$$\sigma^2 = \frac{1}{(2a+1)(2b+1)} \sum\sum \{[g(\gamma) - w\eta(\gamma)] - [\bar{g} - w\bar{\eta}]\}^2$$

γ 為受 summation 影響的項 並且定义常数 k 為 $\frac{1}{(2a+1)(2b+1)}$

而 $$\frac{d\sigma^2}{dw} = k\sum\sum 2\cdot[g(\gamma) - w\eta(\gamma) - \bar{g} + w\bar{\eta}][-\eta(\gamma) + \bar{\eta}] = 0$$

$$= k\sum\sum -g(\gamma)\eta(\gamma) + g(\gamma)\bar{\eta} + w\eta^2(\gamma) - w\eta(\gamma)\bar{\eta}$$

$$+ \bar{g}\eta(\gamma) - \bar{g}\bar{\eta} - w\bar{\eta}\eta(\gamma) + w\bar{\eta}^2$$

$$= -\overline{g\eta} + \bar{g}\bar{\eta} + w\overline{\eta^2} - w\bar{\eta}^2 + \bar{g}\bar{\eta} - \bar{g}\bar{\eta} - w\bar{\eta}^2 + w\bar{\eta}^2$$

$$= -\overline{g\eta} + \bar{g}\bar{\eta} + w(\overline{\eta^2} - \bar{\eta}^2) = 0 \ \#$$

其中，$$\frac{1}{(2a+1)(2b+1)} \sum\sum g(\gamma)\eta(\gamma) = \overline{g\eta}$$

而 $$w = \frac{\overline{g\eta} - \bar{g}\bar{\eta}}{\overline{\eta^2} - \bar{\eta}^2} \qquad w(x,y) = \frac{\overline{g(x,y)\eta(x,y)} - \bar{g}(x,y)\bar{\eta}(x,y)}{\overline{\eta^2}(x,y) - \bar{\eta}^2(x,y)}$$

亦符合 Eq(5.4-21)

Q4.

$$g(x,y)=\iint_{-\infty}^{\infty} f(\alpha,\beta)\, h(x-\alpha, y-\beta)\, d\alpha\, d\beta$$

把 $f(x,y)$ 換成 $=\delta(x-\alpha) \Rightarrow f(\alpha,\beta)=\delta(x-\beta)$

$$\hookrightarrow g(x,y)=\iint_{-\infty}^{\infty} \delta(x-\alpha)\, e^{-[(x-\alpha)^2+(x-\beta)^2]}\, d\alpha\, d\beta$$

$$\cdot \iint_{-\infty}^{\infty} \delta(x-\alpha) e^{-[(x-\alpha)^2]} e^{-[(y-\beta)^2]}\, d\alpha\, d\beta$$

$$=\int_{-\infty}^{\infty} \delta(x-\alpha) e^{-[(x-\alpha)^2]}\, d\alpha \int_{-\infty}^{\infty} e^{-[(y-\beta)^2]}\, d\beta$$

$$=e^{-[x-\alpha]^2}\int_{-\infty}^{\infty} e^{-(y-\beta)^2}\, d\beta$$

where we used the fact that the integral of the impulse is "nonzero" only when $\alpha=a \Rightarrow$

$$\int_{-\infty}^{\infty} e^{-(y-\beta)^2}\, d\beta=\int_{-\infty}^{\infty} e^{-(\beta-y)^2}\, d\beta$$

which is in the form of a constant times a Gaussian density with variance $=\frac{1}{2} \rightsquigarrow \sigma=\frac{1}{\sqrt{2}}$

$$e^{-(\beta-y)^2}=\sqrt{2\pi\times\tfrac{1}{2}}\left[\frac{1}{\sqrt{2\pi\times\frac{1}{2}}}\, e^{(-\frac{1}{2})\left[\frac{(\beta-y)^2}{\frac{1}{2}}\right]}\right]$$

The integral from minus to plus infinity of the quantity inside the brackets is $1 \Rightarrow g(x,y)=\sqrt{\pi}\, e^{-[(x-\alpha)^2]}$ #

which is blurred version of the original image.

$$x_0(t)=\begin{cases}\frac{at}{T_1} & 0\leq t\leq T_1\\ a & T_1<t\leq T_1+T_2\end{cases}\qquad y_0(t)=\begin{cases}0\ , & 0\leq t\leq T_1\\ \frac{b(t-T_1)}{T_1} & T_1<t\leq T_1+T_2\end{cases}$$

$$H(u,v)=\int_0^{T_1} e^{-j2\pi[uat/T_1]}dt+\int_{T_1}^{(T_1+T_2)} e^{-j2\pi\left[ua+\frac{vb(t-T_1)}{T_2}\right]}dt$$

$$=\frac{T_1}{\pi ua}\sin(\pi ua)e^{-j\pi ua}+e^{-j2\pi ua}\int_{T_1}^{(T_1+T_2)} e^{-j2\pi vb(t-T_1)/T_2}dt$$

$$=\frac{T_1}{\pi ua}\sin(\pi ua)e^{-j\pi ua}+e^{-j2\pi ua}\int_0^{T_2} e^{-j2\pi vb\tau/T_2}d\tau \Rightarrow \tau=t-T_1\text{代入}$$

$$=\frac{T_1}{\pi ua}\sin(\pi ua)e^{-j\pi ua}+e^{-j2\pi ua}\frac{T_2}{\pi vb}\sin(\pi vb)e^{-j\pi vb}$$

而 $g(x,y)=\mathfrak{F}^{-1}[H(u,v)F(u,v)] \longrightarrow$ 其中 $F(u,v)$ 是 Fourier transform of the Input Image

Q6.

$$\nabla^2[s(x,y)] = \left[\frac{\partial^2 s(x,y)}{\partial x^2} + \frac{\partial^2 s(x,y)}{\partial y^2}\right] = \frac{x^2+y^2-2\sigma^2}{\sigma^4} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

$$\because H(u,v) = \mathfrak{F}[h(x,y)] = \mathfrak{F}[\nabla^2[s(x,y)]]$$

$$而又 \because \mathfrak{F}[\nabla^2 s(x,y)] = -4\pi^2(u^2+v^2)F(u,v) \longrightarrow F(u,v) = \mathfrak{F}[s(x,y)] = \mathfrak{F}\left[e^{-\frac{x^2+y^2}{2\sigma^2}}\right]$$

$$又\ \mathfrak{F}\left[e^{-\frac{x^2+y^2}{2\sigma^2}}\right] = 2\pi\sigma^2 e^{-2\pi^2\sigma^2(u^2+v^2)}$$

$$H(u,v) = -4\pi^2(u^2+v^2)F(u,v)$$

$$= [-4\pi^2(u^2+v^2)][2\pi\sigma^2 e^{-2\pi^2\sigma^2(u^2+v^2)}]$$

$$= -8\pi^3\sigma^2(u^2+v^2)e^{-2\pi^2\sigma^2(u^2+v^2)}$$

#

Q7.

可能之解

(1) 使用 Image averaging 去降低噪音

(2) 使用 Wiener Filter 並調整 K 值 ⇒ 調到最清晰的 image

(3) 對 image 作 Fourier Transform 並給出 H(u,v)

(4) Blurred Image of a bright, single star to simulate an impulse (模擬脈衝) 而這个 star 愈小愈佳.