Introduction to Input-Output Analysis
Lecture 2: Closed Model and Multipliers

Enno Schröder
Delft University of Technology
Academic Year 2017-2018, Quarter 4

Outline

Today

- Beyond gross output effects
- Closing the IO model
- Multipliers
- Multiregional models
- Info about assessment

The IO model $\boldsymbol{x} = \boldsymbol{L}\,\boldsymbol{f}$ translates exogenous final demands into gross output effects

- What if final demand is $\boldsymbol{f}^{new}$? Then gross output will be $\boldsymbol{x}^{new}$

$$\begin{bmatrix} x_1^{new} \\ x_2^{new} \\ \vdots \\ x_n^{new} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{12} & \cdots & l_{1n} \\ l_{21} & l_{22} & & \\ \vdots & & \ddots & \\ l_{n1} & & & l_{nn} \end{bmatrix} \begin{bmatrix} f_1^{new} \\ f_2^{new} \\ \vdots \\ f_n^{new} \end{bmatrix}$$

IO model can be used to predict other variables than gross output (MB, p.24)

- E.g. assume that ratio of employment to gross output is stable

$$ec_i = \frac{\text{employment in hours}}{\text{gross output in dollars}} = \frac{e_i}{x_i} = const.$$

- E.g. assume that ratio of value added to gross output is stable

$$vc_i = \frac{\text{value added in dollars}}{\text{gross output in dollars}} = \frac{v_i}{x_i} = const.$$

- To calculate value added effects, multiply gross output effects x_i^{new} by **value added coefficients**, $v_i^{new} = vc_i \cdot x_i^{new}$

$$\begin{aligned} v_1^{new} &= vc_1 \cdot x_1^{new} \\ v_2^{new} &= vc_2 \cdot x_2^{new} \\ &\vdots \\ v_n^{new} &= vc_n \cdot x_n^{new} \end{aligned}$$

The IO model $\widehat{\mathbf{vc}}\,\boldsymbol{x} = \widehat{\mathbf{vc}}\,\boldsymbol{L}\,\boldsymbol{f}$ translates exogenous final demands into value added effects

- What if final demand is $\boldsymbol{f}^{new}$? Then value added will be $\boldsymbol{v}^{new}$

$$\begin{bmatrix} v_1^{new} \\ v_2^{new} \\ \vdots \\ v_n^{new} \end{bmatrix} = \begin{bmatrix} vc_1 & 0 & \cdots & 0 \\ 0 & vc_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & vc_n \end{bmatrix} \begin{bmatrix} l_{11} & l_{12} & \cdots & l_{1n} \\ l_{21} & l_{22} & & \\ \vdots & & \ddots & \\ l_{n1} & & & l_{nn} \end{bmatrix} \begin{bmatrix} f_1^{new} \\ f_2^{new} \\ \vdots \\ f_n^{new} \end{bmatrix}$$

Typical element

$$v_i^{new} = vc_i \cdot x_i^{new} = \frac{v_i}{x_i} \cdot x_i^{new}$$

Typical element on diagonal

$$vc_i = \frac{v_i}{x_i}$$

Gross output effect in corn economy

- Interindustry flow matrix $\boldsymbol{Z}$ is scalar z and $z > 0$
- Economy is viable, $x > z$
- Technical coefficients matrix $\boldsymbol{A}$ is scalar $a = z/x$ and $0 < a < 1$
- Leontief inverse $\boldsymbol{L}$ is scalar $l = (1 - a)^{-1}$ and $l > 1$

$$x = z + f$$
$$x = \frac{z}{x} \cdot x + f$$
$$x = a \cdot x + f$$
$$x = (1 - a)^{-1} \cdot f$$
$$x^{new} = (1 - a)^{-1} \cdot f^{new}$$

Value added effect in corn economy

- Define value added coefficient $vc = v/x$
- To compute value added effect, multiply gross output effect by value added coefficient

$$\begin{aligned} v^{new} &= vc \cdot x^{new} \\ &= vc \cdot (1-a)^{-1} \cdot f^{new} \\ &= f^{new} \end{aligned}$$

