COHOMOLOGY OF (φ, Γ)-MODULES OVER PSEUDORIGID SPACES

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ABSTRACT. We study the cohomology of families of (φ, Γ)-modules with coefficients in pseudoaffinoid algebras. Using the finiteness result proved in [Bel20], we deduce an Euler characteristic formula and Tate local duality. We classify rank-1 (φ, Γ)-modules and deduce that triangulations of pseudorigid families of (φ, Γ)-modules can be interpolated, extending a result of [KPX14]. We then apply this to study extended eigenvarieties at the boundary of weight space, proving in particular that the eigencurve is proper at the boundary and that Galois representations attached to certain characteristic p points are trianguline.

CONTENTS

1. Introduction

In our earlier paper [Bel20], we began studying families of Galois representations varying over *pseudorigid* spaces, that is, families of Galois representations where the coefficients have a non-archimedean topology but which (in contrast to the rigid analytic spaces of Tate) are not required to contain a field. Such coefficients arise naturally in the study of eigenvarieties at the boundary of weight space.

The theory of (φ, Γ)-modules is a crucial tool in the study of p-adic Galois representations. At the expense of making the coefficients more complicated, it lets us turn the data of a Galois representation into the data of a Frobenius operator φ and a 1-dimensional p-adic Lie group Γ. Moreover, Galois representations which are irreducible often become reducible on the level of their associated (φ, Γ)-modules. Such (φ, Γ)-modules have played an important role in the p-adic Langlands program.

In our previous paper [Bel20], we constructed (φ, Γ)-modules associated to Galois representations varying over pseudorigid spaces, and we showed that (φ, Γ)-modules over pseudorigid spaces have finite cohomology, whether or not they come from Galois representations, extending the main result of [KPX14]. We were unable to extend the results of that paper on interpolating triangulations of families of (φ, Γ)-modules, because we did not have a classification of rank-1 (φ, Γ)-modules over pseudorigid spaces or a computation of their cohomology. In the present work, we fill that gap.

We begin by proving an Euler characteristic formula:

Theorem 1.0.1. *If D is a (φ, Γ_K)-module with coefficients in a pseudoaffinoid algebra R, then $\chi(D) = -[K : \mathbf{Q}_p]$.*

This extends a result of [Liu07]. However, the method of proof is different: Liu proved finiteness of cohomology and the Euler characteristic formula at the same time, making a close study of t-torsion (φ, Γ)-modules to shift weights. There is no element t in our setting, because p is not necessarily invertible. However, because we proved finiteness of cohomology for pseudorigid families of (φ, Γ)-modules already, we can deduce the Euler characteristic formula by deformation, without studying torsion objects.

We then turn to (φ, Γ)-modules with coefficients in finite extensions of $\mathbf{F}_p(\!(u)\!)$, and we prove Tate local duality:

Theorem 1.0.2. *Tate local duality holds for every (φ, Γ)-module D over $\Lambda_{R,\mathrm{rig},K}$.*

Our proof closely follows that of [Liu07]; we compute the cohomology of many rank-1 (φ, Γ)-modules and then proceed by induction on the degree, using the Euler characteristic formula to produce non-split extensions. We are then able to finish the computation of the cohomology of (φ, Γ)-modules of character type.

With this in hand, we are able to show that all rank-1 (φ,Γ)-modules over pseudorigid spaces are of character type, following [KPX14], and we deduce that triangulations can be interpolated:

Theorem 1.0.3. *Let X be a reduced pseudorigid space, let D be a (φ,Γ_K)-module over X of rank d, and let $\delta_1,\ldots,\delta_d : K^\times \to \Gamma(X,\mathscr{O}_X^\times)$ be a set of continuous characters. Suppose there is a Zariski-dense set $X_{\mathrm{alg}} \subset X$ of maximal points such that for every $x \in X_{\mathrm{alg}}$, D_x is trianguline with parameters $\delta_{1,x},\ldots,\delta_{d,x}$. Then there exists a proper birational morphism $f : X' \to X$ of reduced pseudorigid spaces and an open subspace $U \subset X'$ containing $\{p=0\}$ such that $f^*D|_U$ is trianguline with parameters $f^*\delta_1,\ldots,f^*\delta_d$.*

Unlike the situation in characteristic 0, the triangulation extends over every point of characteristic p, and there are no critical points. This is again because there is no analogue of Fontaine's element t in our positive characteristic analogue of the Robba ring.

Finally, we turn to applications to the extended eigenvarieties constructed in [JN16]. We prove unconditionally that each irreducible component of the extended eigencurve is proper at the boundary of weight space, and that the Galois representations over characteristic p points in the closure of the Coleman–Mazur eigencurve are trianguline at p. The latter answers a question of [AIP18].

We actually prove these results under somewhat abstracted hypotheses, in order to facilitate deducing analogous results for other extended eigenvarieties. However, in most cases the necessary results have not been proven even for Galois representations attached to classical forms, nor have the required families of Galois representations been constructed.

While this manuscript was being prepared, [RZ20] appeared, giving a much more precise description of the extended eigencurve at the boundary of weight space. However, our approach is Galois-theoretic (following the approach of [DL16]), and we believe it is of independent interest.

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2. Background

2.1. **Rings of p-adic Hodge theory.** Let R be a pseudorigid $\mathscr{O}_E$-algebra, for some discretely valued field $E/\mathbf{Q}_p$ with uniformizer ϖ_E, with ring of definition $R_0 \subset R^\circ$ and pseudo-uniformizer $u \in R_0$, and assume that $p \notin R^\times$. Let K be a p-adic field, i.e., a characteristic 0 field complete with respect to a discrete valuation, with perfect residue field, let $\chi_{\mathrm{cyc}} : \mathrm{Gal}_K \to \mathbf{Z}_p^\times$ be the cyclotomic character, let $H_K := \ker\chi_{\mathrm{cyc}}$, and let $\Gamma_K := \mathrm{Gal}_K / H_K$. Given an interval $I \subset [0,\infty]$, we defined rings $(\widetilde{\Lambda}_{R_0,I,K}, \widetilde{\Lambda}^+_{R_0,I,K})$ and $(\Lambda_{R_0,I,K}, \Lambda^+_{R_0,I,K})$ in [Bel20] which (when $I = [0,b]$) are analogues of the characteristic 0 rings $(\widetilde{\mathbf{A}}_K^{(0,b]}, \widetilde{\mathbf{A}}_K^{\dagger,s(b)})$ and $(\mathbf{A}_K^{(0,r]}, \mathbf{A}_K^{\dagger,s(r)})$ defined in [Col08]. Here $s : (0,\infty) \to (0,\infty)$ is defined via $s(r) := \frac{p-1}{pr}$. We briefly recall their definitions here and state some of their properties.

Let $\mathbf{A}_{\mathrm{inf}} := W(\mathscr{O}_{\mathbf{C}_K}^\flat)$, where $\mathscr{O}_{\mathbf{C}_K}^\flat := \varprojlim_{x\mapsto x^p} \mathscr{O}_{\mathbf{C}_K}$ is the tilt of $\mathscr{O}_{\mathbf{C}_K}$. Let $\varepsilon := (\varepsilon^{(0)}, \varepsilon^{(1)}, \ldots) \in \mathscr{O}_{\mathbf{C}_K}^\flat$ be a choice of a compatible sequence of p-power roots of unity, with $\varepsilon^{(0)} = 1$ and $\varepsilon^{(1)} \neq 1$, and let $\pi := [\varepsilon] - 1 \in \mathbf{A}_{\mathrm{inf}}$. Then if $I = [a, b]$ for $0 \leq a \leq b \leq \infty$, we define $(\widetilde{\Lambda}_{R_0,I}, \widetilde{\Lambda}_{R_0,I}^+)$ such that

$$\mathrm{Spa}(\widetilde{\Lambda}_{R_0,I}, \widetilde{\Lambda}_{R_0,I}^+) = \left(\mathrm{Spa}(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}, R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}})\right) \left\langle \frac{[\overline{\pi}]^{s(a)}}{u}, \frac{u}{[\overline{\pi}]^{s(b)}} \right\rangle$$

If $a = 0$, we take $\frac{[\overline{\pi}]^\infty}{u} = 0$, and if $b = 0$, we take $\frac{u}{[\overline{\pi}]^\infty} = \frac{1}{[\overline{\pi}]}$.

The group H_K acts on $(\widetilde{\Lambda}_{R_0,I}, \widetilde{\Lambda}_{R_0,I}^+)$, because Gal_K acts on $\mathbf{A}_{\mathrm{inf}}$ and H_K fixes $[\overline{\pi}]$. Then by [Bel20, Corollary 3.36],

$$\mathrm{Spa}(\widetilde{\Lambda}_{R_0,I}^{H_K}, \widetilde{\Lambda}_{R_0,I}^{+,H_K}) = \left(\mathrm{Spa}(R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K}, R_0 \widehat{\otimes} \mathbf{A}_{\mathrm{inf}}^{H_K})\right) \left\langle \frac{[\overline{\pi}]^{s(a)}}{u}, \frac{u}{[\overline{\pi}]^{s(b)}} \right\rangle$$

If $I \subset I'$, we have injective maps $\widetilde{\Lambda}_{R_0,I'} \to \widetilde{\Lambda}_{R_0,I}$ and $\widetilde{\Lambda}_{R_0,I'}^{H_K} \to \widetilde{\Lambda}_{R_0,I}^{H_K}$. Thus, if I' is an interval with an open endpoint, we may define

$$(\widetilde{\Lambda}_{R_0,I'}, \widetilde{\Lambda}_{R_0,I'}^+) := \cap_{\substack{I\subset I'\\ \text{closed}}} (\widetilde{\Lambda}_{R_0,I}, \widetilde{\Lambda}_{R_0,I}^+)$$

and

$$(\widetilde{\Lambda}_{R_0,I'}^{H_K}, \widetilde{\Lambda}_{R_0,I'}^{+,H_K}) := \cap_{\substack{I\subset I'\\ \text{closed}}} (\widetilde{\Lambda}_{R_0,I}^{H_K}, \widetilde{\Lambda}_{R_0,I}^{+,H_K})$$

The rings $(\Lambda_{R_0,I,K}, \Lambda_{R_0,I,K}^+)$ are imperfect versions of these, defined when $I \subset [0, b]$ with b sufficiently small. Given $\lambda = \frac{m'}{m} \in \mathbf{Q}_{>0}$ with $\gcd(m, m') = 1$, let $(D_\lambda, D_\lambda^\circ)$ denote the pair of rings corresponding to the localization $(\mathscr{O}_E[\![u]\!], \mathscr{O}_E[\![u]\!]) \left\langle \frac{\varpi_E^m}{u^{m'}} \right\rangle$. By [Lou17, Lemma 4.8], there is some sufficiently small λ such that R is topologically of finite type over D_λ, so we may assume that R_0 is strictly topologically of finite type over D_λ°.

For any unramified extension $F/\mathbf{Q}_p$, the choice of ε gives us a natural map $k_F(\!(\overline{\pi})\!) \to \mathbf{C}_K^\flat$; let $\mathbf{E}_F$ denote its image, and let $\mathbf{E} \subset \mathbf{C}_K^\flat$ be its separable closure. Then $\mathrm{Gal}(\mathbf{E}/\mathbf{E}_F) \cong H_F$ (by the theory of the field of norms), and for any extension K/F, we set $\mathbf{E}_K := \mathbf{E}^{H_K}$. Then $\mathbf{E}_K$ is a discretely valued field, and we may choose a uniformizer $\overline{\pi}_K$; if we lift its minimal polynomial to characteristic 0, Hensel's lemma implies that we have a lift $\pi_K \in W(\mathbf{C}_K^\flat)$ which is integral over $\mathscr{O}_F[\![\pi]\!][\frac{1}{\pi}]^\wedge$. We fix a choice π_K for each K, and work with it throughout (when $F/\mathbf{Q}_p$ is unramified, we take π_F to be π).

Assume that $0 \leq a \leq b < r_K \cdot \lambda$, where r_K is a constant defined in [Col08], and that $\frac{1}{a\cdot v_{\mathbf{C}_K^\flat}(\overline{\pi}_K))}, \frac{1}{b\cdot v_{\mathbf{C}_K^\flat}(\overline{\pi}_K))} \in \mathbf{Z}$. Let $F' \subset K_\infty := K(\mu_{p^\infty})$ be the maximal unramified subfield. Then we define $\Lambda_{R_0,[a,b],K}$ to be the evaluation of $\mathscr{O}_{(R_0\otimes\mathscr{O}_{F'})[\![\pi_K]\!]}$ on the affinoid subspace of $\mathrm{Spa}(R_0\otimes\mathscr{O}_{F'})[\![\pi_K]\!]$ defined by the conditions $u \leq \pi_K^{1/(b\cdot v_{\mathbf{C}_K^\flat}(\overline{\pi}_K))}$ and $\pi_K^{1/(a\cdot v_{\mathbf{C}_K^\flat}(\overline{\pi}_K))} \leq u$ (and similarly for $\Lambda_{R_0,[a,b],K}^+$).

If $p = 0$ in R, then we may take λ arbitrarily large, and hence b arbitrarily large. Thus, in this case we additionally define $\Lambda_{R_0,[a,\infty],K} := (R_0 \otimes \mathscr{O}_{F'})[\![\pi_K]\!]$.

We further define $\Lambda_{R,(0,b],K} := \varprojlim_{a\to 0} \Lambda_{R,[a,b],K}$, and $\Lambda_{R,\mathrm{rig},K} := \varinjlim_{b\to 0} \Lambda_{R,(0,b],K}$.

The rings $\widetilde{\Lambda}_{R_0,I}^{H_K}$ and $\Lambda_{R_0,I,K}$ are equipped with actions of Frobenius and Γ_K. More precisely, we have isomorphisms

$$\varphi : \widetilde{\Lambda}_{R_0,I} \xrightarrow{\sim} \widetilde{\Lambda}_{R_0,\frac{1}{p}I}, \quad \varphi : \widetilde{\Lambda}_{R_0,I}^{H_K} \xrightarrow{\sim} \widetilde{\Lambda}_{R_0,\frac{1}{p}I}^{H_K}$$

and ring homomorphisms

$$\varphi : \Lambda_{R_0,[a,b],K} \to \Lambda_{R_0,[a/p,b/p],K}$$

However, the latter are not isomorphisms; φ makes $\Lambda_{R_0,[0,b/p],K}$ into a free $\varphi(\Lambda_{R_0,[0,b],K})$-module, with basis $\{1,[\varepsilon],\ldots,[\varepsilon]^{p-1}\}$.

If L/K is a Galois extension, $\widetilde{\Lambda}_{R_0,I}^{H_L}$ and $\Lambda_{R_0,I,L}$ are also equipped with actions of $H_{L/K} := H_K/H_L$.

Lemma 2.1.1. *If L/K is a finite Galois extension, then $\Lambda_{R_0,[0,b],L}/\Lambda_{R_0,[0,b],K}$ is a finite free extension and $\Lambda_{R_0,I,L}^{H_K} = \Lambda_{R_0,I,K}$.*

Proof. Let $F' \subset K_\infty := K(\mu_{p^\infty}), F'' \subset L_\infty := L(\mu_{p^\infty})$ be the maximal unramified subfields. A basis for $\mathscr{O}_{F''}$ over $\mathscr{O}_{F'}$ provides a basis for $(R_0 \widehat{\otimes} \mathscr{O}_{F''})[\![\pi_K]\!]$ over $(R_0 \widehat{\otimes} \mathscr{O}_{F'})[\![\pi_K]\!]$, so we may assume that $F' = F''$. Then if $e := e_{L_\infty/K_\infty} = [L_\infty : K_\infty]$, the set $\{1,\pi_L,\ldots,\pi_L^{e-1}\}$ is a basis for $\Lambda_{R_0,[0,0],L}$ over $\Lambda_{R_0,[0,0],K}$.

The trace map defines a perfect pairing

$$\begin{aligned} \Lambda_{R_0,[0,0],L} \times \Lambda_{R_0,[0,0],L} &\to \Lambda_{R_0,[0,0],K} \\ (x,y) &\mapsto \mathrm{Tr}(xy) \end{aligned}$$

The dual basis $\{f_1^* = 1,\ldots,f_e^*\}$ with respect to this pairing is the same as that constructed in [Col08, §6.3]. Since $(R_0 \widehat{\otimes} \Lambda_{[0,b/\lambda],L})\left\langle \frac{u}{\pi_K^{1/(b\cdot v_{\mathbf{C}_p^\flat}(\pi_L))}} \right\rangle$ is a ring of definition of $\Lambda_{R_0,[0,b],L}$ by [Bel20, Proposition 3.38], [Col08, Corollaire 6.10] implies that $f_i^* \in \Lambda_{R_0,[0,b],L}$ for all i. Then for any $x \in \Lambda_{R_0,[0,b],L}$, we may uniquely write $x = \sum_i \mathrm{Tr}(x\pi_L^i)f_i^*$, as desired. $\square$

By [Bel20, Proposition 3.10], the formation of $\widetilde{\Lambda}_{R,I}$ behaves well with respect to rational localization $\mathrm{Spa}\, R$, and $\Lambda_{R_0,I,K}$ does, as well, since it is sheafy. Thus, if X is a (not necessarily affinoid) pseudorigid space, we may let $\widetilde{\Lambda}_{X,I}^{H_K}$ and $\Lambda_{X,I,K}$ denote the corresponding sheaves of algebras.

2.2. (φ,Γ)**-modules and cohomology.** We briefly recall the theory of (φ,Γ)-modules over pseudorigid spaces.

Definition 2.2.1. A φ-module over $\Lambda_{R,(0,b],K}$ is a coherent sheaf D of modules over the pseudorigid space $\bigcup_{a\to 0}\mathrm{Spa}(\Lambda_{R,[a,b],K})$ equipped with an isomorphism

$$\varphi_D : \varphi^*D \xrightarrow{\sim} \Lambda_{R,(0,b/p],K} \otimes_{\Lambda_{R,(0,b],K}} D$$

If $a \in (0,b/p]$, a φ-module over $\Lambda_{R,[a,b],K}$ is a finite $\Lambda_{R,[a,b],K}$-module D equipped with an isomorphism

$$\varphi_{D,[a,b/p]} : \Lambda_{R,[a,b/p],K} \otimes_{\Lambda_{R,[a/p,b/p],K}} \varphi^*D \xrightarrow{\sim} \Lambda_{R,[a,b/p],K} \otimes_{\Lambda_{R,[a,b],K}} D$$

A (φ, Γ_K)-module over $\Lambda_{R,(0,b],K}$ (resp. $\Lambda_{R,[a,b],K}$) is a φ-module over $\Lambda_{R,(0,b],K}$ (resp. $\Lambda_{R,[a,b],K}$) equipped with a semi-linear action of Γ_K which commutes with φ_D (resp. $\varphi_{D,[a,b/p]}$).

A (φ, Γ_K)-module over R is a module D over $\Lambda_{R,\mathrm{rig},K}$ which arises via base change from a (φ, Γ_K)-module over $\Lambda_{R,(0,b],K}$ for some $b > 0$.

We also define a category of (φ, Γ)-modules equipped with Galois descent data:

Definition 2.2.2. If L/K is a finite Galois extension and D is a (φ, Γ_L)-module, we say that D is *equipped with an action of* $\mathrm{Gal}_{L/K}$ if the Galois group Gal_K acts on D and in addition

- the subgroup $H_L \subset \mathrm{Gal}_K$ acts trivially on D, and
- the induced action of Gal_L / H_L coincides with the action of Γ_L.

