GALOIS REPRESENTATIONS OVER PSEUDORIGID SPACES

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Abstract. We study p-adic Hodge theory for families of Galois representations over pseudorigid spaces. Such spaces are non-archimedean analytic spaces which may be of mixed characteristic, and which arise naturally in the study of eigenvarieties at the boundary. We construct overconvergent (φ, Γ)-modules for such Galois representations, and we show that (φ, Γ)-modules have finite cohomology. As a consequence, we deduce that the cohomology groups yield coherent sheaves, and we give partial results extending triangulations defined away from closed subspaces.

Contents

1. Introduction

In this article, we study p-adic Hodge theory for families of representations of Gal_K, where $K/\mathbf{Q}_p$ is a finite extension, and the families vary over certain analytic spaces, in the sense of Huber [Hub94]. Such families have been considered by a number of authors in the classical rigid analytic setting, where p is invertible in R, but working in Huber's setting permits us to study Galois representations with coefficients which are characteristic p or mixed characteristic.

Roughly speaking, p-adic Hodge theory is the study of representations of Galois groups, where both the Galois group and the coefficients are p-adic or characteristic p. One of the most powerful tools for studying p-adic Galois representations is the theory of (φ, Γ)-modules, which provide an equivalence between Galois representations and a certain category of modules equipped with an operator φ and the action of a 1-dimensional p-adic Lie group, called *étale* (φ, Γ)*-modules*.

The category of étale (φ,Γ)-modules is a full subcategory of the category of all (φ,Γ)-modules, and it often happens that the (φ,Γ)-module attached to an irreducible Galois representation becomes reducible in this larger category. Moreover, this reducibility is closely related to subtle p-adic Hodge theoretic invariants of the representation. If the (φ,Γ)-module attached to a Galois representations is the successive extension of rank-1 (φ,Γ)-modules, the representation is said to be *trianguline*.

One key feature of (φ,Γ)-modules is that they behave well in rigid analytic families. Given a Galois representation with coefficients in a $\mathbf{Q}_p$-affinoid algebra, the work of Berger and Colmez [BC08] constructs a family of (φ,Γ)-modules. Their construction is functorial, and so globalizes to sheaves of Galois representations over general rigid analytic spaces.

However, in recent years, interest has developed in families of Galois representations parametrized by analytic spaces which are not defined over a field. For example, Andreatta–Iovita–Pilloni constructed the eigencurve in mixed characteristic [AIP18], and their construction was extended to more general eigenvarieties by Johansson–Newton [JN16].

In this note, we study Galois representations with coefficients in similar rings. More precisely, we consider projective modules M over pseudoaffinoid algebras R equipped with a continuous R-linear action of Gal_K (pseudoaffinoid algebras, and their associated pseudorigid adic spaces, are a class of analytic adic spaces studied in [JN16] and [Lou17]).

Before we can construct and study families of (φ,Γ)-modules, we need to define the appropriate overconvergent period rings. In the rigid analytic setting, it was enough to take completed tensor products of $\mathbf{Q}_p$-Banach algebras and the overconvergent period rings $\mathbf{B}_{\mathrm{rig},K}^{\dagger,s}$ defined in e.g. [Ber02]. This is not possible in our setting because neither R nor $\mathbf{B}_{\mathrm{rig},K}^{\dagger,s}$ has the p-adic topology. However, both are affinoid Tate rings, and in this setting it is possible to define fiber products of the associated adic spaces. We provide a construction in Appendix A for the convenience of the reader.

We then define both perfect and imperfect overconvergent rings, which we denote by $\widetilde{\Lambda}_{R,[a,b]}$ and $\Lambda_{R,[a,b],K}$. We study the Galois action on the perfect rings, and show that we have appropriate normalized trace maps $R_{K,n}:\widetilde{\Lambda}_{R,[0,b]}^{H_K}\to\varphi^{-n}\Lambda_{R,[0,p^{-n}b],K}$.

This lets us prove our first main theorem, on the construction of (φ,Γ)-modules attached to Galois representations:

Theorem 1.1. *Let R be a pseudoaffinoid algebra, and let M be a finite projective R-module of rank d equipped with a continuous action of Gal_K, for a finite extension $K/\mathbf{Q}_p$. Then for some finite Galois extension L/K, there is a functorially associated projective $(\varphi,\Gamma_L,\mathrm{Gal}_{L/K})$-module $D_{(0,b],L}(M)$. This (φ,Γ)-module arises from a module which is projective of rank d over $\Lambda_{R,(0,b],L}$ for some $b>0$, and it is equipped with a Gal_L- and φ-equivariant isomorphism $\widetilde{\Lambda}_{R,(0,b]}\otimes_{\Lambda_{R,(0,b],L}} D_{(0,b],L}(M)\xrightarrow{\sim}\widetilde{\Lambda}_{R,(0,b]}\otimes_R M$.*

Here we define a category of $(\varphi,\Gamma_L,\mathrm{Gal}_{L/K})$-modules, which are projective (φ,Γ_L)-modules equipped with descent data.

We then turn to the study of general (φ, Γ)-modules over $\Lambda_{R,(0,b],K}$. We define the Fontaine–Herr–Liu complex $C_{\varphi,\gamma}$, which computes the cohomology of (φ, Γ)-modules, and we use the Cartan–Serre method of [KL] to prove that it has R-finite cohomology:

Theorem 1.2. *Let R be a pseudoaffinoid algebra over $\mathbf{Z}_p$, and let D be a (φ, Γ)-module over $\Lambda_{R,(0,b],K}$ for some $b > 0$. Then the cohomology of $C_{\varphi,\gamma}$ is R-finite.*

As a result, the formation of cohomology commutes with flat base change on $\mathrm{Spa}(R)$ and the ranks of cohomology groups of specializations are upper semi-continuous on $\mathrm{Spa}(R)$.

We also prove the expected comparison between Galois cohomology and (φ, Γ)-cohomology of the associated (φ, Γ)-module, which we expect to be useful in studying the cohomology of rank-1 objects:

Theorem 1.3. *Let R be a pseudoaffinoid algebra, and let M be a finite projective R-module of rank d equipped with a continuous action of Gal_K, for a finite extension $K/\mathbf{Q}_p$. Then for some finite Galois extension L/K, there is a canonical isomorphism $C^\bullet_{\mathrm{cont}}(\mathrm{Gal}_L, M) \xrightarrow{\sim} H_{\varphi,\Gamma_L}(D_{\mathrm{rig}}(M))$ between Galois cohomology and the cohomology of the associated $(\varphi, \Gamma_L, \mathrm{Gal}_{L/K})$-module.*

As a consequence, we can take the first step towards studying trianguline (φ, Γ)-modules over pseudorigid spaces, and show that if a triangulation is defined over a Zariski-dense subspace of $\mathrm{Spa}(R)$, the homomorphisms of (φ, Γ)-modules extend to points in the complement. However, we do not prove a full triangulation result. Such a theorem would require a careful study of the cohomology of (φ, Γ)-modules, and in particular, the classification of rank-1 (φ, Γ)-modules and a calculation of their cohomology. We intend to address these, and related questions, in subsequent work. We hope to obtain in particular a detailed understanding of the structure of extended eigenvarieties towards the boundary of weight space.

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2. Classical rings of p-adic Hodge theory

Let $\mathbf{C}_p^\flat := \varprojlim_{x \to x^p} \mathbf{C}_p$, and let $\mathscr{O}_{\mathbf{C}_p}^\flat$ be the subset of $x \in \mathbf{C}_p^\flat$ such that $x^{(0)} \in \mathscr{O}_{\mathbf{C}_p}$. Then $\mathbf{C}_p^\flat$ is an algebraically closed field of characteristic p with ring of integers $\mathscr{O}_{\mathbf{C}_p}^\flat$; Colmez calls these rings $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{E}}^+$, respectively. There is a valuation v defined by $v((x^{(i)})) = v_p(x^{(0)})$, and $\mathbf{C}_p^\flat$ is complete with respect to this valuation. There is also a Frobenius (given by raising to the pth power).

Let $\varepsilon := (\varepsilon^{(0)}, \varepsilon^{(1)}, \varepsilon^{(2)} \dots) \in \mathscr{O}_{\mathbf{C}_p}^\flat$ be a choice of compatible pth power roots of unity with $\varepsilon^{(0)} = 1$ and $\varepsilon^{(1)} \neq 1$. There is a natural map $k_F(\!(\overline{\pi})\!) \to \mathbf{C}_p^\flat$ given by sending $\overline{\pi}$ to $\varepsilon - 1$; we denote its image by $\mathbf{E}_F$, we denote by $\mathbf{E}$ the separable closure of $\mathbf{E}_F$ inside $\mathbf{C}_p^\flat$, and we denote by $\mathbf{E}^+$ the valuation ring of $\mathbf{E}$.

Let $\mathbf{A}_{\mathrm{inf}} := W(\mathscr{O}_{\mathbf{C}_p}^\flat)$ and $\widetilde{\mathbf{A}} := W(\mathbf{C}_p^\flat)$. There are two possible topologies on $\mathbf{A}_{\mathrm{inf}}$ and $\widetilde{\mathbf{A}}$, the p-adic topology or the weak topology; they are complete for both.

The *p-adic topology* is defined by putting the discrete topology on $W(\mathbf{C}_p^\flat)/p^nW(\mathbf{C}_p^\flat)$ for all n, and taking the projective limit topology on $\widetilde{\mathbf{A}}$; $\mathbf{A}_{\mathrm{inf}}$ is given the subspace topology. The *weak topology* is defined by putting the valuation topology on $\mathbf{C}_p^\flat$ and giving $\widetilde{\mathbf{A}}$ the product topology; $\mathbf{A}_{\mathrm{inf}}$ is again given the subspace topology.

Alternatively, the weak topology on $\widetilde{\mathbf{A}}$ is given by taking the sets

$$U_{k,n} := p^k\widetilde{\mathbf{A}} + [\widetilde{p}]^n\mathbf{A}_{\mathrm{inf}} \text{ for } k,n \geq 0$$

to be a basis of neighborhoods around 0, where $\widetilde{p} \in \mathscr{O}_{\mathbf{C}_p}^\flat$ is any fixed element with $\widetilde{p}^{(0)} = p$ (i.e., $\widetilde{p}$ is a system of compatible p-power roots of p). The weak topology on $\mathbf{A}_{\mathrm{inf}}$ is similarly generated by the sets $U_{k,n}\cap\mathbf{A}_{\mathrm{inf}} = p^k\mathbf{A}_{\mathrm{inf}} + [\widetilde{p}]^n\mathbf{A}_{\mathrm{inf}}$. Equivalently, $\mathbf{A}_{\mathrm{inf}}$ is given the $(p,[\varpi])$-adic topology, for any pseudo-uniformizer $\varpi \in \mathbf{C}_p^\flat$.

Both rings carry continuous bijective actions of Frobenius (for either topology). However, the Galois action is continuous for the weak topology, but it is not continuous for the p-adic topology because the Galois actions on $\mathscr{O}_{\mathbf{C}_p}^\flat$ and $\mathbf{C}_p^\flat$ are not discrete.

Explicitly, Frobenius acts by

$$\varphi\left(\sum_{k=0}^{\infty} p^k[x_k]\right) = \sum_{k=0}^{\infty} p^k[x_k^p]$$

and the Galois group Gal_K acts by

$$\sigma\left(\sum_{k=0}^{\infty} p^k[x_k]\right) = \sum_{k=0}^{\infty} p^k[\sigma(x_k)]$$

Now consider the pre-adic space $\mathrm{Spa}\,\mathbf{A}_{\mathrm{inf}}$ and its analytic adic subspace $\mathcal{Y}$ (i.e., $\mathcal{Y}$ is $\mathrm{Spa}\,\mathbf{A}_{\mathrm{inf}}$ minus the point corresponding to the maximal ideal). If ϖ is a pseudo-uniformizer of $\mathbf{C}_p^\flat$, there is a surjective continuous map $\kappa : \mathcal{Y} \to [0,\infty]$ given by

$$\kappa(x) := \frac{\log|[\varpi](\widetilde{x})|}{\log|p(\widetilde{x})|}$$

where $\widetilde{x}$ is the rank-1 generization of x. If $I \subset [0,\infty]$ is an interval, we let $\mathcal{Y}_I := \kappa^{-1}(I)$. The Frobenius on $\mathbf{A}_{\mathrm{inf}}$ induces isomorphisms $\mathcal{Y}_I \to \mathcal{Y}_{pI}$ (since $\kappa\circ\varphi = p\kappa$). Note that $\log|[\varpi](\widetilde{x})|, \log|p(\widetilde{x})| \in [-\infty, 0)$, since p and $[\varpi]$ are both topologically nilpotent, and therefore $|[\varpi](\widetilde{x})|, |p(\widetilde{x})| < 1$.

Following Scholze, we choose $\varpi = \widetilde{p}$, that is, a compatible sequence of p-power roots of p. Suppose $a, b \in [0,\infty]$ are rational numbers and $a \leq b$. Then $\mathcal{Y}_{[a,b]}$ is an affinoid subspace of $\mathcal{Y}$, and we write $\mathcal{Y}_{[a,b]} = \mathrm{Spa}(\widetilde{\Lambda}_{[a,b]}, \widetilde{\Lambda}_{[a,b]}^\circ)$. The inequalities

$$a \leq \frac{\log|[\varpi](\widetilde{x})|}{\log|p(\widetilde{x})|} \leq b$$

translate to the conditions

$$a\log|p(\widetilde{x})| \geq \log|[\varpi](\widetilde{x})| \geq b\log|p(\widetilde{x})|$$

or equivalently, $|p(\widetilde{x})|^a \geq |[\varpi](\widetilde{x})| \geq |p(\widetilde{x})|^b$. Thus,

$$\begin{aligned}(\widetilde{\Lambda}_{[a,b]}, \widetilde{\Lambda}^\circ_{[a,b]}) &= \left(\mathbf{A}_{\mathrm{inf}}\left\langle \frac{p}{[\varpi]^{1/b}}, \frac{[\varpi]^{1/a}}{p}\right\rangle\left[\frac{1}{[\varpi]}\right], \mathbf{A}_{\mathrm{inf}}\left\langle \frac{p}{[\varpi]^{1/b}}, \frac{[\varpi]^{1/a}}{p}\right\rangle\left[\frac{1}{[\varpi]}\right]^\circ\right) \\ &= \left(\mathbf{A}_{\mathrm{inf}}\left\langle \frac{p}{[\varpi]^{1/b}}, \frac{[\varpi]^{1/a}}{p}\right\rangle\left[\frac{1}{[\varpi]}\right], \mathbf{A}_{\mathrm{inf}}\left\langle \frac{p}{[\varpi]^{1/b}}, \frac{[\varpi]^{1/a}}{p}\right\rangle\right)\end{aligned}$$

Here we take $p/[\varpi]^\infty := 1/[\varpi]$ and we take $[\varpi]^\infty/p := 0$. If $\frac{p-1}{pa}, \frac{p-1}{pb} \in \mathbf{Z}_{\geq 0}[1/p] \cup \{\infty\}$, this is the pair of rings denoted $(\widetilde{\mathbf{A}}_{[s(b),s(a)]}[\frac{1}{[\varpi]}], \widetilde{\mathbf{A}}_{[s(b),s(a)]})$ in [Ber02] (since $v(\widetilde{p}) = 1$ and $v(\pi) = p/p-1$).

In the special case $I = [a,b] = [0,b]$, $(\widetilde{\Lambda}_{[0,b]}, \widetilde{\Lambda}^\circ_{[0,b]})$ is the pair of rings denoted $(\widetilde{\mathbf{A}}^{(0,b]}, \widetilde{\mathbf{A}}^{\dagger, s(b)})$, where $s(b) = (p-1)/pb$. We can write these rings more explicitly as subrings of $\widetilde{\mathbf{A}}$:

$$\widetilde{\Lambda}^\circ_{[0,b]} = \widetilde{\mathbf{A}}^{\dagger,s(b)} = \{x = \sum_{k=0}^{\infty} p^k[x_k] \in \widetilde{\mathbf{A}} \mid x_k \in \mathscr{O}^\flat_{\mathbf{C}_p}, x_k\varpi^{k/b} \to 0\}$$

and

$$\widetilde{\Lambda}_{[0,b]} = \widetilde{\mathbf{A}}^{(0,b]} = \{x = \sum_{k=0}^{\infty} p^k[x_k] \in \widetilde{\mathbf{A}} \mid x_k\varpi^{k/b} \to 0\}$$

If $[a,b] \neq [0,\infty]$, then the pair $(\widetilde{\Lambda}_{[a,b]}, \widetilde{\Lambda}^\circ_{[a,b]})$ is a Tate algebra; if $a \neq 0$, then p is a pseudo-uniformizer, and if $b \neq \infty$, then $[\varpi]$ (and $[\overline{\pi}]$) is a pseudo-uniformizer.

We can equip $\widetilde{\Lambda}_{[0,b]}$ with a valuation

$$\mathrm{val}^{[0,b]}(x) := \inf_{k\geq 0}(v_{\mathbf{C}_p^\flat}(x_k) + k/b)$$

It is separated and complete with respect to this valuation, and $\widetilde{\Lambda}^\circ_{[0,b]}$ is the ring of integers.

Since $\mathbf{C}_p^\flat$ is algebraically closed, we can extract arbitrary roots of ϖ; we may therefore define another valuation v_b on $\widetilde{\Lambda}_{[0,b]}$ by setting

$$v_b(x) := \sup_{r\in\mathbf{Q}:[\varpi]^r x\in\widetilde{\Lambda}^\circ_{[0,b]}} -r$$

We observe that $v_b([\varpi]x) = 1 + v_b(x)$ for any $x \in \widetilde{\Lambda}_{[0,b]}$.

Lemma 2.1. *For $x \in \widetilde{\Lambda}_{[0,b]}$, $\mathrm{val}^{(0,b]}(x) = v_b(x)$.*

Proof. We observe that $[\varpi]^r x \in \widetilde{\Lambda}^\circ_{[0,b]}$ if and only if $\mathrm{val}^{[0,b]}([\varpi]^r x) = r + \mathrm{val}^{[0,b]}(x) \geq 0$, which holds if and only if $\mathrm{val}^{[0,b]}(x) \geq -r$. Since we may approximate $\mathrm{val}^{[0,b)}(x)$ from below by rational numbers, it follows that $\mathrm{val}^{(0,b]}(x) = v_b(x)$. □

There are versions of all of these rings with no tilde; they are imperfect versions of the rings with tildes.

Let $\pi \in \widetilde{\mathbf{A}}_{\mathrm{inf}}$ denote $[\varepsilon] - 1$, where $[\varepsilon]$ denotes the Teichmüller lift of ε. Then there is a well-defined injective map $\mathscr{O}_F[\![X]\!][X^{-1}] \to \widetilde{\mathbf{A}}$ given by sending X to π; we let $\mathbf{A}_F$ denote the p-adic completion of the image. This is a Cohen ring for $\mathbf{E}_F$. We define $\mathbf{A}$ to be the completion of the integral closure of the image of $\mathbf{Z}_p[\![X]\!][X^{-1}]$ in $\widetilde{\Lambda}_{[0,0]}$, and we let $\mathbf{A}^+ := \mathbf{A} \cap \mathbf{A}_{\inf}$.

Because extensions of $\mathbf{E}_F$ correspond to unramified extensions of $\mathbf{A}_F[1/p]$, we get a natural Galois action on $\mathbf{A}$ and $\mathbf{A}^+$. We may therefore define $\mathbf{A}_K := \mathbf{A}^{H_K}$ and $\mathbf{A}_K^+ = (\mathbf{A}^+)^{H_K}$. When K is unramified over $\mathbf{Q}_p$, this agrees with our original definition of these rings.

We define the *overconvergent* subrings of $\mathbf{A}_K$: For $b \in [0,\infty]$, let $\Lambda_{[0,b],K} := \mathbf{A}_K \cap \widetilde{\Lambda}_{[0,b]}$ and let $\Lambda^\circ_{[0,b],K} := \mathbf{A}_K \cap \widetilde{\Lambda}^\circ_{[0,b]}$. These rings are given the topology induced as closed subspaces of $\widetilde{\Lambda}_{[0,b]}$. Thus, $\mathbf{A}_K^+ = \Lambda_{[0,\infty],K}$ and $\mathbf{A}_K = \Lambda_{[0,0],K}$.

Since we have isomorphisms $\varphi : \widetilde{\Lambda}_{[a,b]}^{H_K} \xrightarrow{\sim} \widetilde{\Lambda}_{[a/p,b/p]}^{H_K}$, we have induced Frobenius maps $\varphi : \Lambda_{[0,b],K} \to \Lambda_{[0,b/p],K}$. However, as $\mathbf{E}_K$ is imperfect, these maps are no longer isomorphisms. Indeed, $\Lambda_{[0,0],K}$ is free over $\varphi(\Lambda_{[0,0],K})$ of rank p, with a basis given by $\{1, [\varepsilon], \ldots, [\varepsilon^{p-1}\}$. We may therefore define a left inverse $\psi : \Lambda_{[0,0],K} \to \Lambda_{[0,0],K}$ to φ via $\psi := \frac{1}{p}\varphi^{-1} \circ \mathrm{Tr}_{\Lambda_{[0,0],K}/\varphi(\Lambda_{[0,0],K})}$.

Proposition 2.2. *If $b \in (0,\infty)$, then $\Lambda_{[0,b],K}$ is a Tate ring with ring of definition $\Lambda^\circ_{[0,b],K}$ and pseudo-uniformizer π. If $b = \infty$, then $\Lambda_{[0,\infty],K} = \mathbf{A}_K^+$ is an adic ring topologized by the ideal (p,π).*

Proof. We first observe that the cokernel of the inclusion $\mathbf{A}_K \hookrightarrow \widetilde{\mathbf{A}}^{H_K}$ has no p- or π-torsion, so the same holds for the cokernel of the inclusions $\Lambda^\circ_{[0,b],K} \hookrightarrow (\widetilde{\Lambda}^\circ_{[0,b]})^{H_K}$ for $b > 0$. Thus, the natural map $\Lambda_{[0,\infty],K}/p \to \mathbf{A}_{\inf}^{H_K}/p = \widehat{\mathscr{O}}^\flat_{K_\infty}$ remains injective; since $\Lambda_{[0,\infty],K}$ has the closed subspace topology from $\mathbf{A}_{\inf}^{H_K}$ and the topology on $\widehat{\mathscr{O}}^\flat_{K_\infty}$ is $\overline{\pi}$-adic, the topology on $\Lambda_{[0,\infty],K}$ is (p,π)-adic. Similarly, for $b \in (0,\infty)$, the natural map $\Lambda^\circ_{[0,b],K}/\pi \to (\widetilde{\Lambda}^\circ_{[0,b]})^{H_K}/\pi$ remains injective. Since $\pi \in (\widetilde{\Lambda}^\circ_{[0,b]})^{H_K}$ and π is a topologically nilpotent unit of $\widetilde{\Lambda}^{H_K}_{[0,b]}$, the ideal $(\pi) \subset (\widetilde{\Lambda}^\circ_{[0,b]})^{H_K}$ is an ideal of definition and $(\widetilde{\Lambda}^\circ_{[0,b]})^{H_K}/\pi$ is discrete. It follows that $(\pi) \subset \Lambda^\circ_{[0,b],K}$ is also an ideal of definition. $\square$

We can be more explicit about the structure of $\Lambda_{[0,b],K}$ when b is small. Recall that $\overline{\pi} := \varepsilon - 1 \in \mathscr{O}^\flat_{\mathbf{C}_p}$, so that it is a uniformizer of $\mathbf{E}_F$. Then for any ramified extension K/F, we may choose a uniformizer $\overline{\pi}_K$ of $\mathbf{E}_K$, and we may lift $\overline{\pi}_K$ to $\pi_K \in \mathbf{A}$. We fix a choice of π_K for every K and work with it throughout (when $F/\mathbf{Q}_p$ is unramified, we set $\pi_F = \pi$). Let F' be the maximal unramified extension of F in K_∞ and let $\mathscr{A}^{(0,b]}_{F'} := \{\sum_{m\in\mathbf{Z}} a_m X^m : a_m \in \mathscr{O}_{F'}, v_p(a_m) + mb \to \infty\}$ be the ring of integers of the ring of bounded analytic functions on the half-open annulus $0 < v_p(X) \leq b$ over F'. Then [Col08, Proposition 7.5] states that for $b < r_K$ (where r_K is a constant depending on the ramification of $\mathbf{E}_K/\mathbf{E}_F$), the assignment $f \mapsto f(\pi_K)$ is an isomorphism of topological rings from $\mathscr{A}_{F'}^{(0,bv_{\mathbf{C}_p^\flat}(\pi_K)]}$ to $\Lambda_{[0,b],K}$. Furthermore, if we define a valuation $v^{(0,b]}$ on $\mathscr{A}^{(0,b]}_{F'}$ by $v^{(0,b]}(\sum_{m\in\mathbf{Z}} a_m X^m) := \inf_{m\in\mathbf{Z}}(v_p(a_m) + mb)$, then

$$\mathrm{val}^{(0,b]}(f(\pi_K)) = \frac{1}{b} v^{(0,b]}(f)$$

It follows that after inverting p, we have an isomorphism from the ring of bounded analytic functions on the half-open annulus to $\Lambda_{[0,b],K}[1/p]$, equipped with the valuation $\mathrm{val}^{(0,b]}$. Note that when $K/\mathbf{Q}_p$ is ramified, this isomorphism depends on a choice of uniformizer of $\mathbf{E}_K$.

3. Rings with coefficients

Now we wish to introduce coefficients. We wish to consider Galois representations with coefficients in *pseudoaffinoid algebras*, in the sense of [Lou17] and [JN16] (or more generally, Galois representations on vector bundles over pseudorigid spaces).

Definition 3.1. Let E be a discretely-valued non-archimedean field and let $\mathscr{O}_E$ be its ring of integers. A *pseudoaffinoid $\mathscr{O}_E$-algebra* is a complete Tate $\mathscr{O}_E$-algebra R which has a ring of definition R_0 that is formally of finite type over $\mathscr{O}_E$. A *pseudorigid space* over $\mathscr{O}_E$ is an adic space X over $\mathrm{Spa}(\mathscr{O}_E)$ which is locally of the form $\mathrm{Spa}(R, R^\circ)$ for a pseudoaffinoid $\mathscr{O}_E$-algebra R.

When R is a pseudoaffinoid algebra, we will write $\mathrm{Spa}(R)$ for $\mathrm{Spa}(R, R^\circ)$.

Let R be a pseudoaffinoid algebra over $\mathbf{Z}_p$. Thoughout this section, we fix $R_0 \subset R$ a noetherian ring of definition, and we fix a choice of pseudo-uniformizer $u \in R_0$ and an integer $N \geq 1$ such that $p^N \in uR_0$; if necessary, we replace u by a power of itself. The rings we construct will depend on our choices of both R_0 and u, but for compactness of notation, we suppress u from the notation.

Throughout this section, we also assume that $p \notin R^\times$, since if R has the p-adic topology, we are in the classical setting treated in [BC08]. Under this assumption, we define a descending sequence of ideals $I_j \subset R_0$ via $I_j := r^j R \cap R_0$. Then for all $j \geq 1$, R_0/I_j is a u-torsion-free $(\mathbf{Z}/p^j)[\![u]\!]$-algebra. Since R_0 is noetherian, the I_j are finitely generated, and I_j/I_{j+1} is a finite u-torsion-free R_0/I_1-module.

