Mathematical Thinking and Proof-Writing for Engineers
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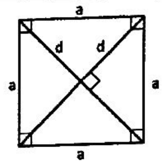
Intersession 2020
Homework 1

The following homework contains proof fundamentals such as direct argument, contradiction, and induction, as well as limits and sequences. Feel free to work with each other. Please write your final submission on paper without lines. It is due on **Friday, January 17**.

Problem 1

An n-gon is a polygon with n sides. A polygon is **convex** provided all of its angles are less than 180 degrees. A **diagonal** of a convex polygon is a line segment that can be drawn between two vertices that are not adjacent. Prove that a convex n-gon has $\frac{n(n-3)}{2}$ diagonals, for $n \geq 3$.

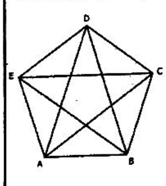


Prove: An n-sided polygon has $\frac{n(n-3)}{2}$ diagonals.

Base Case: Square ($n = 4$): $\frac{n(n-3)}{2} \rightarrow \frac{4((4)-3)}{2} = \frac{4(1)}{2} = 2$, *which is true*

Base case is $n=3$. Is it convex?

Induction: Consider the image of the square shown to the right, where $n = 2$. If we were to add an extra side, i.e. adding 1 to n, all of the vertices of the polygon to congregate to the newly formed vertex to form its diagonals, excluding the two vertices that make up the pentagon's fifth side. However, the side that these two vertices create becomes an extra diagonal. Therefore, it is true that for an n-sided polygon, an $(n+1)$-sided polygon would have $n-2$ more diagonals than before, plus an extra diagonal from the n-sided polygon's side. Is the new polygon convex? Must be



Now assume the relationship is true when $n = k$. Therefore, true for any convex $(k+1)$-gon

$$f(k+1) = f(k) + (k-2) + 1$$

$$\frac{k+1((k+1)-3)}{2} = \frac{k(k-3)}{2} + (k-2) + \frac{2}{2}$$

$$\frac{k+1(k-2)}{2} = \frac{k(k-3) + 2(k-2) + 2}{2}$$

$$(k+1)(k-2) = k(k-3) + 2(k-2) + 2$$

$$k^2 - k - 2 = k^2 - 3k + 2k - 4 + 2$$

$$k^2 - k - 2 = k^2 - k - 2$$

Therefore, since $f(n)$ is true when $f(n+1)$ is true, the relationship is true for all n. ■ ✓

For Problem 2 and 3, we need to define the **supremum**, denoted sup$S$, of set $S \subseteq \mathbb{R}$ as the least upper bound of the set. Precisely, u is an **upper bound** for S if for all $x \in S$, $u \geq x$. A number $s = \sup S$ if it is an upper bound of S, and for any other upper bound u of S, $s \leq u$.

Similarly, the **infimum**, denoted inf S, is the greatest lower bound of S. Neither of these need to exist if the set is unbounded. For example, sup$\mathbb{Z}$ can be considered ∞, as there is no upper bound.

Problem 2

(a) Give an example of a set where the supremum and infimum are **not** members of the set.

Let $S = \left\{\frac{1}{n} : n = 1, 2, .., x\right\}$, where $x \in \mathbb{N}$, then inf(S)=0, since the set approaches (but never falls short of) zero. However, 0 is also not in the set. ✓

Let $S = \left\{-\frac{1}{n} : n = 1, 2, .., x\right\}$, where $x \in \mathbb{N}$, then sup(S)=0, since the set approaches (but never exceeds) zero. However, 0 is also not in the set. ✓

(b) Let S be a set, and $s = \sup S$. Prove that for any $\varepsilon > 0$, there exists $x \in S$ such that

$$s - \epsilon \leq x \leq s$$

What is the analogous condition for the infimum of S?

FSOC, assume that $x < s - \varepsilon$ or $x > s$, where s is the supremum of S. The definition of the supremum of set $S \subseteq \mathbb{R}$ is the least upper bound of the set, u such that $\forall x \in S$, $u \geq x$. Therefore, we can also say that $x \leq s$. $\Rightarrow\Leftarrow$ ■ How did you contradict with $x < s - \varepsilon$?

The analogous condition for the infimum of S can be written as $s \leq x \leq s + \varepsilon$, where s=inf($S$), since the infimum is defined as the greatest lower bound of the set.