We also say that D is a $(\varphi, \Gamma_L, \mathrm{Gal}_{L/K})$-*module.*

In [Bel20] we constructed a functor $M \rightsquigarrow D_{b,L}(M)$ from the category of continuous R-linear Galois representations $\rho : \mathrm{Gal}_K \to \mathrm{GL}(M)$, where M is a projective R-module of rank d, to the category of of projective $(\varphi, \Gamma_L. \mathrm{Gal}_{L/K})$-modules of rank d over $\Lambda_{R_0,[0,b],K}$, for some Galois extension L/K and some $b > 0$.

Proposition 2.2.3. *If D is a projective $(\varphi, \Gamma_L. \mathrm{Gal}_{L/K})$-module of rank d over $\Lambda_{R_0,[0,b],K}$, for some Galois extension L/K and some $b > 0$, then D^{H_K} is a projective (φ, Γ_K)-module over $\Lambda_{R_0,[0,b],K}$.*

Proof. The extension $\Lambda_{R_0,[0,b],K} \to \Lambda_{R_0,[0,b],L}$ is a finite flat cover, by Lemma 2.1.1; descent of modules is effective and D^{H_K} is the descent of D to $\Lambda_{R_0,[0,b],K}$, so the natural map

$$\Lambda_{R_0,[0,b],L} \otimes_{\Lambda_{R_0,[0,b],K}} D^{H_K} \to D$$

is an isomorphism. We can check flatness after an fppf base change, so D^{H_K} is flat over $\Lambda_{R_0,[0,b],K}$. We can also check finiteness of a module after an fppf base change, so D^{H_K} is a finite $\Lambda_{R_0,[0,b],K}$-module. Since $\Lambda_{R_0,[0,b],K}$ is noetherian, it is finitely presented, so projective.

It remains to define the Γ_K-action and show that $\varphi : \varphi^* D \to D$ is an isomorphism. □

As a consequence, $D_{b,K}(M) := D_{b,L}(M)^{H_K}$ is a projective (φ, Γ_K)-module over $\Lambda_{R_0,[0,b],K}$ of rank d, and so $D_{\mathrm{rig},K}(M)$ is a projective (φ, Γ_K)-module over $\Lambda_{R,\mathrm{rig},K}$ of rank d.

Now suppose that $p > 2$. Fix a topological generator $\gamma \in \Gamma_K$. Then for a (φ, Γ_K)-module D, we define the Fontaine–Herr–Liu complex via

$$C^\bullet_{\varphi,\Gamma} : D \xrightarrow{\varphi_D - 1, \gamma - 1} D \oplus D \xrightarrow{(\gamma-1)\oplus(1-\varphi_D)} D$$

(concentrated in degrees 0, 1, and 2). We let $H^i_{\varphi,\Gamma_K}(D)$ denote its cohomology in degree i. If D is the (φ, Γ)-module attached to a Galois representation M, this complex computes the Galois cohomology of M.

The principal result of [Bel20, §5.1] is the following:

Theorem 2.2.4. [Bel20, Corollary 5.13] *If D is a projective (φ,Γ)-module over R, then $C^\bullet_{\varphi,\Gamma_K}(D)\in\mathbf{D}^{[0,2]}_{\mathrm{perf}}(R)$.*

As a corollary, if $R\to R'$ is a homomorphism of pseudoaffinoid algebras, there is a natural quasi-isomorphism

$$R'\otimes^{\mathrm{L}} C^\bullet_{\varphi,\Gamma}(D)\xrightarrow{\sim} C^\bullet_{\varphi,\Gamma}(R'\otimes_R D)$$

and there is a corresponding second-quadrant base-change spectral sequence.

We record the low-degree exact sequences of the base-change spectral sequence here:

Corollary 2.2.5. *Let $R\to R'$ be a morphism of pseudoaffinoid algebras and let D be a (φ,Γ_K)-module over $\Lambda_{R,\mathrm{rig},K}$. Then*

(1) *The natural morphism $R'\otimes_R H^2_{\varphi,\Gamma_K}(D)\to H^2_{\varphi,\Gamma_K}(R'\otimes_R D)$ is an isomorphism.*
(2) *The natural morphism $R'\otimes_R H^1_{\varphi,\Gamma_K}(D)\to H^1_{\varphi,\Gamma_K}(R'\otimes_R D)$ fits into an exact sequence*

$$0\to\mathrm{Tor}_2^R(H^2_{\varphi,\Gamma_K}(D),R')\to R'\otimes_R H^1_{\varphi,\Gamma_K}(D)\to H^1_{\varphi,\Gamma_K}(R'\otimes_R D)\to\mathrm{Tor}_1^R(H^2_{\varphi,\Gamma_K}(D),R')\to 0$$

(3) *There is a filtration*

$$0\subset F^0H^0_{\varphi,\Gamma_K}(R'\otimes_R D)\subset F^{-1}H^0_{\varphi,\Gamma_K}(R'\otimes_R D)\subset H^0_{\varphi,\Gamma_K}(R'\otimes_R D)$$

such that

$$0\to F^{-1}H^0_{\varphi,\Gamma_K}(R'\otimes_R D)\to H^0_{\varphi,\Gamma_K}(R'\otimes_R D)\to\ker\left(\mathrm{Tor}_2^R(H^2_{\varphi,\Gamma_K}(D),R')\to R'\otimes_R H^1_{\varphi,\Gamma_K}(D)\right)\to 0$$

and

$$0\to F^0H^0_{\varphi,\Gamma_K}(R'\otimes_R D)\to F^{-1}H^0_{\varphi,\Gamma_K}(R'\otimes_R D)\to\left(\mathrm{Tor}_1^R(H^1_{\varphi,\Gamma_K}(D),R')/\,\mathrm{Tor}_3^R(H^2_{\varphi,\Gamma_K}(D),R')\right)\to 0$$

are exact, and there is a natural surjection $R'\otimes_R H^0_{\varphi,\Gamma_K}(D)\twoheadrightarrow F^0H^0_{\varphi,\Gamma_K}(R'\otimes_R D)$, with kernel generated by $\mathrm{Tor}_2^R(H^1_{\varphi,\Gamma_K}(D),R')$ and $\ker\left(\mathrm{Tor}_3^R(H^2_{\varphi,\Gamma_K}(D),R')\to\mathrm{Tor}_1(H^1_{\varphi,\Gamma_K}(D),R')\right)$.

Proof. This follows from the convergence of the base-change spectral sequence. ☐

2.3. (φ,Γ)**-modules of character type.** Let $K/\mathbf{Q}_p$ be a finite extension with ramification degree e_K and inertia degree f_K, and let $\mathscr{O}_K$ be its ring of integers, k_K be its residue field, and ϖ_K be a uniformizer. Let $K_0\subset K$ be its maximal unramified subfield. Let R be a pseudoaffinoid algebra over $\mathbf{Z}_p$ with ring of definition $R_0\subset R$ and pseudo-uniformizer $u\in R_0$. By the structure theorem for pseudoaffinoid algebras [Lou17], we may assume R_0 is strictly topologically of finite type over $D^\circ_\lambda:=\mathbf{Z}_p[\![u]\!]\left\langle\frac{p^m}{u^{m'}}\right\rangle$, where $m,m'>0$ and $\lambda:=\frac{m'}{m}$ (we specifically rule out the setting where R is a $\mathbf{Q}_p$-affinoid algebra, since that case was treated in [KPX14]).

We begin by recalling the construction of (φ,Γ_K)-modules of character type from [KPX14].

Lemma 2.3.1. *Let $\alpha\in R^\times$. Up to isomorphism, there is a unique rank-1 $R\otimes\mathscr{O}_{K_0}$-module $D_{f_K,\alpha}$ equipped with a $1\otimes\varphi$-semilinear operator φ_α such that $\varphi_\alpha^{f_K}=\alpha\otimes 1$.*

Proof. This follows exactly as in [KPX14, Lemma 6.2.3]. ☐

Definition 2.3.2. Let $\delta : K^\times \to R^\times$, and write $\delta = \delta_1\delta_2$, where $\delta_1, \delta_2 \rightrightarrows R^\times$ are continuous characters such that δ_1 is trivial on $\mathscr{O}_K^\times$ and δ_2 is trivial on $\langle\varpi_K\rangle$. By local class field theory, δ_2 corresponds to a continuous character $\delta_2' : \mathrm{Gal}_K \to R^\times$. We let $\Lambda_{R,\mathrm{rig},K}(\delta_1) := D_{f_K,\delta_1(\varpi_K)} \otimes_{R\otimes\mathscr{O}_{K_0}} \Lambda_{R,\mathrm{rig},K}$ and $\Lambda_{R,\mathrm{rig},K}(\delta_2) := D_{\mathrm{rig},K}(\delta_2')$, and we define $\Lambda_{R,\mathrm{rig},K}(\delta) := \Lambda_{R,\mathrm{rig},K}(\delta_1) \otimes \Lambda_{R,\mathrm{rig},K}(\delta_2)$.

If D is a (φ,Γ_K)-module and $\delta : K^\times \to R^\times$ is a continuous character, we will let $D(\delta)$ denote $D \otimes \Lambda_{R,\mathrm{rig},K}(\delta)$. We will let $C^\bullet_{\varphi,\Gamma_K}(\delta)$ and $H^i_{\varphi,\Gamma_K}(\delta)$ denote the Fontaine–Herr–Liu complex and the cohomology groups of $\Lambda_{R,\mathrm{rig},K}(\delta)$, respectively.

Lemma 2.3.3. *Suppose L/K is a finite extension, and ϖ_L is a uniformizer of L with $\mathrm{Nm}_{L/K}(\varpi_L) = \varpi_K$. If $\delta : K^\times \to R^\times$ is a continuous character, then $\mathrm{Res}_K^L \Lambda_{R,\mathrm{rig},K}(\delta)$ is of character type, with associated character $\delta \circ \mathrm{Nm}_L/K$.*

Proof. We may consider separately the cases where δ is trivial on $\mathscr{O}_K^\times$ and $\langle\varpi_K\rangle$. If δ is trivial on $\mathscr{O}_K^\times$, then

$$\begin{aligned}\mathrm{Res}_K^L \Lambda_{R,\mathrm{rig},K}(\delta) &= D_{f_K,\delta(\varpi_K)} \otimes_{R\otimes\mathscr{O}_{K_0}} \Lambda_{R,\mathrm{rig},L}\\ &= \left(D_{f_K,\delta(\varpi_K)} \otimes_{R\otimes\mathscr{O}_{K_0}} (R\otimes\mathscr{O}_{L_0})\right) \otimes_{R\otimes\mathscr{O}_{L_0}} \Lambda_{R,\mathrm{rig},L}\end{aligned}$$

But $D_{f_K,\delta(\varpi_K)} \otimes_{R\otimes\mathscr{O}_K} (R\otimes\mathscr{O}_L)$ is a rank-1 $R\otimes\mathscr{O}_{L_0}$-module equipped with a $1\otimes\varphi$-semilinear operator $\varphi_{\delta(\varpi_K)}$ such that $\varphi_{\delta(\varpi_K)}^{f_L} = \delta(\varpi_K)\otimes 1$, so it is isomorphic to $D_{f_L,\delta(\varpi_K)}$. By definition, $D_{f_L,\delta(\varpi_K)} \otimes_{R\otimes\mathscr{O}_{L_0}} \Lambda_{R,\mathrm{rig},L}$ is equal to $\Lambda_{R,\mathrm{rig},L}(\delta\circ\mathrm{Nm}_{L/K})$. On the other hand, if δ is trivial on $\langle\varpi_K\rangle$, the statement follows from functoriality for local class field theory. □

Continuous characters vary in analytic families, and hence (φ,Γ)-modules do, as well:

Proposition 2.3.4. *Let G be a topological group of the form $G_0 \times \mathbf{Z}^{\oplus r_1} \times \mathbf{Z}_p^{\oplus r_2}$, where G_0 is a finite discrete group. Then there is a pseudorigid space X_G and a continuous character $\delta_{\mathrm{univ}} : G \to \mathscr{O}_{X_G}$ such that every continuous character $\delta : G \to R^\times$, where R is a pseudoaffinoid algebra, arises via the pullback of δ_{univ} along a unique morphism $\mathrm{Spa}\, R \to X_G$.*

This result is well-known in the $\mathbf{Q}_p$-affinoid setting (see e.g. [**?**, Lemma 8.2] or [KPX14, Proposition 6.1.1]), and the construction is identical in the pseudorigid setting. In the case $G = \mathbf{Q}_p^\times \cong \mu_{p-1} \times \mathbf{Z} \times \mathbf{Z}_p$, the moduli space X_G has connected components indexed by the elements of μ_{p-1}, each of which is isomorphic to $\mathbf{G}_m^{\mathrm{an}} \times (\mathrm{Spa}\, \mathbf{Z}_p[\![\mathbf{Z}_p]\!])^{\mathrm{an}}$.

As a consequence, we may deduce the Euler characteristic formula for all (φ,Γ_K)-modules:

Theorem 2.3.5. *If D is a (φ,Γ_K)-module with coefficients in a pseudoaffinoid algebra R, then $\chi(D) = -[K : \mathbf{Q}_p]$.*

Proof. Euler characteristics are locally constant, so it suffices to compute $\chi(D_x)$ for a single maximal point x on each connected component of $\mathrm{Spa}\, R$. Thus, we may assume that R is a finite extension of either $\mathbf{Q}_p$ or $\mathbf{F}_p(\!(u)\!)$, and so we have access to the slope filtration theorem of [**?**].

Since Euler characteristics are additive in exact sequences, we may assume that D is pure of slope s; if necessary, replace R by an étale extension so that the slope of D is in the value group of R. The moduli space $X_{\langle\varpi_K\rangle} \cong \mathbf{G}_m^{\mathrm{an}}$ of continuous characters of $\langle\varpi_K\rangle$ has a universal character $\delta_{\mathrm{univ}} : \langle\varpi_K\rangle \to \mathscr{O}^\times_{X_{\langle\varpi_K\rangle}}$, so we may consider the Fontaine–Herr–Liu complex $C^\bullet_{\varphi,\Gamma}$ of the (φ,Γ_K)-module $D(\delta_{\mathrm{univ}})$ over $X_{\langle\varpi_K\rangle}$. Since $C^\bullet_{\varphi,\Gamma}(D(\delta)) \in D^{[0,2]}_{\mathrm{perf}}(R')$ for every affinoid subdomain $\mathrm{Spa}(R') \subset X$, its Euler characteristic is constant on connected components, and it suffices to verify the statement at one point on each component. But each connected component contains a point x such that the slope of $D(\delta)$ at x is 0; then $(D(\delta))(x)$ is étale and we may appeal to the Euler characteristic formula for Galois cohomology. $\square$

3. Positive characteristic function fields

In this section, we closely study overconvergent (φ,Γ)-modules where the coefficients are finite extensions of $\mathbf{F}_p(\!(u)\!)$. This is similar to the situation studied by Hartl [**?**] and Hartl–Pink [**?**], but because we are interested in (φ,Γ)-modules related to representations of characteristic-0 Galois groups, we may work with imperfect coefficients. For this reason, we rely on the slope filtration theorem and φ-cohomology calculations from [**?**], rather than the Dieudonné–Manin classification theorem from [**?**]. We first calculate the cohomology of certain rank-1 $(\varphi,\Gamma_{\mathbf{Q}_p})$-modules (using techniques similar to [Col08]), and then use those calculations to deduce the Tate local duality theorem for all (φ,Γ_K)-modules (following the strategy of [**?**]).

3.1. **Cohomology of rank-1 (φ,Γ)-modules.** We begin by computing the cohomology of (φ,Γ)-modules of character type.

Lemma 3.1.1. *Assume R is a finite extension of $\mathbf{F}_p(\!(u)\!)$, and let $\delta : K^\times \to R^\times$ be a continuous character. Then $H^0_{\varphi,\Gamma_K}(\delta) = 0$ unless δ is trivial, in which case it is a free R-module of rank 1.*

Proof. Write $\delta = \delta_1\delta_2$, as above. We first show that the kernel of $\varphi - 1$ on $\Lambda_{R,\mathrm{rig},K}(\delta_1)$ is trivial unless $\delta_1(\varpi_K) = 1$, in which case it is R, and then compute the elements of $\ker(\varphi-1)$ fixed by Γ_K.

If $f(\overline{\pi}_K) \in \Lambda_{R,\mathrm{rig},K}(\delta_1)$, we may write $f(\overline{\pi}_K)$ uniquely as $f(\overline{\pi}_K) = \sum_{i\in\mathbf{Z}} a_i\overline{\pi}_K^i$, where $a_i \in R\otimes k'$ (for some finite extension k'/k_K). There is some integer $f \geq 1$ such that that $\varphi^{f_K f}$ fixes k'. Using the fact that $\varphi^{f_K f}(\overline{\pi}_K) = \overline{\pi}_K^{f_K f p}$, a straightforward calculation shows that the kernel of $\varphi^{f_K f} - 1$ is trivial unless $\delta(\varpi_K)^f = 1$, in which case it is $R\otimes k'$. We now need to compute the kernel of $\varphi^{f_K} - 1$ on $D_{f_K,\delta(\varpi_K)}\otimes_{k_K} k'$. But there is a basis $\{e_0,\ldots,e_{f-1}\}$ of k'/k_K such that φ^{f_K} acts via $\varphi^{f_K}(e_i) = e_{i-1}$, where the indices are taken modulo f, so the kernel of $\varphi_D^{f_K} - 1$ is trivial unless $\delta(\varpi_K) = 1$, in which case it is $D_{f_K,\alpha}$. We have reduced to computing the kernel of $\varphi - 1$ on $D_{f_K,1}$, but the construction makes clear that this kernel is precisely R.