If we have a Galois representation $\rho : \mathrm{Gal}_K \to \mathrm{GL}(T)$, where T is a finite free $\mathbf{Z}_p$-module, we may consider the module $\mathcal{O}_{\mathcal{Y}} \otimes_{\mathbf{Z}_p} T$. Then we obtain the (φ, Γ)-modules $\mathbf{D}^\dagger(T)$ and $\mathbf{D}^\dagger_{\mathrm{rig}}(T)$ by studying the H_K-invariants of $\mathcal{O}_{\mathcal{Y}} \otimes_{\mathbf{Z}_p} T$ (evaluated on various rational subdomains of $\mathcal{Y}$).

If instead we have a Galois representation $\rho : \mathrm{Gal}_K \to \mathrm{GL}(M)$, where M is a finite free R_0-module, we wish to extend scalars from R_0 to $\mathcal{Y}_{R_0} := \mathrm{Spa}(R_0, R_0) \times_{\mathbf{Z}_p} \mathcal{Y}$ or $\mathcal{Y}_R := \mathrm{Spa}(R) \times_{\mathrm{Spa}\,\mathbf{Z}_p} \mathcal{Y}$ and study the H_K-invariants over rational subdomains.

3.1. **Perfect overconvergent rings.**

The adic space $\mathcal{Y}$ is covered by the two open subspaces $\mathcal{Y}_{(0,\infty]}$ and $\mathcal{Y}_{[0,\infty)}$, which are the subspaces where $p \neq 0$ and $[\varpi] \neq 0$, respectively. Thus, to study $\mathcal{Y}_{R_0}$ or $\mathcal{Y}_R$, it suffices to study the fiber products of $\mathrm{Spa}(R_0, R_0)$ or $\mathrm{Spa}(R)$ with each of these subspaces. We are primarily interested in $\mathrm{Spa}(R_0, R_0) \times_{\mathbf{Z}_p} \mathcal{Y}_{[0,\infty)}$ and $\mathrm{Spa}(R) \times_{\mathbf{Z}_p} \mathcal{Y}_{[0,\infty)}$ (since if p is invertible, the theory reduces to the case of classical rigid analytic spaces).

We must therefore study $\mathrm{Spa}(R_0, R_0) \times_{\mathbf{Z}_p} \mathrm{Spa}(\widetilde{\Lambda}_{[0,r]}, \widetilde{\Lambda}^\circ_{[0,r]})$ and $\mathrm{Spa}(R) \times_{\mathbf{Z}_p} \mathrm{Spa}(\widetilde{\Lambda}_{[0,r]}, \widetilde{\Lambda}^\circ_{[0,r]})$. These will be pre-adic spaces (and in general not affinoid or quasi-compact), and they will be exhausted by pre-adic spaces of the form

$$\mathrm{Spa}^{\mathrm{ind}}\left((R_0 \mathop{\widehat{\otimes}}_{\mathbf{Z}_p} \widetilde{\Lambda}^\circ_{[0,r]})\left\langle \frac{u}{[\varpi]^{1/a}}\right\rangle\left[\frac{1}{[\varpi]}\right], \left((R_0 \otimes_{\mathbf{Z}_p} \widetilde{\Lambda}^\circ_{[0,r]})\left[\frac{u}{[\varpi]^{1/a}}\right]^{\mathrm{int}}\right)^\wedge\right)$$

or

$$\mathrm{Spa}^{\mathrm{ind}}\left((R_0 \mathop{\widehat{\otimes}}_{\mathbf{Z}_p} \widetilde{\Lambda}^\circ_{[0,r]})\left\langle \frac{u}{[\varpi]^{1/a}}, \frac{[\varpi]^a}{u}\right\rangle\left[\frac{1}{[\varpi]}, \frac{1}{u}\right], \left((R^+ \otimes_{\mathbf{Z}_p} \widetilde{\Lambda}^\circ_{[0,r]})\left[\frac{u}{[\varpi]^{1/a}}, \frac{[\varpi]^a}{u}\right]^{\mathrm{int}}\right)^\wedge\right)$$

respectively. Here $a \in \mathbf{Q}_{>0}$, $(R_0 \otimes_{\mathbf{Z}_p} \widetilde{\Lambda}^\circ_{[0,r]})\left[\frac{u}{[\varpi]^{1/a}}\right]^{\mathrm{int}}$ denotes the integral closure of $(R_0 \otimes_{\mathbf{Z}_p} \widetilde{\Lambda}_{[0,r]})\left[\frac{u}{[\varpi]^{1/a}}\right]$ in $(R_0 \widehat{\otimes}_{\mathbf{Z}_p} \widetilde{\Lambda}^\circ_{[0,r]})\left\langle \frac{u}{[\varpi]^{1/a}}, \frac{[\varpi]^a}{u}\right\rangle\left[\frac{1}{[\varpi]}\right]$, and $(R^\circ \otimes_{\mathbf{Z}_p} \widetilde{\Lambda}^\circ_{[0,r]})\left[\frac{u}{[\varpi]^{1/a}}, \frac{[\varpi]^a}{u}\right]^{\mathrm{int}}$ denotes the integral closure of $(R^\circ \otimes_{\mathbf{Z}_p} \widetilde{\Lambda}^\circ_{[0,r]})\left[\frac{u}{[\varpi]^{1/a}}, \frac{[\varpi]^a}{u}\right]$ in $(R_0 \otimes_{\mathbf{Z}_p} \widetilde{\Lambda}^\circ_{[0,r]})\left[\frac{u}{[\varpi]^{1/a}}, \frac{[\varpi]^a}{u}\right]\left[\frac{1}{[\varpi]}, \frac{1}{u}\right]$.

Since $\mathbf{Z}_p \to R^\circ$ is continuous, $\mathrm{ord}_R(p) \geq 0$. In other words, there are integers $m, m' \in \mathbf{Z}_{\geq 0}$ such that $p^m \in (u^{m'})$. Let $s = \frac{m}{m'}$. Then

$$\left(\frac{p^s}{u}\right)\cdot\left(\frac{u}{[\varpi]^{1/a}}\right) = \frac{p^s}{[\varpi]^{1/a}} = \left(\frac{p}{[\varpi]^{1/as}}\right)^s$$

so $\frac{p}{[\varpi]^{1/as}}$ is power-bounded, and we obtain continuous maps

$$(R_0 \mathop{\widehat{\otimes}}_{\mathbf{Z}_p} \widetilde{\Lambda}^\circ_{[0,as]})\left\langle \frac{u}{[\varpi]^{1/a}}\right\rangle\left[\frac{1}{[\varpi]}\right] \to (R_0 \mathop{\widehat{\otimes}}_{\mathbf{Z}_p} \widetilde{\Lambda}^\circ_{[0,r]})\left\langle \frac{u}{[\varpi]^{1/a}}\right\rangle\left[\frac{1}{[\varpi]}\right]$$

and

$$(R_0 \mathop{\widehat{\otimes}}_{\mathbf{Z}_p} \widetilde{\Lambda}^\circ_{[0,as]})\left\langle \frac{u}{[\varpi]^{1/a}}\right\rangle\left[\frac{1}{[\varpi]}\right] \to (R_0 \mathop{\widehat{\otimes}}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}})\left\langle \frac{u}{[\varpi]^{1/a}}\right\rangle\left[\frac{1}{[\varpi]}\right]$$

Proposition 3.2. *The image of* $(R_0 \widehat{\otimes} \widetilde{\Lambda}^\circ_{[0,as]})\left\langle \frac{u}{[\varpi]^{1/a}}\right\rangle$ *in* $(R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}})\left\langle \frac{u}{[\varpi]^{1/a}}\right\rangle\left[\frac{1}{[\varpi]}\right]$ *is a ring of definition. If* $r \geq as$*, then the image of* $(R_0 \widehat{\otimes} \widetilde{\Lambda}^\circ_{[0,as]})\left\langle \frac{u}{[\varpi]^{1/a}}\right\rangle$ *in* $(R_0 \widehat{\otimes}_{\mathbf{Z}_p} \widetilde{\Lambda}^\circ_{[0,r]})\left\langle \frac{u}{[\varpi]^{1/a}}\right\rangle\left[\frac{1}{[\varpi]}\right]$ *is a ring of definition.*

Proof. We have seen that there is a canonical continuous homomorphism $\widetilde{\Lambda}^\circ_{[0,as]} \to (R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}})\left\langle \frac{u}{[\varpi]^{1/a}}\right\rangle\left[\frac{1}{[\varpi]}\right]$, and therefore there is a continuous map

$$(R_0 \mathop{\widehat{\otimes}}_{\mathbf{Z}_p} \widetilde{\Lambda}^\circ_{[0,as]})\left\langle \frac{u}{[\varpi]^{1/a}}\right\rangle\left[\frac{1}{[\varpi]}\right] \to (R_0 \mathop{\widehat{\otimes}}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}})\left\langle \frac{u}{[\varpi]^{1/a}}\right\rangle\left[\frac{1}{[\varpi]}\right]$$

On the other hand, $R_0 \widehat{\otimes}_{\mathbf{Z}_p} \widetilde{\Lambda}^\circ_{[0,as]}\left\langle \frac{u}{[\varpi]^{1/a}}\right\rangle\left[\frac{1}{[\varpi]}\right]$ is an $R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}$-algebra in which $[\varpi]$ is invertible and $\frac{u}{[\varpi]^{1/a}}$ is power-bounded. Therefore, there is a canonical continuous map

$$(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})\left\langle \frac{u}{[\varpi]^{1/a}}\right\rangle \to (R_0 \widehat{\otimes} \widetilde{\Lambda}^\circ_{[0,as]})\left\langle \frac{u}{[\varpi]^{1/a}}\right\rangle\left[\frac{1}{[\varpi]}\right]$$

These maps are clearly inverse to each other, and since they are continuous, $(R_0 \widehat{\otimes}_{\mathbf{Z}_p} \widetilde{\Lambda}^\circ_{[0,as]})\left\langle \frac{u}{[\varpi]^{1/a}}\right\rangle$ and $(R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}})\left\langle \frac{u}{[\varpi]^{1/a}}\right\rangle$ are both rings of definition.

Similarly, since $\frac{p}{[\varpi]^{1/as}}$ is power-bounded in $(R_0 \widehat{\otimes}_{\mathbf{Z}_p} \widetilde{\Lambda}^\circ_{[0,r]}) \left\langle \frac{u}{[\varpi]^{1/a}} \right\rangle \left[\frac{1}{[\varpi]}\right]$, there is a continuous map $\widetilde{\Lambda}^\circ_{[0,as]} \to (R_0 \widehat{\otimes}_{\mathbf{Z}_p} \widetilde{\Lambda}^\circ_{[0,r]}) \left\langle \frac{u}{[\varpi]^{1/a}} \right\rangle \left[\frac{1}{[\varpi]}\right]$, and therefore a continuous map

$$(R_0 \widehat{\underset{\mathbf{Z}_p}{\otimes}} \widetilde{\Lambda}^\circ_{[0,as]}) \left\langle \frac{u}{[\varpi]^{1/a}} \right\rangle \left[\frac{1}{[\varpi]}\right] \to (R_0 \widehat{\underset{\mathbf{Z}_p}{\otimes}} \widetilde{\Lambda}^\circ_{[0,r]}) \left\langle \frac{u}{[\varpi]^{1/a}} \right\rangle \left[\frac{1}{[\varpi]}\right]$$

On the other hand, if $r \geq as$, then $R_0 \widehat{\otimes}_{\mathbf{Z}_p} \widetilde{\Lambda}^\circ_{[0,as]}$ is an $R_0 \widehat{\otimes}_{\mathbf{Z}_p} \widetilde{\Lambda}^\circ_{[0,r]}$-algebra, and we obtain a canonical continuous map

$$(R_0 \widehat{\underset{\mathbf{Z}_p}{\otimes}} \widetilde{\Lambda}^\circ_{[0,r]}) \left\langle \frac{u}{[\varpi]^{1/a}} \right\rangle \to (R_0 \widehat{\otimes} \widetilde{\Lambda}^\circ_{[0,as]}) \left\langle \frac{u}{[\varpi]^{1/a}} \right\rangle$$

□

In other words, for any $r \in \mathbf{Q}_{\geq 0}$ and any choice of $a \leq r \cdot \mathrm{ord}_R(p)$, the pre-adic space

$$\mathrm{Spa}\left((R_0 \widehat{\underset{\mathbf{Z}_p}{\otimes}} \widetilde{\Lambda}^\circ_{[0,r]}) \left\langle \frac{u}{[\varpi]^{1/a}} \right\rangle \left[\frac{1}{[\varpi]}\right], (R_0 \otimes_{\mathbf{Z}_p} \widetilde{\Lambda}^\circ_{[0,r]})^{\mathrm{int},\wedge} \left\langle \frac{u}{[\varpi]^{1/a}} \right\rangle^{\mathrm{int}} \right)$$

is isomorphic to the rational localization $\mathrm{Spa}\left(R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}\right) \left\langle \frac{u}{[\varpi]^{1/a}} \right\rangle$.

Corollary 3.3. *The pre-adic space* $\mathrm{Spa}(R_0, R_0) \times_{\mathrm{Spa}(\mathbf{Z}_p)} \mathcal{Y}_{[0,\infty)}$ *is exhausted by open subspaces of the form* $\mathrm{Spa}\left(R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}\right) \left\langle \frac{u}{[\varpi]^{1/b}} \right\rangle$ *for* $b \in \mathbf{Q}_{>0}$.

Similarly, we prove

Corollary 3.4. *The pre-adic space* $\mathrm{Spa}(R) \times_{\mathrm{Spa}(\mathbf{Z}_p)} \mathcal{Y}_{[0,\infty)}$ *is exhausted by open subspaces of the form* $\mathrm{Spa}\left(R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}\right) \left\langle \frac{u}{[\varpi]^{1/b}}, \frac{[\varpi]^{1/a}}{u} \right\rangle$, *for* $a, b \in \mathbf{Q}_{>0}$ *and* $a \leq b$.

This motivates the following definition:

Definition 3.5. Let R be a pseudoaffinoid algebra over $\mathbf{Z}_p$ with pseudo-uniformizer $u \in R_0$. Fix $a \in \mathbf{Q}_{\geq 0}$ and $b \in \mathbf{Q}_{>0} \cup \{\infty\}$ such that $a \leq b$. Then we define $\widetilde{\Lambda}_{R_0,[a,b]}$ to be the evaluation of $\mathscr{O}_{\mathcal{Y}_{R_0}}$ on the affinoid subspace defined by the conditions $u \leq [\varpi]^{1/b} \neq 0$ and $[\varpi]^{1/a} \leq u \neq 0$. We let $\widetilde{\Lambda}_{R_0,[a,b],0} \subset \widetilde{\Lambda}_{R_0,[a,b]}$ be the ring of definition $(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}) \left\langle \frac{u}{[\varpi]^{1/b}}, \frac{[\varpi]^{1/a}}{u} \right\rangle$, and we let $\widetilde{\Lambda}_{R,[a,b]} := \widetilde{\Lambda}_{R_0,[a,b]} \left[\frac{1}{u}\right]$ and $\widetilde{\Lambda}_{R,[a,b],0} := \widetilde{\Lambda}_{R_0,[a,b],0} \left[\frac{1}{u}\right]$.

We also make auxiliary definitions $\widetilde{\Lambda}_{R_0,[0,0]} := (R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}) \left[\frac{1}{[\varpi]}\right]^\wedge$ and $\widetilde{\Lambda}_{R,[0,0]} := \widetilde{\Lambda}_{R_0,[0,0]} \left[\frac{1}{u}\right]$, where the completion is u-adic. We give $\widetilde{\Lambda}_{R,[0,0]}$ the *weak topology* generated by the basis $\{u^k (R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}) \left\langle \frac{1}{[\varpi]} \right\rangle + [\varpi]^n (R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})\}_{k,n}$.

If $b \in (0, \infty)$, $\widetilde{\Lambda}_{R_0,[a,b]}$ is a Tate ring with pseudo-uniformizer $[\varpi]$. We may define a valuation $v_{R,[a,b]}$ on it via

$$v_{R_0,[a,b]}(x) := - \inf_{\alpha \in \mathbf{C}_p^\flat : [\alpha] x \in \widetilde{\Lambda}_{R_0,[a,b],0}} -v_{\mathbf{C}_p^\flat}(\alpha)$$

When $a = 0$, we also denote this valuation by $v_{R,b}$.

If $[a,b] \subset (0,\infty)$, then the ideal of definition of $\widetilde{\Lambda}_{R_0,[a,b],0}$ is principal and generated by both u and $[\varpi]$. In particular, $\widetilde{\Lambda}_{R_0,[a,b]} = \widetilde{\Lambda}_{R,[a,b]}$ is Tate with pseudouniformizers u and $[\varpi]$.

Remark 3.6. If $(R, R^\circ) = (\mathbf{Q}_p, \mathbf{Z}_p)$ with $u = p$, this definition recovers $\widetilde{\Lambda}_{[a,b]}$.

The rings $\widetilde{\Lambda}_{R_0,[a,b]}$ and $\widetilde{\Lambda}_{R_0,[a,b],0}$ depend on the choice of ring of definition $R_0 \subset R$, but $\widetilde{\Lambda}_{R,[a,b]}$ and $\widetilde{\Lambda}_{R,[a,b],0}$ do not. However, both depend on our choice of $u \in R$.

Proposition 3.7. *Let $f : R \to R'$ be a homomorphism of pseudoaffinoid rings, and let $R_0' := f(R_0)$ and $u' := f(u)$. Then $R_0' \widehat{\otimes}_{R_0} \widetilde{\Lambda}_{R_0,[a,b]} = \widetilde{\Lambda}_{R_0',[a,b]}$.*

Proof. By [JN16, Lemma 2.2.5], f is topologically of finite type , so R_0' is a ring of definition of R' formally of finite type over $\mathbf{Z}_p$. Then

$$\begin{aligned} R_0' \underset{R_0}{\widehat{\otimes}} \widetilde{\Lambda}_{R_0,[a,b]} &= R_0' \underset{R_0}{\widehat{\otimes}} (R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}) \langle X, Y\rangle / ([\varpi]^{1/b} X - u, uY - [\varpi]^{1/a}, XY - [\varpi]^{1/a-1/b}) \\ &= (R_0' \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}) \langle X, Y\rangle / ([\varpi]^{1/b} X - u', u'Y - [\varpi]^{1/a}, XY - [\varpi]^{1/a-1/b}) \\ &= \widetilde{\Lambda}_{R_0',[a,b]} \end{aligned}$$

□

We do not know whether $\mathcal{Y}_{R_0}$ and $\mathcal{Y}_R$ are adic spaces (even though $\mathrm{Spa}(R_0)$, $\mathrm{Spa}(R)$, and $\mathcal{Y}$ are). However, we have the following partial result:

Proposition 3.8. *Suppose $\{\mathrm{Spa}(R_i)\}_i$ is an affinoid cover of $\mathrm{Spa}(R)$, and let $\mathrm{Spa}(R_{ij}) := \mathrm{Spa}(R_i) \cap \mathrm{Spa}(R_j)$. Then if $[a,b] \subset [0,\infty)$ we have a strict exact sequence*

$$0 \to \widetilde{\Lambda}_{R,[a,b]} \to \prod_i \widetilde{\Lambda}_{R_i,[a,b]} \to \prod_{i,j} \widetilde{\Lambda}_{R_{ij},[a,b]}$$

Proof. We may assume that $\{\mathrm{Spa}(R_i)\}_i$ is a finite cover by rational subspaces of $\mathrm{Spa}(R)$, where $(R_i, R_i^\circ) = (R, R^\circ)\left\langle \frac{f_0,\ldots,f_n}{f_i}\right\rangle$ for some finite set $f_0, \ldots, f_n \in R$ which generate the unit ideal. We may further assume that $f_i \in R_0$ for all i. The ring of definition R_0 is admissible in the sense of [BL93], so we may consider the scheme-theoretic blowing up $X \to \mathrm{Spec}\, R_0$ and the admissible formal blowing up $\mathcal{X} \to \mathrm{Spf}(R_0)$ along the ideal $(f_0, \ldots, f_n)$. Then $\mathscr{O}_X \otimes_{R_0} \widetilde{\Lambda}_{R_0,[a,b],0}$ is a quasi-coherent $\mathscr{O}_X$-module, so we have an exact sequence

$$0 \to \Gamma(X, \mathscr{O}_X) \otimes_{R_0} \widetilde{\Lambda}_{R_0,[a,b],0} \to \prod_i R_0\left[\frac{f_0}{f_i}, \ldots, \frac{f_n}{f_i}\right] \otimes_{R_0} \widetilde{\Lambda}_{R_0,[a,b],0} \to \prod_{i,j} R_0\left[\frac{f_0}{f_i f_j}, \ldots, \frac{f_n}{f_i f_j}\right] \otimes_{R_0} \widetilde{\Lambda}_{R_0,[a,b],0}$$

The $[\varpi]$-adic completion

$$0 \to \Gamma(\mathcal{X}, \mathscr{O}_\mathcal{X}) \widehat{\otimes}_{R_0} \widetilde{\Lambda}_{R_0,[a,b],0} \to \prod_i R_0\left\langle\frac{f_0}{f_i}, \ldots, \frac{f_n}{f_i}\right\rangle \widehat{\otimes}_{R_0} \widetilde{\Lambda}_{R_0,[a,b],0} \to \prod_{i,j} R_0\left\langle\frac{f_0}{f_i}, \ldots, \frac{f_n}{f_i}\right\rangle \widehat{\otimes}_{R_0} \widetilde{\Lambda}_{R_0,[a,b],0}$$

is exact, as well, and since $\Gamma(\mathcal{X}, \mathscr{O}_\mathcal{X})$ is R_0-finite and satisfies $\Gamma(\mathcal{X}, \mathscr{O}_\mathcal{X})\left[\frac{1}{u}\right] = R$, the result now follows from inverting u. □

We have a similar sheaf property with respect to rational localization on $\mathcal{Y}$. Since $R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}$ has no u- or $[\varpi]$-torsion, if $[a,b] \supset [a',b']$, there is an injective map

$$(R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}})\left[\frac{u}{[\varpi]^{1/b}}, \frac{[\varpi]^{1/a}}{u}\right] \to (R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}})\left[\frac{u}{[\varpi]^{1/b'}}, \frac{[\varpi]^{1/a'}}{u}\right]$$

(since both rings are subrings of $(R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}})\left[\frac{1}{[\varpi]}, \frac{1}{u}\right]$).

Proposition 3.9. *The above map extends to an injection $(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})\left\langle \frac{u}{[\varpi]^{1/b}}, \frac{[\varpi]^{1/a}}{u}\right\rangle \to (R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})\left\langle \frac{u}{[\varpi]^{1/b'}}, \frac{[\varpi]^{1/a'}}{u}\right\rangle$, and therefore an injection $\widetilde{\Lambda}_{R,[a,b]} \to \widetilde{\Lambda}_{R,[a',b']}$*

Proof. As in the proof of [Ber02, Lemme 2.5], the map $\widetilde{\Lambda}_{R_0,[a,b],0} \to \widetilde{\Lambda}_{R_0,[a',b'],0}$ factors as $\widetilde{\Lambda}_{R_0,[a,b],0} \to \widetilde{\Lambda}_{R_0,[a,b'],0} \to \widetilde{\Lambda}_{R_0,[a',b'],0}$, and we may assume that either $a = a'$ or $b = b'$. We treat the case $a = a' = 0$ here; the other cases follow as in [Ber02].

We need to show that the natural map $(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})\langle X\rangle/([\varpi]^{1/b}X-u) \to (R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})[X]_u^\wedge/([\varpi]X-1)$ is injective. This map carries a power series $f(X) \in (R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})\langle X\rangle$ to $f(u[\varpi]^{1-1/b}X)$; to show it is injective, we need to check that if $f(u[\varpi]^{1-1/b}X) \in ([\varpi]X-1)(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})[X]_u^\wedge$, then $f(uX) \in (u[\varpi]X-u)(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})\langle u[\varpi]^{1-1/b}X\rangle$.

Writing $f(uX) = ([\varpi]X-1)g(X)$, where $g(X) = \sum_{j=0}^\infty c_jX^j$ and $c_j \to 0$ u-adically, we need to show that $c_j \in u^{j+1}[\varpi]^{j(1-1/b)}(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})$ for all $j \geq 0$. We may also write $f(uX) = \sum_{j=0}^\infty d_j(u[\varpi]^{1-1/b}X)^j$, where $d_j \to 0$ u-adically. Then for $j \geq 1$, we have $d_ju^j[\varpi]^{j(1-1/b)} = [\varpi]c_{j-1} - c_j$.

Since the c_j tend to 0 u-adically, for each j there is some $N_j \gg j$ such that c_{N_j} is a multiple of u^j. This implies that $[\varpi]c_{N_j-1}$ is also a multiple of u^j, and since $(R_0/u^j) \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}$ has no $[\varpi]$-torsion, c_{N_j-1} itself is a multiple of u^j. Repeating this argument, we see that c_{j-1} is a multiple of u^j.

We may write $c_j = u^{j+1}c_j'$. Since $R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}$ has no u-torsion, we have $d_j[\varpi]^{j(1-1/b)} = [\varpi]c_{j-1}' - uc_j'$ for all $j \geq 1$, which implies that uc_1' is a multiple of $[\varpi]^{1-1/b}$. But $R_0 \widehat{\otimes}(\mathbf{A}_{\mathrm{inf}}/[\varpi]^{1-1/b})$ has no u-torsion, so c_1' itself is a multiple of $[\varpi]^{1-1/b}$. We now proceed by induction on j; if c_{j-1}' is a multiple of $[\varpi]^{(j-1)(1-1/b)}$, then uc_j' is a multiple of $[\varpi]^{j(1-1/b)}$, which implies that c_j' itself is a multiple of $[\varpi]^{j(1-1/b)}$. $\square$

Proposition 3.10. *Suppose $[a,b] \neq [0,\infty]$. Then there is an exact sequence (of R_0-modules)*

$$0 \to R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}} \to (R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}})\left\langle \frac{u}{[\varpi]^{1/b}}\right\rangle \oplus (R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}})\left\langle \frac{[\varpi]^{1/a}}{u}\right\rangle \to (R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}})\left\langle \frac{u}{[\varpi]^{1/b}}, \frac{[\varpi]^{1/a}}{u}\right\rangle \to 0$$

Proof. This is an adaptation of the proof of [Ber02, Lemme 2.15]. We first treat the case where $a = b$.

The natural map

$$(R_0 \widehat{\underset{\mathbf{Z}_p}{\otimes}} \mathbf{A}_{\mathrm{inf}})\left\langle \frac{u}{[\varpi]^{1/a}}\right\rangle \oplus (R_0 \widehat{\underset{\mathbf{Z}_p}{\otimes}} \mathbf{A}_{\mathrm{inf}})\left\langle \frac{[\varpi]^{1/a}}{u}\right\rangle \to (R_0 \widehat{\underset{\mathbf{Z}_p}{\otimes}} \mathbf{A}_{\mathrm{inf}})\left\langle \frac{u}{[\varpi]^{1/a}}, \frac{[\varpi]^{1/a}}{u}\right\rangle$$

is clearly surjective, so it remains to check that the kernel is exactly $R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}$.

