2 Foundations of Input–Output Analysis

2.1 Introduction

In this chapter we begin to explore the fundamental structure of the input–output model, the assumptions behind it, and also some of the simplest kinds of problems to which it is applied. Later chapters will examine the special features that are associated with regional models and some of the extensions that are necessary for particular kinds of problems – for example, in energy or environmental studies or as part of a broader system of social accounts.

The mathematical structure of an input–output system consists of a set of n linear equations with n unknowns; therefore, matrix representations can readily be used. In this chapter we will start with more detailed algebraic statements of the fundamental relationships and then go on to use matrix notation and manipulations more and more frequently. Appendix A contains a review of matrix algebra definitions and operations that are essential for input–output models. While solutions to the input–output equation system, via an inverse matrix, are straightforward mathematically, we will discover that there are interesting economic interpretations to some of the algebraic results.

2.2 Notation and Fundamental Relationships

An input–output model is constructed from observed data for a particular economic area – a nation, a region (however defined), a state, etc. In the beginning, we will assume (for reasons that will become clear in the next chapter) that the economic area is a country. The economic activity in the area must be able to be separated into a number of segments or producing sectors. These may be industries in the usual sense (e.g., steel) or they may be much smaller categories (e.g., steel nails and spikes) or much larger ones (e.g., manufacturing). The necessary data are the flows of products from each of the sectors (as a producer/seller) to each of the sectors (as a purchaser/buyer); these *interindustry* flows, or transactions (or intersectoral flows – the terms *industry* and *sector* are often used interchangeably in input–output analysis) are measured for a particular time period (usually a year) and in monetary terms – for example, the dollar value of steel sold to automobile manufacturers last year.[1]

The exchanges of goods between sectors are, ultimately, sales and purchases of physical goods – tons of steel bought by automobile manufacturers last year. In accounting for transactions between and among all sectors, it is possible in principle to record all exchanges either in physical or in monetary terms. While the physical measure is perhaps a better reflection of one sector's use of another sector's product, there are substantial measurement problems when sectors actually sell more than one good (a Cadillac CTS and a Ford Focus are distinctly different products with different prices; in physical units, however, both are cars). For these and other reasons, then, accounts are generally kept in monetary terms, even though this introduces problems due to changes in prices that do not reflect changes in the use of physical inputs. (In section 2.6 we will explore the implications of a data set in which transactions are expressed in physical units – for example, tons of steel sold to the automobile sector last year.)

One essential set of data for an input–output model are monetary values of the transactions between pairs of sectors (from each sector i to each sector j); these are usually designated as z_{ij}. Sector j's demand for inputs from other sectors during the year will have been related to the amount of goods produced by sector j over that same period. For example, the demand from the automobile sector for the output of the steel sector is very closely related to the output of automobiles, the demand for leather by the shoe-producing sector depends on the number of shoes being produced, etc.

In addition, in any country there are sales to purchasers who are more external or *exogenous* to the industrial sectors that constitute the producers in the economy – for example, households, government, and foreign trade. The demands of these units – and hence the magnitudes of their purchases from each of the industrial sectors – are generally determined by considerations that are relatively unrelated to the amount being produced. For example, government demand for aircraft is related to broad changes in national policy, budget levels, or defense needs; consumer demand for small cars is related to gasoline availability, and so on. The demand of these external units, since it tends to be much more for goods to be used as such and not to be used as an input to an industrial production process, is generally referred to as *final demand*.

Assume that the economy can be categorized into n sectors. If we denote by x_i the total output (production) of sector i and by f_i the total final demand for sector i's product, we may write a simple equation accounting for the way in which sector i distributes its product through sales to other sectors and to final demand:

$$x_i = z_{i1} + \cdots + z_{ij} + \cdots + z_{in} + f_i = \sum_{j=1}^{n} z_{ij} + f_i \tag{2.1}$$

[1] In Chapters 4 and 5 we will explore more recent distinctions between "commodities" and "industries" and see how these observations lead to alternative representations of the input–output model.

The z_{ij} terms represent *interindustry* sales by sector i (also known as *intermediate* sales) to all sectors j (including itself, when $j = i$). Equation (2.1) represents the distribution of sector i *output*. There will be an equation like this that identifies sales of the output of each of the n sectors:

$$\begin{aligned} x_1 &= z_{11} + \cdots + z_{1j} + \cdots + z_{1n} + f_1 \\ &\vdots \\ x_i &= z_{i1} + \cdots + z_{ij} + \cdots + z_{in} + f_i \\ &\vdots \\ x_n &= z_{n1} + \cdots + z_{nj} + \cdots + z_{nn} + f_n \end{aligned} \tag{2.2}$$

Let

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{Z} = \begin{bmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & \ddots & \vdots \\ z_{n1} & \cdots & z_{nn} \end{bmatrix} \text{ and } \mathbf{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \tag{2.3}$$

Here and throughout this text we use lower-case bold letters for (column) vectors, as in $\mathbf{f}$ and $\mathbf{x}$ (so $\mathbf{x}'$ is the corresponding row vector) and upper case bold letters for matrices, as in $\mathbf{Z}$. With this notation, the information in (2.2) on the distribution of each sector's sales can be compactly summarized in matrix notation as

$$\mathbf{x} = \mathbf{Zi} + \mathbf{f} \tag{2.4}$$

We use $\mathbf{i}$ to represent a column vector of 1's (of appropriate dimension – here n). This is known as a "summation" vector (Section A.8, Appendix A). The important observation is that post-multiplication of a matrix by $\mathbf{i}$ creates a column vector whose elements are the row sums of the matrix. Similarly, $\mathbf{i}'$ is a row vector of 1's, and premultiplication of a matrix by $\mathbf{i}'$ creates a row vector whose elements are the column sums of the matrix. We will use summation vectors often in this and subsequent chapters.

Consider the information in the jth column of z's on the right-hand side:

$$\begin{bmatrix} z_{1j} \\ \vdots \\ z_{ij} \\ \vdots \\ z_{nj} \end{bmatrix}$$

These elements are sales to sector j – j's purchases of the products of the various producing sectors in the country; the column thus represents the sources and magnitudes of sector j's *inputs*. Clearly, in engaging in production, a sector also pays for other items – for example, labor and capital – and uses other inputs as well, such as inventoried items.

Table 2.1 Input–Output Table of Interindustry Flows of Goods

		Buying Sector				
		1	$\cdots$	j	$\cdots$	n
Selling Sector	1	z_{11}	$\cdots$	z_{1j}	$\cdots$	z_{1n}
	$\vdots$	$\vdots$		$\vdots$		$\vdots$
	i	z_{i1}	$\cdots$	z_{ij}	$\cdots$	z_{in}
	$\vdots$	$\vdots$		$\vdots$		$\vdots$
	n	z_{n1}	$\cdots$	z_{nj}	$\cdots$	z_{nn}

All of these *primary inputs* together are termed the *value added* in sector j. In addition, imported goods may be purchased as inputs by sector j. All of these inputs (value added and imports) are often lumped together as purchases from what is called the *payments* sector, whereas the z's on the right-hand side of (2.2) serve to record the purchases from the *processing* sector, the *interindustry inputs* (or *intermediate inputs*). Since each equation in (2.2) includes the possibility of purchases by a sector of its own output as an input to production, these *inter*industry inputs include *intra*industry transactions as well.

The magnitudes of these interindustry flows can be recorded in a table, with sectors of origin (producers) listed on the left and the same sectors, now destinations (purchasers), listed across the top. From the column point of view, these show each sector's inputs; from the row point of view the figures are each sector's outputs; hence the name *input–output table*. These figures are the core of input–output analysis.

2.2.1 Input–Output Transactions and National Accounts

As was suggested by Table 1.1, an input–output transactions (flow) table, such as that shown in Table 2.1, constitutes part of a complete set of income and product accounts for an economy. To emphasize the other elements in a full set of accounts, we consider a small, two-sector economy. We present an expanded flow table for this extremely simple economy in Table 2.2. (We examine more of the details of a system of national accounts in Chapter 4.)

The component parts of the final demand vector for sectors 1 and 2 represent, respectively, consumer (household) purchases, purchases for (private) investment purposes, government (federal, state, and local) purchases, and sales abroad (exports). These are often grouped into *domestic* final demand $(C+I+G)$ and *foreign* final demand (exports, E). Then $f_1 = c_1 + i_1 + g_1 + e_1$ and similarly $f_2 = c_2 + i_2 + g_2 + e_2$.

The component parts of the payments sector are payments by sectors 1 and 2 for employee compensation (labor services, l_1 and l_2) and for all other value-added items – for example, government services (paid for in taxes), capital (interest payments), land

Table 2.2 Expanded Flow Table for a Two-Sector Economy

		Processing Sectors 1	2	Final Demand				Total Output ($\mathbf{x}$)
Processing Sectors	1	z_{11}	z_{12}	c_1	i_1	g_1	e_1	x_1
	2	z_{21}	z_{22}	c_2	i_2	g_2	e_2	x_2
Payments Sectors	Value Added ($\mathbf{v}'$)	l_1	l_2	l_C	l_I	l_G	l_E	L
		n_1	n_2	n_C	n_I	n_G	n_E	N
	Imports	m_1	m_2	m_C	m_I	m_G	m_E	M
Total Outlays ($\mathbf{x}'$)		x_1	x_2	C	I	G	E	X

(rental payments), entrepreneurship (profit), and so on. Denote these other value-added payments by n_1 and n_2; then total value-added payments are $v_1 = l_1 + n_1$, and $v_2 = l_2 + n_2$, for the two sectors.

Finally, assume that some (or perhaps all) sectors use imported goods in producing their outputs. One approach is to record these import amounts in an imports row in the payments sector as m_1 and m_2.[2] Total expenditures in the payments sector by sectors 1 and 2 are $l_1 + n_1 + m_1 = v_1 + m_1$ and $l_2 + n_2 + m_2 = v_2 + m_2$, respectively. However, it is often the case that the exports part of the final demand column is expressed as *net* exports so that the sum of all final demands is equal to traditional definitions of gross domestic product, i.e., net of imports. In that case a distinction is often made between imports of goods that are also domestically produced (competitive imports) and those for which there is no domestic source (noncompetitive imports), and all the competitive imports in the imports row will have been netted out of the appropriate elements in a *gross* exports column. Under these circumstances it is possible for one or more elements in the net export column to be negative, if the value of imports of those goods exceeds the value of exports. (For example, if an economy exported €300 million of agricultural products last year but imported €350 million, the net exports figure for the agricultural sector would be €50 million.) Also, if the federal government *sells* more of a stockpiled item (e.g., wheat) than it buys, a negative entry in the government column of the final demand part of the table could result. If the negative number is large enough, it could swamp the other (positive) final demand purchases of that good, leaving a negative total final demand figure.

The elements in the intersection of the value-added rows and the final demand columns represent payments by final consumers for labor services (for example, l_C includes household payments for, say, domestic help; l_G represents payments to

[2] The treatment of imports in input–output accounts is much more complicated than this, but for the present we prefer to concentrate on the overall structure of a transactions table. We return to imports in section 2.3.4 below, and in more detail in Chapter 4.

government workers) and for other value added (for example, n_C includes tax payments by households). In the imports row and final demand columns are, for example, m_G, which represents government purchases of imported items, and m_E, which represents imported items that are re-exported.

Summing down the total output column, total gross output throughout the economy, X, is found as

$$X = x_1 + x_2 + L + N + M$$

This same value can be found by summing across the total outlays row; namely

$$X = x_1 + x_2 + C + I + G + E$$

These are simply two alternative ways of summing all the elements in the table.

In national income and product accounting, it is the value of total *final* product that is of interest – goods available for consumption, export, and so on. Equating the two expressions for X and subtracting x_1 and x_2 from both sides leaves

$$L + M + N = C + I + G + E$$

or

$$L + N = C + I + G + (E - M)$$

The left-hand side represents gross national income – the total factor payments in the economy – and the right-hand side represents gross national product – the total spent on consumption and investment goods, total government purchases, and the total value of net exports from the economy. Again, national accounts are examined in more detail in Chapter 4.

In most developed economies, consumption is the largest individual component of final demand. For example, in the USA in 2003 the percentages of total final demand were as follows: personal consumption expenditure (PCE), 71 percent; gross private domestic investment (including producers' durable equipment, plant construction, residential construction, and net inventory change), 15 percent; government purchases (federal, state and local), 19 percent; net foreign exports, −5 percent (the value of imports exceeded the value of exports). [However, in the USA during the 1942–1945 period (World War II), PCE was between 40 and 48 percent and for much of the 1950s and 1960s it was under 60 percent.]

2.2.2 *Production Functions and the Input–Output Model*

In input–output work, a fundamental assumption is that the interindustry flows from i to j – recall that these are for a given period, say a year – depend entirely on the total output of sector j for that same time period. Clearly, no one would argue against the idea that the more cars produced in a year, the more steel will be needed during that year by automobile producers. Where argument *does* arise is over the exact nature of this relationship. In input–output analysis it is as follows: Given z_{ij} and x_j – for example, input of aluminum (i) bought by aircraft producers (j) last year and total aircraft production last year – form the ratio of aluminum input to aircraft output, z_{ij}/x_j [the units are (\$/\$)], and denote it by a_{ij}:

$$a_{ij} = \frac{z_{ij}}{x_j} = \frac{\text{value of aluminum bought by aircraft producers last year}}{\text{value of aircraft production last year}} \tag{2.5}$$

This ratio is called a technical coefficient; the terms input–output coefficient and direct input coefficient are also often used. For example, if $z_{14} = \$300$ and $x_4 = \$15,000$ (sector 4 used \$300 of goods from sector 1 in producing \$15,000 of sector 4 output), $a_{14} = z_{14}/x_4 = \$300/\$15,000 = 0.02$. Since a_{14} is actually \$0.02/\$1, the 0.02 is interpreted as the "dollars' worth of inputs from sector 1 per dollar's worth of output of sector 4."

From (2.5), $a_{ij}x_j = z_{ij}$. This is trivial algebra, but it presents the operational form in which the technical coefficients are used. In input–output analysis, once a set of observations has given us the result $a_{14} = 0.02$, this technical coefficient is assumed to be unchanging in the sense that if one asked how much sector 4 would buy from sector 1 if sector 4 were to produce a total output (x_4) of \$45,000, the input–output answer would be $z_{14} = a_{14}x_4 = (0.02)(\$45,000) = \$900$ – when output of sector 4 is tripled, the input from sector 1 is tripled. The a_{ij} are viewed as measuring fixed relationships between a sector's output and its inputs. Economies of scale in production are thus ignored; production in a Leontief system operates under what is known as constant returns to scale.

In addition, input–output analysis requires that a sector use inputs in *fixed proportions*. Suppose, to continue the previous example, that sector 4 also buys inputs from sector 2, and that, for the period of observation, $z_{24} = \$750$. Therefore $a_{24} = z_{24}/x_4 = \$750/\$15,000 = 0.05$. For $x_4 = \$15,000$, inputs from sector 1 and from sector 2 were used in the proportion $p_{12} = z_{14}/z_{24} = \$300/\$750 = 0.4$. If x_4 were \$45,000, z_{24} would be (0.05)(\$45,000) = \$2250; since $z_{14} = \$900$ for $x_4 = \$45,000$, the proportion between inputs from sector 1 and from sector 2 is \$900/\$2250 = 0.4, as before. This reflects the fact that

$$p_{12} = z_{14}/z_{24} = a_{14}x_4/a_{24}x_4 = a_{14}/a_{24} = 0.02/0.05 = 0.4;$$

the proportion is the ratio of the technical coefficients, and since the coefficients are fixed, then the input proportion is fixed.

For the reader with some background in basic microeconomics, we can identify the form of production function inherent in the input–output system and compare it with that in the general neoclassical microeconomic approach. Production functions relate the amounts of inputs used by a sector to the maximum amount of output that could be produced by that sector with those inputs. An illustration is

$$x_j = f(z_{1j}, z_{2j}, \ldots, z_{nj}, v_j, m_j)$$

Using the definition of the technical coefficients in (2.5), we can see that in the Leontief model this becomes

$$x_j = \frac{z_{1j}}{a_{1j}} = \frac{z_{2j}}{a_{2j}} = \cdots = \frac{z_{nj}}{a_{nj}}$$

(This ignores, for the moment, the contributions of v_j and m_j.)

A problem with this extremely simple formulation is that it is meaningless if a particular input i is not used in production of j, since then $a_{ij} = 0$ and hence z_{ij}/a_{ij} is infinitely large. Thus, the more usual specification of the kind of production function that is embodied in the input–output model is

$$x_j = \min\left(\frac{z_{1j}}{a_{1j}}, \frac{z_{2j}}{a_{2j}}, \cdots, \frac{z_{nj}}{a_{nj}}\right)$$

where min (x, y, z) denotes the smallest of the numbers x, y and z. In the input–output model, for those a_{ij} coefficients that are not zero, these ratios will all be the same, and equal to x_j – from the fundamental definition of a_{ij} in (2.5). For those a_{ij} coefficients that are zero, the ratio z_{ij}/a_{ij} will be infinitely large and hence will be overlooked in the process of searching for the smallest among the ratios. This specification of the production function in the input–output model reflects the assumption of constant returns to scale; multiplication of $z_{1j}, z_{2j}, \ldots, z_{nj}$ by any constant will multiply x_j by the same constant. (Tripling all inputs will triple output; cutting inputs in half will halve output, etc.)

For the reader who is acquainted with the economist's production function geometry, we show four alternative representations of production functions in input space for a two-sector economy in Figure 2.1. A *linear production function*, depicted in Figure 2.1(a) assumes that output is a simple linear function of inputs, which means that the inputs are infinitely substitutable for each other for any level of output. The figure shows a set of isoquants (constant output lines) depicting higher and higher levels of output.

A *classical production function*, depicted in Figure 2.1(b), also shows a set of isoquants (now constant output curves) depicting higher and higher levels of output. For a given value of z_{1j} in Figure 2.1(b), increasing z_{2j} leads to increases in x_j – intersections with higher-value isoquants. In this case input substitution is also possible but not linearly, as indicated by the isoquants showing alternative input combinations that generate the same level of output. For example, moving rightward along a particular isoquant in Figure 2.1(b) can be accomplished by reducing the amount of input 2 and increasing the amount of input 1, or leftward by reducing z_{1j} and increasing z_{2j}.

The shape of the isoquants in Figure 2.1(b) reflects two specific classical assumptions about how inputs are combined to produce outputs. The negative slopes of the isoquants represent the fact that as the amount of one input is decreased, the amount of the other input must be increased in order to maintain the level of production indicated by a specific isoquant. The fact that the curves bulge toward the origin (mathematically

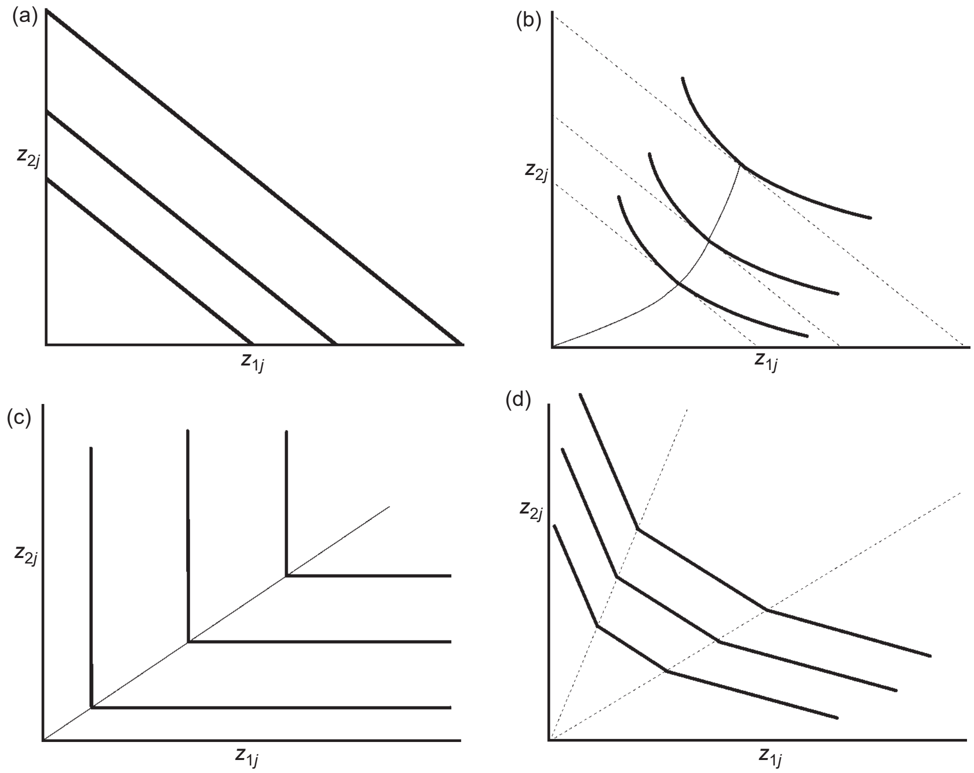

Figure 2.1a–d. Production Functions in Input Space. (a) Linear production function. (b) Classical production function. (c) Leontief production function. (d) Activity analysis production function.

their convexity) reflects the economist's law of diminishing marginal productivity.[3] The "expansion path" representing input combinations that are used for various levels of output is a curve from the origin through the points of tangency between isocost (constant cost) lines – dashed in Figure 2.1(b) – and the isoquants.

