ASTR-4240 — Gravitation & Cosmology
PHYS-4240 — General Relativity

Class 1
Newtonian Gravity

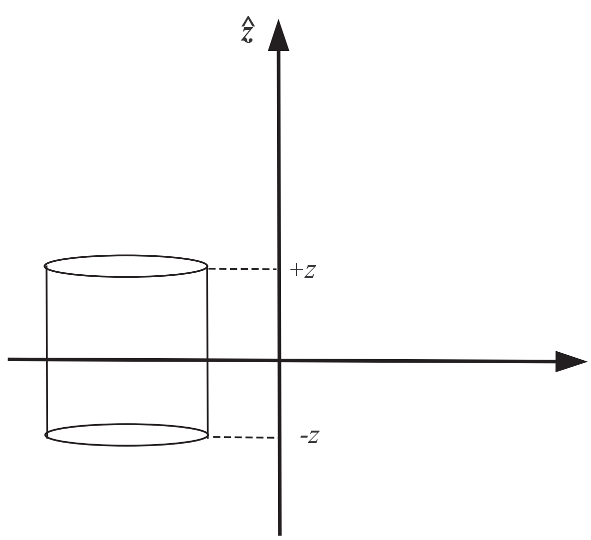
Exercise (30 pts)

Suppose that we model the mass distribution of our Milky Way galaxy as an infinite sheet of thickness H and uniform mass density ρ.

1. (10 pts) — Let z be vertical distance measured from $z = 0$ at the midplane of the disk. Find the gravitational field $\mathbf{g}$ (magnitude and direction) as a function of z.

2. (10 pts) — A star is initially at rest at some position z_0. Find its trajectory $z(t)$ in terms of the quantities given.

3. (10 pts) — What is the period of the motion in years if the disk contains $50\, M_\odot$ in a $1\,\text{pc}^2$ column? Assume that $H = 50\,\text{pc}$.



Solution

1. By symmetry, we must have $\mathbf{g} = -g(z)\,\hat{\boldsymbol{z}}$, where g denotes the magnitude of g (a positive number). Now apply Gauss's Law to a cylindrical surface with endcaps (area A) at $\pm z$:

$$\oint_S \mathbf{g}\cdot\hat{\boldsymbol{n}}\, dA = -4\pi G \int_V \rho\, dV. \tag{1}$$

The sides of the cylinder give no contribution and each endcap gives $-g(z)A$. Thus

$$-2g(z)A = -4\pi G \times 2z\rho A \tag{2}$$

and

$$\boxed{\mathbf{g}(z) = -4\pi G\rho z\ \hat{\boldsymbol{z}}.} \tag{3}$$

2. If

$$\omega \equiv \sqrt{4\pi G\rho}, \tag{4}$$

then the equation of motion for $z(t)$ is

$$\frac{d^2 z}{dt^2} = -\omega^2 z. \tag{5}$$

The solution subject to the boundary conditions $z(0) = z_0$ and $\dot{z}(0) = 0$ is

$$\boxed{z(t) = z_0\ \cos\omega t.} \tag{6}$$

The data imply a mass density of $\rho = 6.8 \times 10^{-23}\text{g cm}^{-3}$.

3. The period is

$$T = \frac{2\pi}{\omega} = 8.3 \times 10^{14}\,\text{s} = \boxed{26\,\text{Myr.}} \tag{7}$$