We first verify this modulo u.

$$\left((R_0 \mathop{\widehat{\otimes}}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}) \left\langle \frac{u}{[\varpi]^{1/a}} \right\rangle\right)/(u) \cong (R_0 \mathop{\widehat{\otimes}}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}})\,[Y]\,/([\varpi]^{1/a}Y, u)$$

$$\left((R_0 \mathop{\widehat{\otimes}}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}) \left\langle \frac{[\varpi]^{1/a}}{u} \right\rangle\right)/(u) \cong (R_0/u) \otimes_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}\,[X]\,/([\varpi]^{1/a})$$

$$\left((R_0 \mathop{\widehat{\otimes}}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}) \left\langle \frac{u}{[\varpi]^{1/a}}, \frac{[\varpi]^{1/a}}{u} \right\rangle\right)/(u) \cong (R_0/u) \otimes_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}\left[X, X^{-1}\right]/([\varpi]^{1/a})$$

Moreover, the map $\left((R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}) \left\langle \frac{u}{[\varpi]^{1/a}} \right\rangle\right)/(u) \to \left((R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}) \left\langle \frac{u}{[\varpi]^{1/a}}, \frac{[\varpi]^{1/a}}{u} \right\rangle\right)/(u)$ is given by $Y \mapsto X^{-1}$, and it factors through $\left((R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}) \left\langle \frac{u}{[\varpi]^{1/a}} \right\rangle\right)/(u, [\varpi]^{1/a}) = (R_0/u) \otimes_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}[Y]/([\varpi]^{1/a})$.

But the intersection

$$\left((R_0/u) \otimes \mathbf{A}_{\mathrm{inf}}/([\varpi]^{1/a})\right)[X] \cap \left((R_0/u) \otimes \mathbf{A}_{\mathrm{inf}}/([\varpi]^{1/a})\right)[X^{-1}] \subset \left((R_0/u) \otimes \mathbf{A}_{\mathrm{inf}}/([\varpi]^{1/a})\right)[X, X^{-1}]$$

is just $(R_0/u) \otimes \mathbf{A}_{\mathrm{inf}}/([\varpi]^{1/a})$. So the image of the intersection of $(R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}) \left\langle \frac{u}{[\varpi]^{1/a}} \right\rangle$ and $(R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}) \left\langle \frac{[\varpi]^{1/a}}{u} \right\rangle$ is contained in the image of $R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}$.

Thus, given $x \in (R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}) \left\langle \frac{u}{[\varpi]^{1/a}} \right\rangle \cap (R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}) \left\langle \frac{[\varpi]^{1/a}}{u} \right\rangle$, there is some $y \in R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}$ such that $x - y \in (u)(R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}) \left\langle \frac{u}{[\varpi]^{1/a}}, \frac{[\varpi]^{1/a}}{u} \right\rangle$. By considering the reductions of all of these rings modulo u, this implies that $x - y \in (u)(R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}) \left\langle \frac{[\varpi]^{1/a}}{u} \right\rangle$ and $x - y \in (u)(R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}) \left\langle \frac{u}{[\varpi]^{1/a}} \right\rangle + [\varpi]^{1/a}(R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}})$.

There is therefore some $z \in [\varpi]^{1/a}(R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}})$ such that $x - y - z \in (u)(R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}) \left\langle \frac{u}{[\varpi]^{1/a}} \right\rangle$. Since $\frac{[\varpi]^{1/a}}{u} \in (R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}) \left\langle \frac{[\varpi]^{1/a}}{u} \right\rangle$ and $(R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}) \left\langle \frac{u}{[\varpi]^{1/a}} \right\rangle \cap (R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}) \left\langle \frac{[\varpi]^{1/a}}{u} \right\rangle$ has no u-torsion, we have

$$x - y - z \in (u)\left((R_0 \mathop{\widehat{\otimes}}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}) \left\langle \frac{u}{[\varpi]^{1/a}} \right\rangle \cap (R_0 \mathop{\widehat{\otimes}}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}) \left\langle \frac{[\varpi]^{1/a}}{u} \right\rangle\right)$$

Since $R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}$ is $(u, [\varpi]^{1/a})$-adically separated and complete, we can iterate this argument, and the conclusion follows.

To handle the general case, we consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}} \arrow[r, hook] \arrow[d, equal] & (R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}) \left\langle \frac{u}{[\varpi]^{1/b}} \right\rangle \oplus (R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}) \left\langle \frac{[\varpi]^{1/a}}{u} \right\rangle \arrow[r, two heads] \arrow[d, hook] & (R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}) \left\langle \frac{u}{[\varpi]^{1/b}}, \frac{[\varpi]^{1/a}}{u} \right\rangle \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}} \arrow[r, hook] & (R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}) \left\langle \frac{u}{[\varpi]^{1/a}} \right\rangle \oplus (R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}) \left\langle \frac{[\varpi]^{1/a}}{u} \right\rangle \arrow[r, two heads] & (R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}) \left\langle \frac{u}{[\varpi]^{1/a}}, \frac{[\varpi]^{1/a}}{u} \right\rangle \arrow[r] & 0
\end{tikzcd}$$

Since the bottom row is exact, and the top row is exact except possibly in the middle, and the vertical arrows are injections, a diagram chase shows that the top row is exact. □

Corollary 3.11. *Suppose $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$ are intervals such that $I_1 \cap I_2 \neq \emptyset$. Then $\widetilde{\Lambda}_{R_0, I_1, 0} \cap \widetilde{\Lambda}_{R_0, I_2, 0} = \widetilde{\Lambda}_{R_0, I_1 \cup I_2, 0}$ (where the intersection is taken inside $\widetilde{\Lambda}_{R_0, I_1 \cap I_2, 0}$).*

Proof. We may assume that $a_1 \leq a_2 \leq b_1 \leq b_2$, so that $I_1 \cap I_2 = [a_2, b_1]$ and $I_1 \cup I_2 = [a_1, b_2]$. If $f \in \widetilde{\Lambda}_{R_0,I_1,0} \cap \widetilde{\Lambda}_{R_0,I_2,0}$, we may write

$$f = g_1 + h_1 = g_2 + h_2$$

with $g_1 \in \widetilde{\Lambda}_{R_0,[a_1,\infty],0}$, $h_1 \in \widetilde{\Lambda}_{R_0,[0,b_1],0}$, $g_2 \in \widetilde{\Lambda}_{R_0,[a_2,\infty],0}$, and $h_2 \in \widetilde{\Lambda}_{R_0,[0,b_2],0}$. Then $(g_1 - g_2) + (h_1 - h_2) = 0$, with $g_1 - g_2 \in \widetilde{\Lambda}_{R_0,[a_2,\infty],0}$ and $h_1 - h_2 \in \widetilde{\Lambda}_{R_0,[0,b_1],0}$. It follows from the previous proposition that $g_1 - g_2 = h_2 - h_1 = f' \in R_0 \widehat{\otimes} \mathbf{A}_{\inf}$. Then $f = (g_1 - f') + (h_1 + f') = (g_1 - f') + h_2$; since $g_1 - f' \in \widetilde{\Lambda}_{R_0,[a_1,\infty],0}$ and $h_2 \in \widetilde{\Lambda}_{R_0,[0,b_2],0}$, we are done. □

3.2. **The action of H_K on $\widetilde{\Lambda}_{R,[a,b]}$.** Since $v_{\mathbf{C}_p^\flat}([\varpi]) = \frac{p-1}{p} v_{\mathbf{C}_p^\flat}([\overline{\pi}])$, we define $s : (0,\infty) \to (0,\infty)$ via $s(a) = \frac{p-1}{pa}$. From now on, we assume that $s(a), s(b) \in p^{\mathbf{Z}}$. Then we can rewrite $\widetilde{\Lambda}_{R_0,[a,b],0}$ as the rational localization $(R_0 \widehat{\otimes} \mathbf{A}_{\inf}) \left\langle \frac{u}{[\overline{\pi}]^{s(b)}}, \frac{[\overline{\pi}]^{s(a)}}{u} \right\rangle$. Since H_K acts trivially on u and $[\overline{\pi}]$, this implies that the action of H_K on $\mathbf{A}_{\inf}$ induces an action on $\widetilde{\Lambda}_{R,[a,b]}$.

If $(R, R^\circ) = (\mathbf{Q}_p, \mathbf{Z}_p)$ and $u = p$, it is easy to read off from the Witt vector description of $(\widetilde{\Lambda}_{[0,r]}, \widetilde{\Lambda}^\circ_{[0,r]})$ that

$$\left(\widetilde{\Lambda}_{[0,r]}, \widetilde{\Lambda}^\circ_{[0,r]}\right)^{H_K} = \left(\mathbf{A}_{\inf}^{H_K} \left\langle \frac{p}{[\overline{\pi}]^{s(r)}} \right\rangle \left[\frac{1}{[\overline{\pi}]}\right], \mathbf{A}_{\inf}^{H_K} \left\langle \frac{p}{[\overline{\pi}]^{s(r)}} \right\rangle\right)$$

Moreover, it follows from Berger's Lemme 2.29 that

$$\left(\widetilde{\Lambda}_{[p^{-n},\infty]}, \widetilde{\Lambda}^\circ_{[p^{-n},\infty]}\right)^{H_K} = \left(\mathbf{A}_{\inf}^{H_K} \left\langle \frac{[\overline{\pi}]^{(p-1)p^{n-1}}}{p} \right\rangle \left[\frac{1}{[\overline{\pi}]}\right], \mathbf{A}_{\inf}^{H_K} \left\langle \frac{[\overline{\pi}]^{(p-1)p^{n-1}}}{p} \right\rangle\right)$$

The same argument as in the proof of Proposition 3.10 shows the following:

Proposition 3.12. *There is an exact sequence (of R_0-modules)*

$$0 \to R_0 \widehat{\otimes} \mathbf{A}_{\inf}^{H_K} \to (R_0 \widehat{\otimes} \mathbf{A}_{\inf}^{H_K}) \left\langle \frac{u}{[\overline{\pi}]^{s(b)}} \right\rangle \oplus (R_0 \widehat{\otimes} \mathbf{A}_{\inf}^{H_K}) \left\langle \frac{[\overline{\pi}]^{s(a)}}{u} \right\rangle \to (R_0 \widehat{\otimes} \mathbf{A}_{\inf}^{H_K}) \left\langle \frac{u}{[\overline{\pi}]^{s(b)}}, \frac{[\overline{\pi}]^{s(a)}}{u} \right\rangle \to 0$$

Lemma 3.13. *There is a connecting homomorphism $\widetilde{\Lambda}^{H_K}_{R_0,[a,b],0} \to \mathrm{H}^1(H_K, R_0 \widehat{\otimes} \mathbf{A}_{\inf})$.*

Proof. If $[a,b] = [0,b]$, we choose the map $(\mathrm{id}, 0)$. If $a \neq 0$ but $b = \infty$, we choose the map $(0, \mathrm{id})$.

Otherwise, $[a,b] \subset (0,\infty)$ and we have an exact sequence

$$0 \to R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\inf} \to (R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\inf}) \left\langle \frac{u}{[\varpi]^{1/b}} \right\rangle \oplus (R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\inf}) \left\langle \frac{[\varpi]^{1/a}}{u} \right\rangle \to (R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\inf}) \left\langle \frac{u}{[\varpi]^{1/b}}, \frac{[\varpi]^{1/a}}{u} \right\rangle \to 0$$

and we need to produce a continuous set-theoretic section

$$s : \widetilde{\Lambda}_{R,[a,b],0} \to \widetilde{\Lambda}_{R,[0,b],0} \oplus \widetilde{\Lambda}_{R,[a,\infty],0}$$

We first construct a continuous set-theoretic map $s_1 : \widetilde{\Lambda}_{R_0,[a,b],0} \to \widetilde{\Lambda}_{R_0,[a,\infty],0}$. Both rings are u-adically separated and complete, so it suffices to construct a set-theoretic map $\overline{s}_1 : \widetilde{\Lambda}_{R_0,[a,b],0} \to \widetilde{\Lambda}_{R_0,[a,\infty],0}$. We may write $\widetilde{\Lambda}_{R_0,[a,b],0}/u = (R_0 \otimes \mathbf{A}_{\inf})[X,Y]/(u, [\varpi]^{1/a}, [\varpi]^{1/b}X, XY - [\varpi]^{1/a - 1/b})$; given $\overline{x} \in \widetilde{\Lambda}_{R_0,[a,b],0}/u$, we may choose a lift to $(R_0 \otimes \mathbf{A}_{\inf}[X,Y]/(u)$ of the form $\sum_{i\geq 1} \alpha_i X^i + \sum_{j \geq 0} \beta_j Y^j$, then project $\sum_{j\geq 0} \beta_j Y^j$ to $\widetilde{\Lambda}_{R_0,[a,\infty],0}/u$.

The resulting map s_1 is not necessarily a section, but it has the property that for $x \in \widetilde{\Lambda}_{R_0,[a,b],0}$, $x - \mathrm{im}(s_1(x)) \in \mathrm{im}(\widetilde{\Lambda}_{R_0,[0,b],0})$. Since $\widetilde{\Lambda}_{R,[0,b]} \to \widetilde{\Lambda}_{R,[a,b]}$ is injective, the map $x \mapsto x - \mathrm{im}(s_1(x))$ defines another map $s_2 : \widetilde{\Lambda}_{R,[a,b]} \to \widetilde{\Lambda}_{R,[0,b]}$. Then $-s_1 \oplus s_2$ is the desired section. □

Lemma 3.14. *There is an exact sequence*

$$0 \to (R_0 \underset{\mathbf{Z}_p}{\widehat{\otimes}} \mathbf{A}_{\mathrm{inf}})^{H_K} \left[\frac{1}{u}\right] \to \widetilde{\Lambda}^{H_K}_{R_0,[0,b],0} \left[\frac{1}{u}\right] \oplus \widetilde{\Lambda}^{H_K}_{R,[a,\infty]} \to \widetilde{\Lambda}^{H_K}_{R,[a,b]} \to 0$$

Proof. This follows as in Berger's Lemme 2.27. If $a = 0$ or $b = \infty$, the result is trivial. If not, we have an exact sequence

$$0 \to (R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}})^{H_K} \left[\tfrac{1}{u}\right] \to \widetilde{\Lambda}^{H_K}_{R_0,[0,b],0} \left[\tfrac{1}{u}\right] \oplus \widetilde{\Lambda}^{H_K}_{R,[a,\infty]} \to \widetilde{\Lambda}^{H_K}_{R,[a,b]} \to \mathrm{H}^1(H_K, (R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}) \left[\tfrac{1}{u}\right])$$

and we need to show that the map $\delta : \widetilde{\Lambda}^{H_K}_{R,[a,b]} \to H^1(H_K, (R_0 \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{A}_{\mathrm{inf}}) \left[\tfrac{1}{u}\right]$ is the zero map.

If $x \in \widetilde{\Lambda}^{H_K}_{R_0,[a,b],0}$, then $x \cdot \left(\frac{u}{[\overline{\pi}]^{s(b)}}\right) \in \widetilde{\Lambda}^{H_K}_{R_0,[a,b],0}$ as well, and after multiplying by a suitable power of u, we may assume that $\delta(x)$ and $\delta(x \cdot \left(\frac{u}{[\overline{\pi}]^{s(b)}}\right))$ are both elements of $\mathrm{H}^1(H_K, R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})$. But then

$$u \cdot \delta(x) = \delta(ux) = \delta(x \cdot \left(\frac{u}{[\overline{\pi}]^{s(b)}}\right) \cdot [\overline{\pi}]^{s(b)}) = [\overline{\pi}]^{s(b)} \delta(x \cdot \left(\frac{u}{[\overline{\pi}]^{s(b)}}\right))$$

Then the result follows from the next lemma. □

Lemma 3.15. *If $\alpha \in \mathscr{O}^\flat_{\mathbf{C}_p}$ is in the maximal ideal, then the homomorphism $\mathrm{H}^1(H_K, R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}) \xrightarrow{\times[\alpha]} \mathrm{H}^1(H_K, R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})$ induced by multiplication by $[\alpha]$ is the zero map.*

Proof. If $c_\tau \in \mathrm{H}^1(H_K, R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})$ is in the image of this map, we may consider its image in $\mathrm{H}^1(H_K, (R_0/u) \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})$. Then a standard Tate–Sen argument shows that this image is the trivial cocycle. Working modulo successive powers of u, it follows that c_τ is itself trivial. □

Thus, to study the H_K-invariants of $\widetilde{\Lambda}_{R,[a,b]} \left[\frac{1}{u}\right]$, it suffices to study the H_K-invariants of $\widetilde{\Lambda}_{R_0,[0,b],0} \left[\frac{1}{u}\right]$ and $\widetilde{\Lambda}_{R_0,[a,\infty]} \left[\frac{1}{u}\right]$.

We first study the ring $\widetilde{\Lambda}^{H_K}_{R_0,[0,\infty],0} = (R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})^{H_K}$.

Lemma 3.16. (1) *If N is a finite module over a noetherian discrete $\mathbf{Z}/p^n$-algebra R_0, then $\left(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}\right)^{H_K} = R_0 \widehat{\otimes} \mathbf{A}^{H_K}_{\mathrm{inf}}$.*
(2) *If R_0 is a Huber ring with principal ideal of definition, then $\left(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}\right)^{H_K} = R_0 \widehat{\otimes} \mathbf{A}^{H_K}_{\mathrm{inf}}$.*
(3) *If N is a finite module over a noetherian discrete $\mathbf{Z}/p^n$-algebra and $[\mathfrak{m}_{\mathbf{C}_p^\flat}] \subset \mathbf{A}_{\mathrm{inf}}$ denotes the ideal generated by $\{[\alpha]\}_{\alpha \in \mathfrak{m}_{\mathbf{C}_p^\flat}}$, then $\left(R_0 \otimes [\mathfrak{m}_{\mathbf{C}_p^\flat}]/[\overline{\pi}]^{s(a)}[\mathfrak{m}_{\mathbf{C}_p^\flat}]\right)^{H_K} = R_0 \otimes [\mathfrak{m}_{\widehat{K}^\flat_\infty}]/[\overline{\pi}]^{s(a)}[\mathfrak{m}_{\widehat{K}^\flat_\infty}]$.*

Proof. In order to compute the H_K-invariants of $N \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}$, we proceed by induction on n. If $n = 1$, then N is a $\mathbf{F}_p$-vector space and we may choose an $\mathbf{F}_p$-basis $\{e_j\}_{j\in J}$. Then $N \widehat{\otimes} \mathbf{A}_{\mathrm{inf}} = \{\sum_j b_j e_j \mid b_j \in \mathscr{O}^\flat_{\mathbf{C}_p}, b_j \to 0\}$, where the coefficients b_j tend to 0 with respect to the cofinite filter. It follows that

$$\left(N \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}\right)^{H_K} = \{\sum_j b_j e_j \mid b_j \in \widehat{\mathscr{O}}^\flat_{K_\infty}, b_j \to 0\} = N \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K}$$

Now assume the result holds for $n-1$. We have an exact sequence

$$0 \to p^{n-1}(N \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}) \to N \widehat{\otimes} \mathbf{A}_{\mathrm{inf}} \to (N \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})/(p^{n-1}) \to 0$$

Since $p^{n-1}N$ is a finite module over the discrete $\mathbf{F}_p$-algebra R_0/p, the inductive hypothesis implies that we have an exact sequence

$$0 \to p^{n-1} N \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K} \to \left(N \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}\right)^{H_K} \to (N \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K})/(p^{n-1})$$

There is furthermore a natural map $N \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K} \to \left(N \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}\right)^{H_K}$ and a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \left(p^{n-1} N \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}\right)^{H_K} \arrow[r] & \left(N \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}\right)^{H_K} \arrow[r] & \left((N/p^{n-1}) \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}\right)^{H_K} & \\
0 \arrow[r] & p^{n-1} N \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K} \arrow[r] \arrow[u, equal] & N \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K} \arrow[r] \arrow[u] & (N/p^{n-1}) \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K} \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

A diagram chase shows that the map $N \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K} \to \left(N \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}\right)^{H_K}$ is an isomorphism.

For the second part, let $u \in R_0$ generate the ideal of definition. Since p is topologically nilpotent in R_0, each quotient ring R_0/u^k is p-power torsion. We observe that $R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}} = \varprojlim_k \varprojlim_{k'} (R_0/u^k) \otimes (\mathbf{A}_{\mathrm{inf}}/[\overline{\pi}]^{k'})$, and so

$$\begin{aligned}
\left(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}\right)^{H_K} = \varprojlim_k \left(\varprojlim_{k'} (R_0/u^k) \otimes (\mathbf{A}_{\mathrm{inf}}/[\overline{\pi}]^{k'})\right)^{H_K} &= \varprojlim_k \left((R_0/u^k) \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}\right)^{H_K} \\
&= \varprojlim_k (R_0/u^k) \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K} = R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K}
\end{aligned}$$

The last part follows similarly, using the fact that $\left(\mathfrak{m}_{\mathbf{C}_p^\flat}/\overline{\pi}^{s(a)}\mathfrak{m}_{\mathbf{C}_p^\flat}\right)^{H_K} = \mathfrak{m}_{\widehat{K}^\flat_\infty}/\overline{\pi}^{s(a)}\mathfrak{m}_{\widehat{K}^\flat_\infty}$. $\square$

To study the rings $\widetilde{\Lambda}^{H_K}_{R_0,[a,b]}$ when $[a, b] \neq [0, \infty]$, we require a number of preparatory results. We will proceed by making a careful study of $(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})/(u - [\overline{\pi}]^{s(a)})$ and bootstrapping from characteristic p to characteristic 0. However, our techniques specifically exclude the classical case, where p is a pseudo-uniformizer of R.

Lemma 3.17. *If R is a pseudoaffinoid algebra over $\mathbf{Z}_p$, $R_0 \subset R$ is a noetherian ring of definition formally of finite type over $\mathbf{Z}_p$ with pseudo-uniformizer u, and N is a finite flat R_0-module, then $(N \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})/(u - [\overline{\pi}]^{s(a)})$ and $(N \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K})/(u - [\overline{\pi}]^{s(a)})$ are flat over R_0. In particular, $(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})/(u - [\overline{\pi}]^{s(a)})$ and $(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K})/(u - [\overline{\pi}]^{s(a)})$ are flat over R_0.*

Proof. We first observe that

$$(R_0 \mathbin{\widehat{\otimes}} \mathbf{A}_{\mathrm{inf}})/(u-[\overline{\pi}]^{s(a)}) \cong \left((R_0 \mathbin{\widehat{\otimes}} \mathbf{Z}_p[\![[\overline{\pi}]^{s(a)}]\!])/(u-[\overline{\pi}]^{s(a)})\right) \mathbin{\widehat{\otimes}}_{\mathbf{Z}_p[\![[\overline{\pi}]^{s(a)}]\!]} \mathbf{A}_{\mathrm{inf}}$$

and

$$(R_0 \mathbin{\widehat{\otimes}} \mathbf{A}_{\mathrm{inf}}^{H_K})/(u-[\overline{\pi}]^{s(a)}) \cong \left((R_0 \mathbin{\widehat{\otimes}} \mathbf{Z}_p[\![[\overline{\pi}]^{s(a)}]\!])/(u-[\overline{\pi}]^{s(a)})\right) \mathbin{\widehat{\otimes}}_{\mathbf{Z}_p[\![[\overline{\pi}]^{s(a)}]\!]} \mathbf{A}_{\mathrm{inf}}^{H_K}$$

Since R_0, $\mathbf{Z}_p[\![[\overline{\pi}]^{s(a)}]\!]$, and $R_0 \mathbin{\widehat{\otimes}} \mathbf{Z}_p[\![[\overline{\pi}]^{s(a)}]\!]$ are all noetherian, we may apply [Mat89, Theorem 22.6] with $A = R_0$ (resp. $\mathbf{Z}_p[\![[\overline{\pi}]^{s(a)}]\!]$), $B = R_0 \mathbin{\widehat{\otimes}} \mathbf{Z}_p[\![[\overline{\pi}]^{s(a)}]\!]$, $M = (N \mathbin{\widehat{\otimes}} \mathbf{Z}_p[\![[\overline{\pi}]^{s(a)}]\!])/(u-[\overline{\pi}]^{s(a)})$, and $b = u-[\overline{\pi}]^{s(a)}$. Since B is flat over A and $\mathfrak{m}\cap A$ is a maximal ideal $\mathfrak{n}\subset R_0$ (resp. $\mathfrak{m}\cap A = ([\overline{\pi}]^{s(a)})$) for every maximal ideal $\mathfrak{m}\subset B$, it is enough to check that the image of $u-[\overline{\pi}]^{s(a)}$ is not a zero-divisor in $M/\mathfrak{n}$ (resp. $M/[\overline{\pi}]^{s(a)}$). But $M/\mathfrak{m} \cong N/\mathfrak{n} \mathbin{\widehat{\otimes}} \mathbf{Z}_p[\![[\overline{\pi}]^{s(a)}]\!]$ (resp. $M/[\overline{\pi}]^{s(a)} \cong N$), the image of $u-[\overline{\pi}]^{s(a)}$ is the image of $[\overline{\pi}]^{s(a)}$ since every maximal ideal of R_0 contains u (resp. the image of $u-[\overline{\pi}]^{s(a)}$ is u), and $(N/\mathfrak{n}) \mathbin{\widehat{\otimes}} \mathbf{Z}_p[\![[\overline{\pi}]^{s(a)}]\!]$ is $[\overline{\pi}]^{s(a)}$-torsion-free (resp. N is u-torsion-free).

Thus, $(N \mathbin{\widehat{\otimes}} \mathbf{Z}_p[\![[\overline{\pi}]^{s(a)}]\!])/(u-[\overline{\pi}]^{s(a)})$ is flat over R_0 and $\mathbf{Z}_p[\![[\overline{\pi}]^{s(a)}]\!]$; if we set $M_n := (N \mathbin{\widehat{\otimes}} \mathbf{Z}_p[\![[\overline{\pi}]^{s(a)}]\!])/(u^n, u-[\overline{\pi}]^{s(a)})$, then M_n is flat over R_0/u^n and $M_{n+1}\to M_n$ is surjective for all $n\geq 1$. Since $\mathbf{A}_{\mathrm{inf}}$ and $\mathbf{A}_{\mathrm{inf}}^{H_K}$ are flat over $\mathbf{Z}_p[\![[\overline{\pi}]^{s(a)}]\!]$, $\{M_n \otimes_{\mathbf{Z}_p[\![[\overline{\pi}]^{s(a)}]\!]} \mathbf{A}_{\mathrm{inf}}\}_{n\geq 1}$ and $\{M_n \otimes_{\mathbf{Z}_p[\![[\overline{\pi}]^{s(a)}]\!]} \mathbf{A}_{\mathrm{inf}}^{H_K}\}_{n\geq 1}$ are projective systems of flat R_0-modules with surjective transition maps. It then follows from [Sta18, Tag 0912] that $(R_0 \mathbin{\widehat{\otimes}} \mathbf{A}_{\mathrm{inf}})/(u-[\overline{\pi}]^{s(a)}) \cong \varprojlim_n \left(M_n \otimes_{\mathbf{Z}_p[\![[\overline{\pi}]^{s(a)}]\!]} \mathbf{A}_{\mathrm{inf}}\right)$ and $(R_0 \mathbin{\widehat{\otimes}} \mathbf{A}_{\mathrm{inf}}^{H_K})/(u-[\overline{\pi}]^{s(a)}) \cong \varprojlim_n \left(M_n \otimes_{\mathbf{Z}_p[\![[\overline{\pi}]^{s(a)}]\!]} \mathbf{A}_{\mathrm{inf}}^{H_K}\right)$ are flat over R_0. □

Corollary 3.18. *If (R,R^+) is topologically finite type over $(\mathbf{F}_p(\!(u)\!), \mathbf{F}_p[\![u]\!])$, $R_0\subset R$ is a ring of definition strictly topologically of finite type over $\mathbf{F}_p[\![u]\!]$, and N is a finite u-torsion-free R_0-module, then $(N \mathbin{\widehat{\otimes}} \mathbf{A}_{\mathrm{inf}})/(u-\overline{\pi}^{s(a)})$ has no u- or $\overline{\pi}$-torsion. In particular, $(R_0 \mathbin{\widehat{\otimes}} \mathbf{A}_{\mathrm{inf}})/(u-\overline{\pi}^{s(a)})$ has no u- or $\overline{\pi}$-torsion.*

Lemma 3.19. *There is a subset $\{e_i\}_{i\in I}\subset \mathbf{A}_{\mathrm{inf}}^{H_K}$ such that the natural map*

$$\{\sum_{i\in I} a_i\mathbf{e}_i \mid a_i\in \mathbf{Z}_p[\![[\overline{\pi}]^{s(a)}]\!], a_i\to 0\}\to \mathbf{A}_{\mathrm{inf}}^{H_K}$$

given by $\mathbf{e}_i\mapsto e_i$ is a topological isomorphism.