In the Leontief model, the isoquant "curves" of constant output appear as in Figure 2.1(c). Once the observed proportion of inputs 1 and 2 is known, as $p_{12} = z_{1j}/z_{2j}$, then additional amounts of either input 1 or input 2 alone are useless from the point of view of increasing the output of j. Only when availabilities of *both* input 1 and input 2 are increased can x_j increase; and only if the amounts of increase of 1 and 2 are in the proportion p_{12} will all the available amounts of both be used up. Of course, the "true" geometric representation should be in n-dimensional input space, with a separate axis for each of the n possible inputs, but the principles are the same when only

3 From basic microeconomics concepts, recall that the slope of an isoquant (assuming that these are smooth functions) at any point is the ratio of the marginal productivities of inputs 1 and 2. These marginal productivities, in turn, are the partial derivatives of the production function (also assumed smooth) with respect to each of the inputs – thus the slope is $\frac{\partial f/\partial x_1}{\partial f/\partial x_2}$. As we move rightward along an isoquant, the amount of input 2 used decreases and the amount of input 1 used increases. By diminishing marginal productivity, then, $\partial f/\partial x_1$ decreases and $\partial f/\partial x_2$ increases; hence the slope decreases, as is true for the isoquants in Figure 2.1(b).

two inputs are considered. From the Leontief production function, if $z_{1j}, z_{2j}, \ldots, z_{(n-1)j}$ were all doubled but z_{nj} were only increased by 50 percent (multiplied by 1.5), then the minimum of the new ratios would be z_{nj}/a_{nj} and the new output of sector j would be 50 percent larger. There would be excess and unused amounts of inputs from sectors 1, 2,. . ., $(n-1)$. But since inputs are not free goods, sector j will not buy more from any sector than is needed for its production, and thus the input combinations chosen by sector j will lie along the ray as represented in Figure 2.1(c). In short, Leontief production functions require inputs in fixed proportions where a fixed amount of each input is required to produce one unit of output.

Figure 2.1(d) shows an *activity analysis production function*, which is a generalization of the Leontief production function and is a piece-wise linear approximation of the classical production function. Each isoquant is represented by a connected set of line segments. Each segment is a linear production function applicable over a limited range of combinations of inputs to produce a given level of output.

Once the notion of a set of fixed technical coefficients is accepted, (2.2) can be rewritten, replacing each z_{ij} on the right by $a_{ij}x_j$:

$$\begin{aligned}
x_1 &= a_{11}x_1 + \cdots + a_{1i}x_i + \cdots + a_{1n}x_n + f_1 \\
&\vdots \\
x_i &= a_{i1}x_1 + \cdots + a_{ii}x_i + \cdots + a_{in}x_n + f_i \\
&\vdots \\
x_n &= a_{n1}x_1 + \cdots + a_{ni}x_i + \cdots + a_{nn}x_n + f_n
\end{aligned} \tag{2.6}$$

These equations serve to make explicit the dependence of interindustry flows on the total outputs of each sector. They also bring us closer to the form needed in input–output *analysis*, in which the following kind of question is asked: If the demands of the exogenous sectors were forecast to be some specific amounts next year, how much output from each of the sectors would be necessary to supply these final demands? From the point of view of this equation, the $f_1, \ldots, f_n$ are known numbers, the a_{ij} are known coefficients, and the $x_1, \ldots, x_n$ are to be found. Therefore, bringing all x terms to the left,

$$\begin{aligned}
x_1 - a_{11}x_1 - \cdots - a_{1i}x_i - \cdots - a_{1n}x_n &= f_1 \\
\vdots \\
x_i - a_{i1}x_1 - \cdots - a_{ii}x_i - \cdots - a_{in}x_n &= f_i \\
\vdots \\
x_n - a_{n1}x_1 - \cdots - a_{ni}x_i - \cdots - a_{nn}x_n &= f_n
\end{aligned}$$

and, grouping the x_1 together in the first equation, the x_2 in the second, and so on,

$$\begin{aligned}
(1 - a_{11})x_1 - \cdots - a_{1i}x_i - \cdots - a_{1n}x_n &= f_1 \\
&\vdots \\
- a_{i1}x_1 - \cdots + (1 - a_{ii})x_i - \cdots - a_{in}x_n &= f_i \\
&\vdots \\
- a_{n1}x_1 - \cdots - a_{ni}x_i - \cdots + (1 - a_{nn})x_n &= f_n
\end{aligned} \tag{2.7}$$

These relationships can be represented compactly in matrix form. In matrix algebra notation, a "hat" over a vector denotes a diagonal matrix with the elements of the vector along the main diagonal, so, for example, $\hat{\mathbf{x}} = \begin{bmatrix} x_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x_n \end{bmatrix}$. From the basic definition of an inverse, $(\hat{\mathbf{x}})(\hat{\mathbf{x}})^{-1} = \mathbf{I}$, it follows that $\hat{\mathbf{x}}^{-1} = \begin{bmatrix} 1/x_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1/x_n \end{bmatrix}$. Also, postmultiplication of a matrix, $\mathbf{M}$, by a diagonal matrix, $\hat{\mathbf{d}}$, creates a matrix in which each element in column j of $\mathbf{M}$ is multiplied by d_j in $\hat{\mathbf{d}}$ (Appendix A, section A.7). Therefore the $n \times n$ matrix of technical coefficients can be represented as

$$\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1} \tag{2.8}$$

Using the definitions in (2.3) and (2.8), the matrix expression for (2.6) is

$$\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{f} \tag{2.9}$$

Let $\mathbf{I}$ be the $n \times n$ identity matrix – ones on the main diagonal and zeros elsewhere;

$$\mathbf{I} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \text{ so then } (\mathbf{I} - \mathbf{A}) = \begin{bmatrix} (1 - a_{11}) & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & (1 - a_{22}) & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & (1 - a_{nn}) \end{bmatrix}.$$

Then the complete $n \times n$ system shown in (2.7) is just[4]

$$(\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{f} \tag{2.10}$$

For a given set of f's, this is a set of n linear equations in the n unknowns, $x_1, x_2, \ldots, x_n$ and hence it may or may not be possible to find a unique solution. In fact, whether or

[4] This is parallel to the form $\mathbf{Ax} = \mathbf{b}$ that is usually used to denote a set of linear equations. The difference is purely notational; since it is standard in input–output analysis to define the technical coefficients matrix as $\mathbf{A}$, then the matrix of coefficients in the input–output equation system becomes $(\mathbf{I} - \mathbf{A})$. Similarly, convention is responsible for denoting the right-hand sides of the input–output equations by $\mathbf{f}$ (for final demand) instead of $\mathbf{b}$.

not there is a unique solution depends on whether or not $(\mathbf{I}-\mathbf{A})$ is singular; that is, whether or not $(\mathbf{I}-\mathbf{A})^{-1}$ exists. The matrix $\mathbf{A}$ is known as the technical (or input–output, or direct input) coefficients matrix. From the basic definition of an inverse for a square matrix (Appendix A), $(\mathbf{I}-\mathbf{A})^{-1}=(1/|\mathbf{I}-\mathbf{A}|)[\mathrm{adj}(\mathbf{I}-\mathbf{A})]$. If $|\mathbf{I}-\mathbf{A}|\neq 0$, then $(\mathbf{I}-\mathbf{A})^{-1}$ can be found, and using standard matrix algebra results for linear equations the unique solution to (2.10) is given by

$$\mathbf{x}=(\mathbf{I}-\mathbf{A})^{-1}\mathbf{f}=\mathbf{L}\mathbf{f} \tag{2.11}$$

where $(\mathbf{I}-\mathbf{A})^{-1}=\mathbf{L}=[l_{ij}]$ is known as the *Leontief inverse* or the *total requirements matrix*.

In more detail, the equations summarized in (2.11) are

$$\begin{aligned}
x_1 &= l_{11}f_1+\cdots+l_{1j}f_j+\cdots+l_{1n}f_n\\
&\vdots\\
x_i &= l_{i1}f_1+\cdots+l_{ij}f_j+\cdots+l_{in}f_n\\
&\vdots\\
x_n &= l_{n1}f_1+\cdots+l_{nj}f_j+\cdots+l_{nn}f_n
\end{aligned} \tag{2.12}$$

This makes clear the dependence of each of the gross outputs on the values of each of the final demands. Readers familiar with differential calculus and partial derivatives will recognize that $\partial x_i/\partial f_j=l_{ij}$.

2.3 An Illustration of Input–Output Calculations

2.3.1 Numerical Example: Hypothetical Figures – Approach I

Impacts on Industry Outputs We now turn to a small numerical example, as presented in Table 2.3. For the moment, the final demand elements and the value-added elements have not been disaggregated into their component parts.

The corresponding table of input–output coefficients, Table 2.4, is found by dividing each flow in a particular column of the producing sectors in Table 2.3 by the total output (row sum) of that sector. Thus, $a_{11}=150/1000=0.15$; $a_{21}=200/1000=0.2$; $a_{12}=500/2000=0.25$; $a_{22}=100/2000=0.05$. In particular,

$$\mathbf{A}=\mathbf{Z}\hat{\mathbf{x}}^{-1}=\begin{bmatrix}150 & 500\\ 200 & 100\end{bmatrix}\begin{bmatrix}1/1000 & 0\\ 0 & 1/2000\end{bmatrix}$$

The $\mathbf{A}$ matrix is shown in Table 2.4. To add specificity for the remainder of this example, we assume sector 1 represents “Agriculture” and sector 2 “Manufacturing.”

The principal way in which input–output coefficients are used for *analysis* is as follows. We assume that the numbers in Table 2.4 represent the structure of production in the economy; the columns are, in effect, the production recipes for each of the sectors, in terms of inputs from all the sectors. To produce one dollar’s worth of manufactured goods, for example, 25 cents’ worth of agricultural products and 5 cents’ worth of

Table 2.3 Flows (z_{ij}) for the Hypothetical Example

		To Processing Sectors 1	2	Final Demand (f_i)	Total Output (x_i)
From	1	150	500	350	1000
Processing Sectors	2	200	100	1700	2000
Payments Sector		650	1400	1100	3150
Total Outlays (x_i)		1000	2000	3150	6150

Table 2.4 Technical Coefficients (the **A** Matrix) for the Hypothetical Example

	Sector 1 (Agriculture)	Sector 2 (Manufacturing)
Sector 1 (Agriculture)	.15	.25
Sector 2 (Manufacturing)	.20	.05

manufactures are needed as intermediate ingredients. These are, of course, only the inputs needed from other producing sectors; there will be inputs of a more "nonproduced" nature as well, such as labor, from the payments sectors. For an analysis of interrelationships among productive sectors, these are not of major importance.

We can now ask the question: If *final demand* for agriculture output were to increase to \$600 next year and that for manufactures were to decrease to \$1500 – for example, because of changes in government spending, consumers' tastes, and so on – how much total output from the two sectors would be necessary in order to meet this new demand? We denote this new demand as $\mathbf{f}^{new} = \begin{bmatrix} f_1^{new} \\ f_2^{new} \end{bmatrix} = \begin{bmatrix} 600 \\ 1500 \end{bmatrix}$. In the year of observation, when $\mathbf{f} = \begin{bmatrix} 350 \\ 1700 \end{bmatrix}$, we saw that $\mathbf{x} = \begin{bmatrix} 1000 \\ 2000 \end{bmatrix}$, precisely because, in producing to satisfy final demands, each sector must also produce to satisfy the demands for inputs into the processes of production themselves. Now we are asking, for $f_1^{new} = 600$ and $f_2^{new} = 1500$, what are the elements of $\mathbf{x}^{new} = \begin{bmatrix} x_1^{new} \\ x_2^{new} \end{bmatrix}$? To satisfy the demands, x_1^{new} can be no less than \$600 and x_2^{new} no less than \$1500. These would be the necessary outputs – the "direct effects" – if neither product were used in production and all output were directly available for final demand. But since both products serve as inputs, in a manner that is reflected in the technical coefficients of Table 2.4, it seems clear that in the end, more than \$600 worth of agriculture goods and more than \$1500 worth of manufactures will have to have been produced in order to meet the new final demands. That is, there will be "indirect effects" as well. Both of these effects are captured in the input–output model.

In the 2×2 case, $|\mathbf{I}-\mathbf{A}| = (1-a_{11})(1-a_{22}) - a_{12}a_{21}$ (Appendix A) and

$$\mathrm{adj}(\mathbf{I}-\mathbf{A}) = \begin{bmatrix} (1-a_{22}) & a_{12} \\ a_{21} & (1-a_{11}) \end{bmatrix}$$

For this example, $\mathbf{A} = \begin{bmatrix} .15 & .25 \\ .20 & .05 \end{bmatrix}$ so $(\mathbf{I}-\mathbf{A}) = \begin{bmatrix} .85 & -.25 \\ -.20 & .95 \end{bmatrix}$; hence $|\mathbf{I}-\mathbf{A}| = 0.7575 \neq 0$ and we know that $\mathbf{L} = (\mathbf{I}-\mathbf{A})^{-1}$can be found. Here we have

$$\mathbf{L} = \begin{bmatrix} 1.2541 & .3300 \\ .2640 & 1.1221 \end{bmatrix}$$

Assuming that technology (as represented in **A**), does not change, the needed total outputs caused by $\mathbf{f}^{new}$ are then found as in (2.11):

$$\mathbf{x}^{new} = \mathbf{L}\mathbf{f}^{new} = \begin{bmatrix} 1.2541 & .3300 \\ .2640 & 1.1221 \end{bmatrix} \begin{bmatrix} 600 \\ 1500 \end{bmatrix} = \begin{bmatrix} 1247.52 \\ 1841.58 \end{bmatrix} \tag{2.13}$$

These values – $x_1^{new} = \$1247.52$ and $x_2^{new} = \$1841.58$ – are one measure of the *impact* on the economy of the new final demands.[5]

With this result for $\mathbf{x}^{new}$, it is straightforward to examine the changes in all elements in the interindustry flows table (as in Table 2.3) caused by $\mathbf{f}^{new}$. From the definition of coefficients in (2.8), $\mathbf{Z} = \mathbf{A}\hat{\mathbf{x}}$. With a constant **A** matrix and new outputs, $\mathbf{x}^{new}$, we find $\mathbf{Z}^{new} = \mathbf{A}\hat{\mathbf{x}}^{new} = \begin{bmatrix} 187.13 & 460.40 \\ 249.50 & 92.08 \end{bmatrix}$; along with $\mathbf{f}^{new} = \begin{bmatrix} 600 \\ 1500 \end{bmatrix}$, we have the results shown in Table 2.5.

The elements in the Payments Sector are found as the difference between new total outputs (total outlays) and new total interindustry inputs for each sector. (For the example we assume no change in payments sector transactions with final demand.) Notice that sector 1's purchases are larger (reflecting an increase in final demand for that sector) and sector 2's purchases are smaller (reflecting smaller demand for that sector).

The input–output model allows us to deal equally easily with *changes* in demands and outputs instead of *levels*. Here and throughout, we use superscripts "0" to represent the initial (base year) situation and "1" for values of variables after the change in demands (instead of "*new*" as we did above). Assuming that technology is unchanged means $\mathbf{A}^0 = \mathbf{A}^1 = \mathbf{A}$ and $\mathbf{L}^0 = \mathbf{L}^1 = \mathbf{L}$, so $\mathbf{x}^0 = \mathbf{L}\mathbf{f}^0$ and $\mathbf{x}^1 = \mathbf{L}\mathbf{f}^1$; letting $\Delta\mathbf{x} = \mathbf{x}^1 - \mathbf{x}^0$ and

[5] Here x_1^{new} and x_2^{new} are shown to two decimals for comparison with results from an alternative approach in section 2.3.2. These $\mathbf{x}^{new}$ values reflect computer calculations carried out with more than four significant digits and hence often will (as here) differ (to the right of the decimal point) from what the reader will produce with a hand calculator using the four-digit elements shown for **A**. In any actual analysis, such detail might be questionable because of the much less accurate data from which the technical coefficients are derived (compare the figures in Table 2.3).

Table 2.5 Flows (z_{ij}) for the Hypothetical Example Associated with $\mathbf{x}^{new}$

		To Processing Sectors			
		1	2	Final Demand (f_i)	Total Output (x_i)
From	1	187.13	460.40	600	1247.52
Processing Sectors	2	249.50	92.08	1500	1841.58
Payments Sector		810.89	1289.11	1100	3200.00
Total Outlays (x_i)		1247.52	1841.58	3200	6289.10

$\Delta\mathbf{f} = \mathbf{f}^1 - \mathbf{f}^0$

$$\Delta\mathbf{x} = \mathbf{L}\mathbf{f}^1 - \mathbf{L}\mathbf{f}^0 = \mathbf{L}\Delta\mathbf{f} \tag{2.14}$$

In this example, $\Delta\mathbf{f} = \begin{bmatrix} 250 \\ -200 \end{bmatrix}$, giving $\Delta\mathbf{x} = \begin{bmatrix} 247.5 \\ -158.4 \end{bmatrix}$ and so

$$\mathbf{x}^1 = \mathbf{x}^0 + \Delta\mathbf{x} = \begin{bmatrix} 1000 \\ 2000 \end{bmatrix} + \begin{bmatrix} 247.5 \\ -158.4 \end{bmatrix} = \begin{bmatrix} 1247.5 \\ 1841.6 \end{bmatrix}$$

This corresponds to the result in (2.13), except for rounding.

Other Impacts In many cases, the dollar value of each sector's gross output may not ultimately be the most important measure of the economic impact following a change in exogenous demands. Gross output requirements could be translated into employment effects (in either dollars or physical terms – for example, person-years), or effects on value-added, or energy consumption (of a particular type, e.g., petroleum), or pollution emissions (again, of a particular type, e.g., CO_2), and so forth. In each instance, we need a set of appropriate coefficients with which to convert outputs into associated effects. For illustration we consider employment in monetary terms. Let the value of employment in the two sectors be denoted as[6]

$$\mathbf{e}' = \begin{bmatrix} e_1 & e_2 \end{bmatrix}$$

A vector of employment *coefficients* contains the base-year employment in each sector divided by that sector's base-year gross output, x_1^0 and x_2^0,

$$\mathbf{e}'_c = \begin{bmatrix} e_1/x_1^0 & e_2/x_2^0 \end{bmatrix} = \begin{bmatrix} e_{c1} & e_{c2} \end{bmatrix}$$

Then $\boldsymbol{\varepsilon} = \hat{\mathbf{e}}'_c\mathbf{x}^1 = \hat{\mathbf{e}}'_c\mathbf{L}\mathbf{f}^1$ produces a vector whose elements are the total labor income in each sector that accompanies the new exogenous final demand;

$$\boldsymbol{\varepsilon} = \begin{bmatrix} e_{c1} & 0 \\ 0 & e_{c2} \end{bmatrix} \begin{bmatrix} x_1^1 \\ x_2^1 \end{bmatrix} = \begin{bmatrix} e_{c1}x_1^1 \\ e_{c2}x_2^1 \end{bmatrix}$$

[6] Later in this chapter (and still later, in Chapter 6 on multipliers) we will need to alter this notation to be able to accommodate additional possibilities.

