PROBLEM SET ON LOCAL FIELDS

(1) Prove that $\alpha = \sum_{i \gg -\infty} a_i p^i \in \mathbf{Q}_p$ is in $\mathbf{Q}$ if and only if the coefficients a_i are eventually periodic.
(2) Prove that $\alpha = \sum_{i \geq 0} a_i p^i \in \mathbf{Z}_p$ is a unit if and only if $a_0 \neq 0$.
(3) Prove that $\mathbf{Q}_p$ is totally disconnected: the only connected subsets are singletons.
(4) Prove that the natural map $\mathbf{Z}[\![X]\!] \to \mathbf{Z}_p$ sending $X \mapsto p$ is surjective with kernel $(X - p)$.
(5) Let K be a field, and $v : K \to \mathbf{R}$ a non-archimedean valuation. Let $\mathcal{O}_v$ be the valuation ring of v, $\mathfrak{m}_v \subset \mathcal{O}_v$ its maximal ideal, and $k_v := \mathcal{O}_v/\mathfrak{m}_v$ its residue field. Define $|\cdot|_v := |k_v|^{-v(\cdot)}$.
    (a) Let $K = \mathbf{Q}$. Prove that for any $\alpha \in \mathbf{Q}$, $|\alpha|_\infty \times \prod_v |\alpha|_v = 1$, where the product runs over non-archimedean valuations corresponding to primes of $\mathbf{Z}$.
    (b) Let $K = \mathbf{F}_q(t)$ be a field of Laurent polynomials. Try to write down an analogous formula. Use this to define a valuation v_∞ on K that does not come from a prime ideal $\mathfrak{p} \subset \mathcal{O}_K = \mathbf{F}_q[t]$. Prove that every valuation on $\mathbf{F}_q(t)$ is equivalent to either $v_\mathfrak{p}$ or v_∞.
(6) (a) Prove that the formal power series $\log(1-x) = \sum_{i \geq 1} \frac{x^i}{i}$ converges on an open p-adic ball.
    (b) Prove that the formal power series $\exp(x) = \sum_{i \geq 0} \frac{x^i}{i!}$ converges on an open p-adic ball. Hint: compute the p-adic valuation of $i!$.
    (c) Deduce that $\mathbf{Z}_p^\times \cong \mu_{p-1} \times \mathbf{Z}_p$, as a group.
(7) Let $f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0 \in \mathbf{Z}_p[x]$, with $v_p(a_0) = 0$. Prove that every root of f (in a suitable extension field) has p-adic valuation 0. What are the valuations of the roots of $x^2 + \lambda x + pu$, where $u \in \mathbf{Z}_p^\times$ and $\lambda \in \mathbf{Z}_p$?
(8) Prove that no extension of $\mathbf{Q}_p$ of infinite degree is complete.