6 Multipliers in the Input–Output Model

6.1 Introduction

One of the major uses of the information in an input–output model is to assess the effect on an economy of changes in elements that are exogenous to the model of that economy. For example,

> Leontief input–output economics derive their significance largely from the fact that output multipliers measuring the combined effects of the direct and indirect repercussions of a change in final demand were readily calculated. (Steenge, 1990, p. 377.)

In Chapters 2 and 3 we presented several numerical illustrations of the ways in which assumed changes in final-demand elements (e.g., federal government spending, household consumption, exports) were translated, via the appropriate Leontief inverse, to corresponding output changes in the industrial sectors of the economy. When the exogenous changes occur because of the actions of only one "impacting agent" (or a small number of such agents) and when the changes are expected to occur in the short run (e.g., next year), this is usually called *impact analysis*. Examples are a change in federal government defense spending or in consumer demand for recreation vehicles.

On the other hand, when longer-term and broader changes are examined, then we are dealing with projections and forecasting. If we project the levels of final demand for outputs of *all* sectors in an economy five years hence, and estimate, using the Leontief inverse, the outputs from all sectors that will be needed to satisfy this demand, this is an exercise in *forecasting*. As the period of projection gets longer, the accuracy of such an exercise tends to decrease, both because our ability to forecast the new final demands accurately (the elements of **f**) will diminish and also because the coefficients matrix – the elements of **A** and hence of **L** – may have become outdated. (The issue of temporal stability of input–output coefficients is examined in Chapter 7.) If the model is built from commodity–industry accounts, then it is the matrices **B**, **C** and/or **D** that may become out of date.

In either impact analysis or forecasting, the general form of the model is $\mathbf{x} = \mathbf{Lf}$ [or $\Delta\mathbf{x} = \mathbf{L}\Delta\mathbf{f}$], and the usefulness of the result, $\mathbf{x}$ (or $\Delta\mathbf{x}$), will depend on the "correctness" of both the Leontief inverse and the final-demand vector. Our primary concern in this section is with the elements a_{ij}, and hence with $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1}$. The $\mathbf{f}$ (or $\Delta\mathbf{f}$) vector incorporates the assumed or projected behavior of one or more final-demand elements, and accuracy in the estimation of these elements is also of paramount importance to generating an accurate result. When the question is one of impact, then the final-demand value or values are usually completely specified – for example, what is the impact, by sector, of a new order for $2.5 million worth of sector j output by the federal government? Then $\Delta\mathbf{f}$ contains 2.5 (million) in the jth row and zeros elsewhere.

Alternatively, to find $\mathbf{x}$ for some future year requires a projection of both $\mathbf{A}$ and $\mathbf{f}$ to that year. We will investigate some of the approaches for changing $\mathbf{A}$ over time in Chapter 7. The projection of $\mathbf{f}$ is a problem that is often approached via econometric models. The input–output forecasts of 1985 industrial outputs (and employment) for the US economy in Almon *et al.* (1974, Chapters 8 and 9) depend on detailed and painstaking projections of each of the components of final demand – personal consumption expenditures, investment in capital equipment, construction, inventories, imports and exports, and government expenditures (1974, Chapters 2 through 7, respectively). In some but by no means all "joined" input–output and econometric models, the econometric model provides a forecast of the final demands, which then "drive" the input–output model. (There is a growing literature on this issue of the interactions between input–output models and econometric models, particularly at the regional level. Some of this is explored in Chapter 14.)

A number of summary measures, derived from the elements of $\mathbf{L}$, are often employed in impact analysis; these are input–output multipliers. We examine multipliers in this chapter.

6.2 General Structure of Multiplier Analysis

Several of the most frequently used types of multipliers are those that estimate the effects of exogenous changes on (a) outputs of the sectors in the economy, (b) income earned by households in each sector because of the new outputs, (c) employment (jobs, in physical terms) that is expected to be generated in each sector because of the new outputs and (d) the value added that is created by each sector in the economy because of the new outputs. We examine these in this section.

The notion of multipliers rests upon the difference between the *initial* effect of an exogenous change and the *total* effects of that change. The total effects can be defined either as the *direct* and *indirect* effects (found from an input–output model that is open with respect to households) or as *direct, indirect* and *induced* effects (found from a model that is closed with respect to households).[1] The multipliers that incorporate direct and indirect effects are also known as *simple* multipliers. When direct, indirect and induced effects are captured, they are often called *total* multipliers.

[1] In some discussions of multipliers in an input–output model, what we have called the *initial* effect is termed the *direct* effect. For later exposition – for example, in looking at shortcut methods for finding multipliers – when the power series approximation

$$(\mathbf{I}-\mathbf{A})^{-1}=\mathbf{I}+\mathbf{A}+\mathbf{A}^2+\mathbf{A}^3+\cdots$$

will be used, it seems to us preferable to associate "initial" with the $\mathbf{I}$ term, "direct" with $\mathbf{A}$, and "indirect" with the remaining terms, $\mathbf{A}^2+\mathbf{A}^3+\cdots$

6.2.1 Output Multipliers

An output multiplier for sector j is defined as the total value of production in all sectors of the economy that is necessary in order to satisfy a dollar's worth of final demand for sector j's output.

Simple Output Multipliers For the simple output multiplier, this total production is obtained from a model with households exogenous. The initial output effect on the economy is defined to be just the initial dollar's worth of sector j output needed to satisfy the additional final demand. Then, formally, the output multiplier is the ratio of the direct and indirect effect to the initial effect alone.

We continue with the small example in Chapter 2, section 2.3, where

$$\mathbf{A} = \begin{bmatrix} .15 & .25 \\ .20 & .05 \end{bmatrix}$$

and

$$\mathbf{L} = \begin{bmatrix} 1.254 & .330 \\ .264 & 1.122 \end{bmatrix}$$

(In the remainder of this book we will sometimes keep three figures to the right of the decimal point and sometimes four, depending on the purposes of the numerical illustration.) Note that $\Delta\mathbf{f}(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ indicates an additional dollar's worth of final demand for the output of sector 1 only, and $\Delta\mathbf{f}(2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ indicates, similarly, an additional dollar's worth of final demand for the output of sector 2 only. Consider $\Delta\mathbf{f}(1)$; the implications for sectors 1 and 2 are found as $\mathbf{L}\Delta\mathbf{f}(1)$. Denote this by $\Delta\mathbf{x}(1)$, so

$$\Delta\mathbf{x}(1) = \begin{bmatrix} 1.254 & .330 \\ .264 & 1.122 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.254 \\ .264 \end{bmatrix} \tag{6.1}$$

This is, of course, just the first column of $\mathbf{L}$ – $\begin{bmatrix} l_{11} \\ l_{21} \end{bmatrix}$.

The additional outputs of \$1.254 from sector 1 and \$0.264 from sector 2 are required for a dollar of new final demand for the output of sector 1 *only*. The \$1.254 from sector 1 represents \$1.00 to satisfy the original new dollar of final demand plus an additional \$0.254 for intra- and interindustry use. The \$0.264 from sector 2 is for intra- and interindustry use only. The sector 1 output multiplier, $m(o)_1$, is defined as the sum of the elements in the $\Delta\mathbf{x}(1)$ column, namely \$1.518, divided by \$1; $m(o)_1 = \$1.518/\$1 = 1.518$, a dimensionless number. The \$1 in the denominator is the initial effect on sector 1 output of the new dollar's worth of final demand for sector 1's product; the dollar's worth of final demand becomes an additional dollar's worth of sector 1 output as the first term in the series assessment of total direct and indirect effects on sector 1 production. Formally, using $\mathbf{i}' = [1\ 1]$ as usual to generate column sums

$$m(o)_1 = \mathbf{i}'\Delta\mathbf{x}(1) = \sum_{i=1}^{n} l_{i1} \tag{6.2}$$

where $n = 2$ in this example.

Similarly,

$$\Delta\mathbf{x}(2) = \begin{bmatrix} 1.254 & .330 \\ .264 & 1.122 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} .330 \\ 1.122 \end{bmatrix} = \begin{bmatrix} l_{12} \\ l_{22} \end{bmatrix}$$

and

$$m(o)_2 = \mathbf{i}'\Delta\mathbf{x}(2) = \sum_{i=1}^{n} l_{i2} \tag{6.3}$$

Here $m(o)_2 = 1.452$. In general, the simple output multiplier for sector j is

$$m(o)_j = \sum_{i=1}^{n} l_{ij} \tag{6.4}$$

Thus, for example, if a government agency were trying to determine the differential effects of spending an additional dollar (or \$100, or \$1,000,000, or whatever amount) on the output of a sector, comparison of output multipliers would show where this spending would have the greatest impact in terms of total dollar value of output generated throughout the economy. Note that when maximum total output effects are the exclusive goal of government spending, it would always be rational to spend all the money in the sector with the largest output multiplier. Even with anticipated expenditures of \$1,000,000, there would be no reason, on the basis of output multipliers alone, to divide that spending between the sectors.

Of course, there might well be other reasons – taking into account strategic factors, equity, capacity constraints for sectoral production, and so on – for using some of the new final-demand dollars on the output of the other sector (or sectors, when $n > 2$). Note also that multipliers of this sort may overstate the effect on the economy in question if some sectors are operating at or near capacity and hence some of the needed new inputs would have to be imported to the economy and/or outputs from some sectors would be shifted from exports and kept in the economy for use as inputs. Phenomena such as these will assume even more importance in regional models.

We see that $\mathbf{L}$ is a matrix of sector-to-sector multipliers, l_{ij}, relating final demand in sector j to output in sector i. Output multipliers (column sums of $\mathbf{L}$) represent sector-to-economy multipliers, relating final demand in sector j to economy-wide output. For an n-sector model, denote the row vector of these multipliers by $\mathbf{m}(o) = [m(o)_1, \ldots, m(o)_n]$.[2]

[2] Strictly speaking, one expects a row vector to include a "prime" in its designation, as with $\mathbf{x}$ and $\mathbf{x}'$ in earlier chapters. However, here and throughout this discussion of multipliers we simply define various rows of multipliers without the prime to save on notational complexity.

With $\underset{(1\times n)}{\mathbf{i}'} = [1, \ldots, 1]$, we have

$$\mathbf{m}(o) = \underbrace{\mathbf{i}' \overbrace{\mathbf{L}}^{\substack{\textit{Sector-demand-}\\ \textit{to-sector-output}\\ \textit{multipliers}}}}_{\substack{\textit{Sector-demand-}\\ \textit{to-economy-wide-}\\ \textit{output multipliers}}} \tag{6.5}$$

We will see that many additional input–output multiplier variations build on this representation. All that is required is to alter the elements in the multiplier matrix so that instead of $(\Delta f_j = 1) \rightarrow (\Delta x_i)$ they represent $(\Delta f_j = 1) \rightarrow$ (some function of Δx_i), such as employment or energy use or pollution emissions.

Total Output Multipliers If we consider the input coefficients matrix closed with respect to households (as described in section 2.5) we capture in the model the additional *induced* effects of household income generation through payments for labor services and the associated consumer expenditures on goods produced by the various sectors. Continuing with the example from section 2.5, the augmented coefficient matrix, with an added household row and column, was

$$\bar{\mathbf{A}} = \begin{bmatrix} .15 & .25 & .05 \\ .20 & .05 & .40 \\ .30 & .25 & .05 \end{bmatrix}$$

and the Leontief inverse, with elements $\bar{l}_{ij}$, was

$$\bar{\mathbf{L}} = (\mathbf{I} - \bar{\mathbf{A}})^{-1} = \begin{bmatrix} 1.365 & 0.425 & 0.251 \\ 0.527 & 1.348 & 0.595 \\ 0.570 & 0.489 & 1.289 \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{L}}_{11} & \bar{\mathbf{L}}_{12} \\ \bar{\mathbf{L}}_{21} & \bar{\mathbf{L}}_{22} \end{bmatrix} \tag{6.6}$$

as in (2.7) but rounded here to three decimals. We have added the partitioned matrix representation because it will be useful in much of what follows in this chapter. Clearly, the elements in $\bar{\mathbf{L}} = [\bar{l}_{ij}]$ also relate final-demand changes to sector outputs, only now these are in a model with households endogenous, and hence the effects tend to be larger.

To assess the impact of a new dollar's worth of final demand for sector 1 output, we would now form the three-element vector $\Delta\bar{\mathbf{f}}(1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ (meaning no exogenous change in demand for sector 2 output or for labor services), and find exactly the first column of $\bar{\mathbf{L}}$, namely

$$\Delta\bar{\mathbf{x}}(1) = \bar{\mathbf{L}}\Delta\bar{\mathbf{f}}(1) = \begin{bmatrix} 1.365 \\ 0.527 \\ 0.570 \end{bmatrix}$$

[Compare (6.1) above.] Adding these elements gives a parallel to (6.2),

$$\bar{m}(o)_1 = \mathbf{i}'\Delta\bar{\mathbf{x}}(1) = \sum_{i=1}^{n+1} \bar{l}_{i1} = 2.462 \tag{6.7}$$

with $n = 2$, as before but now with $\mathbf{i}' = [1, 1, 1]$. (In what follows we assume that $\mathbf{i}$ or $\mathbf{i}'$ always has appropriate dimensions for the multiplication in which it is involved.)

Sums of the first n elements in each of the columns of $\bar{\mathbf{L}}$ ($n = 2$ for our example) represent the total output multiplier effects over the original n sectors only – the *truncated* output multipliers. They can be found as $\mathbf{i}'\bar{\mathbf{L}}_{11}$. When interest is centered on the total output multipliers for the original n sectors (for example, to be compared with the simple output multipliers for these same n sectors), these *truncated* output multipliers are of interest. Denote these *truncated* total output multipliers by $\bar{m}[o(t)]_j$; here $\bar{m}[o(t)]_1 = 1.892$.

The total output multiplier for sector 2 is

$$\bar{m}(o)_2 = \sum_{i=1}^{n+1} \bar{l}_{i2} = 2.262 \tag{6.8}$$

and $\bar{m}[o(t)]_2 = 1.773$. In general, for sector j, the total output multiplier is given by

$$\bar{m}(o)_j = \sum_{i=1}^{n+1} \bar{l}_{ij} \tag{6.9}$$

and the truncated total output multiplier is $\bar{m}[o(t)]_j = \sum_{i=1}^{n} \bar{l}_{ij}$. In compact matrix terms,

$$\bar{\mathbf{m}}(o) = \mathbf{i}'\bar{\mathbf{L}} \text{ and } \bar{\mathbf{m}}[o(t)] = \mathbf{i}'\bar{\mathbf{L}}_{11} \tag{6.10}$$

Example: The US Input–Output Model for 2003 We again use the seven-sector 2003 US model. The Leontief inverse was shown as Table 2.7 in Chapter 2 and is not repeated here. The simple output multipliers are easily found to be

$$\mathbf{m}(o) = [1.9195\ 1.6051\ 1.7218\ 1.9250\ 1.4868\ 1.6081\ 1.5985]$$

In this case, the largest multipliers are associated with manufacturing (4) and agriculture (1). This is hardly surprising, considering the seven-sector level of aggregation.

Table 6.1 Total Requirements Matrices in Commodity–Industry Models

	Industry Technology	Commodity Technology
Commodity-Demand Driven Models		
Commodity-by-Commodity	$(\mathbf{I}-\mathbf{BD})^{-1}$	$(\mathbf{I}-\mathbf{BC}^{-1})^{-1}$
Industry-by-Commodity	$[\mathbf{D}(\mathbf{I}-\mathbf{BD})^{-1}]$	$[\mathbf{C}^{-1}(\mathbf{I}-\mathbf{BC}^{-1})^{-1}]$
Industry-Demand Driven Models		
Industry-by-Industry	$(\mathbf{I}-\mathbf{DB})^{-1}$	$(\mathbf{I}-\mathbf{C}^{-1}\mathbf{B})^{-1}$
Commodity-by-Industry	$[\mathbf{D}^{-1}(\mathbf{I}-\mathbf{DB})^{-1}]$	$[\mathbf{C}(\mathbf{I}-\mathbf{C}^{-1}\mathbf{B})^{-1}]$

Output Multipliers in Commodity–Industry Models With commodity-by-industry models, no new principles are involved. As usual, output multipliers would be found as column sums of the relevant total requirements matrices (open or closed with respect to households). In Table 6.1 we collect the results for total requirements matrices from Tables 5.4 and 5.5 in Chapter 5.

For example, for the commodity-by-commodity total requirements matrix under industry technology, the row vector of these output multipliers is $\mathbf{i}'(\mathbf{I}-\mathbf{BD})^{-1}$. Notice that since $\mathbf{i}'\mathbf{D}=\mathbf{i}'$ (column sums of $\mathbf{D}$ are all 1), the same output multipliers will be found for the industry-by-commodity total requirements matrix: $\mathbf{i}'[\mathbf{D}(\mathbf{I}-\mathbf{BD})^{-1}]=\mathbf{i}'(\mathbf{I}-\mathbf{BD})^{-1}$. The same will be true for any other pair of matrices (vertically) in the table. This is because (1) $\mathbf{i}'\mathbf{C}=\mathbf{i}'$ ($\mathbf{C}$ is constructed so that is true), (2) $\mathbf{i}'\mathbf{D}^{-1}=\mathbf{i}'$ (this is easy to show, given $\mathbf{i}'\mathbf{D}=\mathbf{i}'$) and (3) similarly, $\mathbf{i}'\mathbf{C}^{-1}=\mathbf{i}'$. This result is what we would expect – summing down the columns in a total requirements matrix (over all rows) should give the same result, irrespective of the row labels ("commodities" or "industries").

The results below are for the total requirements matrices in the numerical examples from Chapter 5. They illustrate the identical results for pairs of matrices.

