M3/4/5P12 Solutions #3

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1. Given $M \in \mathrm{GL}_d(\mathbf{C})$ such that $M^n = \mathbf{1}$, we may construct a matrix representation $\rho : C_n = \langle g : g^n = e\rangle \to \mathrm{GL}_d(C)$ by setting $\rho(g) := M$. Since C_n is an abelian group, this representation is isomorphic to the direct sum of d 1-dimensional representations, so it is diagonalizable.

   Alternatively, we may put M in Jordan normal form and assume it has λ's on the diagonal for some $\lambda \in \mathbf{C}$ and 1's on the superdiagonal. Then $(M - \lambda\mathbf{1})^d = 0$ and $M^n = \mathbf{1}$. In other words, if $p(X)$ denotes the minimal polynomial of M, then $(X-\lambda)^d$ and $X^n - 1$ are both multiples of $p(X)$. But $X^n - 1 = \prod_\zeta (X - \zeta)$, where the product runs over nth roots of 1 (and they are all distinct), and the only factors of $(X-\lambda)^d$ are powers of $X - \lambda$. This implies that the minimal polynomial of M is actually $X - \lambda$, so $M = \lambda\mathbf{1}$ with $\lambda^n = 1$.

2. There are a number of approaches to this problem. At this point in the course, the easiest solution is to obsesrve that for each 1-dimensional representation (W, ρ_W) of D_8, the inner product $\langle \chi_{V\otimes V}, \chi_W\rangle = 1$. Thus, in the notation of the solution to problem 7, $V \otimes V \cong V_{\mathrm{triv}} \oplus V_{+-} \oplus V_{-+} \oplus V_{--}$.

   Another character-theoretic approach is to write $V \otimes V \cong V_{\mathrm{triv}}^{\oplus m_1} \oplus V_{+-}^{\oplus m_2} \oplus V_{-+}^{\oplus m_3} \oplus V_{--}^{\oplus m_4} \oplus V_2^{\oplus m_5}$ (in the notation of problem 7) and compute the inner product $\langle \chi_{V\otimes V}, \chi_{V\otimes V}\rangle$:

   $$4 = \langle \chi_{V\otimes V}, \chi_{V\otimes V}\rangle = \sum_i \langle \chi_{V_i^{\oplus m_i}}, \chi_{V_i^{\oplus m_i}}\rangle = \sum_i m_i^2$$

   Therefore, either four of the m_i's are 1, or one of them is 2 and the rest 0. The first case corresponds to $V \otimes V \cong V_{\mathrm{triv}} \oplus V_{+-} \oplus V_{-+} \oplus V_{--}$, and the second corresponds to $V \otimes V \cong V_2 \oplus V_2$. But by problem 9, $V \otimes V$ has at least one 1-dimensional subrepresentation, namely $\wedge^2 V$, so the second possibility is ruled out.

   Another approach is to observe that the eigenvalues of $\rho_V(s)$ are $\pm i$. Therefore, if we let $\{v_1, v_2\}$ be a basis of V diagonalizing $\rho_V(g)$, then $\rho_{V\otimes V}(s^2)(v_i \otimes v_j) = v_i \otimes v_j$ for any $v_i \otimes v_j$. In other words, $\rho_{V\otimes V}(s^2) = \mathbf{1}$, so we may view $\rho_{V\otimes V}$ as a representation of $D_8/\langle s^2\rangle \cong C_2 \times C_2$. Since $C_2 \times C_2$ is abelian, $\rho_{V\otimes V}$ can be diagonalized, and the only question is which 1-dimensional representations of D_8 show up in the decomposition. To find an explicit diagonalization, we restrict to the ± 1 eigenspaces for $\rho_{V\otimes V}(s)$; they are each 2-dimensional and $\rho_{V\otimes V}(t)$ preserves them since $\rho_{V\otimes V}(t)\rho_{V\otimes V}(s) = \rho_{V\otimes V}(s)\rho_{V\otimes V}(t)$ (even though $ts \neq st$ in D_8). But this is a straightforward calculation, and shows that every 1-dimensional representation of D_8 appears.

   We can also start by writing down the matrices for $\rho_{V\otimes V}(s)$ and $\rho_{V\otimes V}(t)$ explicitly. If we change our notation and let $v_1 = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $v_2 = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$, we may choose the basis $(v_1\otimes v_1, v_1\otimes v_2, v_2\otimes v_1, v_2\otimes v_2)$ for $V \otimes V$. Then

   $$\rho_{V\otimes V}(s) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \rho_{V\otimes V}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

   It is clear that $v_1 \otimes v_1 + v_2 \otimes v_2$ and $v_1 \otimes v_2 + v_1 \otimes v_2$ generate 1-dimensional subrepresentations (for the first, s and t both act as 1; for the second, s and t both act as -1).

