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Binomial theorem

In elementary algebra, the **binomial theorem** (or **binomial expansion**) describes the algebraic expansion of powers of a binomial. According to the theorem, it is possible to expand the polynomial $(x+y)^n$ into a sum involving terms of the form ax^by^c, where the exponents b and c are nonnegative integers with $b+c=n$, and the coefficient a of each term is a specific positive integer depending on n and b. For example (for $n = 4$),

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

The coefficient a in the term of ax^by^c is known as the binomial coefficient $\binom{n}{b}$ or $\binom{n}{c}$ (the two have the same value). These coefficients for varying n and b can be arranged to form Pascal's triangle. These numbers also arise in combinatorics, where $\binom{n}{b}$ gives the number of different combinations of b elements that can be chosen from an n-element set. Therefore $\binom{n}{b}$ is often pronounced as "n choose b".

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

The binomial coefficient $\binom{n}{b}$ appears as the bth entry in the nth row of Pascal's triangle (counting starts at 0). Each entry is the sum of the two above it.

Contents

History

Special cases of the binomial theorem were known since at least the 4th century BC when Greek mathematician Euclid mentioned the special case of the binomial theorem for exponent 2.[1][2] There is evidence that the binomial theorem for cubes was known by the 6th century AD in India.[1][2]

Binomial coefficients, as combinatorial quantities expressing the number of ways of selecting *k* objects out of *n* without replacement, were of interest to ancient Indian mathematicians. The earliest known reference to this combinatorial problem is the *Chandaḥśāstra* by the Indian lyricist Pingala (c. 200 BC), which contains a method for its solution.[3]:230 The commentator Halayudha from the 10th century AD explains this method using what is now known as Pascal's triangle.[3] By the 6th century AD, the Indian mathematicians probably knew how to express this as a quotient $\frac{n!}{(n-k)!k!}$,[4] and a clear statement of this rule can be found in the 12th century text *Lilavati* by Bhaskara.[4]

The first formulation of the binomial theorem and the table of binomial coefficients, to our knowledge, can be found in a work by Al-Karaji, quoted by Al-Samaw'al in his "al-Bahir".[5][6][7] Al-Karaji described the triangular pattern of the binomial coefficients[8] and also provided a mathematical proof of both the binomial theorem and Pascal's triangle, using an early form of mathematical induction.[8] The Persian poet and mathematician Omar Khayyam was probably familiar with the formula to higher orders, although many of his mathematical works are lost.[2] The binomial expansions of small degrees were known in the 13th century mathematical works of Yang Hui[9] and also Chu Shih-Chieh.[2] Yang Hui attributes the method to a much earlier 11th century text of Jia Xian, although those writings are now also lost.[3]:142

In 1544, Michael Stifel introduced the term "binomial coefficient" and showed how to use them to express $(1+a)^n$ in terms of $(1+a)^{n-1}$, via "Pascal's triangle".[10] Blaise Pascal studied the eponymous triangle comprehensively in the treatise *Traité du triangle arithmétique* (1665). However, the pattern of numbers was already known to the European mathematicians of the late Renaissance, including Stifel, Niccolò Fontana Tartaglia, and Simon Stevin.[10]

Isaac Newton is generally credited with the generalized binomial theorem, valid for any rational exponent.[10][11]

Theorem statement

According to the theorem, it is possible to expand any power of $x + y$ into a sum of the form

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n,$$

where each $\binom{n}{k}$ is a specific positive integer known as a binomial coefficient. (When an exponent is zero, the corresponding power expression is taken to be 1 and this multiplicative factor is often omitted from the term. Hence one often sees the right side written as $\binom{n}{0}x^n + \ldots$.) This formula is also referred to as the **binomial formula** or the **binomial identity**. Using summation notation, it can be written as