Now suppose δ_1 is trivial, so that $H^0_{\varphi,\Gamma_K}(\Lambda_{R,\mathrm{rig},K}(\delta))$ is $R^{\Gamma_K=1}$. If $\gamma \in \Gamma_K$ is a topological generator of Γ_K, it acts on R via multiplication by β for some $\beta \in \Lambda_{R,(0,b],K}$. This clearly fixes no elements unless $\beta = 1$, in which case it fixes all of R. $\square$

Corollary 3.1.2. *If R is a finite extension of $\mathbf{F}_p((u))$ and D is a rank-1 (φ,Γ)-module over $\Lambda_{R,\mathrm{rig},K}$ of character type, then D has no proper non-trivial sub-(φ,Γ)-module or quotient (φ,Γ)-module.*

Lemma 3.1.3. *Suppose $\alpha \in R^\times$ satisfies $v_R(\alpha) < 0$. Then if $f \in \Lambda_{R,[0,b],\mathbf{Q}_p}$ is in the image of φ, there is some $b' > 0$ such that f is in the image of $\alpha\varphi - 1 : \Lambda_{R,[0,b'],\mathbf{Q}_p} \to \Lambda_{R,[0,b'/p],\mathbf{Q}_p}$.*

Proof. We are looking for a solution to the equation $(\alpha\varphi-1)(g) = \varphi(f')$; applying ψ to both sides, it suffices to show that the sum $\sum_{k\geq 0}(\alpha^{-1}\psi)^k$ converges on $\Lambda_{R,[0,b'],K}$ for b' sufficiently small. But we may write

$$\psi\left(\sum_{i\in\mathbf{Z}}\alpha_i\overline{\pi}^i\right) = \sum_{i\in\mathbf{Z}}\left(\sum_{j=0}^{p-1}\alpha_{pi+j}\right)\overline{\pi}^i$$

so for any $b' > 0$,

$$v_{R,b'}\left(\psi\left(\sum_{i\in\mathbf{Z}}\alpha_i\overline{\pi}^i\right)\right) \geq v_{R,b'/p}\left(\sum_{i\in\mathbf{Z}}\alpha_i\overline{\pi}^i\right) - pb' \geq v_{R,b'}\left(\sum_{i\in\mathbf{Z}}\alpha_i\overline{\pi}^i\right) - pb'$$

where the second inequality follows from the maximum modulus principle. Thus, we may simply choose $b' < \frac{1}{p}v_R(\alpha^{-1})$. $\square$

Lemma 3.1.4. *Suppose $\alpha \in R^\times$. Then $\alpha\varphi - 1 : \overline{\pi}_K\Lambda_{R,(0,\infty],K} \to \overline{\pi}_K\Lambda_{R,(0,\infty],K}$ is surjective. If $\alpha \neq 1$, then $\alpha\varphi - 1 : \Lambda_{R,(0,\infty],K} \to \Lambda_{R,(0,\infty],K}$ is surjective.*

Proof. It suffices to show that $\sum_{k\geq 0}(\alpha\varphi)^k$ converges on $\overline{\pi}_K\Lambda_{R,[a,\infty],K}$ for all $a > 0$. Now for any $f = \sum_{i\geq 1}\alpha_i\overline{\pi}_K^i$, we have

$$\begin{aligned} v_{R,[a,\infty]}((\alpha\varphi)^k(f)) &= k\cdot v_R(\alpha) + \inf_i\left\{v_R(\alpha_i) + \frac{p^{k+1}ai}{p-1}\right\} \\ &\geq k\cdot v_R(\alpha) + a(p^k+\cdots+p) + \inf_i\left\{v_R(\alpha_i) + \frac{pai}{p-1}\right\} \\ &= v_{R,[a,\infty]}(f) + k\cdot v_R(\alpha) + a(p^k+\cdots+p) \end{aligned}$$

Thus, for any $\alpha \in R^\times$ and any $a > 0$, the sum $\sum_{k\geq 0}(\alpha\varphi)^k(f)$ converges to an element of $\overline{\pi}_K\Lambda_{R,[a,\infty],K}$, as desired.

If $\alpha \neq 1$, then $(\alpha\varphi-1)\left(\frac{1}{\alpha-1}\right) = 1$, so R is also in the image of $\alpha\varphi - 1$. $\square$

Corollary 3.1.5. *Suppose $\alpha \in R^\times$ satisfies $v_R(\alpha) < 0$. If $f \in \Lambda_{R,(0,b],\mathbf{Q}_p}$, then (possibly after shrinking b) there is some $g \in \Lambda_{R,(0,b],\mathbf{Q}_p}$ such that $f - (\alpha\varphi-1)g \in \Lambda_{R,[0,b],\mathbf{Q}_p}^{\psi=0}$.*

Proof. We have exact sequences

$$0 \to R_0[\![\overline{\pi}]\!] \to \Lambda_{R,(0,\infty],\mathbf{Q}_p} \oplus \Lambda_{R,[0,b],\mathbf{Q}_p} \to \Lambda_{R,(0,b],\mathbf{Q}_p} \to 0$$

for every $b > 0$, so we may write $f = f_+ + f_-$, where $f_+ \in \overline{\pi}\Lambda_{R,(0,\infty],\mathbf{Q}_p}$ and $f_- \in \Lambda_{R,[0,b],\mathbf{Q}_p}$. Then we can find $g_+ \in \overline{\pi}\Lambda_{R,(0,\infty],\mathbf{Q}_p}$ and $g_- \in \Lambda_{R,[0,b'],\mathbf{Q}_p}$ for some $b' \leq b$ such that $f_+ = (\alpha\varphi-1)(g_+)$ and $f_- - (\alpha\varphi-1)(g_-) \in \Lambda_{R,[0,b'],\mathbf{Q}_p}^{\psi=0}$, so $f - (\alpha\varphi-1)(g_+ + g_-) \in \Lambda_{R,[0,b'],\mathbf{Q}_p}^{\psi=0}$, as desired. $\square$

Corollary 3.1.6. *If $\delta : \mathbf{Q}_p^\times \to R^\times$ is a continuous character trivial on $1+p\mathbf{Z}_p \subset \mathbf{Z}_p^\times$, such that $v_R(\delta(p))<0$, then $H^2_{\varphi,\Gamma_{\mathbf{Q}_p}}(\delta)=0$.*

Proof. Lemma 3.1.5 implies that after subtracting an element of the form $(\alpha\varphi-1)(g)$, any cohomology class of $H^2_{\varphi,\Gamma_{\mathbf{Q}_p}}(\delta)$ has a representative $f\in\Lambda^{\psi=0}_{R,[0,b],\mathbf{Q}_p}$, for sufficiently small b. But if γ is a topological generator of the procyclic part of $\mathbf{Z}_p^\times$, $\gamma-1$ acts invertibly on $\Lambda^{\psi=0}_{R,[0,b],\mathbf{Q}_p}$, for sufficiently small b, and the result follows. □

Now we wish to compute $H^1_{\varphi,\Gamma_{\mathbf{Q}_p}}(\delta)$, where $v_R(\delta(p))<0$ and δ is trivial on $1+p\mathbf{Z}_p$.

Lemma 3.1.7. *If $\alpha\in R^\times$ satisfies $v_R(\alpha)<0$ and $(\alpha\varphi-1)(f)\in\Lambda^{\psi=0}_{R,(0,b],\mathbf{Q}_p}$, then $f\in\Lambda_{R,(0,\infty],\mathbf{Q}_p}$.*

Proof. We may assume $f=\sum_{i\le -1}\alpha_i\overline{\pi}^i$ with $\alpha_i\in R_0$, since ψ commutes with passing to Laurent "tails". Then

$$0=\psi\left((\varphi-\alpha^{-1})(f)\right)=(1-\alpha^{-1}\psi)(f)$$

so $f=\alpha^{-1}\psi(f)$. But for any $b'<\frac{1}{p}v_R(\alpha^{-1})$, we have seen that $v_{R,b'}(\alpha^{-1}\psi(f))>v_{R,b'}(f)$; since $\Lambda_{R,[0,b'],\mathbf{Q}_p}$ is $\overline{\pi}$-adically separated, this implies $f=0$. □

Lemma 3.1.8. *If γ is a topological generator of the procyclic part of $\Gamma_{\mathbf{Q}_p}$, the action of $\gamma-1$ defines a surjective map $\gamma-1:\Lambda^{\psi=0}_{R_0,[0,\infty],\mathbf{Q}_p}\to\oplus_{j=1}^{j-1}\varepsilon^j\varphi\left(\overline{\pi}\Lambda_{R_0,[0,\infty],\mathbf{Q}_p}\right)$.*

Proof. We may assume $\chi(\gamma)=1+p$; then for any $f\in\Lambda_{R_0,[0,\infty],\mathbf{Q}_p}$,

$$(\gamma-1)\left(\varepsilon\varphi(f)\right)=\varepsilon\varphi\left(G_\gamma(f)\right)$$

where $G_\gamma(f)=\varepsilon\gamma(f)-f=\overline{\pi}\left(1-\frac{\varepsilon}{\overline{\pi}}(\gamma-1)\right)(f)$. In addition,

$$\gamma(\overline{\pi})=(1+\overline{\pi})^{1+p}-1=(1+\overline{\pi})(1+\overline{\pi}^p)-1=\overline{\pi}+\overline{\pi}^p+\overline{\pi}^{p+1}=\overline{\pi}+\varphi(\overline{\pi})\varepsilon$$

so $\gamma-1$ carries $\Lambda_{R_0,[0,\infty],\mathbf{Q}_p}$ to $\varphi(\overline{\pi})\Lambda_{R_0,[0,\infty],\mathbf{Q}_p}$, and $G_\gamma(f)\in\overline{\pi}\Lambda_{R_0,[0,\infty],\mathbf{Q}_p}$.

It remains to show that $\overline{\pi}^i$ is in the image of $G_\gamma(f)$ for all $i\ge 1$. To see this, it suffices to show that $\sum_{k\ge0}\left(\frac{\varepsilon}{\overline{\pi}}(\gamma-1)\right)^k(\overline{\pi})^i$ converges for all $i\ge1$. But

$$(\gamma-1)(\overline{\pi}^i)=\sum_{j=0}^{i-1}\binom{i}{j}\overline{\pi}^j\varphi(\overline{\pi})^{i-j}\varepsilon^{i-j}\in\overline{\pi}^{i+(p-1)}>1$$

so $\sum_{k\ge0}\left(\frac{\varepsilon}{\overline{\pi}}(\gamma-1)\right)^k(\overline{\pi})^i$ and $G_\gamma(f)$ is invertible. □

Lemma 3.1.9. *If $\delta:\mathbf{Q}_p^\times\to R^\times$ is a character with $v_R(p)<0$ and $\delta|_{1+p\mathbf{Z}_p}=1$, then $H^1_{\varphi,\Gamma_{\mathbf{Q}_p}}(\delta)$ is 1-dimensional.*

Proof. Let $\alpha:=\delta(p)$. For any cohomology class in $H^1_{\varphi,\Gamma_{\mathbf{Q}_p}}(\delta)$, we may choose a representative pair $(f,g)\in\Lambda_{R,(0,b/p],\mathbf{Q}_p}\oplus\Lambda_{R,(0,b],\mathbf{Q}_p}$ with $(\gamma-1)(f)=(\alpha\varphi-1)(g)$ (for some topological generator γ of the procyclic part of $\mathbf{Z}_p^\times$). By Lemma 3.1.5, there is some $h\in\Lambda_{R,(0,b],\mathbf{Q}_p}$ such that $f-(\alpha\varphi-1)(h)\in\Lambda^{\psi=0}_{R,[0,b/p],\mathbf{Q}_p}$, so we may replace (f,g) with $(f-(\alpha\varphi-1)(h),g-(\gamma-1)(h))$ and assume that $f\in\Lambda^{\psi=0}_{R,[0,b/p],\mathbf{Q}_p}$.

Since $\gamma - 1$ preserves $\Lambda_{R,[0,b],\mathbf{Q}_p}^{\psi=0}$, we have $(\alpha\varphi - 1)(g) \in \Lambda_{R,[0,b],\mathbf{Q}_p}^{\psi=0}$, as well, so $g \in \Lambda_{R,(0,\infty],\mathbf{Q}_p}$ by Lemma 3.1.7. But then $(\alpha\varphi - 1)(g) \in \Lambda_{R,(0,\infty],\mathbf{Q}_p}$, as well as in $\Lambda_{R,[0,b],\mathbf{Q}_p}^{\psi=0}$, so $(\alpha\varphi - 1)(g) \in \Lambda_{R_0,[0,\infty],\mathbf{Q}_p}$.

Let c denote the image of $(\alpha\varphi - 1)(g)$ in $\Lambda_{R_0,[0,\infty],\mathbf{Q}_p}/\overline{\pi}^p$. We claim that (f, g) is a coboundary if and only if $c = 0$. Indeed, if (f, g) represents the 0 class, there is some $h \in \Lambda_{R_0,(0,b],\mathbf{Q}_p}$ such that $f = (\alpha\varphi - 1)(h)$ and $g = (\gamma - 1)(h)$ (possibly after shrinking b). Since $f \in \Lambda_{R_0,[0,b],\mathbf{Q}_p}^{\psi=0}$, Lemma 3.1.7 implies that $h \in \Lambda_{R,(0,\infty],\mathbf{Q}_p}$ and hence $f \in \Lambda_{R_0,[0,b],\mathbf{Q}_p}^{\psi=0} \cap \Lambda_{R,(0,\infty],\mathbf{Q}_p} = \Lambda_{R_0,[0,\infty],\mathbf{Q}_p}^{\psi=0}$. Then Lemma 3.1.8 implies that $c \equiv (\gamma - 1)(f) \equiv 0 \mod \overline{\pi}^p\Lambda_{R_0,[0,\infty],\mathbf{Q}_p}^{\psi=0}$.

On the other hand, suppose $c = 0$. Then Lemma 3.1.8 implies that $f \in \Lambda_{R_0,[0,\infty],\mathbf{Q}_p}^{\psi=0} \subset \Lambda_{R_0,(0,\infty],\mathbf{Q}_p}$; since $\alpha\varphi - 1 : \Lambda_{R,(0,\infty],\mathbf{Q}_p} \to \Lambda_{R,(0,\infty],\mathbf{Q}_p}$ is surjective by Lemma 3.1.4 ($\alpha \neq 1$, by assumption), we may assume $f = 0$. It follows that $(\alpha\varphi - 1)(g) = 0$, so by Lemma 3.1.3, $g = 0$.

Thus, there is an injective map $H^1_{\varphi,\Gamma_{\mathbf{Q}_p}}(\delta) \to \Lambda_{R_0,[0,\infty],\mathbf{Q}_p}^{\psi=0}/\overline{\pi}^p\Lambda_{R_0,[0,\infty],\mathbf{Q}_p}^{\psi=0}$. To compute its image, we consider the Δ-fixed subspace of the target (since we may assume (f, g) is fixed by Δ). The subgroup $\Delta \subset \mathbf{Z}_p^\times$ acts semi-linearly on $\Lambda_{R_0,[0,\infty],\mathbf{Q}_p}$ via $\delta|_\Delta$, and we claim it fixes a subspace of dimension 1 over R. It suffices to consider sums of the form $\sum_{j=1}^{j-1} \varepsilon^j\alpha_j$, where $\alpha_j \in R_0$; we observe that $\sum_{j=1}^{p-1} \varepsilon^j\alpha_j \in \overline{\pi}\Lambda_{R_0,[0,\infty],\mathbf{Q}_p}$ if and only if $\sum_j \alpha_j = 0$. Since Δ is a cyclic group, we may choose a generator $\gamma \in \Delta$ and compute $\gamma\left(\sum_{j=1}^{j-1} \varepsilon^j\alpha_j\right) = \delta(\gamma)\sum_{j=1}^{p-1}\gamma(\varepsilon)^j\alpha_j$. Moreover, $\{\gamma(\varepsilon)^j\} = \{\varepsilon^j\}$ since γ is a generator, so $\{\alpha_j\} = \{\delta(\gamma)^j\alpha_1\}$. Thus, if $\delta(\gamma) = 1$, we see that

$$\sum_j \alpha_j = 0 \Leftrightarrow \alpha_j = 0 \text{ for all } j \Leftrightarrow \sum_{j=1}^{p-1} \varepsilon^j\alpha_j \in \overline{\pi}^p\Lambda_{R_0,[0,\infty],\mathbf{Q}_p}^{\Delta=1,\psi=0}$$

and $H^1_{\varphi,\Gamma_{\mathbf{Q}_p}}(\delta)$ is 1-dimensional over R, with classes in bijection with $(\alpha\varphi - 1)(g)(0)$.

If $\delta|_\Delta$ is non-trivial, we have $\sum_j \alpha_j = \sum_j \delta(\gamma)^j\alpha_1 = (\delta(\gamma) - 1)^{-1}(\delta(\gamma)^{p-1} - 1) = 0$ automatically. Thus, the Δ-fixed subspace of $\Lambda_{R_0,[0,\infty],\mathbf{Q}_p}^{\psi=0}/\overline{\pi}^p\Lambda_{R_0,[0,\infty],\mathbf{Q}_p}^{\psi=0}$ is a 1-dimensional R-vector space which lies in the image of $\left(\overline{\pi}\Lambda_{R_0,[0,\infty],\mathbf{Q}_p}\right)^{\psi=0}$. □

3.2. **Tate local duality.**

Lemma 3.2.1. *If $\delta : \mathbf{Q}_p^\times \to R^\times$ is a continuous character such that $v_R(\delta(p)) < 0$ and $\delta|_{1+p\mathbf{Z}_p}$ is trivial, then Tate duality holds for $\Lambda_{R,\mathrm{rig},\mathbf{Q}_p}(\delta)$ and $\Lambda_{R,\mathrm{rig},\mathbf{Q}_p}(\delta^{-1}\chi_{\mathrm{cyc}})$.*

Proof. Theorem 2.3.5 implies that $\dim_R H^1_{\varphi,\Gamma_{\mathbf{Q}_p}}(\delta^{-2}\chi_{\mathrm{cyc}}) \geq 1$, and so there is a non-split extension of $(\varphi, \Gamma_{\mathbf{Q}_p})$-modules

$$0 \to \Lambda_{R,\mathrm{rig},\mathbf{Q}_p}(\delta^{-1}\chi_{\mathrm{cyc}}) \to D \to \Lambda_{R,\mathrm{rig},\mathbf{Q}_p}(\delta) \to 0$$

Then an argument with slope filtrations shows that D is pure of slope 0, hence comes from a Galois representation, so Tate local duality holds for D. The associated long exact sequence in cohomology, combined with Lemma 3.1.1 shows that $H^0_{\varphi,\Gamma_{\mathbf{Q}_p}}(D) = 0$, so duality implies that $H^2_{\varphi,\Gamma_{\mathbf{Q}_p}}(D^\vee(\chi_{\mathrm{cyc}})) = 0$.

Since $\delta^{-1}\chi_{\mathrm{cyc}}$ and δ are non-trivial, $H^0_{\varphi,\Gamma_{\mathbf{Q}_p}}(\delta^{-1}\chi_{\mathrm{cyc}}) = H^0_{\varphi,\Gamma_{\mathbf{Q}_p}}(\delta) = 0$, and since $v_R(\delta(p)) < 0$, $H^2_{\varphi,\Gamma_{\mathbf{Q}_p}}(\delta) = 0$ and $H^1_{\varphi,\Gamma_{\mathbf{Q}_p}}(\delta)$ is 1-dimensional.