To continue with the numerical example, suppose that $e_{c1} = 0.30$ and $e_{c2} = 0.25$ give the dollars' worth of labor inputs per dollar's worth of output of the two sectors. (We will examine the role of labor inputs and household consumption in an input–output model in some detail in section 2.5, below.) Then

$$\boldsymbol{\varepsilon} = \hat{\mathbf{e}}'_c \mathbf{x}^1 = \begin{bmatrix} 0.30 & 0 \\ 0 & 0.25 \end{bmatrix} \begin{bmatrix} 1247.52 \\ 1841.58 \end{bmatrix} = \begin{bmatrix} 374.26 \\ 460.40 \end{bmatrix}$$

This indicates the values of labor inputs purchased by the two sectors.

If, additionally, we have an occupation-by-industry matrix, $\mathbf{P}$, where p_{ij} is the *proportion* of sector j employment that is in occupation i, then $\tilde{\boldsymbol{\varepsilon}} = \mathbf{P}\hat{\boldsymbol{\varepsilon}}$ gives a matrix of employment by sector by occupation type. For example, with k occupation types and two sectors,

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ \vdots & \vdots \\ p_{k1} & p_{k2} \end{bmatrix}$$

and

$$\tilde{\boldsymbol{\varepsilon}} = \mathbf{P}\hat{\boldsymbol{\varepsilon}} = \begin{bmatrix} p_{11}e_{c1}x_1^1 & p_{12}e_{c2}x_2^1 \\ \vdots & \vdots \\ p_{k1}e_{c1}x_1^1 & p_{k2}e_{c2}x_2^1 \end{bmatrix}$$

Column sums would give total labor use by sector; row sums give total employment of a particular occupational category across all sectors. (The vector $\mathbf{P}\boldsymbol{\varepsilon}$ shows employment by occupational category, aggregated across all sectors.)

Suppose that our economy has three occupational groups: (1) engineers, (2) bankers and (3) farmers, and

$$\mathbf{P} = \begin{bmatrix} 0 & 0.8 \\ 0.6 & 0.2 \\ 0.4 & 0 \end{bmatrix}$$

(For example, this says that 40 percent of the agricultural labor force is farmers; 80 percent of manufacturing labor force is made up of engineers, etc.) Then

$$\tilde{\boldsymbol{\varepsilon}} = \mathbf{P}\hat{\boldsymbol{\varepsilon}} = \begin{bmatrix} 0 & 0.8 \\ 0.6 & 0.2 \\ 0.4 & 0 \end{bmatrix} \begin{bmatrix} 374.26 & 0 \\ 0 & 460.40 \end{bmatrix} = \begin{bmatrix} 0 & 368.32 \\ 224.56 & 92.08 \\ 149.70 & 0 \end{bmatrix}$$

Column sums of $\tilde{\boldsymbol{\varepsilon}}$ are 374.26 and 460.40, as expected (the elements of $\boldsymbol{\varepsilon}$). Row sums give the economy-wide (across both sectors) employment of engineers, farmers and bankers, respectively. If sectoral disaggregation is not necessary, then

$$\mathbf{P}\boldsymbol{\varepsilon} = \begin{bmatrix} 0 & 0.8 \\ 0.6 & 0.2 \\ 0.4 & 0 \end{bmatrix} \begin{bmatrix} 374.26 \\ 460.40 \end{bmatrix} = \begin{bmatrix} 368.32 \\ 316.64 \\ 149.70 \end{bmatrix}$$

gives employment by occupational type, across sectors.

A wide variety of such conversion coefficients vectors (as in $\mathbf{e}'_c$) or matrices (as in $\mathbf{P}$) is possible. For example, in arid regions, water-use coefficients, $\mathbf{w}'_c = \begin{bmatrix} w_{c1} & w_{c2} \end{bmatrix}$, could be used in $\mathbf{w}'_c\mathbf{x}'$ to assess the water consumption associated with new outputs generated by new final demands. We explore these kinds of alternative impacts again in Chapter 6 on input–output multipliers, and in Chapters 9 and 10, some of the energy and environmental repercussions of final demand impacts are discussed in detail.

2.3.2 Numerical Example: Hypothetical Figures – Approach II

Consider the same economy, whose 2×2 technical coefficients matrix is given in Table 2.4 and for which the projected $\mathbf{f}^1$ vector is $\begin{bmatrix} 600 \\ 1500 \end{bmatrix}$. We can examine the question of outputs necessary to satisfy this final demand in a more intuitive way that is less mechanical than finding elements in an inverse matrix.

1. Initially, it is clear that agriculture needs to produce \$600 and manufacturing, \$1500. If the sectors are going to meet the new final demands, they could not get away with producing less than these amounts.
2. However, to produce \$600, agriculture needs, as inputs to that productive process, (0.15)(\$600) = \$90 from itself and (0.20)(\$600) = \$120 from manufacturing. These figures come from the coefficients in column 1 of the **A** matrix – the production recipe for agriculture. Similarly, to produce its \$1500, manufacturing will have to buy (0.25)(\$1500) = \$375 from agriculture and (0.05)(\$1500) = \$75 from itself. Thus agriculture must, in fact, produce the \$600 noted in 1, above, plus another \$(90 + 375) = \$465 more, to satisfy the needs for inputs that it has itself and also that come from manufacturing. Similarly, manufacturing will have to produce an additional \$(120 + 75) = \$195 to satisfy its own need plus that of agriculture for inputs to produce the "original" \$600 and \$1500.
3. In item 2, above, we found the interindustry needs that resulted from production of \$600 in agriculture and \$1500 in manufacturing. These were \$465 and \$195, respectively. But now we realize that this "extra" production, above the \$600 and \$1500, will also generate interindustry needs – in order to engage in the production of \$465, agriculture will need (0.15)(\$465) = \$69.75 from itself and (0.20)(\$465) = \$93 from manufacturing. Similarly, manufacturing will now additionally need (.025)(\$195) = \$48.75 from agriculture and (0.05)(\$195) = \$9.75 from itself. The total new demands for the two sectors are thus \$(69.75 + 48.75) = \$118.50 and \$(93 + 9.75) = \$102.75.
4. At this point we realize that it is necessary to treat the additional \$118.50 for agriculture and \$102.75 for manufacturing in the same fashion as the \$465 and \$195 in item 3. Hence we find additional required outputs of \$43.46 and \$28.84 from the two sectors.
5. Continuing in this way, we find that eventually the numbers become so small that they can be ignored (less than \$0.005).

Table 2.6 Round-by-Round Impacts (in dollars) of $f_1^1 = \$600$ and $f_2^1 = \$1500$

Round	0	1	2	3	4	5	6	7	8 + 9+ 10 + 11	$\mathbf{Lf}^1$
Sec. 1	600	465.00	118.50	43.46	13.73	4.60	1.50	0.50	0.24	1247.52
Sec. 2	1500	195.00	102.75	28.84	10.13	3.25	1.08	0.35	0.17	1841.58
Cumulative Total										
Sec. 1		1065.00	1183.50	1226.96	1240.64	1245.29				1247.52
Sec. 2		1695.00	1797.75	1826.59	1836.72	1839.97				1841.58
Percent of Total Effect Captured										
Sec. 1		85.40	94.90	98.40	99.50	99.80				1247.52
Sec. 2		92.00	97.60	99.20	99.70	99.90				1841.58

Looking at the total impact of a particular set of final demands this way is described as looking at the "round-by-round" effects. The initial demands generate a need for inputs from the productive sectors; this is the "first round" of effects, as found in item 2, above. But these outputs themselves generate a need for additional inputs – "second round" effects – as found in item 3, above; and so forth. For the present example, these figures have been collected in Table 2.6.

For agriculture, the sum of these round-by-round effects, \$647.53, plus the original demand of \$600, is \$1247.53; for manufacturing, the total is $\$341.57 + \$1500 = \$1841.57$. These total outputs (except for small rounding errors) are the same as those found by using the Leontief inverse, where $x_1^1 = \$1247.52$ and $x_2^1 = \$1841.58$. (It was for this comparison that the two-decimal accuracy was kept in the Leontief-inverse approach to this example.)

In this second view of the numerical example we have developed something of a feeling for the way in which external (final) demands are transmitted through the productive sectors of the economic system. In fact, we see that the elements of $(\mathbf{I}-\mathbf{A})^{-1}$ are really very useful and important numbers. Each captures, in a single *number*, an entire *series* of direct and indirect effects. (The equivalence between Approaches I and II is examined for the general case in Appendix 2.1.)

2.3.3 Numerical Example: Mathematical Observations

The inverse in this small example, $\mathbf{L} = \begin{bmatrix} 1.2541 & .3300 \\ .2640 & 1.1221 \end{bmatrix}$, illustrates a general feature of Leontief inverses for input–output models of any size – the diagonal elements are larger than 1. This is entirely consistent with the economic logic of the round-by-round approach. From (2.13)

$$x_1^1 = (1.2541)(600) + (0.3300)(1500)$$

Looking at the first product on the right, the new final demand of \$600 for agriculture output is multiplied by 1.2541. This can be thought of as (1 + 0.2541)(600). The (1)(600) reflects the fact that the \$600 new agriculture demand must be met by producing \$600 more agriculture output. The additional (0.2541)(600) captures the additional agriculture output required because this output is also used as an input to production activity in both agriculture and also manufacturing. Similarly, from (2.13),

$$x_2^1 = (0.2640)(600) + (1.1221)(1500)$$

and the same logic explains why the coefficient (1.1221) relating manufacturing output to new final demand for manufacturing goods, \$1500, must be greater than 1.

We examine why both of the diagonal elements in **L** will be greater than 1 in the two-sector case. (A more complicated derivation can be used for the general n-sector input–output model, and it is also apparent from the power series discussion in section 2.4.) For this 2 × 2 example, as we saw in section 2.3.1, above,

$$\mathbf{L} = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix} = \frac{1}{|\mathbf{I} - \mathbf{A}|}[\text{adj}(\mathbf{I} - \mathbf{A})]$$
$$= \frac{1}{(1 - a_{11})(1 - a_{22}) - a_{12}a_{21}} \begin{bmatrix} (1 - a_{22}) & a_{12} \\ a_{21} & (1 - a_{11}) \end{bmatrix}$$

So, for example,

$$l_{11} = \frac{(1 - a_{22})}{(1 - a_{22})\left[(1 - a_{11}) - \frac{a_{12}a_{21}}{(1 - a_{22})}\right]} = \frac{1}{1 - \left[a_{11} + \frac{a_{12}a_{21}}{(1 - a_{22})}\right]}$$

Assuming that $(1 - a_{22}) > 0$, $l_{11} > 1$ if the denominator on the right-hand side is less than 1, which it will be when $a_{11} > 0$ and/or $a_{12}a_{21} > 0$ – since $(1 - a_{22}) > 0$. Similar reasoning shows that $l_{22} = (1 - a_{11})/\,|\mathbf{I} - \mathbf{A}| > 1$ under similar reasonable conditions on the a_{ij}.

Whether or not the off-diagonal elements are larger than 1 depends entirely on the sizes of a_{12} and a_{21}, relative to $|\mathbf{I} - \mathbf{A}|$. In most actual input–output tables, with a rather detailed breakdown of sectors, the off-diagonal elements in **L** will be less than 1, as in (2.13). However, for example, if a_{21} in Table 2.4 had been 0.70 instead of 0.20, so that the coefficients matrix had been

$$\mathbf{A} = \begin{bmatrix} .15 & .25 \\ .70 & .05 \end{bmatrix}$$

then

$$\mathbf{L} = \begin{bmatrix} 1.5020 & .3953 \\ 1.1067 & 1.3439 \end{bmatrix}$$

Notice that a coefficient as large as $a_{21} = 0.7$ – which says that there is 70 cents' worth of sector 2 output in a dollar's worth of sector 1 output – is not likely to be seen

Table 2.7 The 2003 US Domestic Direct Requirements Matrix, **A**

	Sector	1	2	3	4	5	6	7
1	Agriculture	.2008	.0000	.0011	.0338	.0001	.0018	.0009
2	Mining	.0010	.0658	.0035	.0219	.0151	.0001	.0026
3	Construction	.0034	.0002	.0012	.0021	.0035	.0071	.0214
4	Manufacturing	.1247	.0684	.1801	.2319	.0339	.0414	.0726
5	Trade, Transportation & Utilities	.0855	.0529	.0914	.0952	.0645	.0315	.0528
6	Services	.0897	.1668	.1332	.1255	.1647	.2712	.1873
7	Other	.0093	.0129	.0095	.0197	.0190	.0184	.0228

Table 2.8 The 2003 US Domestic Total Requirements Matrix, $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1}$

	Sector	1	2	3	4	5	6	7
1	Agriculture	1.2616	.0058	.0131	.0576	.0037	.0069	.0072
2	Mining	.0093	1.0748	.0122	.0343	.0193	.0033	.0073
3	Construction	.0075	.0034	1.0047	.0064	.0065	.0111	.0250
4	Manufacturing	.2292	.1192	.2615	1.3419	.0692	.0856	.1261
5	Trade, Transportation & Utilities	.1493	.0850	.1371	.1563	1.0887	.0598	.0853
6	Services	.2383	.2931	.2700	.2918	.2712	1.4116	.3138
7	Other	.0243	.0239	.0231	.0367	.0280	.0297	1.0338

often in real tables. The sizes of the between-sector technical coefficients, a_{ij} $(i \neq j)$, and of the off-diagonal elements in **L**, are related to the level of sectoral detail (that is, the number of sectors) in the model. We will return to this topic in Chapter 4, when we consider the effects of aggregating (combining) sectors in an input–output model. (In Appendix 2.2 we examine the conditions under which a Leontief inverse matrix will always contain only non-negative elements, as logic suggests should always be the case.)

2.3.4 Numerical Example: The US 2003 Data

We present a highly aggregated, seven-sector version of the 2003 US input–output coefficients matrix and its associated Leontief inverse in Tables 2.7 and 2.8. (Appendix B contains a series of such tables over time for the US economy at the seven-sector level of aggregation.) It is important to note that these data for the US represent *domestically produced* inputs; this requires explanation.

Imports are generally divided into two categories: "competitive" and "non-competitive" imports (or "competing" and "non-competing").

Competitive imports are goods that have a domestic counterpart (that is, are also produced in the USA). For example, grapes from Chile that are used to make grape jelly in the USA, where domestically grown grapes are also used in grape jelly recipes.

Non-competitive imports have no domestic counterpart. For example, coffee beans from Brazil used by US coffee roasting firms (coffee beans are not grown in the USA).

Some national tables (the USA is one example) show competitive imports within the transactions table, so that sales of grapes to jelly producers include both domestic and foreign sources. This correctly reflects the total amount of grapes needed by domestic producers. However, it causes problems when input–output models are used for impact analysis. Briefly put, this is because an analyst is usually interested in the economic consequences *on the domestic* (or regional or local) *economy* of an exogenous demand change. With Chilean grapes in a transactions matrix, and hence in the associated **A** and **L** matrices, some of the demand repercussions measured by the model would in fact be felt by Chilean grape growers. For this reason, we present here US data based on a *domestic* transactions matrix ($\mathbf{Z}^D$) in which the transactions matrix ($\mathbf{Z}$) has been purged of "competitive" (or "competing") imports. In matrix terms, $\mathbf{Z}^D = \mathbf{Z} - \mathbf{M}$, where **M** is a matrix of competitive imports. This "scrubbing" of the matrix is not always easy to do if the data are lumped together in a published **Z** table (as is the case in the USA), but it is very important when the question is one of impacts of final demand changes on the domestic economy (and this is usually the question of interest).[7]

Spending on non-competitive imports usually appears in a row in the payments sector (a single value indicating a sector's payments for all non-competitive imports). We return to these issues in Chapter 4.

The effects on US output of various final-demand vectors can be easily quantified using **L** in Table 2.8. For example, suppose that there were increased foreign demand (the export component of the final-demand vector) for agricultural and manufactured items of \$1.2 million and \$6.8 million, respectively. Here (in millions of dollars)

$$\Delta\mathbf{f} = \begin{bmatrix} 1.2 \\ 0 \\ 0 \\ 6.8 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

[7] By contrast, if one is interested in the structure of production ("production recipes") and if or how they have changed over time (structural analysis), it may be more useful to have competitive imports included in the **Z** matrix and hence reflected in **A** and **L**, since such imports are certainly part of those recipes.

and, using (2.14), we find from $\mathbf{L}$ in Table 2.8 that (in millions of dollars)

$$\Delta\mathbf{x} = \begin{bmatrix} 1.9114 \\ 0.2444 \\ 0.0526 \\ 9.1249 \\ 1.2421 \\ 2.2709 \\ 0.2788 \end{bmatrix}$$

As might be expected, the greatest effect, \$9.125 million, is felt in the manufacturing sector. The next-greatest effect, \$2.271 million, is felt in services. Also, agriculture output would increase by \$1.911 million and trade, transportation and utilities would increase by \$1.242 million. Effects on the remaining three sectors are less than \$1 million. The total new output effect throughout the country, obtained by summing the elements in $\Delta\mathbf{x}$, is \$15.125 million; this is generated by a total new exogenous demand of \$8 million. This again illustrates the multiplicative effect in an economy of an exogenous stimulus via an increase in one or more components of final demand. These multiplier effects will be discussed in further detail in Chapter 6.

2.4 The Power Series Approximation of $(\mathbf{I}-\mathbf{A})^{-1}$

In preparing input–output tables for many real-world applications of the model, in which one wants to maintain a reasonable distinction between sectors (e.g., so that sectors producing aluminum storm windows and women's apparel are not lumped together as a single sector labeled "manufacturing"), tables with hundreds of sectors are not unusual. However, early in the history of input–output studies, computer speed and capacity posed real problems for implementation of input–output models – inversion of large matrices was simply not possible.[8] The amount of computer capacity and time needed to invert, say, a 150×150 $(\mathbf{I}-\mathbf{A})$ matrix will vary with the particular computer and the inversion program that is used, and it is quite possible that in some cases the number of sectors that can be accommodated may be limited. One approach is then to aggregate the data into a smaller number of sectors. We will say more about such sectoral aggregation later, but clearly industrial (sectoral) detail is lost in the process. In addition, the inversion calculations themselves can be carried out sequentially on a series of smaller submatrices of $(\mathbf{I}-\mathbf{A})$.[9] However, there is a useful matrix algebra result generally applicable to $(\mathbf{I}-\mathbf{A})$ matrices that makes possible an approximation to $(\mathbf{I}-\mathbf{A})^{-1}$ requiring no inverses at all; moreover, this approximation procedure has a useful economic interpretation.

[8] In 1939 it reportedly took 56 hours to invert a 42-sector table (on Harvard's Mark II computer; see Leontief, 1951a, p. 20). In 1947, 48 hours were needed to invert a 38-sector input–output matrix. However, by 1953 the same operation took only 45 minutes. (Morgenstern, 1954, p. 496; also, see Lahr and Stevens, 2002, p. 478.) By 1969 a 100-sector matrix could be inverted in between 10 and 36 seconds, depending on the computer used. (Polenske, 1980, p. 15.)

[9] This is possible using a partitioned matrix approach; the details need not concern us at this point.

By definition, we know that $\mathbf{A}$ is a non-negative matrix with $a_{ij} \geq 0$ for all i and j. (This characteristic is often written as $\mathbf{A} \geq \mathbf{0}$, where it is understood that not all $a_{ij} = 0$.)[10] The sum of the elements in the jth column of $\mathbf{A}$ indicates the dollars' worth of inputs from other sectors that are used in making a dollar's worth of output of sector j. In an open model, given the economically reasonable assumption that each sector uses some inputs from the payments sector (labor, other value added, etc.), then each of these column sums will be less than one ($\sum_{i=1}^{n} a_{ij} < 1$ for all j). (We will see below, in section 2.6, that this column sum condition need not apply to tables based on physical, not monetary, measures of transactions and outputs.) For input–output coefficients matrices with these two characteristics – $a_{ij} \geq 0$ and $\sum_{i=1}^{n} a_{ij} < 1$ for all j – it is possible to approximate the gross output vector $\mathbf{x}$ associated with any final demand vector $\mathbf{f}$ without finding $(\mathbf{I} - \mathbf{A})^{-1}$.