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

The final expression follows from the previous one by the symmetry of x and y in the first expression, and by comparison it follows that the sequence of binomial coefficients in the formula is symmetrical. A simple variant of the binomial formula is obtained by substituting 1 for y, so that it involves only a single variable. In this form, the formula reads

$$(1+x)^n = \binom{n}{0}x^0 + \binom{n}{1}x^1 + \binom{n}{2}x^2 + \cdots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n,$$

or equivalently

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Examples

The most basic example of the binomial theorem is the formula for the square of $x + y$:

$$(x+y)^2 = x^2 + 2xy + y^2.$$

The binomial coefficients 1, 2, 1 appearing in this expansion correspond to the second row of Pascal's triangle. (The top "1" of the triangle is considered to be row 0, by convention.) The coefficients of higher powers of $x + y$ correspond to lower rows of the triangle:

$$\begin{aligned}
(x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3, \\
(x+y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4, \\
(x+y)^5 &= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5, \\
(x+y)^6 &= x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6, \\
(x+y)^7 &= x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7.
\end{aligned}$$

Several patterns can be observed from these examples. In general, for the expansion $(x + y)^n$:

1. the powers of x start at n and decrease by 1 in each term until they reach 0 (with $x^0 = 1$, often unwritten);
2. the powers of y start at 0 and increase by 1 until they reach n;
3. the nth row of Pascal's Triangle will be the coefficients of the expanded binomial when the terms are arranged in this way;

4. the number of terms in the expansion before like terms are combined is the sum of the coefficients and is equal to 2^n; and
5. there will be $n+1$ terms in the expression after combining like terms in the expansion.

The binomial theorem can be applied to the powers of any binomial. For example,

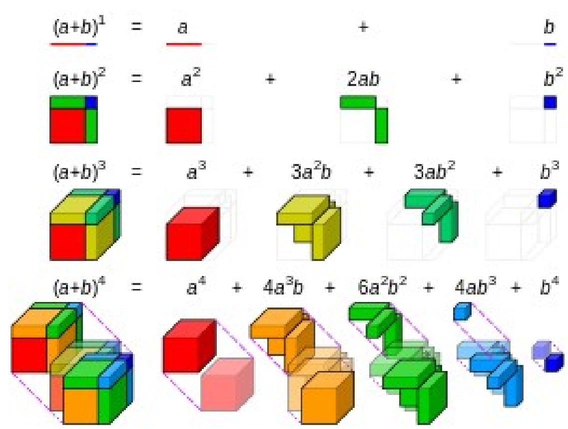
$$\begin{aligned}(x+2)^3 &= x^3+3x^2(2)+3x(2)^2+2^3\\ &= x^3+6x^2+12x+8.\end{aligned}$$

For a binomial involving subtraction, the theorem can be applied by using the form $(x-y)^n = (x+(-y))^n$. This has the effect of changing the sign of every other term in the expansion:

$$(x-y)^3 = (x+(-y))^3 = x^3+3x^2(-y)+3x(-y)^2+(-y)^3 = x^3-3x^2y+3xy^2-y^3.$$

Geometric explanation

For positive values of a and b, the binomial theorem with $n = 2$ is the geometrically evident fact that a square of side $a + b$ can be cut into a square of side a, a square of side b, and two rectangles with sides a and b. With $n = 3$, the theorem states that a cube of side $a + b$ can be cut into a cube of side a, a cube of side b, three $a\times a\times b$ rectangular boxes, and three $a\times b\times b$ rectangular boxes.



Visualisation of binomial expansion up to the 4th power

In calculus, this picture also gives a geometric proof of the derivative $(x^n)' = nx^{n-1}$:[12] if one sets $a = x$ and $b = \Delta x$, interpreting b as an infinitesimal change in a, then this picture shows the infinitesimal change in the volume of an n-dimensional hypercube, $(x+\Delta x)^n$, where the coefficient of the linear term (in Δx) is nx^{n-1}, the area of the n faces, each of dimension $(n-1)$:

$$(x+\Delta x)^n = x^n + nx^{n-1}\Delta x + \binom{n}{2}x^{n-2}(\Delta x)^2 + \cdots.$$

Substituting this into the definition of the derivative via a difference quotient and taking limits means that the higher order terms, $(\Delta x)^2$ and higher, become negligible, and yields the formula $(x^n)' = nx^{n-1}$, interpreted as

> "the infinitesimal rate of change in volume of an *n*-cube as side length varies is the area of *n* of its $(n-1)$-dimensional faces".