		Buying sectors (producers as consumers)		
		$j = 1$	Final demand	Total outputs
Selling sectors	$i = 1$	z	f	x
	Primary inputs	v	\$0	
	Total inputs	x		

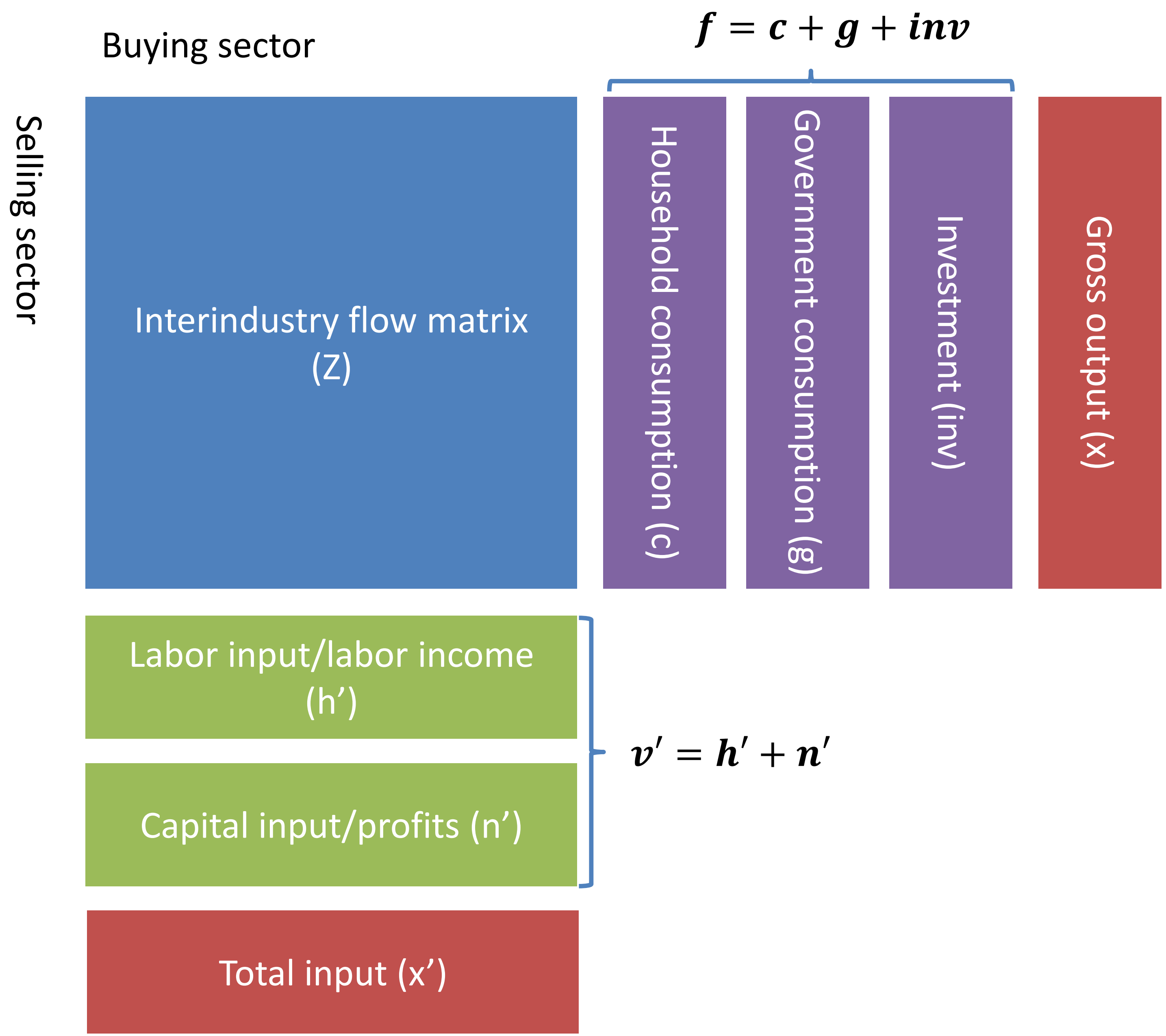
