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Combination

In mathematics, a **combination** is a selection of items from a collection, such that (unlike permutations) the order of selection does not matter. For example, given three fruits, say an apple, an orange and a pear, there are three combinations of two that can be drawn from this set: an apple and a pear; an apple and an orange; or a pear and an orange. More formally, a *k*-**combination** of a set *S* is a subset of *k* distinct elements of *S*. If the set has *n* elements, the number of *k*-combinations is equal to the binomial coefficient

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1},$$

which can be written using factorials as $\frac{n!}{k!(n-k)!}$ whenever $k \leq n$, and which is zero when $k > n$. The set of all *k*-combinations of a set *S* is often denoted by $\binom{S}{k}$.

Combinations refer to the combination of *n* things taken *k* at a time without repetition. To refer to combinations in which repetition is allowed, the terms *k*-selection,^[1] *k*-multiset,^[2] or *k*-combination with repetition are often used.^[3] If, in the above example, it were possible to have two of any one kind of fruit there would be 3 more 2-selections: one with two apples, one with two oranges, and one with two pears.

Although the set of three fruits was small enough to write a complete list of combinations, with large sets this becomes impractical. For example, a poker hand can be described as a 5-combination ($k = 5$) of cards from a 52 card deck ($n = 52$). The 5 cards of the hand are all distinct, and the order of cards in the hand does not matter. There are 2,598,960 such combinations, and the chance of drawing any one hand at random is $1 / 2,598,960$.

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Number of k -combinations

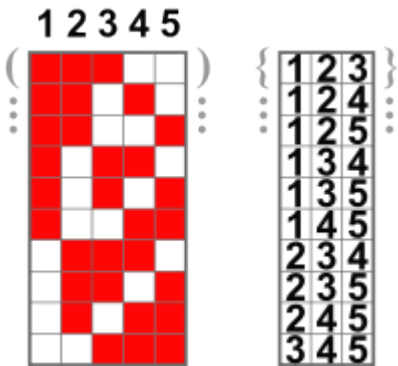
The number of k -combinations from a given set S of n elements is often denoted in elementary combinatorics texts by $C(n, k)$, or by a variation such as C_k^n , ${}_nC_k$, nC_k , $C_{n,k}$ or even C_n^k (the latter form was standard in French, Romanian, Russian, Chinese^[4] and Polish texts). The same number however occurs in many other mathematical contexts, where it is denoted by $\binom{n}{k}$ (often read as " n choose k "); notably it occurs as a coefficient in the binomial formula, hence its name **binomial coefficient**. One can define $\binom{n}{k}$ for all natural numbers k at once by the relation

$$(1 + X)^n = \sum_{k \geq 0} \binom{n}{k} X^k,$$

from which it is clear that

$$\binom{n}{0} = \binom{n}{n} = 1,$$

and further,



3-element subsets of a 5-element set

$$\binom{n}{k} = 0 \text{ for } k > n.$$

To see that these coefficients count k -combinations from S , one can first consider a collection of n distinct variables X_s labeled by the elements s of S , and expand the product over all elements of S :

$$\prod_{s \in S} (1 + X_s);$$

it has 2^n distinct terms corresponding to all the subsets of S , each subset giving the product of the corresponding variables X_s . Now setting all of the X_s equal to the unlabeled variable X , so that the product becomes $(1 + X)^n$, the term for each k -combination from S becomes X^k , so that the coefficient of that power in the result equals the number of such k -combinations.

Binomial coefficients can be computed explicitly in various ways. To get all of them for the expansions up to $(1 + X)^n$, one can use (in addition to the basic cases already given) the recursion relation

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k},$$

for $0 < k < n$, which follows from $(1 + X)^n = (1 + X)^{n-1}(1 + X)$; this leads to the construction of Pascal's triangle.

For determining an individual binomial coefficient, it is more practical to use the formula

$$\binom{n}{k} = \frac{n(n-1)(n-2) \cdots (n-k+1)}{k!}.$$

The numerator gives the number of k -permutations of n , i.e., of sequences of k distinct elements of S , while the denominator gives the number of such k -permutations that give the same k -combination when the order is ignored.