If we dualize our exact sequence and tensor with χ_{cyc}, we get a second exact sequence

$$0 \to \Lambda_{R,\mathrm{rig},\mathbf{Q}_p}(\delta^{-1}\chi_{\mathrm{cyc}}) \to D^\vee(\chi_{\mathrm{cyc}}) \to \Lambda_{R,\mathrm{rig},\mathbf{Q}_p}(\delta) \to 0$$

and its associated long exact sequence in cohomology. Then the cup product gives us a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & H^1_{\varphi,\Gamma_{\mathbf{Q}_p}}(\delta^{-1}\chi_{\mathrm{cyc}}) \arrow[r] \arrow[d] & H^1_{\varphi,\Gamma_{\mathbf{Q}_p}}(D^\vee(\chi_{\mathrm{cyc}})) \arrow[r] \arrow[d] & H^1_{\varphi,\Gamma_{\mathbf{Q}_p}}(\delta) \arrow[r] \arrow[d] & H^2_{\varphi,\Gamma_{\mathbf{Q}_p}}(\delta \arrow[d] \\
H^2_{\varphi,\Gamma_{\mathbf{Q}_p}}(\delta^{-1}\chi_{\mathrm{cyc}})^\vee \arrow[r] & H^1_{\varphi,\Gamma_{\mathbf{Q}_p}}(\delta)^\vee \arrow[r] & H^1_{\varphi,\Gamma_{\mathbf{Q}_p}}(D)^\vee \arrow[r] & H^1_{\varphi,\Gamma_{\mathbf{Q}_p}}(\delta^{-1}\chi_{\mathrm{cyc}})^\vee \arrow[r] & 0
\end{tikzcd}$$

Since $H^1_{\varphi,\Gamma_{\mathbf{Q}_p}}(D^\vee(\chi_{\mathrm{cyc}})) \to H^1_{\varphi,\Gamma_{\mathbf{Q}_p}}(D)^\vee$ is an isomorphism (by the classical theorem), a diagram chase shows that $H^1_{\varphi,\Gamma_{\mathbf{Q}_p}}(\delta^{-1}\chi_{\mathrm{cyc}}) \to H^1_{\varphi,\Gamma_{\mathbf{Q}_p}}(\delta)^\vee$ is injective, so $\dim_R H^1_{\varphi,\Gamma_{\mathbf{Q}_p}}(\delta^{-1}\chi_{\mathrm{cyc}}) \leq \dim_R H^1_{\varphi,\Gamma_{\mathbf{Q}_p}}(\delta) = 1$. But Theorem 2.3.5 implies that $\dim_R H^1_{\varphi,\Gamma_{\mathbf{Q}_p}}(\delta^{-1}\chi_{\mathrm{cyc}}) \geq 1$, so $\dim_R H^1_{\varphi,\Gamma_{\mathbf{Q}_p}}(\delta^{-1}\chi_{\mathrm{cyc}}) = 1$ and the map $H^1_{\varphi,\Gamma_{\mathbf{Q}_p}}(\delta^{-1}\chi_{\mathrm{cyc}}) \to H^1_{\varphi,\Gamma_{\mathbf{Q}_p}}(\delta)^\vee$ is an isomorphism. □

Theorem 3.2.2. *Tate local duality holds for every (φ, Γ)-module D over $\Lambda_{R,\mathrm{rig},K}$.*

Proof. We may replace D by $\mathrm{Ind}_K^{\mathbf{Q}_p} D$ and treat the case of (φ, Γ)-modules over $\Lambda_{R,\mathrm{rig},\mathbf{Q}_p}$. We may also assume that D is pure of slope s, and by replacing it with $D^\vee(\chi_{\mathrm{cyc}})$ if necessary, that $s \geq 0$.

If $s = 0$, D is étale and the result follows from the comparison with Galois cohomology. Otherwise, we proceed by induction on the degree of D, i.e. $\deg(D) := (\mathrm{rk}\, D)s$. Let $\delta : \mathbf{Q}_p^\times \to R^\times$ be a continuous character with $v_R(\delta(p)) = -1$ and $\delta|_{1+p\mathbf{Z}_p}$ trivial. Since $\dim_R H^1_{\varphi,\Gamma_{\mathbf{Q}_p}}(D(\delta^{-1})) \geq \mathrm{rk}\, D \geq 1$ by Theorem 2.3.5, there is a non-split extension

$$0 \to D \to D' \to \Lambda_{R,\mathrm{rig},\mathbf{Q}_p}(\delta) \to 0$$

We will prove that Tate local duality holds for D'; since it also holds for $\Lambda_{R,\mathrm{rig},\mathbf{Q}_p}(\delta)$, we may deduce it for D.

If D' is pure, the result follows, since D' has degree $\deg D - 1$ and slope $(\deg D - 1)/(\mathrm{rk}\, D + 1) < s$. Otherwise, D' has a unique slope filtration $0 = D_0 \subset D_1 \subset \cdots \subset D_k = D'$ by saturated (φ, Γ)-submodules, such that the successive quotients are pure and $\mu(D_1/D_0) < \mu(D_2/D_1) < \cdots < \mu(D_k/D_{k-1})$. Then $\mu(D_1) \leq \mu(D') < \mu(D)$.

We have an exact sequence

$$0 \to D_1 \cap D \to D_1 \to D_1/(D_1 \cap D) \to 0$$

But $D_1/(D_1 \cap D)$ is the image of D_1 in the quotient $\Lambda_{R,\mathrm{rig},\mathbf{Q}_p}(\delta)$, so it is either 0 or all of $\Lambda_{R,\mathrm{rig},\mathbf{Q}_p}(\delta)$. If $D_1 \cap D = 0$, then $D_1 = \Lambda_{R,\mathrm{rig},\mathbf{Q}_p}(\delta)$ and the map $D_1 \to D$ is a section of $D \to \Lambda_{R,\mathrm{rig},\mathbf{Q}_p}(\delta)$. But we constructed our extension to be non-split, so $D_1 \subset D$ and $\mu(D_1) > 0$ (since D is pure of positive slope).

Thus, $\mu(D_i/D_{i-1}) > 0$ for all i. Moreover, $\deg D' = \sum_i \deg(D_i/D_{i-1}) = \sum_i \mu(D_i/D_{i-1})\cdot \mathrm{rk}(D_i/D_{i-1})$, so $\deg(D_i/D_{i-1}) < \deg D$ for all i. Then the inductive hypothesis implies that Tate local duality holds for each D_i/D_{i-1}, so it holds for D', and we are done. $\square$

Now we can complete the computation of the cohomology of (φ, Γ_K)-modules of character type when the coefficients are a finite extension of $\mathbf{F}_p(\!(u)\!)$.

Corollary 3.2.3. *Let R be a finite extension of $\mathbf{F}_p(\!(u)\!)$ and let $\delta : K^\times \to R^\times$ be a continuous character. Then*

(1) $H^0_{\varphi,\Gamma_K}(\delta) = 0$ *unless δ is the trivial character, in which case $H^0_{\varphi,\Gamma_K}(\delta)$ is a 1-dimensional R-vector space.*
(2) $H^2_{\varphi,\Gamma_K}(\delta) = 0$ *unless $\delta = \chi_{\mathrm{cyc}} \circ \mathrm{Nm}_{K/\mathbf{Q}_p}$, in which case $H^2_{\varphi,\Gamma_K}(\delta)$ is a 1-dimensional R-vector space.*
(3) $H^1_{\varphi,\Gamma_K}(\delta)$ *is an R-vector space of dimension $[K : \mathbf{Q}_p]$ unless either $H^0_{\varphi,\Gamma_K}(\delta) \neq 0$ or $H^2_{\varphi,\Gamma_K}(\delta) \neq 0$, in which case it is an R-vector space of dimension $[K : \mathbf{Q}_p] + 1$.*

4. Triangulations

4.1. **Classification of rank-1 (φ, Γ)-modules.** In this section, we show that rank-1 (φ, Γ)-modules over a pseudorigid space X are free locally on X, and up to twisting by a line bundle on X, are of character type. The proof is largely the same as in [KPX14, §6.2]. We first treat the case where the coefficients are a field, where we can exploit the fact that $\Lambda_{R,(0,b],K}$ is Bézout, and then deduce the case where the coefficients are artinian by a deformation argument.

Proposition 4.1.1. *Suppose R is an artin local pseudoaffinoid algebra. If D is a rank-1 (φ, Γ)-module over $\Lambda_{R_0,(0,b],K}$, then there is a unique continuous character $\delta : K^\times \to R^\times$ such that $\mathscr{L} := H^0_{\varphi,\Gamma_K}(D(\delta^{-1})$ is free of rank 1 over R. In addition,*

(1) *The natural map $\Lambda_{R,\mathrm{rig},K}(\delta) \otimes_R \mathscr{L} \to D$ is an isomorphism.*
(2) $H^1_{\varphi,\Gamma_K}(D(\delta^{-1}))$ *is free over R of rank $1 + [K : \mathbf{Q}_p]$.*
(3) $H^2_{\varphi,\Gamma_K}(D(\delta^{-1})) = 0$.

abridge this

Proof. This proof is nearly identical to the proof of [KPX14, Lemma 6.2.13].

We first treat the case where R is a field. Suppose that D is étale. Then there is some (φ, Γ)-module D' over $\Lambda_{R_0,[0,b],K}$ such that $D \cong \Lambda_{R_0,(0,b],K} \otimes_{\Lambda_{R_0,[0,b],K}} D'$, and we may construct a Galois representation $\rho : \mathrm{Gal}_K \to \mathrm{GL}(M)$ via $M := \left(\widetilde{\Lambda}_{R_0,[0,0]} \otimes_{\Lambda_{R_0,[0,b],K}} D'\right)^{\varphi=1}$. Standard arguments show that M is a rank-1 $\mathbf{F}_q[\![u]\!]$-module, and by local class field theory the representation corresponds to a character $\delta : K^\times \to \mathrm{GL}(M)$. The construction of (φ, Γ)-modules of character type shows that D' is of character type, with character δ. To see that δ is unique, it suffices to consider the situation where $\Lambda_{R,\mathrm{rig},K} \cong \Lambda_{R,\mathrm{rig},K}(\delta)$ (as (φ, Γ)-modules). But in this case, local class field theory implies that δ is trivial.

Now suppose D has slope $s \neq 0$. Since R is a discretely valued field, the value group of $\Lambda_{R_0,[0,0],K} \cong (R_0 \otimes_{\mathbf{F}_p} k')[\![\overline{\pi}_K]\!]\left[\frac{1}{\overline{\pi}_K}\right]^\wedge$ is the same as the value group of R.

Thus, R contains an element α with $v_R(\alpha)=s$. Let $\delta:\mathbf{Q}_p^\times\to R^\times$ be the character with $\delta(p)=\alpha$ and $\delta|_{\mathbf{Z}_p^\times}=1$. Then $\delta\circ\mathrm{Nm}_{K/\mathbf{Q}_p}$ is a character $K^\times\to R^\times$ trivial on $\mathscr{O}_K^\times$ and sending a uniformizer of K to α^f, and by construction, the associated (φ,Γ)-module $D(\delta\circ\mathrm{Nm}_{K/\mathbf{Q}_p})$ has slope s. Twisting D by its inverse, we reduce to the étale case.

By construction, $\mathscr{L}:=H^0_{\varphi,\Gamma_K}(D(\delta^{-1}))=R$, so the natural map $\Lambda_{R,\mathrm{rig},K}(\delta)\otimes_R\mathscr{L}\to D$ is an isomorphism. Finally, the claims about $H^1_{\varphi,\Gamma_K}(D(\delta^{-1}))$ and $H^2_{\varphi,\Gamma_K}(D(\delta^{-1}))$ follow from our computations of the cohomology of (φ,Γ_K)-modules of character type.

We now bootstrap to the case where R is an artin local ring. If $\mathfrak{m}_R\subset R$ is the maximal ideal of R, we may factor the extension $R\twoheadrightarrow R/\mathfrak{m}_R$ as a sequence of small extensions, that is, extensions of the form

$$0\to I\to R\to R'\to 0$$

where $I\subset R$ is a principal ideal with $I\mathfrak{m}_R=0$. In this situation, $R\to R'$ is a square-zero thickening, and I is a 1-dimensional $R/\mathfrak{m}_R$-vector space. We will assume the result for rank-1 (φ,Γ_K)-modules over R' and deduce it for rank-1 (φ,Γ_K)-modules over R.

Let D be a rank-1 (φ,Γ_K)-module over R, and let $\delta':K^\times\to {R'}^\times$ be the continuous character corresponding to $R'\otimes_R D$. Since X_{Γ_K} is smooth, there exists a lift $\widetilde{\delta'}:K^\times\to R^\times$; twisting by $\widetilde{\delta'}^{-1}$, we may assume that $R'\otimes_R D$ is trivial as a (φ,Γ_K)-module, and $\delta':K^\times\to {R'}^\times$ is the trivial character. Such extensions of (φ,Γ_K)-modules are classified by $I\otimes_{R/\mathfrak{m}_R}H^1_{\varphi,\Gamma_K}(\Lambda_{R/\mathfrak{m}_R,\mathrm{rig},K})$, characters $\delta:K^\times\to R^\times$ with $\delta \mod I$ trivial are classified by $I\otimes_{R/\mathfrak{m}_R}H^1_{\varphi,\Gamma_K}(\mathrm{Gal}_K,R/\mathfrak{m}_R)$, and the map induced by $\delta\mapsto\Lambda_{R,\mathrm{rig},K}(\delta)$ is precisely the canonical isomorphism between Galois cohomology and Fontaine–Herr–Liu cohomology.

We need to check that $H^0_{\varphi,\Gamma_K}(D(\delta^{-1}))$ is free of rank 1 over R. In fact, $H^0_{\varphi,\Gamma_K}(\Lambda_{R,\mathrm{rig},K})=R$. It is clear that $R\subset H^0_{\varphi,\Gamma_K}(\Lambda_{R,\mathrm{rig},K})$. On the other hand, considering the exact sequence

$$0\to I\Lambda_{R,\mathrm{rig},K}\to\Lambda_{R,\mathrm{rig},K}\to\Lambda_{R',\mathrm{rig},K}\to 0$$

the inductive hypothesis implies that the length of $H^0_{\varphi,\Gamma_K}(\Lambda_{R,\mathrm{rig},K})$ as an R-module is at most $1+\mathrm{length}(R')=\mathrm{length}(R)$. Thus, the claim follows by induction.

To show that $H^2_{\varphi,\Gamma_K}(D(\delta^{-1}))=0$, it suffices to show that $R/\mathfrak{m}_R\otimes_R H^2_{\varphi,\Gamma_K}(D(\delta^{-1}))=H^2_{\varphi,\Gamma_K}(D_{R/\mathfrak{m}_R}(\delta^{-1}))=0$. But $D_{R/\mathfrak{m}_R}(\delta^{-1})$ is trivial as a (φ,Γ_K)-module, by construction, so our calculations show that $H^2_{\varphi,\Gamma_K}(D_{R/\mathfrak{m}_R}(\delta^{-1}))=0$, as desired. Now the base change spectral sequence implies that the formation of $H^1_{\varphi,\Gamma_K}(D(\delta^{-1}))$ commutes with base change on R, since $H^2_{\varphi,\Gamma_K}(D(\delta^{-1}))$ is locally free of rank 0. The Euler characteristic formula implies that $\dim_{R/\mathfrak{m}_R}H^1_{\varphi,\Gamma_K}(D_{R/\mathfrak{m}_R}(\delta^{-1}))=1+[K:\mathbf{Q}_p]$, so by Nakayama's lemma, $H^1_{\varphi,\Gamma_K}(D(\delta^{-1}))$ is free of the same rank. □

In order to give a classification over a general base, we again follow the strategy of the proof of [KPX14, Theorem 6.2.14] and twist our rank-1 (φ,Γ)-module by the universal family of characters. Then we can use the settled case over artin local rings and cohomology and base change to cut out the appropriate character. The difficulty is in verifying that the slopes of a family of (φ,Γ)-modules over a pseudoaffinoid algebra are bounded; this is the essential content of the following proposition, whose proof we do not duplicate here.

Proposition 4.1.2. *Let D be a $(\varphi, \Gamma_{\mathbf{Q}_p})$-module over $\Lambda_{R,\mathrm{rig},\mathbf{Q}_p}$. Then*

(1) *The quotient $D/(\psi - 1)$ is a finitely generated R-module*
(2) *If $n \in \mathbf{Z}$, let $\delta_n : \mathbf{Q}_p^\times \to R^\times$ be the character trivial on $\mathbf{Z}_p^\times$ which sends p to u^n. Then for all $n \gg 0$, the map $\psi - 1 : D(\delta_{-n}) \to D(\delta_{-n})$ is surjective.*

Proof. The proof of [KPX14, Proposition 3.3.2] carries over verbatim. One takes a model of D as a finite projective module over $\Lambda_{R,(0,b],\mathbf{Q}_p}$, considers it as a summand of a free module, and carefully analyzes the actions of φ and ψ. $\square$

Now we give the desired general classification, again following [KPX14, Theorem 6.2.14]:

abridge this proof

Theorem 4.1.3. *Let R be a pseudoaffinoid algebra, and let D be a rank-1 (φ, Γ)-module over $\Lambda_{R,\mathrm{rig},K}$. Then there exists a unique continuous character $\delta : K^\times \to R^\times$ and a unique invertible sheaf $\mathscr{L}$ on $\operatorname{Spa} R$ such that $D \cong \Lambda_{R,\mathrm{rig},K}(\delta) \otimes_R \mathscr{L}$.*

Remark 4.1.4. If such a δ and $\mathscr{L}$ exist, then $\mathscr{L}(U) = H^0_{\varphi,\Gamma_K}(D(\delta^{-1})|_U)$ for every open subspace $U \subset \operatorname{Spa} R$.

Proof. We first treat uniqueness. Since the formation of $H^0_{\varphi,\Gamma_K}(D(\delta^{-1}))$ commutes with flat base change on R, it suffices to show that if $H^0_{\varphi,\Gamma_K}(\Lambda_{R,\mathrm{rig},K}(\delta))$ is locally free of rank 1 over R, then δ is trivial. There is a Zariski-open dense subspace $U \subset \operatorname{Spa} R$ such that $H^i_{\varphi,\Gamma_K}(\Lambda_{R,\mathrm{rig},K}(\delta)|_U)$ is flat for all i; if $x \in U$ and $\mathfrak{m}_x \subset R$ is the corresponding maximal ideal, then the base change spectral sequence implies that $H^i_{\varphi,\Gamma_K}(R/\mathfrak{m}_x^k \otimes_R \Lambda_{R,\mathrm{rig},K}(\delta)) \cong R/\mathfrak{m}_x^k \otimes_R H^i_{\varphi,\Gamma_K}(\delta)$ for all i and all $k \geq 1$. In particular, $H^0_{\varphi,\Gamma_K}(\Lambda_{R/\mathfrak{m}_x^k,\mathrm{rig},K}(\delta))$ is free of rank 1 over $R/\mathfrak{m}_x^k$, which implies that $\delta : K^\times \to (R/\mathfrak{m}_x^k)^\times$ is trivial for all $k \geq 1$. It follows that $\delta : K^\times \to (R_{U'})^\times$ is trivial for all affinoid $U' \subset U$. But the condition $\delta = 1$ defines a Zariski-closed subspace of $\operatorname{Spa} R$; since it contains a Zariski-open dense subspace, it is all of $\operatorname{Spa} R$.

To show existence, we follow [KPX14] and consider the twist of D by the inverse of the universal family of characters $\delta_{\mathrm{univ}} : K^\times \to R^\times$; this is a (φ, Γ_K)-module over $\operatorname{Spa} R \times X_{K^\times}$, and we use Tate local duality to cut out a subspace corresponding to the desired character. More precisely, we let Γ'_D and Γ''_D be the support of $H^2_{\varphi,\Gamma_K}(D^\vee(\delta_{\mathrm{univ}}\chi_{\mathrm{cyc}}))$ and $H^2_{\varphi,\Gamma_K}(D(\delta_{\mathrm{univ}}^{-1}\chi_{\mathrm{cyc}}))$ in $\operatorname{Spa} R \times X_{K^\times}$, respectively, and let $\Gamma_D := \Gamma'_D \times_{\operatorname{Spa} R \times X_{\Gamma_K}} \Gamma''_D$. Since the formation of H^2_{φ,Γ_K} commutes with arbitrary base change on $\operatorname{Spa} R$, the formation of Γ'_D and Γ''_D, and hence Γ_D, commutes with arbitrary base change on $\operatorname{Spa} R$.