Outline

Today

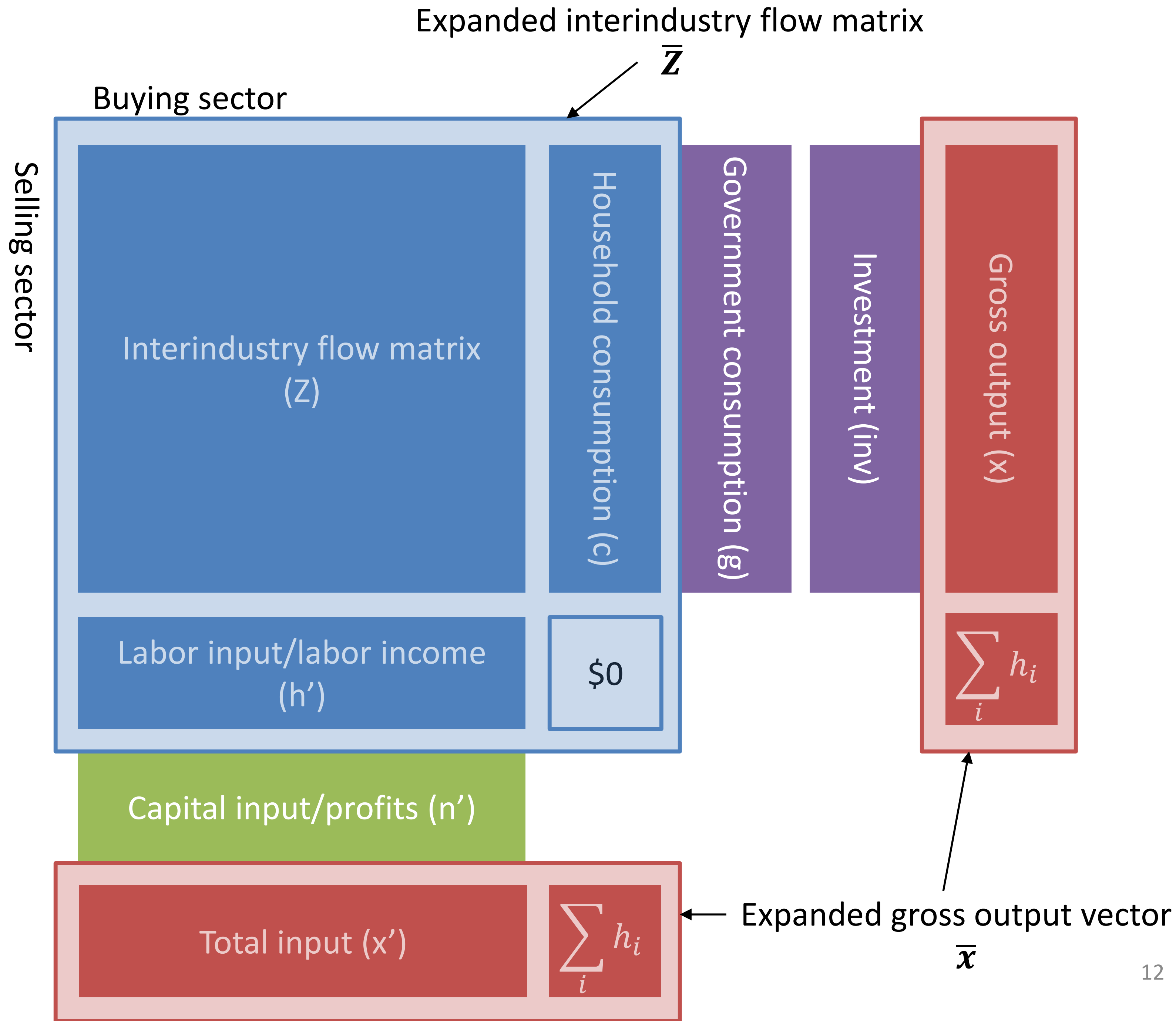
- Beyond gross output effects
- Closing the IO model
- Multipliers
- Multiregional models
- Info about assessment