When k exceeds $n/2$, the above formula contains factors common to the numerator and the denominator, and canceling them out gives the relation

$$\binom{n}{k} = \binom{n}{n-k},$$

for $0 \leq k \leq n$. This expresses a symmetry that is evident from the binomial formula, and can also be understood in terms of k -combinations by taking the complement of such a combination, which is an $(n - k)$ -combination.

Finally there is a formula which exhibits this symmetry directly, and has the merit of being easy to remember:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

where $n!$ denotes the factorial of n . It is obtained from the previous formula by multiplying denominator and numerator by $(n - k)!$, so it is certainly inferior as a method of computation to that formula.

The last formula can be understood directly, by considering the $n!$ permutations of all the elements of S . Each such permutation gives a k -combination by selecting its first k elements. There are many duplicate selections: any combined permutation of the first k elements among each other, and of the final $(n - k)$ elements among each other produces the same combination; this explains the division in the formula.

From the above formulas follow relations between adjacent numbers in Pascal's triangle in all three directions:

$$\binom{n}{k} = \begin{cases} \binom{n}{k-1} \frac{n-k+1}{k} & \text{if } k > 0 \\ \binom{n-1}{k} \frac{n}{n-k} & \text{if } k < n \\ \binom{n-1}{k-1} \frac{n}{k} & \text{if } n, k > 0 \end{cases}.$$

Together with the basic cases $\binom{n}{0} = 1 = \binom{n}{n}$, these allow successive computation of respectively all numbers of combinations from the same set (a row in Pascal's triangle), of k -combinations of sets of growing sizes, and of combinations with a complement of fixed size $n - k$.

Example of counting combinations

As a specific example, one can compute the number of five-card hands possible from a standard fifty-two card deck as:^[5]

$$\binom{52}{5} = \frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1} = \frac{311,875,200}{120} = 2,598,960.$$

Alternatively one may use the formula in terms of factorials and cancel the factors in the numerator against parts of the factors in the denominator, after which only multiplication of the remaining factors is required:

$$\begin{aligned} \binom{52}{5} &= \frac{52!}{5!47!} \\ &= \frac{52 \times 51 \times 50 \times 49 \times 48 \times \cancel{47!}}{5 \times 4 \times 3 \times 2 \times \cancel{4} \times \cancel{47!}} \\ &= \frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2} \\ &= \frac{(26 \times \cancel{2}) \times (17 \times \cancel{3}) \times (10 \times \cancel{5}) \times 49 \times (12 \times \cancel{4})}{\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2}} \\ &= 26 \times 17 \times 10 \times 49 \times 12 \\ &= 2,598,960. \end{aligned}$$

Another alternative computation, equivalent to the first, is based on writing

$$\binom{n}{k} = \frac{(n-0)}{1} \times \frac{(n-1)}{2} \times \frac{(n-2)}{3} \times \cdots \times \frac{(n-(k-1))}{k},$$

which gives

$$\binom{52}{5} = \frac{52}{1} \times \frac{51}{2} \times \frac{50}{3} \times \frac{49}{4} \times \frac{48}{5} = 2,598,960.$$

When evaluated in the following order, $52 \div 1 \times 51 \div 2 \times 50 \div 3 \times 49 \div 4 \times 48 \div 5$, this can be computed using only integer arithmetic. The reason is that when each division occurs, the intermediate result that is produced is itself a binomial coefficient, so no remainders ever occur.

Using the symmetric formula in terms of factorials without performing simplifications gives a rather extensive calculation:

$$\begin{aligned} \binom{52}{5} &= \frac{n!}{k!(n-k)!} = \frac{52!}{5!(52-5)!} = \frac{52!}{5!47!} \\ &= \frac{80,658,175,170,943,878,571,660,636,856,403,766,975,289,505,440,883,277,824,000,000,000,000}{120 \times 258,623,241,511,168,180,642,964,355,153,611,979,969,197,632,389,120,000,000,000} \\ &= 2,598,960. \end{aligned}$$