If one integrates this picture, which corresponds to applying the fundamental theorem of calculus, one obtains Cavalieri's quadrature formula, the integral $\int x^{n-1}\,dx = \tfrac{1}{n}x^n$ – see proof of Cavalieri's quadrature formula for details.[12]

Binomial coefficients

The coefficients that appear in the binomial expansion are called **binomial coefficients**. These are usually written

$\binom{n}{k}$, and pronounced “n choose k”.

Formulae

The coefficient of $x^{n-k}y^k$ is given by the formula

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

which is defined in terms of the factorial function $n!$. Equivalently, this formula can be written

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} = \prod_{\ell=1}^{k} \frac{n-\ell+1}{\ell} = \prod_{\ell=0}^{k-1} \frac{n-\ell}{k-\ell}$$

with k factors in both the numerator and denominator of the fraction. Although this formula involves a fraction, the binomial coefficient $\binom{n}{k}$ is actually an integer.

Combinatorial interpretation

The binomial coefficient $\binom{n}{k}$ can be interpreted as the number of ways to choose k elements from an n-element set. This is related to binomials for the following reason: if we write $(x + y)^n$ as a product

$$(x+y)(x+y)(x+y)\cdots(x+y),$$

then, according to the distributive law, there will be one term in the expansion for each choice of either x or y from each of the binomials of the product. For example, there will only be one term x^n, corresponding to choosing x from each binomial. However, there will be several terms of the form $x^{n-2}y^2$, one for each way of choosing exactly two binomials to contribute a y. Therefore, after combining like terms, the coefficient of $x^{n-2}y^2$ will be equal to the number of ways to choose exactly 2 elements from an n-element set.

Proofs

Combinatorial proof

Example

The coefficient of xy^2 in

$$\begin{aligned}(x+y)^3 &= (x+y)(x+y)(x+y) \\ &= xxx + xxy + xyx + \underline{xyy} + yxx + \underline{yxy} + \underline{yyx} + yyy \\ &= x^3 + 3x^2y + \underline{3xy^2} + y^3.\end{aligned}$$

equals $\binom{3}{2} = 3$ because there are three x,y strings of length 3 with exactly two y's, namely,

$$xyy,\; yxy,\; yyx,$$

corresponding to the three 2-element subsets of { 1, 2, 3 }, namely,

$$\{2,3\},\ \{1,3\},\ \{1,2\},$$

where each subset specifies the positions of the y in a corresponding string.

General case

Expanding $(x+y)^n$ yields the sum of the 2^n products of the form $e_1e_2 \ldots e_n$ where each e_i is x or y. Rearranging factors shows that each product equals $x^{n-k}y^k$ for some k between 0 and n. For a given k, the following are proved equal in succession:

- the number of copies of $x^{n-k}y^k$ in the expansion
- the number of n-character x,y strings having y in exactly k positions
- the number of k-element subsets of $\{1, 2, \ldots, n\}$
- $\binom{n}{k}$ (this is either by definition, or by a short combinatorial argument if one is defining $\binom{n}{k}$ as $\frac{n!}{k!(n-k)!}$).

This proves the binomial theorem.

Inductive proof

Induction yields another proof of the binomial theorem. When $n = 0$, both sides equal 1, since $x^0 = 1$ and $\binom{0}{0} = 1$. Now suppose that the equality holds for a given n; we will prove it for $n + 1$. For $j, k \geq 0$, let $[f(x, y)]_{j,k}$ denote the coefficient of x^jy^k in the polynomial $f(x, y)$. By the inductive hypothesis, $(x + y)^n$ is a polynomial in x and y such that $[(x + y)^n]_{j,k}$ is $\binom{n}{k}$ if $j + k = n$, and 0 otherwise. The identity

$$(x+y)^{n+1} = x(x+y)^n + y(x+y)^n$$

shows that $(x + y)^{n+1}$ also is a polynomial in x and y, and

$$[(x+y)^{n+1}]_{j,k} = [(x+y)^n]_{j-1,k} + [(x+y)^n]_{j,k-1},$$

since if $j + k = n + 1$, then $(j - 1) + k = n$ and $j + (k - 1) = n$. Now, the right hand side is