Commodity-Demand-Driven Models

Industry Technology **Commodity Technology**

Commodity-by-Commodity

$$(\mathbf{I}-\mathbf{BD})^{-1}=\begin{bmatrix}1.1568 & .0898\\ .1314 & 1.0782\end{bmatrix} \qquad (\mathbf{I}-\mathbf{BC}^{-1})^{-1}=\begin{bmatrix}1.1644 & .0825\\ .1375 & 1.0723\end{bmatrix}$$

Output Multipliers $[\,1.2882 \;\; 1.1680\,]$ $[\,1.3019 \;\; 1.1548\,]$

Industry-by-Commodity

$$\mathbf{D}(\mathbf{I}-\mathbf{BD})^{-1}=\begin{bmatrix}1.0411 & .0809\\ .2471 & 1.0871\end{bmatrix} \qquad \mathbf{C}^{-1}(\mathbf{I}-\mathbf{BC}^{-1})^{-1}=\begin{bmatrix}1.1507 & -.0247\\ .1512 & 1.1795\end{bmatrix}$$

Output Multipliers $[\,1.2882 \;\; 1.1680\,]$ $[\,1.3019 \;\; 1.1548\,]$

Industry-Demand-Driven Models

	Industry Technology		**Commodity Technology**
	Industry-by-Industry		
$(\mathbf{I}-\mathbf{DB})^{-1}$	$=\begin{bmatrix} 1.1478 & .0809 \\ .1537 & 1.0871 \end{bmatrix}$	$(\mathbf{I}-\mathbf{C}^{-1}\mathbf{B})^{-1}$	$=\begin{bmatrix} 1.1507 & .0821 \\ .1512 & 1.0861 \end{bmatrix}$
Output Multipliers	[1.3015 1.1680]		[1.3019 1.1682]
	Commodity-by-Industry		
$\mathbf{D}^{-1}(\mathbf{I}-\mathbf{DB})^{-1}$	$=\begin{bmatrix} 1.2753 & .0898 \\ .0262 & 1.0782 \end{bmatrix}$	$\mathbf{C}(\mathbf{I}-\mathbf{C}^{-1}\mathbf{B})^{-1}$	$=\begin{bmatrix} 1.1644 & .1808 \\ .1375 & .9873 \end{bmatrix}$
Output Multipliers	[1.3015 1.1680]		[1.3019 1.1682[3]]

6.2.2 Income/Employment Multipliers

Generally an analyst is more likely to be concerned with the economic impacts of new final demand as measured by jobs created, increased household earnings, value added generated, etc., rather than simply gross output by sector. In this section we explore impacts on households; the approach is exactly the same whether we measure this impact in terms of jobs (physical) or earnings (monetary). In what follows, we illustrate using income, but this applies equally well to jobs.

Income Multipliers One straightforward approach is simply to convert the elements in $\mathbf{L}$ into dollars' worth of employment using labor-input coefficients – either monetary (wages earned per unit of output, as in $[a_{n+1,1}, \ldots, a_{n+1,n}]$) or physical (person-years, or some such measure, per unit of output). We begin with transactions information; let $\mathbf{h}'$ (for households) denote the row vector of these data. In the monetary case, this is $\mathbf{h}' = [z_{n+1,1}, \ldots, z_{n+1,n}]$; in physical terms it would be some measure of numbers of employees in each sector in the base period. Then $\mathbf{h}'_c = \mathbf{h}'\hat{\mathbf{x}}^{-1}$ is the row of associated household input *coefficients*.[4] Again, in monetary terms these are the elements in $[a_{n+1,1}, \ldots, a_{n+1,n}]$, used in the example above to close the model with respect to households ($a_{n+1,j} = z_{n+1,j}/x_j$), indicating household income received per dollar's worth of sector output.

Associated with $\Delta\mathbf{f} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we found output effects in the first column of $\mathbf{L} - \begin{bmatrix} l_{11} \\ l_{21} \end{bmatrix}$, as in (6.1). The conversion of this first column to income terms is accomplished by weighting the first element by $a_{n+1,1}$ and the second element by $a_{n+1,2}$,

[3] This is not equal to 0.1808 + 0.9873 only because of rounding in the total requirements matrix.
[4] We denoted this as $\mathbf{h}_R$ earlier when closing the model with respect to households. Now we modify the notation to emphasize that this is a row vector of coefficients and to allow for generalization to other kinds of multipliers.

giving $\begin{bmatrix} a_{n+1,1}l_{11} \\ a_{n+1,2}l_{21} \end{bmatrix}$. In general, then, using $m(h)_j$ for the simple household income multiplier for sector j,

$$m(h)_j = \sum_{i=1}^{n} a_{n+1,i}l_{ij} \tag{6.11}$$

Again, "simple" refers to the fact that these multipliers are found using elements in the **L** matrix, with households exogenous.

Continuing the same example, we had $a_{n+1,1} = 0.3$ and $a_{n+1,2} = 0.25$. Thus

$$m(h)_1 = (0.3)(1.254) + (0.25)(0.264) = 0.376 + 0.066 = 0.442$$

and

$$m(h)_2 = (0.3)(0.33) + (0.25)(1.122) = 0.099 + 0.281 = 0.380$$

In this illustration, $m(h)_1 = 0.442$ indicates that an additional dollar of final demand for the sector 1 output would generate \$0.442 of new household income, when all direct and indirect effects are converted into dollar estimates of income. If earnings in individual sectors are of interest, we see that \$0.376 would be earned by employees in sector 1 and \$0.066 would be earned by sector 2 employees. And similarly, $m(h)_2 = 0.380$ could be disaggregated into earnings in each of the sectors. From this example, using this measure of effectiveness, dollars of final demand – for example, new government purchases – generate more dollars of new household income when they are spent on the output of sector 1 than when they are spent on the output of sector 2.

If the elements in $\bar{\mathbf{L}}$ are weighted similarly, *total* (direct plus indirect plus induced) income effects or household income multipliers are obtained. As before, using an overbar to denote a multiplier derived from $\bar{\mathbf{L}}$, the parallel to $m(h)_j$ in (6.11) is

$$\bar{m}(h)_j = \sum_{i=1}^{n+1} a_{n+1,i}\bar{l}_{ij} \tag{6.12}$$

For our numerical example, with $a_{n+1,3} = 0.05$,

$$\bar{m}(h)_1 = (0.3)(1.365) + (0.25)(0.527) + (0.05)(0.570) = 0.570$$

and

$$\bar{m}(h)_2 = (0.3)(0.425) + (0.25)(1.348) + (0.05)(0.489) = 0.489$$

These total income multipliers for sectors 1 and 2 are equal to $\bar{l}_{n+1,1}$ and $\bar{l}_{n+1,2}$, the elements of $\bar{\mathbf{L}}_{21}$ [in $\bar{\mathbf{L}}$ (6.6)]. Recall the interpretation of any element $\bar{l}_{ij}$; it measures the total (direct, indirect, and induced) effect on sector i output of a dollar's worth of new demand for sector j output. Thus $\bar{l}_{n+1,j}$ is the total effect on the output of the *household* sector (the total value of labor services needed) when there is a dollar's worth of new final demand for goods of sector j. This is precisely what we mean by the total household income effect or total household income multiplier. So

$$\bar{m}(h)_j = \bar{l}_{n+1,j} \tag{6.13}$$

(In Appendix 6.1, the relationship between the total household income multipliers and the bottom-row elements of $\bar{\mathbf{L}}$ is shown exactly, using matrix algebra results on the inverse of a partitioned matrix.) Again, if we are only interested in household income-generating effects originating in the n original sectors, we would calculate a truncated total household income multiplier, $\bar{m}[h(t)]_1$, by summing down the columns of $\bar{\mathbf{L}}_{11}$ only. For the example, $\bar{m}[h(t)]_1 = 0.541$ and $\bar{m}[h(t)]_2 = 0.465$.

In this and all subsequent discussions in this chapter, all results hold if $\mathbf{A}$ and $\mathbf{L}$ are understood to be direct and total requirements matrices in a commodity–industry model – as for example with $\underset{(c\times c)}{\mathbf{A}_I} = \mathbf{BD}$ and $\underset{(c\times c)}{\mathbf{L}_I} = (\mathbf{I}-\mathbf{BD})^{-1}$. We illustrated the case of output multipliers for various commodity–industry models in section 6.2.1, above.

Type I and Type II Income Multipliers With income multipliers, one has some choice regarding what should logically be termed the initial effect of new final demand. With output multipliers it was fairly clear that the initial effect of a new dollar's worth of final demand for sector j output is that sector j production must increase by one dollar (and eventually, of course, by more than that dollar). With income effects, the same dollar's worth of new demand for sector j becomes, initially, the same dollar's worth of new output by sector j; this is what we considered the initial effect in developing the household income multipliers, above. However, the initial dollar's worth of new output from sector j means an initial additional income payment of $a_{n+1,j}$ to workers in sector j. Hence $a_{n+1,j}$ could be viewed as the initial *income* effect of the new demand for sector j output.

Thus there is another kind of simple income multiplier, usually called the type I income multiplier, for any sector j. This has the direct and indirect income effect, or the simple household income multiplier [(6.11)] as a numerator, and uses as a denominator not the initial dollar's worth of output but rather its initial labor income effect, $a_{n+1,j}$.[5] Let $m(h)_j^I$ represent this type I income multiplier for sector j, so

$$m(h)_j^I = \frac{\sum_{i=1}^{n} a_{n+1,i} l_{ij}}{a_{n+1,j}} = \frac{m(h)_j}{a_{n+1,j}} \tag{6.14}$$

For our numerical example,

$$m(h)_1^I = 0.442/0.3 = 1.473$$

$$m(h)_2^I = 0.380/0.25 = 1.520$$

Again, if the coefficients matrix is closed with respect to households, income effects similar to these type I multipliers can be calculated; these are called type II income

[5] These have also been called "normalized" multipliers; for example, in Oosterhaven (1981).

multipliers:[6]

$$m(h)_j^{II} = \frac{\sum_{i=1}^{n+1} a_{n+1,i}\bar{l}_{ij}}{a_{n+1,j}} = \frac{\bar{m}(h)_j}{a_{n+1,j}} \tag{6.15}$$

Again, for the numerical example,

$$m(h)_1^{II} = \frac{0.570}{0.3} = 1.900$$

$$m(h)_2^{II} = \frac{0.489}{0.25} = 1.956$$

The parallel between this measure and the type I effect in (6.14) is the same as that between the total and simple household income multipliers – $\bar{m}(h)_j$ and $m(h)_j$. The numerator for $m(h)_j^I$ is $m(h)_j$ from (6.11); the numerator for $m(h)_j^{II}$ is $\bar{m}(h)_j$ from (6.12) or from (6.13). Thus, for exactly the same reasons as for $\bar{m}(h)_j$, we can alternatively define $m(h)_j^{II}$ as

$$m(h)_j^{II} = \bar{l}_{n+1,j}/a_{n+1,j} \tag{6.16}$$

These multipliers show by how much the initial *income* effects (0.3 and 0.25) are blown up when direct, indirect, and induced effects (due to household spending because of increased household income) are taken into account, via $\bar{\mathbf{L}}$. Truncated type II income multipliers would be found, as usual, by considering columns in $\bar{\mathbf{L}}_{11}$ only. In this example they are $m[h(t)]_1^{II} = 1.803$ and $m[h(t)]_2^{II} = 1.860$.

It is generally conceded that Type I multipliers probably underestimate economic impacts (since household activity is absent) and Type II multipliers probably give an overestimate (because of the rigid assumptions about labor incomes and attendant consumer spending). For example, Oosterhaven, Piek and Stelder (1986, p. 69) suggest

> These two multipliers [Type II and Type I] may be considered as upper and lower bounds on the true indirect effect of an increase in final demand; a realistic estimate generally lies roughly halfway between the Type I and Type II multipliers.

Relationship Between Simple and Total Income Multipliers or Between Type I and Type II Income Multipliers To the extent that the results of an input–output analysis with households exogenous tend to underestimate total effects, total or type II multipliers may be more useful than simple or type I multipliers in estimating potential impacts. Or some in-between figure might be more realistic, as noted above, but deciding exactly where between these two limits may be problematic. However, if one is primarily interested in *ranking* or ordering the sectors – which sector has the largest multiplier, which has the next largest, and so on – then type I multipliers are just as useful as type II (and usually easier to obtain), because the ratio of type II

[6] The designations "type I" and "type II" seem to have originated with Moore (1955). Calculation of these measures (in a regional setting) was pioneered by Moore and Petersen (1955) for Utah and later by Hirsch (1959) for St. Louis.

to type I income multipliers can be shown to be a constant across all sectors. Since $m(h)_j^{II} = \bar{m}(h)_j/a_{n+1,j}$ and $m(h)_j^{I} = m(h)_j/a_{n+1,j}$, $m(h)_j^{II}/m(h)_j^{I} = \bar{m}(h)_j/m(h)_j$. What is now claimed is that $m(h)_j^{II}/m(h)_j^{I} = k$ (a constant) for all j. Moreover, k can be easily found without any need for $\bar{\mathbf{L}}$. This represents a computational advantage. To show that this ratio is a constant requires that we apply some facts on the inverse of the partitioned matrix $\bar{\mathbf{L}}$. This is done in Appendix 6.2, for the interested reader. In our illustrative example we found $m(h)_1 = 0.442$, $\bar{m}(h)_1 = 0.570$, $m(h)_2 = 0.380$, $\bar{m}(h)_2 = 0.489$, $m(h)_1^I = 1.473$, $m(h)_1^{II} = 1.900$, $m(h)_2^I = 1.520$, and $m(h)_2^{II} = 1.956$. Therefore (to two decimals), $\bar{m}(h)_1/m(h)_1 = 0.570/0.442 = 1.29$, $m(h)_1^{II}/m(h)_1^I = 1.90/1.47 = 1.29$, and the same values can be found for $\bar{m}(h)_2/m(h)_2$, and $m(h)_2^{II}/m(h)_2^I$, so $k = 1.29$ for this example.

Which Multiplier to Use? As a practical matter, the choice between multiplier effects as measured by $m(h)_j$ [and $\bar{m}(h)_j$] or by $m(h)_j^I$ [and $m(h)_j^{II}$] depends on the nature of the exogenous change whose impact is being studied. If that change is, for example, an increase in federal government spending on output of the aircraft sector, then the most useful figures may be those that convert the total dollar value of new government spending into total new income earned by households in the economy – the income multipliers $m(h)_j$ and $\bar{m}(h)_j$. Using $m(h)_1 = 0.442$ and $m(h)_2 = 0.380$ from the example, we would estimate that a tariff policy that would increase foreign demand for sector 1 goods by \$100,000 would ultimately lead to an increase of (0.442)(\$100,000) = \$44,200 in new income earned, while a policy that increased export demand for sector 2 goods by \$100,000 would generate (0.380)(\$100,000) = \$38,000 in new household income earned. If we also attempt to capture the consumer spending that is associated with income earned, in a closed model, we would use $\bar{m}(h)_1$ and $\bar{m}(h)_2$ and find (0.570)(\$100,000) = \$57,000 and (0.489)(\$100,000) = \$48,900, respectively. In either case, we find that stimulation of export demand for sector 1 output generates the larger effect, as expected, because $\bar{m}(h)_j/m(h)_j = k$ (here 1.29), so the largest simple multiplier will be the largest total multiplier.

The impacts of decreases can be assessed just as easily. Suppose that management teams in two different industries, i and j, were considering moving a large assembly plant out of the country because of lower labor costs abroad. If these plants had annual payrolls of \$$p_i$ and \$$p_j$, respectively, then a measure of the total household income lost throughout the national economy because of the contemplated relocations would be given by $m(h)_i^I p_i$ and $m(h)_j^I p_j$ – or by $m(h)_i^{II} p_i$ and $m(h)_j^{II} p_j$, if one wants to include induced households consumption effects. For example, using $m(h)_1^I = 1.473$ and $m(h)_2^I = 1.520$ from our example, if a plant in industry 1 with an annual payroll of \$100,000 were to move out of the country, we would estimate a total income loss of (1.47)(\$100,000) = \$147,300 throughout the economy. Similarly, if a plant in industry 2, with an annual payroll of \$250,000, were to move out of the economy, we could estimate the total loss to household income throughout the economy because of this out-movement as (1.520)(\$250,000) = \$380,000. Again, if we capture consumer spending using a closed model, our estimates, using $m(h)_1^{II} = 1.900$ and $m(h)_2^{II} = 1.956$, would be a (1.900)($100,000) = $190,000 income loss from the out-movement of the plant in industry 1 and a (1.956)($250,000) = $489,000 income decrease from loss of the plant in industry 2.

Even More Income Multipliers As noted above (section 3.2.3), in an important early study of Boulder, Colorado, Miernyk *et al.* (1967) implement a model that distinguishes between consumption propensities of new residents in a region and those of established residents. In addition, current residents were divided into income classes (four in this study), and separate regional consumption functions were estimated for each income class. The results of this approach have been termed type III income multipliers, and they are smaller, sector by sector, than the type II income multipliers. This is to be expected, since *marginal* consumption coefficients, associated with current residents' consumption habits, were smaller than *average* consumption coefficients, associated with new residents' consumption habits and which are the exclusive basis of the type II multipliers.[7]

Although the ratio of type III to type II income multipliers is not constant across sectors, the range was only 0.87–0.91, with an average of 0.88. Since the (constant) ratio of type II to type I income multipliers in this study was 1.34, this means that the ratio of type III to type I income multipliers averaged 1.18. If a similar narrow range of ratios of type III to type II income multipliers were found in other regional studies in which households were similarly disaggregated, it would be possible to approximate type III income multipliers across all sectors by appropriate "inflation" of the type I multiplier. In the Boulder study, the inflating factor would be 1.18.

Further, Madden and Batey (1983 and elsewhere) derive a type IV income multiplier. Like the type III multipliers, these are (generally) larger than type I but smaller than type II income multipliers. The distinction here is between the spending patterns of currently employed local residents and the spending patterns of currently unemployed local residents.[8] The models giving rise to these four kinds of multipliers are discussed and summarized in Batey and Weeks (1989). Table 6.2 provides an overview.

Physical Employment Multipliers All of the above types of multipliers apply equally well if we are interested in counts of jobs, in physical terms. Our initial information, in $\mathbf{h}'$, would be in person-years or some similar unit of measure, and the results

7 In the Boulder study, the *average* (aggregate) household consumption coefficient, for the products of all 31 sectors of the local economy, is 0.40. (This is $\mathbf{i}'\mathbf{h}_C$, using the household column in the Boulder study.) The *marginal* (aggregate) household consumption coefficients for the products of the same 31 sectors, are 0.31, 0.21, 0.16, and 0.02 for the four income classes; their average is 0.1730. (Calculated from Tables IV-2 and V-4a, respectively, in Miernyk *et al.*, 1967.) The type III multipliers in the Boulder study were found not from the Leontief inverse of a model that had been closed with respect to households in this disaggregated way but rather in an iterative, round-by-round fashion.