Proof. We first reduce modulo p. Then $\mathbf{A}_{\mathrm{inf}}^{H_K}/p$ is a complete $\overline{\pi}$-torsion-free $\mathbf{F}_p[\![\overline{\pi}^{s(a)}]\!]$-module, and $(\mathbf{A}_{\mathrm{inf}}^{H_K}/p)\left[\frac{1}{\overline{\pi}^{s(a)}}\right]$ is an $\mathbf{F}_p(\!(\overline{\pi}^{s(a)})\!)$-Banach space. As in the proof of [Sch02, Proposition 10.1], we may replace the natural norm $\|\cdot\|$ on $(\mathbf{A}_{\mathrm{inf}}^{H_K}/p)\left[\frac{1}{\overline{\pi}^{s(a)}}\right]$ with the equivalent norm

$$\|v\|' := \inf\{s\in |\mathbf{F}_p(\!(\overline{\pi}^{s(a)})\!)| \mid s\geq \|v\|\}$$

The unit ball of $(\mathbf{A}_{\mathrm{inf}}^{H_K}/p)\left[\frac{1}{\overline{\pi}^{s(a)}}\right]$ is the same with respect to both norms, i.e., $\mathbf{A}_{\mathrm{inf}}^{H_K}/p$.

Then again as in the proof of [Sch02, Proposition 10.1], we may lift an $\mathbf{F}_p$-basis of $\mathbf{A}_{\mathrm{inf}}^{H_K}/(p,[\overline{\pi}]^{s(a)})$ to obtain a topological $\mathbf{F}_p[\![\overline{\pi}^{s(a)}]\!]$-basis of $\mathbf{A}_{\mathrm{inf}}^{H_K}/p$. That is, there is an index set I and a set $\{e_i\}_{i\in I}\subset \mathbf{A}_{\mathrm{inf}}^{H_K}/p$ such that every element of $\mathbf{A}_{\mathrm{inf}}^{H_K}/p$ can be written uniquely in the form $\sum_{i\in I} a_i e_i$, where the a_i are elements of $\mathbf{F}_p[\![\overline{\pi}^{s(a)}]\!]$ and $a_i \to 0$ with respect to the cofinite filter.

We may lift the set $\{e_i\}_{i\in I}$ to a subset $\{\widetilde{e}_i\}_{i\in I} \subset \mathbf{A}_{\mathrm{inf}}^{H_K}$. Then by a similar argument, we see that the topologically free $\mathbf{Z}_p[\![[\overline{\pi}]^{s(a)}]\!]$-module on $\{\widetilde{e}_i\}_{i\in I}$ surjects onto $\mathbf{A}_{\mathrm{inf}}^{H_K}$. Since $\mathbf{A}_{\mathrm{inf}}^{H_K}$ is flat over $\mathbf{Z}_p[\![[\overline{\pi}]^{s(a)}]\!]$, we see that the natural map is an injection, and we are done. □

Corollary 3.20. *There is a subset $\{e_i\}_{i\in I} \subset (R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K})/(u-[\overline{\pi}]^{s(a)})$ such that the natural map*

$$\{\sum_{i\in I} a_i \mathbf{e}_i \mid a_i \in (R_0 \widehat{\otimes} \mathbf{Z}_p[\![[\overline{\pi}]^{s(a)}]\!])/(u-[\overline{\pi}]^{s(a)}), a_i \to 0\} \to (R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K})/(u-[\overline{\pi}])$$

given by $\mathbf{e}_i \mapsto e_i$ is a topological isomorphism.

Lemma 3.21. *If (R,R^+) is topologically of finite type over $(\mathbf{F}_p(\!(u)\!), \mathbf{F}_p[\![u]\!])$, $R_0 \subset R$ is a ring of definition strictly topologically of finite type over $\mathbf{F}_p[\![u]\!]$, and N is a finite u-torsion-free R_0-module, then the natural map $N \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K} \to \big((N \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})/(u-[\overline{\pi}]^{s(a)})\big)^{H_K}$ is surjective.*

Proof. We have an isomorphism

$$(N \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})/(u-[\overline{\pi}]^{s(a)}) \cong \Big((N \widehat{\otimes} \mathbf{F}_p[\![\overline{\pi}^{s(a)}]\!])/(u-\overline{\pi}^{s(a)})\Big) \widehat{\otimes}_{\mathbf{F}_p[\![\overline{\pi}^{s(a)}]\!]} \mathscr{O}_{\mathbf{C}_p}^{\flat}$$

Since N is u-torsion-free, $(N \widehat{\otimes} \mathbf{F}_p[\![\overline{\pi}^{s(a)}]\!])/(u-\overline{\pi}^{s(a)})$ is $\overline{\pi}$-torsion-free as an $\mathbf{F}_p[\![\overline{\pi}^{s(a)}]\!]$-module. If we reduce modulo $\overline{\pi}^{s(a)}$, it is an $\mathbf{F}_p$-vector space; we may choose a basis $\{\overline{e}_i\}_{i\in I}$ of $(N \widehat{\otimes} \mathbf{F}_p[\![\overline{\pi}^{s(a)}]\!])/(u, \overline{\pi}^{s(a)}) \cong N/u$ and lift it to a subset $\{e_i\}_{i\in I} \subset (N \widehat{\otimes} \mathbf{F}_p[\![\overline{\pi}^{s(a)}]\!])/(u-\overline{\pi}^{s(a)})$. By successive $\overline{\pi}^{s(a)}$-adic approximation, we can write any element of $(N \widehat{\otimes} \mathbf{F}_p[\![\overline{\pi}^{s(a)}]\!])/(u-\overline{\pi}^{s(a)})$ in the form $\sum_{i\in I} a_i e_i$, where the $a_i \in \mathbf{F}_p[\![\overline{\pi}]\!]$ tend to 0 with respect to the cofinite filter. Since $(N \widehat{\otimes} \mathbf{F}_p[\![\overline{\pi}^{s(a)}]\!])/(u-\overline{\pi}^{s(a)})$ has no $\overline{\pi}^{s(a)}$-torsion, this expression is unique and $(N \widehat{\otimes} \mathbf{F}_p[\![\overline{\pi}^{s(a)}]\!])/(u-\overline{\pi}^{s(a)})$ is topologically free over $\mathbf{F}_p[\![\overline{\pi}^{s(a)}]\!]$. It follows that

$$\Big((N \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})/(u-[\overline{\pi}]^{s(a)})\Big)^{H_K} = (N \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K})/(u-[\overline{\pi}]^{s(a)})$$

as desired. □

Now we can begin to bootstrap to the case where (R,R^+) is $\mathbf{Z}_p$-flat. Recall that we defined a sequence of ideals $I_j := p^j R \cap R_0$. Each ideal I_j is finitely generated (since R_0 is noetherian), so there is a sequence of integers $k_j \geq 1$ such that $u^{k_j} I_j \subset p^j R_0$.

Lemma 3.22. *With notation as above, $k_j \leq jk_1$.*

Proof. We proceed by induction on j. The case $j=1$ is trivial, so assume the result holds for $j-1$. If $x \in I_j$, then $u^{k_j}x = u^{k_j-k_1}u^{k_1}x \in p^j R_0$. But since $I_j \subset I_1$, it follows that $u^{k_1}x = px'$, and since p is not a zero-divisor, $u^{k_j-k_1}x' \in p^{j-1}R_0$. By the inductive hypothesis, $k_j - k_1 \leq (j-1)k_1$, and the result follows. □

Lemma 3.23. *With notation as above,*

(1) $\cap_j I_j = \{0\}$,
(2) $\cap_j I_j \big((R_0 \widehat{\otimes} \mathbf{Z}_p[\![[\overline{\pi}]^{s(a)}]\!])/(u-[\overline{\pi}]^{s(a)})\big) = \{0\}$.

Proof. If $x \in \cap_j I_j$, then for all $j \geq 1$, $u^{jk_1}x = p^j x_j$ for some $x_j \in R_0$. This implies that $x \in \left(\frac{p}{u^{k_1}}\right)^j R$ for all j. Since $p \notin R^\times$, $\left(\frac{p}{u^{k_1}}\right) R$ is a proper ideal, and Krull's intersection theorem implies that $\cap_j \left(\frac{p}{u^{k_1}}\right)^j R = \{0\}$. Since $\left((R_0 \widehat{\otimes} \mathbf{Z}_p[\![[\overline{\pi}]^{s(a)}]\!])/(u - [\overline{\pi}]^{s(a)})\right)$ is noetherian, the same argument applies to $\cap_j I_j \left((R_0 \widehat{\otimes} \mathbf{Z}_p[\![[\overline{\pi}]^{s(a)}]\!])/(u - [\overline{\pi}]^{s(a)})\right)$. □

We first treat the case where R is a $\mathbf{Z}/p^n$-algebra (and $I_n = (0)$).

Corollary 3.24. *If (R, R^+) is topologically of finite type over $((\mathbf{Z}/p^n)(\!(u)\!), (\mathbf{Z}/p^n)[\![u]\!])$ and $R_0 \subset R$ is a ring of definition strictly topologically of finite type over $(\mathbf{Z}/p^n)[\![u]\!]$, then the natural map $R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K} \to \left((R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})/(u - [\overline{\pi}]^{s(a)})\right)^{H_K}$ is surjective.*

Proof. We have seen that $\left(((R_0/I_1) \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})/(u - [\overline{\pi}]^{s(a)})\right)^{H_K} \cong ((R_0/I) \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K})/(u - [\overline{\pi}]^{s(a)})$, and we will proceed by induction on j. We have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & (I_j/I_{j+1}) \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K} \arrow[r] \arrow[d] & (R_0/I_{j+1}) \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K} \arrow[r] \arrow[d] & (R_0/I_j) \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \left((I_j/I_{j+1}) \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}/(u - [\overline{\pi}]^{s(a)}\right)^{H_K} \arrow[r] & \left((R_0/I_{j+1}) \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}/(u - [\overline{\pi}]^{s(a)})\right)^{H_K} \arrow[r] & \left((R_0/I_j) \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}/(u - [\overline{\pi}]^{s(a)})\right)^{H_K} &
\end{tikzcd}$$

Then the snake lemma implies that the middle arrow is surjective, as well. □

Now we return to the case of a general pseudoaffinoid algebra R.

Lemma 3.25. *The natural map $(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K})/(u - [\overline{\pi}]^{s(a)}) \to (R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})/(u - [\overline{\pi}]^{s(a)})$ is injective.*

Proof. We need to check that $(u - [\overline{\pi}]^{s(a)})(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}) \cap (R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K}) = (u - [\overline{\pi}]^{s(a)})(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K})$, and it suffices to check that $R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}$ has no $u - [\overline{\pi}]^{s(a)}$-torsion. But $(R_0/u) \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}$ has no $[\overline{\pi}]^{s(a)}$-torsion, so if $x \in R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}$ is annihilated by $u - [\overline{\pi}]^{s(a)}$, it is a multiple of u. In addition, $R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}$ has no u-torsion, so if $x = ux'$ is killed by $u - [\overline{\pi}]^{s(a)}$, so is x'. Replacing x with x' and repeating the argument, we see that $x \in u^n(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})$ for all n, so $x = 0$. □

Lemma 3.26. *If $x \in R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K}$ and the image of x in $(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})/(u - [\overline{\pi}]^{s(a)})$ is a multiple of u, then the image of x in $(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K})/(u - [\overline{\pi}]^{s(a)})$ is a multiple of u.*

Proof. We may write $x = ux' + (u - [\overline{\pi}]^{s(a)})y$ for $x', y \in R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}$. Reducing modulo u, we have $x \equiv -[\overline{\pi}]^{s(a)}y$; since $(R_0/u) \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}$ has no $[\overline{\pi}]^{s(a)}$-torsion, we see that $y \equiv y' \pmod{u}$, where $y' \in R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K}$. In other words,

$$x = ux' + (u - [\overline{\pi}]^{s(a)})(y' + uz)$$

for some $z \in R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}$. But then $x - (u - [\overline{\pi}]^{s(a)})y' \in (R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K}) \cap u(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})$. Since $R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}$ has no u-torsion, $(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K}) \cap u(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}) = u(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K})$, and we are done. □

Lemma 3.27. *The natural map $(R_0/(u, I_j)) \otimes (\mathbf{A}_{\mathrm{inf}}^{H_K}/[\overline{\pi}]^{s(a)}) \to (R_0/(u, I_j)) \otimes (\mathbf{A}_{\mathrm{inf}}/[\overline{\pi}]^{s(a)})$ is injective for all $j \geq 1$.*

Proof. We first show that the natural map $\widehat{\mathcal{O}}^\flat_{K_\infty}/[\overline{\pi}]^{s(a)} \to \mathcal{O}^\flat_{\mathbf{C}_p}/[\overline{\pi}]^{s(a)}$ is injective. But the cokernel of the injection $\widehat{\mathcal{O}}^\flat_{K_\infty} \to \mathcal{O}^\flat_{\mathbf{C}_p}$ is an $\widehat{\mathcal{O}}^\flat_{K_\infty}$-module with no $[\overline{\pi}]^{s(a)}$-torsion, so this follows.

We proceed by induction on j. If $j = 1$, then $R_0/(u, I_1)$ is a discrete $\mathbf{F}_p$-vector space, and therefore the map $\widehat{\mathcal{O}}^\flat_{K_\infty}/[\overline{\pi}]^{s(a)} \to \mathcal{O}^\flat_{\mathbf{C}_p}/[\overline{\pi}]^{s(a)}$ remains injective after tensoring with $R_0/(u, I_1)$. So assume the result for $j-1$. We have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & (I_{j-1}/(u,I_j)) \otimes (\mathbf{A}_{\mathrm{inf}}^{H_K}/[\overline{\pi}]^{s(a)}) \arrow[r] \arrow[d] & (R_0/(u,I_j)) \otimes (\mathbf{A}_{\mathrm{inf}}^{H_K}/[\overline{\pi}]^{s(a)}) \arrow[r] \arrow[d] & (R_0/(u,I_{j-1})) \otimes (\mathbf{A}_{\mathrm{inf}}^{H_K}/[\overline{\pi}]^{s(a)}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (I_{j-1}/(u,I_j)) \otimes (\mathbf{A}_{\mathrm{inf}}/[\overline{\pi}]^{s(a)}) \arrow[r] & (R_0/(u,I_j)) \otimes (\mathbf{A}_{\mathrm{inf}}/[\overline{\pi}]^{s(a)}) \arrow[r] & (R_0/(u,I_{j-1})) \otimes (\mathbf{A}_{\mathrm{inf}}/[\overline{\pi}]^{s(a)}) \arrow[r] & 0
\end{tikzcd}$$

Since $I_{j-1}/(u, I_j)$ is annihilated by p, the left vertical arrow is injective. The right vertical arrow is injective by the inductive hypothesis. A diagram chase then implies that the middle vertical arrow is injective, as desired. □

Lemma 3.28. *For any ring R and any ideals $I_1, I_2 \subset R$, there is an exact sequence*

$$0 \to R/(I_1 \cap I_2) \to R/I_1 \oplus R/I_2 \to R/(I_1 + I_2) \to 0$$

where the map $R/I_1 \oplus R/I_2 \to R/I_1 + I_2$ is given by $(f_1, f_2) \mapsto f_1 - f_2$.

Proof. The map $R/I_1 \oplus R/I_2 \to R/(I_1 + I_2)$ is clearly surjective, and the map $R/I_1 \cap I_2 \to R/I_1 \oplus R/I_2$ is clearly injective. It remains to check exactness in the middle. So suppose we have a pair $(f_1, f_2) \in R/I_1 \oplus R/I_2$ such that $f_1 - f_2 = 0$ in $R/(I_1 + I_2)$. Since the map $R/(I_1 \cap I_2) \to R/I_2$ is surjective, we may assume that $f_2 = 0$, and therefore that $f_1 \in (I_1 + I_2)/I_1$. But $(I_1 + I_2)/I_1 \cong I_2/(I_1 \cap I_2)$ as R-modules; given a representation $f_1 = g_1 + g_2$ with $g_i \in I_i$, this isomorphism sends f_1 to the image of g_2 modulo $I_1 \cap I_2$. Then the natural map $R/(I_1 \cap I_2) \to R/I_1 \oplus R/I_2$ carries g_2 to $(f_1, 0)$, as desired. □

Applying this to $(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K})/(u - [\overline{\pi}]^{s(a)})$ and $(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})/(u - [\overline{\pi}]^{s(a)})$ yields the following:

Lemma 3.29. *For each $j \geq 1$, there are exact sequences*

$$0 \to (R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K})/(u - [\overline{\pi}]^{s(a)}, uI_j) \to (R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K})/(u, [\overline{\pi}]^{s(a)}) \oplus (R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K})/(u - [\overline{\pi}]^{s(a)}, I_j) \to (R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K})/(u, [\overline{\pi}]^{s(a)}, I_j) \to 0$$

and

$$0 \to (R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})/(u - [\overline{\pi}]^{s(a)}, uI_j) \to (R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})/(u, [\overline{\pi}]^{s(a)}) \oplus (R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})/(u - [\overline{\pi}]^{s(a)}, I_j) \to (R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})/(u, [\overline{\pi}]^{s(a)}, I_j) \to 0$$

Proof. By Lemma 3.28, we have exact sequences

$$0 \to R_0/((u) \cap I_j) \to R_0/(u) \oplus R_0/I_j \to R_0/((u) + I_j) \to 0$$

Certainly, $(u) \cap I_j \subset uI_j$. Moreover, if $uf \in I_j$ for some $f \in R_0$, then $u^k uf \in p^j R_0$ for some $k \gg 0$, and so $f \in I_j$. Thus, the inclusion $uI_j \subset (u) \cap I_j$ is actually an equality, and we have exact sequences

$$0 \to R_0/uI_j \to R_0/(u) \oplus R_0/I_j \to R_0/((u) + I_j) \to 0$$

Since $(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K})/(u - [\overline{\pi}]^{s(a)})$ and $(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})/(u - [\overline{\pi}]^{s(a)})$ are flat over R_0 by Lemma 3.17, we may now extend scalars from R_0 to $(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K})/(u - [\overline{\pi}]^{s(a)})$ to obtain the desired result. □

Proposition 3.30. *If* $x \in \left((R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})/(u-[\overline{\pi}]^{s(a)})\right)^{H_K}$ *and* $\alpha \in \mathfrak{m}_{\widehat{K}_\infty^\flat}$, *then* $[\alpha]x$ *is in the image of* $R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K}$.

Proof. We first consider the image of x modulo I_1. Since it is fixed by H_K, it defines an element of $((R_0/I_1) \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K})/(u-[\overline{\pi}]^{s(a)})$, and therefore so does $[\alpha]x$.

Reducing modulo u instead, x defines an element of $\left((R_0/u) \otimes (\mathbf{A}_{\mathrm{inf}}/[\overline{\pi}]^{s(a)})\right)^{H_K}$. There is a sequence of H_K-equivariant maps

$$\mathbf{A}_{\mathrm{inf}}/[\overline{\pi}]^{s(a)} \xrightarrow{\times[\alpha]} [\alpha]/[\overline{\pi}^{s(a)}\alpha] \to [\alpha]/[\overline{\pi}^{s(a)}\mathfrak{m}_{\mathbf{C}_p^\flat}] \to [\mathfrak{m}_{\mathbf{C}_p^\flat}]/[\overline{\pi}^{s(a)}\mathfrak{m}_{\mathbf{C}_p^\flat}] \to [\mathfrak{m}_{\mathbf{C}_p^\flat}]/[\overline{\pi}]^{s(a)}\mathbf{A}_{\mathrm{inf}} \to \mathbf{A}_{\mathrm{inf}}/[\overline{\pi}]^{s(a)}$$

Since $\left([\mathfrak{m}_{\mathbf{C}_p^\flat}]/[\overline{\pi}^{s(a)}\mathfrak{m}_{\mathbf{C}_p^\flat}]\right)^{H_K} = [\mathfrak{m}_{\widehat{K}_\infty^\flat}]/[\overline{\pi}^{s(a)}\mathfrak{m}_{\widehat{K}_\infty^\flat}]$, $[\alpha]x$ defines an element of $(R_0/u) \otimes (\mathbf{A}_{\mathrm{inf}}^{H_K}/[\overline{\pi}]^{s(a)})$.

It follows from Lemma 3.29 and Lemma 3.27 that there is some $a_0 \in (R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K})/(u-[\overline{\pi}]^{s(a)})$ such that $[\alpha]x-a_0 \in uI_1(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})/(u-[\overline{\pi}]^{s(a)})$. In other words, $[\alpha]x-a_0 = ux_1$, where $x_1 \in I_1\left((R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})/(u-[\overline{\pi}]^{s(a)})\right)^{H_K}$.

We may write $ux_1 = [\overline{\pi}]^{s(a)}x_1 = [\overline{\pi}]^{(p-1)^2/p^2a}[\overline{\pi}]^{(p-1)/p^2a}x_1$ and apply the previous argument to $[\overline{\pi}]^{(p-1)/p^2a}x_1$. We obtain some $a_1 \in (R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K})/(u-[\overline{\pi}]^{s(a)})$ such that $[\overline{\pi}]^{(p-1)/p^2a}x_1 - a_1 \in uI_2(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})/(u-[\overline{\pi}]^{s(a)})$ and remains fixed by H_K.

Continuing in this fashion, we obtain a sequence $\{a_j\}_{j\geq 0}$ with $a_j \in (R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K})/(u-[\overline{\pi}]^{s(a)})$ such that

$$[\alpha]x - \sum_{j=0}^{n-1}[\overline{\pi}]^{(p-1)^2j/p^2a}a_j \in uI_n(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})/(u-[\overline{\pi}]^{s(a)})$$

Since the terms $[\overline{\pi}]^{(p-1)^2j/p^2a}a_j$ tend to 0, the sum $\sum_{j\geq 0}[\overline{\pi}]^{(p-1)^2j/p^2a}a_j$ converges in $(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K})/(u-[\overline{\pi}]^{s(a)})$, and $[\alpha]x-\sum_{j\geq 0}[\overline{\pi}]^{(p-1)^2j/p^2a}a_j \in \cap_j I_j(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})/(u-[\overline{\pi}]^{s(a)})$.

To finish, we need to show that $\cap_j I_j(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})/(u-[\overline{\pi}]^{s(a)}) = \{0\}$. But this follows by combining Corollay 3.20 and Lemma 3.23. □

Corollary 3.31. *The natural map* $(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K})/(u-[\overline{\pi}]^{s(a)}) \to \left((R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})/(u-[\overline{\pi}]^{s(a)})\right)^{H_K}$ *is an isomorphism.*

Proof. This follows by combining Proposition 3.30 with Lemma 3.25 and Lemma 3.26. □

We are finally in a position to compute $\widetilde{\Lambda}_{R_0,[a,b]}^{H_K}$.

Corollary 3.32. *Suppose* $[a,b] \subset (0,\infty)$. *If* R *is a pseudoaffinoid algebra and* $R_0 \subset R$ *is a noetherian ring of definition formally of finite type over* $\mathbf{Z}_p$, *then*

$$\widetilde{\Lambda}_{R_0,[a,b]}^{H_K} = (R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K})\left\langle \frac{u}{[\overline{\pi}]^{s(b)}}, \frac{[\overline{\pi}]^{s(a)}}{u}\right\rangle\left[\frac{1}{u}\right]$$

Proof. We consider $\widetilde{\Lambda}_{R_0,[a,\infty]}^{H_K}$ and $\widetilde{\Lambda}_{R_0,[0,b]}^{H_K}$ separately.

There is a natural map $(R_0 \widehat{\otimes} \mathbf{A}_{\inf}) \left\langle \frac{[\overline{\pi}]^{s(a)}}{u} \right\rangle \to (R_0 \widehat{\otimes} \mathbf{A}_{\inf})/(u - \overline{\pi}^{s(a)})$ with kernel $(1 - \frac{[\overline{\pi}]^{s(a)}}{u})$, extending the quotient map $R_0 \widehat{\otimes} \mathbf{A}_{\inf} \twoheadrightarrow (R_0 \widehat{\otimes} \mathbf{A}_{\inf})/(u - [\overline{\pi}]^{s(a)})$. Given $x \in (R_0 \widehat{\otimes} \mathbf{A}_{\inf}) \left\langle \frac{[\overline{\pi}]^{s(a)}}{u} \right\rangle$, there is a non-decreasing sequence $\{\alpha_i\}_{i\geq 1}$ of integers with $0 \leq \alpha_i \leq i-1$ for all i and $\lim \frac{\alpha_i}{i} = 0$ such that $x \in u^{-\alpha_i} R_0 \widehat{\otimes} \mathbf{A}_{\inf} + (\frac{u-[\overline{\pi}]^{s(a)}}{u})^i (R_0 \widehat{\otimes} \mathbf{A}_{\inf}) \left\langle \frac{[\overline{\pi}]^{s(a)}}{u} \right\rangle$ for all i (as in the proof of [Ber02, Lemme 2.29]). If x is fixed by H_K, Corollary 3.31 implies that there is some $a_0 \in R_0 \widehat{\otimes} \mathbf{A}_{\inf}^{H_K}$ such that $x \equiv a_0 \pmod{u - [\overline{\pi}]^{s(a)}}$. Moreover, $a_0 \in u^{-\alpha_1}(R_0 \widehat{\otimes} \mathbf{A}_{\inf}^{H_K})$.

