Gravitation & Cosmology — ASTR-4240
General Relativity — PHYS-4961

Class 5
Relativistic Electrodynamics

Exercise (30 pts)

We want to prove that Maxwell's Equations are equivalent to

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c}\, j^\nu \tag{1}$$

and

$$\partial^\alpha F^{\mu\nu} + \partial^\nu F^{\alpha\mu} + \partial^\mu F^{\nu\alpha} = 0, \tag{2}$$

where

$$j^\nu \equiv \begin{pmatrix} \rho c, & j_x, & j_y, & j_z \end{pmatrix} \tag{3}$$

and

$$F^{\mu\nu} \equiv \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{4}$$

1. (10 pts) — How many independent equations does expression (1) represent? Explain your reasoning briefly.

2. (10 pts) — How many independent equations does expression (2) represent? Explain briefly. **Hint 1:** What happens if any two indices are equal? **Hint 2:** Suppose you choose some particular values for α, μ, and ν. What happens if you permute these values?

3. (10 pts) — Write out equation (2) in terms of $\mathbf{E}$ and $\mathbf{B}$ for the choice $\alpha = 0$, $\mu = 1$, $\nu = 2$. Is this a component of one of the Maxwell equations? Which one?

Solution

1. The only free index, ν, can take on the values $\nu = 0, 1, 2, 3$ so there are 4 equations.

2. Suppose that $\alpha = \mu$. Then equation (2) reduces to

$$\partial^\alpha F^{\alpha\nu} + \partial^\nu F^{\alpha\alpha} + \partial^\alpha F^{\nu\alpha} \;=\; 0. \tag{5}$$

The first and third terms add to zero because F is antisymmetric and the middle term vanishes because the diagonal elements of F are zero. Indeed, if any two indices are equal we get the trivial identity $0 = 0$. We must therefore choose all three indices differently and there are $4 \times 3 \times 2 = 24$ ways to do this. However each choice has $3! = 6$ permutations of the indices which give exactly the same equation. A cyclic permutation simply permutes the order of the terms; an anticyclic permutation permutes the terms and switches the indices on the field tensor. The number of distinct equations is therefore $24/3! = 4$.

3. Using eq. (4) to write

$$\partial^0 F^{12} + \partial^2 F^{01} + \partial^1 F^{20} = 0 \tag{6}$$

in terms of $\mathbf{E}$ and $\mathbf{B}$ gives

$$\frac{1}{c}\frac{\partial}{\partial t}(-B_z) - \frac{\partial}{\partial y}(-E_x) - \frac{\partial}{\partial x}(E_y) = 0. \tag{7}$$

Noting that

$$(\boldsymbol{\nabla}\times\mathbf{E})_z = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}, \tag{8}$$

we see that our equation is equivalent to

$$(\boldsymbol{\nabla}\times\mathbf{E})_z = -\frac{1}{c}\frac{\partial B_z}{\partial t}. \tag{9}$$

This is just the z component of Faraday's Law.