8 Conway (1977) proposed applying the terms "type A" and "type B" multipliers to the numerators of "type I" and "type II" multipliers. The motivation is to facilitate studies of changes in multiplier values over time. When the multiplier is a ratio in which both numerator and denominator elements change over time, a change in a multiplier value can reflect changes in either the numerator or in the denominator or in both.

Table 6.2 Model Closures with Respect to Households

Model	Measured Effects		Model Closure	Income Multiplier
	Direct + Indirect	Induced*		
1	Direct + Indirect	None	None	Type I
2	Direct + Indirect	Intensive	Single household row and column	Type II
3	Direct + Indirect	Intensive + Extensive	Two household rows and columns	Type III
4	Direct + Indirect	Intensive + Extensive + Redistributive	Three household rows and columns	Type IV

*Intensive effects are associated with indigenous workers and marginal consumption coefficients. Extensive effects are associated with in-migrants and average consumption coefficients. Redistributive effects are associated with unemployed residents and their consumption propensities based on benefit payments.

in (6.11) through (6.16) remain valid, with the interpretation in physical rather than monetary terms.

6.2.3 Value-Added Multipliers

Another kind of multiplier relates the new value added created in each sector in response to the initial exogenous shock to that initial shock. The principles are identical, and the results in (6.11) through (6.16) again remain valid. The only new information required is a set of sectoral value-added coefficients – $\mathbf{v}'_c = \mathbf{v}'\hat{\mathbf{x}}^{-1}$. We leave it for the reader to fill in details. It is often argued that value added is a better measure of a sector's contribution to an economy than, say, total output, since it truly captures the value that is added by the sector in engaging in production – the difference between a sector's total output and the cost of its intermediate inputs.

6.2.4 Matrix Representations

Matrix representation provides a compact and efficient way to express multipliers. Output multipliers were represented in (6.5) as

$$\mathbf{m}(o) = \mathbf{i}'\mathbf{L}$$

For income multipliers (simple), with $\mathbf{h}'_c = \mathbf{h}'\hat{\mathbf{x}}^{-1}$, we have

$$\mathbf{m}(h) = [m(h)_1, \ldots, m(h)_n] = \mathbf{h}'_c\mathbf{L} \tag{6.17}$$

Here the summation row, $\mathbf{i}'$ in (6.5), has been replaced by the row of labor-input coefficients, $\mathbf{h}'_c$. We can deconstruct this in the following way:

$$\mathbf{m}(h) = \mathbf{h}'_c\mathbf{L} = \mathbf{h}'\hat{\mathbf{x}}^{-1}\mathbf{L} = \underbrace{\mathbf{i}' \overbrace{\hat{\mathbf{h}}'\hat{\mathbf{x}}^{-1}\mathbf{L}}^{\substack{\textit{Sector-demand-}\\ \textit{to-sector income}\\ \textit{multipliers}\\ [\mathbf{M}(h)]}}}_{\substack{\textit{Sector-demand-}\\ \textit{to-economy-wide}\\ \textit{income multipliers}\\ [\mathbf{m}(h)]}} \tag{6.18}$$

In particular, $\hat{\mathbf{h}}'\hat{\mathbf{x}}^{-1}\mathbf{L}$ converts the inverse matrix of final demand-to-output multipliers in $\mathbf{L}$ into a *matrix* of final demand-to-income multipliers, $\mathbf{M}(h)$. Then $\mathbf{i}'\hat{\mathbf{h}}'\hat{\mathbf{x}}^{-1}\mathbf{L} = \mathbf{h}'\hat{\mathbf{x}}^{-1}\mathbf{L} = \mathbf{h}'_c\mathbf{L}$ generates a *vector* of economy-wide income multipliers, $\mathbf{m}(h)$, the column sums of the converted inverse. Notice that in this generic format, the simple output multipliers in (6.5) can be thought of as

$$\mathbf{m}(o) = \mathbf{i}'\mathbf{L} = \mathbf{x}'\hat{\mathbf{x}}^{-1}\mathbf{L} = \mathbf{i}'\hat{\mathbf{x}}'\hat{\mathbf{x}}^{-1}\mathbf{L}$$

For the closed model, the n-element vector of total income multipliers for the n sectors is

$$\bar{\mathbf{m}}(h) = [\bar{m}(h)_1, \ldots, \bar{m}(h)_n] = \underset{[1\times(n+1)]}{\begin{bmatrix}\mathbf{h}'_c & a_{n+1,n+1}\end{bmatrix}} \underset{[(n+1)\times n]}{\begin{bmatrix}\bar{\mathbf{L}}_{11}\\ \bar{\mathbf{L}}_{21}\end{bmatrix}} = \bar{\mathbf{h}}'_c \begin{bmatrix}\bar{\mathbf{L}}_{11}\\ \bar{\mathbf{L}}_{21}\end{bmatrix} \tag{6.19}$$

This makes clear that $\bar{m}(h)_j > m(h)_j$ for two reasons: (1) even though the weights in $\mathbf{h}'_c$ are the same for both models, the inverse elements in $\bar{\mathbf{L}}_{11}$ are consistently larger than those in $\mathbf{L}$, and (2) each $\bar{m}(h)_j$ includes the additional term $a_{n+1,n+1}\bar{l}_{n+1,j}$. [In the case of truncated multipliers, only (1) is relevant.]

The n-element row vector of type I income multipliers for each sector, $\mathbf{m}(h)^I$, can be compactly represented using $\mathbf{m}(h)$ from (6.18), namely

$$\mathbf{m}(h)^I = \mathbf{m}(h)(\hat{\mathbf{h}}'_c)^{-1} = \mathbf{h}'_c\mathbf{L}(\hat{\mathbf{h}}'_c)^{-1} \tag{6.20}$$

A row vector of type II income multipliers for the original n sectors can be defined using $\bar{\mathbf{L}}_{21} = [\bar{l}_{n+1,1}, \bar{l}_{n+1,2}, \ldots, \bar{l}_{n+1,n}]$, namely

$$\mathbf{m}(h)^{II} = \bar{\mathbf{L}}_{21}(\hat{\mathbf{h}}'_c)^{-1} \tag{6.21}$$

6.2.5 Summary

Table 6.3 presents a summary of the results in sections 6.2.1–6.2.3. Table 6.4 summarizes these multiplier results in a set of generic templates. We use "$\mathbf{z}'_c$" for the

Table 6.3 Input–Output Multipliers

	Output Effects	Income Effects[a]	
Exogenous Change	$\Delta f_j = 1$	$\Delta f_j = 1$	
Initial Effect (N) (sector j)	$\Delta x_j = 1$	$\Delta x_j = 1$	Δ in sector j payments to labor = $a_{n+1,j}$
Total Effect (T) in open model (Direct + Indirect)	$\sum_{i=1}^{n} l_{ij}$	$\sum_{i=1}^{n} a_{n+1,i} l_{ij}$	
Simple Multiplier (T/N) (open model)	Simple output multiplier $m(o)_j = \sum_{i=1}^{n} l_{ij}/\Delta f_j = \sum_{i=1}^{n} l_{ij}$ [(6.4)]	Simple income multiplier $m(h)_j = \sum_{i=1}^{n} a_{n+1,i} l_{ij}/\Delta f_j = \sum_{i=1}^{n} a_{n+1,i} l_{ij}$ [(6.11)]	Type I income multiplier $m(h)_j^I = \sum_{i=1}^{n} a_{n+1,i} l_{ij}/a_{n+1,j} = m(h)_j/a_{n+1,j}$ [(6.14)]
Total Effect ($\bar{T}$) in closed model (Direct + Indirect + Induced)	$\sum_{i=1}^{n+1} \bar{l}_{ij}$	$\sum_{i=1}^{n+1} a_{n+1,i} \bar{l}_{ij}$	
Total Multiplier ($\bar{T}/N$) (closed model)[b]	Total output multiplier $\bar{m}(o)_j = \sum_{i=1}^{n+1} \bar{l}_{ij}/\Delta f_j = \sum_{i=1}^{n+1} \bar{l}_{ij}$ [(6.9)]	Total income multiplier $\bar{m}(h)_j = \sum_{i=1}^{n+1} a_{n+1,i} \bar{l}_{ij}/\Delta f_j = \sum_{i=1}^{n+1} a_{n+1,i} \bar{l}_{ij}$ [(6.12)] $= \bar{l}_{n+1,j}$ [(6.13)]	Type II income multiplier $m(h)_j^{II} = \sum_{i=1}^{n+1} a_{n+1,i} \bar{l}_{ij}/a_{n+1,j} = \bar{m}(h)_j/a_{n+1,j}$ [(6.15)] $= \bar{l}_{n+1,j}/a_{n+1,j}$ [(6.16)]

[a] For income effects, $a_{n+1,j} = z_{n+1,j}/x_j$, where $z_{n+1,j} =$ sector j's payments to households (labor). For employment effects, replace $z_{n+1,j}$ with sector j's employment measured in physical units. For value-added effects, replace $z_{n+1,j}$ with sector j's value-added payments.

[b] For truncated total multiplier effects, sum over $i = 1, \ldots, n$ rather than $i = 1, \ldots, n+1$.

appropriate row vector of coefficients, found from transactions ($\mathbf{z}'$) and output ($\mathbf{x}$) information; $\mathbf{z}'_c = \mathbf{z}'\hat{\mathbf{x}}^{-1}$. When $\mathbf{z}' = \mathbf{x}'$, $\mathbf{z}'_c = \mathbf{i}'$, and we have traditional output multipliers. [Contrary to subsequent notation, we denoted these as $\mathbf{m}(o)$, for "output," rather than $\mathbf{m}(x)$.] Note that Type I and II *output* multipliers are meaningless;

Table 6.4 General Multiplier Formulas

Multiplier	Matrix Definition
Simple	$\mathbf{m}(z) = \mathbf{z}'_c\mathbf{L}$
Total	$\bar{\mathbf{m}}(z) = \bar{\mathbf{z}}'_c \begin{bmatrix} \bar{\mathbf{L}}_{11} \\ \bar{\mathbf{L}}_{21} \end{bmatrix}$
	where
	$\bar{\mathbf{z}}'_c = \begin{bmatrix} \mathbf{z}'_c & z_{n+1,n+1}/x_{n+1} \end{bmatrix}$
Truncated	$\bar{\mathbf{m}}[z(t)] = \mathbf{z}'_c\bar{\mathbf{L}}_{11}$
Type I	$\mathbf{m}(z)^I = \mathbf{z}'_c\mathbf{L}(\hat{\mathbf{z}}'_c)^{-1}$
Type II	$\mathbf{m}(z)^{II} = \bar{\mathbf{L}}_{21}(\hat{\mathbf{z}}'_c)^{-1}$

they are identical to simple and total output multipliers since $\hat{\mathbf{i}}' = \mathbf{I}$. When $\mathbf{z}' = \mathbf{h}'$or $\mathbf{z}' = \mathbf{v}'$ we have household (either income or employment) or value-added multipliers, respectively. Many other kinds of multipliers are possible. For example, if $\mathbf{z}' = \mathbf{e}'$ is a row measuring amounts of pollution emitted by production in each of the sectors, we would have an environmental (pollution-generation) multiplier, or if $\mathbf{z}' = \mathbf{n}'$ is a row indicating energy consumption by sector, we would have energy-use multipliers. Energy-use, pollution-generation, and other such multipliers are frequently found in truncated form, as $\mathbf{z}'_c\bar{\mathbf{L}}_{11}$, which is equivalent to setting $\bar{\mathbf{z}}'_c = \begin{bmatrix} \mathbf{z}'_c & 0 \end{bmatrix}$ in Table 6.4. Some of these energy and environmental extensions are discussed in Chapter 10.

6.3 Multipliers in Regional Models

In section 6.2 we presented the basic concepts of various input–output multipliers. All of these multipliers, which quantify impacts on the economy under study, rely on the fact that the $\mathbf{A}$ matrix (as well as the associated coefficients for income, employment, value added, etc.) must represent interindustry relationships *within that economy*. In particular, if sector i is agriculture and sector j is food processing, a_{ij} must represent the value of inputs of agricultural products *produced within the economy* (not imported) per dollar's worth of output of the food-processing sector in the same economy.

6.3.1 Regional Multipliers

Very often an analyst is interested in impacts at a regional level. For example, the federal government may be trying to decide where to award a new military contract and have as one of its concerns the stimulation of economic development in one or more less-developed regions. A state government may wish to allocate funds for labor skill training in one or more industries among several counties with currently above-average levels of unemployment, and so on. In a single-region input–output model, as in section 3.2, the $\mathbf{A}^r = \hat{\mathbf{p}}^r\mathbf{A}$ matrix represented one way of trying to capture regional interrelationships among sectors, and the various kinds of multipliers discussed above would acquire a spatial dimension by using the elements of $\mathbf{A}^r$ and its associated Leontief inverse.

For example, in section 3.2 a national table, $\mathbf{A} = \begin{bmatrix} .15 & .25 \\ .20 & .05 \end{bmatrix}$, was modified because of the assumption that in region r the basic technology of production in sectors 1 and 2 was essentially the same as that reflected in the two columns of **A**, but the *proportions* of inputs required from sectors 1 and 2 that could be expected to come from within the region were $p_1^r = 0.8$ and $p_2^r = 0.6$, so $\mathbf{p}^r = \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix}$, and

$$\mathbf{A}^r = \hat{\mathbf{p}}^r\mathbf{A} = \begin{bmatrix} .12 & .20 \\ .12 & .03 \end{bmatrix} \quad \text{and} \quad \mathbf{L}^r = (\mathbf{I} - \mathbf{A}^r)^{-1} = \begin{bmatrix} 1.169 & 0.241 \\ 0.145 & 1.061 \end{bmatrix}$$

Hence the regional *simple* output multipliers, as in (6.4), are $m(o)_1^r = 1.314$ and $m(o)_2^r = 1.302$. Recall from section 6.2.1 that the output multipliers in the original **A** matrix were $m(o)_1 = 1.518$ and $m(o)_2 = 1.452$. The difference, of course, is due to the fact that the elements of **A** have been reduced, using the regional percentages in $\mathbf{p}^r$, to reflect the need for imports to supply some of the necessary production. Similarly, external output multipliers (not regional – denoted $\tilde{r}$) are $m(o)_1^{\tilde{r}} = 1.518 - 1.314 = 0.204$ for sector 1 and $m(o)_2^{\tilde{r}} = 1.452 - 1.302 = 0.150$ for sector 2. The interpretation of these is similar to that for other output multipliers: for each dollar's worth of final demand in the region for sector 1 output, 20.4 cents' worth of inputs will be needed from firms in all sectors outside of the region. And for each dollar's worth of final demand in the region for sector 2 output, this figure is 15 cents.

If we have estimates of household inputs, household consumption, and income earned *in the region*, the model can be closed with respect to households, allowing calculation of regional total output multipliers. If we assume that the household input coefficients in the region are the same as those for the nation as a whole and that these represent labor supplied by workers living in the region, then $a_{31}^r = 0.30$, $a_{32}^r = 0.25$, and $a_{33}^r = 0.05$. Also, if we assume that sectors 1 and 2 supply 80 percent and 60 percent, respectively, of consumer needs (the same percentages as they supply of the needs for production), then $a_{13}^r = (0.8)(0.05) = 0.04$ and $a_{23}^r = (0.6)(0.40) = 0.24$ so

$$\bar{\mathbf{A}}^r = \begin{bmatrix} .12 & .20 & .04 \\ .12 & .03 & .24 \\ .30 & .25 & .05 \end{bmatrix} \quad \text{and} \quad \bar{\mathbf{L}}^r = (\mathbf{I} - \bar{\mathbf{A}}^r)^{-1} = \begin{bmatrix} 1.217 & 0.282 & 0.123 \\ 0.263 & 1.164 & 0.305 \\ 0.453 & 0.395 & 1.172 \end{bmatrix}$$

Therefore, the regional *total* output multipliers, as in (6.9), are $\bar{m}(o)_1^r = 1.933$ and $\bar{m}(o)_2^r = 1.841$.

With information on regional labor inputs (in monetary terms) and household consumption coefficients, various income multipliers could be found for the region. Value-added multipliers could also be found in exactly parallel ways. No new principles are involved in assessing multiplier effects with a single-region table instead of a national table. However, with many-region input–output models, a wider variety of multipliers is possible. We examine these in the interregional and multiregional cases in turn.

6.3.2 Interregional Input–Output Multipliers

With interregional and multiregional input–output models output, various multiplier effects can be calculated (a) for a single region (region r), (b) for each of the other regions, (c) for the "rest of the economy" (aggregated over *all* regions outside of r), and (d) for the total, many-region (national) economy.

We illustrate the possibilities using a set of hypothetical data for a two-region model. Consider the following coefficients matrices for an interregional model with (the same) three sectors in each region

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}^{rr} & \mathbf{A}^{rs} \\ \mathbf{A}^{sr} & \mathbf{A}^{ss} \end{bmatrix} = \begin{bmatrix} .150 & .250 & .050 & .021 & .094 & .017 \\ .200 & .050 & .400 & .167 & .125 & .133 \\ .300 & .250 & .050 & .050 & .050 & .000 \\ .075 & .050 & .060 & .167 & .313 & .067 \\ .050 & .013 & .025 & .125 & .125 & .047 \\ .025 & .100 & .100 & .250 & .250 & .133 \end{bmatrix} \tag{6.22}$$

and

$$\mathbf{L} = \begin{bmatrix} \mathbf{L}_{11} & \mathbf{L}_{12} \\ \mathbf{L}_{21} & \mathbf{L}_{22} \end{bmatrix} = \begin{bmatrix} 1.462 & .506 & .332 & .259 & .382 & .147 \\ .721 & 1.514 & .761 & .558 & .629 & .324 \\ .678 & .578 & 1.378 & .318 & .390 & .147 \\ .318 & .253 & .251 & 1.428 & .649 & .190 \\ .177 & .123 & .124 & .268 & 1.315 & .114 \\ .346 & .365 & .365 & .598 & .695 & 1.300 \end{bmatrix} \tag{6.23}$$

Recall from Chapter 3 that we use subscript numbers for elements (submatrices) of a partitioned interregional $\mathbf{L}$ matrix because $\mathbf{L}^{rr}$ and $\mathbf{L}^{ss}$ are used for $(\mathbf{I} - \mathbf{A}^{rr})^{-1}$ and $(\mathbf{I} - \mathbf{A}^{ss})^{-1}$, respectively.