There is a natural projection map $\Gamma_D \to \operatorname{Spa} R$; a section induces a morphism $\operatorname{Spa} R \to X_{\Gamma_K}$, or equivalently, a continuous character $\delta : K^\times \to R^\times$. We will show that $\Gamma_D \to \operatorname{Spa} R$ is actually an isomorphism.

Granting this, we may replace D with $D(\delta_D^{-1})$, where $\delta_D : K^\times \to R^\times$ is the continuous character corresponding to $\operatorname{Spa} R = \Gamma_D \to X_{K^\times}$, so that Γ_D corresponds to the trivial character. Then we need to show that $H^0_{\varphi,\Gamma_K}(D)$ is a line bundle over $\operatorname{Spa} R$, and $D \cong \Lambda_{R,\mathrm{rig},K} \otimes_R H^0_{\varphi,\Gamma_K}(D)$ as a (φ, Γ_K)-module. If R' is a pseudoaffinoid artin local ring and $R \to R'$ is a homomorphism, there is a unique continuous character $\delta' : K^\times \to R'^\times$ such that $H^0_{\varphi,\Gamma_K}(D_{R'}(\delta'^{-1}))$ is free of rank 1 over R', and in addition, $H^1_{\varphi,\Gamma_K}(D_{R'}(\delta'^{-1}))$ is free of rank $1+[K:\mathbf{Q}_p]$ and $H^2_{\varphi,\Gamma_K}(D_{R'}(\delta'^{-1}))=0$. Thus, the formation of $H^0_{\varphi,\Gamma_K}(D_{R'}(\delta'^{-1}))$ commutes with arbitrary base change on R'; in particular, $H^0_{\varphi,\Gamma_K}(D_{R'/\mathfrak{m}_{R'}}(\delta'^{-1}))$ is non-zero. Since $H^2_{\varphi,\Gamma_K}(D^\vee_{R'/\mathfrak{m}_{R'}}(\delta'\chi_{\mathrm{cyc}}))$ and $H^2_{\varphi,\Gamma_K}(D_{R'/\mathfrak{m}_{R'}}(\delta'^{-1}\chi_{\mathrm{cyc}}))$ are dual to $H^0_{\varphi,\Gamma_K}(D_{R'/\mathfrak{m}_{R'}}(\delta'^{-1}))$ and $H^0_{\varphi,\Gamma_K}(D^\vee_{R'/\mathfrak{m}_{R'}}(\delta'))$, respectively, and the formation of H^2_{φ,Γ_K} commutes with arbitrary base change on R, we see that $H^2_{\varphi,\Gamma_K}(D^\vee_{R'}(\delta'\chi_{\mathrm{cyc}}))$ and $H^2_{\varphi,\Gamma_K}(D_{R'}(\delta'^{-1}\chi_{\mathrm{cyc}}))$ are both non-zero. Thus, the graph of the morphism $\operatorname{Spa} R' \to X_{K^\times}$ induced by δ' is contained in Γ_D; since Γ_D corresponds to the trivial character, δ' is trivial.

In other words, for any homomorphism $R\to R'$ with R' a pseudoaffinoid artin local ring, $H^0_{\varphi,\Gamma_K}(D_{R'})$ is free of rank 1 over R', $H^1_{\varphi,\Gamma_K}(D_{R'})$ is free of rank $1+[K:\mathbf{Q}_p]$, and $H^2_{\varphi,\Gamma_K}(D_{R'})=0$; on residue fields, this implies that $H^2_{\varphi,\Gamma_K}(D_{R'/\mathfrak{m}_{R'}})=0$, so by Nakayama's lemma, $H^2_{\varphi,\Gamma_K}(D_{R'})=0$, as well. This implies that $H^2_{\varphi,\Gamma_K}(D)$ is locally free of rank 0, so by the base change spectral sequence, the formation of $H^1_{\varphi,\Gamma_K}(D)$ commutes with arbitrary base change on R. It follows that $H^1_{\varphi,\Gamma_K}(D)$ is locally free of rank $1+[K:\mathbf{Q}_p]$, so the base change spectral sequence again implies that the formation of $H^0_{\varphi,\Gamma_K}(D)$ commutes with arbitrary base change on R, and we conclude that $H^0_{\varphi,\Gamma_K}(D)$ is locally free of rank 1, as desired.

We now prove that $\Gamma_D\to\operatorname{Spa} R$ is an isomorphism. In fact, it suffices to prove that Γ_D is pseudoaffinoid. Indeed, since pseudoaffinoid rings are jacobson, an isomorphism of pseudoaffinoid spaces can be detected on the level of completed local rings at maximal points (in the sense of [JN19, Definition 2.2.7]), and this follows from the result on artin local rings.

Since Γ_D is a Zariski-closed subspace of the quasi-Stein space $\operatorname{Spa} R\times X_{K^\times}$, it is enough to show that that its image in $X_{K^\times}$ is contained in an affinoid subspace, and since $X_{K^\times}\cong \mathbf{G}_m^{\mathrm{an}}\times X_{\mathscr{O}_K^\times}$, it is enough to show that its image in $\mathbf{G}_m^{\mathrm{an}}$ is bounded. There is some $N\geq 0$ such that for all $n\geq N$, $\psi-1 : D^\vee(\delta_{-n}\delta_{\mathrm{univ}}\chi_{\mathrm{cyc}})\to D^\vee(\delta_{-n}\delta_{\mathrm{univ}}\chi_{\mathrm{cyc}})$ and $\psi-1 : D(\delta_{-n}\delta_{\mathrm{univ}}^{-1}\chi_{\mathrm{cyc}})\to D(\delta_{-n}\delta_{\mathrm{univ}}^{-1}\chi_{\mathrm{cyc}})$ are surjective. Surjectivity is preserved under arbitrary base change $R\to R'$, and the isomorphism $H^\bullet_{\varphi,\Gamma_K}\to H^\bullet_{\psi,\Gamma_K}$ implies that $H^2(D^\vee(\delta_{-n}\delta_{\mathrm{univ}}\delta'\chi_{\mathrm{cyc}}))=H^2_{\varphi,\Gamma_K}(D(\delta_{-n}\delta_{\mathrm{univ}}^{-1}\delta'\chi_{\mathrm{cyc}}))=0$ for all continuous characters $\delta' : \mathscr{O}_K^\times\to R'^{-1}$. Thus, if T denotes the coordinate on $\mathbf{G}_m$, the image of Γ'_D is contained in the subspace $T\leq u^N$ and the image of Γ''_D is contained in the subspace $T^{-1}\leq u^N$, and we are done. □

4.2. **Deformations of trianguline (φ,Γ)-modules.** Trianguline (φ,Γ)-modules are those which are extensions of (φ,Γ)-modules of character type. More precisely,

Definition 4.2.1. Let X be a pseudorigid space over $\mathscr{O}_E$ for some finite extension $E/\mathbf{Q}_p$, let $K/\mathbf{Q}_p$ be a finite extension, and let $\underline{\delta}=(\delta_1,\ldots,\delta_d) : (K^\times)^d\to\Gamma(X,\mathscr{O}_X^\times)$ be a d-tuple of continuous characters. A (φ,Γ_K)-module D is *trianguline with parameter* $\underline{\delta}$ if (possibly after enlarging E) there is an increasing filtration $\operatorname{Fil}^\bullet D$ by (φ,Γ_K)-modules and a set of line bundles $\mathscr{L}_1,\ldots,\mathscr{L}_d$ such that $\operatorname{gr}^i D\cong\Lambda_{X,\mathrm{rig},K}(\delta_i)\otimes\mathscr{L}_i$ for all i.

If $X = \operatorname{Spa} R$ where R is a field, we say that D is *strictly trianguline with parameter* $\underline{\delta}$ if for each i, $\operatorname{Fil}^{i+1} D$ is the unique sub-(φ, Γ_K)-module of D containing $\operatorname{Fil}^i D$ such that $\operatorname{gr}^{i+1} D \cong \Lambda_{R,\mathrm{rig},K}(\delta_{i+1})$.

As in the characteristic 0 situation treated in [BC09], we may define and study deformations of trianguline (φ, Γ)-modules:

Definition 4.2.2. Let R be a finite extension of $\mathbf{F}_p(\!(u)\!)$ and let D be a fixed (φ, Γ_K)-module of rank d over $\Lambda_{R,\mathrm{rig},K}$ equipped with a triangulation $\operatorname{Fil}^\bullet D$ with parameter $\underline{\delta}$. Let $\mathcal{C}_R$ denote the category of artin local $\mathbf{Z}_p$-algebras R' equipped with an isomorphism $R'/\mathfrak{m}_{R'} \xrightarrow{\sim} R$. The trianguline deformation functor $\operatorname{Def}_{D,\operatorname{Fil}^\bullet} : \mathcal{C}_R \to \underline{\mathrm{Set}}$ is defined to be the set of isomorphism classes

$$\operatorname{Def}_{D,\operatorname{Fil}^\bullet}(R') := \{(D_{R'}, \operatorname{Fil}^\bullet D_{R'}, \iota)\}/\sim$$

where $D_{R'}$ is a (φ, Γ_K)-module over $\Lambda_{R',\mathrm{rig},K}$, $\operatorname{Fil}^\bullet D_{R'}$ is a triangulation, and $\iota : R \otimes_{R'} D_{R'} \xrightarrow{\sim} D$ is an isomorphism which also defines isomorphisms $R \otimes_{R'} \operatorname{Fil}^i D_{R'} \xrightarrow{\sim} \operatorname{Fil}^i D$.

One of the consequences of the proof of Proposition 4.1.1 is that when $d = 1$, $\operatorname{Def}_{D,\operatorname{Fil}^\bullet}$ is formally smooth. As in the characteristic 0 situation, the same is true for general d, so long as the parameter satisfies a certain regularity condition.

Proposition 4.2.3. *Suppose the parameter $\underline{\delta}$ of $\operatorname{Fil}^\bullet D$ satisfies the property that $\delta_i\delta_j^{-1} \neq \chi_{\mathrm{cyc}}$ for any $i < j$. Then $\operatorname{Def}_{D,\operatorname{Fil}^\bullet}$ is formally smooth.*

Proof. The proof is essentially identical to that of [BC09, Proposition 2.3.10], but we sketch it here for the convenience of the reader. We proceed by induction on d; the case $d = 1$ follows from the proof of Proposition 4.1.1, so we assume the result for trianguline deformations of (φ, Γ)-modules of rank $d-1$. Let $I \subset R'$ be a square-zero ideal. We need to prove that $\operatorname{Def}_{D,\operatorname{Fil}^\bullet}(R') \to \operatorname{Def}_{D,\operatorname{Fil}^\bullet}(R'/I)$ is surjective, so we may factor $R' \twoheadrightarrow R'/I$ into a series of small extensions and assume that I is principal and $I\mathfrak{m}_{R'} = 0$. By the inductive hypothesis, we may find a trianguline deformation D' of $\operatorname{Fil}^{d-1} D$ over $\Lambda_{R',\mathrm{rig},L}$. By twisting, we may assume that δ_d is trivial. Then we need to show that the natural map $H^1_{\varphi,\Gamma}(D') \to H^1_{\varphi,\Gamma}(\operatorname{Fil}^{d-1})$ is surjective. But the cokernel of this map is $H^2_{\varphi,\Gamma}(I \otimes_{R'/\mathfrak{m}_{R'}} \operatorname{Fil}^{d-1} D(\delta_d^{-1})) = I \otimes_{R'/\mathfrak{m}_{R'}} H^2_{\varphi,\Gamma}(\operatorname{Fil}^{d-1} D(\delta_d^{-1}))$, which is 0 by assumption and Corollary 3.2.3. $\square$

This motivates the following definition.

Definition 4.2.4. Let $\mathcal{T} := \widehat{K^\times}$ be the pseudorigid space representing the functor on pseudorigid spaces whose X-points are given by $\mathcal{T}(X) := \operatorname{Hom}_{\mathrm{cts}}(K^\times, \mathscr{O}(X)^\times)$. We say that a continuous character $\kappa : K^\times \to \mathscr{O}(X)^\times$ is *regular* if for all maximal points $x \in X$, the residual character $\kappa_x : K^\times \to k(x)^\times$ is not of the form $\alpha \mapsto \alpha^{-\mathbf{i}}$ or $\alpha \mapsto \alpha^{\mathbf{i}+\mathbf{1}}|\alpha|$ for $\mathbf{i} \in \mathbf{Z}_{\geq 0}^{\operatorname{Hom}(K,k(x))}$ (if x is a characteristic 0 point), or trivial or $\chi_{\mathrm{cyc}} \circ \operatorname{Nm}_{K/\mathbf{Q}_p}$ (if x is a characteristic p point).

The space of *regular parameters* $\mathcal{T}^d_{\mathrm{reg}} \subset \mathcal{T}^d$ is the Zariski-open subspace whose X-points are given by parameters $\underline{\delta} : (K^\times)^d \to \mathscr{O}(X)^\times$ such that $\delta_i\delta_j^{-1} : K^\times \to \mathscr{O}(X)^\times$ is regular for all $i \neq j$.

Consider the functor $\mathcal{S}_d^\square$ on pseudorigid spaces defined via

$$X \rightsquigarrow \{(D, \mathrm{Fil}^\bullet D, \underline{\delta}, \underline{\nu})\}/\sim$$

where D is a trianguline (φ, Γ_K)-module with filtration $\mathrm{Fil}^\bullet D$ and regular parameter $\underline{\delta} \in \mathcal{D}^d_{\mathrm{reg}}$, and $\underline{\nu}$ is a sequence of trivializations $\nu_i : \mathrm{gr}^i D \xrightarrow{\sim} \Lambda_{X,\mathrm{rig},K}$. There is a natural transformation $\mathcal{S}_d^\square \to \mathcal{T}^d_{\mathrm{reg}}$ given on X-points by

$$(D, \mathrm{Fil}^\bullet D, \underline{\delta}, \underline{\nu}) \rightsquigarrow \underline{\delta}$$

Exactly as in [Che13, Théorème 3.3] and [HS16, Theorem 2.4], we have the following:

Proposition 4.2.5. *The functor $\mathcal{S}_d^\square$ is representable by a pseudorigid space, which we also denote $\mathcal{S}_d^\square$, and the morphism $\mathcal{S}_d^\square \to \mathcal{T}^d_{\mathrm{reg}}$ is smooth of relative dimension $\frac{d(d-1)}{2}[K : \mathbf{Q}_p]$.*

One proves by induction on d that if D is a trianguline (φ, Γ_K)-module over X with parameter $\underline{\delta} \in (\mathcal{T}_{\mathrm{reg}})^d$, then $H^1_{\varphi,\Gamma_K}(D)$ is a vector bundle over X of rank $d[K : \mathbf{Q}_p]$ (the regularity assumption ensures that $H^0_{\varphi,\Gamma_K}(D) = H^2_{\varphi,\Gamma_K}(D) = 0$). Now $\mathcal{S}_1^\square = \mathcal{T} = \mathcal{T}^1_{\mathrm{reg}}$, so $S_1^\square$ is representable and is smooth of the correct dimension over $\mathcal{T}^1_{\mathrm{reg}}$. Then one may proceed by induction on d again, and construct $\mathcal{S}_d^\square$ as the moduli space of extensions of the universal (φ, Γ_K)-module of character type $\Lambda_{\mathcal{T},\mathrm{rig},K}(\delta_{\mathrm{univ}}$ by the universal object $D_{d-1,\mathrm{univ}}$ over $\mathcal{S}_{d-1}^\square$. For a specified regular parameter $\underline{\delta} = (\delta_1, \ldots, \delta_d) \in \mathcal{T}^d_{\mathrm{reg}}(X)$, the fiber $\mathcal{S}_d^\square|_{\underline{\delta}}$ is equal to $\mathrm{Ext}^1(\Lambda_{X,\mathrm{rig},K}(\delta_d), D_{d-1,\mathrm{univ}}|_{(\delta_1,\ldots,\delta_{d-1})}) = H^1_{\varphi,\Gamma_K}(D_{d-1,\mathrm{univ}}|_{(\delta_1,\ldots,\delta_{d-1})}(\delta_d^{-1}))$. This is a rank-$(d-1)$ vector bundle over X, and the claim follows.

4.3. **Interpolating triangulations.**

Lemma 4.3.1. *Let $X = \mathrm{Spa}\, R$ be a reduced pseudorigid space with $p \notin R^\times$, let D be a (φ, Γ)-module over $\Lambda_{R,\mathrm{rig},K}$, and let $\delta : K^\times \to R^\times$ be a continuous character such that $H^0_{\varphi,\Gamma_K}(D^\vee(\delta))$ is free of rank 1 over R and $H^i_{\varphi,\Gamma_K}(D^\vee(\delta))$ has Tor-dimension at most 1 for $i = 1, 2$. Then the morphism $D \to \Lambda_{R,\mathrm{rig},K}(\delta)$ corresponding to a basis of $H^0_{\varphi,\Gamma_K}(D^\vee(\delta))$ is surjective over an open subspace $U \subset X$ containing $\{p = 0\} \subset X$.*

Proof. Choose a basis element of $H^0_{\varphi,\Gamma_K}(D^\vee(\delta))$; there is some $b > 0$ such that the corresponding homomorphism $D \to \Lambda_{R,\mathrm{rig},K}(\delta)$ is defined over $\Lambda_{R,(0,b],K}$, and we may view it as a morphism of coherent sheaves over the corresponding quasi-Stein space. Moreover, φ-equivariance means that to check surjectivity, it suffices to check that $\Lambda_{R,[b/p,b],K} \otimes_{\Lambda_{R,(0,b],K}} D \to \Lambda_{R,[b/p,b],K} \otimes_{\Lambda_{R,(0,b],K}} \Lambda_{R,(0,b],K}(\delta)$ is surjective.

The morphism $\Lambda_{R,[b/p,b],K} \otimes_{\Lambda_{R,(0,b],K}} D \to \Lambda_{R,[b/p,b],K} \otimes_{\Lambda_{R,(0,b],K}} \Lambda_{R,(0,b],K}(\delta)$ fails to be surjective on a Zariski-closed subspace $Z \subset \mathrm{Spa}\, \Lambda_{R,[b/p,b],K}$. Since $\mathrm{Spa}\, \Lambda_{R,[b/p,b],K}$ is affinoid, so is Z.

Consider specializations at the characteristic p maximal points $x \in \mathrm{Spa}\, R$. If $H^0_{\varphi,\Gamma_K}(D^\vee(\delta))$ is flat of rank 1 over R, then $k_x \otimes_R H^0_{\varphi,\Gamma_K}(D^\vee(\delta))$ is a 1-dimensional k_x-vector space. If $H^i_{\varphi,\Gamma_K}(D^\vee(\delta))$ has Tor-dimension at most 1 for $i = 1, 2$, then the specialization maps $R \to k_x$ give us exact sequences

$$0 \to k_x \otimes_R H^0_{\varphi,\Gamma_K}(D^\vee(\delta)) \to H^0_{\varphi,\Gamma_K}(k_x \otimes_R D^\vee(\delta)) \to \mathrm{Tor}_1^R(H^1_{\varphi,\Gamma_K}(D^\vee(\delta)), k_x) \to 0$$

Thus, the induced maps $k_x \otimes_R D \to k_x \otimes_R \Lambda_{R,\mathrm{rig},K}(\delta)$ are non-zero, and if k_x has positive characteristic, this implies that the corresponding map is surjective.