Consider the matrix product

$$(\mathbf{I} - \mathbf{A})(\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \cdots + \mathbf{A}^n)$$

where, for square matrices, $\mathbf{A}^2$ denotes $\mathbf{AA}$, $\mathbf{A}^3 = \mathbf{AAA} = \mathbf{AA}^2$, and so on. Premultiplication of the series in parentheses by $(\mathbf{I} - \mathbf{A})$ can be accomplished by first multiplying all terms in the right-hand parentheses by $\mathbf{I}$ and then multiplying all terms by $(-\mathbf{A})$. This leaves only $(\mathbf{I} - \mathbf{A}^{n+1})$; all other terms cancel – for $\mathbf{A}^2$ there is a $-\mathbf{A}^2$, for $\mathbf{A}^3$ there is a $-\mathbf{A}^3$, and so on. Thus

$$(\mathbf{I} - \mathbf{A})(\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \cdots + \mathbf{A}^n) = (\mathbf{I} - \mathbf{A}^{n+1}) \tag{2.15}$$

If it were true that for large n (more formally, as $n \rightarrow \infty$), the elements in $\mathbf{A}^{n+1}$ all become zero, or close to zero (i.e., $\mathbf{A}^{n+1} \rightarrow \mathbf{0}$), then the right-hand side of (2.15) would be simply $\mathbf{I}$, and the matrix series that postmultiplies $(\mathbf{I} - \mathbf{A})$ in (2.15) would constitute the inverse to $(\mathbf{I} - \mathbf{A})$, from the fundamental defining property of an inverse.

For any matrix, $\mathbf{M}$, if we sum the absolute values of the elements in each column, the largest sum is called the norm of $\mathbf{M}$ – denoted $N(\mathbf{M})$ or $\|\mathbf{M}\|$.[11] For example, for the coefficients matrix $\mathbf{A}$ given in Table 2.4, $N(\mathbf{A}) = 0.35$, the sum of the elements in the first column. (The sum of the elements in column 2 is 0.30.) For a pair of matrices, $\mathbf{A}$ and $\mathbf{B}$, that are conformable for the multiplication $\mathbf{AB}$, there is a theorem stating that the product of the norms of $\mathbf{A}$ and $\mathbf{B}$ is no smaller than the norm of the product $\mathbf{AB}$ – $N(\mathbf{A})N(\mathbf{B}) \geq N(\mathbf{AB})$. By replacing $\mathbf{B}$ with $\mathbf{A}$, it follows that $N(\mathbf{A})N(\mathbf{A}) \geq N(\mathbf{A}^2)$

[10] A more exact characterization of vectors and matrices is often needed for more advanced matrix algebra results. See section A.9 in Appendix A, where $\mathbf{A} > \mathbf{0}$ is used for the case when $\mathbf{A} \geq \mathbf{0}$ and $\mathbf{A} \neq \mathbf{0}$.

[11] A *norm* is just a measure of the general size of the elements in a matrix. (A measure of the size of the matrix itself is given by the dimensions of the matrix.) For example, a non-negative $m \times n$ matrix that has all elements smaller than 0.1 will have a smaller norm than one that has all elements larger than 10. There are many possible definitions of the norm of a matrix. The one used here (maximum column sum of absolute values) is one of the simplest.

or $[N(\mathbf{A})]^2 \geq N(\mathbf{A}^2)$ and finally, continuing similarly,

$$[N(\mathbf{A})]^n \geq N(\mathbf{A}^n) \tag{2.16}$$

As was noted above, all column sums of an open and "reasonable" value-based $\mathbf{A}$ matrix will be less than one, so we know that $N(\mathbf{A}) < 1$. Moreover, since $a_{ij} \geq 0$, we also know that $a_{ij} \leq N(\mathbf{A})$; no element in a non-negative matrix can be larger than the largest column sum. Thus: (1) since $N(\mathbf{A}) < 1$, $[N(\mathbf{A})]^n \to 0$ as $n \to \infty$; (2) from (2.16), this means that $N(\mathbf{A}^n) \to 0$ also as $n \to \infty$; (3) finally, then, all elements in $\mathbf{A}^n$ must approach zero, since no single element in a non-negative matrix can be larger than the norm of that matrix. This is the result that we are interested in. The right-hand side of (2.15) becomes simply $\mathbf{I}$ as n gets large and so

$$\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1} = (\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \cdots) \tag{2.17}$$

[This is analogous to the series result in ordinary algebra that $1/(1-a) = 1 + a + a^2 + a^3 + \cdots$, for $|a| < 1$.] Notice that the terms on the right-hand side of (2.17) are all positive. Even if some a_{ij} are zero, the increasing number of products of $\mathbf{A}$ virtually guarantees that no zeros will be in evidence at the end of the summation.[12] This means that $\mathbf{L}$ will contain only positive elements. (Appendix 2.2 looks into the issue of positivity of $\mathbf{L}$ in more detail.)

Then $\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{f}$ can be found as

$$\mathbf{x} = (\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \cdots)\mathbf{f} \tag{2.18}$$

Removing parentheses, this is

$$\mathbf{x} = \mathbf{f} + \mathbf{Af} + \mathbf{A}^2\mathbf{f} + \mathbf{A}^3\mathbf{f} + \cdots = \mathbf{f} + \mathbf{Af} + \mathbf{A}(\mathbf{Af}) + \mathbf{A}(\mathbf{A}^2\mathbf{f}) + \cdots \tag{2.19}$$

Each term after the first can be found as the preceding term premultiplied by $\mathbf{A}$. In many applications it has been found that after about $\mathbf{A}^7$ or $\mathbf{A}^8$, the terms multiplying $\mathbf{f}$ become insignificantly different from zero. Even with modern-day computer capacity and speed, there still may be times when the approximation in (2.18) or (2.19) may prove useful (for example, since matrix multiplications are much more straightforward than inversion, especially of a large matrix).[13]

Returning to the original $\mathbf{A}$ matrix and the $\mathbf{f}$ vector of the example in section 2.3 (and dropping the "0" superscripts for simplicity), where $\mathbf{A} = \begin{bmatrix} .15 & .25 \\ .20 & .05 \end{bmatrix}$ and

[12] As mentioned, the elements in any particular $\mathbf{A}^k$ *do* approach zero – which is the whole point.

[13] Alternatively, some analysts have used the power series approximation as a framework for introducing "dynamic" concepts into input–output models. We explore these ideas briefly in section 13.4.7.

$\mathbf{f} = \begin{bmatrix} 600 \\ 1500 \end{bmatrix}$, we have

$$\mathbf{If} = \begin{bmatrix} 600 \\ 1500 \end{bmatrix}$$

$$\mathbf{Af} = \begin{bmatrix} .15 & .25 \\ .20 & .05 \end{bmatrix} \begin{bmatrix} 600 \\ 1500 \end{bmatrix} = \begin{bmatrix} 465 \\ 195 \end{bmatrix}$$

$$\mathbf{A}^2\mathbf{f} = \begin{bmatrix} .0725 & .0500 \\ .0400 & .0525 \end{bmatrix} \begin{bmatrix} 600 \\ 1500 \end{bmatrix} = \begin{bmatrix} 118.50 \\ 102.75 \end{bmatrix}$$

$$\mathbf{A}^3\mathbf{f} = \begin{bmatrix} .0209 & .0206 \\ .0165 & .0126 \end{bmatrix} \begin{bmatrix} 600 \\ 1500 \end{bmatrix} = \begin{bmatrix} 43.44 \\ 28.80 \end{bmatrix}$$

$$\mathbf{A}^4\mathbf{f} = \begin{bmatrix} .0073 & .0063 \\ .0050 & .0048 \end{bmatrix} \begin{bmatrix} 600 \\ 1500 \end{bmatrix} = \begin{bmatrix} 13.83 \\ 10.20 \end{bmatrix}$$

$$\mathbf{A}^5\mathbf{f} = \begin{bmatrix} .0024 & .0021 \\ .0017 & .0015 \end{bmatrix} \begin{bmatrix} 600 \\ 1500 \end{bmatrix} = \begin{bmatrix} 4.59 \\ 3.27 \end{bmatrix}$$

$$\mathbf{A}^6\mathbf{f} = \begin{bmatrix} .0008 & .0007 \\ .0006 & .0005 \end{bmatrix} \begin{bmatrix} 600 \\ 1500 \end{bmatrix} = \begin{bmatrix} 1.53 \\ 1.11 \end{bmatrix}$$

$$\mathbf{A}^7\mathbf{f} = \begin{bmatrix} .0003 & .0002 \\ .0002 & .0002 \end{bmatrix} \begin{bmatrix} 600 \\ 1500 \end{bmatrix} = \begin{bmatrix} 0.48 \\ 0.42 \end{bmatrix}$$

We see that the individual terms in the power series approximation (except for rounding errors) simply represent the magnitudes of the round-by-round effects, as recorded in Table 2.6. (The reader should reconsider the algebra of the round-by-round calculations to be convinced that, in fact, they were equivalent to premultiplication of $\mathbf{f}$ by a series of powers of the $\mathbf{A}$ matrix.) Thus it is possible that one may capture "most" of the effects associated with a given final demand by using the first few terms in the power series. As illustrated in Table 2.6, for our small example more than 98 percent of the total effects in both sectors was captured in three rounds.

2.5 Open Models and Closed Models

The model that we have dealt with thus far, $\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{f}$, depends on the existence of an exogenous sector, disconnected from the technologically interrelated productive sectors, since it is here that the important final demands for outputs originate. The basic kinds of transactions that constitute the activity of this sector, as we have seen, are consumption purchases by households, sales to government, gross private domestic investment, and shipments in foreign trade (either gross exports or net exports – exports from a sector less the value of imports of the same goods). In the case of households, especially, this "exogenous" categorization is something of a strain on basic economic theory. Households (consumers) earn incomes (at least in part) in payment for their

Table 2.9 Input–Output Table of Interindustry Flows with Households Endogenous

		Buying Sector					
		1	$\cdots$	j	$\cdots$	n	*Households (Consumers)*
Selling Sector	1	z_{11}	$\cdots$	z_{1j}	$\cdots$	z_{1n}	$z_{1,n+1}$
	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
	i	z_{i1}	$\cdots$	z_{ij}	$\cdots$	z_{in}	$z_{i,n+1}$
	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
	n	z_{n1}	$\cdots$	z_{nj}	$\cdots$	z_{nn}	$z_{n,n+1}$
	Households (Labor)	$z_{n+1,1}$	$\cdots$	$z_{n+1,j}$	$\cdots$	$z_{n+1,n}$	$z_{n+1,n+1}$

labor inputs to production processes, and, as consumers, they spend their income in rather well patterned ways. And in particular, a *change* in the amount of labor needed for production in one or more sectors – say an increase in labor inputs due to increased output – will lead to a change (here an increase) in the amounts spent by households as a group for consumption. Although households tend to purchase goods for "final" consumption, the amount of their purchases is related to their income, which depends on the outputs of each of the sectors. Also, as we have seen, consumption expenditures constitute possibly the largest single element of final demand; at least in the US economy they have frequently constituted more than two-thirds of the total final-demand figure.

Thus one could move the household sector from the final-demand column and labor-input row and place it inside the technically interrelated table, making it one of the *endogenous* sectors. This is known as closing the model with respect to households. Input–output models can be "closed" with respect to other exogenous sectors as well (for example, government sales and purchases); however, closure with respect to households is more usual. It requires a row and a column of transactions for the new household sector – the former showing the distribution of its output (labor services) among the various sectors and the latter showing the structure of its purchases (consumption) distributed among the sectors. It is customary to add the household row and column at the bottom and to the right of the transactions and coefficients tables. Dollar flows *to* consumers, representing wages and salaries received by households from the n sectors in payment for their labor services, would fill an $(n+1)$st row – $[z_{n+1,1}, \ldots, z_{n+1,n}]$. Dollar flows *from* consumers, representing the values of household purchases of the goods of the n sectors, would fill an $(n+1)$st column: $\begin{bmatrix} z_{1,n+1} \\ \vdots \\ z_{n,n+1} \end{bmatrix}$. Finally, the element in the $(n+1)$st row and the $(n+1)$st column, $z_{n+1,n+1}$, would represent household purchases of labor services. Thus Table 2.1 would have one new row, at the bottom, and one new column, at the right, as indicated in Table 2.9.

The ith equation, as shown in (2.1), would now be modified to

$$x_i = z_{i1} + \cdots + z_{ij} + \cdots + z_{in} + z_{i,n+1} + f_i^* \tag{2.20}$$

where f^* is understood to represent the remaining final demand for sector i output – exclusive of that from households, which is now captured in $z_{i,n+1}$. In addition to this kind of modification on each of the equations in set (2.2), there would be one new equation for the total "output" of the household sector, defined to be the total value of its sale of labor services to the various sectors – total earnings. Thus

$$x_{n+1} = z_{n+1,1} + \cdots + z_{n+1,j} + \cdots + z_{n+1,n} + z_{n+1,n+1} + f_{n+1}^* \tag{2.21}$$

The last term on the right in (2.21) would include, for example, payments to government employees.

Household input coefficients are found in the same manner as any other element in an input–output coefficients table: The value of sector j purchases of labor (for a given period), $z_{n+1,j}$, divided by the value of total output of sector j (for the same period), x_j, gives the value of household services (labor) used per dollar's worth of j's output; $a_{n+1,j} = z_{n+1,j}/x_j$. For the elements of the household purchases (consumption) column, the value of sector i sales to households (for a given period), $z_{i,n+1}$, is divided by the total output (measured by income earned) of the household sector, x_{n+1}. Thus, household "consumption coefficients" are $a_{i,n+1} = z_{i,n+1}/x_{n+1}$. A drawback to this approach is that now household behavior is "frozen" in the model in the same way as producer behavior (constant coefficients).

The ith equation in the fundamental set given in (2.6), above, becomes

$$x_i = a_{i1}x_1 + \cdots + a_{in}x_n + a_{i,n+1}x_{n+1} + f_i^* \tag{2.22}$$

and the added equation which relates household output to output of all of the sectors is

$$x_{n+1} = a_{n+1,1}x_1 + \cdots + a_{n+1,n}x_n + a_{n+1,n+1}x_{n+1} + f_{n+1}^* \tag{2.23}$$

Similarly, parallel to the equations in (2.7), we now have, rewriting (2.22) for the ith equation,

$$-a_{i1}x_1 - \cdots + (1 - a_{ii})x_i - \cdots - a_{in}x_n - a_{i,n+1}x_{n+1} = f_i^*$$

And, for the household equation, rewriting (2.23),

$$-a_{n+1,1}x_1 - \cdots - a_{n+1,n}x_n + (1 - a_{n+1,n+1})x_{n+1} = f_{n+1}^*$$

Let the row vector of labor input coefficients, $a_{n+1,j} = z_{n+1,j}/x_j$, be denoted by $\mathbf{h}_R = [a_{n+1,1}, \ldots, a_{n+1,n}]$, the column vector of household consumption coefficients, $a_{i,n+1} = z_{i,n+1}/x_{n+1}$, be $\mathbf{h}_C = \begin{bmatrix} a_{1,n+1} \\ \vdots \\ a_{n,n+1} \end{bmatrix}$ and let $h = a_{n+1,n+1}$.[14] Denote by $\bar{\mathbf{A}}$

[14] In the initial numerical illustration in section 2.3.1, above, for simplicity we used $\mathbf{e}'_c$ for the vector of employment coefficients. These are seen to be the elements in $\mathbf{h}_R$, which is the notation frequently used in closed models. Strictly speaking, we should use a "prime" to denote a row vector, but the subscript "R" reminds us that this is a *row* of coefficients.

the $(n+1)\times(n+1)$ technical coefficients matrix with households included. Using partitioning to separate the old **A** matrix from the new sector,

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{h}_C \\ \mathbf{h}_R & h \end{bmatrix}$$

Let $\bar{\mathbf{x}}$ denote the $(n+1)$-element column vector of gross outputs

$$\bar{\mathbf{x}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{x} \\ x_{n+1} \end{bmatrix}$$

Also, let $\mathbf{f}^*$ be the n-element vector of remaining final demands for output of the original n sectors and $\bar{\mathbf{f}}$ the $(n+1)$-element vector of final demands, including that for the output of households

$$\bar{\mathbf{f}} = \begin{bmatrix} f_1^* \\ \vdots \\ f_n^* \\ f_{n+1}^* \end{bmatrix} = \begin{bmatrix} \mathbf{f}^* \\ f_{n+1}^* \end{bmatrix}$$

Then the new system of $n+1$ equations, with households endogenous, can be represented as

$$(\mathbf{I} - \bar{\mathbf{A}})\bar{\mathbf{x}} = \bar{\mathbf{f}} \tag{2.24}$$

or

$$\begin{bmatrix} \mathbf{I} - \mathbf{A} & -\mathbf{h}_C \\ -\mathbf{h}_R & (1-h) \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{f}^* \\ f_{n+1}^* \end{bmatrix} \tag{2.25}$$

That is, we have the set of n equations

$$(\mathbf{I} - \mathbf{A})\mathbf{x} - \mathbf{h}_C x_{n+1} = \mathbf{f}^*$$

[a matrix rearrangement of (2.22)] and the added one for households

$$-\mathbf{h}_R\mathbf{x} + (1-h)x_{n+1} = f_{n+1}^*$$

[a matrix rearrangement of (2.23)]. Together these determine the values of outputs for the n original sectors – $x_1, \ldots, x_n$ – and the value of household services used (wages paid) to produce those outputs – x_{n+1}. If the $(n+1)\times(n+1)$ coefficients matrix is nonsingular, the unique solution can be found using an inverse matrix in the usual way:

$$\begin{bmatrix} \mathbf{x} \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{I} - \mathbf{A} & -\mathbf{h}_C \\ -\mathbf{h}_R & (1-h) \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{f}^* \\ f_{n+1}^* \end{bmatrix} \tag{2.26}$$

Table 2.10 Flows (z_{ij}) for Hypothetical Example, with Households Endogenous

From \ To	1	2	Household Consumption (C)	Other Final Demand (f^*)	Total Output ($\mathbf{x}$)
1	150	500	50	300	1000
2	200	100	400	1300	2000
Labor Services (L)	300	500	50	150	1000
Other Domestic Payments (N)	325	800	300	250	1675
Imports (M)	25	100	200	150	475
Total Outlays ($\mathbf{x}'$)	1000	2000	1000	2150	6150

or

$$\bar{\mathbf{x}} = (\mathbf{I} - \bar{\mathbf{A}})^{-1}\bar{\mathbf{f}} = \bar{\mathbf{L}}\bar{\mathbf{f}}$$

Consider again the information given in Table 2.3. Suppose that the household consumption part of final demand and the household labor input part of the payments sector are as shown in Table 2.10. Of the \$650 bought by sector 1 from the payments sectors (Table 2.3), \$300 was for labor services; of the \$1400 bought by sector 2, \$500 was for labor inputs. Also, of the \$1100 which represented purchases of final-demand sectors from the payments sectors, \$50 was paid out by households for labor services (e.g., domestic help); government purchases of labor was \$150. The \$300 would record household payments to government (taxes), and so forth.

