M3/4/5P12: Problems about induced representations

1. Let $G = D_{2n} = \langle s, g : s^n = t^2 = e, tst = s^{-1} \rangle$ and let $H = C_n = \langle s \rangle \subset G$. Classify the 1-dimensional representations (V, ρ_V) of H such that $\operatorname{Ind}_H^G V$ is an irreducible representation of G.

2. Suppose (V, ρ_V) is a representation of H and (W, ρ_W) is a representation of G. If $\operatorname{Res}_H^G W$ is the restriction of W to H, prove that $W \otimes \operatorname{Ind}_H^G V \cong \operatorname{Ind}_H^G(\operatorname{Res}_H^G W \otimes V)$ as representations of G.

3. Let $G = S_4$ and let $H = S_3 \subset G$, where we view S_3 as the set of permutations of $\{1, 2, 3, 4\}$ which fix 4. For each irreducible representation (V, ρ_V) of S_3, compute the irreducible decomposition of $\operatorname{Ind}_H^G V$ as a representation of G.

4. Let $H \subset G$ be a subgroup of G. Let (V, ρ_V) be an irreducible representation of H and let $\chi_1, \ldots, \chi_r$ be the irreducible characters of G. If $\chi_{\operatorname{Ind}_H^G V} = \sum_i d_i \chi_i$ is the irreducible decomposition of $\chi_{\operatorname{Ind}_H^G V}$, prove that $\sum_i d_i^2 \leq [G : H]$.

5. Let $H \lhd G$ be a normal subgroup and let $g_1, \ldots, g_s$ be coset representatives. If (V, ρ_V) is a representation of H, prove that $\operatorname{Ind}_H^G V_{g_i} \cong \operatorname{Ind}_H^G V$.