M3/4/5P12 Solutions #4

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1. (a) $Z(G) = \{\pm 1\}$, so two conjugacy classes are $\{1\}$ and $\{-1\}$. In addition, $i^{-1} = -1$, $j^{-1} = -j$, and $k^{-1} = -k$. Thus, $(-j)i(-j)^{-1} = jij^{-1} = -ijj^{-1} = -i$ and $(-k)i(-k)^{-1} = kik^{-1} = -ikk^{-1} = -i$, so $\{\pm i\}$ is another conjugacy class. Similar reasoning shows that the other conjugacy classes are $\{\pm j\}$ and $\{\pm k\}$.

   (b) There are many ways to see this. For example, Q_8 has only one element of order 2, namely -1, while D_8 has five, namely s^2, t, st, s^2t, s^3t.

   (c) Let (V, ρ_V) be a 1-dimensional representation of Q_8. A 1-dimensional representation must be constant on conjugacy classes of Q_8, so $\rho_V(i) = \rho_V(-i)$, and $\rho_V(1) = \rho_V(i \cdot (-i)) = \rho_V(i)^2$, so $\rho_V(i) = \pm 1$. Similarly, $\rho_V(j) = \pm 1$ and since $ij = k$, $\rho_V(k) = \rho_V(i)\rho_V(j)$. Thus, there are four 1-dimensional representations of Q_8.

   (d) Q_8 has order 8 and we have found four 1-dimensional representations. Since $8 = \sum_i (\dim V_i)^2$, where the sum runs over the irreducible representations V_i of Q_8, and Q_8 has five conjugacy classes, there is one more representation and it is 2-dimensional. The character table so far is

	$\{1\}$	$\{-1\}$	$\{\pm i\}$	$\{\pm j\}$	$\{\pm k\}$
conjugacy class size	1	1	2	2	2
χ_{++}	1	1	1	1	1
χ_{+-}	1	1	1	-1	-1
χ_{-+}	1	1	-1	1	-1
χ_{--}	1	1	-1	-1	1
χ_5	2	?	?	?	?

   Column orthogonality implies that $\chi_{++}(1)\overline{\chi_{++}(-1)} + \chi_{+-}(1)\overline{\chi_{+-}(-1)} + \chi_{-+}(1)\overline{\chi_{-+}(-1)} + \chi_{--}(1)\overline{\chi_{--}(-1)} + \chi_5(1)\overline{\chi_5(-1)} = 0$, so $\chi_5(-1) = -2$. Column orthogonality further implies that

$$4 = \chi_{++}(i)^2 + \chi_{+-}(i)^2 + \chi_{-+}(i)^2 + \chi_{--}(i)^2 + \chi_5(i)\overline{\chi_5(-i)}$$
$$4 = \chi_{++}(j)^2 + \chi_{+-}(j)^2 + \chi_{-+}(j)^2 + \chi_{--}(j)^2 + \chi_5(j)\overline{\chi_5(-j)}$$
$$4 = \chi_{++}(k)^2 + \chi_{+-}(k)^2 + \chi_{-+}(k)^2 + \chi_{--}(k)^2 + \chi_5(k)\overline{\chi_5(-k)}$$

   But this implies that $\chi_5(i) = \chi_5(j) = \chi_5(k) = 0$. Thus, our character table is

	$\{1\}$	$\{-1\}$	$\{\pm i\}$	$\{\pm j\}$	$\{\pm k\}$
conjugacy class size	1	1	2	2	2
χ_{++}	1	1	1	1	1
χ_{+-}	1	1	1	-1	-1
χ_{-+}	1	1	-1	1	-1
χ_{--}	1	1	-1	-1	1
χ_5	2	-2	0	0	0

2. Since $|G| = \sum_i (\dim V_i)^2 = \sum_i 1$, there are $|G|$ irreducible representations of G. Therefore, $|G| = \dim C(G) \geq \dim C_{cl}(G) \geq |G|$ and so each element of G is its own conjugacy class. It follows that $G = Z(G)$ and G is abelian.

