M3/4/5P12 Problem sheet #3

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1. Let $M \in \mathrm{GL}_d(\mathbf{C})$ be a matrix such that $M^n = \mathbf{1}$. Prove that the eigenvalues of M are nth roots of 1. Prove that M can be diagonalized. *Hint*: You can either consider the Jordan normal form, or you can consider $g \mapsto M$ as a representation of the finite cyclic group $C_n = \langle g : g^n = e \rangle$.

2. Let (V, ρ) be the unique irreducible 2-dimensional representation of D_8, and consider the 4-dimensional representation $(V \otimes V, \rho_{V \otimes V})$. We have previously seen that irreducible representations of D_8 have dimension 1 or 2, so this representation is reducible. Write down its irreducible components.

3. Let (V, ρ_V) and (W, ρ_W) be representations of G. Construct a G-linear isomorphism between $V \otimes W$ and $W \otimes V$ by choosing bases for V and W. Without choosing bases, construct a G-linear isomorphism between $\mathrm{Hom}(V^*, W)$ and $\mathrm{Hom}(W^*, V)$.

4. Let G be a finite group. Prove that the regular representation V_{reg} is isomorphic to its dual V_{reg}^*.

5. Let $(V_{\mathrm{reg}}, \rho_{\mathrm{reg}})$ be the regular representation of a finite group G. What is the character χ_{reg} associated to this representation? If (W, ρ_W) is an irreducible representation of G, compute the inner product $\langle \chi_{\mathrm{reg}}, \chi_W \rangle$.

6. Let (V, ρ_V) and (W, ρ_W) be representations of a finite group G, with $\dim V = 1$. Prove that W is irreducible if and only if $V \otimes W$ is irreducible.

7. Write down all the characters of representations of D_8. *Note*: You need to find the conjugacy classes of D_8.

8. Let X be a finite set with an action $G \times X \to X$ of a finite group G. Let (V, ρ_V) be the associated representation of G.
   (a) Show that $\chi_V(g) = |\{x \in X | g \cdot x = x\}|$.
   (b) Show that the function $\xi : G \to \mathbf{C}$ with $\xi(g) := \chi_V(g) - 1$ is also a character of G.

9. Let (V, ρ_V) be a representation of a finite group G.
   (a) Define $f : V \otimes V \to V \otimes V$ by setting $f(v_1 \otimes v_2) = v_2 \otimes v_1$ for $v_1, v_2 \in V$. Show that f is G-linear.
   (b) Define $S^2V := \ker(\mathbf{1} - f) = \{v \in V \otimes V | f(v) = f\}$. Define $\wedge^2 V := \ker(\mathbf{1} + f) = \{v \in V \otimes V | f(v) = -v\}$. Show that S^2V and $\wedge^2 V$ are complementary subrepresentations of $V \otimes V$. *Note*: You need to check that they are subrepresentations in the first place.
   (c) Prove that $\frac{1}{2}(\mathbf{1} + f)$ is a projection with image S^2V, and prove that $\frac{1}{2}(\mathbf{1} - f)$ is a projection with image $\wedge^2 V$.
   (d) If $(v_1, \ldots, v_d)$ is a basis for V, write down bases for S^2V and $\wedge^2 V$. What are the dimensions of these vector spaces?
   (e) Suppose the eigenvalues of $\rho_V(g)$ are $\lambda_1, \ldots, \lambda_d$. Show that the eigenvalues of $\rho_{\wedge^2 V}(g)$ are $\{\lambda_i \lambda_j | i < j\}$.
   (f) Show that the characters $\chi_{\wedge^2 V}$ and $\chi_{S^2 V}$ are given by

$$\chi_{\wedge^2 V}(g) = \frac{\chi_V(g)^2 - \chi_V(g^2)}{2} \qquad \chi_{S^2 V}(g) = \frac{\chi_V(g)^2 + \chi_V(g^2)}{2}$$

10. Prove that if $g \in G$ is not the identity, there is some irreducible character χ of G so that $\chi(x) \neq \chi(1)$.