In IO analysis, closing the model means endogenizing (parts of) final demand

- In open IO model, intermediate goods demand is endogenous and final demand ($\boldsymbol{f}$) is exogenous
- Final demand -> production -> household income -> household consumption
- To reflect the **induced** response of household consumption, we add consumption functions to IO model



$\boldsymbol{f} = \boldsymbol{c} + \boldsymbol{g} + \boldsymbol{inv}$
Buying sector
Selling sector
Interindustry flow matrix (Z)
Household consumption (c)
Government consumption (g)
Investment (inv)
Gross output (x)
Labor input/labor income (h')
Capital input/profits (n')
$\boldsymbol{v}' = \boldsymbol{h}' + \boldsymbol{n}'$
Total input (x')



Expanded interindustry flow matrix
$\overline{Z}$
Buying sector
Selling sector
Interindustry flow matrix
(Z)
Household consumption (c)
Government consumption (g)
Investment (inv)
Gross output (x)
Labor input/labor income
(h')
$0
$\sum_i h_i$
Capital input/profits (n')
Total input (x')
$\sum_i h_i$
Expanded gross output vector
$\overline{x}$

Closing the IO model with respect to households

- $\boldsymbol{c} =$ household consumption vector, $\boldsymbol{h} =$ household income vector
- $\overline{\boldsymbol{Z}} = \begin{bmatrix} \boldsymbol{Z} & \boldsymbol{c} \\ \boldsymbol{h}' & 0 \end{bmatrix}$ = expanded interindustry flow matrix
- $h = \boldsymbol{i}'\boldsymbol{h} = \sum_i \boldsymbol{h}_i =$ "output" of household sector
- $\overline{\boldsymbol{x}} = \begin{bmatrix} \boldsymbol{x} \\ h \end{bmatrix}$ = expanded gross output vector
- $\overline{\boldsymbol{A}} = \overline{\boldsymbol{Z}}\,\widehat{\overline{\boldsymbol{x}}}^{-1} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{cc} \\ \boldsymbol{hc}' & 0 \end{bmatrix}$ = expanded technical coefficients matrix
- Typical element of income coefficient vector $\boldsymbol{hc}$ is h_i/x_i
- Typical element of consumption coefficient vector $\boldsymbol{cc}$ is c_i/h
- $\overline{\boldsymbol{x}}^{new} = (\boldsymbol{I} - \overline{\boldsymbol{A}})^{-1}\,\overline{\boldsymbol{f}}^{new} = \begin{bmatrix} \boldsymbol{L}_{11} & \boldsymbol{L}_{12} \\ \boldsymbol{L}_{21} & \boldsymbol{L}_{22} \end{bmatrix} \overline{\boldsymbol{f}}^{new}$

Closing the corn economy with respect to households

- $c =$ household consumption, $h =$ household income
- Expanded interindustry flow matrix $\bar{\boldsymbol{Z}} = \begin{bmatrix} z & c \\ h & 0 \end{bmatrix}$
- Expanded gross output vector $\bar{\boldsymbol{x}} = \begin{bmatrix} x \\ h \end{bmatrix}$
- Expanded technical coefficients matrix $\bar{\boldsymbol{A}} = \bar{\boldsymbol{Z}}\,\hat{\bar{\boldsymbol{x}}}^{-1} = \begin{bmatrix} a & \frac{c}{h} \\ \frac{h}{x} & 0 \end{bmatrix}$
- Leontief inverse of closed model $\bar{\boldsymbol{L}} = (\boldsymbol{I} - \bar{\boldsymbol{A}})^{-1} =$

$$\begin{bmatrix} \boldsymbol{L}_{11} & \boldsymbol{L}_{12} \\ \boldsymbol{L}_{21} & \boldsymbol{L}_{22} \end{bmatrix} = \begin{bmatrix} 1-a & -\frac{c}{h} \\ -\frac{h}{x} & 1 \end{bmatrix}^{-1} = \frac{1}{1-a-c/x} \begin{bmatrix} 1 & \frac{c}{h} \\ \frac{h}{x} & 1-a \end{bmatrix}$$