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k},$$

by Pascal's identity.[13] On the other hand, if $j + k \neq n + 1$, then $(j - 1) + k \neq n$ and $j + (k - 1) \neq n$, so we get $0 + 0 = 0$. Thus

$$(x+y)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} x^{n+1-k} y^k,$$

which is the inductive hypothesis with $n + 1$ substituted for n and so completes the inductive step.

Generalizations

Newton's generalized binomial theorem

Around 1665, Isaac Newton generalized the binomial theorem to allow real exponents other than nonnegative integers. (The same generalization also applies to complex exponents.) In this generalization, the finite sum is replaced by an infinite series. In order to do this, one needs to give meaning to binomial coefficients with an arbitrary upper index, which cannot be done using the usual formula with factorials. However, for an arbitrary number r, one can define

$$\binom{r}{k} = \frac{r(r-1)\cdots(r-k+1)}{k!} = \frac{(r)_k}{k!},$$

where $(\cdot)_k$ is the Pochhammer symbol, here standing for a falling factorial. This agrees with the usual definitions when r is a nonnegative integer. Then, if x and y are real numbers with $|x| > |y|$,[Note 1] and r is any complex number, one has

$$\begin{aligned}(x+y)^r &= \sum_{k=0}^{\infty} \binom{r}{k} x^{r-k} y^k \\ &= x^r + r x^{r-1} y + \frac{r(r-1)}{2!} x^{r-2} y^2 + \frac{r(r-1)(r-2)}{3!} x^{r-3} y^3 + \cdots.\end{aligned}$$

When r is a nonnegative integer, the binomial coefficients for $k > r$ are zero, so this equation reduces to the usual binomial theorem, and there are at most $r + 1$ nonzero terms. For other values of r, the series typically has infinitely many nonzero terms.

For example, $r = 1/2$ gives the following series for the square root:

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 - \cdots$$

Taking $r = -1$, the generalized binomial series gives the geometric series formula, valid for $|x| < 1$:

$$(1+x)^{-1} = \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$

More generally, with $r = -s$:

$$\frac{1}{(1-x)^s} = \sum_{k=0}^{\infty} \binom{s+k-1}{k} x^k.$$

So, for instance, when $s = 1/2$,

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \frac{35}{128}x^4 - \frac{63}{256}x^5 + \cdots$$

Further generalizations

The generalized binomial theorem can be extended to the case where x and y are complex numbers. For this version, one should again assume $|x| > |y|$[Note 1] and define the powers of $x + y$ and x using a holomorphic branch of log defined on an open disk of radius $|x|$ centered at x. The generalized binomial theorem is valid also for elements x and y of a Banach algebra as long as $xy = yx$, x is invertible, and $\|y/x\| < 1$.

A version of the binomial theorem is valid for the following Pochhammer symbol-like family of polynomials: for a given real constant c, define $x^{(0)} = 1$ and $x^{(n)} = \prod_{k=1}^{n}[x + (k-1)c]$ for $n > 0$. Then[14]

$$(a+b)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} a^{(n-k)} b^{(k)}.$$

The case $c = 0$ recovers the usual binomial theorem.

More generally, a sequence $\{p_n\}_{n=0}^{\infty}$ of polynomials is said to be **binomial** if

- $\deg p_n = n$ for all n,
- $p_0(0) = 1$, and
- $p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y)$ for all x, y, and n.

An operator Q on the space of polynomials is said to be the *basis operator* of the sequence $\{p_n\}_{n=0}^{\infty}$ if $Qp_0 = 0$ and $Qp_n = np_{n-1}$ for all $n \geqslant 1$. A sequence $\{p_n\}_{n=0}^{\infty}$ is binomial if and only if its basis operator is a Delta operator.[15] Writing E^a for the shift by a operator, the Delta operators corresponding to the above "Pochhammer" families of polynomials are the backward difference $I - E^{-c}$ for $c > 0$, the ordinary derivative for $c = 0$, and the forward difference $E^{-c} - I$ for $c < 0$.