The total output of the household sector, as in (2.16), is (here $n+1=3$), $x_3 = z_{31} + z_{32} + z_{33} + f_3^* = 300 + 500 + 50 + 150 = 1000$. The household input coefficients, $a_{n+1,j} = z_{n+1,j}/x_j$, are: $a_{31} = 300/1000 = 0.3$, $a_{32} = 500/2000 = 0.25$ and $a_{33} = 50/1000 = 0.05$; $\mathbf{h}_R = \begin{bmatrix} 0.3 & 0.25 \end{bmatrix}$ and $h = 0.05$. Similarly, household consumption coefficients, $a_{i,n+1} = z_{i,n+1}/x_{n+1}$ are $a_{13} = 50/1000 = 0.05$ and $a_{23} = 400/1000 = 0.4$; thus $\mathbf{h}_C = \begin{bmatrix} 0.05 \\ 0.4 \end{bmatrix}$. Therefore,

$$\bar{\mathbf{A}} = \begin{bmatrix} .15 & .25 & .05 \\ .2 & .05 & .4 \\ .3 & .25 & .05 \end{bmatrix}, \quad (\mathbf{I} - \bar{\mathbf{A}}) = \begin{bmatrix} .85 & -.25 & -.05 \\ -.2 & .95 & -.4 \\ -.3 & -.25 & .95 \end{bmatrix}$$

and

$$\bar{\mathbf{L}} = (\mathbf{I} - \bar{\mathbf{A}})^{-1} = \begin{bmatrix} 1.3651 & .4253 & .2509 \\ .5273 & 1.3481 & .5954 \\ .5698 & .4890 & 1.2885 \end{bmatrix} \tag{2.27}$$

Consider again the numerical example in section 2.3 (again we ignore the "0" and "1" superscripts for simplicity). There we assumed a change in the final-demand vector such that f_1 went from 350 to 600 and f_2 from 1700 to 1500. Referring now to Table 2.10, simply for illustration, suppose that this entire final-demand change was concentrated in the Other Final Demand sector. In fact, let it represent a change in the demands of the federal government [which are a part of the Other Final Demand column (f_i^*) in Table 2.10]. These new demands of \$600 and \$1500 represent increases in both cases, from the current levels of \$300 and \$1300 for all nonhousehold final-demand categories.

The most straightforward comparison is now to use the 3×3 Leontief inverse $(\mathbf{I} - \bar{\mathbf{A}})^{-1}$ in (2.27) in conjunction with $\bar{\mathbf{f}} = \begin{bmatrix} 600 \\ 1500 \\ 0 \end{bmatrix}$ to find the impact of these changes in the final demands for the outputs of sectors 1 and 2 on the two original sectors plus the added impact due to closure of the model with respect to households. We have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \bar{\mathbf{x}} = \begin{bmatrix} 1.3651 & .4253 & .2509 \\ .5273 & 1.3481 & .5954 \\ .5698 & .4890 & 1.2885 \end{bmatrix} \begin{bmatrix} 600 \\ 1500 \\ 0 \end{bmatrix} = \begin{bmatrix} 1456.94 \\ 2338.51 \\ 1075.48 \end{bmatrix}$$

In the earlier example of section 2.3, with households exogenous to the model, the new outputs were $x_1 = \$1247.46$ and $x_2 = \$1841.55$. The new (larger) values – \$1456.94 and \$2338.51, respectively – reflect the fact that *additional* outputs are necessary to satisfy the anticipated increase in consumer spending, as reflected in the household consumption coefficients column, expected because of the increased household earnings due to increased outputs from sectors 1 and 2 and hence increased wage payments. Using the labor input coefficients $a_{31} = 0.3$ and $a_{32} = 0.25$, the necessary household inputs for the original gross outputs (when households were exogenous) would be

$$a_{31}x_1 + a_{32}x_2 = (0.3)\,(1247.46) + (0.25)(1841.55) = 834.63$$

As would be expected, outputs are increased for all three sectors, due to the introduction of the formerly exogenous household sector into the model. The example serves to illustrate an expected outcome – namely that when the added impact of more household consumption spending due to increased wage income is explicitly taken into consideration in the model, the outputs of the original sectors in the interindustry model (here sectors 1 and 2) are larger than is the case when consumer spending is ignored.

In this section we have introduced the basic considerations involved in moving households from final demand into the model as an endogenous sector – closing the model with respect to households. Similar kinds of data and algebraic extensions would be needed if other exogenous sectors – for example, federal, or state and/or local government activities – were to be made endogenous in the model. However, because the value of consumption tends to be the largest component of final demand and because of the relatively direct linkage between earned income and consumption and between consumption and output, the household sector is the one final-demand sector that is most often moved inside the model.

In practice, however, the issue is far more subtle, and the procedure is more complicated than might be suggested by the discussion in this section. All of the previous reservations about the a_{ij} apply here as well, if not with greater force. For *each* additional dollar of received earnings, households are assumed to spend 5 cents on the output of sector 1, 40 cents on the output of sector 2, and so on. Those coefficients, which reflect *average* behavior during the observation period when household income was \$1000 ($a_{13} = 50/1000$ and $a_{23} = 400/1000$), are assumed to hold for the additional, or *marginal*, amounts of household earnings associated with the new outputs of sectors 1 and 2. One approach, particularly at the regional level, is to divide consumers into two groups: established residents, for whom the new income associated with new production would represent an addition to current earnings, and new residents (in-migrants), who may have moved in search of employment and for whom the new income represents total earnings. For the former group, a set of marginal consumption coefficients might be appropriate, while for the latter group average consumption coefficients would be relevant.

In addition, spending patterns of consumers, especially out of additions to (or reductions in) disposable income, will depend on the income category in which a particular consumer is located. An addition of \$100 to the spendable income of a worker earning \$20,000 per year is likely to be spent differently than an additional \$100 in the hands of an engineer with an annual income of \$150,000, and both will no doubt differ from the way in which the \$100 would be spent by a previously unemployed person. In effect, this is simply noting that inputs to the household sector (consumption) per dollar of output (household income) will not be independent of the level of that output. Yet such independence is assumed in the way that the direct input coefficients are used in an input–output model; each sector's production function (column of direct input coefficients) is assumed to represent inputs per dollar's worth of output, regardless of the amount (level) of that output.

Another approach, then, is to disaggregate "the" household sector into several sectors, distinguished by total income. For example, \$0–\$10,000, \$10,001–\$20,000, \$20,001–\$30,000; and so on. Consumption coefficients, by sector, could then be derived for each income class. We will return to this issue in a regional context in Chapter 3 and in Chapter 10 when examining social accounting matrices. A very thorough discussion of an approach for incorporating a disaggregated household sector into the endogenous part of an input–output model, using a good deal of matrix algebra, can be found in Miyazawa (1976). We explore that model in more detail in Chapter 6.

Further disaggregations of the household sector have been proposed and incorporated in input–output analysis. These frameworks fall into the category of what are known as "extended" input–output models. (For a concise overview see Batey, Madden and Weeks, 1987 or Batey and Weeks, 1989.) The idea is to separate income payments to and consumption patterns of different household groups – for example, established vs. new residents (noted above) and currently employed vs. unemployed.

One could imagine a process of moving, one by one, each of the remaining sectors from the final-demand vector into the interindustry coefficients matrix, constructing rows of input coefficients and columns of purchase coefficients until there were no exogenous sectors at all. This is termed a *completely closed model*. However, the economic logic behind fixed coefficients in the case, say, of a government sector is less easy to accept than for the productive sectors, and completely closed models are seldom implemented in practice.[15]

2.6 The Price Model

2.6.1 Overview

Leontief originally developed the input–output model in physical units (bushels of wheat, yards of cloth, man-years of labor, etc.).[16] In particular, he assumed that direct input coefficients, **A**, are based on *physical quantities* of inputs divided by *physical quantities* of output. These data were then converted to a table of (base year) transactions in value terms by using (base year) unit prices – for a bushel of wheat, a yard of cloth and a man-year of labor. He writes (Leontief, 1986, pp. 22–23):

> All figures [in the *value* transactions table]…can also be interpreted as representing *physical quantities* of the goods or services to which they refer. This only requires that the physical unit in which the entries…are measured be redefined as being equal to that amount of output of that particular sector that can be purchased for $1 at [base year] prices… In practice the structural matrices are usually computed from input-output tables described in value terms…In any case, the input coefficients [**A**] – for analytical purposes…must be interpreted as ratios of two quantities measured in *physical units* [emphasis added].

As already noted, input–output data are usually assembled and input–output studies are generally carried out in monetary (value) units.

However, with the emergence of energy and environmental concerns, mixed-units models have been developed, where economic transactions are recorded in monetary terms and ecological and/or energy transactions are recorded in physical terms (tons, BTUs, joules, etc.).[17] Another line of inquiry has led to input–output tables in common physical units (e.g., all transactions and outputs measured in tons). Stahmer (2000) gives an overview of this kind of work, including tables for Germany for 1990 in both monetary and physical units – sometimes designated MIOTs and PIOTs, respectively. (There are problems in trying to measure outputs of services in physical units.)[18] We explore a small illustration in section 2.6.8, below, using an aggregation of the German data.

[15] The original work done by Leontief, however, was in the framework of a completely closed model of the United States for 1919. See Leontief (1951b).
[16] See, for example, Leontief (1951a, 1951b, 1986), Leontief *et al.* (1953).
[17] These issues are explored further in Chapters 9 and 10.
[18] Stahmer (2000) also introduces the notion of data measured in time units, leading to TIOTs.

Table 2.11 Transactions in Physical Units

	1	2	d_i	q_i	Physical units of measure
1	75	250	175	500	bushels
2	40	20	340	400	tons

Table 2.12 Transactions in Monetary Units (see Table 2.3)

	1	2	f_i	x_i	\$ Price per physical unit
1	150	500	350	1000	2
2	200	100	1700	2000	5

Table 2.13 Transactions in Revised Physical Units

	1	2	d_i	q_i	Revised physical units of measure
1	150	500	350	1000	1/2 bushels
2	200	100	1700	2000	1/5 tons

2.6.2 Physical vs. Monetary Transactions

We return to the illustration in section 2.3. Suppose the *physical* unit measures for outputs are bushels for sector 1 (agriculture) and tons for sector 2 (manufacturing) and that transactions measured in these physical units are shown in Table 2.11, where we now use d_i for physical amounts delivered to final demand and q_i for physical amounts of total output.

If we know the per-unit prices of the two products, the information in Table 2.11 can be converted to monetary units. For example, if the price per bushel is \$2.00 and the price per ton is \$5.00, then the *monetary* transactions table is exactly as shown in Table 2.3. Now, redefine the physical units of measurement for each sector to be the amount that can be bought for \$1.00; that is, so that the per-unit price for each sector's output is \$1.00. This simply means that we measure the physical output of sector 1 in *half bushel* units and the physical output of sector 2 in *fifths of a ton*. Then, in these revised units, the information in Table 2.12 can be reinterpreted as recording transactions in physical units, as in Table 2.13 – for example, 500 half-bushels of sector 1 output were bought by sector 2 (for \$500), 2000 fifths of a ton of sector 2 output were delivered to final demand (for \$2000), etc.

Table 2.14 Transactions in Monetary Terms

	Sectors						
Sectors	1	$\cdots$	j	$\cdots$	n	Final Demand	Total Output
1	z_{11}	$\cdots$	z_{1j}	$\cdots$	z_{1n}	f_1	x_1
2	z_{21}	$\cdots$	z_{2j}	$\cdots$	z_{2n}	f_2	x_2
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$	$\vdots$
n	z_{n1}	$\cdots$	z_{nj}	$\cdots$	z_{nn}	f_n	x_n
Labor	v_1	$\cdots$	v_j	$\cdots$	v_n	f_{n+1}	x_{n+1}

In practice, sectors produce more than one good, and the assumption of one price for a sector's output is unrealistic. And in any case, monetary tables are assembled on the basis of recorded values of transactions; price and quantity are generally not recorded separately.

2.6.3 The Price Model based on Monetary Data

Monetary transactions are arranged as usual, where for notational simplicity we assume that all value added is represented by labor (Table 2.14). As we saw in section 2.2.1, when *all* inputs are accounted for in the processing *and* payments sectors, then the jth column sum (total outlays) is equal to the jth row sum (total output). Thus, summing down the jth column in Table 2.14,

$$x_j = \sum_{i=1}^{n} z_{ij} + v_j \tag{2.28}$$

or

$$\mathbf{x}' = \mathbf{i}'\mathbf{Z} + \mathbf{v}' \tag{2.29}$$

where, as earlier, $\mathbf{v}' = [v_1, \ldots, v_n]$, total value-added expenditures by each sector.

Substituting $\mathbf{Z} = \mathbf{A}\hat{\mathbf{x}}$, $\mathbf{x}' = \mathbf{i}'\mathbf{A}\hat{\mathbf{x}} + \mathbf{v}'$, and postmultiplying by $\hat{\mathbf{x}}^{-1}$,

$$\mathbf{x}'\hat{\mathbf{x}}^{-1} = \mathbf{i}'\mathbf{A}\hat{\mathbf{x}}\hat{\mathbf{x}}^{-1} + \mathbf{v}'\hat{\mathbf{x}}^{-1}$$

or

$$\mathbf{i}' = \mathbf{i}'\mathbf{A} + \mathbf{v}'_c \tag{2.30}$$

where $\mathbf{v}'_c = \mathbf{v}'\hat{\mathbf{x}}^{-1} = [v_1/x_1, \ldots, v_n/x_n]$. The right-hand side of (2.30) is the cost of inputs per unit of output. Output prices are set equal to total cost of production (in the general case, this will include an allocation for profit and other primary inputs in $\mathbf{v}'$ and hence in $\mathbf{v}'_c$), so each price is equal to 1 [the left-hand side of (2.30)]. This illustrates the unique measurement units in the base year table – amounts that can be purchased for \$1.00. If we denote these base year index prices by $\tilde{p}_j$, so $\tilde{\mathbf{p}}' = [\tilde{p}_1, \ldots, \tilde{p}_n]$, then the input–output price model is:

$$\tilde{\mathbf{p}}' = \tilde{\mathbf{p}}'\mathbf{A} + \mathbf{v}'_c \tag{2.31}$$

which leads to $\tilde{\mathbf{p}}'(\mathbf{I} - \mathbf{A}) = \mathbf{v}'_c$ and

$$\tilde{\mathbf{p}}' = \mathbf{v}'_c(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{v}'_c\mathbf{L} \tag{2.32}$$

Frequently the model is transposed and expressed in terms of column vectors rather than row vectors. In that case,

$$\tilde{\mathbf{p}} = (\mathbf{I} - \mathbf{A}')^{-1}\mathbf{v}_c = \mathbf{L}'\mathbf{v}_c \tag{2.33}$$

[The interested reader can show that, given $(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{L}$, then $(\mathbf{I} - \mathbf{A}')^{-1} = \mathbf{L}'$.]

From (2.32), index prices, $\tilde{\mathbf{p}}$, are determined by the exogenous values (costs) of primary inputs. For a two-sector model,

$$\tilde{p}_1 = l_{11}v_{c1} + l_{21}v_{c2}$$
$$\tilde{p}_2 = l_{12}v_{c1} + l_{22}v_{c2}$$

The logic is that changes in labor input prices (or, more generally, primary input price changes) lead to changes in sectoral unit costs (and therefore output prices, not output quantities) via the fixed production recipes in $\mathbf{A}$, and hence in $\mathbf{L}$ and $\mathbf{L}'$. For example, cost increases are passed along completely as intermediate input price increases to all purchasers, who in turn pass on these increases by raising their output prices accordingly, etc. As opposed to the *demand-pull input–output quantity* model earlier in this chapter, the price model in (2.32) or (2.33) is more completely known as the *cost-push input–output price model* (Oosterhaven, 1996; Dietzenbacher, 1997). In it, quantities are fixed and prices change. Table 2.15 summarizes the two (dual) models where, again, superscripts "0" and "1" indicate values before and after accounting for the exogenous change. Examples in the following section illustrate the workings of this model.

2.6.4 Numerical Examples Using the Price Model based on Monetary Data

Example 1: Base Year Prices Table 2.16 contains data from Table 2.10 to construct an added row to reflect labor as the only primary input. The corresponding direct inputs matrix is

$$\bar{\mathbf{A}} = \begin{bmatrix} .15 & .25 & .11 \\ .20 & .05 & .54 \\ .65 & .70 & .35 \end{bmatrix} \tag{2.34}$$

Using $\mathbf{A}$ for the 2×2 submatrix of sector 1 and sector 2 coefficients,

$$(\mathbf{L}^0)' = (\mathbf{I} - \mathbf{A}')^{-1} = \begin{bmatrix} 1.254 & .264 \\ .330 & 1.122 \end{bmatrix} \tag{2.35}$$

Table 2.15 The Leontief Quantity and Price Models

Leontief Quantity Model (Demand-pull) [Prices fixed; quantities change]	Exogenous Variables	$\mathbf{f}^1 = [f_i^1]$ or $\Delta\mathbf{f} = [\Delta f_i]$
	Endogenous Variables	$\mathbf{x}^1 = \mathbf{L}^0\mathbf{f}^1$ or $\Delta\mathbf{x} = \mathbf{L}^0(\Delta\mathbf{f})$
Leontief Price Model (Cost-push) [Quantities fixed; prices change]	Exogenous Variables	$\mathbf{v}_c^1 = (\hat{\mathbf{x}}^0)^{-1}\mathbf{v}^1 = [v_j^1/x_j^0]$ or $\Delta\mathbf{v}_c = (\hat{\mathbf{x}}^0)^{-1}(\Delta\mathbf{v}) = [\Delta v_j/x_j^0]$
	Endogenous Variables	$\tilde{\mathbf{p}}^1 = (\mathbf{L}^0)'\mathbf{v}_c^1$ or $\Delta\tilde{\mathbf{p}} = (\mathbf{L}^0)'(\Delta\mathbf{v}_c)$

Table 2.16 Transactions for Hypothetical Example with One Primary Input

	1	2	f_i	x_i
1	150	500	350	1000
2	200	100	1700	2000
3 (Labor)	650	1400	1100	3150

From the base year data, $\mathbf{v}_c^0 = \begin{bmatrix} .65 \\ .70 \end{bmatrix} = \begin{bmatrix} \bar{a}_{31} \\ \bar{a}_{32} \end{bmatrix}$ – from the bottom row of $\bar{\mathbf{A}}$ in (2.34). Thus, in (2.33),

$$\tilde{\mathbf{p}}^0 = (\mathbf{L}^0)'\mathbf{v}_c^0 = \begin{bmatrix} 1.254 & .264 \\ .330 & 1.122 \end{bmatrix} \begin{bmatrix} .65 \\ .70 \end{bmatrix} = \begin{bmatrix} 1.00 \\ 1.00 \end{bmatrix} \tag{2.36}$$

This reproduces the base year index prices, as expected.

Example 2: Changed Base Year Prices The value-based cost-push price model is generally used to measure the impact on prices throughout the economy of new primary-input costs (or a change in those costs) in one or more sectors. Again, suppose that these costs consist entirely of wage payments and that wages in sector 1 increase by 30 percent (from 0.65 to 0.845) while those in sector 2 remain unchanged.

The vector of new labor costs is

$$\mathbf{v}_c^1 = \begin{bmatrix} .845 \\ .700 \end{bmatrix}$$

and, from (2.33),

$$\tilde{\mathbf{p}}^1 = (\mathbf{L}^0)'\mathbf{v}_c^1 = \begin{bmatrix} 1.254 & .264 \\ .330 & 1.122 \end{bmatrix}\begin{bmatrix} .845 \\ .700 \end{bmatrix} = \begin{bmatrix} 1.245 \\ 1.064 \end{bmatrix} \tag{2.37}$$

Relative to the original index prices ($\tilde{p}_1^0 = 1.00$ and $\tilde{p}_2^0 = 1.00$), sector 1's price has gone up to 1.245 (a 24.5 percent increase), and sector 2's price has increased by 6.4 percent.

As with the demand-driven input–output model in earlier sections, this exercise can just as well be carried out in the "Δ" form of the model, namely

$$\Delta\tilde{\mathbf{p}} = (\mathbf{L}^0)'\Delta\mathbf{v}_c \tag{2.38}$$

In this case, $\Delta\mathbf{v}_c = \begin{bmatrix} .195 \\ 0 \end{bmatrix}$, where $(0.195) = (0.30)(0.65)$, and using (2.38),

$$\Delta\tilde{\mathbf{p}} = (\mathbf{L}^0)'\Delta\mathbf{v}_c = \begin{bmatrix} 1.254 & .264 \\ .330 & 1.122 \end{bmatrix}\begin{bmatrix} .195 \\ 0 \end{bmatrix} = \begin{bmatrix} .245 \\ .064 \end{bmatrix} \tag{2.39}$$

The results in either (2.37) or (2.39) convey the same information – the economy-wide effect of the 30 percent wage increase in sector 1 is that the price of sector 1 output goes up by 24.5 percent and that of sector 2 increases by 6.4 percent. In this cost-push input–output price model, we find *relative* price impacts – the absolute values of those prices, even in the base year, are not explicit in the model.

