

idea

If we have a point, (x_1, \dots, x_n) , distributed within the n -ball of radius R , and we measure its “radius”, $r = \sqrt{\sum_i^n x_i^2}$, what is $P(R|r)$? If we know $P(R|r)$ and we measure r for a lot of points, can we come up with a good estimate for R ? Intuitively, it seems like as $n \rightarrow \infty$, $P(R|r)$ should approach $\delta(R - r)$, since all of the volume of the n -ball will be basically at the surface.

work

According to <http://mathworld.wolfram.com/BallPointPicking.html>, a point chosen from the uniform distribution over the unit N -ball is distributed like

$$\frac{(X_1, \dots, X_n)}{\sqrt{Y + \sum_i^N X_i^2}}$$

Where X_i are drawn independently from a standard normal, and Y is drawn independently from an exponential distribution with $\lambda = 1$.

Thus, the radius of the point is distributed like

$$R \sim \sqrt{\frac{\sum_i^N X_i^2}{Y + \sum_i^N X_i^2}}$$

or

$$R \sim \frac{1}{\sqrt{\frac{Y}{\sum_i^N X_i^2} + 1}}$$

This is basically an exponential distribution over a χ^2 distribution.

Let's call the quotient part Q . That is, $Q \sim \frac{Y}{\sum_i^N X_i^2}$. What is the distribution of Q ? According to [dx.doi.org/10.1214/aoms/1177731679](https://doi.org/10.1214/aoms/1177731679), it should be

$$p_Q(q) = \int_{-\infty}^{\infty} |y| p_{exp}(qy) p_{\chi^2}(y) dy$$

$$p_Q(q) = \frac{\lambda}{2^{N/2} \Gamma(\frac{N}{2})} \int_{-\infty}^{\infty} |y| e^{-\lambda q y} y^{N/2-1} e^{-y/2} dy$$

$$p_Q(q) = \frac{\lambda}{2^{N/2} \Gamma(\frac{N}{2})} \int_{-\infty}^{\infty} |y| y^{N/2-1} e^{-(\lambda q + 1/2)y} dy$$

With some help from Wolfram alpha, I think that we can do this integral. Wolfram gives the identity

$$\int x^n e^{-cx} dx = -\frac{x^n (cx)^{-n} \Gamma(n+1, cx)}{c} + \text{constant}$$

Where $\Gamma(a, x)$ is the incomplete gamma function.

I don't think we need the part of the integral below zero. Because the exponential random variable $p_{\text{exp}}(qy)$ is only nonzero for $qy > 0$. I think this implies that $y > 0$, since obviously q is going to be greater than zero too.

The indefinite integral given implies

$$\int_0^\infty x^n e^{-cx} dx = c^{-n-1} \Gamma(n+1)$$

Which would give

$$p_Q(q) = \frac{\lambda}{2^{N/2} \Gamma(\frac{N}{2})} \left(\lambda q + \frac{1}{2} \right)^{-\frac{N}{2}-1} \Gamma(N/2+1)$$

$$p_Q(q) = \frac{N\lambda}{2^{\frac{N}{2}+1}} \left(\lambda q + \frac{1}{2} \right)^{-\frac{N}{2}-1}$$

Since $R(q)$ is invertible, we get (?)

$$p_R(r) = \frac{N\lambda}{2^{\frac{N}{2}+1}} \left(\frac{\lambda}{r^2} - \lambda + \frac{1}{2} \right)^{-\frac{N}{2}-1}$$

λ was really assumed to be one from the beginning (it was the unit-ball), so

$$p(r) = \frac{N}{2^{\frac{N}{2}+1}} \left(\frac{1}{r^2} - \frac{1}{2} \right)^{-\frac{N}{2}-1}$$

Next step: What if it's *not* a unit-ball? Then we'll get $p(r)$ as a function of the ball's radius. We dealt with the λ separately, but there is also a term from the χ^2 distributions (which should be λ^2 that I didn't carry through).