Multinomial theorem

The binomial theorem can be generalized to include powers of sums with more than two terms. The general version is

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{k_1+k_2+\cdots+k_m=n} \binom{n}{k_1, k_2, \ldots, k_m} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}.$$

where the summation is taken over all sequences of nonnegative integer indices k_1 through k_m such that the sum of all k_i is n. (For each term in the expansion, the exponents must add up to n). The coefficients $\binom{n}{k_1,\cdots,k_m}$ are known as multinomial coefficients, and can be computed by the formula

$$\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1! \cdot k_2! \cdots k_m!}.$$

Combinatorially, the multinomial coefficient $\binom{n}{k_1,\cdots,k_m}$ counts the number of different ways to partition an n-element set into disjoint subsets of sizes k_1, ..., k_m.

Multi-binomial theorem

It is often useful when working in more dimensions, to deal with products of binomial expressions. By the binomial theorem this is equal to

$$(x_1+y_1)^{n_1}\cdots(x_d+y_d)^{n_d}=\sum_{k_1=0}^{n_1}\cdots\sum_{k_d=0}^{n_d}\binom{n_1}{k_1}x_1^{k_1}y_1^{n_1-k_1}\;\dots\;\binom{n_d}{k_d}x_d^{k_d}y_d^{n_d-k_d}.$$

This may be written more concisely, by multi-index notation, as

$$(x+y)^\alpha=\sum_{\nu\le\alpha}\binom{\alpha}{\nu}x^\nu y^{\alpha-\nu}.$$

General Leibniz rule

The general Leibniz rule gives the nth derivative of a product of two functions in a form similar to that of the binomial theorem:[16]

$$(fg)^{(n)}(x)=\sum_{k=0}^{n}\binom{n}{k}f^{(n-k)}(x)g^{(k)}(x).$$

Here, the superscript (n) indicates the nth derivative of a function. If one sets $f(x) = e^{ax}$ and $g(x) = e^{bx}$, and then cancels the common factor of $e^{(a+b)x}$ from both sides of the result, the ordinary binomial theorem is recovered.

Applications

Multiple-angle identities

For the complex numbers the binomial theorem can be combined with de Moivre's formula to yield multiple-angle formulas for the sine and cosine. According to De Moivre's formula,

$$\cos(nx)+i\sin(nx)=(\cos x+i\sin x)^n.$$

Using the binomial theorem, the expression on the right can be expanded, and then the real and imaginary parts can be taken to yield formulas for $\cos(nx)$ and $\sin(nx)$. For example, since

$$(\cos x+i\sin x)^2=\cos^2 x+2i\cos x\sin x-\sin^2 x,$$

De Moivre's formula tells us that

$$\cos(2x)=\cos^2 x-\sin^2 x\quad\text{and}\quad\sin(2x)=2\cos x\sin x,$$

which are the usual double-angle identities. Similarly, since

$$(\cos x+i\sin x)^3=\cos^3 x+3i\cos^2 x\sin x-3\cos x\sin^2 x-i\sin^3 x,$$

De Moivre's formula yields

$$\cos(3x) = \cos^3 x - 3\cos x \sin^2 x \quad \text{and} \quad \sin(3x) = 3\cos^2 x \sin x - \sin^3 x.$$

In general,

$$\cos(nx) = \sum_{k \text{ even}} (-1)^{k/2} \binom{n}{k} \cos^{n-k} x \sin^k x$$

and

$$\sin(nx) = \sum_{k \text{ odd}} (-1)^{(k-1)/2} \binom{n}{k} \cos^{n-k} x \sin^k x.$$

Series for *e*

The number *e* is often defined by the formula

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

Applying the binomial theorem to this expression yields the usual infinite series for *e*. In particular:

$$\left(1 + \frac{1}{n}\right)^n = 1 + \binom{n}{1}\frac{1}{n} + \binom{n}{2}\frac{1}{n^2} + \binom{n}{3}\frac{1}{n^3} + \cdots + \binom{n}{n}\frac{1}{n^n}.$$