   Now we can use the results of problem 9: both of these are subrepresentations of S^2V, which is a 3-dimensional representation. But then S^2V must be the direct sum of three 1-dimensional representations of D_8. In fact, $v_1 \otimes v_1 - v_2 \otimes v_2$ generates another subrepresentation of S^2V, with

s acting by -1 and t acting by 1. In addition, $\wedge^2 V$ is a 1-dimensional subrepresentation of $V \otimes V$, and since the eigenvalues of $\rho_V(s)$ are $\pm i$ and the eigenvalues of $\rho_V(t)$ are ± 1, s acts on $\wedge^2 V$ by 1 and t acts on $\wedge^2 V$ by -1.

3. (a) Let $\{v_i\}$ be a basis for V and let $\{w_s\}$ be a basis for W. Then $\{v_i \otimes w_s\}$ is a basis for $V \otimes W$ and $\{w_s \otimes v_i\}$ is a basis for $W \otimes V$. Thus, we define a linear transformation $f : V \otimes W \to W \otimes V$ by setting $f(v_i \otimes w_s) := w_s \otimes v_i$ and extending by linearity. Since $\mathrm{im}(f)$ contains a basis of $W \otimes V$, f is surjective, and since $V \otimes W$ and $W \otimes V$ have the same dimension, f is an isomorphism.

   To check that this map is G-linear, suppose that the matrix for $\rho_V(g)$ with respect to $\{v_i\}$ is M and suppose the matrix for $\rho_W(g)$ with respect to $\{w_s\}$ is N. We need to check that the image of $\rho_{V\otimes W}(g)(v_i \otimes w_s) = (M \cdot v_i) \otimes (N \cdot w_s)$ is equal to $\rho_{W\otimes V}(g)(w_s \otimes v_i) = (N \cdot w_s) \otimes (M \cdot v_i)$:

   $$\rho_{V\otimes W}(g)(v_i \otimes w_s) = (M \cdot v_i) \otimes (N \cdot w_s) = \left(\sum_j M_{ji} v_j\right) \otimes \left(\sum_t N_{ts} w_t\right) = \sum_{j,t} M_{ji} N_{ts} v_j \otimes w_t$$

   and its image in $W \otimes V$ is $\sum_{j,t} M_{ji} N_{ts} w_t \otimes v_j$. On the other hand,

   $$\rho_{W\otimes V}(g)(w_s \otimes v_i) = (N \cdot w_s) \otimes (M \cdot v_i) = \left(\sum_t N_{ts} w_t\right) \otimes (M_{ji} v_j) = \sum_{t,j} N_{ts} M_{ji} w_t \otimes v_j$$

   so the two tensors are equal.

   (b) Consider $f \in \mathrm{Hom}(V^*, W)$. Then $f : V^* \to W$, and we define the dual $f^* : W^* \to V^{**} \xrightarrow{\sim} V$ via $f^*(h) := h \circ f$ (for $h \in W^*$). Thus, we can define a map $\mathrm{Hom}(V^*, W) \to \mathrm{Hom}(W^*, V)$ via $f \mapsto f^*$. This is evidently linear. I claim it is injective: if $f^* = 0$, then $h \circ f = 0$ for all $h \in W^*$, so $f = 0$ (if not, we could take h to be the projection onto a 1-dimensional subspace of $\mathrm{im}(f)$). Since $\mathrm{Hom}(V^*, W)$ and $\mathrm{Hom}(W^*, V)$ both have dimension $\dim V \cdot \dim W$, this implies that our map is an isomorphism.

   To check that this map is G-linear, we need to check that

   $$\rho_{\mathrm{Hom}(W^*,V)}(g) \circ f^* = \left(\rho_{\mathrm{Hom}(V^*,W)}(g)(f)\right)^*$$

   (as elements of $\mathrm{Hom}(W^*, V)$). To see this, we evaluate them on elements $h \in W^*$:

   $$\begin{aligned}
   \left(\rho_{\mathrm{Hom}(W^*,V)}(g)(f^*)\right)(h) &= \left(\rho_V(g) \circ f^* \circ \rho_{W^*}(g^{-1})\right)(h) = (\rho_V(g) \circ f^*)(h \circ \rho_W(g)) \\
   &= \rho_V(g)(h \circ \rho_W(g) \circ f) = \rho_{V^{**}}(g)(h \circ \rho_W(g) \circ f) \\
   &= h \circ \rho_W(g) \circ f \circ \rho_{V^*}(g^{-1}) \\
   \left(\rho_{\mathrm{Hom}(V^*,W)}(g)(f)\right)^*(h) &= h \circ \left(\rho_{\mathrm{Hom}(V^*,W)}(g)(f)\right) = h \circ \rho_W(g) \circ f \circ \rho_{V^*}(g^{-1})
   \end{aligned}$$

   Note that we have used the fact that the isomorphism $V \xrightarrow{\sim} V^{**}$ is G-linear.