Intraregional Effects For exogenous changes in final demands for region r goods (the first three elements in a six-element $\mathbf{f}$ vector), the elements in the 3×3 submatrix $\mathbf{L}_{11}$ represent impacts on the outputs of sectors in region r. Here

$$\mathbf{L}_{11} = \begin{bmatrix} 1.462 & .506 & .332 \\ .721 & 1.514 & .761 \\ .678 & .578 & 1.378 \end{bmatrix} \tag{6.24}$$

Simple intraregional output multipliers for region r are found as the column sums of $\mathbf{L}_{11}$;

$$\mathbf{m}(o)^{rr} = \mathbf{i}'[\mathbf{L}_{11}] = \begin{bmatrix} 2.861 & 2.598 & 2.471 \end{bmatrix} \tag{6.25}$$

Similarly, for region s,

$$\mathbf{m}(o)^{ss} = \mathbf{i}'[\mathbf{L}_{22}] = \begin{bmatrix} 2.294 & 2.659 & 1.604 \end{bmatrix} \tag{6.26}$$

If we had household input coefficients in monetary terms for regions r ($a^{rr}_{n+1,j}$) and s ($a^{ss}_{n+1,j}$), we could find simple intraregional household income multipliers and type I income multipliers. Note that finding total intraregional output multipliers, household income multipliers, or type II income multipliers requires that we have labor input coefficients (in monetary terms) and household consumption coefficients for four different matrices. Initially, the input coefficients matrix for region r – $\mathbf{A}^{rr}$ in (6.22) – must be closed with respect to households. This then adds a row to $\mathbf{A}^{rs}$ and a column to $\mathbf{A}^{sr}$. The former represents inputs of labor from region r to sector 1, 2, and 3 production in region s (for example, commuters). The latter represents purchases of outputs of sectors 1, 2, and 3 in region s by consumers located in region r (imports of consumer goods). For complete consistency, in order to capture income-generating effects throughout the entire (here, two-region) system, the input coefficients matrix for region s – $\mathbf{A}^{ss}$ in (6.22) – should also be closed with respect to households. This then additionally requires a new row in $\mathbf{A}^{sr}$ and a new column in $\mathbf{A}^{rs}$. These new coefficients represent inputs of labor from region s to production in r and purchases by consumers in s of goods made in r, respectively. Thus the $\bar{\mathbf{A}}$ and $\bar{\mathbf{L}}$ matrices, for our numerical example, would grow from 6×6 to 8×8.

Given this $\bar{\mathbf{L}}$ matrix, total intraregional output multipliers, household income multipliers, and type II income multipliers for region r would be found using the elements from the upper left submatrix – now 4×4 – in $\bar{\mathbf{L}}$. Similarly, using intraregional physical labor input coefficients or value-added coefficients for both regions, total intraregional employment or value-added multipliers and type II multipliers could be found.

Interregional Effects The essence of an interregional (or multiregional) input–output model is that it includes impacts in one region that are caused by changes in another region; these are often termed the interregional spillover effects. In our example, these are reflected in the $\mathbf{L}_{12}$ and $\mathbf{L}_{21}$ matrices; here

$$\mathbf{L}_{21} = \begin{bmatrix} .318 & .253 & .251 \\ .177 & .123 & .124 \\ .346 & .365 & .365 \end{bmatrix} \tag{6.27}$$

Consider, $(l_{21})_{23} = 0.124$; this indicates that for each dollar's worth of final demand for the output of sector 3 in region r, 12.4 cents' worth of output from sector 2 in region s is required as input.

Thus, in an interregional input–output model, we can calculate simple interregional multipliers, $m(o)^{sr}_{\cdot j}$ – the total value of output from all sectors in region s used to satisfy a dollar's worth of final demand for sector j in region r. Here,

$$\mathbf{m}(o)^{sr} = \mathbf{i}'[\mathbf{L}_{21}] = \begin{bmatrix} 0.841 & 0.741 & 0.740 \end{bmatrix} \tag{6.28}$$

These are output impacts that are transmitted across regional boundaries – here from r (where the exogenous change occurs) to s (where production occurs). As the reader can perhaps imagine by now, we have the same set of possibilities for measuring various interregional income effects, interregional employment effects, and total interregional effects using the same kinds of calculations as for intraregional effects, now using $\mathbf{L}_{21}$ (and $\bar{\mathbf{L}}_{21}$ if the regions were closed with respect to households). Interregional effects whose origins are in new final demand in region s would be calculated using the elements of $\mathbf{L}_{12}$ (or $\bar{\mathbf{L}}_{12}$). Here

$$\mathbf{m}(o)^{rs} = \mathbf{i}'[\mathbf{L}_{12}] = \begin{bmatrix} 1.135 & 1.401 & 0.618 \end{bmatrix} \tag{6.29}$$

National Effects Assuming, once again, that there are exogenous increases in final demands for region r goods and hence in outputs of region r sectors, we can denote as national effects the sums of columns in both $\mathbf{L}_{11}$ and $\mathbf{L}_{21}$. (These could logically also be termed *total* effects, but we have used *total*, as contrasted with simple, for effects that are calculated from a matrix that has households endogenous.) Arranged as row vectors,

$$\begin{aligned} \mathbf{m}(o)^{r} &= \mathbf{i}'\begin{bmatrix} \mathbf{L}_{11} \\ \mathbf{L}_{21} \end{bmatrix} = \begin{bmatrix} 3.702 & 3.339 & 3.211 \end{bmatrix} \\ \mathbf{m}(o)^{s} &= \mathbf{i}'\begin{bmatrix} \mathbf{L}_{12} \\ \mathbf{L}_{22} \end{bmatrix} = \begin{bmatrix} 3.429 & 4.060 & 2.222 \end{bmatrix} \end{aligned} \tag{6.30}$$

For the two-region interregional system, let $\mathbf{m}(o) = \begin{bmatrix} \mathbf{m}(o)^{r} & \mathbf{m}(o)^{s} \end{bmatrix}$. Here

$$\mathbf{m}(o) = \mathbf{i}'\mathbf{L} = \begin{bmatrix} 3.702 & 3.339 & 3.211 & 3.429 & 4.060 & 2.222 \end{bmatrix} \tag{6.31}$$

A policy implication from these figures is that a dollar's worth of government spending on the output of sector 2 in region s would have the greatest impact throughout the two-region economy, as measured by total output (direct plus indirect) required from all sectors in both regions. Similarly, if the government is interested in acquiring goods from sector 1 or sector 3, the greatest *national* (both regions) economic impact will occur if the purchases are made from firms in region r.

Again, using information on labor inputs or value added in each region, simple and type I income, employment and value-added effects could be calculated at the national (all regions) level. Similarly, for a system in which all regions have been closed with respect to households, total national output, income, employment and value-added effects and type II multipliers can be found.

Sectoral Effects As a final kind of multiplier, we can find the impact on sector i throughout the entire country, because of a dollar's worth of final demand for sector j in either region. (Since this crosses regional boundaries, it is also a kind of "national" effect.) Denote this simple output multiplier as $m(o)_{ij}^{\cdot r}$ and $m(o)_{ij}^{\cdot s}$. For our example,

$$\begin{aligned} m(o)_{13}^{\cdot r} &= (l_{11})_{13} + (l_{21})_{13} = 0.332 + 0.251 = 0.583 \\ m(o)_{21}^{\cdot s} &= (l_{22})_{21} + (l_{12})_{21} = 0.268 + 0.558 = 0.826 \end{aligned}$$

and so on. With additional region-specific information (labor input or value-added coefficients) we could find various simple or type I effects; with elements from $\bar{\mathbf{L}}$, we would find total multipliers and type II effects. (These kinds of sectoral effects are only meaningful when each region contains the same sectors.)

More Than Two Regions With models of more than two regions, there are no new principles involved, although the possibilities increase. For example, with three regions, one can trace interregional effects in now six different ways: (1) exogenous changes in region 1 affecting outputs in region 2 and/or region 3, (2) exogenous changes in 2 affecting outputs in 1 and/or 3, and (3) exogenous changes in 3 affecting outputs in 1 and/or 2.

6.3.3 Multiregional Input–Output Multipliers

All of the multipliers found in the interregional input–output model have their counterparts in the multiregional model. This is to be expected, since the multiregional model is an attempt to capture all of the connections in the interregional model using a simpler set of data. Each of the components in the interregional case – for example, $\mathbf{A}^{rr}$ and $\mathbf{A}^{rs}$ – has its counterpart estimate – $\hat{\mathbf{c}}^{rr}\mathbf{A}^{r}$ and $\hat{\mathbf{c}}^{rs}\mathbf{A}^{s}$ – in the multiregional case. A thorough exploration of multipliers in the multiregional input–output model can be found in DiPasquale and Polenske (1980).

The final form of the multiregional model was

$$\mathbf{x} = (\mathbf{I} - \mathbf{CA})^{-1}\mathbf{Cf} \tag{6.32}$$

Here $\mathbf{A} = \begin{bmatrix} \mathbf{A}^{r} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^{s} \end{bmatrix}$ is a block diagonal matrix whose submatrices represent regional technical (not regional input) coefficients and $\mathbf{C} = \begin{bmatrix} \hat{\mathbf{c}}^{rr} & \hat{\mathbf{c}}^{rs} \\ \hat{\mathbf{c}}^{sr} & \hat{\mathbf{c}}^{ss} \end{bmatrix}$, where the components of the submatrices in $\mathbf{C}$ represent flows between regions in the form of proportions of a commodity in a region that come from within the region and from each of the other regions.

The important point to be recalled is that in the interregional model the exogenous sectors represent final demands, wherever located, for goods made by producers in a particular region. In the multiregional model, the $\mathbf{f}$'s represent demands exercised by exogenous sectors located in a given region for goods, wherever produced. For a two-region multiregional model, it is the $\hat{\mathbf{c}}^{rr}$ and $\hat{\mathbf{c}}^{sr}$ matrices that spatially distribute the final demand in region r between producers in r and producers in s.

For example, assume that there are two sectors in each of the two regions and that we want to assess the impact throughout the two-region system of an increase of \$100 in final demand for good 1 by households located in region r, so $\mathbf{f}^r = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$ and $\mathbf{f}^s = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$;

$$\mathbf{f} = \begin{bmatrix} \mathbf{f}^r \\ \mathbf{f}^s \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Let

$$\hat{\mathbf{c}}^{rr} = \begin{bmatrix} 0.7 & 0 \\ 0 & 0.4 \end{bmatrix}, \hat{\mathbf{c}}^{rs} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix}, \hat{\mathbf{c}}^{sr} = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.6 \end{bmatrix}, \hat{\mathbf{c}}^{ss} = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.7 \end{bmatrix}$$

Then

$$\mathbf{C} = \begin{bmatrix} \hat{\mathbf{c}}^{rr} & \hat{\mathbf{c}}^{rs} \\ \hat{\mathbf{c}}^{sr} & \hat{\mathbf{c}}^{ss} \end{bmatrix} = \begin{bmatrix} 0.7 & 0 & 0.2 & 0 \\ 0 & 0.4 & 0 & 0.3 \\ 0.3 & 0 & 0.8 & 0 \\ 0 & 0.6 & 0 & 0.7 \end{bmatrix}$$

and the $\mathbf{Cf}$ term that postmultiplies $(\mathbf{I} - \mathbf{CA})^{-1}$ in (6.32) is

$$\mathbf{Cf} = \begin{bmatrix} 0.7 & 0 & 0.2 & 0 \\ 0 & 0.4 & 0 & 0.3 \\ 0.3 & 0 & 0.8 & 0 \\ 0 & 0.6 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} 100 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 70 \\ 0 \\ 30 \\ 0 \end{bmatrix}$$

The impact of the new \$100 is not felt exclusively in region r, rather only \$70 (70 percent) is presented as new demand for good 1 made in region r, and \$30 (30 percent) turns out to be new demand for good 1 in region s.

The $\mathbf{C}$ matrix distributes the final demands in the multiregional model across supplying regions in accordance with the percentages embodied in the components of $\mathbf{C}$. Premultiplication of $\mathbf{Cf}$ by $(\mathbf{I} - \mathbf{CA})^{-1}$ then converts these distributed final demands into necessary outputs from each sector in each region in the usual way. Thus the matrix from which the various multipliers are derived in the multiregional model is $(\mathbf{I} - \mathbf{CA})^{-1}\mathbf{C}$.

In the numerical illustration in section 3.4.4, with two regions of three sectors each, we found

$$(\mathbf{I}-\mathbf{CA})^{-1}\mathbf{C}=\begin{bmatrix} 1.127 & .447 & .300 & .478 & .418 & .153 \\ .628 & 1.317 & .606 & .552 & 1.115 & .323 \\ .512 & .526 & 1.101 & .335 & .470 & .247 \\ \\ .625 & .369 & .250 & 1.224 & .456 & .216 \\ .238 & .385 & .205 & .278 & .650 & .167 \\ .472 & .445 & .589 & .594 & .529 & 1.232 \end{bmatrix}$$

in (3.31). This matrix plays the same role for multiplier analysis in the multiregional model that $\mathbf{L}=(\mathbf{I}-\mathbf{A})^{-1}=\begin{bmatrix}\mathbf{L}_{11} & \mathbf{L}_{12} \\ \mathbf{L}_{21} & \mathbf{L}_{22}\end{bmatrix}$ in (6.23) did for the interregional case. We examine some of these possibilities; the parallels with the interregional case should be clear, so the illustrations need not be exhaustive. To emphasize the parallel, we define

$$\boldsymbol{\mathcal{L}}=(\mathbf{I}-\mathbf{CA})^{-1}\mathbf{C}=\begin{bmatrix}\boldsymbol{\mathcal{L}}_{11} & \boldsymbol{\mathcal{L}}_{12} \\ \\ \boldsymbol{\mathcal{L}}_{21} & \boldsymbol{\mathcal{L}}_{22}\end{bmatrix}$$

Intraregional Effects Column sums of elements in $\boldsymbol{\mathcal{L}}_{11}$ and $\boldsymbol{\mathcal{L}}_{22}$ are simple intraregional output multipliers. These multipliers correspond to (6.25) and (6.26), above; here

$$\begin{aligned}\mathbf{m}(o)^{rr}&=\mathbf{i}'\boldsymbol{\mathcal{L}}_{11}=\begin{bmatrix}2.267 & 2.290 & 2.007\end{bmatrix}\\ \mathbf{m}(o)^{ss}&=\mathbf{i}'\boldsymbol{\mathcal{L}}_{22}=\begin{bmatrix}2.096 & 1.635 & 1.615\end{bmatrix}\end{aligned}\tag{6.33}$$

As before, income, employment or value-added multipliers could be found if we had the requisite additional data. Closing the multiregional model with respect to households, in order to be able to calculate total and type II multipliers, requires the addition of regional labor input coefficient rows and household consumption coefficient columns to each of the regional input matrices in $\mathbf{A}$, and it requires estimates of $c^{rr}_{n+1,n+1}$, $c^{rs}_{n+1,n+1}$, and so on – these are the proportions of household demands for labor services that are expected to be supplied from within and from outside of each region. These coefficients would be added to the lower right of each diagonal matrix $\hat{\mathbf{c}}^{rr}$, $\hat{\mathbf{c}}^{rs}$, etc. Given $(\mathbf{I}-\bar{\mathbf{C}}\bar{\mathbf{A}})^{-1}\bar{\mathbf{C}}$, using overbars to indicate a model in which households have been made endogenous, we could find these various intraregional multipliers in the usual way, from the upper left and lower right submatrices. Also, with information on value added in each sector in each region, value-added multipliers could be found as in the interregional case.

Interregional Effects As in the interregional model, these effects are derived from $\mathcal{L}_{12}$ and $\mathcal{L}_{21}$. Here, corresponding to (6.28) and (6.29), we have

$$\begin{aligned} \mathbf{m}(o)^{sr} &= \mathbf{i}'[\mathcal{L}_{21}] = \begin{bmatrix} 1.335 & 1.199 & 1.044 \end{bmatrix} \\ \mathbf{m}(o)^{rs} &= \mathbf{i}'[\mathcal{L}_{12}] = \begin{bmatrix} 1.365 & 2.003 & 0.723 \end{bmatrix} \end{aligned} \tag{6.34}$$

National Effects Corresponding to (6.30), we have the following simple output multipliers that reflect production in all sectors in all (here, the two) regions to support a dollar's worth of new final demand for a particular good. Here

$$\begin{aligned} \mathbf{m}(o)^{r} &= \mathbf{i}'\begin{bmatrix} \mathcal{L}_{11} \\ \mathcal{L}_{21} \end{bmatrix} = \begin{bmatrix} 3.602 & 3.489 & 3.051 \end{bmatrix} \\ \mathbf{m}(o)^{s} &= \mathbf{i}'\begin{bmatrix} \mathcal{L}_{12} \\ \mathcal{L}_{22} \end{bmatrix} = \begin{bmatrix} 3.461 & 3.638 & 2.338 \end{bmatrix} \end{aligned} \tag{6.35}$$

Thus, a new dollar's worth of demand from households located in r for good 2 generates a total of \$3.49 new output throughout the entire multiregional system. Arranged in a single row vector, and parallel to (6.31), we have

$$\mathbf{m}(o) = \mathbf{i}'\mathcal{L} = \begin{bmatrix} 3.602 & 3.489 & 3.501 & \quad 3.461 & 3.638 & 2.338 \end{bmatrix} \tag{6.36}$$

and similar kinds of policy implications can be drawn from these figures. For example, assume that the government could stimulate consumer demand in a particular region for a particular product (e.g., through tax credits, as for insulation and storm windows in cold regions). The greatest overall (national) effect, as measured by these simple national output multipliers, would come from consumer demand in region s for good 2.

Sectoral Effects Finally, as with the interregional model, we can assess the impact on sector i throughout the economy of one dollar's worth of new final demand in region r for good j. For example, $m(o)_{13}^{\cdot r} = (\ell_{11})_{13} + (\ell_{21})_{13} = 0.300 + 0.250 = 0.550$, $m(o)_{21}^{\cdot s} = (\ell_{22})_{21} + (\ell_{12})_{21} = 0.278 + 0.552 = 0.830$, and so on.

