

Gravitation & Cosmology — ASTR-4240
General Relativity — PHYS-4961

Class 7
Weak Gravitational Fields I

Exercise (10 pts)

Show that the definition of the new field variable,

$$\phi^{\mu\nu} = h^{\mu\nu} - \frac{1}{2} h \eta^{\mu\nu}, \quad (1)$$

can be inverted to give

$$h^{\mu\nu} = \phi^{\mu\nu} - \frac{1}{2} \phi \eta^{\mu\nu}. \quad (2)$$

This expression will be used frequently: we will solve the field equations for $\phi^{\mu\nu}$ but $h^{\mu\nu}$ is what describes the spacetime geometry. *Hint:* Take the trace of eq. (1).

Solution

The trace of (1) is

$$\eta_{\nu\mu} \phi^{\mu\nu} = \eta_{\nu\mu} h^{\mu\nu} - \frac{1}{2} h \eta_{\nu\mu} \eta^{\mu\nu}. \quad (3)$$

Noting that $\eta_{\nu\mu} \phi^{\mu\nu} = \phi$, $\eta_{\nu\mu} h^{\mu\nu} = h$, and $\eta_{\nu\mu} \eta^{\mu\nu} = 4$, we see that (3) is the same as

$$\phi = h - \frac{1}{2} h (4) = -h. \quad (4)$$

Substituting $h = -\phi$ into (1) and solving for $h^{\mu\nu}$ gives (2).