Enumerating *k*-combinations

One can enumerate all *k*-combinations of a given set *S* of *n* elements in some fixed order, which establishes a bijection from an interval of $\binom{n}{k}$ integers with the set of those *k*-combinations. Assuming *S* is itself ordered, for instance $S = \{ 1, 2, \dots, n \}$, there are two natural possibilities for ordering its *k*-combinations: by comparing their smallest elements first (as in the illustrations above) or by comparing their largest elements first. The latter option has the advantage that adding a new largest element to *S* will not change the initial part of the enumeration, but just add the new *k*-combinations of the larger set after the previous ones. Repeating this process, the enumeration can be extended indefinitely with *k*-combinations of ever larger sets. If moreover the intervals of the integers are taken to start at 0, then the *k*-combination at a given place *i* in the enumeration can be computed easily from *i*, and the bijection so obtained is known as the combinatorial number system. It is also known as "rank"/"ranking" and "unranking" in computational mathematics.^{[6][7]}

There are many ways to enumerate *k* combinations. One way is to visit all the binary numbers less than 2^n . Choose those numbers having *k* nonzero bits, although this is very inefficient even for small *n* (e.g. $n = 20$ would require visiting about one million numbers while the maximum number of allowed *k* combinations is about 186 thousand for $k = 10$). The positions of these 1 bits in such a number is a specific *k*-combination of the set $\{ 1, \dots, n \}$.^[8] Another simple, faster way is to track *k* index numbers of the elements selected, starting with $\{0 \dots k-1\}$ (zero-based) or $\{1 \dots k\}$ (one-based) as the first allowed *k*-combination and then repeatedly moving to

the next allowed k -combination by incrementing the last index number if it is lower than $n-1$ (zero-based) or n (one-based) or the last index number x that is less than the index number following it minus one if such an index exists and resetting the index numbers after x to $\{x+1, x+2, \dots\}$.

Number of combinations with repetition

A **k -combination with repetitions**, or **k -multicombination**, or **multisubset** of size k from a set S is given by a sequence of k not necessarily distinct elements of S , where order is not taken into account: two sequences define the same multiset if one can be obtained from the other by permuting the terms. In other words, the number of ways to sample k elements from a set of n elements allowing for duplicates (i.e., with replacement) but disregarding different orderings (e.g. $\{2,1,2\} = \{1,2,2\}$). Associate an index to each element of S and think of the elements of S as *types* of objects, then we can let x_i denote the number of elements of type i in a multisubset. The number of multisubsets of size k is then the number of nonnegative integer solutions of the Diophantine equation:^[9]

$$x_1 + x_2 + \dots + x_n = k.$$

If S has n elements, the number of such k -multisubsets is denoted by,

$$\left(\binom{n}{k}\right),$$

a notation that is analogous to the binomial coefficient which counts k -subsets. This expression, n multichoose k ,^[10] can also be given in terms of binomial coefficients:

$$\left(\binom{n}{k}\right) = \binom{n+k-1}{k}.$$

This relationship can be easily proved using a representation known as stars and bars.^[11]

Proof

A solution of the above Diophantine equation can be represented by x_1 stars, a separator (a *bar*), then x_2 more stars, another separator, and so on. The total number of stars in this representation is k and the number of bars is $n-1$ (since no separator is needed at the very end). Thus, a string of $k+n-1$ symbols (stars and bars) corresponds to a solution if there are k stars in the string. Any solution can be represented by choosing k out of $k+n-1$ positions to place stars and filling the remaining positions with bars. For example, the solution $x_1 = 3, x_2 = 2, x_3 = 0, x_4 = 5$ of the equation $x_1 + x_2 + x_3 + x_4 = 10$ can be represented by

$$\star\star\star|\star\star||\star\star\star\star\star.\text{^[12]}}$$

The number of such strings is the number of ways to place 10 stars in 13 positions, $\binom{13}{10} = \binom{13}{3} = 286$, which is the number of 10-multisubsets of a set with 4 elements.

As with binomial coefficients, there are several relationships between these multichoose expressions. For