Suppose we have a sequence $a_0, \ldots, a_{n-1}$ of elements of $R_0 \widehat{\otimes} \mathbf{A}_{\inf}^{H_K}$ such that $a_i \in u^{i-\alpha_{i+1}}(R_0 \widehat{\otimes} \mathbf{A}_{\inf}^{H_K})$ and $x - \left(\sum_{i=0}^{n-1} a_i (\frac{u-[\overline{\pi}]^{s(a)}}{u})^i\right) \in (\frac{u-[\overline{\pi}]^{s(a)}}{u})^n (R_0 \widehat{\otimes} \mathbf{A}_{\inf}) \left\langle \frac{[\overline{\pi}]^{s(a)}}{u} \right\rangle$. Then it follows from Corollary 3.31 that there is some $a_n' \in R_0 \widehat{\otimes} \mathbf{A}_{\inf}^{H_K}$ such that

$$x - \left(\sum_{i=0}^{n-1} a_i (\frac{u - [\overline{\pi}]^{s(a)}}{u})^i\right) - a_n' (\frac{u - [\overline{\pi}]^{s(a)}}{u})^n \in (\frac{u - [\overline{\pi}]^{s(a)}}{u})^{n+1} (R_0 \widehat{\otimes} \mathbf{A}_{\inf}) \left\langle \frac{[\overline{\pi}]^{s(a)}}{u} \right\rangle^{H_K}$$

Since the sequence $\{\alpha_i\}$ is non-decreasing, $a_n'(\frac{u-[\overline{\pi}]^{s(a)}}{u})^n \in u^{-\alpha_{n+1}}(R_0 \widehat{\otimes} \mathbf{A}_{\inf}) + (\frac{u-[\overline{\pi}]^{s(a)}}{u})^{n+1}(R_0 \widehat{\otimes} \mathbf{A}_{\inf}) \left\langle \frac{[\overline{\pi}]^{s(a)}}{u} \right\rangle$, and we may write

$$ua_n'(u - [\overline{\pi}]^{s(a)})^n = u^{n+1-\alpha_{n+1}} b_n + (u - [\overline{\pi}]^{s(a)})^{n+1} c_n$$

with $b_n \in R_0 \widehat{\otimes} \mathbf{A}_{\inf}$ and $c_n \in (R_0 \widehat{\otimes} \mathbf{A}_{\inf}) \left\langle \frac{[\overline{\pi}]^{s(a)}}{u} \right\rangle$. This implies that $u^{n+1-\alpha_{n+1}} b_n \in (u - [\overline{\pi}]^{s(a)})^n (R_0 \widehat{\otimes} \mathbf{A}_{\inf})$; since $(R_0 \widehat{\otimes} \mathbf{A}_{\inf})/(u - [\overline{\pi}]^{s(a)})$ has no u-torsion, it follows that $b_n \in (u - [\overline{\pi}]^{s(a)})^n (R_0 \widehat{\otimes} \mathbf{A}_{\inf})$. Thus,

$$a_n' = u^{n-\alpha_{n+1}} \left(\frac{b_n}{(u - [\overline{\pi}]^{s(a)})^n}\right) + \left(\frac{u - [\overline{\pi}]^{s(a)}}{u}\right) c_n$$

If we consider the image of a_n' in $\left((R_0 \widehat{\otimes} \mathbf{A}_{\inf})/(u - [\overline{\pi}]^{s(a)})\right)^{H_K}$, we see that it is equal to the image of $u^{n-\alpha_{n+1}} \left(\frac{b_n}{(u-[\overline{\pi}]^{s(a)})^n}\right)$. Thus, there is some $a_n \in u^{n-\alpha_{n+1}}(R_0 \widehat{\otimes} \mathbf{A}_{\inf}^{H_K})$ such that $a_n \equiv a_n' \pmod{u - [\overline{\pi}]^{s(a)}}$. It follows that $ux - \left(\sum_{i=0}^{n} a_i (\frac{u-[\overline{\pi}]^{s(a)}}{u})^i\right) \in (\frac{u-[\overline{\pi}]^{s(a)}}{u})^{n+1} \widetilde{\Lambda}_{R_0,[a_0,\infty]}^{H_K}$.

By induction, we obtain a sequence $\{a_i\}$ of elements of $R_0 \widehat{\otimes} \mathbf{A}_{\inf}^{H_K}$ such that the sum $\sum_{i=0}^{\infty} a_i (\frac{u-[\overline{\pi}]^{s(a)}}{u})^i$ converges in $\widetilde{\Lambda}_{R_0,[a,\infty]}^{H_K}$ to x.

A similar argument applies to elements of $\widetilde{\Lambda}_{R_0,[0,b]}^{H_K}$. □

3.3. **Imperfect overconvergent rings.** We now define *imperfect* period rings, which will be noetherian pseudoaffinoid algebras over R. As in the case of perfect overconvergent rings, we would like to consider the fiber products of $\operatorname{Spa} R_0$ and $\operatorname{Spa} R$ with analytic subspaces of $\operatorname{Spa} \mathbf{A}_K^+$. However, because we only have an explicit description of $\Lambda_{[0,b]}^{H_K}$ and $\Lambda_{[0,b]}^{\circ,H_K}$ for sufficiently small b, we restrict our definitions to that setting.

Let $K/\mathbf{Q}_p$ be a finite extension and let $F' \subset K_\infty$ be its maximal unramified subextension. Recall that we defined

$$\mathscr{A}_{F'}^{(0,b]} := \{\sum_{m\in\mathbf{Z}} a_m X^m : a_m \in \mathscr{O}_{F'}, v_p(a_m) + mb \to \infty \text{ as } m \to -\infty\}$$

to be the ring of integers of the ring of bounded analytic functions on the half-open annulus $0 < v_p(X) \leq b$ over F'. Let

$$r_K := \begin{cases} (2v_{\mathbf{C}_p^\flat}(\mathfrak{d}_{\mathbf{E}_K/\mathbf{E}_F}))^{-1} & \text{if } \mathbf{E}_K/\mathbf{E}_F \text{ is ramified} \\ 1 & \text{otherwise} \end{cases}$$

Then we have the following:

Proposition 3.33. [Col08, Proposition 7.5] *For $b < r_K$, the assignment $f \mapsto f(\pi_K)$ is an isomorphism of topological rings from $\mathscr{A}_{F'}^{(0,bv_{\mathbf{C}_p^\flat}(\pi_K)]}$ to $\Lambda_{[0,b],K}$. Furthermore, if we define a valuation $v^{(0,b]}$ on $\mathscr{A}_{F'}^{(0,b]}$ by*

$$v^{(0,b]}(\sum_{m\in\mathbf{Z}} a_m X^m) := \inf_{m\in\mathbf{Z}}(v_p(a_m) + mb)$$

then

$$\mathrm{val}^{[0,b]}(f(\pi_K)) = \frac{1}{b}v^{(0,b]}(f)$$

where $\mathrm{val}^{[0,b]}$ is the restriction of the corresponding valuation on $\widetilde{\Lambda}_{[0,b]}$.

In the special case $b = 0$, $\Lambda_{[0,0]}^{H_K} = \mathbf{A}_K \cong \mathscr{O}_{F'}[\![\pi_K]\!]\left[\frac{1}{\pi_K}\right]^\wedge$, where the completion is p-adic.

With this in mind, we may reason as in the perfect case and show:

Proposition 3.34. *Suppose $b \in (0, r_K)$. Then*

(1) *The pre-adic space $\operatorname{Spa} R_0 \times \operatorname{Spa} \Lambda_{[0,b]}^{H_K}$ is exhausted by affinoid adic spaces of the form $\operatorname{Spa}\left(R_0 \widehat{\otimes} \mathscr{O}_{F'}[\![\pi_K]\!]\right)\left\langle \frac{u}{\pi_K^{1/(b'\cdot v_{\mathbf{C}_p^\flat}(\overline{\pi}_K))}}\right\rangle$ for $b' \in (0,b]$ satisfying $b' \leq \frac{b\cdot \mathrm{ord}_R(p)}{v_{\mathbf{C}_p^\flat}(\overline{\pi}_K)}$ and $\frac{1}{b'\cdot v_{\mathbf{C}_p^\flat}(\overline{\pi}_K)} \in \mathbf{N}$.*

(2) *The pre-adic space $\operatorname{Spa} R \times \operatorname{Spa} \Lambda_{[0,b]}^{H_K}$ is exhausted by affinoid adic spaces of the form $\operatorname{Spa}\left(R_0 \widehat{\otimes} \mathscr{O}_{F'}[\![\pi_K]\!]\right)\left\langle \frac{\pi_K^{1/(a\cdot v_{\mathbf{C}_p^\flat}(\overline{\pi}_K))}}{u}, \frac{u}{\pi_K^{1/(b'\cdot v_{\mathbf{C}_p^\flat}(\overline{\pi}_K))}}\right\rangle$ for a, b' satisfying $0 < a \leq b' \leq \frac{b\cdot\mathrm{ord}_R(p)}{v_{\mathbf{C}_p^\flat}(\overline{\pi}_K)}$ and $\frac{1}{a\cdot v_{\mathbf{C}_p^\flat}(\overline{\pi}_K)}, \frac{1}{b'\cdot v_{\mathbf{C}_p^\flat}(\overline{\pi}_K)} \in \mathbf{N}$.*

This motivates the following definition.

Definition 3.35. Let R be a pseudoaffinoid $\mathbf{Z}_p$-algebra with pseudo-uniformizer $u \in R_0$, and let $K/\mathbf{Q}_p$ be a finite extension. Fix $a, b \in \mathbf{Q}_{>0}$ with $a \leq b < \frac{r_K\cdot\mathrm{ord}_R(p)}{v_{\mathbf{C}_p^\flat}(\overline{\pi}_K)}$ such that $\frac{1}{a\cdot v_{\mathbf{C}_p^\flat}(\overline{\pi}_K))}, \frac{1}{b\cdot v_{\mathbf{C}_p^\flat}(\overline{\pi}_K))} \in \mathbf{Z}$. Then we define the $\mathbf{Z}_p$-algebra $\Lambda_{R,[a,b],K}$ to be the evaluation of $\mathscr{O}_{(R_0\otimes\mathscr{O}_{F'})[\![\pi_K]\!]}$ on the affinoid subspace of $\operatorname{Spa}(R_0\otimes \mathscr{O}_{F'})[\![\pi_K]\!]$ defined by the conditions $u \leq \pi_K^{1/(b\cdot v_{\mathbf{C}_p^\flat}(\overline{\pi}_K))}$ and $\pi_K^{1/(a\cdot v_{\mathbf{C}_p^\flat}(\overline{\pi}_K))} \leq u$.

This is a pseudoaffinoid algebra with ring of definition $\Lambda_{R_0,[a,b],0,K} := (R_0 \otimes \mathscr{O}_{F'})[\![\pi_K]\!]\left\langle \frac{u}{\pi_K^{1/(b\cdot v_{\mathbf{C}_p^\flat}(\overline{\pi}_K))}}, \frac{\pi_K^{1/(a\cdot v_{\mathbf{C}_p^\flat}(\overline{\pi}_K))}}{u}\right\rangle$ and pseudo-uniformizers u and π_K.

We make an auxiliary definition $\Lambda_{R_0,[0,0],K} := (R_0 \widehat{\otimes}\, \mathscr{O}_{F'})[\![\pi_K]\!]\left[\frac{1}{\pi_K}\right]^\wedge_u$, where the completion is u-adic.

If $I \subset (0,\infty)$ is any interval (with either open or closed endpoints), we set

$$\Lambda_{R,I,K} := \varprojlim_{[a,b]\subset I} \Lambda_{R,[a,b],K}$$

Remark 3.36. Since $\Lambda_{R,[a,b],K}$ has noetherian ring of definition, $\operatorname{Spa}\Lambda_{R,[a,b],K}$ is an adic space, not merely a pre-adic space. Thus, the sheaf property with respect to covers of $\operatorname{Spa} R$ or with respect to change of intervals is automatic.

Proposition 3.37. *For $0 \leq a \leq b < \infty$, the ring $\Lambda_{R,[a,b],K}$ is flat as an R-module.*

Proof. The set $\left\{\left(\frac{u}{\pi_K^{1/(b\cdot v_{\mathbf{C}_p^\flat}(\overline{\pi}_K))}}\right)^n\right\} \cup \left\{\left(\frac{\pi_K^{1/(a\cdot v_{\mathbf{C}_p^\flat}(\overline{\pi}_K))}}{u}\right)^n\right\}$ provides a topological basis for $\Lambda_{R,[a,b],0,K}$ as an R_0-module. To check that $\Lambda_{R,[a,b],K}$ is R-flat, it suffices to check that $\Lambda_{R,[a,b],K} \otimes_R N \to \Lambda_{R,[a,b],K} \otimes_R N'$ is injective for any injection of finite R-modules $N \to N'$. But $\Lambda_{R,[a,b],K} \otimes_R N \cong \oplus_n \left(\frac{u}{\pi_K^{1/(b\cdot v_{\mathbf{C}_p^\flat}(\overline{\pi}_K))}}\right)^n \cdot N \bigoplus \oplus_n \left(\frac{\pi_K^{1/(a\cdot v_{\mathbf{C}_p^\flat}(\overline{\pi}_K))}}{u}\right)^n \cdot N$, and similarly for $\Lambda_{R,[a,b],K} \otimes_R N'$. Thus, if $N \to N'$ is injective, so is $\Lambda_{R,[a,b],K} \otimes_R N \to \Lambda_{R,[a,b],K} \otimes_R N'$, and the result follows. $\square$

Since the Frobenius on $\widetilde{\Lambda}_{R,[0,0]}$ acts via $\varphi(\pi) = (1+\pi)^p - 1$ and Γ acts via $\gamma(\pi) = (1+\pi)^{\chi(\gamma)} - 1$, we see that $\Lambda_{R_0,[0,0],F}$ is stable under the actions of φ and Γ. Since $\mathbf{A}_K$ is also stable under the actions of φ and Γ, we see that φ and Γ act on $\Lambda_{R_0,[0,0],K}$, as well. Since we have isomorphisms $\varphi : \widetilde{\Lambda}_{R,[a,b]}^{H_K} \to \widetilde{\Lambda}_{R,[a/p,b/p]}$, we have induced maps

$$\varphi : \Lambda_{R,[a,b],K} \to \Lambda_{R,[a/p,b/p],K}$$

However, $\varphi : \Lambda_{R_0,[0,0],K} \to \Lambda_{R_0,[0,0],K}$ is no longer surjective. Indeed, $\Lambda_{R_0,[0,0],K}$ is free over $\varphi(\Lambda_{R_0,[0,0],K})$ of rank p, and a basis is given by $\{1, [\varepsilon], \ldots, [\varepsilon]^{p-1}\}$. We may therefore define a left inverse $\psi : \Lambda_{R_0,[0,0],K} \to \varphi(\Lambda_{R_0,[0,0],K})$ of φ via $\psi(\varphi(a_0) + \varphi(a_1)[\varepsilon] + \ldots + \varphi(a_{p-1})[\varepsilon]^{p-1}) = a_0$, where $a_i \in \Lambda_{R_0,[0,0],K}$. Note that as p may not be invertible in R, we cannot use the definition $\psi = \frac{1}{p}\varphi^{-1} \circ \operatorname{Tr}_{\Lambda_{R_0,[0,0],K}/\varphi(\Lambda_{R_0,[0,0],K})}$ from the classical case.

4. The Tate–Sen axioms for families

Given a Galois representation with coefficients in $\mathbf{Z}_p$, base extension gives us a vector bundle over $\mathcal{Y}$. The various (φ,Γ)-modules associated to the representation are constructed by studying the H_K-invariants of restrictions of this vector bundle to various rational subdomains of $\mathcal{Y}$.

Now let R be a pseudoaffinoid $\mathbf{Z}_p$-algebra with noetherian ring of definition R_0 and pseudo-uniformizer $u \in R_0$. If we have a Galois representation with coefficients in R which admits a Galois-stable R_0-lattice, it is therefore natural to study vector bundles over $\mathrm{Spa}(R) \times_{\mathrm{Spa}(\mathbf{Z}_p)} \mathcal{Y}$ and $\mathrm{Spa}(R_0) \times_{\mathrm{Spa}(\mathbf{Z}_p)} \mathcal{Y}$. Now $\mathcal{Y}$ is covered by the open subspaces $\mathcal{Y} \smallsetminus \{x : |p(x)| = 0\}$ and $\mathcal{Y} \smallsetminus \{x : |[\varpi](x)| = 0\}$. The former is exhausted by the affinoid subspaces $\mathrm{Spa}(\mathbf{A}_{\mathrm{inf}}) \langle (p, [\varpi])^n/p \rangle$ (which are Tate, with pseudo-uniformizer p) and the latter is exhausted by the affinoid subspaces $\mathrm{Spa}(\mathbf{A}_{\mathrm{inf}}) \langle (p, [\varpi])^n/[\varpi] \rangle$ (which are Tate, with pseudo-uniformizer $[\varpi]$).

Since H_K acts isometrically on $\mathbf{C}_p^\flat$, it preserves these affinoid subspaces. Thus, to construct the fiber products $\mathrm{Spa}(R, R^+) \times_{\mathrm{Spa}(\mathbf{Z}_p)} \mathcal{Y}$ and $\mathrm{Spa}(R^+) \times_{\mathrm{Spa}(\mathbf{Z}_p)} \mathcal{Y}$ and study the H_K-action on them, it suffices to study the H_K-action on rational localizations $\mathrm{Spa}(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}) \left\langle \frac{u}{[\varpi]^{1/b}}, \frac{[\varpi]^{1/a}}{u} \right\rangle$. This motivated our definition of $\widetilde{\Lambda}_{R,[a,b]}$ in the previous section.

The Tate–Sen axioms concern a profinite group G_0, an open normal subgroup $H_0 \subset G_0$ such that G_0/H_0 contains $\mathbf{Z}_p$ as an open subgroup, a valued ring $\widetilde{\Lambda}$ with a continuous action of G_0, and a collection of subrings $\{\Lambda_{H,k}\}_{k \gg 0}$ of $\widetilde{\Lambda}^H$, where H is any open subgroup of H_0. These axioms permit us to descend continuous 1-cocycles of G_0 from $\widetilde{\Lambda}$ to some $\Lambda_{H,k}$.

The axioms are as follows:

TS1 There is a constant $c_1 \in \mathbf{R}_{>0}$ such that for all open subgroups $H_1 \subset H_2$ in H_0 that are normal in G, there is some $\alpha \in \widetilde{\Lambda}^{H_1}$ satisfying $v_\Lambda > -c_1$ and $\sum_{\tau \in H_2/H_1} \tau(\alpha) = 1$.

TS2 There is a constant $c_2 \in \mathbf{R}_{>0}$ such that for all open subgroups $H \subset H_0$ that are normal in G, there is a collection $\{\Lambda_{H,k}, R_{H,k}\}_{k \geq n(H)}$, where $\Lambda_{H,k} \subset \widetilde{\Lambda}^H$ is a closed subalgebra and $R_{H,k} : \widetilde{\Lambda}^H \to \Lambda_{H,k}$ is a $\Lambda_{H,k}$-linear map such that
 (a) if $H_{L_1} \subset H_{L_2}$, and $k \geq \max\{n(H_{L_1}), n(H_{L_2})\}$, then $\Lambda_{H_2,k} \subset \Lambda_{H_1,k}$ and $R_{H_1,k}|_{\Lambda_{H_2,k}} = R_{H_2,k}$
 (b) $R_{H,k}$ is a $\Lambda_{H,k}$-linear section to the inclusion $\Lambda_{H,k} \hookrightarrow \widetilde{\Lambda}^H$
 (c) $g(\Lambda_{H,k}) = \Lambda_{H,k}$ and $g(R_{H,k}(x)) = R_{H,k}(gx)$ for all $x \in \widetilde{\Lambda}^H$ and $g \in G_0$
 (d) $v_\Lambda(R_{H,k}(x)) \geq v_\Lambda(x) - c_2$ for all $x \in \widetilde{\Lambda}^H$
 (e) $\lim_{k\to\infty} R_{H,k}(x) = x$ for all $x \in \widetilde{\Lambda}^H$

TS3 There is a constant $c_3 \in \mathbf{R}_{>0}$ such that for every open normal subgroup $G \subset G_0$ (setting $H := G \cap H_0$) there is an integer $n(G) \geq \max\{n_1(G), n(H)\}$ such that
 (a) $\gamma - 1$ is invertible on $X_{H,k} := \ker(R_{H,k})$
 (b) $v_\Lambda(x) \geq v_\Lambda((\gamma - 1)(x)) - c_3$ for all $x \in X_{H,k}$
 for all $k \geq n(G)$ and all $\gamma \in G_0/H$ with $n(\gamma) \leq k$.

In other words, $\Lambda_{H,k}$ is a summand of $\widetilde{\Lambda}^H$ (as a $\Lambda_{H,k}$-module), and a topological generator of Γ acts invertibly (with continuous inverse) on its complement.

Colmez showed that if we take $G_0 := \mathrm{Gal}_{\mathbf{Q}_p}$, $H_0 := \ker(\chi)$, $\widetilde{\Lambda} := \widetilde{\Lambda}_{[0,1]}$ (with the valuation $\mathrm{val}^{(0,1]}$), and $\Lambda_{H_K,k} := \varphi^{-k}\left(\Lambda_{[0,1]}^{H_K}\right)$,then the Tate–Sen axioms are satisfied for any choices $c_1 > 0$, $c_2 > 0$, and $c_3 > 1/(p-1)$ [BC08, Proposition 4.2.1]. Here K is a finite extension of $\mathbf{Q}_p$.

Recall that we have valuations $\mathrm{val}^{(0,b]}(x)$ and $v_b(x)$ on $\widetilde{\Lambda}_{[0,b]}$ defined by

$$\mathrm{val}^{(0,b]}(x) := \inf_{k\geq 0}\{v_{\widetilde{\mathbf{E}}}(x_k) + \frac{k}{b}\} = \sup_{r\in\mathbf{Q}:[\varpi]^r x\in\widetilde{\mathbf{A}}^{\dagger,s(b)}} -r =: v_b(x)$$

where $x = \sum_{k\geq 0}[x_k]p^k$.

Suppose that $p^N \in uR_0$, so that there is a homomorphism $\widetilde{\Lambda}^{\circ}_{[0,bN]} \to \widetilde{\Lambda}_{R_0,[0,b]}$. By Proposition 3.2, the image of $(R_0 \widehat{\otimes} \widetilde{\Lambda}^{\circ}_{[0,bN]})\left\langle \frac{u}{[\overline{\pi}]^{s(b)}}\right\rangle$ is a ring of definition of $\widetilde{\Lambda}_{R_0,[0,b]}$. In particular, the image of $\widetilde{\Lambda}^{\circ}_{[0,bN]}$ in $\widetilde{\Lambda}_{R_0,[0,b]}$ is bounded; if $p^N \in uR_0$, then $v_{bN}(x) \geq v_{R,b}(x) \geq v_{bN}(x) - \frac{N-1}{bN}$.

Proposition 4.1. *If $p^N \in uR_0$, the ring $\widetilde{\Lambda}_{R_0,[0,b]}$ satisfies the first Tate–Sen axiom for any $b > 0$ and any $c_1 > \frac{N-1}{bN}$.*

Proof. Choose $c_1 > 0$. Then for any appropriate subgroups $H_{L_1} \subset H_{L_2}$ of H, the proof of [Col08, Lemme 10.1] constructs $\beta \in \widehat{L^\flat_\infty}$ such that $\mathrm{Tr}_{\widehat{L^\flat_\infty}/\widehat{K^\flat_\infty}}(\beta) = 1$, with $v_{\mathbf{C}_p^\flat}(\beta)$ arbitrarily close to 0. This implies that $\mathrm{Tr}_{\widehat{L^\flat_\infty}/\widehat{K^\flat_\infty}}([\beta]) = \sum_{i\geq 0} p^i[x_i]$ is a unit of $\widetilde{\Lambda}^{\circ,H_K}_{[0,bN]}$, and therefore that $(\mathrm{Tr}_{\widehat{L^\flat_\infty}/\widehat{K^\flat_\infty}}([\beta]))^{-1}[\beta] \in \widetilde{\Lambda}^{\circ,H_K}_{[0,bN]}$ satisfies $v_{R,b}\left((\mathrm{Tr}_{\widehat{L^\flat_\infty}/\widehat{K^\flat_\infty}}([\beta]))^{-1}[\beta]\right) \geq v_{\mathbf{C}_p^\flat}(\beta) - \frac{N-1}{bN}$. Thus, we merely need to choose β such that $v_{\mathbf{C}_p^\flat}(\beta) > \inf\{\frac{N-1}{bN} - c_1, -\frac{1}{bN}\}$. □

Corollary 4.2. *Suppose M is a finite free R_0-module of rank d equipped with a continuous R_0-linear action of G_K. Then there is some finite extension K'/K such that $\widetilde{D}_{K'}(M) := \left(\widetilde{\Lambda}_{R_0,[0,1]} \otimes_{R_0} M\right)^{H_{K'}}$ is free over $\widetilde{\Lambda}^{H_{K'}}_{R_0,[0,1]}$ of rank d.*

Proof. Choose a basis of M and let $\rho : \mathrm{Gal}_K \to \mathrm{GL}_d(R_0)$ denote the Galois representation corresponding to M. Let $c_\tau \in \mathrm{H}^1(H_K, \mathrm{GL}_d(\widetilde{\Lambda}_{R_0,[0,1]}))$ be the corresponding cocycle. If we let K'/K be the finite extension corresponding to the kernel of the homomorphism $\overline{\rho} : \mathrm{Gal}_K \to \mathrm{GL}(M/u)$, the proofs of [BC08, Lemme 3.2.1] and [BC08, Corollaire 3.2.2] applied to the image of c in $\mathrm{H}^1(H_{K'}, \mathrm{GL}_d(\widetilde{\Lambda}_{R_0,[0,1]}))$ carry over nearly verbatim (we need to work modulo powers of u rather than p, since p might not be a pseudo-uniformizer), and we conclude that the restriction of c is trivial. The result follows. □

4.1. **Normalized trace maps.** The next step is to construct so-called normalized trace maps. In the classsical setting, this has the following form:

Proposition 4.3 ([Col08, Cor. 8.11])**.** *Suppose $0 < b$ and $p^{-n}b < r_K$. Then there is a constant $c_K(b)$ (depending on K and b) and a $\varphi^{-n}\left(\Lambda^{H_K}_{[0,p^{-n}b]}\right)$-linear map $R_{K,n} : \widetilde{\Lambda}^{H_K}_{[0,b]} \to \varphi^{-n}\left(\Lambda^{H_K}_{[0,p^{-n}b]}\right)$ such that*

1. *$R_{K,n}$ is a section to the inclusion $\varphi^{-n}\left(\Lambda^{H_K}_{[0,p^{-n}b]}\right) \to \widetilde{\Lambda}^{H_K}_{[0,b]}$*
2. *$R_{K,n}(x) \to x$ as $n \to \infty$ and $v_b(R_{K,n}(x)) \geq v_b(x) - p^{-n}c_K(b)$.*

(3) *$R_{K,n}$ commutes with the action of Γ_K.*

To construct $R_{K,n}$, we use the fact that $\{[\varepsilon]^i \mid i \in \mathbf{Z}[\frac{1}{p}] \cap [0,1)\}$ provides a topological basis for $\widetilde{\Lambda}^{H_K}_{[0,0]}$ over $\Lambda^{H_K}_{[0,0]}$ (and in fact for $\widetilde{\Lambda}^{H_F}_{[0,\infty]}$ over $\Lambda^{H_F}_{[0,\infty]}$ when $F/\mathbf{Q}_p$ is unramified). In other words, for $x \in \widetilde{\Lambda}^{H_K}_{[0,0]}$, we can write $x = \sum_i a_i(x)[\varepsilon]^i$ for unique $a_i(x)$ tending to 0 with respect to the cofinite filter. Then we bound $a_i(x)$ in terms of x and show that $a_i(x)$ has the correct analyticity properties when $x \in \widetilde{\Lambda}^{H_K}_{[0,b]}$ for $b > 0$, and define $R_{K,n}(x) = \sum_{v_p(i) \geq -n} a_i(x)[\varepsilon]^i$.