Notice that if labor costs are only a part of the value-added component for sector 1, then a 30 percent increase in wages in sector j will generate a less than 30 percent increase in v_{cj} – for example, if wages comprise 40 percent of sector j's value-added payments, and no other value-added costs increase, a 30 percent wage increase translates into a 12 percent increase in v_{cj}. The effects of primary input price *decreases* can also be quantified in the same way by the models in (2.32) [or (2.33)] or (2.38).

2.6.5 Applications

An early example of the use of this input–output price model is provided by Melvin (1979), where the price effects of changes in corporate income taxes are estimated for both the United States and Canada, using an 82-sector US table for 1965 and a 110-sector Canadian table for 1966. Another illustration is provided by Duchin and Lange (1995) who use the price model framework to assess price effects of alternative technologies in the US economy. Based on US 1963 and 1977 data, they use 1977 technology with 1963 factor prices to assess the price effects of the change in technology over that period. Similarly, using projections to 2000, they examine the price effects of technology change over 1977 to 2000. (They also change technology in the **A** matrix one column

Table 2.17 Flows in Physical Units

	Sectors					
Sectors	1	2	$\cdots$	n	Final Demand	Total Output
1	s_{11}	s_{12}	$\cdots$	s_{1n}	d_1	q_1
2	s_{21}	s_{22}	$\cdots$	s_{2n}	d_2	q_2
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
n	s_{n1}	s_{n2}	$\cdots$	s_{nn}	d_n	q_n
Labor	$s_{n+1,1}$	$s_{n+1,2}$	$\cdots$	$s_{n+1,n}$	d_{n+1}	q_{n+1}

at a time; and this is done in the context of a *dynamic* price model. We explore dynamic input–output models in Chapter 13.) A few additional examples include Lee, Blakesley and Butcher (1977) at a regional level, Polenske (1978) for a multiregional example, Marangoni (1995) for Italy and Dietzenbacher and Velázquez (2007) who include an analysis of cost-push effects of changes in water prices.

2.6.6 The Price Model based on Physical Data

In this section we examine the implications of an input–output model based on a set of data in *physical* units, as was shown in Table 2.11. Here, in Table 2.17, we let s_{ij} represent the physical quantity of i goods shipped to j [e.g., bushels of agricultural products (i) sold to manufacturers (j)], d_i is deliveries to final demand (e.g., in bushels for agricultural demand) and q_i is total sector i production (e.g., total bushels produced by agriculture). Again, for simplicity, let the exogenous payments (value added) sector consist exclusively of labor inputs (measured in person-days).

Reading across any row in Table 2.17 we have the basic accounting relationships in physical units:

$$q_i = s_{i1} + \cdots + s_{ij} + \cdots + s_{in} + d_i = \sum_{j=1}^{n} s_{ij} + d_i \tag{2.40}$$

[Compare with (2.1) in value terms.] Using obvious matrix definitions, this is

$$\mathbf{q} = \mathbf{S}\mathbf{i} + \mathbf{d} \tag{2.41}$$

This is the physical-units parallel to (2.4).

Direct input coefficients in *physical* terms are defined as

$$c_{ij} = \frac{s_{ij}}{q_j} \quad \text{or} \quad \mathbf{C} = \mathbf{S}\hat{\mathbf{q}}^{-1} \tag{2.42}$$

For the example of agricultural input into manufacturing (Table 2.11), this would be 250/400 = 0.625 (bushels per ton). Then, in a series of steps that parallel the earlier development of the value-based model in section 2.2, substitution into (2.41) gives

$$\mathbf{q} = \mathbf{C}\hat{\mathbf{q}}\mathbf{i} + \mathbf{d} = \mathbf{C}\mathbf{q} + \mathbf{d}$$

from which

$$\mathbf{q} = (\mathbf{I} - \mathbf{C})^{-1}\mathbf{d} \tag{2.43}$$

This is the physical-units model parallel to (2.11).

Introduction of Prices Suppose also that we know the per-unit price for each sector's output, p_i, and the labor cost per person-hour, p_{n+1}. Then, as Leontief observes in the quotation above, we can easily convert the basic data to the value units from earlier in this chapter:

$$x_i = p_i q_i \tag{2.44}$$

$$z_{ij} = p_i s_{ij} \tag{2.45}$$

$$f_i = p_i d_i \tag{2.46}$$

Multiplying (2.40) on both sides by p_i gives

$$x_i = p_i q_i = \sum_{j=1}^{n} p_i s_{ij} + p_i d_i = \sum_{j=1}^{n} z_{ij} + f_i \tag{2.47}$$

or $\mathbf{x} = \mathbf{Z}\mathbf{i} + \mathbf{f}$. These are, of course, the original accounting relationships in (2.1) and (2.3) in value terms.

In section 2.6.1, the representation of total outputs in terms of column sums of Table 2.14 was given in monetary terms in (2.28), namely $x_j = \sum_{i=1}^{n} z_{ij} + v_j$. Column sums are not meaningful in Table 2.17 since elements in each row are measured in different units. The objective now is to introduce the results from (2.44) and (2.45) into (2.28). Assume, for now, that the wage rate is p_{n+1}(dollars per person-hour) across all sectors. Then $z_{n+1,j} = p_{n+1}s_{n+1,j} = v_j$; this represents sector j's total expenditure on labor – the price, p_{n+1}, times total person-hours of labor, $s_{n+1,j}$. Then (2.28) becomes

$$p_j q_j = \sum_{i=1}^{n} p_i s_{ij} + p_{n+1} s_{n+1,j} \tag{2.48}$$

Dividing by q_j (which we assume is not zero),

$$p_j = \sum_{i=1}^{n} p_i s_{ij}/q_j + p_{n+1} s_{n+1,j}/q_j = \sum_{i=1}^{n} p_i c_{ij} + p_{n+1} c_{n+1,j} \tag{2.49}$$

In matrix form, this is

$$\mathbf{p}' = \mathbf{p}'\mathbf{C} + \boldsymbol{\upsilon}'_c \tag{2.50}$$

where $\mathbf{p}' = [p_1, \ldots, p_n]$, $\mathbf{C}$ is defined in (2.42) and $\boldsymbol{\upsilon}'_c = p_{n+1}[c_{n+1,1}, \ldots, c_{n+1,n}]$. So $\boldsymbol{\upsilon}'_c$ represents the labor cost (price) per unit of physical output – for example, labor costs per ton of output [$/ton = ($/person-hour) × (person-hours/ton)].

Labor costs were assumed to be uniform across all sectors; thus we have only p_{n+1} and not $p_{n+1,j}$. This can easily be extended to encompass differing labor costs (perhaps reflecting labor of differing skills) among sectors. Equation (2.50) defines the unit price for each sector's output as equal to the total costs (interindustry plus primary inputs) of producing a unit of that output. (In general there will be more than one component to primary input costs for each sector, but the principles remain the same.)

From (2.50),

$$\mathbf{p}' = \boldsymbol{\upsilon}'_c(\mathbf{I} - \mathbf{C})^{-1} \tag{2.51}$$

As before, we can transpose both sides of (2.50) and (2.51) to have the prices in a column vector instead of a row vector,

$$\mathbf{p} = \mathbf{C}'\mathbf{p} + \boldsymbol{\upsilon}_c \quad \text{and} \quad \mathbf{p} = (\mathbf{I} - \mathbf{C}')^{-1}\boldsymbol{\upsilon}_c \tag{2.52}$$

This is the Leontief price model based on physical units. These structures are completely parallel to those in (2.32) and (2.33) for the monetary-based index-price model, For the $n = 2$ case, we have

$$\begin{aligned} p_1 &= p_1 c_{11} + p_2 c_{21} + \upsilon_{c1} \\ p_2 &= p_1 c_{12} + p_2 c_{22} + \upsilon_{c2} \end{aligned} \tag{2.53}$$

and

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} (1 - c_{11}) & -c_{21} \\ -c_{12} & (1 - c_{22}) \end{bmatrix}^{-1} \begin{bmatrix} \upsilon_{c1} \\ \upsilon_{c2} \end{bmatrix} \tag{2.54}$$

Relationship between **A** *and* **C** Direct input coefficients in value terms are

$$a_{ij} = \frac{z_{ij}}{x_j} \quad \text{or} \quad \mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1}$$

Therefore [from (2.44) and (2.45)]

$$a_{ij} = \frac{p_i s_{ij}}{p_j q_j} = c_{ij}\left(\frac{p_i}{p_j}\right) \tag{2.55}$$

In matrix terms[19]

$$\mathbf{A} = \hat{\mathbf{p}}\mathbf{S}(\hat{\mathbf{p}}\hat{\mathbf{q}})^{-1} = \hat{\mathbf{p}}\mathbf{C}\hat{\mathbf{q}}(\hat{\mathbf{q}}^{-1}\hat{\mathbf{p}}^{-1}) = \hat{\mathbf{p}}\mathbf{C}\hat{\mathbf{p}}^{-1} \tag{2.56}$$

[19] When two matrices, **M** and **N,** satisfy the relationship $\mathbf{M} = \hat{\mathbf{v}}\mathbf{N}\hat{\mathbf{v}}^{-1}$, they are said to be *similar*.

Either the value-based coefficients, **A**, or the physical coefficients, **C**, are assumed fixed in applications of the input–output model. However, assuming fixed c_{ij} (in effect, a fixed "engineering" production function) has been seen by many as less restrictive than fixed a_{ij} (a fixed "economic" production function), because in the latter case both a physical coefficient, c_{ij}, and a price ratio, p_i/p_j, are assumed unchanging.[20]

2.6.7 Numerical Examples Using the Price Model based on Physical Data

Example 1: Base Year Prices Consider again the two-sector economy (agriculture and manufacturing) in Table 2.11 closed with an added row showing labor inputs and final demand (consumption). From that table we can find the physical technical coefficients [as in (2.42)]

$$\bar{\mathbf{C}} = \begin{bmatrix} .15 & .625 & .556 \\ .08 & .05 & 1.079 \\ .13 & .35 & .349 \end{bmatrix} \tag{2.57}$$

We use $\bar{\mathbf{C}}$ for the (closed) technical coefficients matrix that includes households; **C** will represent the 2×2 matrix in the upper-left corner – technical coefficients connecting the two producing sectors in the economy. Note that $\bar{c}_{23} > 1$; column sums in $\bar{\mathbf{C}}$ are meaningless, since each row is measured in different units.

The relationships in (2.54) are

$$\begin{aligned} 2 &= (2)(.15) + (5)(.08) + (10)(.13) = .30 + .40 + 1.30 \\ 5 &= (2)(.625) + (5)(.05) + (10)(.35) = 1.25 + .25 + 3.50 \end{aligned} \tag{2.58}$$

If we use the base-period value-added-per-unit-of-output figures,

$$\upsilon_{c1}^0 = p_3\bar{c}_{31} = (10)(.13) \ = 1.30 \text{ and } \upsilon_{c2}^0 = p_3\bar{c}_{32} = (10)(.35) = 3.50$$

along with

$$(\mathbf{I} - \mathbf{C}')^{-1} = \begin{bmatrix} 1.243 & .106 \\ .825 & 1.122 \end{bmatrix} \tag{2.59}$$

(from the 2×2 upper-left submatrix of $\bar{\mathbf{C}}$) in $\mathbf{p} = (\mathbf{I} - \mathbf{C}')^{-1}\boldsymbol{\upsilon}_c$,

$$\begin{bmatrix} p_1^0 \\ p_2^0 \end{bmatrix} = (\mathbf{I} - \mathbf{C}')^{-1}\boldsymbol{\upsilon}_c^0 = \begin{bmatrix} 1.254 & .106 \\ .825 & 1.122 \end{bmatrix}\begin{bmatrix} 1.3 \\ 3.5 \end{bmatrix} = \begin{bmatrix} 2.00 \\ 5.00 \end{bmatrix} \tag{2.60}$$

This generates the base year prices, as expected.

[20] Economists have held differing opinions on this question of the plausibility of the assumption of stability for physical vs. value-based coefficients. For early examples, see Klein (1953) who suggests that a_{ij}'s may be more stable than c_{ij}'s, and Moses (1974) who argues the opposite.

Example 2: Changed Base Year Prices Continuing with this physical coefficients model, suppose that the wage costs in sector 1 increase from \$10.00 to \$13.00 (a 30 percent increase) while those in sector 2 remain unchanged ($p_{31}^1 = \$13.00$ and $p_{32}^1 = p_{32}^0 = 10.00$), so $\boldsymbol{v}_c^1 = \begin{bmatrix} (13)(.13) \\ (10)(.35) \end{bmatrix} = \begin{bmatrix} 1.69 \\ 3.50 \end{bmatrix}$. Then

$$\begin{bmatrix} p_1^0 \\ p_2^0 \end{bmatrix} = (\mathbf{I} - \mathbf{C}')^{-1} \boldsymbol{v}_c^0 = \begin{bmatrix} 1.254 & .106 \\ .825 & 1.122 \end{bmatrix} \begin{bmatrix} 1.69 \\ 3.50 \end{bmatrix} = \begin{bmatrix} 2.49 \\ 5.32 \end{bmatrix} \tag{2.61}$$

Specifically, $p_1^1 = \$2.49$ (an increase of 24.5 percent over $p_1^0 = \$2.00$) and $p_2^1 = \$5.32$ (a 6.4 percent increase over $p_2^0 = \$5.00$). This illustrates the operation of the cost-push input–output price model based on physical input coefficients. It generates the new prices directly (from which percentage changes can easily be found). In section 2.6.4 we found these percentage increases directly from the index-price model in (2.37).

2.6.8 *The Quantity Model based on Physical Data*

Data in physical units can also form the core of an input–output quantity model, as in $\mathbf{q} = (\mathbf{I} - \mathbf{C})^{-1}\mathbf{d}$ in (2.43) – before the introduction of prices. Using the data from the two numerical examples immediately above,

$$\mathbf{C} = \begin{bmatrix} .150 & .625 \\ .080 & .050 \end{bmatrix} \quad \text{and} \quad (\mathbf{I} - \mathbf{C})^{-1} = \begin{bmatrix} 1.254 & .825 \\ .106 & 1.122 \end{bmatrix}$$

Base year outputs are correctly generated by

$$\mathbf{q}^0 = (\mathbf{I} - \mathbf{C})^{-1}\mathbf{d}^0 \Rightarrow \begin{bmatrix} 1.254 & .825 \\ .106 & 1.122 \end{bmatrix} \begin{bmatrix} 175 \\ 340 \end{bmatrix} = \begin{bmatrix} 500 \\ 400 \end{bmatrix}$$

and, for example, doubling demand doubles outputs,

$$\mathbf{q}^1 = (\mathbf{I} - \mathbf{C})^{-1}\mathbf{d}^1 \Rightarrow \begin{bmatrix} 1.254 & .825 \\ .106 & 1.122 \end{bmatrix} \begin{bmatrix} 350 \\ 680 \end{bmatrix} = \begin{bmatrix} 1000 \\ 800 \end{bmatrix}$$

This is completely parallel to the demand-driven model in monetary terms, except that units of measurement are consistent only across each row. This means that the new demands (350 bushels and 680 tons) lead to production of 1000 bushels and 800 tons. Notice the units in $(\mathbf{I} - \mathbf{C})^{-1}$. For example, in the first column, 1.254 represents direct and indirect bushels of output per bushel of final demand, and 0.106 is direct and indirect output of tons per bushel of final demand.

A real-world illustration of this kind of model based on physical units appears in Stahmer (2000)[21]. This consists of a 12-sector input–output data set in physical terms for Germany in 1990 (an aggregation of a 91-sector model), where all transactions and outputs are measured in a common physical unit – tons. Hubacek and Giljum (2003)

[21] Also available as: "The Magic Triangle of Input–Output Tables," paper presented to the 13th International Input–Output Association Conference on Input–Output Techniques, Macerata, Italy, August, 2000.

Table 2.18 Transactions in Physical Terms (Germany, 1990) (millions of tons)

	Primary	Secondary	Tertiary	Final Demand	Total Output
Primary	2248	1442	336	84	4110
Secondary	27	1045	206	708	1986
Tertiary	5	69	51	36	161

generate a three-sector aggregation of these data for the illustrations in their study. In particular, transactions are shown in Table 2.18, above.

As in the illustration in (2.57), the associated direct inputs matrix, here **C**, has coefficients that are larger than 1:

$$\mathbf{C} = \begin{bmatrix} .5470 & .7261 & 2.0870 \\ .0066 & .5262 & 1.2795 \\ .0012 & .0347 & .3168 \end{bmatrix}$$

As we saw above, this does not pose any problems for the usual input–output calculations; here the Leontief inverse is easily found to be

$$(\mathbf{I} - \mathbf{C})^{-1} = \begin{bmatrix} 2.3185 & 4.7204 & 15.9220 \\ .0502 & 2.5486 & 4.9262 \\ .0067 & .1380 & 1.7425 \end{bmatrix}$$

As we see, some elements are large; these are associated with the large elements in **C**, but they are not inappropriate in the context of this PIOT. The reader can easily check the validity of this inverse from the base-case data, namely

$$\mathbf{x} = \begin{bmatrix} 4110 \\ 1986 \\ 161 \end{bmatrix} = (\mathbf{I} - \mathbf{C})^{-1}\mathbf{f} = \begin{bmatrix} 2.3185 & 4.7204 & 15.9220 \\ .0502 & 2.5486 & 4.9262 \\ .0067 & .1380 & 1.7425 \end{bmatrix} \begin{bmatrix} 84 \\ 708 \\ 36 \end{bmatrix}$$

Despite the unusual elements in **C**, the power series approximation to the Leontief inverse – $\mathbf{I} + \mathbf{C} + \mathbf{C}^2 + \mathbf{C}^3 + \cdots$ – works just fine, although slowly; it requires 37 terms to come within four-digit accuracy. Here are some of the terms:

$$\mathbf{C}^{10} = \begin{bmatrix} 0.0077 & 0.0971 & 0.3551 \\ 0.0011 & 0.0167 & 0.0616 \\ 0.0001 & 0.0018 & 0.0067 \end{bmatrix}, \quad \mathbf{C}^{20} = \begin{bmatrix} 0.0002 & 0.0030 & 0.0111 \\ 0.0000 & 0.0005 & 0.0018 \\ 0.0000 & 0.0001 & 0.0002 \end{bmatrix},$$

$$\mathbf{C}^{30} = \begin{bmatrix} 0 & .0001 & .0003 \\ 0 & 0 & .0001 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{C}^{37} = \mathbf{0}$$

and

$$\left(\mathbf{I} + \sum_{k=1}^{37} \mathbf{C}^k\right) = \begin{bmatrix} 2.3185 & 4.7204 & 15.9220 \\ .0502 & 2.5486 & 4.9262 \\ .0067 & .1380 & 1.7425 \end{bmatrix} = (\mathbf{I} - \mathbf{C})^{-1}$$

The interested reader with access to combinatorial algebra software on a computer might check that for this illustration, with $(\mathbf{I}-\mathbf{C}) = \begin{bmatrix} .4530 & -.7261 & -2.0870 \\ -.0066 & .4738 & -1.2795 \\ -.0012 & -.0347 & .6832 \end{bmatrix}$, the Hawkins–Simon conditions are satisfied, meaning that all seven principal minors of $(\mathbf{I}-\mathbf{C})$ are positive (Appendix 2.2).