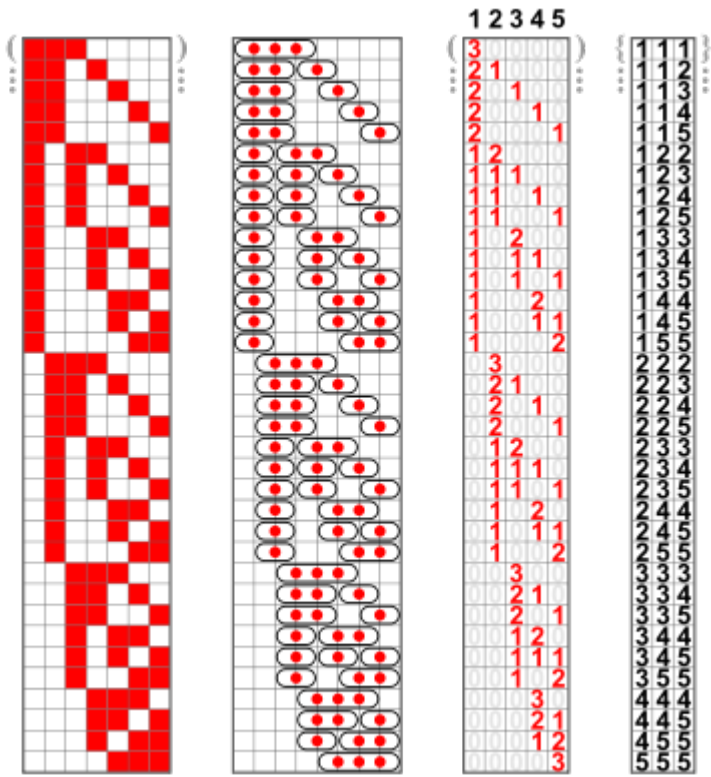
example, for $n \geq 1, k \geq 0$,

$$\binom{n}{k} = \binom{k+1}{n-1}.$$

This identity follows from interchanging the stars and bars in the above representation.^[13]

Example of counting multisubsets

For example, if you have four types of donuts ($n = 4$) on a menu to choose from and you want three donuts ($k = 3$), the number of ways to choose the donuts with repetition can be calculated as



Bijection between 3-subsets of a 7-set (left) and 3-multisubsets with elements from a 5-set (right). This illustrates that $\binom{7}{3} = \binom{5}{3}$.

$$\binom{4}{3} = \binom{4+3-1}{3} = \binom{6}{3} = \frac{6 \times 5 \times 4}{3 \times 2 \times 1} = 20.$$

This result can be verified by listing all the 3-multisubsets of the set $S = \{1,2,3,4\}$. This is displayed in the following table.^[14] The second column shows the nonnegative integer solutions $[x_1, x_2, x_3, x_4]$ of the equation $x_1 + x_2 + x_3 + x_4 = 3$ and the last column gives the stars and bars representation of the solutions.^[15]

No.	3-Multiset	Eq. Solution	Stars and Bars
1	{1,1,1}	[3,0,0,0]	★★★
2	{1,1,2}	[2,1,0,0]	★★ ★
3	{1,1,3}	[2,0,1,0]	★★ ★
4	{1,1,4}	[2,0,0,1]	★★ ★
5	{1,2,2}	[1,2,0,0]	★ ★★
6	{1,2,3}	[1,1,1,0]	★ ★ ★
7	{1,2,4}	[1,1,0,1]	★ ★ ★
8	{1,3,3}	[1,0,2,0]	★ ★★
9	{1,3,4}	[1,0,1,1]	★ ★ ★
10	{1,4,4}	[1,0,0,2]	★ ★★
11	{2,2,2}	[0,3,0,0]	★★★
12	{2,2,3}	[0,2,1,0]	★★ ★
13	{2,2,4}	[0,2,0,1]	★★ ★
14	{2,3,3}	[0,1,2,0]	★ ★★
15	{2,3,4}	[0,1,1,1]	★ ★ ★
16	{2,4,4}	[0,1,0,2]	★ ★★
17	{3,3,3}	[0,0,3,0]	★★★
18	{3,3,4}	[0,0,2,1]	★★ ★
19	{3,4,4}	[0,0,1,2]	★ ★★
20	{4,4,4}	[0,0,0,3]	★★★

Number of k -combinations for all k

The number of k -combinations for all k is the number of subsets of a set of n elements. There are several ways to see that this number is 2^n . In terms of combinations, $\sum_{0 \leq k \leq n} \binom{n}{k} = 2^n$, which is the sum of the n th row (counting from 0) of the binomial coefficients in Pascal's triangle. These combinations (subsets) are enumerated by the 1 digits of the set of base 2 numbers counting from 0 to $2^n - 1$, where each digit position is an item from the set of n .