Problem 3

Let (x_n) be a bounded sequence in $\mathbb{R}$. Prove that if (x_n) is monotone, then it converges, in two parts.

(a) Let $S = \{x_n : n = 1, 2, ...\}$ be the set that contains all the elements of (x_n). Prove that if (x_n) is monotone non-decreasing (i.e. $x_1 \leq x_2 \leq x_3 ...$), then $x_n \to \sup S$. Similarly, show that if (x_n) is monotone non-increasing, then $x_n \to \inf S$ (use Problem 2).

(b) Use the previous part to prove the original theorem.

WTS: $\forall \varepsilon > 0 \ \exists N(\varepsilon): \forall n \geq N, |x_n - x| < \varepsilon$, where $x = \sup(S)$ *or* $\inf(S)$

Let $S = \{x_n : n = 1, 2, \ldots\}$ *be a bounded sequence*, $\Rightarrow \exists M \ St. |x_n| \leq M$

Is this true?

- Let S be monotone non-decreasing, $\Rightarrow M = \sup(S) \Rightarrow x_n \leq M \Rightarrow \exists \varepsilon > 0: x \geq M - \varepsilon$, since M is the upper bound of $x_n \Rightarrow \ M - \varepsilon < x_N \leq x_n \leq M < M + \varepsilon, \ \forall n \geq N \Rightarrow$ $M - \varepsilon < x_n < M + \varepsilon \Rightarrow |x_n - M| < \varepsilon$, where $M = \sup(S)$

~~You showed one x_N with $|x_N - M| < \varepsilon$. What about for $n \geq N$?~~

- Let S be monotone non-increasing, $\Rightarrow M = \inf(S) \Rightarrow x_n \geq M \Rightarrow \exists \varepsilon > 0: x \leq M + \varepsilon$, since M is the lower bound of $x_n \Rightarrow M - \varepsilon < M \leq x_n \leq x_N < M + \varepsilon, \ \forall n \geq N \Rightarrow$ $M - \varepsilon < x_n < M + \varepsilon \Rightarrow |x_n - M| < \varepsilon$, where $M = \inf(S)$

■

Problem 4 (optional)

Prove that the closed interval [0,1] is uncountable.

A countable set is one where every element of the set, whether finite or arbitrary, can be associated with a natural number. While the interval [0,1] contains an infinite amount of rational numbers from 0 to 1, it also contains an infinite number of irrational numbers (such as the root of fractions like $\frac{1}{3}, \frac{1}{2}$, etc.) which contradicts the definition of a countable set. ■

Why does this contradict the definition?
You can combine two countable sets to make a countable set.

1. Base case: $n=3$. A convex 3-gon has no diagonals, since all vtcs. are adjacent. $\frac{n(n-3)}{2} = \frac{3(3-3)}{2} = 0$, which checks out.

Inductive Hypothesis: $\forall k \geq 3$, a convex k-gon has $\frac{k(k-3)}{2}$ diagonals

Inductive step: WTS $\frac{(k+1)(k+2)}{2} = \frac{k^2-k-2}{2}$ = # diagonals in k+1-gon

Start from a (k+1)-gon.

[A k-gon has k sides, k vertices, and $\frac{k(k-3)}{2}$ diagonals. Say a vertex is inserted between vertex k and 1, meaning instead of k vertices, there are k+1 vertices, indexed from 1 - k+1. All vertices but the k+1th have $\frac{k(k-3)}{2}$ diagonals. Vertex k+1 has k-2 nonadjacent vertices, and formerly adjacent vertices k and 1 are no longer adjacent, meaning k-2+1 diagonals can be newly drawn.

The k+1-gon has $\frac{k(k-3)}{2} + k-1$ diagonals $= \frac{k^2-3k}{2} + \frac{2k-2}{2}$

$= \frac{k^2-k-2}{2}$ ■

2. a. A set $(0,1) \in \mathbb{R}$ has infimum 0 and supremum 1, but neither are contained in the set. ✓

b. $s = \sup S$. $\forall x \in S$, $s \geq x$, and for any other upper bound u of S, $s \leq u$.