Thus, p is a nowhere-vanishing function on Z, and since Z is affinoid, the minimum modulus principle implies that $p|_Z$ is bounded away from 0. That is, there is some λ such that $\{|p| \leq \lambda\} \cap Z$ is empty. Setting $U := \{|p| \leq \lambda\} \subset X$ yields the desired subspace. □

state and prove minimum modulus principle for pseudorigid spaces explicitly

Theorem 4.3.2. *Let X be a reduced pseudorigid space, let D be a (φ, Γ_K)-module over X of rank d, and let $\delta : K^\times \to \Gamma(X, \mathscr{O}_X^\times)$ be a continuous character. Suppose there is a Zariski-dense set $X_{\mathrm{alg}} \subset X$ of maximal points such that for every $x \in X_{\mathrm{alg}}$, $H^0_{\varphi,\Gamma_K}(D_x^\vee(\delta_x))$ is 1-dimensional and the image of $\Lambda_{k_x,\mathrm{rig},K}$ under any basis of this space is saturated in $D_x^\vee(\delta_x)$. Then there exists a proper birational morphism $f : X' \to X$ of reduced pseudorigid spaces, a homomorphism $\lambda : f^*D \to \Lambda_{X',\mathrm{rig},K}(\delta) \otimes_{X'} \mathscr{L}$ of (φ, Γ_K)-modules, and an open subspace $U \subset X'$ containing $\{p = 0\}$ such that*

1. *$\lambda|_U : f^*D|_U \to \Lambda_{U,\mathrm{rig},K}(\delta|_U) \otimes_U \mathscr{L}|_U$ is surjective*
2. *the kernel of $\lambda|_U$ is a (φ, Γ_K)-module of rank $d-1$*

Proof. We may replace X with its normalization (using the theory of normalizations of pseudorigid spaces developed in [JN19]), and we may consider the connected components of X separately.

Using perfectness of $C^\bullet_{\varphi,\Gamma_K}(D^\vee(\delta))$, we may use [KPX14, Corollary 6.3.6(2)] to construct a proper birational morphism $f_0 : X' \to X$ such that $D' := f_0^*(D^\vee(\delta))$ has $H^0_{\varphi,\Gamma_K}(D')$ flat and $H^i_{\varphi,\Gamma_K}(D')$ with Tor-dimension at most 1 for $i = 1, 2$. Then for any maximal point $x \in X'$, the base change spectral sequence gives us a short exact sequence

$$0 \to k_x \otimes_R H^0_{\varphi,\Gamma_K}(D') \to H^0_{\varphi,\Gamma_K}(k_x \otimes_R D') \to \mathrm{Tor}_1^R(H^1_{\varphi,\Gamma_K}(D'), k_x) \to 0$$

By [KPX14, Lemma 6.3.7], the set of points $x \in X'$ such that the last term is non-zero is a Zariski-closed subspace $Z_0' \subset X'$ whose complement is open and dense. Thus, $H^0_{\varphi,\Gamma_K}(D')$ is flat of rank 1. Letting $\mathscr{L} := H^0_{\varphi,\Gamma_K}(D')^\vee$, we obtain a homomorphism $\lambda_0 : f^*D \to \Lambda_{X',\mathrm{rig},K}(\delta) \otimes_{X'} \mathscr{L}$.

The formation of $H^0_{\varphi,\Gamma_K}(D')$ commutes with flat base change on X; we may find a collection $\{X_i'\}$ of open pseudoaffinoid subspaces of X' such that $H^0_{\varphi,\Gamma_K}(D')|_{X_i'}$ is free, $\{p = 0\} \subset \cup_i X_i'$, and p is not invertible on X_i'. Then we may apply Lemma 4.3.1 to conclude that $\lambda_0|_{X_i'}$ is surjective (possibly after shrinking X_i'). Setting $U := \cup X_i'$, we see that X', $U \subset X'$, and λ_0 satisfy the first of our desired properties.

To check the second claim, observe that for some $b > 0$ we have an exact sequence over U

$$0 \to P \to \Lambda_{U,(0,b],K} \otimes D'|_U \to \Lambda_{U,(0,b],K}(\delta) \otimes_U \mathscr{L}|_U \to 0$$

Since $\Lambda_{U,(0,b],K}(\delta) \otimes_{X'} \mathscr{L}$ is R'-flat, this sequence remains exact after specializing at any point $x \in U$, so $k_x \otimes P$ is a (φ, Γ_K)-module of rank $d-1$. It follows that P is a vector bundle of rank $d-1$ over the quasi-Stein space associated to $\Lambda_{U,(0,b],K}$, and hence is a (φ, Γ_K)-module of the correct rank. □

5. Applications to eigenvarieties

5.1. **Set-up.** Extended eigenvarieties have been constructed by [AIP18], [JN16], and [?] for various groups; these extended eigenvarieties are expected to (and in some cases known to) carry families of Galois representations such that local Galois-theoretic data matches certain Hecke-theoretic data. At places away from p and the level, this compatibility specifies that the local Galois representation is unramified and gives a characteristic polynomial for Frobenius. At places dividing p, this compatibility specifies that the local Galois representation is trianguline and gives the parameters of the triangulation.

In this subsection, we use our results on trianguline (φ, Γ)-modules to study extended eigenvarieties at the boundary of weight space, in order to address two questions:

(1) Are irreducible components proper (in the sense of [?]) at the boundary of weight space?
(2) Are Galois representations at characteristic p points trianguline at p?

We will give partial affirmative answers to both questions.

Before stating our assumptions more precisely, we recall the construction of [JN16]. Let F be a number field, let $\mathbf{H}$ be a reductive group over F split at all places above p, and set $\mathbf{G} := \operatorname{Res}_{F/\mathbf{Q}} \mathbf{H}$. Fix a tame level by choosing a compact open subgroup $K_\ell \subset \mathbf{G}(\mathbf{Q}_\ell)$ for each prime $\ell \neq p$, such that K_ℓ is hyperspecial for all but finitely many ℓ, and let $K_p \subset \mathbf{G}(\mathbf{Q}_p)$ be an Iwahori subgroup. Let S' denote the set of places w of $\mathbf{Q}$ such that either $w = \infty$, or K_w is not hyperspecial, and let S denote the set of places of F lying above the places in S'. Then [JN16] proved the following:

Theorem 5.1.1. [JN16, Theorems A and B] *The eigenvarieties for $\mathbf{G}$ constructed in [?] naturally extend to pseudorigid spaces $\mathscr{X}_{\mathbf{G}}$ equipped with a weight map $\mathrm{wt} : \mathscr{X}_{\mathbf{G}} \to \mathcal{W}$ to extended weight space $\mathcal{W} := (\operatorname{Spa} \mathbf{Z}_p[\![T_0']\!])^{\mathrm{an}}$, where T_0' is a certain quotient of the $\mathbf{Z}_p$-points of a (split) maximal torus of a model of $\mathbf{G}$ over $\mathbf{Z}_p$. Moreover, if F is totally real or CM and $\mathrm{H} = \mathrm{GL}_d$, there is a continuous d-dimensional determinant $D : \mathscr{O}(\mathscr{X}_{\mathbf{G}})[\mathrm{Gal}_{F,S}] \to \mathscr{O}^+(\mathscr{X}_{\mathbf{G}}^{\mathrm{red}})$ such that $D(1 - X \cdot \mathrm{Frob}_v) = P_v(X)$ for all $v \notin S$, where $P_v(X)$ is the Hecke polynomial.*

When F is totally real and $d = 2$, the characteristic 0 eigenvariety $\mathscr{X}_{\mathbf{G}}^{\mathrm{rig}}$ contains a Zariski very dense set of classical points (using [Che04, Lemme 6.2.10], [Che04, Lemme 6.2.8], and a "small slope implies classical" criterion). Furthermore, local-global compatibility at places dividing p is known for classical Hilbert modular forms, and so in this case $\mathscr{X}_{\mathbf{G}}^{\mathrm{rig}}$ contains a dense set of points at which D corresponds to a trianguline Galois representation.

When H is a totally definite quaternion algebra over a totally real field, split at p, a similar argument shows that $\mathscr{X}_{\mathbf{G}}^{\mathrm{rig}}$ contains a Zariski very dense set of classical points. Moreover, the p-adic Jacquet–Langlands correspondance of [Che05], [?] can be extended to the pseudorigid setting. This identifies each irreducible component of a quaternionic eigenvariety with an irreducible component of an eigenvariety for Hilbert modular forms; it follows that a Galois determinant can be pulled back to

$\mathscr{X}_{\mathbf{G}}$, and it corresponds to a trianguline representation at a dense set of points of $\mathscr{X}_{\mathbf{G}}^{\mathrm{rig}}$.

There is a similar story when $\mathbf{G}$ is a definite unitary group over $\mathbf{Q}$ split at p. Characteristic 0 eigenvarieties have been constructed [**?**], [BC09] which interpolate classical automorphic forms and carry a family of Galois determinants:

Theorem 5.1.2. [BC09, Chapter 7] *Let $F/\mathbf{Q}$ be an imaginary quadratic field and let $\mathbf{G}$ be a definite unitary group associated to F, split at p. Then the characteristic 0 eigenvariety $\mathscr{X}_{\mathbf{G}}^{\mathrm{rig}}$ contains a Zariski very dense set of classical points (corresponding to p-refined automorphic representations), and there is a continuous determinant $D : \mathscr{O}^+(\mathscr{X}_{\mathbf{G}}^{\mathrm{rig}})[\mathrm{Gal}_{F,S}] \to \mathscr{O}^+(\mathscr{X}_{\mathbf{G}}^{\mathrm{rig}})$ such that $D(1 - X \cdot \mathrm{Frob}_v) = P_v(X)$ for all $v \notin S$, where $P_v(X)$ is the Hecke polynomial.*

Moreover, the corresponding Galois representation is known to be trianguline at classical points; thus, there is a continuous Galois determinant $\overline{D} : \mathscr{O}^+(\overline{\mathscr{X}_{\mathbf{G}}^{\mathrm{rig}}})[\mathrm{Gal}_{F,S}] \to \mathscr{O}^+(\overline{\mathscr{X}_{\mathbf{G}}^{\mathrm{rig}}})$ defined on the closure of $\mathscr{X}_{\mathbf{G}}^{\mathrm{rig}}$ in $\mathscr{X}_{\mathbf{G}}$, and it corresponds to a trianguline Galois representation at a Zariski very dense of points.

has enough of the Paris book project been published yet?

We will make more precise what kind of trianguline conditions we have (or hope for) at places dividing p. If $\mathbf{T}$ is a split maximal torus of a model of $\mathbf{G}$ over $\mathbf{Z}_p$, consider a splitting of the inclusion $\mathbf{T}(\mathbf{Z}_p) \hookrightarrow \mathbf{T}(\mathbf{Q}_p)$, and let Σ denote the kernel. There are two submonoids $\Sigma^{\mathrm{cpt}} \subset \Sigma^+ \subset \Sigma$; we refer the reader to [JN16, §3.3] for precise definitions, but we note that when $\mathbf{G}(\mathbf{Z}_p) \cong \prod_{v|p} \mathrm{GL}_d(\mathscr{O}_{F_v})$, we may take $\mathbf{T}$ to be the standard torus and

$$\begin{aligned}
\Sigma &= \prod_{v|p} \{\mathrm{diag}(\varpi_v^{a_1}, \ldots, \varpi_v^{a_d}) \mid a_i \in \mathbf{Z}\} \\
\Sigma^+ &= \prod_{v|p} \{\mathrm{diag}(\varpi_v^{a_1}, \ldots, \varpi_v^{a_d}) \mid a_{i+1} \geq a_i\} \\
\Sigma^{\mathrm{cpt}} &= \prod_{v|p} \{\mathrm{diag}(\varpi_v^{a_1}, \ldots, \varpi_v^{a_d}) \mid a_{i+1} > a_i\}
\end{aligned}$$

The construction of $\mathscr{X}_{\mathbf{G}}$ depends on a choice of $t \in \Sigma^{\mathrm{cpt}}$, which in the above case we take to be $\prod_{v|p} \mathrm{diag}(1, \ldots, \varpi_v^{d-1})$; the authors construct a spectral variety $\mathscr{Z} \subset \mathcal{W} \times \mathbf{G}_m$ using the Fredholm series of the corresponding controlling Hecke operator U_t, and then construct $\mathscr{X}_{\mathbf{G}} \to \mathscr{Z}$ finite, such that there is a homomorphism $\psi : \mathbf{T}(\Delta^p, K^p) \to \mathscr{O}(\mathscr{X}_{\mathbf{G}})$. Here $\mathbf{T}(\Delta^p, K^p)$ is a Hecke algebra with no Hecke operators at places above p.

However, it is possible to make the same construction using other choices of Hecke algebras, and we will need to do so (this is discussed in greater detail in [JN19, §3.4]. In particular, let $\mathscr{A}_p^+ \subset \mathbf{Z}_p[\mathbf{G}(\mathbf{Q}_p)//I]$ be the subring generated by the characteristic functions $\mathbf{1}_{[IsI]}$ for $s \in \Sigma^+$. Then there is an extended eigenvariety $\mathscr{X}_{\mathbf{G}}^{\mathscr{A}_p^+}$ equipped with a homomorphism $\mathbf{T}(\Delta^p, K^p) \otimes_{\mathbf{Z}_p} \mathscr{A}_p^+ \to \mathscr{O}(\mathscr{X}_{\mathbf{G}}^{\mathscr{A}_p^+})$ and a finite morphism $\mathscr{X}_{\mathbf{G}}^{\mathscr{A}_p^+} \to \mathscr{X}_{\mathbf{G}}$. There is a surjective finite map $\mathscr{X}_{\mathbf{G}}^{\mathscr{A}_p^+} \to \mathscr{X}_{\mathbf{G}}$, and we obtain a Galois determinant $\mathscr{O}(\mathscr{X}_{\mathbf{G}}^{\mathscr{A}_p^+,\mathrm{red}})[\mathrm{Gal}_{F,S}] \to \mathscr{O}(\mathscr{X}_{\mathbf{G}}^{\mathscr{A}_p^+,\mathrm{red}})$ by pulling back the determinant on $\mathscr{X}_{\mathbf{G}}^{\mathrm{red}}$.

Let $\mathbf{G}_{m,\mathcal{W}}$ denote the pseudorigid space $\mathbf{G}_m^{\mathrm{an}}\times\mathcal{W}$, so that we have finite morphisms $\mathscr{X}_{\mathbf{G}}^{\mathscr{A}_p^+}\to\mathscr{X}_{\mathbf{G}}\to\mathbf{G}_{m,\mathcal{W}}$. By [JN19, Lemma 3.4.1], the image of $[IsI]$ in $\mathscr{O}(\mathscr{X}_{\mathbf{G}}^{\mathscr{A}_p^+})$ is invertible for all $s\in\Sigma^+$, and so for $s\in\Sigma$, we can write $s=s's''^{-1}$ for $s',s''\in\Sigma^+$ and obtain $\psi([Is'I])\psi([Is''I]^{-1})\in\mathscr{O}(\mathscr{X}_{\mathbf{G}}^{\mathscr{A}_p^+})^\times$. Thus, we have a morphism $\mathscr{X}_{\mathbf{G}}^{\mathscr{A}_p^+}\to\widehat{\Sigma}:=\mathrm{Hom}(\Sigma,\mathbf{G}_{m,\mathcal{W}})$ such that the diagram

$$\begin{tikzcd}
\mathscr{X}_{\mathbf{G}}^{\mathscr{A}_p^+} \arrow[r] \arrow[d] & \widehat{\Sigma} \arrow[d] \\
\mathscr{X}_{\mathbf{G}} \arrow[r] & \mathbf{G}_{m,\mathcal{W}}
\end{tikzcd}$$

commutes and has finite horizontal maps. Here the right vertical map is induced by evaluation at U_t. Any choice of a basis of Σ will give us parameters $\delta_{i,v}:F_v^\times\to\mathscr{O}(\mathscr{X}_{\mathbf{G}}^{\mathscr{A}_p^+})^\times$.

When F is a number field and $\mathbf{G}=\mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_d$ with the standard maximal torus, there is a natural ordered basis of Σ, namely $\{\mathrm{diag}(1,\ldots,\varpi_v,\ldots,1)\}_{i,v}$, where $v|p$ and ϖ_v is placed in the $d-i$ slot. Then (restricting to non-critical points for simplicity), [**?**, Conjecture 1.2.2(iii)] predicts that if $x\in\mathscr{X}_{\mathbf{G}}^{\mathscr{A}_p^+}$ is non-critical, the Galois representation corresponding to $D_x|_{\mathrm{Gal}_{F_v}}$ is trianguline with parameters $\{\delta_{i,v}\}$ such that $\{\delta_{i,v}(\varpi_v)=\psi(s_{i,v})\}$ (and $\delta_{i,v}|_{\mathscr{O}_{F_v}^\times}$ corresponds to the automorphic weight). Similarly, in the unitary case sketched above, [BC09, Proposition 7.5.13] implies that at non-critical points $x\in\mathscr{X}_{\mathbf{G}}^{\mathscr{A}_p^+}$, the Galois representation corresponding to $D_x|_{\mathrm{Gal}_{F_v}}$ is trianguline with parameter $\{\delta_{i,v}\}$, where $\delta_{i,v}(p)=\psi(\mathrm{diag}(1,\ldots,p,\ldots,1))$.

