M3/4/5P12 Problem sheet #1

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January 23, 2017

1. (a) Let $G = C_4 \times C_2 = \langle s, t : s^4 = t^2 = e, st = ts \rangle$. Let $V = \mathbf{C}^2$ (with the standard basis), and consider the matrices

$$S := \left(\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right) \text{and } T := \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$$

   Verify that sending s to S and t to T defines a representation of G. Is this representation faithful?

   (b) Now let

$$Q := \left(\begin{smallmatrix} i & 0 \\ 1 & 1 \end{smallmatrix}\right) \text{and } R := \left(\begin{smallmatrix} -1 & 0 \\ i+1 & 1 \end{smallmatrix}\right)$$

   Verify that sending s to Q and t to R also defines a representation of G. Is this representation faithful?

   (c) Show that S and Q are conjugate, and so are T and R.

   (d) Is there a basis $\mathcal{B}$ such that Q and R are both diagonal with respect to $\mathcal{B}$?

   (e) Are the two representations isomorphic?

2. Let $C_n = \langle g : g^n = e \rangle$ denote the cyclic group of order n, with generator g.

   (a) Write down a matrix for the regular representation of C_n with respect to its standard basis (it is enough to write down the matrix for g).

   (b) What are the eigenvalues for the matrix of g?

   (c) Write down a basis of eigenvectors for the matrix of g.

3. *(Advanced question)* Let (V, ρ) be a representation of a finite abelian group G. Is there a basis $\mathcal{B}$ of V such that the matrix representation of $\rho(g)$ with respect to $\mathcal{B}$ is diagonal for *every* $g \in G$?

4. Recall that $S_3 \cong D_6$, where D_6 denotes the symmetry group of a regular triangle. Embed an equilateral triangle in the real plane by placing vertices at $(1, 0)$, $(-1/2, \sqrt{3}/2)$, and $(-1/2, -\sqrt{3}/2)$, and number these 1, 2, and 3, respectively.

   (a) Which permutation corresponds to "counterclockwise rotation"? What about "reflection over the x-axis"?

   (b) Write down the matrix for "counterclockwise rotation" with respect to the standard basis of $\mathbf{R}^2$.

   (c) Find a basis $\mathcal{B}$ so that "counterclockwise rotation" is diagonal. Write down the matrix for "reflection over the x-axis" with respect to $\mathcal{B}$.

5. Recall that D_8, the symmetry group of the square, is a group of order 8 with a presentation $\langle s, t : s^4 = t^2 = e, tst = s^{-1} \rangle$. Write down two 1-dimensional representations of D_8. Are there any others?

6. (a) Write down the permutation representation (V, ρ) for S_3 with respect to the standard basis, i.e., the one indexed by elements of $\{1, 2, 3\}$ (it is enough to write it down for generators).

(b) Write down a 1-dimensional subrepresentation $W \subset V$. Are there any other 1-dimensional subrepresentations?

(c) With the "usual" inner product on V (i.e., $(ae_1+be_2+ce_3)\cdot(a'e_1+b'e_2+c'e_3) = aa'+bb'+cc'$), write down a subspace $W' \subset V$ orthogonal to your W. Does the action of S_3 on V stabilize W'?

(d) Find a basis $\mathcal{B}$ of W' so that the matrix of $\rho(123)$ with respect to $\mathcal{B}$ is diagonal. What is the matrix of $\rho(23)$ with respect to $\mathcal{B}$?

7. Let (V, ρ_V) and (W, ρ_W) be representations of a finite group G, and let $f : V \to W$ be a G-linear map.

   (a) If f is invertible, prove that f^{-1} is also G-linear.

   (b) Prove that the composition of two G-linear maps is again G-linear.

   (c) Deduce that "is isomorphic" is an equivalence relation on representations of G.

8. Let G, H be finite groups, and let $f : G \to H$ be a homomorphism. If (V, ρ) is a representation of H, prove that $\rho \circ f : G \to \mathrm{GL}(V)$ is a representation of G. We call this representation the *restriction* of V from H to G along f.

9. *(Advanced question)*

   (a) Suppose that $H \subset G$ are finite groups. Describe how to construct a representation (V, ρ) of G from the set of cosets G/H.

   (b) Now suppose that $G = S_n$ and $H = A_n$. Is there a basis $\mathcal{B}$ of V such that the matrix for $\rho(g)$ with respect to $\mathcal{B}$ is diagonal for every $g \in S_n$?

   (c) More generally, suppose $H \lhd G$, so that G/H is also a group. Describe the representation V in terms of a representation of G/H.