FSOC, $\nexists$ $x \in S$ for some $\varepsilon > 0$: $s-\varepsilon \leq x \leq s$. This means $\exists$ an ε st. no element $x \in S$: $s-\varepsilon \leq x \leq s$, meaning that $\forall x \in S$, either $x > s$, or $x < s-\varepsilon$. $x \in S$, and s is the supremum, so $x \ngtr s$ since $s = \sup S$. This must mean $x < s-\varepsilon$. But for any upper bound u of S, $s \leq u$. This means that $s-\varepsilon$ is an upper bound for S, yet $s \leq u$ $\forall$ u of S. $s \nleq s-\varepsilon$, which is absurd! Thus, $\forall \varepsilon > 0$ $\exists$ $x \in S$ st $s-\varepsilon$ st. $s-\varepsilon \leq x \leq s$. ■ Excellent!

Inf S: Let S be a set, and $s = \inf S$. $\forall$ $\varepsilon > 0$, $\exists$ $x \in S$ st. $s+\varepsilon \geq x \geq s$. ✓

3. a. Let (x_n) be a monotone, non-decreasing, bounded sequence in $\mathbb{R}$.

$\exists s = \sup(x_n)$. This means $\forall \varepsilon > 0$, $\exists N: \forall n \geq N$, $s - \varepsilon \leq x_N$, since we know $s - \varepsilon \not> x_N$ otherwise $s - \varepsilon$ would be an upper bound $s - \varepsilon \leq s$, which cannot be as s is the supremum and $\varepsilon \neq 0$. By part 2, $\exists x \in (x_n)$ st. $s - \varepsilon \leq x \leq s$. This is equivalent to $s \leq \cancel{x + \varepsilon \leq} s + \varepsilon$.

Since $x_N \leq s$, by definition of the supremum, $s - \varepsilon \leq x_N \leq s \cancel{\leq x + \varepsilon} \leq s + \varepsilon$. We see that $s - \varepsilon \leq x_N \leq s + \varepsilon$, meaning a monotone, non-decreasing, bonded sequence in $\mathbb{R}$ converges to $\sup(x_n)$, meaning $(x_n) \to \sup(x_n)$. ■

Careful! You showed this for one N, not all $x_n : n \geq N$.

Let (x_n) be a monotone, non-increasing, bounded sequence in $\mathbb{R}$. This means that $\exists s = \inf(x_n)$. This means $\forall \varepsilon > 0$, $\exists N: \forall n \geq N$, $s + \varepsilon \geq x_N$, since we know $s + \varepsilon < x_N$ cannot be true otherwise $s + \varepsilon$ would be a lower bound $s + \varepsilon \geq s$, which cannot true as s is the infimum and $\varepsilon \neq 0$. By part 2, $\exists x \in (x_n)$ st. $s + \varepsilon \geq x_N \geq s$, equivalent to $s \geq x - \varepsilon \geq s - \varepsilon$. $x_N \geq s$. This means $s + \varepsilon \geq x_N \geq s \geq x_N - \varepsilon \geq s - \varepsilon$, or $s + \varepsilon \geq x_N \geq s - \varepsilon$. We see that a monotone, non-increasing, bonded sequence in $\mathbb{R}$ converges to $\inf(x_n)$, meaning $(x_n) \to \inf(x_n)$. ■

Same feedback

✓ b. Let (x_n) be a bounded sequence in $\mathbb{R}$. (x_n) is monotone, and there are two cases:

1. non-increasing. (x_n) is bounded, monotone, and non-increasing. Thus, (x_n) converges to $\inf(x_n)$, meaning (x_n) converges. ✓

2. non-decreasing. (x_n) is bounded, monotone, and non-decreasing. Thus, (x_n) converges to $\sup(x_n)$, meaning (x_n) converges. ■

#1)

I will use induction on the number of sides of a convex n-gon to show that a convex n-gon will have $\frac{n\cdot(n-3)}{2}$ diagonals.

<u>Base Case:</u> A triangle, which has 3 sides will have $\frac{3\cdot(3-3)}{2}=0$ diagonals as it has no non-adjacent vertices.

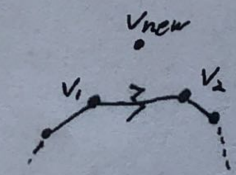
Is a triangle convex?