5.2. **Properness at the boundary.** We follow the strategy of [DL16] to show extended eigenvarieties are proper at the boundary. We assume we have a Galois representation, and sufficiently many classical points where it is known to be trianguline at p, with parameters compatible with the Hecke algebra at p:

Theorem 5.2.1. *Suppose $\mathscr{X}_{\mathbf{G}}^{\mathscr{A}_p^+}$ is an extended eigenvariety such that there is a continuous determinant $D:\mathscr{O}(\mathscr{X}_{\mathbf{G}}^{\mathscr{A}_p^+\mathrm{red}})[\mathrm{Gal}_{F,S}]\to\mathscr{O}(\mathscr{X}_{\mathbf{G}}^{\mathscr{A}_p^+,\mathrm{red}})$ for some number field F, and let $\mathscr{X}\hookrightarrow\mathscr{X}_{\mathbf{G}}^{\mathscr{A}_p^+,\mathrm{red}}$ be an irreducible component. Suppose in addition there is an ordered basis $\{s_{i,v}\}$ of Σ with $\prod_{j=1}^i s_{j,v}\in\Sigma^+$ such that for a Zariski-dense set of points $Z\subset\mathscr{X}$ the Galois representation attached to D is trianguline at all places $v\mid p$, with parameters $\{\delta_{i,v}\}$ induced by $\{s_{i,v}\}$. Then if we have a commutative diagram*

$$\begin{tikzcd}
\mathrm{Spa}\,R\smallsetminus\{p=0\} \arrow[r] \arrow[d] & \mathscr{X}_{\mathbf{G}}^{\mathscr{A}_p^+,\mathrm{red}} \arrow[d] \\
\mathrm{Spa}\,R \arrow[r] \arrow[ur, dashed] & \mathcal{W}
\end{tikzcd}$$

with $\operatorname{Spa} R \to \mathcal{W}$ *corresponding to an open weight* $\kappa : \mathbf{T}(\mathbf{Z}_p) \to R^\times$ *and* R *a normal pseudoaffinoid algebra flat over* $\mathbf{Z}_p$ *with* $p \notin R^\times$ *and* R/p *a domain, the dashed arrow can be filled in.*

We first treat the case of a finite morphism:

Lemma 5.2.2. *Suppose* $\mathscr{X} \to \mathscr{Y}$ *is a finite morphism of pseudorigid spaces and we have a commutative diagram*

$$\begin{tikzcd} \operatorname{Spa} R \smallsetminus \{p=0\} \arrow[r] \arrow[d] & \mathscr{X} \arrow[d] \\ \operatorname{Spa} R \arrow[r] \arrow[ur, dashed] & \mathscr{Y} \end{tikzcd}$$

where R *is a* $\mathbf{Z}_p$*-flat normal pseudoaffinoid algebra with* $p \notin R^\times$*. Then the dashed arrow can be filled in.*

Proof. By [Hub13, 1.4.4], the pre-image in $\mathscr{X}$ of any affinoid subspace of $\mathscr{Y}$ is itself affinoid, so we may assume that $\mathscr{X}$ and $\mathscr{Y}$ are both affinoid. Then if $\mathscr{X} = \operatorname{Spa} A$, the morphism $\operatorname{Spa} R \smallsetminus \{p=0\} \to \mathscr{X}$ is induced by a compatible sequence of continuous homomorphisms $(A, A^\circ) \to (R_0 \left\langle \frac{u^k}{p} \right\rangle \left[\frac{1}{u}\right], R_0 \left\langle \frac{u^k}{p} \right\rangle \left[\frac{1}{u}\right]^\circ)$ for some noetherian ring of definition $R_0 \subset R$ and $k \geq 1$. By [Lou17, Theorem 5.1], $R^\circ = \cap_k R \left\langle \frac{u^k}{p} \right\rangle^\circ$, so we have a continuous homomorphism $A^\circ \to R^\circ$. Since A° contains a ring of definition of A, we obtain a continuous homomorphism $A \to R$, as well. Since the composition $\operatorname{Spa} R \to \mathscr{X} \to \mathscr{Y}$ agrees with the specified morphism $\operatorname{Spa} R \to \mathscr{P}\mathscr{Y}$ after restricting to $\operatorname{Spa} R \smallsetminus \{p=0\}$ and $\mathscr{Y}$ is separated, it agrees on all of $\operatorname{Spa} R$. $\square$

Combined with the theory of determinants B, the assumption that a determinant $D : \mathscr{O}(\mathscr{X}_{\mathbf{G}})[\mathrm{Gal}_{F,S}] \to \mathscr{O}(\mathscr{X}_{\mathbf{G}})$ exists implies that there is a natural map $\mathscr{X}_{\mathbf{G}}^{\mathrm{red}} \to X_p^{\mathrm{an}}$, where X_p is the adic space associated to the deformation rings of all of the determinants attached to isomorphism classes of d-dimensional modular residual representations of $\mathrm{Gal}_{F,S}$.

set of isomorphism classes is finite? can avoid, but find a reference

Lemma 5.2.3. *Let* R *be an integral normal pseudoaffinoid algebra flat over* $\mathbf{Z}_p$ *with pseudouniformizer* u*. Then any morphism* $\operatorname{Spa} R \smallsetminus \{p=0\} \to X_p^{\mathrm{an}}$ *extends uniquely to a morphism* $\operatorname{Spa} R \to X_p^{\mathrm{an}}$.

Proof. We use the Hebbarkeitssätze of [Lou17]. The pseudorigid space $\operatorname{Spa} R \smallsetminus \{p = 0\}$ is connected, so its image in X_p^{an} has constant residual determinant $\overline{D}$. Thus, the morphism $\operatorname{Spa} R \smallsetminus \{p=0\} \to X_p^{\mathrm{an}}$ is induced by a series of homomorphisms $R_{\overline{D}} \to R \left\langle \frac{u^k}{p} \right\rangle^\circ$ for $k \geq 1$, where $R_{\overline{D}}$ is the pseudodeformation ring parametrizing lifts of $\overline{D}$. But by [Lou17, Theorem 5.1], $R^\circ = \cap_k R \left\langle \frac{u^k}{p} \right\rangle^\circ$, so we get a continuous homomorphism $R_{\overline{D}} \to R^\circ$ and a morphism $\operatorname{Spa} R \to (\operatorname{Spa} R_{\overline{D}})^{\mathrm{an}}$. $\square$

Lemma 5.2.4. *For any* $s \in \Sigma^+$*,* $\psi(s) \in \mathscr{O}(\mathscr{X}_{\mathbf{G}}^{\mathscr{A}_p^+,\mathrm{red}})$ *is power-bounded.*

Proof. This follows from the construction of [JN16] and we use the notation of that paper freely. By [JN16, Corollary 3.3.10], the action of s on $\mathcal{D}_\kappa^r$ is norm-decreasing, for any weight $\kappa : T_0 \to R^\times$ and any $r \gg 1/p$ (depending on κ). It follows that the action of s is power-bounded, hence power-bounded on $C^*(K, \mathcal{D}_\kappa^r)$, hence power-bounded on $\mathscr{K}^\bullet := \ker^\bullet Q^*(U_t)$, and hence power-bounded on $H^*\mathscr{K}^\bullet$. $\square$

Now we are in a position to prove Theorem 5.2.1:

Proof of Theorem 5.2.1. Since $\mathscr{X}_{\mathbf{G}}^{\mathscr{A}_p^+,\mathrm{red}} \to \widehat{\Sigma}$ is finite, Lemma 5.2.2 implies that it therefore suffices to lift κ to a morphism $\widetilde{\kappa} : \operatorname{Spa} R \to \widehat{\Sigma}$ (compatibly with the given map $\operatorname{Spa} R \smallsetminus \{p = 0\} \to \widehat{\Sigma}$). In other words, we need to show that the image of $\psi(s_{i,v})$ in $\cap_k R\left\langle \frac{u^k}{p} \right\rangle$ is an element of $R^\times$ for all i and all $v \mid p$.

Let $U_{i,v} := \prod_{j=1}^{i} s_{i,v} \in \Sigma^+$; by Lemma 5.2.4, $\psi(U_{i,v})$ is power-bounded for all i and all $v \mid p$, and by [Lou17, Theorem 5.1], the image of $\psi(U_{i,v})$ in $\cap_k R\left\langle \frac{u^k}{p} \right\rangle$ lands in $R^+ \subset R$.

It suffices to show that the image of $\psi(U_{i,v})$ is a unit of R, i.e. that it does not vanish at any point in the locus $\{p = 0\}$. We proceed by induction on i. Let $F_{1,v}$ denote the image of $\psi(s_{1,v}) = \psi(U_{1,v})$ in R. By Lemma 5.2.3, the determinant $D : \mathscr{O}(\mathscr{X}_{\mathbf{G}}^{\mathscr{A}_p^+,\mathrm{red}})[\mathrm{Gal}_{F,S}] \to \mathscr{O}(\mathscr{X}_{\mathbf{G}}^{\mathscr{A}_p^+,\mathrm{red}})$ extends to a determinant $R[\mathrm{Gal}_{F,S}] \to R$. By Proposition **??** there is a morphism $f : X' \to \operatorname{Spa} R$ and a family of Galois representations M' over X' such that M' induces the pullback f^*D_R of D_R to X', and we may assume that $X' \to \operatorname{Spa} R$ is the composition of a blow-up and a finite morphism. We may make a further blow-up, and assume that $H^1_{\varphi,\Gamma}(M')$ and $H^2_{\varphi,\Gamma}(M')$ have Tor-dimension at most 1.

Let M'_v denote the restriction of M' to the local Galois group at $v \mid p$. For each point $x \in \operatorname{Spa} R$, the morphism $f^{-1}(x) \to \operatorname{Spa} k(x)$ is flat, and so $H^i_{\varphi,\Gamma}(M'_v|_{f^{-1}(x)}) = H^i_{\varphi,\Gamma}(M_v|_x)$, where by abuse of notation, we let $M_v|_x$ refer to the Galois representation at x (possibly after extending scalars). In particular, there is a Zariski-dense set of points $x' \in X'$ such that $D_{\mathrm{rig}}(M'_v \otimes k(x'))^{\Gamma=1,\varphi^{f_v}=f^{\#}(F_{1,v})}$ is free of rank 1 over $k(x') \otimes F_{v,0}$. It follows that there is a Zariski-dense open subspace $U' \subset X' \smallsetminus \{p = 0\}$ such that $D_{\mathrm{rig}}(M_v|_{U'})^{\Gamma=1,\varphi^{f_v}=f^{\#}(F_{1,v})}$ is a rank-1 vector bundle over $\mathscr{O}(U') \otimes F_{v,0}$.

There is a finite affinoid cover $\{U'_j\}$ of X' and a finite extension L_v/F_v such that $D_{\mathrm{rig}}^{L_v}(M'_v|_{U'_j})$ is a free $\Lambda_{U'_j,\mathrm{rig},L_V}$-module. Fix an open affinoid $\operatorname{Spa} R'_j \subset U'_j \smallsetminus \{p = 0\}$ such that $D_{\mathrm{rig}}(M_v|_{\operatorname{Spa} R'_j})^{\Gamma=1,\varphi^{f_v}=f^{\#}(F_{1,v})}$ is free of rank 1 and fix a generator $\mathbf{e}_{v,j}$ of $D_{\mathrm{rig}}(M_v|_{\operatorname{Spa} R'_j})^{\Gamma=1,\varphi^{f_v}=f^{\#}(F_{1,v})}$; after dividing by a suitable power of p, we may assume that $\mathbf{e}_{v,j}$ does not vanish on the entire locus $\{p = 0\}$. Since $\varphi^{f_v}(\mathbf{e}_{v,j}) = f^{\#}(F_{1,v})\mathbf{e}_{v,j}$ holds on a Zariski-dense subset of U'_j, it holds on all of U'_j.

By construction, $H^1_{\varphi,\Gamma}(M')$ and $H^2_{\varphi,\Gamma}(M')$ have Tor-dimension at most 1, so Proposition **??** implies that for every maximal point $x' \in U'_j$, we have an injection $k(x') \otimes D_{\mathrm{rig}}^L(M') \hookrightarrow D_{\mathrm{rig}}^L(k(x') \otimes M')$. If $\mathbf{e}_{v,j}$ is non-zero at x', injectivity of φ on $D_{\mathrm{rig}}^L(k(x') \otimes M')$ implies that the image of $F_{1,v}$ does not vanish at x'. But $F_{1,v}$ vanishes at a maximal point $x \in \operatorname{Spa} R$ if and only if its image vanishes at every point of $f^{-1}(x)$. Its vanishing locus is therefore a proper closed subspace of $\operatorname{Spa} R/p$; since we assumed $\operatorname{Spec} R/p$ irreducible, Krull's Hauptidealsatz implies that $F_{1,v}$ is actually a unit of R.

Let $\delta_{1,v} : F_v^\times \to R^\times$ be the character which sends the uniformizer ϖ_v to $F_{1,v}$ and is defined by κ on $\mathscr{O}_{F_v}^\times$. We have a morphism $f^*\Lambda_{X',\mathrm{rig},L}(\delta_{1,v}) \to D_{\mathrm{rig}}^L(M')$ over a Zariski-open subspace containing the locus $\{p = 0\}$ which is injective with projetive quotient. Then we may repeat this argument with $\psi(s_{2,v}) = \psi(U_{2,v})/F_{1,v}$, and eventually conclude that $\psi(s_{i,v})$ is a unit of R for all i, as desired. □

In particular, we may deduce that every irreducible component of the extended eigencurve is proper at the boundary of weight space.

5.3. **Trianguline points.** In this section, we show that the Galois representations attached to certain characteristic p points of $X_{\mathbf{G}}$ in the closure of the characteristic 0 eigenvariety are trianguline, partially answering a question of [AIP18] and [JN16].

Our setup is similar to the previous section:

Theorem 5.3.1. *Suppose $\mathscr{X}_{\mathbf{G}}^{\mathscr{A}_p^+}$ is an extended eigenvariety such that there is a continuous determinant $D : \mathscr{O}(\mathscr{X}_{\mathbf{G}}^{\mathscr{A}_p^+\mathrm{red}})[\mathrm{Gal}_{F,S}] \to \mathscr{O}(\mathscr{X}_{\mathbf{G}}^{\mathscr{A}_p^+,\mathrm{red}})$ for some number field F, and let $\mathscr{X} \hookrightarrow \mathscr{X}_{\mathbf{G}}^{\mathscr{A}_p^+,\mathrm{red}}$ be an irreducible Zariski-closed subspace. Suppose in addition there is an ordered basis $\{s_{i,v}\}$ of Σ with $\prod_{j=1}^i s_{j,v} \in \Sigma^+$ for all i such that for a very Zariski-dense set of points $Z \subset \mathscr{X}$ the Galois representation attached to D is trianguline at all places $v \mid p$, with parameters $\{\delta_{i,v}\}$ induced by $\{s_{i,v}\}$. If $x \in \mathscr{X}$ is a maximal point whose residue field has positive characteristic, then the Galois representation associated to the restriction $D|_x$ is also trianguline at all places $v \mid p$, with parameters $\{\delta_{i,v}\}$ induced by $\{s_{i,v}\}$.*

Proof. Let $U = \operatorname{Spa} R \subset X_{\mathbf{G}}^{\mathrm{red}}$ be an irreducible affinoid pseudorigid subspace containing x, with $U \smallsetminus \{p = 0\}$ non-empty. By [**?**, ???], there is a topologically finite-type cover $f' : U' := \operatorname{Spa} R' \to U$ and a Galois representation $\rho' : \mathrm{Gal}_{F,S} \to \mathrm{GL}_n(R'^\circ)$ such that the determinant associated to ρ' is equal to $R'^\circ \otimes_{R^\circ} D$. By [Bel20] and Proposition 2.2.3, for each place $v \mid p$ of F, there is a projective (φ, Γ_{F_v})-module $D_{\mathrm{rig}}(\rho'_v)$ associated to ρ'_v.

By assumption, there is a Zariski-dense set of points $\{x_i\} \subset U \smallsetminus \{p = 0\}$ and continuous characters $\delta_{v,j} : E_v \to R^\times$ such that the $(\varphi, \Gamma_{K_v}, \mathrm{Gal}_{K_v/E_v})$-module attached to $D_{x_i}|_{\mathrm{Gal}_{E_v}}$ is trianguline with parameters $(\delta_{v,j})_{x_i}$. Thus, after passing to a further cover $f'' : U'' \to U$, there is an open subspace $V \subset U''$ containing $\{p = 0\} \subset U''$ such that $f''^*\rho'|_V$ is trianguline with parameters $\delta_{v,j}$.

In particular, $f''^*\rho'|_{(f'\circ f'')^{-1}(x)}$ is trianguline. Since $(f' \circ f'')^{-1}(x) \to \{x\}$ is faithfully flat, the triangulation descends to a triangulation on $D_{\mathrm{rig}}(\rho_x)$ with the desired parameters. □

This argument applies strictly to positive characteristic points lying in the closure of points where the Galois representation is already known to be trianguline. In particular, if there are irreducible components supported entirely in positive characteristic, we can say nothing at all. However, the recent work [RZ20] shows that for Hilbert modular eigenvarieties, this can never happen. Indeed, they give a precise description of the structure of these eigenvarieties in a neighborhood of the boundary of weight space.

5.4. **Patching and modularity.** Following a suggestion of Vincent Pilloni, we study the image of $\mathscr{X}_{\mathbf{G}} \to (\operatorname{Spa} R_p)^{\mathrm{an}}$ via patching. We first prove the following extension of the results of [BHS17].

Let $K/\mathbf{Q}_p$ be a finite extension, and let $\overline{\rho} : \mathrm{Gal}_K \to \mathrm{GL}_d(k)$ be a continuous representation, where k is a finite field containing the residue field of K. Let $X^{\square}_{\mathrm{tri},\overline{\rho}} \subset (\operatorname{Spa} R^{\square}_{\overline{\rho}})^{\mathrm{an}} \times \mathcal{T}^d$ be the Zariski closure of the set of maximal points $x = \{(\rho_x, \underline{\delta}_x)\}$, where ρ_x is a (framed) lift of $\overline{\rho}$ and $\underline{\delta}_x \in \mathcal{T}^d_{\mathrm{reg}}(L)$ is a regular parameters of $D_{\mathrm{rig}}(\rho_x)$.

Proposition 5.4.1. *The space $X^{\square}_{\mathrm{tri},\overline{\rho}}$ (equipped with its underlying reduced structure) is equidimensional of dimension $d^2 + [K : \mathbf{Q}_p]\frac{d(d+1}{2}$.*

Proof. The proof is very similar to that of [BHS17, Théorème 2.6]. By construction, there is a universal framed deformation $\rho_{\mathrm{univ}} : \mathrm{Gal}_K \to \mathrm{GL}_d(R^{\square}_{\overline{\rho}})$ of $|overline\rho$, and we may pull it back to $X^{\square}_{\mathrm{tri},\overline{\rho}}$. Then there is a sequence of blow-ups and normalizations $f : \widetilde{X} \to X^{\square}_{\mathrm{tri},\overline{\rho}}$ and an open subspace $U \subset \widetilde{X}$ containing the characteristic p locus such that $f^*\rho_{\mathrm{univ}}|_U$ is trianguline with parameters $f^*\underline{\delta}$. Furthermore, there is a Zariski-dense and open subspace $V \subset X^{\square}_{\mathrm{tri},\overline{\rho}}$ such that $f^{-1}(V) \subset U$ and f defines an isomorphism $f^{-1}(V) \xrightarrow{\sim} V$.

Over U, the (φ,Γ_K)-module $D := D_{\mathrm{rig},K}(f^*\rho_{\mathrm{univ}})$ is euipped with an increasing filtration $\mathrm{Fil}^\bullet D$ such that $\mathrm{gr}^i D \cong \Lambda_{U,\mathrm{rig},K}(f^*\delta_i) \otimes \mathscr{L}_i$ for some line bundle $\mathscr{L}_i$ on U. We may therefore construct a $\mathbf{G}^d_{\mathrm{m}}$-torsor $U^{\square} \to U$ trivializing each of the $\mathscr{L}_i$; since $U^{\square}$ carries the data $(D, \mathrm{Fil}^\bullet D, f^*\underline{\delta}, \underline{\nu})$, where $\underline{\nu}$ is the set of trivializations $\nu_i : \mathrm{gr}^i D \xrightarrow{\sim} \Lambda_{U,\mathrm{rig},K}(f^*\delta_i)$, there is a morphism $U^{\square} \to \mathcal{S}^{\square}_d$.