2.6.9 A Basic National Income Identity

From (2.43), $\mathbf{q} = (\mathbf{I}-\mathbf{C})^{-1}\mathbf{d}$; from (2.51), $\mathbf{p}' = \boldsymbol{\upsilon}'_c(\mathbf{I}-\mathbf{C})^{-1}$, and postmultiplying this by $\mathbf{d}$,

$$\mathbf{p}'\mathbf{d} = \boldsymbol{\upsilon}'_c(\mathbf{I}-\mathbf{C})^{-1}\mathbf{d} = \boldsymbol{\upsilon}'_c\mathbf{q}$$

The total value of spending (exogenous final demand, $\mathbf{p}'\mathbf{d}$) equals the total value of earnings (payments to exogenous primary inputs, $\boldsymbol{\upsilon}'_c\mathbf{q}$), or national income spent equals national income received.

2.7 Summary

We have introduced the basic structure of the input–output model in this chapter. After investigating the special features of sectoral production functions that are assumed in the Leontief system, we examined its mathematical features. Importantly, the model is expressed in a set of linear equations, and we have tried to indicate the connection between the purely algebraic solution to the input–output equations, using the Leontief inverse matrix, and the logical, economic content of the round-by-round view of production interrelationships in an economy. Both the algebraic details as well as the economic assumptions needed to close the model with respect to households were discussed. Some of the special problems associated with the concept of household consumption coefficients have been addressed in applications, especially at the regional level. We also introduced the Leontief price model, a logical (and mathematical) companion to the quantity model, and we explored alternatives to both models when the underlying data are measured in physical rather than monetary terms. Table 2.19 summarizes the alternatives. (Information in the monetary row is in Table 2.15.)

We turn to regional input–output models in the next chapter. It is important to add the regional dimension; many if not most important policy questions are not purely national in scope. Rather, analysts (even at the national level) are interested in differential regional effects of, say, a change in national government policy regarding exports. It is important to know not only the total magnitudes of the new outputs, by sector, that come about because of stimulation of exports, but also to know something of their geographical incidence – is a particularly depressed area helped by such export stimulation, or does the increased output occur largely in regions that are economically more healthy? Extensions of the basic model to deal with issues of this sort will occupy us in Chapter 3.

Table 2.19 Alternative Input–Output Price and Quantity Models

Measurement Units	Quantity Model	Price Model
Monetary	$\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{f}$ (2.11)	$\tilde{\mathbf{p}}' = \mathbf{v}'_c(\mathbf{I} - \mathbf{A})^{-1}$ (2.32) or $\tilde{\mathbf{p}} = (\mathbf{I} - \mathbf{A}')^{-1}\mathbf{v}_c$ (2.33)
Physical	$\mathbf{q} = (\mathbf{I} - \mathbf{C})^{-1}\mathbf{d}$ (2.43)	$\mathbf{p}' = \boldsymbol{\upsilon}'_c(\mathbf{I} - \mathbf{C})^{-1}$ (2.51) or $\mathbf{p} = (\mathbf{I} - \mathbf{C}')^{-1}\boldsymbol{\upsilon}_c$ (2.52)

Appendix 2.1 The Relationship between Approaches I and II

To examine the connection between the two alternative approaches to the numerical example in section 2.3, we consider a general two-sector economy with $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and let f_1 and f_2 represent values of the new final demands.[22]

A2.1.1 Approach I

Using the Leontief-inverse, we find $(\mathbf{I} - \mathbf{A}) = \begin{bmatrix} (1-a_{11}) & -a_{12} \\ -a_{21} & (1-a_{22}) \end{bmatrix}$ and, provided that $|\mathbf{I} - \mathbf{A}| \neq 0$, which means that $(1-a_{11})(1-a_{22}) - (-a_{12})(-a_{21}) \neq 0$ (Appendix A)

$$(\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{|\mathbf{I} - \mathbf{A}|}[\text{adj}(\mathbf{I} - \mathbf{A})] = \begin{bmatrix} \dfrac{(1-a_{22})}{|\mathbf{I} - \mathbf{A}|} & \dfrac{a_{12}}{|\mathbf{I} - \mathbf{A}|} \\ \dfrac{a_{21}}{|\mathbf{I} - \mathbf{A}|} & \dfrac{(1-a_{11})}{|\mathbf{I} - \mathbf{A}|} \end{bmatrix} \tag{A2.1.1}$$

The associated gross outputs are found from $\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{f}$, namely

$$\begin{aligned} x_1 &= \left[\frac{(1-a_{22})}{|\mathbf{I} - \mathbf{A}|}\right]f_1 + \left[\frac{a_{12}}{|\mathbf{I} - \mathbf{A}|}\right]f_2 \\ x_2 &= \left[\frac{a_{21}}{|\mathbf{I} - \mathbf{A}|}\right]f_1 + \left[\frac{(1-a_{11})}{|\mathbf{I} - \mathbf{A}|}\right]f_2 \end{aligned} \tag{A2.1.2}$$

[22] As elsewhere in this chapter, we ignore the “0” and “1” superscripts for notational simplicity when the intended meaning is clear from the context.

A2.1.2 Approach II

The round-by-round calculation of total impacts requires only the elements of the **A** matrix. The first-round impact on sector 1 – in terms of what it must produce to satisfy its own and sector 2's needs for inputs – is $\underbrace{a_{11}f_1 + a_{12}f_2}_{\text{Sector 1, Round 1}}$. For sector 2, the first-round impact is $\underbrace{a_{21}f_1 + a_{22}f_2}_{\text{Sector 2, Round 1}}$. (These were \$465 and \$195 in the numerical example.)

The second-round impacts result from production that is required to take care of first-round needs. These are easily seen to be

$$\text{For sector 1: } a_{11}\underbrace{(a_{11}f_1 + a_{12}f_2)}_{\text{Sector 1, Round 1}} + a_{12}\underbrace{(a_{21}f_1 + a_{22}f_2)}_{\text{Sector 2, Round 1}}$$

$$\text{For sector 2: } a_{21}\underbrace{(a_{11}f_1 + a_{12}f_2)}_{\text{Sector 1, Round 1}} + a_{22}\underbrace{(a_{21}f_1 + a_{22}f_2)}_{\text{Sector 2, Round 1}}$$

(These were \$118.50 and \$102.75 in the numerical example.)

The nature of the expansion is now clear. For sector 1 in round 3, we will have

$$\begin{aligned} &a_{11}\underbrace{[a_{11}(a_{11}f_1 + a_{12}f_2) + a_{12}(a_{21}f_1 + a_{22}f_2)]}_{\text{Sector 1, Round 2}} \\ &\quad + a_{12}\underbrace{[a_{21}(a_{11}f_1 + a_{12}f_2) + a_{22}(a_{21}f_1 + a_{22}f_2)]}_{\text{Sector 2, Round 2}} \end{aligned}$$

and for sector 2 in round 3:

$$\begin{aligned} &a_{21}\underbrace{[a_{11}(a_{11}f_1 + a_{12}f_2) + a_{12}(a_{21}f_1 + a_{22}f_2)]}_{\text{Sector 1, Round 2}} \\ &\quad + a_{22}\underbrace{[a_{21}(a_{11}f_1 + a_{12}f_2) + a_{22}(a_{21}f_1 + a_{22}f_2)]}_{\text{Sector 2, Round 2}} \end{aligned}$$

(These were \$43.46 and \$28.84 in the numerical example.)

Without going further, we can develop an expression for an approximation to x_1 in terms of f_1 and f_2 and the technical coefficients on the basis of only three rounds of effects. Collecting the terms for round-by-round effects on sector 1, we have

$$\begin{aligned} x_1 \cong\, & f_1 + a_{11}f_1 + a_{11}^2 f_1 + a_{12}a_{21}f_1 + a_{11}^3 f_1 + a_{11}a_{12}a_{21}f_1 \\ & + a_{12}a_{21}a_{11}f_1 + a_{12}f_2 + a_{11}a_{12}f_2 + a_{12}a_{22}f_2 + a_{11}a_{11}a_{12}f_2 \\ & + a_{11}a_{12}a_{22}f_2 + a_{12}a_{21}a_{12}f_2 + a_{12}a_{22}a_{22}f_2 \end{aligned}$$

or

$$\begin{aligned} x_1 \cong\ & (1 + a_{11} + a_{11}^2 + a_{12}a_{21} + a_{11}^3 + a_{11}a_{12}a_{21} + a_{12}a_{21}a_{11})f_1 \\ & + (a_{12} + a_{11}a_{12} + a_{12}a_{22} + a_{11}a_{11}a_{12} + a_{11}a_{12}a_{22} \\ & + a_{12}a_{21}a_{12} + a_{12}a_{22}a_{22})f_2 \end{aligned} \tag{A2.1.3}$$

A similar expression can be derived for x_2.

The object of this algebra is to make clear that in round 2, the effect is found in products of *pairs* of coefficients (e.g., a_{11}^2 and $a_{11}a_{12}$); in round 3, the effect comes from products of *triples* of coefficients (e.g., a_{11}^3 and $a_{11}a_{12}a_{21}$). Similarly, in round 4, sets of four coefficients will be multiplied together, ... and in round n, sets of n coefficients will be multiplied. In monetary terms, all $a_{jj} < 1$ and $a_{ij} < 1$ since producer j must buy, from himself and each supplier i, less than one dollar's worth of inputs per dollar's worth of output. Therefore it is clear that eventually the effects in the "next" round will be essentially negligible. Mathematically, the expression for x_1 has the form

$$\begin{aligned} x_1 =\ & (1 + \text{infinite series of terms involving products of pairs, triples, \ldots, of } a_{ij})f_1 \\ & + (\text{similar infinite series})f_2 \end{aligned} \tag{A2.1.4}$$

There would be a parallel expression for x_2. If we denote these two parenthetical series terms for x_1 by s_{11} and s_{12}, and in the similar expression for x_2 by s_{21} and s_{22}, we have gross outputs related to final demands by

$$\begin{aligned} x_1 &= s_{11}f_1 + s_{12}f_2 \\ x_2 &= s_{21}f_1 + s_{22}f_2 \end{aligned} \tag{A2.1.5}$$

The evaluation of the s terms as four different infinite series would be a difficult and tedious task.

Alternatively, we could think of the new total output x_1 as composed of two parts: (a) the new final demands for sector 1's output, f_1, and (b) all direct and indirect effects on sector 1 generated by f_1 and f_2. (This approach was suggested in Dorfman, Samuelson and Solow, 1958, section 9.3.) To this end, define $F_1 = a_{11}f_1 + a_{12}f_2$, the first-round response from sector 1, and, similarly, let $F_2 = a_{21}f_1 + a_{22}f_2$ for sector 2. These first-round outputs will similarly generate second-round outputs, and so on, exactly as did f_1 and f_2 above. The suggestion is that the final outputs can be looked at as (1) a series of round-by-round effects on f_1 and f_2 *or* as (2) f_1 and f_2, plus a series of round-by-round effects on F_1 and F_2. In this alternative view, a complete derivation similar to that preceding (A2.1.5) would lead to

$$\begin{aligned} x_1 &= f_1 + s_{11}F_1 + s_{12}F_2 \\ x_2 &= f_2 + s_{21}F_1 + s_{22}F_2 \end{aligned} \tag{A2.1.6}$$

Substituting $F_1 = a_{11}f_1 + a_{12}f_2$ and $F_2 = a_{21}f_1 + a_{22}f_2$ and collecting terms,

$$\begin{aligned} x_1 &= (1 + s_{11}a_{11} + s_{12}a_{21})f_1 + (s_{11}a_{12} + s_{12}a_{22})f_2 \\ x_2 &= (s_{21}a_{11} + s_{22}a_{21})f_1 + (1 + s_{21}a_{12} + s_{22}a_{22})f_2 \end{aligned} \tag{A2.1.7}$$

Both (A2.1.5) and (A2.1.7) show x_1 and x_2 as linear functions of f_1 and f_2, so the coefficients in corresponding positions must be equal. That is,

$$\begin{array}{ll} s_{11} = 1 + s_{11}a_{11} + s_{12}a_{21} & s_{12} = s_{11}a_{12} + s_{12}a_{22} \\ s_{21} = s_{21}a_{11} + s_{22}a_{21} & s_{22} = 1 + s_{21}a_{12} + s_{22}a_{22} \end{array}$$

The top two are linear equations in the unknowns s_{11} and s_{12}, and the bottom two are linear equations in s_{21} and s_{22}. Rearranging to emphasize that the s are unknowns and the a are known coefficients,

$$\begin{aligned} (1 - a_{11})s_{11} - a_{21}s_{12} &= 1 \\ -a_{12}s_{11} + (1 - a_{22})s_{12} &= 0 \\ (1 - a_{11})s_{21} - a_{21}s_{22} &= 0 \\ -a_{12}s_{21} + (1 - a_{22})s_{22} &= 1 \end{aligned}$$

or

$$\begin{bmatrix} (1 - a_{11}) & -a_{21} \\ -a_{12} & (1 - a_{22}) \end{bmatrix} \begin{bmatrix} s_{11} \\ s_{12} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{A2.1.8a}$$

$$\begin{bmatrix} (1 - a_{11}) & -a_{21} \\ -a_{12} & (1 - a_{22}) \end{bmatrix} \begin{bmatrix} s_{21} \\ s_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{A2.1.8b}$$

Both sets of equations have the same coefficient matrix. Since

$$\begin{bmatrix} (1 - a_{11}) & -a_{21} \\ -a_{12} & (1 - a_{22}) \end{bmatrix}^{-1} = \frac{1}{(1 - a_{11})(1 - a_{22}) - a_{12}a_{21}} \begin{bmatrix} (1 - a_{22}) & a_{21} \\ a_{12} & (1 - a_{11}) \end{bmatrix}$$

and since $(1-a_{11})(1-a_{22})-a_{12}a_{21} = |\mathbf{I} - \mathbf{A}|$ [in (A2.1.1) and (A2.1.2)], the solutions to the two pairs of linear equations in (A2.1.8) are

$$\begin{bmatrix} s_{11} \\ s_{12} \end{bmatrix} = \begin{bmatrix} \dfrac{(1 - a_{22})}{|\mathbf{I} - \mathbf{A}|} & \dfrac{a_{21}}{|\mathbf{I} - \mathbf{A}|} \\ \dfrac{a_{12}}{|\mathbf{I} - \mathbf{A}|} & \dfrac{(1 - a_{11})}{|\mathbf{I} - \mathbf{A}|} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and}$$

$$\begin{bmatrix} s_{21} \\ s_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{(1 - a_{22})}{|\mathbf{I} - \mathbf{A}|} & \dfrac{a_{21}}{|\mathbf{I} - \mathbf{A}|} \\ \dfrac{a_{12}}{|\mathbf{I} - \mathbf{A}|} & \dfrac{(1 - a_{11})}{|\mathbf{I} - \mathbf{A}|} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

That is,

$$s_{11} = \frac{(1 - a_{22})}{|\mathbf{I} - \mathbf{A}|}, \quad s_{12} = \frac{a_{12}}{|\mathbf{I} - \mathbf{A}|}, \quad s_{21} = \frac{a_{21}}{|\mathbf{I} - \mathbf{A}|}, \quad s_{22} = \frac{(1 - a_{11})}{|\mathbf{I} - \mathbf{A}|}$$

These algebraic expressions equate the four infinite series terms, whose complex form was suggested in (A2.1.3) and (A2.1.4), to very simple functions of the elements of **A**.

Moreover, these four simple functions are precisely the four elements of the Leontief inverse, as found in (A2.1.1). In economic terms, the $(\mathbf{I}-\mathbf{A})^{-1}$ matrix captures in each of its elements all of the infinite series of round-by-round direct and indirect effects that the new final demands have on the outputs of the two sectors. (A demonstration along these lines is much more complex for a three-sector input–output model and unwieldy for more than three sectors.)

The elements of this Leontief inverse matrix are often termed *multipliers*. With $(\mathbf{I}-\mathbf{A})^{-1}=\mathbf{L}=\begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix}$ and forecasts for f_1 and f_2, the total effect on x_1 is given by $l_{11}f_1+l_{12}f_2$, the sum of the multiplied effects of each of the individual final demands. And similarly for x_2. Input–output multipliers are explored in Chapter 6.

Appendix 2.2 The Hawkins–Simon Conditions

No matter how many terms we use in the series approximation to $(\mathbf{I}-\mathbf{A})^{-1}$in (2.17), it is clear that each of the terms contains only non-negative elements, since all $a_{ij} \geq 0$. As noted in section 2.4, not only is $\mathbf{A} \geq \mathbf{0}$, but $\mathbf{A}^2 \geq \mathbf{0}, \ldots, \mathbf{A}^n \geq \mathbf{0}$; therefore $(\mathbf{I}+\mathbf{A}+\mathbf{A}^2+\cdots)$ is a matrix of non-negative terms. If the elements of $\mathbf{f}$ are all non-negative, then the associated $\mathbf{x}$ will contain non-negative elements also. This is what one would expect; when faced with a set of non-negative final demands it would be meaningless in an economy to find that one or more of the necessary gross outputs were negative.[23] For a Leontief system with $\mathbf{A} \geq \mathbf{0}$ and $N(\mathbf{A}) < 1$ [so that the results in (2.17) hold], we know that negative outputs will never be required from any sector to satisfy non-negative final demands.

One could also explore conditions under which $\mathbf{f} \geq \mathbf{0}$ would always generate $\mathbf{x} \geq \mathbf{0}$ by examining the general definition $(\mathbf{I}-\mathbf{A})^{-1} = \frac{1}{|\mathbf{I}-\mathbf{A}|}[\mathrm{adj}(\mathbf{I}-\mathbf{A})]$ (Appendix A). For the simplest, two-sector case,

$$(\mathbf{I}-\mathbf{A})^{-1} = \begin{bmatrix} \dfrac{(1-a_{22})}{|\mathbf{I}-\mathbf{A}|} & \dfrac{a_{12}}{|\mathbf{I}-\mathbf{A}|} \\ \dfrac{a_{21}}{|\mathbf{I}-\mathbf{A}|} & \dfrac{(1-a_{11})}{|\mathbf{I}-\mathbf{A}|} \end{bmatrix}$$

and all of the elements in $(\mathbf{I}-\mathbf{A})^{-1}$ must be non-negative – the numerators must all be non-negative and the denominator must be positive (the denominator must not be zero, either). Or, all numerators could be non-positive and the denominator negative.

We have already noted that $a_{ij} \geq 0$ and that $N(\mathbf{A}) < 1$ and (also by their definition) all $a_{ij} < 1$.[24] Thus all numerators in $(\mathbf{I}-\mathbf{A})^{-1}$ are non-negative. Therefore, if $|\mathbf{I}-\mathbf{A}| > 0$, all elements in the 2×2 Leontief inverse will be non-negative.

[23] In some models, as we have seen, negative values could have meaning. When both $\mathbf{x}$'s and $\mathbf{f}$'s are defined as "changes in", namely $\Delta\mathbf{x}$ and $\Delta\mathbf{f}$, then a result like $\Delta x_3 = -400$ is interpreted as a *decrease* of \$400 in sector 3's output.

[24] As we saw in section 2.6, this need not be the case in input–output tables denominated in *physical* rather than monetary terms – for example, liters of input per kilogram of output. See also Chapters 9 and 10.

Hawkins and Simon (1949) investigated the issue of non-negative solutions to more general equation systems. For a system in which $\mathbf{A} \geq \mathbf{0}$ (as in the input–output case) but in which no restriction is placed on the column sums of $\mathbf{A}$, they found for the 2×2 case that necessary and sufficient conditions to assure $\mathbf{x} \geq \mathbf{0}$ are[25]

$$\begin{aligned} &\text{(a) } (1 - a_{11}) > 0 \text{ and } (1 - a_{22}) > 0 \\ &\text{(b) } |\mathbf{I} - \mathbf{A}| > 0 \end{aligned} \tag{A2.2.1}$$

These conditions have a straightforward geometrical interpretation for the 2×2 case. We examine the solution-space representation. The fundamental relations

$$\begin{aligned} &\text{(a) } (1 - a_{11})x_1 - a_{12}x_2 = f_1 \\ &\text{(b) } -a_{21}x_1 + (1 - a_{22})x_2 = f_2 \end{aligned} \tag{A2.2.2}$$

define a pair of linear equations in x_1x_2 space. By setting one variable at a time equal to zero in each equation, it is easy to find the intercepts of each line on each axis. These are shown in Figure A2.2.1a, for arbitrary (but positive) f_1 and f_2. (Assume that both a_{12} and a_{21} are strictly positive, i.e., that each sector sells some inputs to the other. In a highly aggregated model this is virtually certain to be the case.)

