CSE 421
Algorithms

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Lecture 19

Longest Common Subsequence

Longest Common Subsequence

- $C=c_1 \ldots c_g$ is a subsequence of $A=a_1 \ldots a_m$ if C can be obtained by removing elements from A (but retaining order)
- LCS(A, B): A maximum length sequence that is a subsequence of both A and B

`ocurranec` **`attacggct`**

`occurrence` **`tacgacca`**

Determine the LCS of the following strings

BARTHOLEMEWSIMPSON

KRUSTYTHECLOWN

String Alignment Problem

- Align sequences with gaps

```
CAT TGA  AT

CAGAT AGGA
```

- Charge δ_x if character x is unmatched
- Charge γ_{xy} if character x is matched to character y

Note: the problem is often expressed as a minimization problem, with $\gamma_{xx} = 0$ and $\delta_x > 0$

LCS Optimization

- $A = a_1a_2 \ldots a_m$
- $B = b_1b_2 \ldots b_n$

- Opt[j, k] is the length of $LCS(a_1a_2 \ldots a_j, b_1b_2 \ldots b_k)$

Optimization recurrence

If $a_j = b_k$, Opt[j,k] = 1 + Opt[j-1, k-1]

If $a_j \neq b_k$, Opt[j,k] = max(Opt[j-1,k], Opt[j,k-1])

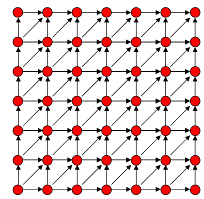
Give the Optimization Recurrence for the String Alignment Problem

- Charge δ_x if character x is unmatched
- Charge γ_{xy} if character x is matched to character y

Opt[j, k] =

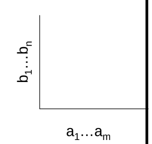
Let a_j = x and b_k = y
Express as minimization

Dynamic Programming Computation



Code to compute Opt[j,k]

Storing the path information




```
A[1..m],  B[1..n]
for i := 1 to m     Opt[i, 0] := 0;
for j := 1 to n     Opt[0,j] := 0;
Opt[0,0] := 0;
for i := 1 to m
        for j := 1 to n
                if A[i] = B[j]  {  Opt[i,j] := 1 + Opt[i-1,j-1];  Best[i,j] := Diag; }
                else if Opt[i-1, j] >= Opt[i, j-1]
                        {  Opt[i, j] := Opt[i-1, j], Best[i,j] := Left; }
                else        {  Opt[i, j] := Opt[i, j-1], Best[i,j] := Down; }
```

How good is this algorithm?

- Is it feasible to compute the LCS of two strings of length 100,000 on a standard desktop PC? Why or why not.

Observations about the Algorithm

- The computation can be done in O(m+n) space if we only need one column of the Opt values or Best Values
- The algorithm can be run from either end of the strings

Algorithm Performance

- O(nm) time and O(nm) space
- On current desktop machines
  - n, m < 10,000 is easy
  - n, m > 1,000,000 is prohibitive
- Space is more likely to be the bounding resource than time

Observations about the Algorithm

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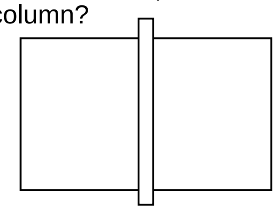
Computing LCS in O(nm) time and O(n+m) space

- Divide and conquer algorithm
- Recomputing values used to save space

Divide and Conquer Algorithm

- Where does the best path cross the middle column?



- For a fixed i, and for each j, compute the LCS that has a_i matched with b_j

Constrained LCS

- $LCS_{i,j}(A,B)$: The LCS such that
  - $a_1,\ldots,a_i$ paired with elements of $b_1,\ldots,b_j$
  - $a_{i+1},\ldots a_m$ paired with elements of $b_{j+1},\ldots,b_n$
- $LCS_{4,3}$(abbacbb, cbbaa)

A = RRSSRTTRTS
B=RTSRRSTST

Compute $LCS_{5,0}(A,B)$, $LCS_{5,1}(A,B)$,…,$LCS_{5,9}(A,B)$

A = RRSSRTTRTS
B=RTSRRSTST

Compute $LCS_{5,0}(A,B)$, $LCS_{5,1}(A,B)$,…,$LCS_{5,9}(A,B)$

j	left	right
0	0	4
1	1	4
2	1	3
3	2	3
4	3	3
5	3	2
6	3	2
7	3	1
8	4	1
9	4	0

Computing the middle column

- From the left, compute $LCS(a_1…a_{m/2},b_1…b_j)$
- From the right, compute $LCS(a_{m/2+1}…a_m,b_{j+1}…b_n)$
- Add values for corresponding j's

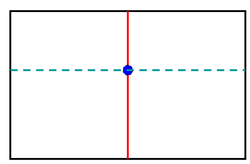
- Note – this is space efficient

Divide and Conquer

- $A = a_1,…,a_m$ $B = b_1,…,b_n$
- Find j such that
  - $LCS(a_1…a_{m/2}, b_1…b_j)$ and
  - $LCS(a_{m/2+1}…a_m,b_{j+1}…b_n)$ yield optimal solution
- Recurse

Algorithm Analysis

- T(m,n) = T(m/2, j) + T(m/2, n-j) + cnm



Prove by induction that T(m,n) <= 2cmn

Memory Efficient LCS Summary

- We can afford O(nm) time, but we can't afford O(nm) space
- If we only want to compute the length of the LCS, we can easily reduce space to O(n+m)
- Avoid storing the value by recomputing values
  - Divide and conquer used to reduce problem sizes