Gravitation & Cosmology — ASTR-4240
General Relativity — PHYS-4240

Class 8
Weak Gravitational Fields II

Exercise (20 pts)

Calculate the trace of the field equation,

$$\partial_\lambda \partial^\lambda h^{\mu\nu} + \partial^\mu \partial^\nu h - \left(\partial_\lambda \partial^\nu h^{\mu\lambda} + \partial_\lambda \partial^\mu h^{\nu\lambda}\right) - \eta^{\mu\nu} \partial_\lambda \partial^\lambda h + \eta^{\mu\nu} \partial_\lambda \partial_\sigma h^{\lambda\sigma} = -\kappa T^{\mu\nu}. \tag{1}$$

Solution

Each term in the field equation is a second-rank tensor of the form $A^{\mu\nu}$. The trace of such a tensor is

$$A = A_\nu^{\ \nu} = \eta_{\nu\mu} A^{\mu\nu} \tag{2}$$

or

$$A = A^\mu_{\ \mu} = \eta_{\mu\nu} A^{\mu\nu}. \tag{3}$$

So all we need to do is multiply each term of the field equation by $\eta_{\nu\mu}$ or $\eta_{\mu\nu}$ and carry out the summation. Here are the results, term by term:

First term on LHS:

$$\eta_{\nu\mu} \partial_\lambda \partial^\lambda h^{\mu\nu} = \partial_\lambda \partial^\lambda \eta_{\nu\mu} h^{\mu\nu} = \partial_\lambda \partial^\lambda h. \tag{4}$$

Second term on LHS:

$$\eta_{\nu\mu} \partial^\mu \partial^\nu h = \partial_\nu \partial^\nu h = \partial_\lambda \partial^\lambda h. \tag{5}$$

Third term on LHS:

$$-\eta_{\mu\nu} \partial_\lambda \partial^\nu h^{\mu\lambda} = -\partial_\lambda \partial_\mu h^{\mu\lambda} = -\partial_\lambda \partial_\sigma h^{\sigma\lambda}. \tag{6}$$

Fourth term on LHS:

$$-\eta_{\nu\mu} \partial_\lambda \partial^\mu h^{\nu\lambda} = -\partial_\lambda \partial_\nu h^{\nu\lambda} = -\partial_\lambda \partial_\sigma h^{\sigma\lambda}. \tag{7}$$

Fifth term on LHS:

$$-\eta_{\nu\mu} \eta^{\mu\nu} \partial_\lambda \partial^\lambda h = -\eta_\nu^\nu \partial_\lambda \partial^\lambda h = -4 \partial_\lambda \partial^\lambda h \tag{8}$$

(remember that η^{ν}_{ν} is the Kronecker delta).

Sixth term on LHS:

$$\eta_{\nu\mu}\eta^{\mu\nu}\partial_\lambda\partial_\sigma h^{\lambda\sigma} = 4\partial_\lambda\partial_\sigma h^{\lambda\sigma} = 4\partial_\lambda\partial_\sigma h^{\sigma\lambda}. \tag{9}$$

RHS:

$$-\eta_{\nu\mu}\kappa T^{\mu\nu} = -\kappa T. \tag{10}$$

Combining all the terms gives

$$-2\partial_\lambda\partial^\lambda h + 2\partial_\lambda\partial_\sigma h^{\sigma\lambda} = -\kappa T, \tag{11}$$

which agrees with eq. [3.106].