Mathematical Thinking and Proof-Writing for Engineers **Intersession 2020**
Instructor: Ronak Mehta Homework 2

The following homework contains topology concepts such as open, closed, and compact sets, as well as continuity. Feel free to work with each other. Please write your final submission on paper without lines. It is due during class **Thursday, January 23**, but is preferred by email beforehand.

Problem 1

Understand every step of every definition and proof shown in the course. If not, come to the final class with questions about anything you do not understand, or discuss with Ronak during normal lab hours. If everything is clear, then state on the homework "I understand everything from class perfectly."

- Overview of subcovers
- Reiteration of compact definition regarding its converging subsequences
- Quick summary of Cantor Diagonal Argument

Problem 2

Come up with one aspect of the course that was successful, and one aspect that could use improvement. Did the course mostly agree with or differ from your expectations?

I enjoyed the wide variety of proof-based mathematics taught in the course, however I felt as if the pace was a bit faster than I had initially thought (which is understandable since it's an intersession course).

Problem 3

Prove that the intersection of two closed sets is closed.

A set $A \subseteq \mathbb{R}$ is closed if it contains all of its APs. Let A and B be two sets that are closed. $Therefore,$ $x \in S = A \cap B$, which means that x exists in both A and B, which is contained in set S. If A and B are both closed, it is true that their complements are open. ~~Without loss of generalization,~~ we can state that the complement of S = the intersection of the complements of A and B. Now we can prove if $A^c \cap B^c$ is open. If A^c is open, $\exists \mathcal{E}_1 > 0: (x - \mathcal{E}_1, x + \mathcal{E}_1) \sqsubseteq A^c$. If B^c is open, $\exists \mathcal{E}_2 > 0: (x - \mathcal{E}_2, x + \mathcal{E}_2) \sqsubseteq B^c$. If we set $c = \min(\mathcal{E}_1, \mathcal{E}_2)$, this will guarantee there exists an AP $(x - c, x + c) \sqsubseteq A^c \cap B^c$, which proves that $A^c \cap B^c$ is open. ■ You need to show tha $A^c \cup B^c$

is open, as $(A \cap B)^c = A^c \cup B^c$.
That you can take from class.

Problem 4

If A is a set and f is a function, let $f(A) = \{f(x) : x \in A\}$, i.e. all outputs generated from inputs in A. Provide examples of the following. Provide examples of the following.

(a) Continuous f and open A such that $f(A)$ is not open.

$f(x) = x^2$, A: $\mathbb{R}$ ✓

(b) Continuous f and closed A such that $f(A)$ is not closed.

$f(x) = \tan^{-1}(x)$, A: $\mathbb{R}$ ✓

(c) Continuous f and compact $f(A)$ such that A is not compact. $f(A)$: $\mathbb{R}$ is not compact

$f(x) = \ln(x)$ A: $\mathbb{R}$

The function and the sets need not be the same between examples, and you do not need to prove that the function is continuous or that the sets are open, closed, or compact.

1. I understand how the proofs are constructed step-by-step (except for the Cauchy → converges one), but I think I need to sit down and think about the implications the definitions have before I understand all the concepts + feel confident applying them. (also, not sure if this HW is right)

2. The content of the course was really interesting, and I enjoyed learning it. It was presented well and thoughtfully. I think we could have spent more focused time on determining what proof techniques are best for which types of problems.

3. FSOC Assume $A \cap B$ not closed, meaning $\exists$ some AP y of $A \cap B$, $y \notin A \cap B$. A is closed and B is closed, so all APs of $A \in A$, and all APs of $B \in B$. ✗ If y is an AP of $A \cap B$, and $A \cap B \in A$ and $\in B$, then $y \in A$ and $y \in B$. However, by assumption $y \notin A \cap B$, so either $y \notin A$ ($\Rightarrow \Leftarrow$), $y \notin B$ ($\Rightarrow \Leftarrow$), or both ($\Rightarrow \Leftarrow$). Thus, $A \cap B$ is closed.