4. It suffices to prove that $\dim \mathrm{Hom}(V_{\mathrm{reg}}^*, W)^G = \dim W$ for all irreducible representations W. But by the previous exercise

   $$\mathrm{Hom}(V_{\mathrm{reg}}^*, W) \cong \mathrm{Hom}(W^*, V_{\mathrm{reg}})$$

   as representations, so their G-fixed subspaces are isomorphic:

   $$\mathrm{Hom}(V_{\mathrm{reg}}^*, W)^G \cong \mathrm{Hom}(W^*, V_{\mathrm{reg}})^G$$

   Since W^* is irreducible, the dimension of the right side is $\dim W^* = \dim W$, so we are done.

5. $\chi_{\mathrm{reg}}(e) = \mathrm{Tr}(\mathbf{1}) = |G|$ and $\chi_{\mathrm{reg}}(g) = 0$ if $g \neq e$: If $\mathrm{Tr}(\rho_{\mathrm{reg}}(g)) \neq 0$, there is a non-zero entry on the diagonal of the matrix for $\rho_{\mathrm{reg}}(g)$. But if we take the basis $\{b_h\}_{h\in G}$ for V_{reg}, this would imply that $\rho_{\mathrm{reg}}(g)(b_h) = b_{gh} = b_h$ for some h. But this is impossible unless $g = e$, so $\chi_{\mathrm{reg}}(g) = 0$ if $g \neq e$.

   We compute

   $$\langle \chi_{\mathrm{reg}}, \chi_W \rangle = \frac{1}{|G|} \sum_{g\in G} \chi_{\mathrm{reg}}(g) \overline{\chi_W(g)} = \frac{1}{|G|} \cdot |G| \cdot \dim W = \dim W$$

6. It is enough to prove that if W is irreducible, then $V \otimes W$ is irreducible. So suppose $W' \subset V \otimes W$ is a proper non-zero subrepresentation (so $0 \neq \dim W' < \dim W$). Recall that (V^*, ρ_{V^*}) is a 1-dimensional representation with $\rho_{V^*}(g) = \rho_V(g^{-1})$. We claim that $V^* \otimes W'$ is a proper non-zero subrepresentation of W. Certainly $V^* \otimes W'$ has dimension $\dim W'$, so if it is a subrepresentation of W it is proper and non-zero.

   Let $v \in V$ and $v' \in V^*$ be basis elements, and let $\{w_i\} \subset W$ be a basis of W. Then we may write an element of W' uniquely in the form $v \otimes (\sum_i a_i w_i)$ for some $a_i \in \mathbf{C}$, and we may write an element of $V^* \otimes W'$ uniquely in the form $v' \otimes (v \otimes (\sum_i a_i w_i))$. Now we define a map $V^* \otimes W' \to W$ via

$$v' \otimes \left(v \otimes (\sum_i a_i w_i) \right) \mapsto \sum_i a_i w_i$$

   This is G-linear because

$$\begin{aligned} \rho_{V^* \otimes W'}(g) \left(v' \otimes \left(v \otimes (\sum_i a_i w_i) \right) \right) &= \rho_{V^*}(g)(v') \otimes \left(\rho_V(g)(v) \otimes \sum_i a_i \rho_W(g)(w_i) \right) \\ &= v' \otimes \left(v \otimes (\sum_i a_i \rho_W(g)(w_i)) \right) \end{aligned}$$

   so $\rho_{V^* \otimes W'}(g) \left(v' \otimes (v \otimes (\sum_i a_i w_i)) \right) \mapsto \sum_i a_i \rho_W(g)(w_i)$, which is $\rho_W(g)$ applied to the image of $v' \otimes (v \otimes (\sum_i a_i w_i))$.