As long as $(1 - a_{11}) > 0$ and $(1 - a_{22}) > 0$ – the first Hawkins–Simon condition in the 2×2 case – for $f_1 > 0$ and $f_2 > 0$, the intercept of (A2.2.2)(a) on the x_1-axis will be to the right of the origin and the intercept of (A2.2.2)(b) on the x_2-axis will be above the origin. Therefore, for non-negative total outputs, it is required that these two equations intersect in the first quadrant; this means that the slope of equation (a) must be greater

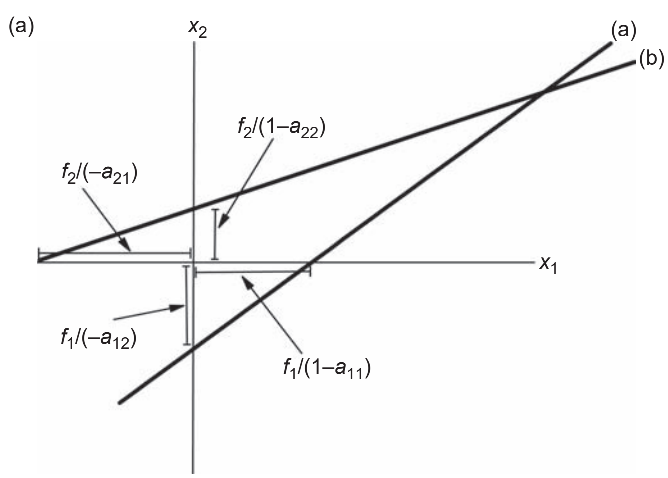

Figure A2.2.1a Solution Space Representation of (A2.2.2); $a_{12} > 0$ and $a_{21} > 0$

[25] The matrix algebra requirement for a unique solution to $(\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{f}$ is that $|\mathbf{I} - \mathbf{A}| \neq 0$. Now we are further restricting this determinant to only positive values.

than the slope of equation (b). These slopes are:

$$\text{For equation (a)}\ \frac{\dfrac{f_1}{a_{12}}}{\dfrac{f_1}{(1-a_{11})}} = \frac{(1-a_{11})}{a_{12}}$$

$$\text{For equation (b)}\ \frac{\dfrac{f_2}{(1-a_{22})}}{\dfrac{f_2}{a_{21}}} = \frac{a_{21}}{(1-a_{22})}$$

and thus the slope requirement is $(1-a_{11})/a_{12} > a_{21}/(1-a_{22})$. Multiplying both sides of the inequality by $(1-a_{22})$ and by a_{12} – both of which are assumed to be strictly positive – does not alter the direction of the inequality, giving $(1-a_{11})(1-a_{22}) > a_{12}a_{21}$ or $(1-a_{11})(1-a_{22}) - a_{12}a_{21} > 0$, which is just $|\mathbf{I} - \mathbf{A}| > 0$, the second Hawkins–Simon condition in the 2×2 case.

The effects of less interdependence in the two-sector economy are illustrated in Figures A2.2.1b and A2.2.1c. If $a_{21} = 0$, meaning that $z_{21} = 0$ (sector 1 uses no inputs from sector 2), then the slope of the line labeled (b) is zero. It is a horizontal line intersecting the x_2-axis at the height $f_2/(1-a_{22})$. This is to be expected; the gross output necessary from sector 2 depends only on final demand for the output of sector 2, f_2, and the amount of *intra*industry input that sector 2 buys from itself, a_{22} (Figure A2.2.1b). Similarly, if $a_{12} = 0$ – sector 2 buys no inputs from sector 1 – line (a) in the figure will have an infinite slope; it will be vertical through the point $f_1/(1-a_{11})$ on the x_1-axis (Figure A2.2.1c).

The geometry of the 2×2 case does not generalize easily, at least for $n > 3$. For this, we need some matrix terminology. The *minor* of an element a_{ij} in an $n \times n$ square

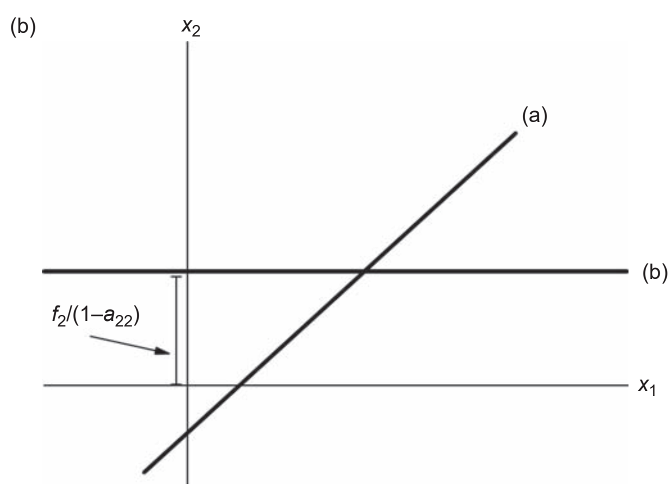

Figure A2.2.1b Solution Space Representation of (A2.2.2); $a_{21} = 0$

(c)

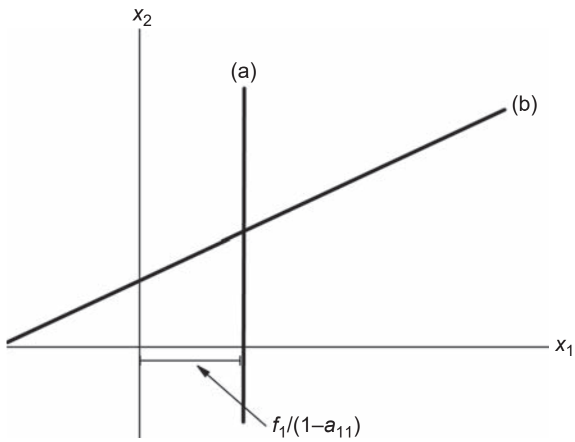

Figure A2.2.1c Solution Space Representation of (A2.2.2); $a_{12} = 0$

matrix, **A**, is defined as the determinant of the $(n-1) \times (n-1)$ matrix remaining when row i and column j are removed from **A** (Appendix A). Another kind of minor that is associated with a matrix (not with a particular element in a matrix) is a *principal minor*. If none, or one, or more than one row *and* the same columns are removed from **A**, the determinant of the remaining square matrix is a principal minor of **A**. Using the concept of principal minors, the Hawkins–Simon conditions for the 2×2 case in (A2.2.1) can be expressed compactly as the requirement that *all principal minors* of $(\mathbf{I} - \mathbf{A})$ be strictly positive – (a) in (A2.2.1) results from removing row and column 1 or row and column 2, (b) in (A2.2.1) results from removing no rows and columns. (It is impossible to remove more than $n-1$ rows and columns; if all n are gone, there is no matrix left.)

For a 3×3 matrix **A**, removal of row and column 1, *or* row and column 2, *or* row and column 3 leaves, in each case, a square 2×2 matrix. The determinants of those three matrices are all principal minors of **A** (sometimes called second-order principal minors, because they are determinants of 2×2 matrices). Moreover, removal of rows and columns 1 and 2, *or* rows and columns 1 and 3, *or* rows and columns 2 and 3 leaves, in each case, a square 1×1 matrix (the determinant of a 1×1 matrix is defined simply as the value of the element itself); these are the three first-order principal minors of **A**. By extension, the third-order principal minor in this case is just the determinant of the entire 3×3 matrix, when no rows and columns are removed. Thus there are seven principal minors in a 3×3 matrix.

This principal minor rule can be generalized; namely, regardless of the size of n, the parallel to (A2.2.1) is that *all* principal minors of $(\mathbf{I}-\mathbf{A})$ – first-order, second-order, . . ., nth-order – should be positive. The interested reader might try writing out the seven principal minors of a 3×3 $(\mathbf{I}-\mathbf{A})$ matrix. In the 4×4 case there are 15 principal minors. (For the reader familiar with the mathematics of combinations, this number is found as $C_0^4 + C_1^4 + C_2^4 + C_3^4 = 1+4+6+4 = 15$.) This gives some idea of the way in which the complexity of these rules increases with the number of sectors in the input–output model, and extension and application of the results in (A2.2.1) to conditions for an $n \times n$ system with n even modestly large would be cumbersome and tedious, even though the definition of principal minors of a matrix presents a simple way of expressing the rule for the general case. These conditions are totally impractical to check for large, real-world input–output systems. [For example, for a 10-sector model, the number of principal minors is 1023 (!).]

However, there is a large amount of published work on alternative sets of conditions on $\mathbf{A}$ and $\mathbf{f}$ that serve to identify when non-negative final demands will generate non-negative outputs. Dietzenbacher (2005) provides an extremely simple sufficient condition. If the original data are $\mathbf{Z}^0 > \mathbf{0}$ and $\mathbf{f}^0 \geq \mathbf{0}$ (with at least one $f_i^0 > 0$), then $\mathbf{L}^0 = (\mathbf{I} - \mathbf{A}^0)^{-1} > \mathbf{0}$ and $\mathbf{x}^1 = \mathbf{L}^0\mathbf{f}^1 \geq \mathbf{0}$ for any $\mathbf{f}^1 \geq \mathbf{0}$. These requirements on $\mathbf{Z}^0$ and $\mathbf{f}^0$ are easily checked by inspection, bypassing the need for the Hawkins–Simon principal minors. In fact, as noted in Dietzenbacher (2005), the positivity condition, $\mathbf{Z}^0 > \mathbf{0}$, can be relaxed to the requirement of non-negativity, $\mathbf{Z}^0 \geq \mathbf{0}$, using an assumption that allows $\mathbf{Z}^0$ to contain many zeros.[26] This allows for the more realistic case, especially in highly disaggregated models, of zero-valued intermediate flows between some sectors. An additional benefit is that derivation of these results does not depend on $a_{ij}^0 < 1$. When tables are based on transactions measured in physical terms it is entirely possible that some coefficients will be larger than 1 and hence that $N(\mathbf{A}) > 1$ – as we saw in section 2.6, above.

Problems

2.1 Dollar values of last year's interindustry transactions and total outputs for a two-sector economy (agriculture and manufacturing) are as shown below:

$$\mathbf{Z} = \begin{bmatrix} 500 & 350 \\ 320 & 360 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1000 \\ 800 \end{bmatrix}$$

a. What are the two elements in the final-demand vector $\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$?

b. Suppose that f_1 increases by \$50 and f_2 decreases by \$20. What new gross outputs would be necessary to satisfy the new final demands?

i. Find an approximation to the answer by using the first five terms in the power series, $\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^n$.

ii. Find the exact answer using the Leontief inverse.

[26] Both the assumption and the further analysis are considerably more complex and beyond the scope of this text – involving, for example, Frobenius theorems, indecomposable (irreducible) matrices, eigenvectors, and eigenvalues. The interested reader is referred to the thorough discussion of these and other mathematical issues in input–output analysis in Takayama (1985, Chapter 4).

2.2 Interindustry sales and total outputs in a small three-sector national economy for year t are given in the following table, where values are shown in thousands of dollars. (S_1, S_2 and S_3 represent the three sectors.)

	Interindustry Sales			
	S_1	S_2	S_3	Total Output
S_1	350	0	0	1000
S_2	50	250	150	500
S_3	200	150	550	1000

a. Find the technical coefficients matrix, **A**, and the Leontief inverse matrix, **L**, for this economy.
b. Suppose that because of government tax policy changes, final demands for the outputs of sectors 1, 2 and 3 are projected for next year (year $t + 1$) to be 1300, 100 and 200, respectively (also measured in thousands of dollars). Find the total outputs that would be necessary from the three sectors to meet this projected demand, assuming that there is no change in the technological structure of the economy (that is, assuming that the **A** matrix does not change from year t to year $t + 1$).
c. Find the original (year t) final demands from the information in the table of data. Compare with the projected (year $t + 1$) final demands. Also, compare the original total outputs with the outputs found in part b. What basic feature of the input–output model do these two comparisons illustrate?

2.3 Using the data of Problem 2.1, above, suppose that the household (consumption) expenditures part of final demand is \$90 from sector 1 and \$50 from sector 2. Suppose, further, that payments from sectors 1 and 2 for household labor services were \$100 and \$60, respectively; that total household (labor) income in the economy was \$300; and that household purchases of labor services were \$40. Close the model with respect to households and find the impacts on sectors 1 and 2 of a final demand of \$200 and \$1000 for sectors 1 and 2, respectively, using the Leontief inverse for the new 3×3 coefficient matrix. Compare the outputs of sectors 1 and 2 with those obtained without closing the model to households. How do you explain the differences?

2.4 Consider an economy organized into three industries: lumber and wood products, paper and allied products, and machinery and transportation equipment. A consulting firm estimates that last year the lumber industry had an output valued at \$50 (assume all monetary values are in units of \$100,000), 5 percent of which it consumed itself; 70 percent was consumed by final demand; 20 percent by the paper and allied products industry; 5 percent by the equipment industry. The equipment industry consumed 15 percent of its own products, out of a total of \$l00; 25 percent went to final demand; 30 percent to the lumber industry; 30 percent to the paper and allied products industry. Finally, the paper and allied products industry produced \$50, of which it consumed

10 percent; 80 percent went to final demand; 5 percent went to the lumber industry; and 5 percent to the equipment industry.

a. Construct the input–output transactions matrix for this economy on the basis of these estimates from last year's data. Find the corresponding matrix of technical coefficients, and show that the Hawkins–Simon conditions are satisfied.
b. Find the Leontief inverse for this economy.
c. A recession in the economy this year is reflected in decreased final demands, reflected in the following table:

Industry	% Decrease in Final Demand
Lumber & Wood Products	25
Machinery & Transportation Equipment	10
Paper & Allied Products	5

d. What would be the total production of all industries required to supply this year's decreased final demand? Compute the value-added and intermediate output vectors for the new transactions table.

2.5 Consider a simple two-sector economy containing industries A and B. Industry A requires \$2 million worth of its own product and \$6 million worth of Industry B's output in the process of supplying \$20 million worth of its own product to final consumers. Similarly, Industry B requires \$4 million worth of its own product and \$8 million worth of Industry A's output in the process of supplying \$20 million worth of its own product to final consumers.

a. Construct the input–output transactions table describing economic activity in this economy.
b. Find the corresponding matrix of technical coefficients and show that the Hawkins–Simon conditions are satisfied.
c. If in the year following the one in which the data for this model was compiled there were no changes expected in the patterns of industry consumption, and if a final demand of \$15 million worth of good A and \$18 million worth of good B were presented to the economy, what would be the total production of all industries required to supply this final demand as well as the interindustry activity involved in supporting deliveries to this final demand?

2.6 Consider the following transactions table, $\mathbf{Z}$, and total outputs vector, $\mathbf{x}$, for two sectors, A and B:

$$\mathbf{Z} = \begin{bmatrix} 6 & 2 \\ 4 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 20 \\ 15 \end{bmatrix}$$

a. Compute the value-added and final-demand vectors. Show that the Hawkins–Simon conditions are satisfied.

b. Consider the r-order round-by-round approximation of $\mathbf{x} = \mathbf{Lf}$ to be $\tilde{\mathbf{x}} = \sum_{i=0}^{r} A^i f$ (remember that $\mathbf{A}^0 = \mathbf{I}$). For what value of r do all the elements of $\tilde{\mathbf{x}}$ come within 0.2 of the actual values of $\mathbf{x}$?

c. Assume that the cost of performing impact analysis on the computer using the round-by-round method is given by $C_r = c_1 r + c_2(r - 1.5)$ where r is the order of the approximation (c_1 is the cost of an addition operation and c_2 is the cost of a multiplication operation). Also, assume that $c_1 = 0.5c_2$, that the cost of computing $(\mathbf{I}-\mathbf{A})^{-1}$ exactly is given by $C_e = 20c_2$ and the cost of using this inverse in impact analysis (multiplying it by a final-demand vector) is given by $C_f = c_2$. If we wish to compute the impacts (total outputs) of a particular (arbitrary) final-demand vector to within at least 0.2 of the actual values of $\mathbf{x} = \mathbf{Lf}_a$, where $\mathbf{f}_a$ is an arbitrary final-demand vector, should we use the round-by-round method or should we compute the exact inverse and then perform impact analysis? The idea is to find the least-cost method for computing the solution.

d. Suppose we had five arbitrary final-demand vectors whose impact we wanted to assess. How would you now answer part c?

e. For what number of final-demand vectors does it not make any difference which method we use (in answer to the question in part c)?

2.7 Given the following transactions table for industries a, b, and c, and the total output as shown, compute the final-demand vectors and show that the inverse of $(\mathbf{I} - \mathbf{A})$ exists.

Industries	*a*	*b*	c	Total Output
a	3	8	6	22
b	2	4	5	18
c	7	3	9	31

Use the power series to approximate **x** to within 0.1 of the actual output values shown above. What was the highest power of **A** required?

2.8 Consider the following transactions and total output data for an eight-sector economy.

$$\mathbf{Z} = \begin{bmatrix} 8,565 & 8,069 & 8,843 & 3,045 & 1,124 & 276 & 230 & 3,464 \\ 1,505 & 6,996 & 6,895 & 3,530 & 3,383 & 365 & 219 & 2,946 \\ 98 & 39 & 5 & 429 & 5,694 & 7 & 376 & 327 \\ 999 & 1,048 & 120 & 9,143 & 4,460 & 228 & 210 & 2,226 \\ 4,373 & 4,488 & 8,325 & 2,729 & 29,671 & 1,733 & 5,757 & 14,756 \\ 2,150 & 36 & 640 & 1,234 & 165 & 821 & 90 & 6,717 \\ 506 & 7 & 180 & 0 & 2,352 & 0 & 18,091 & 26,529 \\ 5,315 & 1,895 & 2,993 & 1,071 & 13,941 & 434 & 6,096 & 46,338 \end{bmatrix}$$

$$\mathbf{x}' = \begin{bmatrix} 37,610 & 45,108 & 46,323 & 41,059 & 209,403 & 11,200 & 55,992 & 161,079 \end{bmatrix}$$

a. Compute **A** and **L**.
b. If final demands in sectors 1 and 2 increase by 30 percent while that in sector 5 decreases by 20 percent (while all other final demands are unchanged), what new total outputs will be necessary from each of the eight sectors in this economy?

2.9 Consider the following two-sector input–output table measured in millions of dollars:

	Manuf.	Services	Final Demand	Total Output
Manufacturing	10	40	50	100
Services	30	25	85	140
Value Added	60	75	135	
Total Output	100	140		240

If labor costs in the services sector increase, causing a 25 percent increase in value added inputs required per unit of services and labor costs in manufacturing decrease by 25 percent, what are the resulting changes in relative prices of manufactured goods and services?

2.10 For the US direct requirements table given in Table 2.7, what would be the impact on relative prices if a national corporate income tax increased total value added of primary industries (agriculture and mining) by 10 percent, construction and manufacturing by 15 percent, and all other sectors by 20 percent?

2.11 Consider an input–output economy with three sectors: agriculture, services, and personal computers. The matrix of interindustry transactions and vector of total outputs are given, respectively, by $\mathbf{Z} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}$ so that $\mathbf{f} = \mathbf{x} - \mathbf{Zi} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. Notice that this is a closed economy where all industry outputs become inputs. In other words, with the given $\mathbf{x}$, the vector of total value added is found by $\mathbf{v}' = \mathbf{x}' - \mathbf{i}'\mathbf{Z} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ and, of course, gross domestic product is $\mathbf{v}'\mathbf{i} = \mathbf{i}'\mathbf{f} = 0$. Does **L** exist for this economy? Suppose we determine that all of the inputs for the personal computers sector are imported and we seek to create a domestic transactions matrix by “opening” the economy to imports, i.e., transfer the value of all inputs to personal computers to final demand. What are the modified values of **Z**, **f** and **v**? What is the new value of gross domestic product? Does **L** exist for this modified representation of the economy? If so, compute it.

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