Let $V^{\square} \subset U^{\square}$ denote the pullback of $U^{\square} \to U$ to V. We claim that $V^{\square} \to \mathcal{S}^{\square}_d$ is smooth of relative dimension d^2. To see this, we first observe that if D is a projective (φ,Γ_K)-module over a pseudoaffinoid algebra R, and $I \subset R$ is a square-zero ideal such that $(R/I)\otimes_R D$ arises from a family of Galois representations, then D also arises from a family of Galois representations. Indeed, we have a short exact sequence

$$0 \to ID \to D \to (R/I) \otimes_R D \to 0$$

By assumption, $D' := (R/I) \otimes_R D$ arises from a family of Gal_K representations M' over R/I, and since $D'' := ID = (R/\operatorname{ann}_R I) \otimes_R D = (R/\operatorname{ann}_R I) \otimes_{R/I} D'$, it arises from a family of Gal_K representations M'' over $R/\operatorname{ann}_R I$. Since D has a model D_b over $\Lambda_{R,(0,b],K}$, we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{\Lambda}_{R,(0,b/p]} \otimes_R D'' & \longrightarrow & \widetilde{\Lambda}_{R,(0,b/p]} \otimes_R D & \longrightarrow & \widetilde{\Lambda}_{R,(0,b/p]} \otimes_R D' & \longrightarrow & 0 \\
 & & \Big\uparrow{\scriptstyle \varphi-1} & & \Big\uparrow{\scriptstyle \varphi-1} & & \Big\uparrow{\scriptstyle \varphi-1} & & \\
0 & \longrightarrow & \widetilde{\Lambda}_{R,(0,b]} \otimes_R D'' & \longrightarrow & \widetilde{\Lambda}_{R,(0,b]} \otimes_R D & \longrightarrow & \widetilde{\Lambda}_{R,(0,b]} \otimes_R D' & \longrightarrow & 0
\end{array}$$

By construction, $\widetilde{\Lambda}_{R,(0,b]}\otimes_R D'' \cong \widetilde{\Lambda}_{R,(0,b]}\otimes\left(\widetilde{\Lambda}_{R_0,[0,b]}\otimes_{R_0} M_0''\right)$ and $\widetilde{\Lambda}_{R,(0,b]}\otimes_R D' \cong \widetilde{\Lambda}_{R,(0,b]}\otimes\left(\widetilde{\Lambda}_{R_0,[0,b]}\otimes_{R_0} M_0'\right)$, for some integral models M_0'' and M' (perhaps after localizing on $\operatorname{Spa} R$ and shrinking b). Therefore, we have quasi-isomorphisms

$$[M''] \xrightarrow{\sim} [\widetilde{\Lambda}_{R,[0,b]}\otimes_{R_0} M_0'' \xrightarrow{\varphi-1} \widetilde{\Lambda}_{R,[0,b/p]}\otimes_{R_0} M_0''] \xrightarrow{\sim} [\widetilde{\Lambda}_{R,(0,b]}\otimes_R D'' \xrightarrow{\varphi-1} \widetilde{\Lambda}_{R,(0,b/p]}\otimes_R D'']$$

and

$$[M'] \xrightarrow{\sim} [\widetilde{\Lambda}_{R,[0,b]}\otimes_{R_0} M_0' \xrightarrow{\varphi-1} \widetilde{\Lambda}_{R,[0,b/p]}\otimes_{R_0} M_0'] \xrightarrow{\sim} [\widetilde{\Lambda}_{R,(0,b]}\otimes_R D' \xrightarrow{\varphi-1} \widetilde{\Lambda}_{R,(0,b/p]}\otimes_R D']$$

The snake lemma therefore implies that we have an exact sequence

$$0 \to M'' \to \left(\widetilde{\Lambda}_{R,\mathrm{rig},K}\otimes D\right)^{\varphi=1} \to M' \to 0$$

of R-modules equipped with continuous R-linear actions of Gal_K, with M' finite projective over R/I and $M'' = (R/\operatorname{ann}_R I)\otimes_{R/I} M'$. It follows that $M := \left(\widetilde{\Lambda}_{R,\mathrm{rig},K}\otimes D\right)^{\varphi=1}$ is a projective R module of the same rank and $D_{\mathrm{rig},K}(M) = D$.

In our situation, M' is equipped with an integral model $M_0' \subset M$ which is free over some ring of definition $R_0' \subset R/I$ and a basis of M_0', and M_0' and its basis lift to a free module M_0 over some ring of definition $R_0 \subset R$ which is an integral model of M. Moreover, M' is residually a lift of $\overline{\rho}$ at every maximal point of $\operatorname{Spa} R$, so M is, as well. Thus, M corresponds to a $\operatorname{Spa} R$-point $\operatorname{Spa} R \to X_{\mathrm{tri},\overline{\rho}}^{\square}$ deforming M'. Since M' corresponds to a $Spa(R/I)$-point of the Zariski-open subspace $V \subset X_{\mathrm{tri},\overline{\rho}}^{\square}$, the image of the morphism $\operatorname{Spa} R \to X_{\mathrm{tri},\overline{\rho}}^{\square}$ also lands in V. Since D is trianguline with regular parameter and trivialized quotients, the morphism $\operatorname{Spa} R \to V$ lifts to a morphism $\operatorname{Spa} R \to V^{\square}$.

The dimension computation follows because "changing the framing" makes $V^{\square}$ a GL_d-torsor over its image in $\mathcal{S}_d^{\square}$, and our claim follows.

Now by Proposition 4.2.5, we see that $V^{\square}$ is equidimensional of dimension $d^2 + \frac{d(d-1)}{2}[K:\mathbf{Q}_p] + d[K:\mathbf{Q}_p] + d$. Since $V^{\square} \to V$ is a $\mathbf{G}_{\mathrm{m}}^d$-torsor, it follows that V is equidimensional of dimension $d^2 + \frac{d(d+1)}{2}[K:\mathbf{Q}_p]$. Finally, $V \subset X_{\mathrm{tri},\overline{\rho}}^{\square}$ is Zariski-dense, so we are done. $\square$

Now we assume that $K = \mathbf{Q}_p$ and $d = 2$. Suppose that $\overline{\rho} : \mathrm{Gal}_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{F})$ is an absolutely irreducible representation, unramified outside a finite set of primes S, such that $\overline{\rho}|_{\mathrm{Gal}_{\mathbf{Q}_p}}$ has scalar endomorphisms.

Appendix A. Analytic loci of formal schemes

Let E be a complete discretely valued field with ring of integers $\mathscr{O}_E$, uniformizer ϖ_E, and residue field k_E. We briefly recall the theory of notion of a pseudorigid space over $\mathscr{O}_E$, before discussing pseudorigid generalizations of the generic fiber constructions of Bosch–Lütkebohmert [BL93] and Berthelot [dJ95].

Definition A.0.1. Let R be a Tate ring. We say that R is a *pseudoaffinoid $\mathscr{O}_E$-algebra* if it has a noetherian ring of definition $R_0 \subset R$ which is formally of finite type over $\mathscr{O}_E$. If X is an adic space over $\operatorname{Spa}\mathscr{O}_E$, we say that X is *pseudorigid* if it is locally of the form $\operatorname{Spa} R := \operatorname{Spa}(R, R^\circ)$, where R is a pseudoaffinoid $\mathscr{O}_E$-algebra.

Example A.0.2. Let $\lambda = \frac{n}{m} \in \mathbf{Q}_{>0}$ be a positive rational number with $(n,m) = 1$, and set $D_\lambda^\circ := \mathscr{O}_E[\![u]\!]\left\langle \frac{\varpi_E^m}{u^n}\right\rangle$ and $D_\lambda := D_\lambda^\circ\left[\frac{1}{u}\right]$. Then D_λ is a pseudoaffinoid algebra.

Every pseudoaffinoid algebra R is a topologically finite type D_λ-algebra for some sufficiently small $\lambda > 0$, by [Lou17, Lemma 4.8].

Recall that a point of a pre-adic space $\mathrm{Spa}(A, A^+)$ is said to be *analytic* if the kernel of the corresponding valuation is not open. We can describe the analytic locus in certain $\mathscr{O}_E$-formal schemes as explicit pseudorigid spaces, following [dJ95]. Let $\mathfrak{X} = \mathrm{Spf}\, A$ be a noetherian affine formal scheme over $\mathscr{O}_E$, and let $X = \mathrm{Spa}\, A$ be to corresponding adic space; let $I \subset A$ be the ideal of topologically nilpoent elements, and assume in addition that A/I is a finitely generated k_E-algebra.

Proposition A.0.3. *With X, A, and I as above, $X^{\mathrm{an}} = \cup_{f\in I}\cup_{n\in\mathbf{N}} X\left\langle \frac{I^n}{f}\right\rangle$. If $I = (f_1, \ldots, f_r)$, we have $X^{\mathrm{an}} = \cup_{f_i}\cup_{n\in\mathbf{N}} X\left\langle \frac{I^n}{f}\right\rangle$.*

Proof. Each rational subset $X\left\langle \frac{I^n}{f}\right\rangle$ is Tate, hence analytic, so $X^{\mathrm{an}} \supset \cup_{f\in I}\cup_{n\in\mathbf{N}}$ $X\left\langle \frac{I^n}{f}\right\rangle$.

On the other hand, point $x \in X$ is analytic if and only if there is some $f \in I$ which does not vanish at x. For such an x and f, the set $\{g \in A \,|\, |g(x)| \leq |f(x)|\} \subset A$ is open, and so contains I^n for some $n \gg 0$. Thus, every analytic point of X is contained in $\cup_{f\in I}\cup_{n\in\mathbf{N}} (\mathrm{Spa}\, A)\left\langle \frac{I^n}{f}\right\rangle$. $\square$

In the special case when $A = R_0$ is a ring of definition of a pseudoaffinoid algebra, we obtain the following:

Corollary A.0.4. *There is a natural morphism $\mathrm{Spa}\, R \to \mathrm{Spa}\, R_0$ which identifies $\mathrm{Spa}\, R$ with the analytic locus of $\mathrm{Spa}\, R_0$.*

Proof. Since the topology on R_0 is u-adic, there is some $n_0 \gg 0$ such that $I^n \subset uR_0$ for all $n \geq n_0$. Then for all sufficiently large n and all $f \in I^n$, $|f| \leq |u|$ holds on all of $\mathrm{Spa}\, R_0$, and $(\mathrm{Spa}\, R_0)\left\langle \frac{I^n}{u}\right\rangle \subset \mathrm{Spa}\, R_0$ is universal for morphisms $\mathrm{Spa}(R', R'^+) \to \mathrm{Spa}\, R_0$ such that the image of u is invertible in R'. But this is the universal property satisfied by the morphism $\mathrm{Spa}\, R \to \mathrm{Spa}\, R_0$. $\square$

As in [dJ95, Proposition 7.1.7], we can give a functor-of-points characterization of X^{an}:

Proposition A.0.5. *Let $\mathfrak{X}$ and X be as above, and let $Y := \mathrm{Spa}\, R$ be an affinoid pseudorigid space. Then*

$$\varinjlim_{\substack{R_0\subset R \\ \text{ring of definition}}} \mathrm{Hom}_{\mathcal{FS}/\mathscr{O}_E}(\mathrm{Spf}\, R_0, \mathfrak{X}) \xrightarrow{\sim} \mathrm{Hom}(Y, X^{\mathrm{an}}) \tag{A.0.1}$$

is an isomorphism.

Proof. Given any two rings of definition of R, there is a third which contains both of them. Moreover, suppose $R_0 \subset R$ is a ring of definition, and $g : R \to R'$ is a continuous homomorphism of pseudoaffinoid algebras. By [Hub93, Proposition 1.10], g is adic; it therefore carries $R_0 \subset R$ to a ring of definition of R'.

Thus, for a fixed $\mathfrak{X}$, we can view $\varinjlim_{R_0 \subset R \text{ ring of definition}} \operatorname{Hom}_{\mathcal{FS}/\mathscr{O}_E}(\operatorname{Spf} R_0, \mathfrak{X})$ as a covariant functor evaluated on R, and equation A.0.1 as a natural transformation. We will construct an inverse. Suppose we have a morphism $Y \to X^{\mathrm{an}}$; it is induced by a continuous ring homomorphism $g : A \to R$. Using the description of the analytic locus of X from Proposition A.0.3 and the quasi-compactness of Y, we see that the image of Y is contained in $U := \cup_i U_i$, where $U_i := X\left\langle \frac{I^{n_i}}{f_i} \right\rangle$, for some finite set $\{f_i\} \subset I$.

Let $V_i \subset Y$ denote the rational subset $\operatorname{Spa} R\left\langle \frac{g(i)^{n_i}}{g(f_i)} \right\rangle$, and let R_i denote its coordinate ring. Each morphism $V_i \to U_i$ is induced by a continuous ring homomorphism $A\left\langle \frac{I^{n_i}}{f_i} \right\rangle \to R_{0,i}$, for some ring of definition $R_{0,i} \subset R_i$.

Let $R_0 \subset R^\circ$ be the equalizer of $\prod_i R_{0,i} \rightrightarrows \prod_{i,j} R_{i,j}^\circ$; we claim that the map $A \to \Gamma(U, \mathscr{O}_X) \to R$ factors through R_0, and R_0 is a ring of definition of R. For the first claim, we consider the diagram

$$\begin{tikzcd}
0 \arrow[rr] & & \Gamma(U,\mathscr{O}_X) \arrow[rr] \arrow[dd] \arrow[dr, dashed] & & \prod_i \Gamma(U_i,\mathscr{O}_X) \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dd] \arrow[dr] & & \prod_{i,j} \Gamma(U_{i,j},\mathscr{O}_X) \arrow[dd] \\
& 0 \arrow[rr] & & R_0 \arrow[rr] \arrow[dl, hook] & & \prod_i R_{0,i} \arrow[dl, hook] \arrow[dr, shift left] \arrow[dr, shift right] & \\
0 \arrow[rr] & & R^\circ \arrow[rr] & & \prod_i R_i^\circ \arrow[rr, shift left] \arrow[rr, shift right] & & \prod_{i,j} R_{i,j}^\circ
\end{tikzcd}$$

It is commutative with exact rows, and a diagram chase shows that the dotted arrow exists.

For the second claim, let $u \in R$ be a pseudouniformizer; we check that $R_0[u^{-1}] = R$ and R_0 is a bounded subring of R. Given $r \in R$, we may write $\prod_i r_i$ for its image in $\prod_i R_i$. Since $R_{0,i} \subset R_i$ is a ring of definition, there is some $n \in \mathbf{N}$ such that $u^n\left(\prod_i r_i\right) \in \prod_i R_{0,i}$; by construction, $u^n\left(\prod_i r_i\right)$ is in the kernel of $\prod_i R_{0,i} \rightrightarrows \prod_{i,j} R_{i,j}^\circ$, so it defines the desired element of R_0.

To see that $R_0 \subset R$ is bounded, we let $R_0' \subset R$ be a ring of definition. It induces rings of definition $R_{0,i}' \subset R_i$; since $R_{0,i} \subset R_i$ is bounded, there is some $n' \in \mathbf{N}$ such that $u^{n'} \prod_i R_{0,i} \subset \prod_i R_{0,i}'$, and a diagram chase shows that $u^{n'} R_0 \subset R_0'$.

We have constructed a continuous homomorphism $A \to R_0$ inducing the morphism $Y \to X^{\mathrm{an}}$, where $R_0 \subset R$ is a ring of definition. The corresponding morphism $\operatorname{Spf} R_0 \to \operatorname{Spf} A$ is the desired element of the left side of equation A.0.1. It is straightforward to verify that this defines a natural transformation. $\square$

Appendix B. Pseudorigid determinants

We need to extend some of the results of [Che14] on moduli spaces of Galois determinants from the rigid analytic setting to the pseudorigid setting. Recall that for any topological group G and $d \in \mathbf{N}$, [Che14] defines functors $\widetilde{E}_d : \mathcal{FS}/\mathbf{Z}_p \to \underline{\mathrm{Set}}$ and $\widetilde{E}_{d,z} : \mathcal{FS}/\mathbf{Z}_p \to \underline{\mathrm{Set}}$ on the category of formal schemes over $\mathbf{Z}_p$, where z is a

d-dimensional determinant $G \to k$ for some finite field k. More precisely,

$$\widetilde{E}_d(\mathfrak{X}) := \{\text{continuous determinants } \mathscr{O}(\mathfrak{X})[G] \to \mathscr{O}(\mathfrak{X}) \text{ of dimension } d\}$$

and $\widetilde{E}_{d,z}(\mathfrak{X}) \subset \widetilde{E}_d(\mathfrak{X})$ is the subset of continuous determinants which are residually constant and equal to z.

Suppose that G is a topological group satisfying the following property:

For any open subgroup $H \subset G$, there are only finitely many continuous group homomorphisms $H \to \mathbf{Z}/p\mathbf{Z}$.

Under this condition (which is satisfied by absolute Galois groups of characteristic 0 local fields, and by groups $\mathrm{Gal}_{F,S}$, where F is a number field and S is a finite set of places of F), [Che14, Corollary 3.14] implies that $\widetilde{E}_d$ and $\widetilde{E}_{d,z}$ are representable. Moreover, every continuous determinant is residually locally constant, and so $\widetilde{E}_d = \coprod_z \widetilde{E}_{d,z}$.

We may define an analogous functor $\widetilde{E}_d^{\mathrm{an}}$ on the category of pseudorigid spaces, and we wish to prove the following:

Theorem B.0.1. *The functor $\widetilde{E}_d^{\mathrm{an}}$ is representable by a pseudorigid space X_d, and X_d is canonically isomorphic to the analytic locus of $\widetilde{E}_d$. The functor $\widetilde{E}_{d,z}^{\mathrm{an}}$ is representable by a pseudorigid space $X_{d,z}$, and $X_{d,z}$ is canonically isomorphic to the analytic locus of $\widetilde{E}_{d,z}$. Moreover, $\widetilde{E}_d^{\mathrm{an}}$ is the disjoint union of the $\widetilde{E}_{d,z}^{\mathrm{an}}$.*

Remark B.0.2. This is a direct analogue of [Che14, Theorem 3.17], and the proof is virtually identical. However, we sketch it here for the convenience of the reader.

If R is a pseudoaffinoid algebra, it contains a noetherian ring of definition $R_0 \subset R$, and we have $R^\circ = \varinjlim R_0$, where $R^\circ \subset R$ is the subring of power-bounded elements of R and the colimit is taken over all rings of definition of R . We have an injective map

$$\iota : \varinjlim_{\substack{R_0 \subset R \\ \text{ring of definition}}} \widetilde{E}(\mathrm{Spf}\, R_0) \to \widetilde{E}^{\mathrm{an}}(R)$$

Exactly as in [Che14], we have the following:

Lemma B.0.3. *Let R be a pseudoaffinoid algebra, and let $D \in \widetilde{E}_d^{\mathrm{an}}(R)$. Then*

1. *For all $g \in G$, the coefficients of $D(1+gt) \in R[t]$ lie in R°.*
2. *The map $\iota : \varinjlim_{R_0 \subset R} \widetilde{E}(\mathrm{Spf}\, R_0) \to \widetilde{E}^{\mathrm{an}}(R)$ is bijective.*
3. *If R is reduced, then $\widetilde{E}_d(\mathrm{Spf}\, R^\circ) = \widetilde{E}_d^{\mathrm{an}}(R)$.*

In particular, if L is the residue field at a maximal point of $\mathrm{Spa}\, R$ and $\mathscr{O}_L$ is its ring of integers, $\widetilde{E}_d(\mathrm{Spf}\, \mathscr{O}_L) = \widetilde{E}_d^{\mathrm{an}}(L)$. Every L-valued point of $\mathrm{Spa}\, R$ therefore defines a map

$$\widetilde{E}_d(\mathrm{Spa}\, R) \to \widetilde{E}_d(L) = \widetilde{E}_d(\mathrm{Spf}\, \mathscr{O}_L) \to \widetilde{E}_d(k_L)$$

where k_L is the residue field of $\mathscr{O}_L$.

Thus, we may talk about residual determinants of determinants $R[G] \to R$, and define $\widetilde{E}_{d,z}^{\mathrm{an}}$ for any continuous determinant z valued in a finite field.

Now the proof of Theorem B.0.1 follows by combining Proposition A.0.5 and Lemma B.0.3.

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