<u>Inductive Hypothesis:</u> To start, we assume that a convex n-gon has $\frac{n\cdot(n-3)}{2}$ diagonals and we want to show that a convex $(n+1)$-gon has $\frac{(n+1)\cdot(n-2)}{2}$ diagonals. To form an $(n+1)$-gon from an n-gon, we can consider 2 steps:

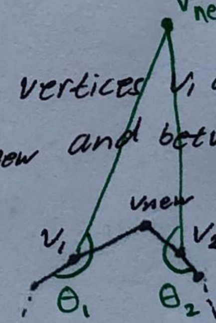
(1) A side of the n-gon breaks between vertices v_1 and v_2

(2) Two new sides form between v_1 and v_{new} and between v_2 and v_{new}

This is visualized here:



(1)



(2)

v_1, v_2 are θ_1 and θ_2 $< 180°$?

When we do this, v_{new} can form diagonals with all n original vertices except for v_1 and v_2, so $n-2$ new diagonals are formed. v_1 and v_2, however, are now non-adjacent so there are a total of $n-2+1=n-1$ new diagonals.

Doing the algebra, we get:

$$\underbrace{\frac{n\cdot(n-3)}{2}}_{\text{n-gon diagonals}} + \underbrace{(n-1)}_{\text{New diagonals}} = \frac{n^2-3n}{2}+\frac{2n-2}{2}=\frac{n^2-n-2}{2}=\underbrace{\frac{(n+1)\cdot(n-2)}{2}}_{\text{(n+1)-gon diagonals.}}$$

So the inductive step is complete and we are done. ∎

#2 a) Define $S = \{ \frac{e^n}{e^n+1} , \forall n \in \mathbb{R} \}$

We can see that any value in $(0,1)$ can be found in S but neither 0 nor 1 can be. Additionally, no value outside of $(0,1)$ is found in S, so S is bounded below by 0 and above by 1. Because we can get infinitely close to 0 or 1 but never reach them we can say:

$$\sup S = 1 \quad \text{and} \quad \inf S = 0$$

b) FSOC, assume that $s - \varepsilon \leq x \leq s$ is false. Because we have $s - \varepsilon < s$, there are only two other cases to consider:

$\forall x$, either

$x > s$ <u>Case 1:</u> $s - \varepsilon < s \leq x$. $x \leq s$ and $s \leq x$ can both happen. You mean $s < x$.

Within this case we have two subcases: either $s = x$ (i) or $s \neq x$ (ii)

(i) If $s = x$ then the statement '$s - \varepsilon < s \leq x$' is logically equivalen to '$s - \varepsilon \leq x \leq s$' which we assumed was false $\Rightarrow\Leftarrow$

(ii) If $s \neq x$ then we have that $\sup S \neq s$ as $x \in S$ yet $s \leq x$ so s cannot bound S $\Rightarrow\Leftarrow$

$x < s - \varepsilon$ <u>Case 2:</u> $x \leq s - \varepsilon < s$

With this case we again get that $\sup S \neq s$ as $s - \varepsilon$ is clearly a lower upper bound on S $\Rightarrow\Leftarrow$

Because every possible case arrives at a contradiction, we can conclude that $s - \varepsilon \leq x \leq s$ is true. The equivalen condition for $\inf S = a$ would be $a \leq x \leq a + \varepsilon$. ∎

#3

Let us say $\underbrace{s = \sup S}$. We want to show $x_n \to s$ as $n \to \infty$

We are given that S is bounded so we can assume that $\sup S$ exists in $\mathbb{R}$

We know from problem 2 that $\forall \varepsilon > 0, \exists x_k \in S$ s.t. $s - \varepsilon \leq x_k \leq s$. (*)

We also know that since (x_n) is monotone non-decreasing that for every $l \geq k$, $x_l \geq x_k$.

To show $x_n \to s$, we must show that $\forall \varepsilon > 0\ \exists N$ s.t. $\forall n > N$, $|x_n - s| < \varepsilon$

Forgot to write here that we set our N to be k from before

We can expand this absolutevalue to reach the logically equivalent statement $s - \varepsilon < x_n < s + \varepsilon$. This is itself equivalent to (*) from above as the left equality of (*) can be removed as a smaller ε can be chosen and the right equality of (*) can be removed by adding ε to s as $a \leq b \Rightarrow a < b + \varepsilon\ \forall \varepsilon > 0$

The proof for monotone non-increasing sequences is identical except that (*) is replaced by the statement $a \leq x_k \leq a + \varepsilon$.