We wish to extend this to our setting and construct maps $R_{K,n} : \widetilde{\Lambda}^{H_K}_{R,[0,b]} \to \varphi^{-n}\left(\Lambda_{R,[0,p^{-n}b],K}\right)$ for sufficiently small b.

We first consider the case where $K = F$ is unramified over $\mathbf{Q}_p$. In this case, $\mathbf{A}^+_F = \mathscr{O}_F[\![\pi]\!]$ and $a_i : \widetilde{\Lambda}^{H_F}_{[0,\infty]} \to \Lambda^{H_K}_{[0,\infty]}$ is $\Lambda^{H_K}_{[0,\infty]}$-linear for all i. We may extend a_i by linearity to a map $a_i : \widetilde{\Lambda}^{H_K}_{R_0,[0,\infty]} \to R_0 \mathbin{\widehat{\otimes}} \Lambda^{H_K}_{[0,\infty]}$. Now suppose $p^N \in uR_0$ and $bN < r_F = 1$, so $\frac{\pi}{[\overline{\pi}]}$ is a unit of $\widetilde{\Lambda}^{H_F}_{[0,b]}$ by [Col08, Lemme 6.5]. Then $\mathrm{Spa}\left(\widetilde{\Lambda}^{H_F}_{R_0,[0,b]}\right)$ can also be constructed as the rational localization $\mathrm{Spa}(R_0 \mathbin{\widehat{\otimes}} \mathbf{A}^{H_F}_{\mathrm{inf}})\left\langle \frac{u}{\pi^{s(b)}}\right\rangle$, since the image of $\widetilde{\Lambda}^{H_F}_{[0,bN]}$ is bounded in both $\widetilde{\Lambda}^{H_F}_{R_0,[0,b]}$ and $(R_0 \mathbin{\widehat{\otimes}} \mathbf{A}^{H_F}_{\mathrm{inf}})\left\langle \frac{u}{\pi^{s(b)}}\right\rangle\left[\frac{1}{\pi}\right]$. Thus, we can use the formula

$$a_i\left(\sum_{j=0}^{\infty} c_j \left(\frac{u}{\pi^{1/(b\cdot v_{\mathbf{C}_p^\flat}(\overline{\pi}))}}\right)^j\right) := \sum_{j=0}^{\infty} a_i(c_j)\left(\frac{u}{\pi^{1/(b\cdot v_{\mathbf{C}_p^\flat}(\overline{\pi}))}}\right)^j$$

to extend a_i to a map $a_i : \widetilde{\Lambda}^{H_F}_{R_0,[0,b]} \to \Lambda_{R_0,[0,b],F}$.

Since $(R_0 \mathbin{\widehat{\otimes}} \mathbf{A}^{H_F}_{\mathrm{inf}})\left\langle \frac{u}{\pi^{1/(b\cdot v_{\mathbf{C}_p^\flat}(\overline{\pi}))}}\right\rangle$ and $(R_0 \mathbin{\widehat{\otimes}} \mathscr{O}_F[\![\pi]\!])\left\langle \frac{u}{\pi^{1/(b\cdot v_{\mathbf{C}_p^\flat}(\overline{\pi}))}}\right\rangle$ are rings of definition of $\widetilde{\Lambda}^{H_F}_{R_0,[0,b]}$ and $\Lambda_{R_0,[0,b],F}$, respectively, both are bounded. More precisely, for $x \in \widetilde{\Lambda}^{H_F}_{[0,bN]}$, $v_{bN}(x) \geq v_{R,b}(x) \geq v_{bN}(x) - \frac{N-1}{bN}$, so

$$v_{R,b}(a_i(x)) \geq v_{R,b}(x) - c(R_0, b, F)$$

for $c(R_0, b, F) = \frac{N-1}{bN}$ (which depends only on $R_0 \subset R$ and b).

Now we wish to extend this construction to the case where $K/\mathbf{Q}_p$ is ramified; let $F' \subset K_\infty$ be the maximal unramified subextension. Then if $b < r_K$, the extension $\Lambda_{R_0,[0,b],K}/\Lambda_{R_0,[0,b],F'}$ is free of rank $e_K := [K_\infty : F'_\infty]$, so we may choose a basis $\{e_j\}$, define the trace $\mathrm{Tr}_{K/F'} : \Lambda_{R_0,[0,b],K} \to \Lambda_{R_0,[0,b],F'}$, and obtain the dual basis $\{e_j^*\}$ with respect to the pairing $(x, y) \mapsto \mathrm{Tr}(xy)$. For $x \in \widetilde{\Lambda}^{H_K}_{R,[0,b]}$, we may then write

$$x = \sum_j \mathrm{Tr}_{K/F'}(xe_j^*)e_j$$

and define

$$a_i(x) := \sum_j a_i(\mathrm{Tr}_{K/F'}(xe_j^*))e_j$$

This construction is independent of the choice of $\{e_j\}$ because a_i is $\Lambda_{R_0,[0,p^{-n}b],F'}$-linear. Since $v_{R,b}(e_j^*) \geq -v_{\mathbf{C}_p^\flat}(\mathfrak{d}_{\widehat{K}^\flat_\infty/\widehat{F}^\flat_\infty})$, we have

$$v_{R,b}(a_i(x)) \geq v_{R,b}(x) - c(R_0, b, K)$$

where $c(R_0, b, K) = c(R_0, b, F') - v_{\mathbf{C}_p^\flat}(\mathfrak{d}_{\widehat{K}_\infty^\flat/\widehat{F}_\infty^\flat})$.

As in the classical setting, we now define $R_{K,n} : \widetilde{\Lambda}_{R_0,[0,b]}^{H_K} \to \varphi^{-n}\left(\Lambda_{R_0,[0,p^{-n}b],K}\right)$ when $0 < bN < r_K$ and $p^{-n}b < r_K$, by setting

$$R_{K,n}(x) = \sum_{v_p(i) \geq -n} a_i(x)[\varepsilon]^i$$

It is clear from the construction that $R_{K,n}$ is a continuous $\varphi^{-n}\left(\Lambda_{R_0,[0,p^{-n}b],K}\right)$-linear section to the inclusion $\varphi^{-n}\left(\Lambda_{R_0,[0,p^{-n}b],K}\right) \to \Lambda_{R_0,[0,b]}^{H_F}$. In addition, the construction of a_i makes clear that $a_{\chi(\gamma)}(\gamma(x)) = \gamma(a_i(x))$ for $\gamma \in \Gamma_K$ and $x \in \widetilde{\Lambda}_{R_0,[0,b]}^{H_K}$, so $R_{K,n}$ commutes with the action of Γ_K.

Moreover, $R_{K,n} = \varphi^{-n} \circ R_{K,0} \circ \varphi^n$, so

$$\begin{aligned} v_{R,b}(R_{K,n}(x)) &\geq p^{-n} v_{R,p^n b}(a_0(\varphi^n(x)) \\ &\geq p^{-n}(v_{R,p^n b}(\varphi^n(x)) - c(R,b,K)) \\ &\geq v_{R,b}(x) - p^{-n} c(R,b,K) \end{aligned}$$

In summary, we have shown the following:

Proposition 4.4. *Suppose $0 < bN < r_K$ and $p^{-n}b < r_K$. Then there are constants $c(R_0, b, K)$ and a continuous $\varphi^{-n}\left(\Lambda_{R_0,[0,p^{-n}b],K}\right)$-linear map $R_{K,n} : \widetilde{\Lambda}_{R_0,[0,b]}^{H_K} \to \varphi^{-n}\left(\Lambda_{R_0,[0,p^{-n}b],K}\right)$ such that*

1. *$R_{K,n}$ is a section to the inclusion $\varphi^{-n}\left(\Lambda_{R_0,[0,p^{-n}b],K}\right) \to \widetilde{\Lambda}_{R_0,[0,b]}^{H_K}$*
2. *$R_{K,n}(x) \to x$ as $n \to \infty$ and $v_{R,b}(R_{K,n}(x)) \geq v_{R,b}(x) - p^{-n}c(R_0, b, K)$*
3. *$R_{K,n}$ commutes with the action of Γ_K.*

Corollary 4.5. *Suppose $p^N \in uR_0$ and $b > 0$ satisfies $bN < r_K$. Then for any $c_2 > 0$, the collection $\{\varphi^{-n}(\Lambda_{R_0,[0,b],K}), R_{K,n}\}$ satisfies the second Tate–Sen axiom for sufficiently large n (depending on the choice of c_2).*

4.2. **The action of Γ.** For any finite extension K/F, the cyclotomic character defines a homomorphism $\chi : \Gamma_K \to \mathbf{Z}_p^\times$. For any $\gamma \in \Gamma_K$ of infinite order, we let $n(\gamma) := v_p(\chi(\gamma) - 1) \in \mathbf{Z}_{\geq 0} \cup \{\infty\}$. Throughout this section, we let $N \geq 1$ be an integer such that $p^N \in uR_0$.

Lemma 4.6. *If $\gamma \in \Gamma_K$ has infinite order, then $\Lambda_{R_0,[0,0],K}^{\gamma=1} = \widetilde{\Lambda}_{R_0,[0,0]}^{H_K,\gamma=1} = R_0 \otimes \mathscr{O}_{F'}^{\gamma=1}$.*

Proof. We first consider the case where $p = 0$ in R. Then we can reduce modulo u and compute the subspaces of $(R_0/u) \widehat{\otimes} k_{F'}(\!(\pi_K)\!)$ and $(R_0/u) \widehat{\otimes} \widehat{K}_\infty^\flat$ fixed by γ. Since R_0/u is an $\mathbf{F}_p$-vector space, we may choose a basis $\{e_i\}_{i \in I}$ and write

$$(R_0/u) \widehat{\otimes} k_{F'}(\!(\pi_K)\!) \cong \{\sum_{i \in I} a_i e_i \mid a_i \in k_{F'}(\!(\pi_K)\!), a_i \to 0\}$$

and

$$(R_0/u) \widehat{\otimes} \widehat{K}_\infty^\flat \cong \{\sum_{i \in I} a_i e_i \mid a_i \in \widehat{K}_\infty^\flat, a_i \to 0\}$$

The action of Γ_K on R_0/u is trivial, so [Col08, Proposition 9.1] implies that

$$\left(\Lambda_{R_0,[0,0],K}/u\right)^{\gamma=1} = \left(\widetilde{\Lambda}_{R_0,[0,0]}^{H_K}/u\right)^{\gamma=1} = (R_0/u)\otimes k_{F'}^{\gamma=1}$$

Then we may approximate elements of $\left(\Lambda_{R_0,[0,0],K}\right)^{\gamma=1}$ and $\left(\widetilde{\Lambda}_{R_0,[0,0]}\right)^{H_K,\gamma=1}$ u-adically to obtain the desired result.

To bootstrap to the case where R_0 is $\mathbf{Z}_p$-flat, we may assume that $p\notin R^\times$ and again filter R_0 by the ideals $\{I_j\}$ where $I_j := p^jR\cap R_0$. Then (I_j/I_{j+1}) is a finite u-torsion-free $\mathbf{F}_p[\![u]\!]$-module, so we may apply the previous argument to calculate the γ-invariants of $(I_j/I_{j+1})\,\widehat{\otimes}\,k_{F'}(\!(\pi_K)\!)$ and $(I_j/I_{j+1})\,\widehat{\otimes}\,\widehat{K}_\infty^\flat$. Since $\cap_j I_j = (0)$, the result follows. ☐

Proposition 4.7. *If $\gamma\in\Gamma_K\smallsetminus\{1\}$ satisfies $\chi(\gamma)\in 1+2p\mathbf{Z}_p$, and if $bN<\inf\{p^{-1}r_K, p^{-n(\gamma)}\}$, then $1-\gamma$ admits a continuous inverse on $\Lambda_{R_0,[0,b],K}^{\psi=0}$.*

Proof. If $z\in\Lambda_{R_0,[0,b],K}^{\psi=0}$, it can be written uniquely in the form $z=\sum_{i=1}^{p-1}[\varepsilon]^i\varphi(z_i)$, where $z_i = a_{i/p}(\varphi^{-1}(z))\in\Lambda_{R_0,[0,0],K}$. In fact, $\varphi(z_i)\in\Lambda_{R_0,[0,b],K}$ and

$$\begin{aligned} v_{R,b}(\varphi(a_{i/p}(\varphi^{-1}(z)))) &\geq p\cdot v_{R,pb}(a_{i/p}(\varphi^{-1}(z)))\\ &\geq p(v_{R,pb}(\varphi^{-1}(z)) - c(R_0,b,K))\\ &\geq p(\frac{1}{p}v_{R,b}(z) - c(R_0,b,K)) = v_{R,b}(z) - pc(R_0,b,K)\end{aligned}$$

Since the image of $(R_0\,\widehat{\otimes}\,\Lambda_{[0,pbN]}^{\circ,H_K})\left\langle\frac{u}{\pi_K^{1/(pb\cdot v_{\mathbf{C}_p^\flat}(\overline{\pi}_K))}}\right\rangle$ is a ring of definition for $\Lambda_{R,[0,pb],K}$, we may extend $(1-\gamma)^{-1}$ to $[\varepsilon]^i\varphi\left(\Lambda_{R_0,[0,b],K}\right)$ by linearity, and therefore to $\Lambda_{R_0,[0,b],K}^{\psi=0}$. Using the continuity of $(1-\gamma)^{-1}$ on $\Lambda_{[0,b]}^{H_K,\psi=0}$ proven in [Col08, Proposition 9.6] and the bound above, we conclude that

$$v_{R,b}((1-\gamma)^{-1}(z)\geq v_{R,b}(z) - c'(R_0,b,K,n(\gamma))$$

for some constant $c'(R_0,b,K,n(\gamma))$, and is therefore continuous on $\Lambda_{R_0,[0,b],K}^{\psi=0}$. ☐

Remark 4.8. Using our explicit calculation of $c(R_0,b,K)$ and [Col08, Remarque 9.7], we see that when $n(\gamma)>n_0(K)$ and $p^{n(\gamma)}>2c_K(bN)$ (where $c_K(r)$ is a family of constants defined in [Col08]), we can take $c'(R_0,b,K,n(\gamma)) = pc(R_0,b,K) + p^{n(\gamma)}v_{\mathbf{C}_p^\flat}(\overline{\pi}) + \frac{N-1}{bN}$. In particular, we can conclude that $p^{-j}c'(R_0,p^{-j}b,K,n(\gamma))$ is bounded independent of j when $n(\gamma)\gg 0$.

Proposition 4.9. *If $b>0$ and $p^{-n}bN<\inf\{r_K,p^{1-n(\gamma)}\}$, and if $\gamma\in\Gamma_K$ satisfies $n(\gamma)\leq m$, then $\gamma-1$ is invertible on $X_{R_0,[0,b],K}^n := \ker(R_{K,n})$, and its inverse is continuous.*

Proof. As in the proof of [Col08, Proposition 9.9], we first observe that $\gamma-1$ is injective on $X_{R_0,[0,b],K}^m$, since $\widetilde{\Lambda}_{R_0,[0,0],K}^{\gamma=1} = \Lambda_{R_0,[0,0],K}^{\gamma=1}$.

If $x\in X_{R_0,[0,b],K}^m$, we can write $x=\sum_{j=m+1}^\infty\sum_{i\in I_j\smallsetminus I_{j+1}}a_i(x)$, and $v_{R,b}\left(\sum_{i\in I_j\smallsetminus I_{j+1}}a_i(x)\right)\geq v_{R,b}(x) - p^{1-j}(c(R_0,b,K))$. Furthermore, if $i\in I_j\smallsetminus I_{j-1}$, then $\varphi^j\left(a_i(x)[\varepsilon^i]\right)\in$

$[\varepsilon^{p^j i}]\varphi(\Lambda_{R_0,[0,0],K}) \subset \Lambda^{\psi=0}_{R_0,[0,0],K}$. This implies that

$$\varphi^j\left(\sum_{i\in I_j\smallsetminus I_{j+1}} a_i(x)\right) = (\gamma-1)z_j$$

where

$$v_{R,p^{-j}b}(z_j) \geq p^j v_{R,b}\left(\sum_{i\in I_j\smallsetminus I_{j+1}} a_i(x)\right) - c'(R_0,p^{-j}b,K,n(\gamma))$$

Thus, $v_{R,b}(\varphi^{-j}(z_j)) \geq v_{R,b}\left(\sum_{i\in I_j\smallsetminus I_{j+1}} a_i(x)\right) - p^{-j}c'(R_0,p^{-j}b,K,n(\gamma))$. Since $\sum_{i\in I_j\smallsetminus I_{j+1}} a_i(x)$ tends to 0 as $j\to\infty$ and $p^{-j}(c'(R_0,p^{-j}b,K,n(\gamma))$ is bounded independently of j, $\varphi^{-j}(z_j)\to 0$ as $j\to\infty$.

Thus, $z := \sum_{j=m+1}^{\infty}\varphi^{-j}(z_i)$ is an element of $\widetilde{\Lambda}^{H_K}_{R_0,[0,b]}$ satisfying $(\gamma-1)(z) = x$. Moreover,

$$v_{R,b}(z) \geq \inf_{j\geq n+1} v_{R,b}(\varphi^{-j}(z_j)) \geq v_{R,b}(x) - \sup_{j\geq n+1}\{p^{1-j}c(R_0,b,K) + p^{-j}c'(R_0,p^{-j}b,K,n(\gamma))\}$$

and the result follows. □

Corollary 4.10. *Suppose $p^n \in uR_0$ and $0 < bN < r_K$. Then for some $c_3 > 0$ (depending on $R_0 \subset R$), the third Tate–Sen axiom holds for the ring $\widetilde{\Lambda}^{H_K}_{R,[0,b]}$, the maps $R_{H,n}$, and the natural action of Γ_K.*

4.3. **The construction of (φ,Γ)-modules.** Now we may apply the arguments of [BC08] (working u-adically rather than p-adically) to construct (φ,Γ)-modules over rings $\Lambda_{R,[0,b],K}$. Let (R,R^+) be a pseudorigid Tate ring, and let M be a free R-module of rank d equipped with a continuous R-linear action of Gal_K.

Following [Che, Lemme 3.18], we first find a Galois-stable lattice in M.

Lemma 4.11. *Let R be a pseudoaffinoid $\mathbf{Z}_p$-algebra with noetherian ring of definition $R_0 \subset R$ and pseudo-uniformizer $u \in R$, and let M be a free R-module of rank d equipped with a continuous R-linear action of a compact topological group G. Then there is a formal scheme $\mathcal{Y} \to \mathrm{Spf}(R_0)$ and a $\mathscr{O}_{\mathcal{Y}}$-module $\mathscr{M}$ equipped with a continuous $\mathscr{O}_{\mathcal{Y}}$-linear action of G such that the natural map $\mathrm{Spa}(R) \to \mathrm{Spa}(R_0)$ factors through a morphism $f : \mathrm{Spa}(R) \to \mathcal{Y}^{\mathrm{ad}}$ and $M \cong f^*\mathscr{M}$.*

Proof. We first observe that there is a finitely generated R_0-module $M_0 \subset M$ such that $R \otimes_{R_0} M_0 = M_0[1/u] = M$. Indeed, we may simply consider a basis of M and let M_0 be the R_0-module it generates inside M.

Since R_0 is noetherian and u-adically complete, $\mathrm{Spf}\, R_0$ is an admissible formal scheme in the sense of [BL93], and the argument of [Che, Lemme 3.18] goes through verbatim to produce an admissible formal blow-up $\mathcal{Y} \to \mathrm{Spf}\, R_0$ and a locally free $\mathscr{O}_{\mathcal{Y}}$-module $\mathscr{M}$ equipped with a continuous $\mathscr{O}_{\mathcal{Y}}$-linear action of G, such that $M = \Gamma(\mathcal{Y},\mathscr{M})\left[\frac{1}{u}\right]$.

It remains to see that $\mathrm{Spa}(R) \to \mathrm{Spa}(R_0)$ factors through a morphism $f : \mathrm{Spa}(R) \to \mathcal{Y}$. If $\mathcal{Y}$ is the blow-up of $\mathrm{Spf}\, R_0$ along $I = (f_1,\ldots,f_r) \subset R_0$, then $\mathcal{Y}$ has a cover of the form $\left\{\mathrm{Spf}\, R_0\left\langle\frac{f_1,\ldots,f_r}{f_i}\right\rangle\right\}_i$. The ring $R_0\left\langle\frac{f_1,\ldots,f_r}{f_i}\right\rangle$ is a noetherian ring of definition for the rational localization $U_i := \mathrm{Spa}(R)\left\langle \frac{f_1,\ldots,f_r}{f_i}\right\rangle \subset \mathrm{Spa}(R)$, so the natural morphism $\mathrm{Spa}(R)|_{U_i} \to \mathrm{Spa}(R_0)$ factors through $\mathcal{Y}^{\mathrm{ad}}$ for each i. Since these morphisms agree on overlaps by construction, we are done. $\square$

Now we can construct (φ,Γ)-modules, exactly as in [BC08].

Theorem 4.12. *Suppose $p^N \in uR_0$ and choose $b > 0$ such that $b < 1/N$ and constants c_1, c_2, c_3 as in the statement of the Tate–Sen axioms. Then there is some finite Galois extension L/K and some integer $n \geq 0$ such that $\widetilde{\Lambda}_{R_0,[0,b]} \otimes_R M$ contains a unique projective sub-$\varphi^{-n}\left(\Lambda_{R,[0,p^{-n}b],L}\right)$-module $D_{b,L,n}(M)$ such that*

- *$D_{b,L,n}(M)$ is stable by Gal_L and fixed by H_L,*
- *The natural map $\widetilde{\Lambda}_{R_0,[0,b]} \otimes_{\varphi^{-n}\left(\Lambda_{R,[0,p^{-n}b],L}\right)} D_{b,L,n}(M) \to \widetilde{\Lambda}_{R_0,[0,b]} \otimes_R M$ is an isomorphism*
- *Locally on $\mathrm{Spa}(R)$, $D_{b,L,n}(M)$ admits a basis which is c_3-fixed.*

Proof. After localizing on $\mathrm{Spa}(R)$, we may assume that M has a Galois-stable R_0-lattice $M_0 \subset M$. Let k be an integer such that $v_{R,b}(u^k) > c_1+2c_2+2c_3$, and let L/K be a finite Galois extension such that Gal_L acts trivially on M_0/u^kM_0. Then by Corollary 4.2, $(\widetilde{\Lambda}_{R_0,[0,1]} \otimes_{R_0} M_0)^{H_L}$ is a free $\widetilde{\Lambda}^{H_L}_{R_0,[0,1]}$-module of rank d. We choose a basis and let $\sigma \mapsto U_\sigma$ denote the corresponding cocycle. The proof of [BC08, Proposition 3.2.6] carries over nearly verbatim (working modulo powers of u rather than p), and yields $B \in 1 + u^k \mathrm{Mat}_d(\widetilde{\Lambda}_{R_0,[0,b]})$ such that $v_{R,b}(B-1) > c_2 + c_3$ and $\sigma \mapsto B^{-1}U_\sigma\sigma(B)$ is trivial on H_L and valued in $\mathrm{Mat}(\varphi^{-n(G)}(\Lambda_{R_0,[0,p^{-n(G)}b],L}))$. This shows the existence of $D_{b,L,n}(M)$; it remains to check that it is unique.

Suppose there are two such submodules; we may choose bases and obtain corresponding cocycles $\sigma \mapsto W_\sigma$ and $\sigma \mapsto W'_\sigma$ valued in $\mathrm{Mat}(\varphi^{-n(G)}(\Lambda_{R_0,[0,p^{-n(G)}b],L}))$. Since these submodules generate the same $\widetilde{\Lambda}_{R_0,[0,b]}$-module, there is some matrix $C \in \mathrm{Mat}\,\widetilde{\Lambda}_{R_0,[0,b]}$ such that $W'_\sigma = C^{-1}W_\sigma(C)$. But [BC08, Proposition 3.2.5] also carries over nearly verbatim, and shows that C actually has coefficients in $\varphi^{-n(G)}(\Lambda_{R_0,[0,p^{-n}b],L}))$. $\square$

Definition 4.13. Let M be a rank-d representation of Gal_K with coefficients in R which admits a Gal_K-stable R_0-lattice, and choose $0 < b < 1/N$, where $p^N \in uR_0$. Then we define

(1) $D_{b,L}(M) := \varphi^{n(L)}\left(D_{p^{n(L)}b,L,n}\right)$
(2) If $I \subset [0,b]$ is an interval (which may have open endpoints), we define $D_{I,L}(M) := \Lambda_{R,I,L} \otimes_{\Lambda_{R,[0,b],L}} D_{b,L}(M)$
(3) $D_{\mathrm{rig},L}(M) := \varinjlim_{b\to 0} D_{(0,b]}(M)$

If L is clear from context, we often drop it from the notation.

Remark 4.14. The modules $D_{I,L}(M)$ are equipped with descent data for $\mathrm{Gal}_{L/K}$; to construct $D_{b,L,n}(M)$, we had to restrict from Gal_K to Gal_L, and the uniqueness of $D_{b,L,n}(M)$ ensures that the construction is functorial.

5. Cohomology of (φ, Γ)-modules

We now assume $p > 2$. Given a representation $\rho : \mathrm{Gal}_K \to \mathrm{GL}_d(R)$, we wish to study the cohomology of the associated (φ, Γ)-module.

Definition 5.1. A φ-module over $\Lambda_{R,(0,b],K}$ is a coherent sheaf D of modules over the pseudorigid space $\bigcup_{a\to 0} \mathrm{Spa}(\Lambda_{R,[a,b],K}, \Lambda^+_{R,[a,b],K})$ equipped with an isomorphism

$$\varphi_D : \varphi^* D \xrightarrow{\sim} \Lambda_{R,(0,b/p],K} \otimes_{\Lambda_{R,(0,b],K}} D$$

If $a \in (0, b/p]$, a φ-module over $\Lambda_{R,[a,b],K}$ is a finite $\Lambda_{R,[a,b],K}$-module D equipped with an isomorphism

$$\varphi_{D,[a,b/p]} : \Lambda_{R,[a,b/p],K} \otimes_{\Lambda_{R,[a/p,b/p],K}} \varphi^* D \xrightarrow{\sim} \Lambda_{R,[a,b/p],K} \otimes_{\Lambda_{R,[a,b],K}} D$$

A (φ, Γ_K)-module over $\Lambda_{R,(0,b],K}$ (resp. $\Lambda_{R,[a,b],K}$) is a φ-module over $\Lambda_{R,(0,b],K}$ (resp. $\Lambda_{R,[a,b],K}$) equipped with a semi-linear action of Γ_K which commutes with φ_D (resp. $\varphi_{D,[a,b/p]}$).

A (φ, Γ_K)-module over R is a module D over $\varinjlim_{b\to 0} \Lambda_{R,(0,b],K}$ which arises via base change from a (φ, Γ_K)-module over $\Lambda_{R,(0,b],K}$ for some $b > 0$.