3. (a) The conjugacy classes of S_3 are $\{e\}$, $\{(1\ 2), (1\ 3), (2\ 3)\}$, and $\{(1\ 2\ 3), (1\ 3\ 2)\}$. There are two 1-dimensional representations, namely the trivial representation and the sign representation, and one 2-dimensional representation. The character table is

	$\{1\}$	$(1\ 2)$	$(1\ 2\ 3)$
conjugacy class size	1	3	2
χ_{triv}	1	1	1
χ_{sign}	1	-1	1
χ_V	2	0	-1

(b) $\chi_{V^*} = \overline{\chi_V}$ so $\chi_{V^*}(e) = 2$, $\chi_{V^*}(1\ 2) = 0$, and $\chi_{V^*}(1\ 2\ 3) = -1$. $\chi_{V\otimes V} = \chi_V \cdot \chi_V$, so $\chi_{V\otimes V}(e) = 4$, $\chi_{V\otimes V}(1\ 2) = 0$, and $\chi_{V\otimes V}(1\ 2\ 3) = 1$. $\chi_{\mathrm{Hom}(V,V)} = \overline{\chi_V} \cdot \chi_V$, so $\chi_{\mathrm{Hom}(V,V)}(e) = 4$, $\chi_{\mathrm{Hom}(V,V)}(1\ 2) = 0$, and $\chi_{\mathrm{Hom}(V,V)}(1\ 2\ 3) = 1$.

(c) $V \otimes V$ and $\mathrm{Hom}(V, V)$ have the same character χ. Furthermore,

$$\begin{aligned}
\langle \chi, \chi_{triv} \rangle &= \frac{1}{6}\left(4 + 3 \cdot 0 + 2 \cdot 1\right) = 1 \\
\langle \chi, \chi_{sign} \rangle &= \frac{1}{6}\left(4 + 3 \cdot 0 + 2 \cdot 1\right) = 1 \\
\langle \chi, \chi_V \rangle &= \frac{1}{6}\left(8 + 3 \cdot 0 + 2 \cdot (-1)\right) = 1
\end{aligned}$$

so $\chi = \chi_{triv} + \chi_{sign} + \chi_V$.

(d) We compute

$$\begin{aligned}
\langle \phi, \chi_{triv} \rangle &= \frac{1}{6}\left(4 + 3 \cdot 0 + 2 \cdot (-5)\right) = -1 \\
\langle \phi, \chi_{sign} \rangle &= \frac{1}{6}\left(4 + 3 \cdot 0 + 2 \cdot (-5)\right) = -1 \\
\langle \phi, \chi_V \rangle &= \frac{1}{6}\left(8 + 3 \cdot 0 + 2 \cdot (-5)(-1)\right) = 3
\end{aligned}$$

so $\phi = -\chi_{triv} - \chi_{sign} + 3\chi_V$. But ϕ is not the character of a representation of S_3 because the trivial and sign representation can't appear with negative multiplicity.

4. (a) This function is the character of the restriction of either 3-dimensional irreducible representation of S_4 to A_4. It is irreducible because

$$\langle \chi_U, \chi_U \rangle = \frac{1}{12}\left(9 + 4 \cdot 0^2 + 4 \cdot 0^2 + 3 \cdot (-1)^2\right) = 1$$

Since $\chi_U(e) = 3$, the corresponding representation is 3-dimensional.

(b) Since $|A_4| = 12 = 1^2 + 3^2 + \sum_i (\dim V_i)^2$, the squares of the remaining representations of A_4 add to 2. Therefore, there are two more 1-dimensional representations.

(c) We need $\langle \chi_3, \chi_U \rangle = \langle \chi_4, \chi_U \rangle = 0$. But

$$\begin{aligned}
0 = \langle \chi_3, \chi_U \rangle &= \frac{1}{12}\left(3 + 4 \cdot 0 + 4 \cdot 0 + 3 \cdot \chi_3((1\ 2)(3\ 4))(-1)\right) = \frac{1}{12}\left(3 - 3\chi_3((1\ 2)(3\ 4))\right) \\
0 = \langle \chi_4, \chi_U \rangle &= \frac{1}{12}\left(3 + 4 \cdot 0 + 4 \cdot 0 + 3 \cdot \chi_4((1\ 2)(3\ 4))(-1)\right) = \frac{1}{12}\left(3 - 3\chi_4((1\ 2)(3\ 4))\right)
\end{aligned}$$

It follows that $\chi_3((1\ 2)(3\ 4)) = \chi_4((1\ 2)(3\ 4)) = 1$.

(d) We also need $\langle \chi_3, \chi_{triv} \rangle = 0$, so

$$0 = \langle \chi_3, \chi_{triv} \rangle = \frac{1}{12}\left(1 + 4 \cdot \chi_3(1\ 2\ 3) + 4 \cdot \chi_3(1\ 3\ 2) + 3 \cdot 1\right) = \frac{1}{12}\left(4 + 4 \cdot \chi_3(1\ 2\ 3) + 4 \cdot \chi_3(1\ 3\ 2)\right)$$

Since χ_3 and χ_4 correspond to 1-dimensional representations, their values are roots of unity. If $\chi_3(1\ 2\ 3) = a+bi$, then since $1+\chi_3(1\ 2\ 3)+\chi_3(1\ 3\ 2) = 0$, we have $\chi_3(1\ 3\ 2) = -1-a-bi$. It follows that

$$1 = |\chi_3(1\ 3\ 2)|^2 = (1+a)^2 + b^2 = 1 + 2a + a^2 + b^2 = 2 + 2a$$

so $a = -1/2$ and $b = \pm\sqrt{3}/2$.

