5 10-24 MCCORMICK RYAN

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Proposition 8.19.

(**i**) $\forall m \in \mathbb{R}$, $-(-m) = m$.

(**ii**) $-0 = 0$.

Proof.

(**i**) As shown in class, $-m = (-1)m$.
Therefore $(-(-m)) = (-1)(-m)$.
So we need to show that $(-1)(-m) = m$.
Adding $-m$ to both sides,
$(-1)(-m) + (-m) = m + (-m)$
For the right side,
$m + (-m) = 0$ (AXIOM 8.4)
For the left side,
$(-1)(-m) + (-m) = (-m)(-1 + 1)$ (AXIOM 8.1(iii))
$(-1 + 1) = (1 + -1) = 0$ (AXIOM 8.1(i) and Axiom 8.4)
So now we have,
$(-m)(-1 + 1) = (-m)(0) = 0$ (PROP 8.15)
Which leaves us with both sides equaling 0,
$0 = 0$, so both sides of the equation are equal to each other. ✓

(**ii**) As shown in class, $-m = (-1)m$.
Therefore, $-0 = (-1)0$ and since -1 is the additive inverse of 1, $-1 \in \mathbb{R}$ so $(-1)0 = 0$ by Prop 8.15, so $-0 = 0$. □

Proposition 8.20. *Given $m, n \in \mathbb{R}$ there exists one and only one $x \in \mathbb{R}$ such that $m + x = n$.*

Proof. If $m = n$, then $x = 0$ by Axiom 8.2, and 0 is unique by Proposition 8.11.
If $m \neq n$, then let there be two equations, $m + x_1 = n$ and $m + x_2 = n$.

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For the first equation, $x_1 = n + (-m)$ by adding $(-m)$ to both sides, and simplifying the equation using Axiom 8.1(i), Axiom 8.4, and Axiom 8.2.
The same thing applies for the second equation, leaving us with $x_2 = n + (-m)$, therefore $x_1 = x_2$, so there only exists one $x \in \mathbb{R}$ that satisfies this equation for a given $m, n \in \mathbb{R}$. □

Project 8.25. *Think about why division by 0 ought not to be defined. Come up with an argument that will convince a friend.*

Proof. Division is defined as y/x or $y \cdot x^{-1}$.
For a number divided by 0, $x = 0$.
However, in Axiom 8.5 which defines multiplicative inverses, 0 is not defined to have a multiplicative inverse. Therefore it follows that division by 0 is not defined because you would be multiplying a number by the multiplicative inverse of 0, which is not defined. □

Proposition 8.40.

(i) $x \in R_{>0} \iff 1/x \in \mathbb{R}_{>0}$.

Proof. (=>)
Since $x \in \mathbb{R}_{>0}, x \in \mathbb{R}$, so $x^{-1} \in \mathbb{R}$ by Axiom 8.5.
Also, $x \cdot x^{-1} = 1$ by Axiom 8.5 and $1 \in \mathbb{R}_{>0}$ by Prop 8.28.
Therefore since $x \in \mathbb{R}_{>0}, x^{-1} \in \mathbb{R}$, and $1 \in \mathbb{R}_{>0}$,then $x^{-1} \in \mathbb{R}_{>0}$ by Proposition 8.36.

(<=)
$(1/x)^{-1} = x$ because $(1/x) \cdot (1/x)^{-1} = 1 = (1/x) \cdot x$. (Axiom 8.5)
Since $1/x \in \mathbb{R}_{>0}, 1/x \in \mathbb{R}$, so $(1/x)^{-1} = x \in \mathbb{R}$ by Axiom 8.5.
Also, $x^{-1} \cdot x = x \cdot x^{-1} = 1$ by Axiom 8.5 and $1 \in \mathbb{R}_{>0}$ by Prop 8.28.
Therefore since $1/x \in \mathbb{R}_{>0}, (x) \in \mathbb{R}$, and $1 \in \mathbb{R}_{>0}$, then $x \in \mathbb{R}_{>0}$ by Proposition 8.36. □

Proposition 8.41. $x^2 < x^3 \iff x > 1$

Proof. (=>)
$x^2 < x^3$
$x \cdot x < x \cdot x \cdot x$
Multiplying by x^{-1} on both sides,
$x \cdot x \cdot x^{-1} < x \cdot x \cdot x \cdot x^{-1}$
$x \cdot 1 < x \cdot x \cdot 1$ (AXIOM 8.5)
$x < x \cdot x$ (AXIOM 8.3)

Multiplying by x^{-1} on both sides again,
$x \cdot x^{-1} < x \cdot x \cdot x^{-1}$ (AXIOM 8.3)
$1 < x \cdot 1$ (AXIOM 8.5)
$1 < x$ (AXIOM 8.3)
$x > 1$ ✓

(<=)
$x > 1 \implies 1 < x$
Multiplying by x^2 on both sides,
$1 \cdot x^2 < x \cdot x^2$
$x^2 < x^3$ (AXIOM 8.3) □

Proposition 8.43. *Let $x, y \in \mathbb{R}$ such that $x < y$. There exists $z \in \mathbb{R}$ such that $x < z < y$.*