Definition 5.2. If L/K is a finite extension, and D is a (φ, Γ_L)-module over $\Lambda_{R,[a,b],L}$ (resp. $\Lambda_{R,(0,b],L}$), we define the induced (φ, Γ_K)-module

$$\mathrm{Ind}_L^K(D) := \mathrm{Ind}_{\Gamma_L}^{\Gamma_K}(D) = \{f : \Gamma_K \to D \mid f(hg) = h \cdot f(g) \text{ for } h \in \Gamma_L\}$$

If we equip $\mathrm{Ind}_L^K(D)$ with a $\Lambda_{R,[a,b],K}$-module (resp. $\Lambda_{R,(0,b],K}$-module) structure via $(a \cdot f)(g) := g(a)f(g)$ and an action of φ via $(\varphi(f))(g) := \varphi(f(g))$ (for $f \in \mathrm{Ind}_L^K(D)$ and $g \in \Gamma_K$), $\mathrm{Ind}_L^K(D)$ becomes a (φ, Γ_K)-module.

We further define a category of $(\varphi, \Gamma_L, \mathrm{Gal}_{L/K})$-modules.

Definition 5.3. If L/K is a finite Galois extension and D is a (φ, Γ_L)-module, we say that D is *equipped with an action of* $\mathrm{Gal}_{L/K}$ if the Galois group Gal_K acts on D and in addition

- the subgroup $H_L \subset \mathrm{Gal}_K$ acts trivially on D, and
- the induced action of Gal_L / H_L coincides with the action of Γ_L.

We also say that D is a $(\varphi, \Gamma_L, \mathrm{Gal}_{L/K})$-*module*.

We will be particularly interested in projective $(\varphi, \Gamma_L, \mathrm{Gal}_{L/K})$-modules; as p need not be invertible in our coeffcient rings, we may encounter (φ, Γ_K)-modules which are H_K-invariants of projective (φ, Γ_L)-modules, but are not themselves projective.

Let $\Delta_K \subset \Gamma_K$ be its torsion subgroup (which has order prime to p, since $p > 2$), and let $\gamma \in \Gamma_K$ be a topological generator of the quotient Γ_K/Δ_K. Then for a (φ, Γ)-module D, we define the Fontaine–Herr–Liu complex via

$$C^\bullet_{\varphi,\Gamma} : D^{\Delta_K} \xrightarrow{\varphi_D - 1, \gamma - 1} D^{\Delta_K} \oplus D^{\Delta_K} \xrightarrow{(\gamma-1)\oplus(1-\varphi_D)} D^{\Delta_K}$$

(concentrated in degrees 0, 1, and 2). We let $H^i_{\varphi,\Gamma_K}(D)$ denote its cohomology in degree i. If D is the (φ, Γ)-module attached to a Galois representation M, this complex computes the Galois cohomology of M.

5.1. **Finiteness of cohomology.** We wish to show that the cohomology of $C^\bullet_{\varphi,\Gamma}$ is R-finite. To do this, we will apply [KL, Lemma 1.10] to the morphisms of complexes

$$\begin{tikzcd} D^{\Delta_K}_{[a,b]} \arrow[r] \arrow[d] & D^{\Delta_K}_{[a,b/p]} \oplus D^{\Delta_K}_{[a,b]} \arrow[r] \arrow[d] & D^{\Delta_K}_{[a,b/p]} \arrow[d] \\ D^{\Delta_K}_{[a',b']} \arrow[r] & D^{\Delta_K}_{[a',b'/p]} \oplus D^{\Delta_K}_{[a',b']} \arrow[r] & D^{\Delta_K}_{[a',b'/p]} \end{tikzcd}$$

induced by the natural homomorphisms $\Lambda_{R,[a,b],K'} \to \Lambda_{R,[a',b'],K'}$ (where $[a',b'] \subset (a,b)$). More precisely, Kedlaya–Liu show that if the morphisms $D^{\Delta_K}_{[a,b]} \to D^{\Delta_K}_{[a',b']}$ are completely continuous and induce isomorphisms on cohomology groups, then both complexes have R-finite cohomology. Since $C^\bullet_{\varphi,\Gamma}$ is the direct limit (as $b \to 0$) of the inverse limit of these complexes (as $a \to 0$), with transition maps which are quasi-isomorphisms, this will imply that $C^\bullet_{\varphi,\Gamma}$ has R-finite cohomology.

Definition 5.4. Let A be a Banach algebra, and let $f : M \to N$ be a morphism of Banach A-modules (equipped with norms $|\cdot|_M$ and $|\cdot|_N$, respectively). We say that f is *completely continuous* if there exists a sequence of finite A-submodules N_i of N such that the operator norms of the compositions $M \to N \to N/N_i$ tend to 0 (where N/N_i is equipped with the quotient semi-norm)

Definition 5.5. Let $f : (A, A^+) \to (A', A'^+)$ be a localization of complete Tate rings over a complete Tate ring (B, B^+). We say that f is *inner* if there is a strict B-linear surjection $B\langle \underline{X}\rangle \twoheadrightarrow A$ such that each element of $\underline{X}$ maps to a topologically nilpotent element of A'. Here $\underline{X}$ is a (possibly infinite) collection of formal variables.

If B is a nonarchimedean field of mixed characteristic and A and A' are topologically of finite type over B, Kiehl proved that inner homomorphisms are completely continuous. We prove the analogous result, using the definition of complete continuity found in [KL].

Proposition 5.6. *If $[a',b'] \subset (a,b)$ and $[a,b] \subset (0,\infty)$, then the map $\Lambda_{R,[a,b],K} \to \Lambda_{R,[a',b'],K}$ induced by restriction is completely continuous.*

Proof. The pairs $(\Lambda_{R,[a,b],K}, \Lambda^+_{R,[a,b],K})$ and $(\Lambda_{R,[a',b'],K}, \Lambda^+_{R,[a',b'],K})$ are localizations of $(R_0 \otimes \mathscr{O}_{F'}[\![\pi_K]\!], R_0 \otimes \mathscr{O}_{F'}[\![\pi_K]\!])$; since $[a',b'], [a,b] \subset (0,\infty)$, they are adic affinoid algebras over (R, R^+). Since $[a',b'] \subset (a,b)$, the natural restriction map is inner. Then [KL, Lemma 5.7] implies that it is completely continuous. □

Lemma 5.7. *Suppose $a \in (0, b/p]$. Then the functor $D \rightsquigarrow \Lambda_{R,[a,b],K} \otimes_{\Lambda_{R,(0,b],K}} D =: D_{[a,b]}$ induces an equivalence of categories between φ-modules over $\Lambda_{R,(0,b],K}$ and φ-modules over $\Lambda_{R,[a,b],K}$.*

Proof. Suppose we have a φ-module $D_{[a,b]}$ over $\Lambda_{R,[a,b],K}$. Then the Frobenius pullback $\varphi^* D_{[a,b]}$ is a finite module over $\Lambda_{R,[a/p,b/p],K}$, and the isomorphism $\varphi_{D,[a,b/p]}$: $\Lambda_{R,[a,b/p],K} \otimes_{\Lambda_{R,[a/p,b/p],K}} \varphi^* D_{[a,b]} \xrightarrow{\sim} \Lambda_{R,[a,b/p],K} \otimes_{\Lambda_{R,[a,b],K}} D_{[a,b]}$ (and the assumption that $a \leq b/p$) provides a descent datum. Thus, we may construct a finite module $D_{[a/p,b]}$ over $\Lambda_{R,[a/p,b],K}$ which restricts to $D_{[a,b]}$.

To show that $D_{[a/p,b]}$ is a φ-module over $\Lambda_{R,[a/p,b],K}$, we need to construct an isomorphism

$$\varphi_{D,[a/p,b/p]} : \Lambda_{R,[a/p,b/p],K} \otimes_{\Lambda_{R,[a/p^2,b/p],K}} \varphi^* D_{[a/p,b]} \xrightarrow{\sim} \Lambda_{R,[a/p,b/p],K} \otimes_{\Lambda_{R,[a/p,b],K}} D_{[a/p,b]}$$

By construction, we have an isomorphism

$$\varphi_{D,[a,b/p]} : \varphi^* D_{[a,b]} \xrightarrow{\sim} \Lambda_{R,[a/p,b/p],K} \otimes_{\Lambda_{R,[a/p,b],K}} D_{[a/p,b]}$$

and if we pull $\varphi_{D,[a,b/p]}$ back by Frobenius, we obtain an isomorphism

$$\varphi_{D,[a/p,b/p^2]} : \Lambda_{R,[a/p,b/p^2],K} \otimes_{\Lambda_{R,[a/p^2,b/p^2],K}} \varphi^* D_{[a/p,b/p]} \xrightarrow{\sim} \Lambda_{R,[a/p,b/p^2],K} \otimes_{\Lambda_{R,[a/p,b/p],K}} D_{[a/p,b/p]}$$

On the overlap, they induce the same isomorphism $\varphi^* D_{[a,b/p]} \to \Lambda_{R,[a/p,b/p^2],K} \otimes_{\Lambda_{R,[a/p,b/p],K}} D_{[a/p,b/p]}$ (by construction), so we obtain the desired isomorphism $\varphi_{D,[a/p,b/p]}$.

Iterating this construction lets us construct a φ-module over $\Lambda_{R,(0,b],K}$.

This proves essential surjectivity; full faithfulness follows because the natural maps $\Lambda_{R,(0,b],K} \to \Lambda_{R,[a,b],K}$ have dense image. □

Corollary 5.8. *If D is a φ-module over $\Lambda_{R,(0,b],K}$, the morphism of complexes*

$$[D \xrightarrow{\varphi-1} D_{(0,b]}] \to [D_{[a,b]} \xrightarrow{\varphi-1} D_{[a,b/p]}]$$

is a quasi-isomorphism for any $a \in (0, b/p]$.

Proof. This follows from the previous lemma because we may interpret the cohomology groups as Yoneda Ext groups. □

In order to prove that the restriction map $D_{(0,b]} \to D_{(0,b/p]}$ induces an isomorphism on cohomology, we will need the ψ operator. The isomorphism $\varphi^* D_{(0,b]} \xrightarrow{\sim} D_{(0,b/p]}$ induces an isomorphism

$$\Lambda_{R,(0,b/p],K} \otimes_{\varphi(\Lambda_{R,(0,b],K})} \varphi(D_{(0,b]}) \xrightarrow{\sim} D_{(0,b/p]}$$

We therefore have a surjective homomorphism $\psi : D_{(0,b/p]} \to D_{(0,b]}$ defined by setting $\psi(a \otimes \varphi(d)) = \psi(a)d$, where $a \in \Lambda_{R,(0,b/p],K}$ and $d \in D_{(0,b]}$.

Lemma 5.9. *Let D be a (φ, Γ)-module over $\Lambda_{R,(0,b],K}$ for some $b > 0$. Then there is some $0 < b' \leq b$ such that the action of $\gamma - 1$ on $(D_{(0,b']})^{\psi=0}$ admits a continuous inverse.*

Proof. We may replace D with $\mathrm{Ind}_K^{\mathbf{Q}_p}(D)$. Since $(D_{(0,b/p^n]})^{\psi=0} = \oplus_{j\in(\mathbf{Z}/p^n)^\times} [\varepsilon]^{\tilde{j}} \varphi^n(D)$, it suffices to show that $\gamma - 1$ has a continuous inverse on $[\varepsilon]^j \varphi^n(D)$ for j prime to p and sufficiently large n. Moreover, since $\gamma^n - 1 = (\gamma - 1)(\gamma^{n-1} + \cdots + 1)$, we may replace $\Gamma_{\mathbf{Q}_p}$ with a finite-index subgroup.

If $\gamma_n \in \Gamma_{\mathbf{Q}_p}$ is such that $\chi(\gamma_n) = 1 + p^n$, then

$$\begin{aligned}
\gamma_n \left([\varepsilon]^j \varphi^n(x)\right) - [\varepsilon]^j \varphi^n(x) &= [\varepsilon]^j [\varepsilon]^{p^n j} \varphi^n(\gamma_n(x)) - [\varepsilon]^j \varphi^n(x) \\
&= [\varepsilon]^j \varphi^n([\varepsilon]^j \gamma_n(x) - x) \\
&= [\varepsilon]^j \varphi^n(G_{\gamma_n}(x))
\end{aligned}$$

where $G_{\gamma_n}(x) := [\varepsilon]^j\gamma_n(x) - x = ([\varepsilon]^j - 1)\cdot\left(1 + \frac{[\varepsilon]^j}{[\varepsilon]^j-1}(\gamma_n - 1)\right)(x)$. Thus, if we can choose n such that $\sum_{k=0}^{\infty}\left(-\frac{[\varepsilon]^j}{[\varepsilon]^j-1}(\gamma_n-1)\right)^k$ converges on $D_{(0,b]}$, we will be done.

The action of $\Gamma_{\mathbf{Q}_p}$ on $D_{[b/p,b]}$ is continuous, so we may choose n_0 such that for $n \geq n_0$, the sum above converges in $\mathrm{End}(D_{[b/p,b]})$. But $D_{[b/p^{k+1},b/p^k]} \cong \varphi^* D_{[b/p^k,b/p^{k-1}]}$ for all $k \geq 0$ and the action of Γ commutes with the action of φ, so the sum converges in $\mathrm{End}(D_{[b/p^{k+1},b/p^k]})$ for all $k \geq 0$ and $\gamma - 1$ acts invertibly on $D_{(0,b/p^n]}^{\psi=0}$ for $n \geq n_0$. $\square$

Proposition 5.10. *If D is a (φ,Γ)-module over R for some $b > 0$, then the cohomology of D is computed by*

$$D_{[a,b]}^{\Delta_K} \xrightarrow{\varphi-1,\gamma-1} D_{[a,b/p]}^{\Delta_K} \oplus D_{[a,b]}^{\Delta_K} \xrightarrow{(\gamma-1)\oplus(1-\varphi)} D_{[a,b/p]}^{\Delta_K}$$

for some sufficiently small b and any $a \in (0, b/p]$.

Proof. We may assume that D is a (φ,Γ)-module over $\Lambda_{R,(0,b],K}$ for some $b > 0$. Since $[D_{(0,b]}^{\Delta_K} \xrightarrow{\varphi-1} D_{(0,b/p]}^{\Delta_K}] \to [D_{[a,b]}^{\Delta_K} \xrightarrow{\varphi-1} D_{[a,b/p]}^{\Delta_K}]$ induces an isomorphism on cohomology, we see that the cohomology of $D_{(0,b]}$ is computed by the above complex.

Since the cohomology of $C^\bullet_{\varphi,\Gamma}(D)$ is computed by the direct limit of the cohomology groups of $C^\bullet_{\varphi,\Gamma}(D_{(0,b/p^n]})$ as $n \to \infty$, it suffices to show that the natural morphism

$$\begin{tikzcd}
C^\bullet_{(0,b]}: & D_{(0,b]}^{\Delta_K} \arrow[r] \arrow[d] & D_{(0,b/p]}^{\Delta_K} \oplus D_{(0,b]}^{\Delta_K} \arrow[r] \arrow[d] & D_{(0,b/p]}^{\Delta_K} \arrow[d] \\
C^\bullet_{(0,b/p]}: & D_{(0,b/p]}^{\Delta_K} \arrow[r] & D_{(0,b/p^2]}^{\Delta_K} \oplus D_{(0,b/p]}^{\Delta_K} \arrow[r] & D_{(0,b/p^2]}^{\Delta_K}
\end{tikzcd}$$

induces an isomorphism on cohomology groups for sufficiently small b.

The maps $\psi : D_{(0,b/p]} \to D_{(0,b]}$ and $\psi : D_{(0,b/p^2]} \to D_{(0,b/p]}$ induce a surjection of complexes $\Psi : C^\bullet_{(0,b/p]} \to C^\bullet_{(0,b]}$ such that the composition $C^\bullet_{(0,b]} \xrightarrow{\varphi} C^\bullet_{(0,b/p]} \xrightarrow{\Psi} C^\bullet_{(0,b]}$ is the identity. By Lemma 5.9, if b is sufficiently small, $\gamma - 1$ has a continuous inverse on $D_{(0,b/p]}^{\psi=0}$. Thus, we have a decomposition $C^\bullet_{(0,b/p]} \cong \varphi(C^\bullet_{(0,b]}) \oplus K^\bullet$, where $K^\bullet := \ker\Psi$, and $K^\bullet$ has vanishing cohomology.

In order to show that the base change map $C^\bullet_{(0,b]} \to C^\bullet_{(0,b/p]}$ is a quasi-isomorphism, it therefore suffices to show that $\mathrm{id}, \varphi : C^\bullet_{(0,b]} \rightrightarrows C^\bullet_{(0,b/p]}$ are homotopic. But this follows by considering the diagram

$$\begin{tikzcd}
D_{(0,b]}^{\Delta_K} \arrow[r, "(\varphi-1,\gamma-1)"] \arrow[d, shift right, "\mathrm{id}"'] \arrow[d, shift left, "\varphi"] & D_{(0,b/p]}^{\Delta_K} \oplus D_{(0,b]}^{\Delta_K} \arrow[r, "(\gamma-1)\oplus(1-\varphi)"] \arrow[d, shift right, "\mathrm{id}"'] \arrow[d, shift left, "\varphi"] \arrow[dl, "\mathrm{pr}_1"'] & D_{(0,b/p]}^{\Delta_K} \arrow[d, shift right, "\mathrm{id}"'] \arrow[d, shift left, "\varphi"] \arrow[dl, "(0,-\mathrm{id})"'] \\
D_{(0,b/p]}^{\Delta_K} \arrow[r, "(\varphi-1,\gamma-1)"] & D_{(0,b/p^2]}^{\Delta_K} \oplus D_{(0,b/p]}^{\Delta_K} \arrow[r, "(\gamma-1)\oplus(1-\varphi)"] & D_{(0,b/p^2]}^{\Delta_K}
\end{tikzcd}$$

$\square$

Corollary 5.11. *If D is a (φ,Γ)-module over $\Lambda_{R,(0,b],K}$ and $[a',b']\subset[a,b]$, the restriction map*

$$\begin{tikzcd} D_{[a,b]} \arrow[r] \arrow[d] & D_{[a,b/p]}\oplus D_{[a,b]} \arrow[r] \arrow[d] & D_{[a,b/p]} \arrow[d] \\ D_{[a',b']} \arrow[r] & D_{[a',b'/p]}\oplus D_{[a',b']} \arrow[r] & D_{[a',b'/p]} \end{tikzcd}$$

induces an isomorphism on cohomology.

Proof. We may assume that $b'\in[b/p,b]$, so that we have induced homomorphisms

$$H^i_{\varphi,\Gamma}(D_{(0,b]})\to H^i_{\varphi,\Gamma}(D_{(0,b']})\to H^i_{\varphi,\Gamma}(D_{(0,b/p]})\to H^i_{\varphi,\Gamma}(D_{(0,b'/p]})$$

Since the compositions $H^i_{\varphi,\Gamma}(D_{(0,b]})\to H^i_{\varphi,\Gamma}(D_{(0,b/p]})$ and $H^i_{\varphi,\Gamma}(D_{(0,b']})\to H^i_{\varphi,\Gamma}(D_{(0,b'/p]})$ are isomorphisms, the homomorphism $H^i_{\varphi,\Gamma}(D_{(0,b']})\to H^i_{\varphi,\Gamma}(D_{(0,b/p]})$ is also an isomorphism, and we are done. □

Now we can finally prove that (φ,Γ)-modules have finite cohomology.

Theorem 5.12. *If D is a (φ,Γ)-module over $\Lambda_{R,(0,b],K}$, its cohomology is R-finite.*

Proof. If $[a',b']\subset(a,b)$, the restriction map induces a quasi-isomorphism

$$\begin{tikzcd} D_{[a,b]} \arrow[r] \arrow[d] & D_{[a,b/p]}\oplus D_{[a,b]} \arrow[r] \arrow[d] & D_{[a,b/p]} \arrow[d] \\ D_{[a',b']} \arrow[r] & D_{[a',b'/p]}\oplus D_{[a',b']} \arrow[r] & D_{[a',b'/p]} \end{tikzcd}$$

which is completely continuous. Then the result follows, by [KL, Lemma 1.10] □

Corollary 5.13. *If D is a projective (φ,Γ)-module over R, then $C^\bullet_{\varphi,\Gamma_K}(D)\in\mathbf{D}^{[0,2]}_{\mathrm{perf}}(R)$.*

Proof. Finiteness of the cohomology of $C^\bullet_{\varphi,\Gamma_K}(D)$ implies that $C^\bullet_{\varphi,\Gamma_K}(D)\in\mathbf{D}^-_{\mathrm{perf}}(R)$, and by Proposition 3.37, the complex $C^\bullet_{\varphi,\Gamma_K}(D)$ consists of flat A-modules. Then as in the proof of [KPX14, Theorem 4.4.5(1)], it follows that $C^\bullet_{\varphi,\Gamma_K}(D)\in\mathbf{D}^{[0,2]}_{\mathrm{perf}}(R)$. □

Corollary 5.14. *If D is a projective (φ,Γ)-module over R, then the cohomology groups $H^i_{\varphi,\Gamma}(D)$ are coherent sheaves on* $\mathrm{Spa}(R)$.

Proof. Since $C^\bullet_{\varphi,\Gamma}(D)\in D^b_{\mathrm{coh}}(R)$, we have a quasi-isomorphism $R'\otimes^{\mathbf{L}}_R C^\bullet_{\varphi,\Gamma}(D)\xrightarrow{\sim} C^\bullet_{\varphi,\Gamma}(R'\otimes_R D)$ for any homomorphism $R\to R'$ of pseudoaffinoid algebras. If $R\to R'$ defines an affinoid subspace of $\mathrm{Spa}(R)$, the morphism is flat and the derived tensor product is an ordinary tensor product. On the other hand, we have a natural homomorphism $C^\bullet_{\varphi,\Gamma}(R'\otimes_R D)\to C^\bullet_{\varphi,\Gamma}(R'\widehat{\otimes}_R D)$, and it is a quasi-isomorphism after every specialization $R'\twoheadrightarrow S$ to a finite-length algebra (since D is flat over R). Since quasi-isomorphisms can be checked on finite-length specializations, the result follows. □

5.2. **Galois cohomology.** If M is a $\mathbf{Q}_p$-linear representation of Gal_K and $D_{\mathrm{rig},K}(M)$ is the associated Galois representation over $\varinjlim_{b\to 0}\Lambda_{(0,b],K}$, then we have a canonical quasi-isomorphism $R\Gamma(\mathrm{Gal}_K, M)\xrightarrow{\sim} C^\bullet_{\varphi,\Gamma}$ between (continuous) Galois cohomology and Fontaine–Herr–Liu cohomology [Liu07, Theorem 2.3]. The same result holds for families of projective Galois representations with coefficients in classical $\mathbf{Q}_p$-affinoid algebras [Pot13, Theorem 2.8]; we wish to prove the corresponding result for Galois representations with pseudoaffinoid coefficients. The key is an Artin–Schreier calculation:

Proposition 5.15. *Let R be a pseudoaffinoid algebra with ring of definition R_0 and pseudo-uniformizer $u\in R_0$. Then for any $b>0$, there is an exact sequence of R_0-modules*

$$0\to R_0\to \widetilde{\Lambda}_{R_0,[0,b],0}\xrightarrow{\varphi-1}\widetilde{\Lambda}_{R_0,[0,b/p],0}\to 0$$

We first compute modulo u:

Lemma 5.16. *If R is a pseudoaffinoid algebra and $R_0\subset R$ is a ring of definition formally of finite type over $\mathbf{Z}_p[\![u]\!]$, then for any $b\in(0,\infty]$ we have an exact sequence*

$$0\to R_0/u\to \widetilde{\Lambda}_{R_0,[0,b],0}/u\xrightarrow{\varphi-1}\widetilde{\Lambda}_{R_0,[0,b/p],0}/u\to 0$$

Proof. We may write $\widetilde{\Lambda}_{R_0,[0,b],0}/u\cong(R_0\widehat{\otimes}\mathbf{A}_{\mathrm{inf}})[Y]/([\varpi]^{1/b}Y,u)$ and $\widetilde{\Lambda}_{R_0,[0,b/p],0}/u\cong(R_0\widehat{\otimes}\mathbf{A}_{\mathrm{inf}})[Y']/([\varpi]^{p/b}Y',u)$; the map $\varphi:\widetilde{\Lambda}_{R_0,[0,b],0}/u\to\widetilde{\Lambda}_{R_0,[0,b/p],0}/u$ carries Y to Y', and the identity map carries Y to $[\varpi]^{(p-1)/b}Y'$. There is some $N\geq 1$ such that $p^N\in uR_0$, and we may filter R_0/u by powers of p; if we reduce modulo p, $R_0/(u,p)$ is an $\mathbf{F}_p$-vector space, and it suffices to prove that the sequence

$$0\to\mathbf{F}_p\to\mathscr{O}^\flat_{\mathbf{C}_p}[Y]/(\varpi^{1/b}Y)\xrightarrow{\varphi-1}\mathscr{O}^\flat_{\mathbf{C}_p}[Y']/(\varpi^{p/b}Y')\to 0$$

is exact.

Given a polynomial $f(Y):=\sum_i a_iY^i\in\mathscr{O}^\flat_{\mathbf{C}_p}[Y]$,

$$(\varphi-1)(f(Y))=\sum_i(\varphi(a_i)-\varpi^{i(p-1)/b}a_i)Y'^i$$

To compute the kernel of $\varphi-1$, we may assume that $v_{\mathbf{C}_p^\flat}(a_i)<\frac{1}{b}$ for all i with $a_i\neq 0$, and that $v_{\mathbf{C}_p^\flat}(\varphi(a_i)-\varpi^{i(p-1)/b}a_i)\geq\frac{p}{b}$ for $i\geq 1$. We have $v_{\mathbf{C}_p^\flat}(\varphi(a_i)-\varpi^{i(p-1)/b}a_i)\geq\min\{pv_{\mathbf{C}_p^\flat}(a_i),\frac{i(p-1)}{b}+v_{\mathbf{C}_p^\flat}(a_i)\}$, with equality unless $v_{\mathbf{C}_p^\flat}(a_i)=\frac{i}{b}$. If $i\geq 1$, this contradicts the assumption that $v_{\mathbf{C}_p^\flat}(a_i)<\frac{1}{b}$, so in that case $v_{\mathbf{C}_p^\flat}(\varphi(a_i)-\varpi^{i(p-1)/b}a_i)\geq\frac{p}{b}$ implies $\min\{pv_{\mathbf{C}_p^\flat}(a_i),\frac{i(p-1)}{b}+v_{\mathbf{C}_p^\flat}(a_i)\}\geq\frac{p}{b}$. But $pv_{\mathbf{C}_p^\flat}(a_i)<\frac{p}{b}$ by assumption, so this is impossible. Thus, if $f(Y)$ represents an element of the kernel of $\varphi-1$, its coefficients in positive degree have valuation at least $\frac{1}{b}$, and therefore vanish in $\mathscr{O}^\flat_{\mathbf{C}_p}[Y]/(\varpi^{1/b}Y)$. Thus, we may assume that $f(Y)\in\mathscr{O}^\flat_{\mathbf{C}_p}$, and therefore that it is an element of $\mathbf{F}_p$.

To see that $\varphi-1:\mathscr{O}^\flat_{\mathbf{C}_p}[Y]/(\varpi^{1/b}Y)\to\mathscr{O}^\flat_{\mathbf{C}_p}[Y']/(\varpi^{p/b}Y')$ is surjective, we may lift an element of $\mathscr{O}^\flat_{\mathbf{C}_p}[Y']/(\varpi^{p/b}Y')$ to a polynomial $g(Y'):=\sum_i b_iY'^i\in\mathscr{O}^\flat_{\mathbf{C}_p}[Y']$, and choose a_i such that $a_i^p-\varpi^{i(p-1)/b}a_i=b_i$. Then if $f(Y):=\sum_i a_iY^i$, we have $(\varphi-1)(f(Y))=g(Y)$, as desired.