It follows that up to switching χ_3 and χ_4, $\chi_3(1\ 2\ 3) = \omega$, $\chi_3(1\ 3\ 2) = \omega^2$, $\chi_4(1\ 2\ 3) = \omega^2$, and $\chi_4(1\ 3\ 2) = \omega$.

(e) The kernel of χ_3 and χ_4 is the subgroup $N := \{e, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$, so χ_3 and χ_4 arise from homomorphisms $\rho_3, \rho_4 : A_4 \to A_4/N \cong C_3 \to \mathrm{GL}_1(\mathbf{C})$.

5. We draw a table of the restrictions, where the notation for the representations of S_4 is as in the notes:

	$\{e\}$	$\{s, s^{-1}\}$	$\{s^2\}$	$\{t, s^2t\}$	$\{st, s^3t\}$
conjugacy class size	1	2	1	2	2
$\chi_{triv}\|_{D_8}$	1	1	1	1	1
$\chi_{sign}\|_{D_8}$	1	-1	1	1	-1
$\chi_W\|_{D_8}$	3	-1	-1	-1	1
$\chi_{W'}\|_{D_8}$	3	1	-1	-1	-1
$\chi_U\|_{D_8}$	2	0	2	2	0

The character table for D_8 is

	$\{e\}$	$\{s, s^{-1}\}$	$\{s^2\}$	$\{st, s^{-1}t\}$	$\{t, s^2t\}$
size of conjugacy class	1	2	1	2	2
$\chi_{\mathrm{triv}}(g)$	1	1	1	1	1
$\chi_{+-}(g)$	1	1	1	-1	-1
$\chi_{-+}(g)$	1	-1	1	-1	1
$\chi_{--}(g)$	1	-1	1	1	-1
$\chi_2(g)$	2	0	-2	0	0

We compute the inner products of the restricted characters with the irreducible characters of D_8:

	χ_{triv}	χ_{+-}	χ_{-+}	χ_{--}	χ_2
$\chi_{triv}\|_{D_8}$	1	0	0	0	0
$\chi_{sign}\|_{D_8}$	0	0	0	1	0
$\chi_W\|_{D_8}$	0	0	1	0	1
$\chi_{W'}\|_{D_8}$	0	1	0	0	1
$\chi_U\|_{D_8}$	1	0	0	1	0

Note that since all of the class functions here are real, we don't have to keep track of order.

To summarize:

$$\begin{aligned}
\chi_{triv}|_{D_8} &= \chi_{\mathrm{triv}} \\
\chi_{sign}|_{D_8} &= \chi_{--} \\
\chi_W|_{D_8} &= \chi_{-+} + \chi_2 \\
\chi_{W'}|_{D_8} &= \chi_{+-} + \chi_2 \\
\chi_U|_{D_8} &= \chi_{triv} + \chi_{--}
\end{aligned}$$

6. Write $C_3 := \langle g : g^3 = e\rangle$. Then $\mathbf{C}[C_3]$ has basis $[e], [g], [g^2]$, and by multiplicativity, it is enough to decide where to send $[g]$. We define a map

$$\begin{aligned}
\mathbf{C}[C_3] &\to \mathbf{C} \oplus \mathbf{C} \oplus \mathbf{C} \\
[e] &\mapsto (1, 1, 1) \\
[g] &\mapsto (1, \omega, \omega^2) \\
[g^2] &\mapsto (1, \omega^2, \omega)
\end{aligned}$$

where $\omega = e^{2\pi i/3}$.

Similarly, if we set $\zeta := e^{2\pi i/n}$, we may define

$$\begin{aligned} \mathbf{C}[C_n] &\to \mathbf{C}^{\oplus n} \\ [g] &\mapsto (1, \zeta, \ldots, \zeta^{n-1}) \end{aligned}$$

7. We have $\pi(a,b) := a$, so

$$\pi((a,b)(a',b')) = \pi(aa',bb') = aa' = \pi(a,b)\pi(a',b')$$

On the other hand, $\sigma(a) := (a,0)$ so $\sigma(1) = (1,0) \neq 1_{A\oplus B}$.