Gross output effects in the closed corn economy

- What if final demand is $\bar{\boldsymbol{f}}^{new} = \begin{bmatrix} f^{new} \\ 0 \end{bmatrix}$? Then gross output will be

$$\begin{aligned}
\bar{\boldsymbol{x}}^{new} &= \bar{\boldsymbol{L}}\,\bar{\boldsymbol{f}}^{new} \\
&= \frac{1}{1-a-c/x}\begin{bmatrix} 1 & \frac{c}{h} \\ \frac{h}{x} & 1-a \end{bmatrix}\begin{bmatrix} f^{new} \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \left(1-a-\frac{c}{x}\right)^{-1} \cdot f^{new} \\ \frac{h}{x} \cdot \left(1-a-\frac{c}{x}\right)^{-1} \cdot f^{new} \end{bmatrix}
\end{aligned}$$

If focus is on original sectors, use **truncated Leontief inverse** of closed model

- $\overline{\boldsymbol{L}} = (\boldsymbol{I} - \overline{\boldsymbol{A}})^{-1} = \begin{bmatrix} \boldsymbol{L}_{11} & \boldsymbol{L}_{12} \\ \boldsymbol{L}_{21} & \boldsymbol{L}_{22} \end{bmatrix}$
- $\boldsymbol{L}_{11}$ is truncated Leontief inverse
- In general $\boldsymbol{L}_{11} \gg \boldsymbol{L}$
- In closed model production must satisfy final demand, intermediate input demand, and **induced consumption** demand

- Output effect in open corn economy:

$$x^{new} = \boldsymbol{L}\, f^{new} = (1 - a)^{-1} \cdot f^{new}$$

- Output effect in closed corn economy:

$$x^{new} = \boldsymbol{L}_{11}\, f^{new} = (1 - a - c/x)^{-1} \cdot f^{new}$$

Value added effect in closed corn economy

- Define value added coefficient $vc = v/x$
- Remember $vc = 1 - a$
- To compute value added effect, multiply gross output effect by value added coefficient

$$\begin{aligned} v^{new} &= vc \cdot x^{new} \\ &= vc \cdot \left(1 - a - \frac{c}{x}\right)^{-1} \cdot f^{new} \\ &= \frac{1 - a}{1 - a - c/x} \cdot f^{new} \end{aligned}$$

Outline

Today

- Beyond gross output effects
- Closing the IO model
- Multipliers
- Multiregional models
- Info about assessment

Multipliers

- Multipliers are defined as $\frac{\text{Total change in some variable}}{\text{Initial exog. change in some variable}}$
- Initial effect is $\boldsymbol{f}^{new}$ and total effect is $\boldsymbol{x}^{new}$ or $\boldsymbol{v}^{new}$ (or any other impact)
- **Simple multipliers** are computed from open IO model and include
  - Direct/initial effect (exogenous final demand change)
  - Indirect effect (endogenous intermediate input demand)
- **Total multipliers** are computed from closed IO model and include
  - Direct/initial effect
  - Indirect effect
  - Induced effect (endogenous final demand)

Multipliers in the corn economy

- Simple output multiplier $\frac{x^{new}}{f^{new}} = (1 - a)^{-1} > 1$
- Simple value added multiplier $\frac{x^{new}}{f^{new}} = 1$
- Total output multiplier $\frac{x^{new}}{f^{new}} = (1 - a - c/x)^{-1} > 1$
- Total value added multiplier $\frac{x^{new}}{f^{new}} = (1 - a)/(1 - a - c/x) > 1$

$$
\begin{array}{ccccccccc}
x_1^{new} & = & l_{11} \cdot f_1^{new} & + & l_{12} \cdot f_2^{new} & + & \cdots & + & l_{1n} \cdot f_n^{new} \\
x_2^{new} & = & l_{21} \cdot f_1^{new} & + & l_{22} \cdot f_2^{new} & + & \cdots & + & l_{2n} \cdot f_n^{new} \\
\vdots & & & & & & & & \\
x_n^{new} & = & l_{n1} \cdot f_1^{new} & + & l_{n2} \cdot f_2^{new} & + & \cdots & + & l_{nn} \cdot f_n^{new}
\end{array}
$$

