idea

If we have a point, $(x_1, \ldots, x_n)$, distributed within the n-ball of radius R, and we measure its "radius", $r = \sqrt{\sum_i^n x_i^2}$, what is $P(R|r)$? If we know $P(R|r)$ and we measure r for a lot of points, can we come up with a good estimate for R? Intuitively, it seems like as $n \to \infty$, $P(R|r)$ should approach $\delta(R - r)$, since all of the volume of the n-ball will be basically at the surface.

work

According to `http://mathworld.wolfram.com/BallPointPicking.html`, a point chosen from the uniform distribution over the unit N-ball is distributed like

$$\frac{(X_1, \ldots X_n)}{\sqrt{Y + \sum_i^N X_i^2}}$$

Where X_i are drawn independently from a standard normal, and Y is drawn independently from an exponential distribution with $\lambda = 1$.

Thus, the radius of the point is distributed like

$$R \sim \sqrt{\frac{\sum_i^N X_i^2}{Y + \sum_i^N X_i^2}}$$

or

$$R \sim \frac{1}{\sqrt{\frac{Y}{\sum_i^N X_i^2} + 1}}$$

This is basically an exponential distribution over a χ^2 distribution.

Let's call the quotient part Q. That is, $Q \sim \frac{Y}{\sum_i^N X_i^2}$. What is the distribution of Q? According to `dx.doi.org/10.1214/aoms/1177731679`, it should be

$$p_Q(q) = \int_{-\infty}^{\infty} |y| p_{exp}(qy) p_{\chi^2}(y) dy$$

$$p_Q(q) = \frac{1}{2^{N/2}\Gamma(\frac{N}{2})} \int_{-\infty}^{\infty} |y| e^{-qy} y^{N/2-1} e^{-y/2} dy$$

$$p_Q(q) = \frac{1}{2^{N/2}\Gamma(\frac{N}{2})} \int_{-\infty}^{\infty} |y| y^{N/2-1} e^{-(q+1/2)y} dy$$

With some help from Wolfram alpha, I think that we can do this integral. Wolfram gives the identity

$$\int x^n e^{-cx}\, dx = -\frac{x^n (cx)^{-n}\, \Gamma(n+1, cx)}{c} + \text{constant}$$

Where $\Gamma(a, x)$ is the incomplete gamma function.

I don't think we need the part of the integral below zero. Because the exponential random variable $p_{\exp}(qy)$ is only nonzero for $qy > 0$. I think this implies that $y > 0$, since obviously q is going to be greater than zero too.

The indefinite integral given implies

$$\int_0^\infty x^n e^{-cx}\, dx = c^{-n-1}\Gamma(n+1)$$

Which would give

$$p_Q(q) = \frac{1}{2^{N/2}\Gamma(\frac{N}{2})}\left(q + \frac{1}{2}\right)^{-\frac{N}{2}-1}\Gamma(N/2+1)$$

$$p_Q(q) = \frac{N}{2^{\frac{N}{2}+1}}\left(q + \frac{1}{2}\right)^{-\frac{N}{2}-1}$$

Since $R(q)$ is invertible, we get (?)

$$p(r) = \frac{N}{2^{\frac{N}{2}+1}}\left(\frac{1}{r^2} - \frac{1}{2}\right)^{-\frac{N}{2}-1}$$

Next step: What if it's *not* a unit-ball? If the ball of of radius R, then that just sets the unit system, so we have something like: (need a new normalization condition?)

$$p(\frac{r}{R}) \propto \frac{N}{2^{\frac{N}{2}+1}}\left(\frac{R^2}{r^2} - \frac{1}{2}\right)^{-\frac{N}{2}-1}$$

So how do we go from here to the quantity we want, which is $p(r|R)$?

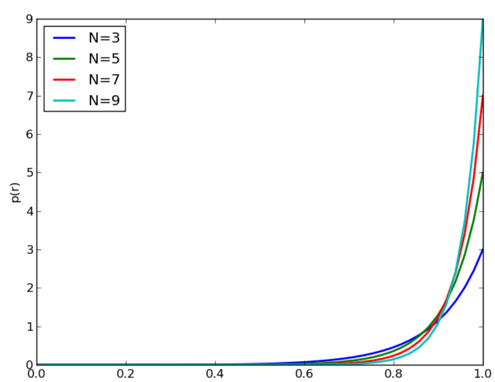

Figure 1: The $p(r)$ expression looks reasonable. In higher dimensions, the distribution is more peaked towards the surface at $r \approx 1$. But I think the normalization is off

We're going to have a lot of observations of r. If we assume that they're independent, then our log likelihood is

$$\mathcal{L} = -\left(\frac{N}{2} - 1\right) \sum_i \log\left(\frac{R^2}{r_i^2} - \frac{1}{2}\right)$$

Taking the derivative of the log likelihood with respect to R, we have

$$\frac{\partial \mathcal{L}}{\partial R} = -\sum_i \frac{4R}{r_i^2 - 2R^2} = 0$$

How do I solve this for R?