7. Write $D_8 = \langle s, t : s^4 = t^2 = e, tst = s^{-1} \rangle$. Then each element of D_8 can be written uniquely in the form $s^i t^j$ where $i \in \{0, 1, 2, 3\}$ and $j \in \{0, 1\}$. The conjugacy classes are $\{e\}$, $\{s, s^{-1}\}$, s^2, $\{st, s^{-1}t\}$, and $\{t, s^2 t\}$. Indeed,

$$\begin{aligned} (s^i t^j) e (s^i t^j)^{-1} &= e \text{ for all } i, j \\ (s^i t^j) s (s^i t^j)^{-1} &= s^i s^{(-1)^j} s^{-i} = s^{\pm 1} \\ (s^i t^j) s^2 (s^i t^j)^{-1} &= s^i (t^j s t^j)(t^j s t^j) s^{-i} = s^2 = s^2 \\ (s^i t^j)(st)(s^i t^j)^{-1} &= s^i t^j s t t^j s^{-i} = s^i s^{(-1)^j} t s^{-i} = s^i s^{(-1)^j} s^i t = s^{\pm 1} t \\ (s^i t^j) t (s^i t^j)^{-1} &= s^i t s^{-i} = s^{2i} t \end{aligned}$$

   There are four 1-dimensional representations of D_8, which we denote $(V_{\text{triv}}, \rho_{\text{triv}})$, (V_{+-}, ρ_{+-}), (V_{-+}, ρ_{-+}), and (V_{--}, ρ_{--}), given by

$$\begin{aligned} \rho_{\text{triv}}(s) &= 1 \quad &\text{and} \quad && \rho_{\text{triv}}(t) &= 1 \\ \rho_{+-}(s) &= 1 \quad &\text{and} \quad && \rho_{+-}(t) &= -1 \\ \rho_{-+}(s) &= -1 \quad &\text{and} \quad && \rho_{-+}(t) &= 1 \\ \rho_{--}(s) &= -1 \quad &\text{and} \quad && \rho_{--}(t) &= -1 \end{aligned}$$

   We write the irreducible 2-dimensional representation (V_2, ρ_2); there is a basis of V_2 such that

$$\rho_2(s) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \rho_2(t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

   Thus, we can write the characters for all of the irreducible representations of D_8:

	$\{e\}$	$\{s, s^{-1}\}$	$\{s^2\}$	$\{st, s^{-1}t\}$	$\{t, s^2 t\}$
$\chi_{\text{triv}}(g)$	1	1	1	1	1
$\chi_{+-}(g)$	1	1	1	-1	-1
$\chi_{-+}(g)$	1	-1	1	-1	1
$\chi_{--}(g)$	1	-1	1	$+1$	-1
$\chi_2(g)$	2	0	-2	0	0

8. Let (V, ρ) be the representation constructed from the action $G \times X \to X$.

(a) Recall that V has a basis $\{b_x\}_{x\in X}$ indexed by elements of X, and the representation is given by $\rho(g)(b_x) = b_{g\cdot x}$. Thus, the matrix for $\rho(g)$ has exactly one 1 in each column, in the row corresponding to $b_{g\cdot x}$. This is a diagonal entry if and only if $g \cdot x$, so

$$\chi_\rho(g) = \mathrm{Tr}(\rho(g)) = |\{x \in X : g \cdot x = x\}|$$

(b) The element $\sum_{x\in X} b_x$ generates a 1-dimensional subrepresentation of V, and it is isomorphic to the trivial representation, because

$$\rho(g)(\sum_{x\in X} b_x) = \sum_{x\in X} b_{g\cdot x}$$

and $\{g \cdot x\}_{x\in X} = X$ because the action of g permutes the elements of X: if $g \cdot x = g \cdot x'$, then

$$x = (g^{-1}g) \cdot x) = g^{-1} \cdot (g \cdot x) = g^{-1} \cdot (g \cdot x') = (g^{-1}g) \cdot x' = x'$$

Thus, there is some subrepresentation $W \subset V$ such that $V \cong W \oplus V_{\mathrm{triv}}$. This implies that $\chi_W = \chi_\rho - 1 = \xi$, as desired.

9. (a) This follows from the solution to problem (3).

(b) Since $\mathbf{1}$ and f are both G-linear maps $V \otimes V \to V \otimes V$, so are $\mathbf{1} - f$ and $\mathbf{1} + f$. Thus, their kernels are subrepresentations of $V \otimes V$ (by §2.5 of the notes).

We first check that $\ker(\mathbf{1} - f) \cap \ker(\mathbf{1} + f) = \{0\}$. If $v \in \ker(\mathbf{1} - f) \cap \ker(\mathbf{1} + f) = \{0\}$, then $f(v) = v$ and $f(v) = -v$, so $v = -v$ and therefore $v = 0$.