- What if final demand is $\boldsymbol{f}^{new} = [50, 25, \dots, 30]'$?

$$
\begin{array}{ccccccccc}
x_1^{new} & = & l_{11} \cdot 50 & + & l_{12} \cdot 25 & + & \cdots & + & l_{1n} \cdot 30 \\
x_2^{new} & = & l_{21} \cdot 50 & + & l_{22} \cdot 25 & + & \cdots & + & l_{2n} \cdot 30 \\
\vdots & & & & & & & & \\
x_n^{new} & = & l_{n1} \cdot 50 & + & l_{n2} \cdot 25 & + & \cdots & + & l_{nn} \cdot 30
\end{array}
$$

- What if final demand is $\boldsymbol{f}^{new} = [50, 0, \dots, 0]'$?

$$
\begin{array}{ccc}
x_1^{new} & = & l_{11} \cdot 50 \\
x_2^{new} & = & l_{21} \cdot 50 \\
\vdots & & \\
x_n^{new} & = & l_{n1} \cdot 50
\end{array}
$$

Column sums of $\boldsymbol{L}$ are simple output multipliers

- What if final demand is $\boldsymbol{f}^{new} = [1, 0, \dots, 0]'$?

$$
\begin{aligned}
x_{i=1}^{new} &= l_{11} \\
x_{i=2}^{new} &= l_{21} \\
&\vdots \\
x_{i=n}^{new} &= l_{n1}
\end{aligned}
$$

- Elements of $\boldsymbol{L}$ are **sector-to-sector** multipliers, the output effect in sector i when final demand for sector j increases by one unit
- Columns sums of $\boldsymbol{L}$ are **sector-to-economy** multipliers, the economy-wide output effect when final demand for sector j increases by one unit
- They represent the total value of output in all sectors of the economy that is necessary in order to satisfy a dollar's worth of final demand for sector j's output

Remember?

- The IO model $\widehat{\mathbf{vc}}\,\boldsymbol{x} = \widehat{\mathbf{vc}}\,\boldsymbol{L}\,\boldsymbol{f}$ translates exogenous final demands into value added effects

$$\begin{bmatrix} v_1^{new} \\ v_2^{new} \\ \vdots \\ v_n^{new} \end{bmatrix} = \begin{bmatrix} vc_1 & 0 & \cdots & 0 \\ 0 & vc_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & vc_n \end{bmatrix} \begin{bmatrix} l_{11} & l_{12} & \cdots & l_{1n} \\ l_{21} & l_{22} & & \\ \vdots & & \ddots & \\ l_{n1} & & & l_{nn} \end{bmatrix} \begin{bmatrix} f_1^{new} \\ f_2^{new} \\ \vdots \\ f_n^{new} \end{bmatrix}$$

Column sums of $\widehat{\mathbf{vc}}\,\boldsymbol{L}$ are simple value added multipliers

- What if final demand is $\boldsymbol{f}^{new} = [1, 0, \dots, 0]'$?

$$
\begin{aligned}
vc_1 \cdot x_{i=1}^{new} &= vc_1 \cdot l_{11} \\
vc_2 \cdot x_{i=2}^{new} &= vc_2 \cdot l_{21} \\
&\vdots \\
vc_n \cdot x_{i=n}^{new} &= vc_n \cdot l_{n1}
\end{aligned}
$$

- Elements of $\widehat{\mathbf{vc}}\,\boldsymbol{L}$ are sector-to-sector multipliers, the value added effect in sector i when final demand for sector j increases by one unit
- Columns sums of $\widehat{\mathbf{vc}}\,\boldsymbol{L}$ are sector-to-economy multipliers, the economy-wide value added effect when final demand for sector j increases by one unit
- They represent the total value added in all sectors of the economy that is generated by one dollar's worth of final demand for sector j's output