Now suppose that we have an exact sequence

$$0 \to R_0/(u,p^k) \to \widetilde{\Lambda}_{R_0,[0,b],0}/(u,p^k) \xrightarrow{\varphi-1} \widetilde{\Lambda}_{R_0,[0,b/p],0}/(u,p^k) \to 0$$

and consider the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & p^kR_0/(u,p^{k+1}) \arrow[r] \arrow[d] & p^k\widetilde{\Lambda}_{R_0,[0,b],0}/(u,p^{k+1}) \arrow[r, "\varphi-1"] \arrow[d] & p^k\widetilde{\Lambda}_{R_0,[0,b/p],0}/(u,p^{k+1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R_0/(u,p^{k+1}) \arrow[r] \arrow[d] & \widetilde{\Lambda}_{R_0,[0,b],0}/(u,p^{k+1}) \arrow[r, "\varphi-1"] \arrow[d] & \widetilde{\Lambda}_{R_0,[0,b/p],0}/(u,p^{k+1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R_0/(u,p^k) \arrow[r] \arrow[d] & \widetilde{\Lambda}_{R_0,[0,b],0}/(u,p^k) \arrow[r, "\varphi-1"] \arrow[d] & \widetilde{\Lambda}_{R_0,[0,b/p],0}/(u,p^k) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The bottom row is exact by assumption, and the columns are exact by construction. Moreover, $p^k\widetilde{\Lambda}_{R_0,[0,b],0}/(u,p^{k+1}) \cong p^kR_0/(u,p^{k+1})\,\widehat{\otimes}\,\mathscr{O}_{\mathbf{C}_p}^\flat[Y]/(\varpi^{1/b}Y)$ and $p^k\widetilde{\Lambda}_{R_0,[0,b/p],0}/(u,p^{k+1}) \cong p^kR_0/(u,p^{k+1})\,\widehat{\otimes}\,\mathscr{O}_{\mathbf{C}_p}^\flat[Y']/(\varpi^{p/b}Y')$; since $p^kR_0/(u,p^{k+1})$ is an $\mathbf{F}_p$-vector space, the preceding calculation shows that the top row is exact, as well. A diagram chase then shows that the middle row is exact, as desired. $\square$

of Proposition 5.15. We work modulo successive powers of u; we claim that for any $k \geq 1$, the sequence

$$0 \to R_0/u^k \to \widetilde{\Lambda}_{R_0,[0,b],0}/u^k \xrightarrow{\varphi-1} \widetilde{\Lambda}_{R_0,[0,b/p],0}/u^k \to 0$$

is exact. We have proved the result for $k=1$, so we proceed by induction on k. Assume the result for k and consider the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & u^kR_0/u^{k+1} \arrow[r] \arrow[d] & u^k\widetilde{\Lambda}_{R_0,[0,b],0}/u^{k+1} \arrow[r, "\varphi-1"] \arrow[d] & u^k\widetilde{\Lambda}_{R_0,[0,b/p],0}/u^{k+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R_0/u^{k+1} \arrow[r] \arrow[d] & \widetilde{\Lambda}_{R_0,[0,b],0}/u^{k+1} \arrow[r, "\varphi-1"] \arrow[d] & \widetilde{\Lambda}_{R_0,[0,b/p],0}/u^{k+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R_0/u^k \arrow[r] \arrow[d] & \widetilde{\Lambda}_{R_0,[0,b],0}/u^k \arrow[r, "\varphi-1"] \arrow[d] & \widetilde{\Lambda}_{R_0,[0,b/p],0}/u^k \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The bottom row is exact by assumption and the columns are exact by construction. Since R_0 has no u-torsion, multiplication by u^k defines an isomorphism $R_0/u \xrightarrow{\times u^k}$

$u^k R_0/u^{k+1}$, and the top row is isomorphic to

$$0 \to R_0/u \to \widetilde{\Lambda}_{R_0,[0,b],0}/u \xrightarrow{\varphi-1} \widetilde{\Lambda}_{R_0,[0,b/p],0}/u \to 0$$

which is exact. Then a diagram chase shows that the middle row is exact, as well. Now we consider the inverse limit as $k \to \infty$; $\widetilde{\Lambda}_{R_0,[0,b],0}$ and $\widetilde{\Lambda}_{R_0,[0,b/p],0}$ are u-adically separated and complete (since $u \in \varpi^{1/b}$), so we have an exact sequence

$$0 \to R_0 \to \widetilde{\Lambda}_{R_0,[0,b],0} \xrightarrow{\varphi-1} \widetilde{\Lambda}_{R_0,[0,b/p],0}$$

Moreover, the transition maps $R_0/u^{k+1} \to R_0/u^k$ are surjective, so the Mittag-Leffler condition ensures that $\varphi - 1 : \widetilde{\Lambda}_{R_0,[0,b],0} \to \widetilde{\Lambda}_{R_0,[0,b/p],0}$ is surjective, so we are done. □

Lemma 5.17. *For any finite extension $K/\mathbf{Q}_p$ and all sufficiently small $b > 0$, there is a quasi-isomorphism*

$$[\widetilde{\Lambda}^{H_K}_{R_0,[0,b]}] \xrightarrow{\sim} C^\bullet_{\mathrm{cont}}(H_K, \widetilde{\Lambda}_{R_0,[0,b]}$$

where $C^\bullet_{\mathrm{cont}}(H_K, \widetilde{\Lambda}_{R_0,[0,b]})$ is the continuous Galois cohomology.

Proof. We need to prove that $H^i_{\mathrm{cont}}(H_K, \widetilde{\Lambda}_{R_0,[0,b]}) = 0$ for $i \geq 1$. But this follows from the first Tate–Sen axiom, as in [BC, Proposition 14.3.2]. □

Now we are in a position to show that the Galois cohomology of a family of Galois representations is computed by the associated (φ, Γ)-module.

Theorem 5.18. *Let R be a pseudoaffinoid algebra, and let M be a finite projective R-module equipped with a continuous R-linear action of Gal_K. Then there is some finite Galois extension L/K and some $b > 0$ such that the Galois cohomology $H^\bullet(\mathrm{Gal}_L, M)$ and the Fontaine–Herr–Liu cohomology $H^\bullet_{\varphi,\Gamma_L}(D_{L,[0,b]}(M))$ of the associated $(\varphi, \Gamma_L, \mathrm{Gal}_{L/K})$-module are canonically isomorphic.*

Remark 5.19. Both $H^\bullet(\mathrm{Gal}_L, M)$ and $H^\bullet_{\varphi,\Gamma_L}(D_{L,[0,b]}(M))$ have actions of $\mathrm{Gal}_{L/K}$, by functoriality, and the isomorphism is compatible with this action.

Proof. After localizing on $\mathrm{Spa}(R)$, we may assume that M is a free R-module and contains a free Gal_K-stable R_0-lattice M_0. There is a finite Galois extension L/K such that the (φ, Γ_L)-module $D_{b,L}(M_0)$ is a free $\Lambda_{R_0,[0,b],L}$-module, for some b, and the natural comparison map $\widetilde{\Lambda}_{R_0,[0,b]} \otimes_{\Lambda_{R_0,[0,b],L}} D_{b,L}(M_0) \to \widetilde{\Lambda}_{R_0,[0,b]} \otimes_{R_0} M_0$ is an isomorphism, equivariantly for the actions of Gal_L, φ, and Γ_L. We deduce that there is a natural exact sequence

$$0 \to M_0 \to \widetilde{\Lambda}_{R_0,[0,b]} \otimes_{\Lambda_{R_0,[0,b],L}} D_{b,L}(M_0) \xrightarrow{\varphi-1} \widetilde{\Lambda}_{R_0,[0,b/p]} \otimes_{\Lambda_{R_0,[0,b/p],L}} D_{b/p,L}(M_0) \to 0$$

Applying continuous H_L-cohomology, Lemma 5.17 implies that we have a quasi-isomorphism

$$C^\bullet_{\mathrm{cont}}(H_L, M_0) \xrightarrow{\sim} \left[\widetilde{\Lambda}^{H_L}_{R_0,[0,b]} \otimes_{\Lambda_{R_0,[0,b],L}} D_{b,L}(M_0) \xrightarrow{\varphi-1} \widetilde{\Lambda}^{H_L}_{R_0,[0,b/p]} \otimes_{\Lambda_{R_0,[0,b/p],L}} D_{b/p,L}(M_0)\right]$$

A Hochschild–Serre argument shows that we have a quasi-isomorphism

$$C^\bullet_{\mathrm{cont}}(\mathrm{Gal}_L, M_0) \xrightarrow{\sim} C^\bullet_{\varphi,\Gamma_L}(\widetilde{\Lambda}^{H_L}_{R_0,[0,b]} \otimes_{\Lambda_{R_0,[0,b],L}} D_{b,L}(M_0))$$

so we need to show that the natural map

$$C^\bullet_{\varphi,\Gamma_L}(D_{b,L}(M)) \to C^\bullet_{\varphi,\Gamma_L}(\widetilde{\Lambda}^{H_L}_{R_0,[0,b]} \otimes_{\Lambda_{R_0,[0,b],L}} D_{b,L}(M))$$

is a quasi-isomorphism. It suffices to show that the cohomology of Γ_L acting on $X^n_{R_0,[0,b],L} \otimes_{\varphi^{-n}(\Lambda_{R_0,[0,p^{-n}b],L,0})} D_{b,L,n}(M_0)$ is trivial for sufficiently small $b > 0$, or more concretely, if γ is a topological generator of the procyclic part of Γ_L, that the action of $\gamma - 1$ is continuously invertible. But we have computed an explicit topological basis for $X^n_{R_0,[0,b],L}$, so an argument as in Lemma 5.9 gives the desired result. □

We also deduce the analogous result for $D_{\mathrm{rig}}(M)$:

Corollary 5.20. *Let R be a pseudoaffinoid algebra, and let M be a finite projective R-module equipped with a continuous R-linear action of Gal_K. Then there is some finite Galois extension L/K such that the Galois cohomology $H^\bullet(\mathrm{Gal}_L, M)$ and the Fontaine–Herr–Liu cohomology $H^\bullet_{\varphi,\Gamma_L}(D_{\mathrm{rig},L}(M))$ of the associated $(\varphi, \Gamma_L, \mathrm{Gal}_{L/K})$-module are canonically isomorphic.*

Proof. After localizing on $\mathrm{Spa}(R)$, we may again assume that M is a free R-module containing a free Gal_K-stable R_0-lattice, and we may again choose a finite Galois extension L/K and construct a free $(\varphi, \mathrm{Gal}_L, \mathrm{Gal}_{L/K})$-module $D_{b,L}(M)$, for some $b > 0$. Copying the proof of [Ked08, Proposition 1.2.6] verbatim, we see that the natural morphism

$$\left[\varinjlim_{b\to 0} D_{b,L}(M) \xrightarrow{\varphi-1} \varinjlim_{b\to 0} D_{b/p,L}(M)\right] \to \left[D_{\mathrm{rig},L}(M) \xrightarrow{\varphi-1} D_{\mathrm{rig},L}(M)\right]$$

is a quasi-isomorphism. Then the result follows from Theorem 5.18. □

5.3. **Trianguline (φ, Γ)-modules.** We wish to show that we can interpolate triangulations of families of (φ, Γ)-modules. We first need to study rank-1 (φ, Γ)-modules.

Lemma 5.21. *Let A be a $\mathbf{Z}_p$-algebra, and let $a \in A^\times$. There is a free rank-1 $W(\mathbf{F}_{p^f}) \otimes_{\mathbf{Z}_p} A$-module $D_{f,a}$ equipped with a $\varphi \otimes 1$-semilinear operator φ satisfying $\varphi^f = 1 \otimes a$. This module is unique up to isomorphism, and $D_{f,ab} \cong D_{f,a} \otimes_{W(\mathbf{F}_{p^f})\otimes_{\mathbf{Z}_p} A} D_{f,b}$ for all $a, b \in A^\times$.*

Proof. The proof follows as in [KPX14, Lemma 6.2.3]. □

Definition 5.22. Let $K/\mathbf{Q}_p$ be a finite extension with $f := [K_0 : \mathbf{Q}_p]$, let R be a pseudoaffinoid algebra, and let $\delta : K^\times \to R^\times$ be a continuous character. Writing $K^\times \cong \mathscr{O}_K^\times \times \langle \varpi_K \rangle$, we may write $\delta = \delta_1\delta_2$, where δ_1 is trivial on $\langle \varpi_K \rangle$ and δ_2 is trivial on $\mathscr{O}_K^\times$, and this decomposition is unique. The Artin map from local class field theory (normalized so that ϖ_K is sent to geometric Frobenius) implies that δ_1 extends to a character $\widehat{\delta}_1 : \mathrm{Gal}_K^{\mathrm{ab}} \to R^\times$; let $D_1 := D_{\mathrm{rig}}(\widehat{\delta}_1)$. We let $D_2 := D_{f,\delta_2(\varpi_K)} \otimes_{W(\mathbf{F}_{p^f})\otimes_{\mathbf{Z}_p} R} \Lambda_{R,\mathrm{rig},K}$, with the obvious actions of φ and Γ_K. Then we set $D(\delta) := D_1 \otimes D_2$.

Now we can study extensions of homomorphisms of (φ, Γ)-modules over subspaces of $\mathrm{Spa}(R)$.

Proposition 5.23. *Suppose R is a reduced pseudoaffinoid algebra, D is a family of projective $(\varphi, \Gamma_L, \mathrm{Gal}_{L/K})$-modules, and $\delta : K^\times \to R^\times$ is a continuous character.*

1. *Suppose there is a Zariski-dense set of closed points $Z \subset \mathrm{Spa}(R)$ such that $H^0_{\varphi,\Gamma_K}(D_z^\vee \otimes D(\delta)_z)$ is 1-dimensional for every $z \in Z$. Then $H^0_{\varphi,\Gamma_K}(D_z^\vee \otimes D(\delta)_z)$ is positive-dimensional for every $z \in \mathrm{Spa}(R)$.*
2. *Suppose there is a Zariski-dense set of closed points $Z \subset \mathrm{Spa}(R)$ such that $H^0_{\varphi,\Gamma_K}(D_z \otimes D(\delta^{-1})_z)$ is 1-dimensional for every $z \in Z$. Then $H^0_{\varphi,\Gamma_K}(D_z^\vee \otimes D(\delta)_z)$ is positive-dimensional for every $z \in \mathrm{Spa}(R)$.*

Proof. We only prove the first, as the statements are dual. Since Fontaine–Herr cohomology is R-finite, the individual cohomology groups form coherent sheaves on $\mathrm{Spa}(R)$. Therefore, the rank of $H^0_{\varphi,\Gamma_K}(D_z^\vee \otimes D(\delta)_z)$ jumps on Zariski-closed subspaces of $\mathrm{Spa}(R)$. Since it is assumed to be 1-dimensional for a Zariski-closed subset, it has dimension at least 1 everywhere. ☐

Appendix A. Fiber products

We explain how to construct fiber products $\mathrm{Spa}(R, R^+) \times_{\mathrm{Spa}(S,S^+)} \mathrm{Spa}(R', R'^+)$ of Tate pre-adic spaces. Huber [Hub13, Proposition 1.2.2] explains how to do this for Huber rings under various sets of hypotheses: either (R, R^+) is "locally of weakly finite type" over (S, S^+) and $(S, S^+) \to (R', R'^+)$ is an adic morphism, or (R, R^+) can be "locally of finite type" over (S, S^+). However, $\mathbf{Z}_p[\![u]\!]\left[\frac{p}{u}\right]^\wedge\left[\frac{1}{u}\right]$ is not locally of finite type over $\mathbf{Z}_p$, and neither are rings like $\widetilde{\mathbf{A}}^{(0,r]}$ and $\mathbf{A}_K^{(0,p^{-n}r]}$, and none of these rings have rings of definition which are adic $\mathbf{Z}_p$-algebras.

Let (R, R^+) and (R', R'^+) be complete admissible Huber pairs over (S, S^+). That is, R, R', and S are finitely generated over rings of definition $R_0 \subset R^+$, ${R'}_0 \subset R'^+$, and $S_0 \subset S^+$, respectively; let $I \subset R_0$ and $I' \subset R'_0$ be ideals of definition. It is asserted in [SW18] that the fiber product $X := \mathrm{Spa}(R, R^+) \times_{\mathrm{Spa}(S,S^+)} \mathrm{Spa}(R', R'^+)$ can be constructed in the category of pre-adic spaces; we provide a construction here for the convenience of the reader.

Recall the definition of a pre-adic space from [SW18]:

Definition A.1. Let $(V)^{\mathrm{ind}}$ be the category of triples $(X, \mathscr{O}_X, (|\cdot(x)|)_{x\in X})$, where X is a topological space, $\mathscr{O}_X$ is a sheaf of ind-topological rings, and for each $x \in X$, $|\cdot(x)|$ is an equivalence class of continuous valuations on $\mathscr{O}_{X,x}$. For a Huber pair (A, A^+), we define $\mathrm{Spa}^{\mathrm{ind}}(A, A^+) \in (V)^{\mathrm{ind}}$ to be $(X, \mathscr{O}_X^{\mathrm{ind}}, (|\cdot(x)|)_{x\in X})$, where $X = \mathrm{Spa}(A, A^+)$, $\mathscr{O}_X^{\mathrm{ind}}$ is the sheafification of the presheaf $\mathscr{O}_X$ in the category of ind-topological rings, and the valuations stay the same.

A *pre-adic space* is an object of $(V)^{\mathrm{ind}}$ which is locally isomorphic to $\mathrm{Spa}^{\mathrm{ind}}(A, A^+)$ for some complete Huber pair (A, A^+).

By [SW18, Proposition 3.4.2],

$$\mathrm{Hom}_{(V)^{\mathrm{ind}}}(\mathrm{Spa}^{\mathrm{ind}}(A, A^+), \mathrm{Spa}^{\mathrm{ind}}(B, B^+)) = \mathrm{Hom}_{\mathrm{CAff}}((B, B^+), (A, A^+))$$

for all complete Huber pairs (A, A^+), (B, B^+) (where CAff denotes the category of complete Huber pairs). This permits us to study pre-adic spaces via a functor of points approach.

We wish to construct the fiber product $X := \mathrm{Spa}^{\mathrm{ind}}(R,R^+)\times_{\mathrm{Spa}^{\mathrm{ind}}(S,S^+)}\mathrm{Spa}^{\mathrm{ind}}(R',R'^+)$. This fiber product must be final among spaces Y equipped with maps $Y\to\mathrm{Spa}(R,R^+)$ and $Y\to\mathrm{Spa}(R',R'^+)$ such that the compositions $Y\to\mathrm{Spa}(R,R^+)\to\mathrm{Spa}(S,S^+)$ and $Y\to\mathrm{Spa}(R',R'^+)\to\mathrm{Spa}(S,S^+)$ agree.

Proposition A.2. *Let notation be as above. The fiber product* $\mathrm{Spa}^{\mathrm{ind}}(R,R^+)\times_{\mathrm{Spa}^{\mathrm{ind}}(S,S^+)}\mathrm{Spa}^{\mathrm{ind}}(R',R'^+)$ *is representable in the category of pre-adic spaces.*

Proof. If $R=R^+=R_0$, $R'=R'^+=R'_0$, and $S=S^+=S_0$, then R^+, R'^+, and S^+ are rings of definitions, and we may simply take $R_0\otimes_{S_0}R'_0$ and complete $(I\otimes R'_0+R_0\otimes I')$-adically.

To handle the general case, we write $R=R_0[\underline{X}]/J$ and $R'=R'_0[\underline{X}']/J'$, where $\underline{X}$ and $\underline{X}'$ are finite collections of elements generating R and R', and $J\subset R_0[\underline{X}]$ and $J'\subset R'_0[\underline{X}']$ are ideals; let $\underline{r},\underline{r}'$ be the images of $\underline{X},\underline{X}'$ in R,R', respectively. Let $g:(S,S^+)\to(R,R^+)$ and $g':(S,S^+)\to(R',R'^+)$ be the structure maps. If necessary, we replace S_0 with $S'_0:=S_0\cap g^{-1}(R_0)\cap g'^{-1}(R'_0)$; S'_0 is an open and bounded subring of S, hence a ring of definition, and it satisfies $g(S'_0)\subset R_0$ and $g'(S'_0)\subset R'_0$.

Let (T,T^+) be a complete affinoid Huber pair over (S,S^+), and suppose that there are homomorphisms $f:(R,R^+)\to(T,T^+)$ and $f':(R',R'^+)\to(T,T^+)$ over (S,S^+). Therefore, there are homomorphisms $R^+\otimes_{S^+}R'^+\to T^+$ and $R\otimes_S R'\to T$. For $r\in R$, consider $f(r)\in T$. Since $T^+\subset T$ is an open subring, there is some open neighborhood $V\subset T^+$ of 0 such that $f(r)\cdot V\subset T^+$. Since $f'(I')$ must consist of topologically nilpotent elements, there is some $n\gg 0$ such that $f'(I'^n)\subset V$, and therefore $f(r)\cdot f'(I'^n)\subset T^+$ consists of integral elements. Similarly, for each $r'\in R'$, there is some $n'\gg 0$ such that $f'(r')\cdot f(I^{n'})$ consists of integral elements. Since R and R' are finitely generated over R_0 and R'_0, respectively, we may choose $n,n'\gg 0$ such that that $f(\underline{r})\cdot f'(I'^n), f'(\underline{r}')\cdot f(I^{n'})\subset T^+$.

We topologize $R\otimes_S R'$ so that $(R_0\otimes_{S_0}R'_0)\left[(\underline{r}\otimes 1)(R_0\otimes I')^n,(1\otimes\underline{r}')(I\otimes R'_0)^n\right]$ is a ring of definition: We let $(R_0\otimes_{S_0}R'_0)[\underline{X},\underline{X}']_{(n)}$ be the polynomial ring $(R_0\otimes_{S_0}R'_0)[\underline{X},\underline{X}']$ and we equip it with the $R_0\otimes I'+I\otimes R'_0$-adic topology. That is,

$$U_m:=\left\{\sum_{\nu,\nu'}a_{\nu,\nu'}\underline{X}^{\nu}\underline{X}'^{\nu'}\mid a_{\nu,\nu'}\in(R_0\otimes I'+I\otimes R'_0)^{n(\nu+\nu')+m}\text{ for all }\nu,\nu'\right\}$$

is a basis of neighborhoods of 0. We set

$$(R\otimes_S R')_{(n)}:=(R_0\otimes_{S_0}R'_0)[\underline{X},\underline{X}']_{(n)}/(J\otimes R'_0,R_0\otimes J')$$

so that $(R\otimes_S R')_{(n),0}:=U_0/(J\otimes R'_0,R_0\otimes J')$ is a ring of definition, and we let $(R\otimes_S R')^+_{(n)}$ be the integral closure of the image of $(R^+\otimes_{S^+}R'^+)\left[(\underline{r}\otimes 1)(R_0\otimes I')^n,(1\otimes\underline{r}')(I\otimes R'_0)^n\right]$ in $(R\otimes_S R')_{(n)}$.

We let $((R\widehat{\otimes}_S R')_{(n)},(R\widehat{\otimes}_S R')^+_{(n)})$ be the completion of $(R\otimes_S R')_{(n)},(R\otimes_S R')^+_{(n)})$, and we define $X_{(n)}:=\mathrm{Spa}^{\mathrm{ind}}((R\widehat{\otimes}_S R')_{(n)},(R\widehat{\otimes}_S R')^+_{(n)})$. By construction, the homomorphisms $g:(R,R^+)\to(T,T^+)$ and $g':(R',R'^+)\to(T,T^+)$ induce a unique morphism $\mathrm{Spa}^{\mathrm{ind}}(T,T^+)\to X_{(n)}$.

Furthermore, there are natural maps $(R\otimes_S R')_{(n+1)} \to (R\otimes_S R')_{(n)}$, and they are compatible with the natural maps $R, R' \rightrightarrows (R\otimes_S R')_{(n)}$ and $S \to (R\otimes_S R')_{(n)}$. The induced maps $X_{(n)} \to X_{(n+1)}$ make $X_{(n)}$ into a rational subset of $X_{(n+1)}$ for all n, so we may define a pre-adic space $X := \cup_n X_{(n)}$. Then X is the pre-adic space representing $\mathrm{Spa}^{\mathrm{ind}}(R, R^+) \times_{\mathrm{Spa}^{\mathrm{ind}}(S,S^+)} \mathrm{Spa}^{\mathrm{ind}}(R', R'^+)$. □

Example A.3. Let $R = R^+ = \mathbf{Z}_p[\![u]\!]$, $R' = \mathbf{Q}_p$, $R'^+ = \mathbf{Z}_p$, and $S = S^+ = \mathbf{Z}_p$, so that we may take $R_0 = \mathbf{Z}_p[\![u]\!]$, $R'_0 = R'^+ = \mathbf{Z}_p$, and $S_0 = S^+ = \mathbf{Z}_p$. Then we need $u^n \cdot \left(\frac{1}{p}\right)$ to be bounded for varying n, so we need to consider quotients of

$$R\langle X\rangle_{(p,u)^n} = \{\sum_{\nu\geq 0} a_\nu X^\nu \mid a_\nu \in \mathbf{Z}_p[\![u]\!];\ \text{for all } m, a_\nu \in (p,u)^{n\nu+m} \text{ for almost all } \nu\}$$

and its subring

$$\{\sum_{\nu\geq 0} a_\nu X^\nu \mid a_\nu \in \mathbf{Z}_p[\![u]\!], a_\nu \in (p,u)^{n\nu} \text{ for all } \nu\}$$

After we quotient by the ideal $(pX-1)$, the latter ring becomes $\mathbf{Z}_p[\![u]\!]\left[\frac{u^n}{p}\right]^\wedge$ and the former becomes $\mathbf{Z}_p[\![u]\!]\left[\frac{u^n}{p}\right]^\wedge\left[\frac{1}{p}\right]$. Thus, we get the standard construction for the generic fiber of $\mathrm{Spf}\,\mathbf{Z}_p[\![u]\!]$.

Proposition A.4. *Suppose (R, R^+) and (R', R'^+) are Tate, with pseudo-uniformizers $u \in R^+$ and $u' \in R'^+$, respectively, and suppose $S = S^+$. Then $(R\widehat{\otimes}_S R')_{(n)}$ is also Tate, and the images of u and u' are pseudo-uniformizers.*

Proof. There are natural continuous maps $R_0 \widehat{\otimes}_{S_0} R'_0 \to (R\widehat{\otimes}_S R')_{(n),0}$ and $R\otimes_S R' \to (R\widehat{\otimes}_S R')_{(n)}$ for all $n \geq 0$. Since $u\otimes 1$ and $1\otimes u'$ are topologically nilpotent in $R_0 \widehat{\otimes} R'_0$ and invertible in $R\otimes_S R'$, they are topologically nilpotent units in $(R\widehat{\otimes}_S R')_{(n)}$. □

Remark A.5. We do not know whether this fiber product preserves properties such as being noetherian or being sheafy. In particular, we do not know whether the fiber product of two adic spaces is again an adic space (or merely a pre-adic space).

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