The *k*th term of this sum is

$$\binom{n}{k}\frac{1}{n^k} = \frac{1}{k!} \cdot \frac{n(n-1)(n-2)\cdots(n-k+1)}{n^k}$$

As $n \to \infty$, the rational expression on the right approaches one, and therefore

$$\lim_{n\to\infty} \binom{n}{k}\frac{1}{n^k} = \frac{1}{k!}.$$

This indicates that *e* can be written as a series:

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots.$$

Indeed, since each term of the binomial expansion is an increasing function of *n*, it follows from the monotone convergence theorem for series that the sum of this infinite series is equal to *e*.

Probability

The binomial theorem is closely related to the probability mass function of the negative binomial distribution. The probability of a (countable) collection of independent Bernoulli trials $\{X_t\}_{t\in S}$ with probability of success $p \in [0,1]$ all not happening is

$$P\left(\bigcap_{t\in S} X_t^C\right) = (1-p)^{|S|} = \sum_{n=0}^{|S|} \binom{|S|}{n} (-p)^n$$

A useful upper bound for this quantity is e^{-pn}. [17]

The binomial theorem in abstract algebra

Formula (1) is valid more generally for any elements x and y of a semiring satisfying $xy = yx$. The theorem is true even more generally: alternativity suffices in place of associativity.

The binomial theorem can be stated by saying that the polynomial sequence $\{ 1, x, x^2, x^3, \ldots \}$ is of binomial type.

In popular culture

- The binomial theorem is mentioned in the Major-General's Song in the comic opera The Pirates of Penzance.
- Professor Moriarty is described by Sherlock Holmes as having written a treatise on the binomial theorem.
- The Portuguese poet Fernando Pessoa, using the heteronym Álvaro de Campos, wrote that "Newton's Binomial is as beautiful as the Venus de Milo. The truth is that few people notice it."[18]
- In the 2014 film The Imitation Game, Alan Turing makes reference to Isaac Newton's work on the Binomial Theorem during his first meeting with Commander Denniston at Bletchley Park.

See also

- Binomial approximation
- Binomial distribution
- Binomial inverse theorem
- Stirling's approximation

Notes

1. This is to guarantee convergence. Depending on r, the series may also converge sometimes when $|x| = |y|$.

References

1. Weisstein, Eric W. "Binomial Theorem" (http://mathworld.wolfram.com/BinomialTheorem.html). *Wolfram MathWorld*.
2. Coolidge, J. L. (1949). "The Story of the Binomial Theorem". *The American Mathematical Monthly*. **56** (3): 147–157. doi:10.2307/2305028 (https://doi.org/10.2307%2F2305028). JSTOR 2305028 (https://www.jstor.org/stable/2305028).
3. Jean-Claude Martzloff; S.S. Wilson; J. Gernet; J. Dhombres (1987). *A history of Chinese mathematics*. Springer.
4. Biggs, N. L. (1979). "The roots of combinatorics". *Historia Math*. **6** (2): 109–136. doi:10.1016/0315-0860(79)90074-0 (https://doi.org/10.1016%2F0315-0860%2879%2990074-0).
5. "THE BINOMIAL THEOREM : A WIDESPREAD CONCEPT IN MEDIEVAL ISLAMIC MATHEMATICS" (https://core.ac.uk/download/pdf/82000184.pdf) (PDF). *core.ac.uk*. p. 401. Retrieved 2019-01-08.

