollary:

Corollary 5.2.5. *The support of* $M^{\mathrm{an}}_\infty/(y_1, \ldots, y_h)$ *in the trianguline variety over* $X^{\square,\psi,\kappa}_{\mathrm{tri},\overline{\rho},\infty,\leq h}|_U/(y_1, \ldots, y_h) \cong X^{\square,\psi,\kappa}_{\mathrm{tri},\overline{\rho}}|_U$ *is a union of irreducible components.*

Remark 5.2.6. We carried out this construction locally, because the requirement that the norm on R be slope adapted to κ introduced an auxiliary choice of integral structure; we have not checked that the analytic patched module M^{an}_∞ is independent of this choice.

5.3. **Modularity.** We are now in a position to prove Theorem 1.

Proposition 5.3.1. *Let $F/\mathbf{Q}$ be a real quadratic extension split at p. Then ρ : $\mathrm{Gal}_{\mathbf{Q}} \to \mathrm{GL}_2(L)$ is modular if and only if $\rho|_{\mathrm{Gal}_F}$ is modular.*

Choose $F/\mathbf{Q}$ a real quadratic extension split at p, such that the class group of F has order prime to p (for example, we may choose $F = \mathbf{Q}(\sqrt{2})$, $\mathbf{Q}(\sqrt{3})$, or $\mathbf{Q}(\sqrt{6})$). Now let D/F be a totally definite quaternion algebra, split at all finite places. The Jacquet–Langlands correspondence gives us a morphism of eigenvarieties $\mathscr{X}_{D^\times} \to \mathscr{X}_{\mathrm{GL}_2/F}$, so it suffices to show that $\rho|_{\mathrm{Gal}_F}$ corresponds to a point on $\mathscr{X}_{\underline{D}^\times}$.

Theorem 5.3.2. *$\rho|_{\mathrm{Gal}_F}$ corresponds to a point on $\mathscr{X}_{\underline{D}^\times}$.*

Proof. Let $\rho_0 := \rho|_{\mathrm{Gal}_F}$. It is enough to show that the point of $X_{\mathrm{tri},\overline{\rho}_0}^{\square,\psi,\kappa}$ corresponding to ρ is in the support of $M_\infty^{\mathrm{an}}/(y_1, \ldots, y_h)$ for some slope datum (U, h). We choose h depending on $\delta_1(p)$, and we fix a weight κ_0 according to $(\delta_1|_{\mathbf{Z}_p^\times}, \delta_2|_{\mathbf{Z}_p^\times})$. Then we use Proposition 3.1.1 to guarantee the existence of a slope datum for an irreducible neighborhood of κ_0.

We claim that U is contained in the support of $M_\infty^{\mathrm{an}}/(y_1, \ldots, y_h)$. Since the parameters of $D_{\mathrm{rig}}(\rho)$ were assumed regular, ρ_0 corresponds to a smooth point of $X_{\mathrm{tri},\overline{\rho}_0,\mathrm{loc},\leq h}^{\square,\psi,\kappa}$. Thus, there is an open neighborhood of ρ_0 contained in a single irreducible component V of $X_{\mathrm{tri},\overline{\rho}_0,\mathrm{loc},\leq h}^{\square,\psi,\kappa}$, and ρ_0 can be analytically deformed to characteristic 0.

The image of the morphism $X_{\mathrm{tri},\overline{\rho}_0,\infty,\leq h}^{\square,\psi,\kappa}|_U \to X_{\mathrm{tri},\overline{\rho}_0,\mathrm{loc},\leq h}^{\square,\psi,\kappa}|_U$ is the union of irreducible components, including V. Furthermore, V contains characteristic 0 crystalline points, and dimension considerations imply that the crystalline locus of any fiber of V over U is a (possibly empty) union of irreducible components.

Inspection of the patching construction shows that the image of $\cup_n X_{\mathrm{tri},\overline{\rho}_0,Q_n',\leq h}^{\square,\psi,\kappa}$ yields a Zariski dense subset of any fiber of V over U. If we choose a regular weight $\kappa_1 \in U$ such that $V|_{\kappa_1}$ contains crystalline points, we obtain a representation ρ_1 of Gal_F, unramified outside $Q_n \cup \Sigma_p$ for some n, and crystalline of regular Hodge–Tate weights at places above p, corresponding to a point of $V|_{\kappa_1}$. But then ρ_1 is known to be modular, so V is in the support of M_∞ and we are done. $\square$

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