Thus, we have an injective linear transformation

$$\ker(\mathbf{1} - f) \oplus \ker(\mathbf{1} + f) \to V \otimes V$$

given by $(v, v') \mapsto v + v'$. We need to check that it is surjective. To see this, we first observe that $f \circ f = \mathbf{1}$. Then for any $v \in V \otimes V$, we may write

$$v = \frac{1}{2}(v - f(v)) + \frac{1}{2}(v + f(v))$$

Then

$$(\mathbf{1} + f)(v - f(v)) = 0 \qquad \text{and} \qquad (\mathbf{1} - f)(v + f(v)) = 0$$

so v is in the image of $\ker(\mathbf{1} - f) \oplus \ker(\mathbf{1} + f)$ and we are done.

(c) In the previous part, we showed that $\mathrm{im}(\mathbf{1} + f) \subset S^2V$ and $\mathrm{im}(\mathbf{1} - f) \subset \wedge^2 V$. Moreover, if $v \in S^2V$, then

$$\frac{1}{2}(\mathbf{1} + f)(v) = \frac{1}{2}(v + v) = v$$

and if $v \in \wedge^2 V$, then

$$\frac{1}{2}(\mathbf{1} - f)(v) = \frac{1}{2}(v - (-v)) = v$$

Thus, $\frac{1}{2}(\mathbf{1} + f)$ and $\frac{1}{2}(\mathbf{1} - f)$ are projections onto S^2V and $\wedge^2 V$, respectively.

(d) Let $\{v_i\}$ be a basis for V. Then the set $\{v_i \otimes v_j + v_j \otimes v_i\}_{j\geq i}$ is a linearly independent subset of S^2V, of size $\frac{d(d+1)}{2}$. The set $\{v_i \otimes v_j - v_j \otimes v_i\}_{j>i}$ is a linearly independent subset of $\wedge^2 V$, of size $\frac{d(d-1)}{2}$. Thus,

$$\{v_i \otimes v_j + v_j \otimes v_i\}_{j\geq i} \cup \{v_i \otimes v_j - v_j \otimes v_i\}_{j>i}$$

is a subset of $V \otimes V$ of size d^2 whose span includes each basis element $v_i \otimes v_j$. It follows that $\{v_i \otimes v_j + v_j \otimes v_i\}_{j\geq i}$ is a basis of S^2V and $\{v_i \otimes v_j - v_j \otimes v_i\}_{j>i}$ is a basis of $\wedge^2 V$, so they have dimensions $\frac{d(d+1)}{2}$ and $\frac{d(d-1)}{2}$, respectively.

(e) Choose a basis $\{v_i\}$ of V such that the matrix for $\rho_V(g)$ is diagonal, with eigenvalues $\lambda_1, \ldots, \lambda_d$. Then

$$\rho_{V\otimes V}(g)(v_i \otimes v_j - v_j \otimes v_i) = \lambda_i\lambda_j(v_i \otimes v_j - v_j \otimes v_i)$$

and

$$\rho_{V\otimes V}(g)(v_i \otimes v_j + v_j \otimes v_i) = \lambda_i\lambda_j(v_i \otimes v_j + v_j \otimes v_i)$$

Thus, the eigenvalues for $\wedge^2 V$ are $\{\lambda_i\lambda_j : j > i\}$.

(f) From the previous part, we know that

$$\chi_{\wedge^2 V}(g) = \mathrm{Tr}(\rho_{\wedge^2 V}(g)) = \sum_{j>i} \lambda_i\lambda_j$$

Therefore,

$$\frac{\chi_V(g)^2 - \chi_V(g^2)}{2} = \frac{1}{2}\sum_{i\neq j} \lambda_i\lambda_j = \chi_{\wedge^2 V}(g)$$

Furthermore, we proved in lecture that $\chi_{V\otimes V}(g) = \chi_V(g)^2$. Since $V \otimes V \cong S^2V \oplus \wedge^2 V$,

$$\chi_{S^2V}(g) = \chi_V(g)^2 - \chi_{\wedge^2 V}(g) = \frac{\chi_V(g)^2 + \chi_V(g^2)}{2}$$

10. We proved in lecture that $\chi(g) = \chi(e)$ if and only if $\rho(g) = \rho(e)$. Thus, the claim amounts to showing that there is some irreducible representation (V, ρ_V) such that $\rho_V(g) \neq \mathbf{1}_V$. However, recall that every irreducible representation appears as a subrepresentation of the regular representation $(V_{\mathrm{reg}}, \rho_{\mathrm{reg}})$. If we had $\rho_V(g) = \mathbf{1}_V$ for every irreducible representation, then we would have $\rho_{\mathrm{reg}}(g) = \mathbf{1}$. But this is impossible unless $g = e$.