6. "Taming the unknown. A history of algebra from antiquity to the early twentieth century" (http://www.ams.org/journals/bull/2015-52-04/S0273-0979-2015-01491-6/S0273-0979-2015-01491-6.pdf) (PDF). *Bulletin of the American Mathematical Society*: 727. "However, algebra advanced in other respects. Around 1000, al-Karaji stated the binomial theorem"
7. Rashed, R. (1994-06-30). *The Development of Arabic Mathematics: Between Arithmetic and Algebra* (https://books.google.com/?id=vSkClSvU_9AC&pg=PA62#v=onepage&q&f=true). Springer Science & Business Media. p. 63. ISBN 9780792325659.
8. O'Connor, John J.; Robertson, Edmund F., "Abu Bekr ibn Muhammad ibn al-Husayn Al-Karaji" (http://www-history.mcs.st-andrews.ac.uk/Biographies/Al-Karaji.html), *MacTutor History of Mathematics archive*, University of St Andrews.
9. Landau, James A. (1999-05-08). "Historia Matematica Mailing List Archive: Re: [HM] Pascal's Triangle" (http://archives.math.utk.edu/hypermail/historia/may99/0073.html) (mailing list email). *Archives of Historia Matematica*. Retrieved 2007-04-13.
10. Kline, Morris (1972). *History of mathematical thought*. Oxford University Press. p. 273.
11. Bourbaki, N. (18 November 1998). *Elements of the History of Mathematics Paperback*. J. Meldrum (Translator). ISBN 978-3-540-64767-6.
12. Barth, Nils R. (2004). "Computing Cavalieri's Quadrature Formula by a Symmetry of the *n*-Cube". *The American Mathematical Monthly*. **111** (9): 811–813. doi:10.2307/4145193 (https://doi.org/10.2307%2F4145193). ISSN 0002-9890 (https://www.worldcat.org/issn/0002-9890). JSTOR 4145193 (https://www.jstor.org/stable/4145193), author's copy (http://nbarth.net/math/papers/barth-01-cavalieri.pdf), further remarks and resources (http://nbarth.net/math/papers/)
13. Binomial theorem (http://proofs.wiki/Binomial_theorem) – inductive proofs Archived (https://web.archive.org/web/20150224130932/http://proofs.wiki/Binomial_theorem) February 24, 2015, at the Wayback Machine
14. Sokolowsky, Dan; Rennie, Basil C. (February 1979). "Problem 352" (https://cms.math.ca/crux/backfile/Crux_v5n02_Feb.pdf#page=26) (PDF). *Crux Mathematicorum*. **5** (2): 55–56.
15. Aigner, Martin (1997) [Reprint of the 1979 Edition]. *Combinatorial Theory*. Springer. p. 105. ISBN 3-540-61787-6.
16. Seely, Robert T. (1973). *Calculus of One and Several Variables*. Glenview: Scott, Foresman. ISBN 978-0-673-07779-0.
17. Cover, Thomas M.; Thomas, Joy A. (2001-01-01). *Data Compression*. John Wiley & Sons, Inc. p. 320. doi:10.1002/0471200611.ch5 (https://doi.org/10.1002%2F0471200611.ch5). ISBN 9780471200611.
18. "Arquivo Pessoa: Obra Édita - O binómio de Newton é tão belo como a Vénus de Milo" (http://arquivopessoa.net/textos/224). arquivopessoa.net.

Further reading

- Bag, Amulya Kumar (1966). "Binomial theorem in ancient India". *Indian J. History Sci*. **1** (1): 68–74.
- Graham, Ronald; Knuth, Donald; Patashnik, Oren (1994). "(5) Binomial Coefficients". *Concrete Mathematics* (2nd ed.). Addison Wesley. pp. 153–256. ISBN 978-0-201-55802-9. OCLC 17649857 (https://www.worldcat.org/oclc/17649857).

External links

- Solomentsev, E.D. (2001) [1994], "Newton binomial" (https://www.encyclopediaofmath.org/index.php?title=Newton_binomial), in Hazewinkel, Michiel (ed.), *Encyclopedia of Mathematics*, Springer Science+Business Media B.V. / Kluwer Academic Publishers, ISBN 978-1-55608-010-4

- Binomial Theorem (http://demonstrations.wolfram.com/BinomialTheorem/) by Stephen Wolfram, and "Binomial Theorem (Step-by-Step)" (http://demonstrations.wolfram.com/BinomialTheoremStepByStep/) by Bruce Colletti and Jeff Bryant, Wolfram Demonstrations Project, 2007.