Final Demand for Goods Made in a Particular Region If one is using the version of the multiregional input–output model in which impacts of new region-specific final demands are being assessed (as in the example of a foreign airline's new order for Boeing jetliners made in the state of Washington), where

$$\mathbf{x} = (\mathbf{I} - \mathbf{CA})^{-1}\mathbf{f}^*$$

as in (3.32) in Chapter 3, then all of the multiplier calculations outlined above would be found from the elements in $(\mathbf{I} - \mathbf{CA})^{-1}$ rather than $(\mathbf{I} - \mathbf{CA})^{-1}\mathbf{C}$. The $(\mathbf{I} - \mathbf{CA})^{-1}$ matrix for this numerical example was given in (3.33) in that chapter. The interested reader may wish to find the various multipliers, as in (6.33) through (6.35).

More Than Two Regions As before, with models of more than two regions, there are no new principles involved, although the possibilities for multiplier calculations increase. For example, with three regions, there are three possible settings in which to calculate various intraregional multiplier effects and six in which to calculate interregional effects.

In section 3.4.6 we introduced a three-sector, three-region aggregation of the Chinese 2000 multiregional model. The $\mathcal{L} = (\mathbf{I} - \mathbf{CA})^{-1}\mathbf{C}$ matrix for that model is repeated below, in Table 6.5. (This was Table 3.9 in Chapter 3.) The regional aggregations used in this table result in very large geographic aggregates, and the relative uniformity of the simple output multipliers across regions, as indicated in the tables to follow, reflects this. Simple intra- and interregional output multipliers for this Chinese model are presented in Table 6.6. In addition, simple national (all-region) multipliers are shown.

For example, a ¥1 change in final demand in the North for manufacturing and construction (sector 2) requires ¥0.41 from all sectors in the South and ¥0.04 from the Rest of China. In view of the sectoral breakdown used in this model it is not surprising that manufacturing and construction (sector 2) has the largest simple output multiplier in each region and in the nation as a whole, or that the services sector has the second-largest multipliers with natural resources a rather distant third.

In terms of regional dependencies, we see that the South is much more dependent on the North than on the Rest of China for the inputs that would be needed to satisfy one unit of final demand in each of the sectors in the South – from the sums of the three elements in the North row for the South, 0.4683, vs. the sums of the three elements in the Rest of China row, 0.1467. Similar aggregate measures can be derived for the other regions.

Sector-specific simple output multipliers, $m(o)_{ij}^{\cdot r}$, are shown in Table 6.7. There is a great deal of uniformity across regions. For example, ¥1 of new demand for manufacturing and construction output by households located in the North, South or Rest of China regions generates a national impact in terms of ¥ worth of new output in sector 1 of 0.3321, 0.3249, or 0.3413 in the three regions. Similarly, ¥1 worth of new final demand for services generates a need for inputs of ¥0.5801, 0.6157 or 0.5033 worth of new manufacturing and construction output in the three regions. The figures are generally similar in other rows of Table 6.7. Again, this is primarily because of the very large sizes of the three regions in this model illustration.

Hioki (2005) presents an empirical analysis for the Chinese economy, using the same Chinese MRIO data but at greater levels of disaggregation. This is an analysis of the magnitude of interregional spread or "trickle down" effects, especially from eastern Chinese (coastal) regions to the less developed western (inland) regions. The study calculated intraregional and interregional simple output multipliers for an eight-region, 17-sector version of the CMRIO model. Illustrative of the kinds of conclusions drawn in this study is the observation that around 20 percent of the total output in the Central region is induced by final demands of the coastal regions (p. 170). This suggests that the government's strategy, begun during the 1980s, favoring development of the coastal

Table 6.5 Leontief Inverse Matrix, $\mathcal{L}$, for the Chinese Multiregional Economy, 2000

	North			South			Rest of China		
	Nat. Res.	Manuf. & Const.	Services	Nat. Res.	Manuf. & Const.	Services	Nat. Res.	Manuf. & Const.	Services
North									
Natural Resources	1.1631	0.2561	0.0965	0.0227	0.0582	0.0268	0.0064	0.0161	0.0085
Manuf. & Const.	0.3008	1.7275	0.4080	0.0537	0.1596	0.0849	0.0191	0.0529	0.0314
Services	0.0840	0.1686	1.1794	0.0115	0.0306	0.0202	0.0035	0.0093	0.0054
South									
Natural Resources	0.0325	0.0681	0.0321	1.1919	0.2504	0.1114	0.0245	0.0459	0.0232
Manuf. & Const.	0.1194	0.2943	0.1588	0.3258	1.9193	0.5036	0.0742	0.2010	0.1187
Services	0.0193	0.0447	0.0284	0.0848	0.1920	1.1965	0.0142	0.0375	0.0252
ROC									
Natural Resources	0.0034	0.0079	0.0039	0.0062	0.0164	0.0082	1.1958	0.2793	0.1061
Manuf. & Const.	0.0098	0.0245	0.0133	0.0176	0.0478	0.0272	0.2068	1.5681	0.3532
Services	0.0021	0.0051	0.0030	0.0045	0.0114	0.0075	0.0730	0.1916	1.1716

Table 6.6 Simple Intra- and Interregional Output Multipliers for the Chinese Multiregional Input–Output System, 2000

	Region and Sector Experiencing a One-Unit Change in Final Demand								
	North			South			Rest of China		
	1	2	3	1	2	3	1	2	3
Total Output to Satisfy the Final Demand Change									
North	1.5479	2.1522	1.6840	0.0879	0.2485	0.1319	0.0289	0.0783	0.0454
South	0.1711	0.4071	0.2193	1.6024	2.3616	1.8115	0.1128	0.2844	0.1670
RoC	0.0154	0.0375	0.0202	0.0283	0.0756	0.0428	1.4755	2.0389	1.6309
Nation	1.7344	2.5967	1.9234	1.7187	2.6856	1.9862	1.6173	2.4016	1.8433

Table 6.7 Sector-Specific Simple Output Multipliers for the Chinese Multiregional Input–Output System, 2000

	Sector and Region Experiencing a One-Unit Change in Final Demand								
	Natural Resources			Manufacturing and Construction			Services		
	North	South	RoC	North	South	RoC	North	South	RoC
1	1.1990	1.2208	1.2267	0.3321	0.3249	0.3413	0.1325	0.1464	0.1378
2	0.4300	0.3970	0.3000	2.0462	2.1267	1.8220	0.5801	0.6157	0.5033
3	0.1054	0.1008	0.0906	0.2184	0.2340	0.2384	1.2108	1.2241	1.2022

regions (which it was thought would then lead to spillovers inland) has “actually started to work to a certain extent” (p. 171).[9]

6.4 Miyazawa Multipliers

The important work of Miyazawa (1976) on endogenizing households in an input–output model generates various multiplier matrices.[10] A comprehensive overview of the explicit demographic-economic interactions in the Miyazawa structure and its applications can be found in the collection of papers in Hewings *et al.* (1999). In this section we depart from some of the notation used elsewhere in this book, in order to be consistent with that used by Miyazawa, since virtually all subsequent discussion and application of the Miyazawa framework has continued to use his notation. Specifically, this means that we will now define $\mathbf{B} = (\mathbf{I} - \mathbf{A})^{-1}$ (instead of $\mathbf{L}$, since Miyazawa uses $\mathbf{L}$ for another purpose, as we will see below).

6.4.1 Disaggregated Household Income Groups

We assume that households can be separated into q distinct income-bracket groups and that payments by producers to wage earners in each of those groups can be identified. Let $\underset{(q\times n)}{\mathbf{V}} = [v_{gj}]$, where v_{gj} represents income paid to a wage earner in income bracket g $(g = 1, \ldots, q)$ per dollar's worth of output of sector j. This is a generalization (to q rows) of the single row of household input coefficients or labor input coefficients in Chapter 2, $\mathbf{h}_R = [a_{n+1,1}, \ldots, a_{n+1,n}]$. Similarly, let $\underset{(n\times q)}{\mathbf{C}} = [c_{ih}]$, where c_{ih} is the amount of sector i's product consumed per dollar of income of households in income group h $(h = 1, \ldots, q)$; this is a generalization (to q columns) of the single

9 We examine some of the details of construction of this multiregional model in section 8.7.

10 The definitive work is Miyazawa (1976), although there were several articles preceding that monograph. Most of these were in the *Hitotsubashi Journal of Economics* in the 1960s and early 1970s and were not widely known outside of Japan. More recent work by Sonis and Hewings (1993, 1995) on extended multiregional Miyazawa multipliers can also be found in that journal, as well as elsewhere (e.g., Sonis and Hewings, 2000).

column of household consumption coefficients in Chapter 2, $\mathbf{h}_C = \begin{bmatrix} a_{1,n+1} \\ \vdots \\ a_{n,n+1} \end{bmatrix}$, and yet another use for **C** in input–output discussions. So the augmented matrix of coefficients is $\bar{\mathbf{A}} = \begin{bmatrix} \underset{(n\times n)}{\mathbf{A}} & \underset{(n\times q)}{\mathbf{C}} \\ \underset{(q\times n)}{\mathbf{V}} & \underset{(q\times q)}{\mathbf{0}} \end{bmatrix}$, and the expanded input–output system is

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{V} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} + \begin{bmatrix} \mathbf{f}^* \\ \mathbf{g} \end{bmatrix} \tag{6.37}$$

where $\underset{(q\times 1)}{\mathbf{y}}$ is a vector of total income for each of the income groups, $\underset{(n\times 1)}{\mathbf{f}^*}$ is a vector of final demands excluding household consumption (now endogenized) and $\underset{(q\times 1)}{\mathbf{g}}$ is a vector of exogenous income (if any) for the income groups.

Assume that $\underset{(q\times 1)}{\mathbf{g}} = \mathbf{0}$; then the two matrix equations in the system in (6.37) are

$$\mathbf{x} = \mathbf{Ax} + \mathbf{Cy} + \mathbf{f}^* \text{ and } \mathbf{y} = \mathbf{Vx} \tag{6.38}$$

From (6.37),

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{I} - \mathbf{A} & -\mathbf{C} \\ -\mathbf{V} & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{f}^* \\ \mathbf{0} \end{bmatrix} \tag{6.39}$$

Using results on inverses of partitioned matrices (Appendix A) it is not difficult to show that the elements of the partitioned inverse in (6.39) can be expressed as

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{B}[\mathbf{I} + \mathbf{C}(\mathbf{I} - \mathbf{VBC})^{-1}\mathbf{VB} & \mathbf{BC}(\mathbf{I} - \mathbf{VBC})^{-1} \\ (\mathbf{I} - \mathbf{VBC})^{-1}\mathbf{VB} & (\mathbf{I} - \mathbf{VBC})^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{f}^* \\ \mathbf{0} \end{bmatrix} \tag{6.40}$$

where, as noted, $\mathbf{B} = (\mathbf{I} - \mathbf{A})^{-1}$.

This can be simplified if, following Miyazawa, we define $\mathbf{VBC} = \mathbf{L}$ and $\mathbf{K} = (\mathbf{I} - \mathbf{L})^{-1} = (\mathbf{I} - \mathbf{VBC})^{-1}$, so that

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \underset{(n\times n)}{\mathbf{B}(\mathbf{I} + \mathbf{CKVB})} & \underset{(n\times q)}{\mathbf{BCK}} \\ \underset{(q\times n)}{\mathbf{KVB}} & \underset{(q\times q)}{\mathbf{K}} \end{bmatrix} \begin{bmatrix} \mathbf{f}^* \\ \mathbf{0} \end{bmatrix} \tag{6.41}$$

Miyazawa defines **L** as the matrix of "inter-income-group coefficients" and **K** as the "interrelational income multiplier" matrix. A typical element of **L** is $l_{gh} = v_{gi}b_{ij}c_{jh}$; this shows the direct increase in the income of group g resulting from expenditure of an additional unit of income by group h. Reading from right to left, household demand (expenditure) of c_{jh} by group h for the output of sector j requires $b_{ij}c_{jh}$ in output from sector i and this, in turn, means income payments from sector i in the amount of $v_{gi}b_{ij}c_{jh}$ to households in group g. Similarly, each element in $\mathbf{K} = (\mathbf{I} - \mathbf{L})^{-1}$ indicates the total increase (direct, indirect and induced) in the income of one group that results from expenditure of an additional unit of income by another group. (An illustration of this approach can be found in the matrix of interrelational income multipliers, **K**, for 11 income groups in the USA for 1987 that is shown in Rose and Li, 1999.)

From (6.41),

$$\mathbf{x} = \mathbf{B}(\mathbf{I} + \mathbf{CKVB})\mathbf{f}^* \tag{6.42}$$

and

$$\mathbf{y} = \mathbf{KVBf}^* \tag{6.43}$$

In (6.42), the effect of final demands on outputs is seen to be the product of two distinct matrices. The first is the Leontief inverse of the open model, **B**. The second is $(\mathbf{I} + \mathbf{CKVB})$; this augments the final demand stimulus, $\mathbf{If}^*$, by $\mathbf{CKVBf}^*$, which endogenizes the total income spending effect. Again, starting at the right, $\mathbf{Bf}^*$ generates the initial output (without household spending), $\mathbf{VBf}^*$ indicates the resultant initial income payments to each group, $\mathbf{KVBf}^*$ multiplies that into total income received in each group – this is exactly what is described by the result in (6.43) – and, finally, $\mathbf{CKVBf}^*$ translates that received income into consumption (demand) by each group on each sector's output. Miyazawa denotes **KVB** the "multi-sector income multiplier" matrix (or the "matrix multiplier of income formation"), indicating the direct, indirect and induced incomes for each income group generated by the initial final demand.

6.4.2 Miyazawa's Derivation

Miyazawa first derives the results on the interrelational multiplier matrix without reference to partitioned matrices [in Miyazawa, 1976, Chapter 1, sections II(2)–III(1); the partitioned inverse structure appears later in Chapter 1, section III(3)]. He makes extensive use of partitioned matrices later in the book – especially in Part 2 on internal and external matrix multipliers. This is a direction that has been explored and expanded considerably in much of the work of Sonis, Hewings and others (summarized in Sonis and Hewings, 1999, which also contains an extensive set of references to their work). A second direction of research that extends the input–output framework to incorporate interactions between economic and demographic components is associated with the many publications of Batey, Madden and others (summarized in Batey and Madden, 1999, again with many references).

We present Miyazawa's initial approach here primarily for completeness, and because the results are often discussed (briefly) in this form in the literature. He begins with

$$\mathbf{x} = \mathbf{Ax} + \mathbf{CVx} + \mathbf{f}^*$$

from (6.38). From this,

$$\mathbf{x} = (\mathbf{I} - \mathbf{A} - \mathbf{CV})^{-1}\mathbf{f}^* \tag{6.44}$$

and with $\mathbf{B} = (\mathbf{I} - \mathbf{A})^{-1}$, straightforward matrix algebra gives

$$(\mathbf{I} - \mathbf{A} - \mathbf{CV}) = (\mathbf{B}^{-1} - \mathbf{CV})\mathbf{BB}^{-1} = (\mathbf{I} - \mathbf{CVB})\mathbf{B}^{-1}$$

Substituting into (6.44),

$$\mathbf{x} = [(\mathbf{I} - \mathbf{CVB})\mathbf{B}^{-1}]^{-1}\mathbf{f}^*$$

and, from the rule for inverses of products,

$$\mathbf{x} = \mathbf{B}(\mathbf{I} - \mathbf{CVB})^{-1}\mathbf{f}^* \tag{6.45}$$

In this form, we find the original Leontief inverse, $\mathbf{B}$, postmultiplied by $(\mathbf{I} - \mathbf{CVB})^{-1}$, which Miyazawa termed the "subjoined inverse matrix."

A further variation is possible and is sometimes used. Starting with (6.45) and, as earlier, with $\mathbf{VBC} = \mathbf{L}$ and $\mathbf{K} = (\mathbf{I} - \mathbf{L})^{-1}$, then

$$\mathbf{K}(\mathbf{I} - \mathbf{VBC}) = \mathbf{I}$$

Premultiply both sides by $\mathbf{C}$ and postmultiply both sides by $\mathbf{VB}$,

$$\mathbf{CK}(\mathbf{I} - \mathbf{VBC})\mathbf{VB} = \mathbf{CVB} \text{ or } \mathbf{CK}(\mathbf{VB} - \mathbf{VBCVB}) = \mathbf{CVB}$$

Factor out $\mathbf{VB}$ to the left and then subtract both sides from $\mathbf{I}$, giving

$$\mathbf{I} - \mathbf{CKVB}(\mathbf{I} - \mathbf{CVB}) = \mathbf{I} - \mathbf{CVB} \text{ or } \mathbf{I} = \mathbf{CKVB}(\mathbf{I} - \mathbf{CVB}) + \mathbf{I} - \mathbf{CVB}$$

Regrouping terms

$$\mathbf{I} = (\mathbf{I} + \mathbf{CKVB})(\mathbf{I} - \mathbf{CVB})$$

and so, from the fundamental definition of an inverse,

$$(\mathbf{I} - \mathbf{CVB})^{-1} = (\mathbf{I} + \mathbf{CKVB})$$

Putting this result into (6.45) gives

$$\mathbf{x} = \mathbf{B}(\mathbf{I} + \mathbf{CKVB})\mathbf{f}^* \tag{6.46}$$

as in (6.42).

Miyazawa suggests that if labor input coefficients, in $\mathbf{V}$, and household consumption coefficients, in $\mathbf{C}$, are less stable than interindustry coefficients (in $\mathbf{A}$ and consequently in $\mathbf{B}$), there is an advantage to using the format in (6.46) instead of (6.45). Namely, a revised subjoined inverse, $(\mathbf{I} - \mathbf{CVB})^{-1}$, whose order is n, can be found by using $\mathbf{K}$, whose order is q "… [which] in most cases is very much smaller than n …" (Miyazawa, 1976, p. 7). However, inverting large matrices is no longer the concern that it was in the 1970s.