4. a ✓ $A = (-1, 1)$ $f(x) = x^2 \to [0, 1)$ b ✗ $A = [1, 2]$ $f(x) = 1$ $\{1\}$ is closed.

c ✗ $A = [0, 1]$ $f(x) = \frac{1}{x}$ This is not true.

$A = [0, 1]$ is compact.

Problem 1

What are some ways to know when to use each type of proof?
(direct, contradiction, induction)

Problem 2

The course agreed with my expectations and I enjoyed it. It was taught well with the definitions and examples tying together. Every so often, a proof was done a little quickly but the pace of the class was usually easy to follow.

Problem 3

Prove that the intersection of two closed sets is closed

Let $\{x_n : n \in \mathbb{R}\}$ be an arbitrary collection of closed sets

WTS The intersection $\bigcap x_n$ is closed

$(\bigcap x_n)^c = \bigcup x_n^c$

The complement of an intersection of closed sets is the union of the complements of the closed sets

The complement of a closed set is open $\rightarrow \bigcup x_n^c$ is a union of open sets

Union of open sets is open

$(\bigcap x_n)^c = \bigcup x_n^c$ is open

Thus, its complement $\bigcap x_n$ is closed

The intersection $\bigcap x_n$ is closed ■ ✓

Problem 4

$f(A) = \{f(x) : x \in A\}$

(a) Continuous f and open A such that f(A) is not open

$A = (0,1)$ open $f(x) = 3$ continuous

$f(A) = 3$ not open $\{3\}$ ✓

(b) Continuous f and closed A such that f(A) is not closed

$A = [0,\infty)$ closed $f(x) = \frac{x}{1+x}$ continuous

$f(A) = [0,1)$ not closed ✓

(c) Continuous f and compact f(A) such that A is not compact

$A = \mathbb{R}$ not compact $f(x) = 3$ continuous

$f(A) = 3$ compact $\{3\}$ one point set ✓

#1 I understand everything from class perfectly.

#2 One successful aspect was how all the concepts built on one another and that felt satisfying when doing problems.

One improvement that could be made is providing more HW to make sure everyone is keeping up. Overall, the course agreed with my expectations.

#3 We want to show that A closed and B closed $\Rightarrow A \cap B$ closed.

We showed in class that A closed $\Leftrightarrow A^c$ open so we know both A^c and B^c are open. We also know that A open and B open $\Rightarrow A \cup B$ is open so we can say $A^c \cup B^c$ is open. In class we were told (though we didn't prove) that $A^c \cup B^c = (A \cap B)^c$ so we know $(A \cap B)^c$ is open. Combining this with the first fact gives us that $((A \cap B)^c)^c = A \cap B$ is closed. $\blacksquare$

#4 a) $f : (0,1) \longrightarrow \{0\}$ $\quad f(x) = \lfloor x \rfloor$

b) $f : \mathbb{R} \longrightarrow (0,1)$ $\quad f(x) = \frac{e^x}{e^x + 1}$

c) $f : \mathbb{R} \longrightarrow [-1,1]$ $\quad f(x) = \sin(x)$

#2 addendum I think it would've been nice to have a daily HW problem to turn in that you/one of the students do at the beginning of class.

[redacted]

Problem 1:

Jack Krew[illegible]

Proving that functions are/aren't uniformly continuous

Proving (x_n) Cauchy $\Rightarrow$ (x_n) converges

Problem 2:

Successful: You're good at explaining new ideas in a simple way. The class felt organized; everything seemed to build on the prior class's material

Room for improvement: I often would understand the more-complicated/less-obvious proofs after you gave them, but I don't know if I got any better at coming up with the right steps on my own

The homework didn't feel entirely comprehensive (like it covered everything we talked about in class).

Expectations: I hadn't really had much exposure to proofs prior to this course, so I didn't have many expectations.