Because we've shown that $(x_n) \to \sup S$ or $(x_n) \to \inf S$, we can conclude that any bounded monotone sequence will converge. ■

#4

$(0,1) \subseteq [0,1]$ so if $(0,1)$ is uncountable then so is $[0,1]$.

The set $S = \{ \frac{1}{x} + \frac{1}{x-1}\ \forall x \in (0,1) \}$ provides a mapping from $(0,1)$ to $\mathbb{R}$. Thus $(0,1)$ is the same size as $\mathbb{R}$ so it is uncountable and so is $[0,1]$. ■

Cool idea! Checks out. Should show that the correspondence $\frac{1}{x} + \frac{1}{x-1} \Longleftrightarrow x$ is one-to-one and onto!

Homework 1

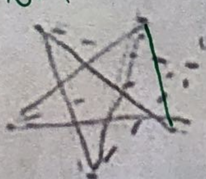
1) Base case: This is true for $n=3$

a convex 3-gon (triangle) has $\frac{3(3-3)}{2} = 0$ diagonals. ✓

Induction step: Assume for some $n=k$, $k \geq 3$,

an n-gon has $\frac{k(k-3)}{2}$ diagonals.

- Must start from (k+1)-gon.



Now add one more vertex so there are $k+1$ vertices.

- Need to count the diagonal drawn

Since the new vertex has only 2 adjacent vertices, there will be diagonals formed with the other $k-2$ vertices. So in total a $k+1$-gon will have $\frac{k(k-3)}{2} + k - 2$ diagonals.

$$\frac{k(k-3)}{2} + k - 2 = \frac{k(k-3) + 2k - 2}{2} = \frac{k^2 - k - 2}{2} = \frac{(k+1)(k-2)}{2} = \frac{(k+1)((k+1)-3)}{2}$$

+1, 2k−4+2

diagonals

By induction, since the problem is true for $k=3$ and for any number $k+1$, for which it holds true for k, and n-gon, n will have $\frac{k(k-3)}{2}$ diagonals. $\blacksquare$

2.) a.) $A = \{-\frac{1}{x} - 1 : x \in \mathbb{Z}, x \geq 1\} \cup \{\frac{-1}{x} + 1 : x \in \mathbb{Z}, x \geq 1\}$

$\sup(A) = 1$ but $1, -1 \notin A$.
$\inf(A) = -1$

b.) S is a set, $s = \sup(S)$. For all $x \in S$, $s \geq x$. $\nexists t$ st $t < s$ and $t \geq x$

FSOC, $\exists\, \varepsilon$ st. $\forall x \in S$, $x > s$ or $x < s - \varepsilon$.

For any x: Case 1: $x > s$ but this can't be true b/c $s \geq x$. You can say this directly.

Case 2. $x < s - \varepsilon$. So all x must be below $s - \varepsilon$. but then Let $s - \varepsilon = t$. So $\exists t$ s.t. $t < s$ and $t \geq x\ \forall x$, or t is a lower upper bound than s. $\Rightarrow \Leftarrow$ ✓

So by contradiction for any $\varepsilon > 0$, $\exists\, x \in S$ st. $s - \varepsilon \leq x \leq s$.

3.) $(x_n) \subseteq \mathbb{R}$, bounded. x_n monotone $\Rightarrow$ converg...
$(x_n) \subseteq \mathbb{R}$, bounded. x_n monotone non decreasing. $x_1 \leq x_2 \leq x_3$ ...
$\Rightarrow S = \{x_n \ n = 1, 2, \ldots\}$. x_n monotone non decreasing
Let $s = \sup S$. For any $\varepsilon > 0$, $\exists\, x \in S$ st. $s - \varepsilon \leq x \leq s$.
Find this x. So $s - \varepsilon \leq x \leq s < s + \varepsilon$. For Any element y that comes after
in the sequence, $y \geq x$ since monotonically non-decreasing. So all
for these elements, $s - \varepsilon \leq y < s + \varepsilon$. Since ε is arbitrary, $s - \varepsilon < y <$
$s - \varepsilon \leq x \leq y \leq s \leq s + \varepsilon$
So $\forall \varepsilon > 0\ \exists N(\varepsilon)$ st. $\forall n \geq N$ $|x_n - x| < \varepsilon$.