From (6.46), household income, $\mathbf{y} = \mathbf{Vx}$, is seen to be

$$\mathbf{y} = \mathbf{VB}(\mathbf{I} + \mathbf{CKVB})\mathbf{f}^* = (\mathbf{I} + \mathbf{VBCK})\mathbf{VBf}^* = (\mathbf{I} + \mathbf{LK})\mathbf{VBf}^*$$

But since $\mathbf{K} = (\mathbf{I} - \mathbf{L})^{-1}$, $(\mathbf{I} - \mathbf{L})\mathbf{K} = \mathbf{I}$, $\mathbf{LK} = \mathbf{K} - \mathbf{I}$, so $(\mathbf{I} + \mathbf{LK}) = \mathbf{K}$, and

$$\mathbf{y} = \mathbf{KVBf}^* \tag{6.47}$$

as in (6.43).

6.4.3 Numerical Example

We expand the numerical example from Chapter 2, assuming a three-sector economy with households divided into two income groups. Let the augmented coefficients matrix be

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{V} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 0.15 & 0.25 & 0.05 & 0.1 & 0.05 \\ 0.2 & 0.05 & 0.4 & 0.2 & 0.1 \\ 0.3 & 0.25 & 0.05 & 0.01 & 0.1 \\ 0.05 & 0.1 & 0.08 & 0 & 0 \\ 0.12 & 0.05 & 0.1 & 0 & 0 \end{bmatrix}$$

In particular, labor income coefficients for the two household groups are given in the two rows of $\mathbf{V} = \begin{bmatrix} 0.05 & 0.1 & 0.08 \\ 0.12 & 0.05 & 0.1 \end{bmatrix}$, and consumption coefficients for those same two groups are given in the two columns of $\mathbf{C} = \begin{bmatrix} 0.1 & 0.05 \\ 0.2 & 0.1 \\ 0.01 & 0.1 \end{bmatrix}$.

Given $\mathbf{V}$, $\mathbf{C}$, and $\mathbf{B} = (\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} 1.3651 & .4253 & .2509 \\ .5273 & 1.3481 & .5954 \\ .5698 & .4890 & 1.2885 \end{bmatrix}$, the relevant Miyazawa matrices are easily found to be

$$\mathbf{VBC} = \begin{bmatrix} .0574 & .0454 \\ .0601 & .0480 \end{bmatrix} \quad \text{and} \quad \mathbf{K} = (\mathbf{I} - \mathbf{VBC})^{-1} = \begin{bmatrix} 1.0642 & .0507 \\ .0671 & 1.0536 \end{bmatrix}$$

For example, in this illustration, a direct increase of \$1 in income to households in group 1 leads to a 6.7 cent (k_{21}) increase in income payments to households in group 2. Similarly,

$$\mathbf{KVB} = \begin{bmatrix} .1898 & .2162 & .1960 \\ .2716 & .1894 & .2106 \end{bmatrix}$$

In this case, for example, an additional unit of final demand for the goods of sector 1 generates 27.16 cents in new income for group 2. Furthermore,

$$\mathbf{B}(\mathbf{I} - \mathbf{CVB})^{-1} = \begin{bmatrix} 1.4445 & .4994 & .3234 \\ .6496 & 1.4609 & .7062 \\ .6577 & .5644 & 1.3648 \end{bmatrix} \quad \text{and} \quad \mathbf{BCK} = \begin{bmatrix} .2476 & .1545 \\ .3642 & .2492 \\ .1923 & .2258 \end{bmatrix}$$

(The reader can make appropriate interpretations of the elements in each of these matrices.)

In this case, the Leontief inverse for the augmented system can easily be found directly; it is[11]

$$(\mathbf{I}-\bar{\mathbf{A}})^{-1}=\bar{\mathbf{B}}=\begin{bmatrix}\bar{\mathbf{B}}_{11} & \bar{\mathbf{B}}_{12}\\ \bar{\mathbf{B}}_{21} & \bar{\mathbf{B}}_{22}\end{bmatrix}=\begin{bmatrix}1.4445 & .4994 & .3234 & .2476 & .1545\\ .6496 & 1.4609 & .7062 & .3642 & .2492\\ .6577 & .5644 & 1.3648 & .1923 & .2258\\ .1898 & .2162 & .1960 & 1.0642 & .0507\\ .2716 & .1894 & .2106 & .0671 & 1.0536\end{bmatrix}$$

and the correspondences with elements in $\bar{\mathbf{B}}$ are exactly as expected, namely $\mathbf{K}=\bar{\mathbf{B}}_{22}$, $\mathbf{KVB}=\bar{\mathbf{B}}_{21}$, $\mathbf{BCK}=\bar{\mathbf{B}}_{12}$ and $\mathbf{B}(\mathbf{I}-\mathbf{CVB})^{-1}=\bar{\mathbf{B}}_{11}$.

6.4.4 Adding a Spatial Dimension

We saw in Chapter 3 that interregional or multiregional input–output models were conveniently represented in partitioned matrix form. To incorporate the Miyazawa structure into an IRIO- or MRIO-style model, assume that we have p regions ($k,l=1,\ldots,p$) with n sectors ($i,j=1,\ldots,n$) each and that we have identified q household income groups ($g,h=1,\ldots,q$) in each region. Then the augmented $\mathbf{A}$ matrix would be

$$\bar{\mathbf{A}}=\begin{bmatrix}\underset{(np\times np)}{\mathbf{A}} & \underset{(np\times pq)}{\mathbf{C}}\\ \underset{(pq\times np)}{\mathbf{V}} & \underset{(pq\times pq)}{\mathbf{0}}\end{bmatrix}$$

where

$$\underset{(np\times np)}{\mathbf{A}}=\begin{bmatrix}\underset{(n\times n)}{\mathbf{A}^{11}} & \cdots & \underset{(n\times n)}{\mathbf{A}^{1p}}\\ \vdots & \ddots & \vdots\\ \underset{(n\times n)}{\mathbf{A}^{p1}} & \cdots & \underset{(n\times n)}{\mathbf{A}^{pp}}\end{bmatrix}=\left[a_{ij}^{kl}\right],\quad \underset{(np\times qp)}{\mathbf{C}}=\begin{bmatrix}\underset{(n\times q)}{\mathbf{C}^{11}} & \cdots & \underset{(n\times q)}{\mathbf{C}^{p1}}\\ \vdots & \ddots & \vdots\\ \underset{(n\times q)}{\mathbf{C}^{p1}} & \cdots & \underset{(n\times q)}{\mathbf{C}^{pp}}\end{bmatrix}=\left[c_{ih}^{kl}\right],$$

and

$$\underset{(pq\times np)}{\mathbf{V}}=\begin{bmatrix}\underset{(q\times n)}{\mathbf{V}^{11}} & \cdots & \underset{(q\times n)}{\mathbf{V}^{1p}}\\ \vdots & \ddots & \vdots\\ \underset{(q\times n)}{\mathbf{V}^{p1}} & \cdots & \underset{(q\times n)}{\mathbf{V}^{pp}}\end{bmatrix}=\left[v_{gj}^{kl}\right].$$

Notice that consumption coefficients require knowledge of the spending habits of consumers in each income group in each region on goods from each sector in each region. Similarly, the labor input coefficients require knowledge on payments to laborers in each income group in each region by each sector in each region.

[11] Again, we use $\bar{\mathbf{B}}$ rather than $\bar{\mathbf{L}}$ to be consistent with the Miyazawa literature.

Table 6.8 Interrelational Interregional Income Multipliers

	Region of Income Origin				
Region of Income Receipt	1	2	3	4	Row Total
1	1.23	0.12	0.16	0.07	1.57
2	0.11	1.28	0.13	0.05	1.57
3	0.11	0.03	1.06	0.01	1.14
4	0.44	0.56	0.50	1.77	3.28
Column Total	1.81	1.99	1.85	1.90	

Source: Hewings, Okuyama and Sonis, 2001, Table 9.

The elements in the partitioned inverse in (6.41) will have the same dimensions as $\bar{\mathbf{A}}$, namely

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \underset{(np \times np)}{\mathbf{B}(\mathbf{I} + \mathbf{CKVB})} & \underset{(np \times pq)}{\mathbf{BCK}} \\ \underset{(pq \times np)}{\mathbf{KVB}} & \underset{(pq \times pq)}{\mathbf{K}} \end{bmatrix} \begin{bmatrix} \mathbf{f}^* \\ \mathbf{0} \end{bmatrix}$$

Clearly, this is potentially very demanding of data. However, an illustrative application can be found in Hewings, Okuyama and Sonis (2001) for a 53-sector, four-region model (Chicago and three surrounding suburbs), without division into income groups – that is, $n = 53$, $p = 4$, and $q = 1$. In this case the income formation impacts are across regions rather than income groups. In particular, $\mathbf{K}$ is a 4×4 matrix; it is shown in Table 6.8.[12]

Reading down column 1 for illustration, we find that from an increase of $1 in income in Region 1, an additional $0.23 is generated in Region 1, $0.11 in Regions 2 and 3, and $0.44 in Region 4. Column sums have an interpretation similar to the more usual output multipliers; they indicate the new income generated throughout the four-region system (Chicago metropolitan area) of an additional $1 in income in the region at the top of the column. Row sums are a measure of additional income in each region at the left as a result of a $1 income increase in each region. (As with row sums of the usual Leontief inverse, these are generally less useful results than the column sums.) Often, results in empirically derived interrelational multiplier matrices are normalized in some way to account, for example, for differences in sizes of the regions being studied. A complete interregional Miyazawa analysis would require that we distinguish several income brackets in each region (that is, $q > 1$) and then create consumption coefficients and labor input coefficients for each of those brackets (in each region).

[12] For additional data and details on this application, see Hewings and Parr (2007).

6.5 Gross and Net Multipliers in Input–Output Models

6.5.1 Introduction

Leontief's earliest formulations (for the USA in 1919, 1929, and 1939) were in terms of "net" accounts. The fundamental balance equations had no z_{ii} or a_{ii} terms; in the empirical tables the on-diagonal elements were zero.

> [The interindustry transactions table] would naturally have many empty squares. Those lying along the main diagonal are necessarily left open because our accounting principle does not allow for registration of any transaction within the same firm …" (Leontief, 1951, p. 13)

> The output of an industry … is defined with exclusion of the products consumed by the same industry in which they have been produced. Thus $a_{11} = a_{22} = \cdots = a_{ii} = \cdots = a_{mm} = 0$ by definition. (Leontief, 1951, p. 189)

The 1947 US input–output tables discussed and published in Evans and Hoffenberg (1952) include on-diagonal transactions, coefficients, and inverse elements; in that sense these tables are "gross." They point out that the inverse figures can be adjusted to exclude intra-sector transactions but they do not suggest that as a preferable alternative.[13] In Leontief *et al.* (1953, Chapter 2 by Leontief) the equations in the text are gross but the tables and the equations in the Mathematical Note to Chapter 2 are net. In virtually all later publications (for example, Leontief, 1966, Chapters 2 and 7) on-diagonal elements are included.[14] (For a thoughtful discussion of net and gross input–output accounts, see Jensen, 1978.) This net/gross distinction led to the concept of input–output "net" multipliers, which we explore below.

6.5.2 Multipliers in the Net Input–Output Model

We consider only square systems. Generating a net model simply means that the main diagonals of $\mathbf{Z}$ and $\mathbf{A}$ contain only zeros, and that the gross output vector is reduced by the amount of each sector's intraindustry transactions. As usual, denote by $\hat{\mathbf{Z}}$ the diagonal matrix containing the elements z_{ii}. Then let $\mathbf{Z}_{net} = \mathbf{Z} - \hat{\mathbf{Z}}$, and $\hat{\mathbf{x}}_{net} = \hat{\mathbf{x}} - \hat{\mathbf{Z}}$; this latter is a diagonal matrix of sectoral outputs in the *net* system from which on-diagonal (intrasectoral) transactions have been removed.[15] As usual, input coefficients are found for the net system as

$$\mathbf{A}_{net} = \mathbf{Z}_{net}(\hat{\mathbf{x}}_{net})^{-1} = (\mathbf{Z} - \hat{\mathbf{Z}})(\hat{\mathbf{x}} - \hat{\mathbf{Z}})^{-1}$$

and

$$(\mathbf{I} - \mathbf{A}_{net}) = \mathbf{I} - (\mathbf{Z} - \hat{\mathbf{Z}})(\hat{\mathbf{x}} - \hat{\mathbf{Z}})^{-1}$$

[13] In contrast, Georgescu-Roegen (1971) argues that diagonal elements in an input–output model ("internal flows") must be suppressed.

[14] Early input–output tables in the UK (for example, for 1954 and 1963) were presented in "net" form (UK, Central Statistical Office, 1961 and 1970). Fifteen-sector versions of these tables appear in Allen and Lecomber (1975) and Barker (1975).

[15] Alternative notation uses $\check{\mathbf{Z}}$ instead of $\mathbf{Z}_{net}$, and similarly for $\mathbf{A}_{net}$ and $\mathbf{x}_{net}$. We avoid that convention because it becomes cumbersome when the vector $\mathbf{x}_{net}$ needs a hat to indicate the associated diagonal matrix – and a "$\wedge$" on top of a "$\vee$" is just too much.

We now examine an alternative expression for the right-hand side. [This demonstration appears to have originated in Weber, 1998 (in German). It is apparently not widely known, at least outside the German-speaking world.] Using the observation that $(\hat{\mathbf{x}} - \hat{\mathbf{Z}})(\hat{\mathbf{x}} - \hat{\mathbf{Z}})^{-1} = \mathbf{I}$, it can be shown that[16]

$$(\mathbf{I} - \mathbf{A}_{net}) = [(\mathbf{I} - \mathbf{A})\hat{\mathbf{x}}](\hat{\mathbf{x}} - \hat{\mathbf{Z}})^{-1}$$

Taking the inverse of both sides,

$$\mathbf{L}_{net} = (\mathbf{I} - \mathbf{A}_{net})^{-1} = \{[(\mathbf{I} - \mathbf{A})\hat{\mathbf{x}}](\hat{\mathbf{x}} - \hat{\mathbf{Z}})^{-1}\}^{-1}$$

and using the matrix algebra rule for inverses of products (for appropriately sized matrices) that $(\mathbf{MNP})^{-1} = \mathbf{P}^{-1}\mathbf{N}^{-1}\mathbf{M}^{-1}$,

$$\mathbf{L}_{net} = (\hat{\mathbf{x}} - \hat{\mathbf{Z}})\hat{\mathbf{x}}^{-1}(\mathbf{I} - \mathbf{A})^{-1} = \hat{\mathbf{x}}_{net}\hat{\mathbf{x}}^{-1}\mathbf{L} \tag{6.48}$$

from which

$$(\hat{\mathbf{x}}_{net})^{-1}\mathbf{L}_{net} = \hat{\mathbf{x}}^{-1}\mathbf{L} \tag{6.49}$$

[Notice from (6.48) that $\mathbf{L}_{net} = (\hat{\mathbf{x}} - \hat{\mathbf{Z}})\hat{\mathbf{x}}^{-1}\mathbf{L} = (\mathbf{I} - \hat{\mathbf{A}})\mathbf{L}$, where $\hat{\mathbf{A}} = \hat{\mathbf{Z}}\hat{\mathbf{x}}^{-1}$.][17]

Consider household income multipliers for the two systems. Given a vector of total household income by sector, $\mathbf{z}_h = [z_{n+1,1}, \ldots, z_{n+1,n}]$, then $\mathbf{h} = \mathbf{z}_h\hat{\mathbf{x}}^{-1}$ and $\mathbf{h}_{net} = \mathbf{z}_h(\hat{\mathbf{x}}_{net})^{-1}$ are the vectors of earnings coefficients in the gross and net systems, respectively. From (6.49),

$$\mathbf{z}_h(\hat{\mathbf{x}}_{net})^{-1}\mathbf{L}_{net} = \mathbf{z}_h\hat{\mathbf{x}}^{-1}\mathbf{L}$$

or

$$\mathbf{h}_{net}\mathbf{L}_{net} = \mathbf{hL}$$

Thus, the income multipliers in the two systems are equal, and therefore for studies in which these kinds of multiplier results are of interest, it makes no difference which model is used.

This result is equally valid for most other multipliers – value-added, household income, pollution-generation, energy use, etc. – associated with productive activity (Table 6.4). The only exception is for output multipliers – $\mathbf{m}(o) = \mathbf{i}'\mathbf{L}$ and $\mathbf{m}(o)_{net} = \mathbf{i}'\mathbf{L}_{net}$; they will not be equal,[18] since from (6.48) $\mathbf{L}_{net} = \hat{\mathbf{x}}_{net}\hat{\mathbf{x}}^{-1}\mathbf{L}$. However, the transformation from one to the other is straightforward, namely

$$\mathbf{m}(o)_{net} = \mathbf{i}'\mathbf{L}_{net} = \mathbf{i}'\hat{\mathbf{x}}_{net}\hat{\mathbf{x}}^{-1}\mathbf{L}$$

[16] This particular expression for the identity matrix may seem unmotivated, but it cleverly allows for a significant rewriting of the expression for $(\mathbf{I} - \mathbf{A}_{net})$. For the interested reader, the derivation is:
$(\mathbf{I} - \mathbf{A}_{net}) = (\hat{\mathbf{x}} - \hat{\mathbf{Z}})(\hat{\mathbf{x}} - \hat{\mathbf{Z}})^{-1} - (\mathbf{Z} - \hat{\mathbf{Z}})(\hat{\mathbf{x}} - \hat{\mathbf{Z}})^{-1} = [(\hat{\mathbf{x}} - \hat{\mathbf{Z}}) - (\mathbf{Z} - \hat{\mathbf{Z}})](\hat{\mathbf{x}} - \hat{\mathbf{Z}})^{-1} = (\hat{\mathbf{x}} - \mathbf{Z})(\hat{\mathbf{x}} - \hat{\mathbf{Z}})^{-1} = [(\mathbf{I} - \mathbf{Z}\hat{\mathbf{x}}^{-1})\hat{\mathbf{x}}](\hat{\mathbf{x}} - \hat{\mathbf{Z}})^{-1} = [(\mathbf{I} - \mathbf{A})\hat{\mathbf{x}}](\hat{\mathbf{x}} - \hat{\mathbf{Z}})^{-1}$.

[17] This fact was noted by Evans and Hoffenberg (1952, p. 140) who used a verbal argument and not a matrix algebra demonstration.