Total multipliers are larger than simple multipliers

- They include the induced consumption effect
- To calculate total (truncated) multipliers, replace Leontief inverse of open model $\boldsymbol{L}$ with truncated Leontief inverse of closed model $\boldsymbol{L}_{11}$

Compact matrix notation

- $\boldsymbol{i}'\,\boldsymbol{L}$ gives row vector of simple output multipliers
- $\boldsymbol{i}'\,\widehat{\boldsymbol{vc}}\,\boldsymbol{L}$ gives row vector of simple value added multipliers
- $\boldsymbol{i}'\,\boldsymbol{L}_{11}$ gives row vector of total truncated output multipliers
- $\boldsymbol{i}'\,\widehat{\boldsymbol{vc}}\,\boldsymbol{L}_{11}$ gives row vector of total truncated value added multipliers

Outline

Today

- Beyond gross output effects
- Closing the IO model
- Multipliers
- Multiregional models
- Info about assessment

Multiregional input-output (MRIO) models

- Interindustry flow matrix: $\boldsymbol{Z} = \begin{bmatrix} \boldsymbol{Z}^{11} & \boldsymbol{Z}^{21} & \boldsymbol{Z}^{31} \\ \boldsymbol{Z}^{21} & \boldsymbol{Z}^{22} & \boldsymbol{Z}^{23} \\ \boldsymbol{Z}^{31} & \boldsymbol{Z}^{32} & \boldsymbol{Z}^{33} \end{bmatrix}$
- If there are $m = 3$ countries and $n = 2$ industries per country, $\dim(\boldsymbol{Z}) = (nm, nm) = (6{,}6)$
- Gross output vector: $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}^{1} \\ \boldsymbol{x}^{2} \\ \boldsymbol{x}^{3} \end{bmatrix}$
- World final demand vector:

$$\boldsymbol{f} = \boldsymbol{f}^1 + \boldsymbol{f}^2 + \boldsymbol{f}^3 = \begin{pmatrix} \boldsymbol{f}^{11} \\ \boldsymbol{f}^{21} \\ \boldsymbol{f}^{31} \end{pmatrix} + \begin{pmatrix} \boldsymbol{f}^{12} \\ \boldsymbol{f}^{22} \\ \boldsymbol{f}^{32} \end{pmatrix} + \begin{pmatrix} \boldsymbol{f}^{13} \\ \boldsymbol{f}^{23} \\ \boldsymbol{f}^{33} \end{pmatrix}$$

Intra-regional and inter-regional multipliers

- Leontief matrix: $\boldsymbol{L} = \begin{bmatrix} \boldsymbol{L}^{11} & \boldsymbol{L}^{21} & \boldsymbol{L}^{31} \\ \boldsymbol{L}^{21} & \boldsymbol{L}^{22} & \boldsymbol{L}^{23} \\ \boldsymbol{L}^{31} & \boldsymbol{L}^{32} & \boldsymbol{L}^{33} \end{bmatrix}$
- To get the intra-regional multiplier of country 1's industry 1, sum up the red values in $\boldsymbol{L}^{11} = \begin{bmatrix} \color{red}{l_{11}^{11}} & l_{12}^{11} \\ \color{red}{l_{21}^{11}} & l_{22}^{11} \end{bmatrix}$
- To get the inter-regional multiplier of country 1's industry 1, sum up the red values in $\begin{bmatrix} \boldsymbol{L}^{21} \\ \boldsymbol{L}^{31} \end{bmatrix} = \begin{bmatrix} \color{red}{l_{11}^{21}} & l_{12}^{21} \\ \color{red}{l_{21}^{21}} & l_{22}^{21} \\ \color{red}{l_{11}^{31}} & l_{12}^{31} \\ \color{red}{l_{21}^{31}} & l_{12}^{31} \end{bmatrix}$

Assignment

- You will work with a particular country and a particular industry (check your email account later)
- Submission deadline is Monday, June 4, 23.59. No deadline extension under any circumstances
- If you email me, I will post your question and my reply here: Brightspace -> Collaboration -> Discussion -> Q&A Forum