If (x_n) monotone non-increasing. Let $i = \inf S$. For any $\varepsilon > 0$, $\exists x \in S$ st $i \leq x < i$
since (x_n) non-increasing any element $y \in S$, that comes after x, $y \geq x$.
So for any element after x $s - \varepsilon < y < s + \varepsilon$. So converges to $\inf(S)$.

b.) $(x_n) \subseteq \mathbb{R}$, bounded, monotone.
either x_n nonincreasing or x_n non decreasing
Case 1: non increasing so converges to $\inf S$.
Case 2: non decreasing so converges to $\sup S$ ∎

Prove that
4.) $A = [0, 1]$ is uncountable.
FSOC assume A is countable. So $\forall a \in A$ $\exists$ unique $n \in \mathbb{N}$ to which it corresponds.
So lay out all binary representations of the decimals to correspond elements in $\mathbb{N}$. Take elements along the diagonal of this matrix and negate them. Then put them in order to form a member of A
what does it mean to negate a decimal?
is not in the list. $\Rightarrow \Leftarrow$
are you sure it is an element of the set A?
So by contradiction $[0, 1]$ is uncountable.

An n-gon is a polygon with n sides. A polygon is convex provided all of its angles are less than 180 degrees. A diagonal of a convex polygon is a line segment that can be drawn between two vertices that are not adjacent. Prove that a convex n-gon has $\frac{n(n-3)}{2}$ diagonals, for $n \geq 3$.

WTS: An n-gon has $\frac{n(n-3)}{2}$ diagonals for $n \geq 3$

Direct argument

$\forall n \; \exists \; n-3$ diagonals as the total number of vertices is n and the diagonals cannot be drawn between a vertice and itself or a vertice and the two adjacent vertices

Since the number of diagonals $(n-3)$ holds true for each of the n vertices of the n-gon, the number of diagonals from one vertice can be multiplied by n for the number of diagonals that can be drawn from all vertices:

$n(n-3)$

While $n(n-3)$ accounts for all diagonals drawn from each vertice, the amount will be doubled as the diagonals will be drawn in both directions between two vertices. Therefore, to find the actual number of diagonals, the total number drawn must be divided by 2

Thus, an n-gon has $\frac{n(n-3)}{2}$ diagonals for $n \geq 3$ ■

This is actually a strategy called combinatorial proof, which we did not cover! Still works though - good job!

Problem 2

(a) Give an example of a set where the supremum and infimum are not members of the set

$A = (0,1)$ $\quad sup(A) = 1$ $\quad inf(A) = 0$

This set includes all numbers between 0 and 1 but does not include 0 and 1 so 0 acts as the greatest lower bound and 1 as the least upper bound

(b) Let S be a set, and $s = \sup S$. Prove that for any $\varepsilon > 0$, there exists $x \in S$ s.t. $s - \varepsilon \leq x \leq s$

WTS: For any $\varepsilon > 0$, $\exists \; x \in S$: $s - \varepsilon \leq x \leq s$

$x \in S$

By definition of an upper bound, s is an upper bound for S if for all $x \in S$, $s \geq x$

Thus, $x \leq s$ ✓

By definition of an upper bound, if $\sup S = s$ and u is an upper bound of S, $s \leq u$

Contrapositive: If $u < s$, u is not an upper bound of A

Take $0 < \varepsilon < s$

$s - \varepsilon < s$

$s - \varepsilon$ is not an upper bound of S

$\rightarrow \exists \; x \in S$ s.t. $s - \varepsilon \leq x$ ✓

Thus, for any $\varepsilon > 0$, $\exists \; x \in S$: $s - \varepsilon \leq x \leq s$ ■

What is the analogous condition for the infimum of S?