[18] Except for the trivial and uninteresting case when $\mathbf{x} = \mathbf{x}_{net}$.

or

$$\mathbf{m}(o) = \mathbf{i}'\mathbf{L} = \mathbf{i}'\hat{\mathbf{x}}(\hat{\mathbf{x}}_{net})^{-1}\mathbf{L}_{net}$$

(Recall that order of multiplication of diagonal matrices makes no difference.)

Numerical Example[19] Let $\mathbf{Z} = \begin{bmatrix} 150 & 500 & 50 \\ 200 & 100 & 400 \\ 300 & 500 & 50 \end{bmatrix}$ so $\mathbf{Z}_{net} = \mathbf{Z} - \hat{\mathbf{Z}} = \begin{bmatrix} 0 & 500 & 50 \\ 200 & 0 & 400 \\ 300 & 500 & 0 \end{bmatrix}$. If $\mathbf{x} = \begin{bmatrix} 1000 \\ 2000 \\ 1000 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} .15 & .25 & .05 \\ .2 & .05 & .4 \\ .3 & .25 & .05 \end{bmatrix}$; $\mathbf{x}_{net} = \begin{bmatrix} 850 \\ 1900 \\ 950 \end{bmatrix}$, $\mathbf{A}_{net} = \mathbf{Z}_{net}(\hat{\mathbf{x}}_{net})^{-1} = \begin{bmatrix} 0 & .2632 & .0526 \\ .2353 & 0 & .4211 \\ .3529 & .2632 & 0 \end{bmatrix}$.
Then $\mathbf{L} = \begin{bmatrix} 1.3651 & .4253 & .2509 \\ .5273 & 1.3481 & .5954 \\ .5698 & .4890 & 1.2885 \end{bmatrix}$ and $\mathbf{L}_{net} = (\mathbf{I} - \mathbf{A}_{net})^{-1} = \begin{bmatrix} 1.1603 & .3615 & .2133 \\ .5010 & 1.2807 & .5656 \\ .5414 & .4646 & 1.2241 \end{bmatrix}$.
In this case,

$$\mathbf{m}(o) = \mathbf{i}'\mathbf{L} = \begin{bmatrix} 2.4623 & 2.2624 & 2.1348 \end{bmatrix}$$

$$\mathbf{m}(o)_{net} = \mathbf{i}'\mathbf{L}_{net} = \begin{bmatrix} 2.2026 & 2.1067 & 2.0030 \end{bmatrix}$$

Here $\hat{\mathbf{x}}(\hat{\mathbf{x}}_{net})^{-1} = \begin{bmatrix} 1.1765 & 0 & 0 \\ 0 & 1.0526 & 0 \\ 0 & 0 & 1.0526 \end{bmatrix}$ and so $\mathbf{m}(o) = \mathbf{i}'\hat{\mathbf{x}}(\hat{\mathbf{x}}_{net})^{-1}\mathbf{L}_{net} =$

$$\begin{bmatrix} 1.1765 & 1.0526 & 1.0526 \end{bmatrix} \begin{bmatrix} 1.1603 & .3615 & .2133 \\ .5010 & 1.2807 & .5656 \\ .5414 & .4646 & 1.2241 \end{bmatrix} = \begin{bmatrix} 2.4623 & 2.2624 & 2.1348 \end{bmatrix}$$

as expected.

Finally, let $\mathbf{z}_h = [100,\ 120, 80]$ (household income payments); then

$$\mathbf{h} = \begin{bmatrix} 0\ .10 & 0.06 & 0.08 \end{bmatrix} \text{ and } \mathbf{h}_{net} = \begin{bmatrix} 0\ .1176 & 0.0632 & 0.0842 \end{bmatrix}$$

from which

$$\mathbf{hL} = \mathbf{h}_{net}\mathbf{L}_{net} = \begin{bmatrix} 0.2137 & 0.1625 & 0.1639 \end{bmatrix}$$

again as expected.

6.5.3 Additional Multiplier Variants

(Indirect Effects)/(Direct Effects) A number of analysts have taken the view that multipliers should not include the initial stimulus, as they do when the basic

[19] We use the 3 × 3 example from earlier but now disregard the fact that sector 3 is households and simply treat this as a general three-sector model illustration.

definition is "total effects"/"direct effects." For example, for output multipliers this means the \$1 of new final-demand for sector j which turns into \$1 of new sector j output. The usual resolution is simply to subtract 1 from each of the elements in $\mathbf{m}(o)$. This is equivalent to replacing $\mathbf{L}$ by $(\mathbf{L} - \mathbf{I})$ in the formula for $\mathbf{m}(o)$, since $\mathbf{i}'(\mathbf{L} - \mathbf{I}) = \mathbf{i}'\mathbf{L} - \mathbf{i}'\mathbf{I} = \mathbf{m}(o) - \mathbf{i}'$. (For example, see Oosterhaven, Piek and Stelder, 1986.)[20] Of course this will not change the *rankings* of the sectors, but it certainly has implications for other kinds of calculations in which the multipliers are used.

The same adjustment [subtracting 1 or using $(\mathbf{L} - \mathbf{I})$] is appropriate for any Type I or Type II multiplier (Table 6.3). As an example, when $\mathbf{r} = \mathbf{h}$, the Type I multiplier, $\mathbf{m}(h) = \mathbf{hL}$ would be converted to $\mathbf{h}(\mathbf{L} - \mathbf{I})\hat{\mathbf{h}}^{-1} = \mathbf{hL}\hat{\mathbf{h}}^{-1} - \mathbf{hI}\hat{\mathbf{h}}^{-1} = \mathbf{m}(h) - \mathbf{i}'$.

"Growth Equalized" Multipliers Policy makers may wish to know the impact on a particular sector of a general expansion in final demand in all sectors (for example, to help identify "bottlenecks") or of changing patterns of final demand. One approach involves what have been called "growth-equalized" multipliers. (See, for example, Gray *et al.*, 1979, and Gowdy, 1991, for these and many additional multipliers.) The motivation is clear: "… size variation among economic sectors prevents meaningful comparisons of multipliers … to add \$1 of output to some sectors represents a much larger rate of growth than it would for other sectors" (Gray *et al.*, 1979, pp. 68, 72, respectively).

Consider output multipliers; again, the principles are the same for all the other possible multipliers. The idea begins with the multiplier *matrix* $\mathbf{M}(o) = \mathbf{L}$. Row sums, $\mathbf{M}(o)\mathbf{i} = \mathbf{Li}$, indicate output effects in each sector when final demand for each sector increases by \$1.00. This is generally considered an unlikely scenario; an obvious variation is to posit an unequal increase in final demand across sectors. For example, instead of $\mathbf{Li}$ one could use $\mathbf{L}\langle\mathbf{f}\langle\mathbf{i}'\mathbf{f}\rangle^{-1}\rangle\mathbf{i}$, where $\langle\mathbf{f}\langle\mathbf{i}'\mathbf{f}\rangle^{-1}\rangle$ is a diagonal matrix showing each sector's final demand as a *proportion* of total final demand, $f_j/\sum_j f_j$; that is, a measure of relative sector size (or importance). (Base-year output proportions, $x_j/\sum_j x_j$, could also be used.) Element (i, j) in the matrix $\mathbf{L}\langle\mathbf{f}\langle\mathbf{i}'\mathbf{f}\rangle^{-1}\rangle$ shows the effect on sector i output of a $\$(f_j/\sum_j f_j)$ increase in j's final demand. Then $\mathbf{L}\langle\mathbf{f}\langle\mathbf{i}'\mathbf{f}\rangle^{-1}\rangle\mathbf{i}$ shows the multiplier effect on each sector's output of a \$1 final-demand increase distributed across sectors according to their proportion of total final demand.

Another possibility is to use equal percentage, not absolute, demand increases across sectors. This is the "growth equalization." For example, elements of the column vector $[\mathbf{M}(o)](0.01)\mathbf{f} = (0.01)\mathbf{Lf}$ indicate output effects in each sector when final demand for each sector increases by one percent, and $(0.01)\mathbf{i}'\mathbf{Lf} = (0.01)[\mathbf{m}(o)]\mathbf{f}$ indicates the economy-wide total output generated. We illustrate with the same three-sector figures.

[20] Since $(\mathbf{L} - \mathbf{I}) = \mathbf{L}(\mathbf{I} - \mathbf{L}^{-1}) = \mathbf{LA}$ or $(\mathbf{L} - \mathbf{I}) = (\mathbf{I} - \mathbf{L}^{-1})\mathbf{L} = \mathbf{AL}$, these modified multipliers could also be found as $\mathbf{i}'\mathbf{AL}$ or $\mathbf{i}'\mathbf{LA}$ (see de Mesnard, 2002, or Dietzenbacher, 2005).

For the example,

$$\mathbf{f} = \begin{bmatrix} 300 \\ 1300 \\ 150 \end{bmatrix} \text{ and } \langle \mathbf{f} \langle \mathbf{i}'\mathbf{f} \rangle^{-1} \rangle = \left[f_j / \sum_j f_j \right] = \begin{bmatrix} 0.1714 & 0 & 0 \\ 0 & 0.7429 & 0 \\ 0 & 0 & 0.0857 \end{bmatrix}$$

In this case,

$$\mathbf{L}\langle \mathbf{f} \langle \mathbf{i}'\mathbf{f} \rangle^{-1} \rangle = \begin{bmatrix} 0.2340 & 0.3159 & 0.0215 \\ 0.0904 & 1.0015 & 0.0510 \\ 0.0977 & 0.3633 & 0.1104 \end{bmatrix} \text{ and } [\mathbf{L}\langle \mathbf{f} \langle \mathbf{i}'\mathbf{f} \rangle^{-1} \rangle]\mathbf{i} = \begin{bmatrix} 0.5714 \\ 1.1429 \\ 0.5714 \end{bmatrix}$$

Using a one percent increase for the growth equalization illustration,

$$\mathbf{L}\langle (0.01)\mathbf{f} \rangle = \begin{bmatrix} 4.0953 & 5.5284 & 0.3764 \\ 1.5820 & 17.5250 & 0.8930 \\ 1.7095 & 6.3576 & 1.9328 \end{bmatrix}$$

and

$$\mathbf{i}'\mathbf{L}\langle (0.01)\mathbf{f} \rangle = [\, 7.3868 \quad 29.4110 \quad 3.2022 \,]$$

Recall that for this example the simple output multipliers were

$$\mathbf{m}(o) = \mathbf{i}'\mathbf{L} = \begin{bmatrix} 2.4623 & 2.2624 & 2.1348 \end{bmatrix}$$

and we see that the relative importance of the sectors is altered (now it is final demand for sector 2 that is the most stimulative; previously – in $\mathbf{m}(o)$ – it was sector 1).

Another Kind of Net Multiplier Standard input–output multipliers (Tables 6.3 and 6.4) are designed to be used with (multiplied by) final demand. Oosterhaven and Stelder (2002a, 2002b) have observed that in the real world, "practitioners" sometimes (perhaps often) use them incorrectly, to multiply total sectoral output (or value added or employment). So they propose *net* multipliers (the terminology could be confusing; these are not multipliers in a net model, as in section 6.5.2). Essentially, they simply convert a standard multiplier so that it can be used in conjunction with total outputs. For example, their Type I *net* output multipliers are $\mathbf{i}'\mathbf{L}\hat{\mathbf{f}}_c$, where $\mathbf{f}_c = [f_j/x_j]$; in their terms, f_j/x_j is the fraction of j's output that may "rightfully be considered exogenous" (Oosterhaven and Stelder, 2002a, p. 536). Specifically, they "decompose" $\mathbf{i}'\mathbf{L}\mathbf{f}$ as follows:

$$\mathbf{i}'\mathbf{L}\mathbf{f} = \mathbf{m}(o)\mathbf{f} = \mathbf{m}(o)\hat{\mathbf{f}}\mathbf{i} = \mathbf{m}(o)\hat{\mathbf{f}}\hat{\mathbf{x}}^{-1}\hat{\mathbf{x}}\mathbf{i} = \mathbf{m}(o)\hat{\mathbf{f}}_c\mathbf{x} = \mathbf{i}'\mathbf{L}\hat{\mathbf{f}}_c\mathbf{x}$$

The *net* multiplier *matrix* is thus $\mathbf{L}\hat{\mathbf{f}}_c$ and the associated *vector* of economy-wide multipliers is $\mathbf{i}'\mathbf{L}\hat{\mathbf{f}}_c = \mathbf{m}(o)\hat{\mathbf{f}}_c$. Other multipliers can be similarly modified.

This work generated considerable discussion and a lengthy and elaborate exchange (de Mesnard, 2002, 2007a, 2007b; Dietzenbacher, 2005, Oosterhaven, 2007), with a variety of interpretations and alternative terminology. In the end, "net contribution" or "net backward linkage" indicators were suggested as a more appropriate label than "multiplier." We will return to an aspect of this in Chapter 12 on linkage measures in input–output models.

6.6 Multipliers and Elasticities

6.6.1 Output Elasticity

Another approach to compensating for differences in industry size is one step further from simply considering percentage increases in final demand (as above, in growth equalized multipliers). The idea is to measure both the stimulus *and* its effect in percentage terms – in this case the percentage change in total output (or income or employment, etc.) due to a percentage change in a given industry's final demand. (See, for example, Mattas and Shrestha, 1991 or Ciobanu, Mattas and Psaltopoulos, 2004.) These (percentage change)/(percentage change) measures are "elasticities" in economics terms.

In particular, consider a one percent change in f_j only, so $(\Delta\mathbf{f})' = [0, \ldots, (0.01)f_j, \ldots, 0]$. Then $\Delta\mathbf{x} = \mathbf{L}\Delta\mathbf{f} = \begin{bmatrix} l_{1j} \\ \vdots \\ l_{nj} \end{bmatrix}(0.01)f_j$. The economy-wide output change is $\mathbf{i}'\Delta\mathbf{x} = \mathbf{i}'\begin{bmatrix} l_{1j} \\ \vdots \\ l_{nj} \end{bmatrix}(0.01)f_j = \mathbf{m}(o)_j(0.01)f_j$. This percentage change in total output (across all industries) that is generated by $(0.01)f_j$ has been labeled the *output elasticity* of industry j (oe_j) and is defined as

$$oe_j = 100 \times (\mathbf{i}'\Delta\mathbf{x}/\mathbf{i}'\mathbf{x}) = 100 \times \mathbf{m}(o)_j[(0.01)f_j/\mathbf{i}'\mathbf{x}] = \mathbf{m}(o)_j[f_j/\mathbf{i}'\mathbf{x}]$$

(It would be more precise to call this an *output-to-final demand* elasticity, to distinguish it from other elasticities, below.)

Modification of any of the other multipliers in section 6.2.2 – through multiplication by $[f_j/\mathbf{i}'\mathbf{x}]$ – produces exactly parallel results, giving income, employment, etc., elasticities to final demand. Note that these are very similar to the "growth-equalized" multipliers above; in that case, the modification was produced by $\left[f_j \Big/ \sum\limits_j f_j\right]$ while here it is $\left[f_j \Big/ \sum\limits_j x_j\right]$.

6.6.2 Output-to-Output Multipliers and Elasticities

Direct Effects Starting with $z_{ij} = a_{ij}x_j$, consider the direct effect of an exogenous change in industry j's output (Δx_j) – $\Delta x_j \rightarrow \Delta z_{ij} = a_{ij}\Delta x_j$. This Δz_{ij} represents new i output *directly* required by j, so $\Delta x_i = \Delta z_{ij}$, and thus $\Delta x_i = a_{ij}\Delta x_j$ or $\Delta x_i/\Delta x_j = a_{ij}$. Now consider a one percent increase in j's output $\Delta x_j = (0.01)x_j$; this means

$\Delta x_i = (0.01)a_{ij}x_j$. So the (i, j)th element of the matrix $(0.01)\mathbf{A}\hat{\mathbf{x}}$ measures the direct effect of j's one percent increase in output on industry i. Expressed as a percentage of i's output, we have $100(\Delta x_i/x_i) = 100(0.01)a_{ij}x_j/x_i = a_{ij}x_j/x_i$. And in matrix form, this is the (i, j)th element of the matrix $\hat{\mathbf{x}}^{-1}\mathbf{A}\hat{\mathbf{x}}$, showing the *direct* effect on industry i's output (percentage change) resulting from a one percent change in industry j's output. This is a *direct output-to-output* elasticity. We will meet the matrix $\hat{\mathbf{x}}^{-1}\mathbf{A}\hat{\mathbf{x}}$ again in Chapter 12, where we explore supply-side input–output models.

Total Effects Elements of the Leontief inverse matrix translate final demand changes into *total* output changes – $\Delta x_i = l_{ij}\Delta f_j$ and $l_{ij} = \Delta x_i/\Delta f_j$. These encompass direct and indirect effects, and they are at the heart of the multipliers explored in previous sections in this chapter. Again, it would be slightly cumbersome but completely accurate to call l_{ij} an *output-to-final-demand multiplier*. Consider l_{jj}, the on-diagonal element in the jth column of $\mathbf{L}$ – $l_{jj} = \Delta x_j/\Delta f_j$ or $\Delta x_j = l_{jj}\Delta f_j$. Define l^*_{ij} as l_{ij}/l_{jj}; then

$$l^*_{ij} = l_{ij}/l_{jj} = [\Delta x_i/\Delta f_j]/[\Delta x_j/\Delta f_j] = \Delta x_i/\Delta x_j$$

or $\Delta x_i = l^*_{ij}\Delta x_j$. Thus, l^*_{ij} could be (and has been) viewed as a *total output-to-output* multiplier.

The matrix of these multipliers, $\mathbf{L}^* = [l^*_{ij}]$, is created by dividing each element in a column of $\mathbf{L}$ by the on-diagonal element for that column – $\mathbf{L}^* = \mathbf{L}(\hat{\mathbf{L}})^{-1}$(as usual, $\hat{\mathbf{L}}$ is a diagonal matrix created from the on-diagonal elements in $\mathbf{L}$). Then each of the elements in column j of $\mathbf{L}^*$ indicates the amount of change in industry i output (the row label) that would be required if the *output* of industry j were increased by one dollar.[21]

Suppose, then, that industry j is projected to increase its output to some new amount, $\bar{x}_j$. Postmultiplication of $\mathbf{L}^*$ by a vector, $\bar{\mathbf{x}}$, with $\bar{x}_j$ as its jth element and zeros elsewhere, will generate a vector of total new outputs, $\mathbf{x}^*$, necessary from each industry in the economy because of the exogenously determined output in industry j. That is,

$$\mathbf{x}^* = \mathbf{L}^*\bar{\mathbf{x}} \tag{6.50}$$

We return to this matrix in Chapter 13 in the context of "mixed" input–output models in which final demands (for some industries) and gross outputs (for the other industries) are specified exogenously.