Proof. Assume for contradiction that $m = min(\mathbb{R})$. Then $m - 1 \in \mathbb{R}$, but $m - 1 < m$ because $m - (m - 1) = 1 \in \mathbb{R}_{>0}$. Therefore $\mathbb{R}$ has no minimum element.
So for $x, y, z \in \mathbb{R}$, there exists a $z < y$ let's say $z = y - 1$ for example since there is no minimum element. Similarly, there exists an $x < z$, let's say $x = z - 1 = y - 1 - 1$ for example for the same reason that there is always an real number smaller than another real number.
The same thing could've been done in the opposite direction, showing that $\mathbb{R}$ has no maximum element, and that there exists a $z > x$ and a $y > z$ such that $x < z < y$ in the real numbers due to it's properties. □

Proposition 8.49. *Let $A \subset \mathbb{R}$ be nonempty. If* $\sup(A) \in A$ *then* $\sup(A)$ *is the largest element of A,* $\sup(A) = \max(A)$*. Conversely, if A has a largest element then* $\max(A) = \sup(A)$ *and* $\sup(A) \in A$*.*

Proof. $b = \sup(A) \in A \implies b \in A$ and
$\sup(A)$ is an upperbound for $A \implies b$ satisfies $a \in A \implies a \leq b$
so $b = \max(A) = \sup(A)$.

Conversely, if $b = \max(A)$ then $a \in A \implies a \leq b$ by definition of a maximum.
Also if b' is an upperbound for A, $b \leq b'$, because $b = \max(A) \implies b \in A$ and $b \leq b'$ by definition of an upperbound. Therefore $\max(A) = \sup(A)$ and $sup(A) \in A$. □

Proposition 8.50. *Suppose* $A \subset B \subset \mathbb{R}$*, A and B are bounded above. Then* $\sup(A) \leq \sup(B)$*.*

Proof. $\sup(A)$ is a number b such that $a \leq b$ for all $a \in A$, and if b' is an upper bound for A, then $b \leq b'$.

$A \subset B \implies$ every $a \in A$ is an element of B.
$\sup(B)$ is an upper bound for $B \implies$ every element $c \in B$ follows $c \leq \sup(B)$.
So $\sup(B)$ is an upperbound for A because every element of A is an element of B.
Also, $\sup(A)$ is the least upper bound for A, so $\sup(A) \leq \sup(B)$ by the definition of supremum. $\square$

Project 8.51. *For a nonempty set $B \subset \mathbb{R}$, one can define the greatest lower bound,* $\inf(B)$ *of B. Give the precise definition for* $\inf(B)$ *and prove that it is unique if it exists. Also define* $\min(B)$ *and prove the analogue of Proposition 8.49 for greatest lower bounds and minima.*

Proof. B is bounded below if there exists $a \in \mathbb{R}$ such that $b \in B \implies b \geq a$.
The $\inf(B)$ for B is a lower bound c such that if c' is a lower bound, $c \geq c'$.
If $\inf(B)$ exists, let x_1 and x_2 be greatest lower bounds for B.
x_1 is a lower bound and x_2 is a greatest lower bound $\implies x_2 \geq x_1$.
x_2 is a lower bound and x_1 is a greatest lower bound $\implies x_1 \geq x_2$.
Therefore, $x_1 \leq x_2 \leq x_1$
So $x_1 = x_2$ by Proposition 8.31 so the greatest lower bound, or $\inf(B)$ is unique if it exists.

$\min(B)$: An element $a \in B$ is the minimum/smallest element ($\min(B)$) of B if for all $b \in B$, $b \in B \implies a \leq b$.

Analogue to Proposition 8.49:
$a = \inf(B) \in B \implies a \in B$.
$\inf(B)$ is a lowerbound for B $\implies a$ satisfies $b \in B \implies a \leq b$.
So $a = \min(B)$.

Conversely, assuming there exists $a = \min(B) \in R$.
$b \in B \implies a \leq b$ by definition of a minimum.
$a = \min(B) \implies a \in B \implies a \geq a'$ for a lower bound a', by definition of a lower bound. Therefore, $a = \inf(B) \in B$. So, $\min(B) = \inf(B) \in B$. $\square$

Proposition 8.53. *Every nonempty subset of* $\mathbb{R}$ *that is bounded below has a greatest lower bound.*

Proof. Let X be every nonempty subset of $\mathbb{R}$ that is bounded below and has a greatest lower bound.
Let Y be the set y = -x : $x \in X$. Since X is bounded below, there exists a lower bound z such that $z \leq x$ for all $x \in X$. Therefore since $z \leq x$, by subtracting x from both sides and subtracting z from both sides, we get $-x \leq -z$.
So $y \leq -z$ for all $y \in Y$. So Y is bounded above by $-z$, so by Axiom 8.52, $\sup(Y)$ exists.
Let $m = \sup(Y)$. Then we need to show that $-m = \inf(X)$.

$-m \leq x, \forall x \in X$ by Axiom 8.52
$y \leq x, \forall x \in X \implies y \leq -m$.

Since m is the least upper bound of Y,
$-x \leq m, \forall x \in X$ and
$-x \leq -y, \forall x \in X \implies m \leq -y$.
So $-x \leq -y, \forall x \in X \implies m \leq -y$.
Therefore, $y \leq x, \forall x \in X \implies y \leq -m$.
So $-m = \inf(X)$. $\square$