Problem 3:

[redacted]

Unravel assumptions:

A closed $\Rightarrow$ A contains all its accumulation points (APs)
B closed $\Rightarrow$ B contains all its APs

WTS: $A \cap B$ closed, meaning $A \cap B$ contains all its APs

Pf// Let y be an AP of $A \cap B$

$\Rightarrow \forall \varepsilon > 0 \ \exists x \in A \cap B : x \in (y-\varepsilon, y+\varepsilon)$ and $x \neq y$

$A \cap B \subseteq A \Rightarrow \forall \varepsilon > 0 \ \exists x \in A : x \in (y-\varepsilon, y+\varepsilon)$ and $x \neq y$

$\Rightarrow y$ is an AP of A

A closed $\Rightarrow$ A contains all its accumulation points $\Rightarrow y \in A$

$A \cap B \subseteq B \Rightarrow \forall \varepsilon > 0 \ \exists x \in B : x \in (y-\varepsilon, y+\varepsilon)$ and $x \neq y$

$\Rightarrow y$ is an AP of B

B closed $\Rightarrow$ B contains all its accumulation points $\Rightarrow y \in B$

$y \in A$ and $y \in B \Rightarrow y \in A \cap B$

Since any AP y of $A \cap B$ is, thus, an element of $A \cap B$ (i.e., $A \cap B$ contains all its APs)

$A \cap B$ is closed ■ Good job!

P[redacted]

$A = (0, 2\pi)$ open because $\forall x \in A\ \exists \varepsilon > 0\ (x-\varepsilon, x+\varepsilon) \subseteq A$ ✓

$f(x) = \sin(x)$ $f(A) = [-1, 1]$ not open because $\exists x \in f(A) : \forall \varepsilon > 0\ (x-\varepsilon, x+\varepsilon) \nsubseteq f(A)$
namely, $x = -1$ and $x = 1$

Problem 4b

$A = [1, \infty)$ closed because contains all its accumulation points ✓

$f(x) = \frac{1}{x}$ $f(A) = (0, 1]$ not closed because $\exists y \notin f(A) : \forall \varepsilon > 0\ \exists x \in A\ x \in (y-\varepsilon, y+\varepsilon),\ y \neq x$
namely, $y = 0$

Problem 4c

$A = \mathbb{R}$ not compact because not bounded: $\nexists M \in \mathbb{R} : \forall x \in \mathbb{R}\ |x| \leq M$ ✓

$f(x) = 3$ $f(A) = \{3\}$ compact because any open cover of $f(A)$ will contain at least 1 set with the element 3; this set alone is a finite subcover of $f(A)$, as it contains all of $f(A)$'s elements

$f(x) = \sin(x)$ $f(A) = [-1, 1]$ compact because closed and bounded

HW #2

not (AP of A AND AP of B) = (not AP of A) OR (not AP of B) [redacted]

1.) A closed, B closed. Prove $A \cap B$ closed.

Take some pt. x s.t. x not AP of A or of B. You seemed to have assumed AND.

It is possible that only one of these is true.

$\exists\ \varepsilon_1 : (x-\varepsilon_1, x+\varepsilon_1)$ doesn't contain any $y_1 \in A$ st $y_1 \neq x$.

$\exists\ \varepsilon_2 : (x-\varepsilon_2, x+\varepsilon_2)$ doesn't contain any $y_2 \in B$ st $y_2 \neq x$.

Let $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$.

So ~~within~~ $(x-\varepsilon, x+\varepsilon)$ $\nexists\ y_3 \in A \cup B$ s.t. $y_3 \neq x$

so $\nexists\ y_3 \in A \cap B$ s.t. $y_3 \neq x$. so x not AP of $A \cap B$.

So contrapositive true as well x AP $A \cap B \Rightarrow x$ AP A, x AP B.

So take any AP $A \cap B$. Call it z. $z \in A, B$ since z AP A and AP B and A, B closed.

So $z \in A \cap B$ ■

2.) a.) A open, $f(A)$ not open ✓

$f(x) = \sin(x)$

$A = (-\infty, \infty)$

$f(A) = [-1, 1]$

b.) A closed, $f(A)$ not closed ✓

$f(x) = \tan^{-1}(x)$

$A = (-\infty, \infty)$

$f(A) = (-\frac{\pi}{2}, \frac{\pi}{2})$

c.) $f(A)$ compact, A not compact ✓

$f(x) = \sin(x)$

$A = (-\infty, \infty)$

$f(A) = [-1, 1]$