If $s = \inf(S)$ | If u is lower bound of S, by definition $u < s$
By definition, $x \geq s$ | Therefore $s \leq s + u$
Since $s+u$ is not a lower bound, $\exists\, x \in S$: $x \leq s + \varepsilon$

For any $\varepsilon > 0$, $\exists\, x \in S$ s.t. $s \leq x \leq s + \varepsilon$

Problem 3

Let (x_n) be a bounded sequence in $\mathbb{R}$. Prove that if (x_n) is monotone, then it converges, in two parts

(a) Let $S = \{x_n : n = 1, 2, \dots\}$ be the set that contains all the elements of (x_n). Prove that if (x_n) is monotone non-decreasing, then $x_n \to \sup S$

WTS If (x_n) is monotone non-decreasing, then $x_n \to \sup S$

Monotone non-decreasing: $x_1 \leq x_2 \leq x_3$

Let $s = \sup(S)$
$\varepsilon > 0$

$s - \varepsilon$ is not an upper bound (Problem 2)

Let $s - \varepsilon < x_N$ (Problem 2: For any $\varepsilon > 0$, $\exists\, x \in S$ s.t. $s - \varepsilon \leq x \leq s$)

Since x_n is non-decreasing, $x_N \leq x_n$ for all $n \geq N$

s is an upper bound

$s - \varepsilon < x_N \leq x_n \leq s < s + \varepsilon$ for all $n \geq N$

$|x_n - s| < \varepsilon$ for all $n \geq N$

Thus, $x_n \to \sup(S)$ ■

if (x_n) is monotone non-decreasing

Similarly, show that if (x_n) is monotone non-increasing, then $x_n \to \inf S$

WTS If (x_n) is monotone non-increasing, then $x_n \to \inf S$

Monotone non-increasing: $x_1 \geq x_2 \geq x_3$

Let $s = \inf(S)$
$\varepsilon > 0$

$s + \varepsilon$ is not a lower bound (Problem 2)

Let $s + \varepsilon > x_N$ (Problem 2: For any $\varepsilon > 0$, $\exists\, x \in S$ s.t. $s \leq x \leq s + \varepsilon$)

Since x_n is non-increasing, $x_N \geq x_n$ for all $n \geq N$

s is a lower bound

$x_n \leq x_N < s + \varepsilon$

$s - \varepsilon < \varepsilon \leq x_n \leq x_N < s + \varepsilon$

$s - \varepsilon < x_n < s + \varepsilon$

$|x_n - s| < \varepsilon$

Thus, If (x_n) is monotone non-increasing, then $x_n \to \inf S$ ■

(b) Prove that if (x_n) is monotone, then it converges

WTS: If (x_n) is monotone, then it converges

x_n is bounded

From part (a):

If x_n is monotone non-decreasing, $x_n \to \sup(S)$. Shown that $|x_n - s| < \varepsilon$

If x_n is monotone non-increasing, $x_n \to \inf(S)$. Shown that $|x_n - s| < \varepsilon$

already have convergence from here.

~~A sequence is called Cauchy if $\forall\, \varepsilon > 0$: $N(\varepsilon)$ st $\forall\, k, l \geq N$ $|x_k - x_l| < \varepsilon$.~~

~~Therefore, x_n bounded, monotone $\Rightarrow$ Cauchy~~

~~By theorem: (x_n) in $\mathbb{R}$ Cauchy $\Rightarrow$ (x_n) converges (Proved in class)~~

~~Thus, if the bounded sequence (x_n) is monotone, then it converges ■~~

Problem 4

Prove that the closed interval $[0,1]$ is uncountable

FSOC Assume $[0,1]$ is countable

If the interval was countable, it would be possible to write all the decimal real numbers in the interval in a list:

$0.s_{11} s_{12} s_{13} s_{14} \cdots$

$0.s_{21} s_{22} s_{23} s_{24} \cdots$

$0.s_{31} s_{32} s_{33} s_{34} \cdots$

$0.s_{41} s_{42} s_{43} s_{44} \cdots$

$0.s_{51} s_{52} s_{53} s_{54} \cdots$

$\vdots$

Let there be a decimal $x = 0.x_1 x_2 x_3 x_4 \ldots$ where

$x_1 \neq s_{11}$ or 9

$x_2 \neq s_{22}$ or 9

$x_3 \neq s_{33}$ or 9

$x_4 \neq s_{44}$ or 9

$\vdots$

Therefore, the decimal x does not consist of reoccurring 9's and is different from the n^{th} element of the list in the n^{th} decimal place

While x is an element in the interval $[0,1]$, it is not in the list containing the countable numbers

But this is absurd

Thus, the closed interval $[0,1]$ is uncountable ■ ✓