Given 3 cards numbered 1 to 3, there are 8 distinct combinations (subsets), including the empty set:

$$|\{\{\}; \{1\}; \{2\}; \{1, 2\}; \{3\}; \{1, 3\}; \{2, 3\}; \{1, 2, 3\}\}| = 2^3 = 8$$

Representing these subsets (in the same order) as base 2 numerals:

0 – 000
1 – 001
2 – 010
3 – 011
4 – 100
5 – 101
6 – 110
7 – 111

Probability: sampling a random combination

There are various [algorithms](#) to pick out a random combination from a given set or list. [Rejection sampling](#) is extremely slow for large sample sizes. One way to select a k -combination efficiently from a population of size n is to iterate across each element of the population, and at each step pick that element with a dynamically changing probability of $\frac{k - \text{\#samples chosen}}{n - \text{\#samples visited}}$. (see [reservoir sampling](#)).

See also

- [Binomial coefficient](#)
- [Combinatorial number system](#)
- [Combinatorics](#)
- [Kneser graph](#)
- [List of permutation topics](#)
- [Multiset](#)
- [Pascal's triangle](#)
- [Permutation](#)
- [Probability](#)
- [Subset](#)

Notes

1. [Ryser 1963](#), p. 7 also referred to as an *unordered selection*.
2. [Mazur 2010](#), p. 10
3. When the term *combination* is used to refer to either situation (as in ([Brualdi 2010](#))) care must be taken to clarify whether sets or multisets are being discussed.
4. *High School Textbook for full-time student (Required) Mathematics Book II B* (in Chinese) (2nd ed.). China: People's Education Press. June 2006. pp. 107–116. [ISBN 978-7-107-19616-4](#).
5. [Mazur 2010](#), p. 21
6. Lucia Moura. "Generating Elementary Combinatorial Objects" (<http://www.site.uottawa.ca/~lucia/courses/5165-09/GenCombObj.pdf>) (PDF). *Site.uottawa.ca*. Retrieved 2017-04-10.

7. "SAGE : Subsets" (<http://www.sagemath.org/doc/reference/sage/combinat/subset.html>) (PDF). *Sagemath.org*. Retrieved 2017-04-10.
8. "Combinations - Rosetta Code" (<http://rosettacode.org/wiki/Combinations>).
9. *Brualdi 2010*, p. 52
10. *Benjamin & Quinn 2003*, p. 70
11. In the article *Stars and bars (combinatorics)* the roles of n and k are reversed.
12. *Benjamin & Quinn 2003*, pp. 71 –72
13. *Benjamin & Quinn 2003*, p. 72 (identity 145)
14. *Benjamin & Quinn 2003*, p. 71
15. *Mazur 2010*, p. 10 where the stars and bars are written as binary numbers, with stars = 0 and bars = 1.

References

- Benjamin, Arthur T.; Quinn, Jennifer J. (2003), *Proofs that Really Count: The Art of Combinatorial Proof*, The Dolciani Mathematical Expositions 27, The Mathematical Association of America, ISBN 978-0-88385-333-7
- Brualdi, Richard A. (2010), *Introductory Combinatorics* (5th ed.), Pearson Prentice Hall, ISBN 978-0-13-602040-0
- Erwin Kreyszig, *Advanced Engineering Mathematics*, John Wiley & Sons, INC, 1999.
- Mazur, David R. (2010), *Combinatorics: A Guided Tour*, Mathematical Association of America, ISBN 978-0-88385-762-5
- Ryser, Herbert John (1963), *Combinatorial Mathematics*, The Carus Mathematical Monographs 14, Mathematical Association of America

External links

- Topcoder tutorial on combinatorics (<https://www.topcoder.com/community/data-science/data-science-tutorials/basics-of-combinatorics/>)
- C code to generate all combinations of n elements chosen as k (<http://compprog.wordpress.com/2007/10/17/generating-combinations-1/>)
- Many Common types of permutation and combination math problems, with detailed solutions (http://mathforum.org/library/drmath/sets/high_perms_combs.html)
- The Unknown Formula (<http://www.murderousmaths.co.uk/books/unknownform.htm>) For combinations when choices can be repeated and order does NOT matter
- Combinations with repetitions (by: Akshatha AG and Smitha B) (<https://dl.dropbox.com/u/7951257/easymath/Combinations%20with%20Repetitions.pdf>)
- The dice roll with a given sum problem (<http://www.lucamoroni.it/the-dice-roll-sum-problem/>) An application of the combinations with repetition to rolling multiple dice

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