Moving to elasticity terms, the (i, j)th element of $(0.01)\mathbf{L}\hat{\mathbf{x}}$ gives the (total) new output in industry i caused by a one-percent output increase in industry j. So, exactly parallel to the direct elasticity case, above, the (i, j)th element of $\hat{\mathbf{x}}^{-1}\mathbf{L}\hat{\mathbf{x}}$ gives the percent increase in industry i total output due to an initial exogenous one percent increase in industry j output – the "direct and indirect output elasticity of industry i with respect to the output

[21] This is equivalent to the "total flow" approach of Szyrmer (for example, Szyrmer, 1992). He makes a case for the unsuitability of the usual output multipliers (from the standard demand-driven input–output model) for a wide variety of real-world impact studies. Some analysts argue that the *initial* exogenous one-dollar stimulus should be removed from the "total effect" calculation. As was seen above (section 6.5.3), this can be accomplished by replacing $\mathbf{L}$ by $(\mathbf{L} - \mathbf{I})$. The interested reader should see de Mesnard (2002) and Dietzenbacher (2005) for details.

in industry *j*" (Dietzenbacher, 2005, p. 426). We will also meet this matrix, $\hat{\mathbf{x}}^{-1}\mathbf{L}\hat{\mathbf{x}}$, again in Chapter 12 in the discussion of supply-side input–output models.

6.7 Multiplier Decompositions

A number of approaches have been suggested for analyzing the economic "structure" that is portrayed in input–output data. Multiplier decompositions are a prominent part of this research, and we explore two of these in this section.[22]

6.7.1 Fundamentals

We start with the fundamental input–output accounting relationship

$$\underset{(n\times 1)}{\mathbf{x}} = \underset{(n\times n)}{\mathbf{A}}\,\underset{(n\times 1)}{\mathbf{x}} + \underset{(n\times 1)}{\mathbf{f}} \tag{6.51}$$

from which $\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{f} = \mathbf{Lf}$. We now introduce some algebra that initially appears unmotivated but it will soon be clear what is accomplished. Given some $\underset{(n\times n)}{\tilde{\mathbf{A}}}$, adding and subtracting $\tilde{\mathbf{A}}\mathbf{x}$ to (6.51) and rearranging produces

$$\mathbf{x} = \mathbf{Ax} - \tilde{\mathbf{A}}\mathbf{x} + \tilde{\mathbf{A}}\mathbf{x} + \mathbf{f} \Rightarrow (\mathbf{I} - \tilde{\mathbf{A}})\mathbf{x} = (\mathbf{A} - \tilde{\mathbf{A}})\mathbf{x} + \mathbf{f} \tag{6.52}$$

and, solving[23] for $\mathbf{x}$,

$$\mathbf{x} = (\mathbf{I} - \tilde{\mathbf{A}})^{-1}(\mathbf{A} - \tilde{\mathbf{A}})\mathbf{x} + (\mathbf{I} - \tilde{\mathbf{A}})^{-1}\mathbf{f}$$

Let $\mathbf{A}^* = (\mathbf{I} - \tilde{\mathbf{A}})^{-1}(\mathbf{A} - \tilde{\mathbf{A}})$; then this is

$$\mathbf{x} = \mathbf{A}^*\mathbf{x} + (\mathbf{I} - \tilde{\mathbf{A}})^{-1}\mathbf{f} \tag{6.53}$$

Next, premultiply both sides of (6.53) by $\mathbf{A}^*$

$$\mathbf{A}^*\mathbf{x} = (\mathbf{A}^*)^2\mathbf{x} + \mathbf{A}^*(\mathbf{I} - \tilde{\mathbf{A}})^{-1}\mathbf{f} \tag{6.54}$$

and substitute this for $\mathbf{A}^*\mathbf{x}$ in the right-hand side of (6.53)

$$\mathbf{x} = (\mathbf{A}^*)^2\mathbf{x} + \mathbf{A}^*(\mathbf{I} - \tilde{\mathbf{A}})^{-1}\mathbf{f} + (\mathbf{I} - \tilde{\mathbf{A}})^{-1}\mathbf{f} = (\mathbf{A}^*)^2\mathbf{x} + (\mathbf{I} + \mathbf{A}^*)(\mathbf{I} - \tilde{\mathbf{A}})^{-1}\mathbf{f} \tag{6.55}$$

Again, solving for $\mathbf{x}$,

$$\mathbf{x} = \underbrace{[\mathbf{I} - (\mathbf{A}^*)^2]^{-1}}_{\mathbf{M}_3}\underbrace{(\mathbf{I} + \mathbf{A}^*)}_{\mathbf{M}_2}\underbrace{(\mathbf{I} - \tilde{\mathbf{A}})^{-1}}_{\mathbf{M}_1}\mathbf{f} \tag{6.56}$$

In this way the usual Leontief inverse (multiplier) matrix, $(\mathbf{I} - \mathbf{A})^{-1}$, has been decomposed into the product of three matrices.

22 For an overview of these and several others, see Sonis and Hewings (1988) or additional references noted in section 14.2, below.
23 Here and throughout we assume nonsingularity of the matrices whose inverses are shown.

This algebra can be continued. Premultiply both sides of (6.55) by $\mathbf{A}^*$,

$$\mathbf{A}^*\mathbf{x} = (\mathbf{A}^*)^3\mathbf{x} + [\mathbf{A}^* + (\mathbf{A}^*)^2](\mathbf{I} - \tilde{\mathbf{A}})^{-1}\mathbf{f} \tag{6.57}$$

and, again, substitute for $\mathbf{A}^*\mathbf{x}$ in the right-hand side of (6.53)

$$\mathbf{x} = (\mathbf{A}^*)^3\mathbf{x} + [\mathbf{I} + \mathbf{A}^* + (\mathbf{A}^*)^2](\mathbf{I} - \tilde{\mathbf{A}})^{-1}\mathbf{f} \tag{6.58}$$

Solving for $\mathbf{x}$, we now find

$$\mathbf{x} = \underbrace{[\mathbf{I} - (\mathbf{A}^*)^3]^{-1}}_{\mathbf{M}_3}\underbrace{[\mathbf{I} + \mathbf{A}^* + (\mathbf{A}^*)^2]}_{\mathbf{M}_2}\underbrace{(\mathbf{I} - \tilde{\mathbf{A}})^{-1}}_{\mathbf{M}_1}\mathbf{f} \tag{6.59}$$

[Compare with the results in (6.56).]

In the context of social accounting matrices (Chapter 11), where much of the fundamental work on multiplier decompositions originated, $\mathbf{M}_1$ is said to capture a "transfer" effect, $\mathbf{M}_2$ embodies "open-loop" effects and $\mathbf{M}_3$ contains "closed-loop" effects. (For example, see Pyatt and Round, 1979.) The logic of these labels will be clear in the interregional context, below.

These iterations can continue any number of times. After k steps, the parallel to (6.58) is

$$\mathbf{x} = (\mathbf{A}^*)^k\mathbf{x} + [\mathbf{I} + \mathbf{A}^* + (\mathbf{A}^*)^2 + \cdots + (\mathbf{A}^*)^{k-1}](\mathbf{I} - \tilde{\mathbf{A}})^{-1}\mathbf{f} \tag{6.60}$$

and the parallel to (6.59) is

$$\mathbf{x} = \underbrace{[\mathbf{I} - (\mathbf{A}^*)^k]^{-1}}_{\mathbf{M}_3}\underbrace{[\mathbf{I} + \mathbf{A}^* + (\mathbf{A}^*)^2 + \cdots + (\mathbf{A}^*)^{k-1}]}_{\mathbf{M}_2}\underbrace{(\mathbf{I} - \tilde{\mathbf{A}})^{-1}}_{\mathbf{M}_1}\mathbf{f} \tag{6.61}$$

6.7.2 Decompositions in an Interregional Context

For a two-region interregional model (section 3.3) the input–output accounting relationship $\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{f}$ becomes

$$\begin{bmatrix} \mathbf{x}^r \\ \mathbf{x}^s \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{rr} & \mathbf{A}^{rs} \\ \mathbf{A}^{sr} & \mathbf{A}^{ss} \end{bmatrix}\begin{bmatrix} \mathbf{x}^r \\ \mathbf{x}^s \end{bmatrix} + \begin{bmatrix} \mathbf{f}^r \\ \mathbf{f}^s \end{bmatrix}$$

With a view toward decompositions, we can isolate the intraregional and interregional elements in $\mathbf{A}$; let

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}^{rr} & \mathbf{A}^{rs} \\ \mathbf{A}^{sr} & \mathbf{A}^{ss} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{rr} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^{ss} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{A}^{rs} \\ \mathbf{A}^{sr} & \mathbf{0} \end{bmatrix}$$

Define $\tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{A}^{rr} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^{ss} \end{bmatrix}$ from which $(\mathbf{I} - \tilde{\mathbf{A}}) = \begin{bmatrix} \mathbf{I} - \mathbf{A}^{rr} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} - \mathbf{A}^{ss} \end{bmatrix}$. Then, using the decomposition in (6.56), for example,

$$\mathbf{M}_1 = (\mathbf{I} - \tilde{\mathbf{A}})^{-1} = \begin{bmatrix} (\mathbf{I} - \mathbf{A}^{rr})^{-1} & \mathbf{0} \\ \mathbf{0} & (\mathbf{I} - \mathbf{A}^{ss})^{-1} \end{bmatrix}$$

(from the rule that the inverse for a block-diagonal matrix is made up of the inverses of the matrices on the main diagonal). Also,

$$\begin{aligned} \mathbf{A}^* &= (\mathbf{I} - \tilde{\mathbf{A}})^{-1}(\mathbf{A} - \tilde{\mathbf{A}}) \\ &= \begin{bmatrix} (\mathbf{I} - \mathbf{A}^{rr})^{-1} & \mathbf{0} \\ \mathbf{0} & (\mathbf{I} - \mathbf{A}^{ss})^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{A}^{rs} \\ \mathbf{A}^{sr} & \mathbf{0} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{0} & (\mathbf{I} - \mathbf{A}^{rr})^{-1}\mathbf{A}^{rs} \\ (\mathbf{I} - \mathbf{A}^{ss})^{-1}\mathbf{A}^{sr} & \mathbf{0} \end{bmatrix} \end{aligned}$$

and so, again from (6.56),

$$\mathbf{M}_2 = \mathbf{I} + \mathbf{A}^* = \begin{bmatrix} \mathbf{I} & (\mathbf{I} - \mathbf{A}^{rr})^{-1}\mathbf{A}^{rs} \\ (\mathbf{I} - \mathbf{A}^{ss})^{-1}\mathbf{A}^{sr} & \mathbf{I} \end{bmatrix}$$

Finally, from straightforward matrix multiplication,

$$(\mathbf{A}^*)^2 = \begin{bmatrix} (\mathbf{I} - \mathbf{A}^{rr})^{-1}\mathbf{A}^{rs}(\mathbf{I} - \mathbf{A}^{ss})^{-1}\mathbf{A}^{sr} & \mathbf{0} \\ \mathbf{0} & (\mathbf{I} - \mathbf{A}^{ss})^{-1}\mathbf{A}^{sr}(\mathbf{I} - \mathbf{A}^{rr})^{-1}\mathbf{A}^{rs} \end{bmatrix}$$

and so

$$\mathbf{M}_3 = [\mathbf{I} - (\mathbf{A}^*)^2]^{-1} = \begin{bmatrix} [\mathbf{I} - (\mathbf{I} - \mathbf{A}^{rr})^{-1}\mathbf{A}^{rs}(\mathbf{I} - \mathbf{A}^{ss})^{-1}\mathbf{A}^{sr}]^{-1} & \mathbf{0} \\ \mathbf{0} & [\mathbf{I} - (\mathbf{I} - \mathbf{A}^{ss})^{-1}\mathbf{A}^{sr}(\mathbf{I} - \mathbf{A}^{rr})^{-1}\mathbf{A}^{rs}]^{-1} \end{bmatrix}$$

(again from the rule for the inverse of a block-diagonal matrix).

In terms of intra- and interregional effects, the matrices in $\mathbf{M}_1$ are seen to capture *intraregional* (Leontief inverse or "transfer") effects, those in $\mathbf{M}_2$ contain *interregional spillover* ("open-loop") effects, and the matrices in $\mathbf{M}_3$ record *interregional feedback* ("closed-loop") effects (Round, 1985, 2001; Dietzenbacher, 2002).[24] As usual, define

$$\mathbf{L}^{rr} = (\mathbf{I} - \mathbf{A}^{rr})^{-1} \quad \text{and} \quad \mathbf{L}^{ss} = (\mathbf{I} - \mathbf{A}^{ss})^{-1}$$

[24] There have been other definitions of these various effects in the input–output literature, beginning perhaps with Miller (1966, 1969) but also including, among others, Yamada and Ihara (1969), Round (1985, 2001), or Sonis and Hewings (2001).

These are the intraregional effects in each region ($\mathbf{M}_1$). The two spillover matrices in $\mathbf{M}_2$ may be represented as

$$\mathbf{S}^{rs} = \mathbf{L}^{rr}\mathbf{A}^{rs} \text{ and } \mathbf{S}^{sr} = \mathbf{L}^{ss}\mathbf{A}^{sr}$$

and the two feedback matrices in $\mathbf{M}_3$ can be defined as

$$\mathbf{F}^{rr} = [\mathbf{I} - \mathbf{L}^{rr}\mathbf{A}^{rs}\mathbf{L}^{ss}\mathbf{A}^{sr}]^{-1} \text{ and } \mathbf{F}^{ss} = [\mathbf{I} - \mathbf{L}^{ss}\mathbf{A}^{sr}\mathbf{L}^{rr}\mathbf{A}^{rs}]^{-1}$$

or

$$\mathbf{F}^{rr} = [\mathbf{I} - \mathbf{S}^{rs}\mathbf{S}^{sr}]^{-1} \text{ and } \mathbf{F}^{ss} = [\mathbf{I} - \mathbf{S}^{sr}\mathbf{S}^{rs}]^{-1}$$

Therefore, in the two-region interregional context, $\mathbf{x} = \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1\mathbf{f}$ becomes

$$\begin{bmatrix} \mathbf{x}^r \\ \mathbf{x}^s \end{bmatrix} = \begin{bmatrix} \mathbf{F}^{rr} & \mathbf{0} \\ \mathbf{0} & \mathbf{F}^{ss} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{S}^{rs} \\ \mathbf{S}^{sr} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{L}^{rr} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}^{ss} \end{bmatrix} \begin{bmatrix} \mathbf{f}^r \\ \mathbf{f}^s \end{bmatrix} \tag{6.62}$$

or, carrying out the multiplications,

$$\begin{bmatrix} \mathbf{x}^r \\ \mathbf{x}^s \end{bmatrix} = \begin{bmatrix} \mathbf{F}^{rr}\mathbf{L}^{rr} & \mathbf{F}^{rr}\mathbf{S}^{rs}\mathbf{L}^{ss} \\ \mathbf{F}^{ss}\mathbf{S}^{sr}\mathbf{L}^{rr} & \mathbf{F}^{ss}\mathbf{L}^{ss} \end{bmatrix} \begin{bmatrix} \mathbf{f}^r \\ \mathbf{f}^s \end{bmatrix} \tag{6.63}$$

6.7.3 Stone's Additive Decomposition

An alternative decomposition isolates *net* effects. Starting with the multiplicative result in (6.56) [or (6.59), or (6.61)], namely $\mathbf{x} = \mathbf{M}\mathbf{f}$, where $\mathbf{M} = \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1$, Stone (1985) proposed the additive form

$$\mathbf{M} = \mathbf{I} + \underbrace{(\mathbf{M}_1 - \mathbf{I})}_{\tilde{\mathbf{M}}_1} + \underbrace{(\mathbf{M}_2 - \mathbf{I})\mathbf{M}_1}_{\tilde{\mathbf{M}}_2} + \underbrace{(\mathbf{M}_3 - \mathbf{I})\mathbf{M}_2\mathbf{M}_1}_{\tilde{\mathbf{M}}_3}$$

(This is easily seen to be true by simply carrying out the algebra on the right-hand side.) Therefore,

$$\mathbf{x} = \mathbf{M}\mathbf{f} = \mathbf{I}\mathbf{f} + \underbrace{(\mathbf{M}_1 - \mathbf{I})}_{\tilde{\mathbf{M}}_1}\mathbf{f} + \underbrace{(\mathbf{M}_2 - \mathbf{I})\mathbf{M}_1}_{\tilde{\mathbf{M}}_2}\mathbf{f} + \underbrace{(\mathbf{M}_3 - \mathbf{I})\mathbf{M}_2\mathbf{M}_1}_{\tilde{\mathbf{M}}_3}\mathbf{f} \tag{6.64}$$

To paraphrase Stone (p. 162) – in the context of an interregional model – we start with a matrix of initial injections, $\mathbf{If}$. The second term ($\tilde{\mathbf{M}}_1\mathbf{f}$) adds on the *net* intraregional effects captured in $\mathbf{M}_1$. Next (in $\tilde{\mathbf{M}}_2\mathbf{f}$) we add in the net interregional spillover effects in $\mathbf{M}_2$. Finally, the fourth term ($\tilde{\mathbf{M}}_3\mathbf{f}$) captures the net interregional feedback effects in

$\mathbf{M}_3$. In the two-region example, these are

$$\tilde{\mathbf{M}}_1 = \mathbf{M}_1 - \mathbf{I} = \begin{bmatrix} \mathbf{L}^{rr} - \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}^{ss} - \mathbf{I} \end{bmatrix}$$

$$\tilde{\mathbf{M}}_2 = (\mathbf{M}_2 - \mathbf{I})\mathbf{M}_1 = \begin{bmatrix} \mathbf{0} & \mathbf{S}^{rs} \\ \mathbf{S}^{sr} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{L}^{rr} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}^{ss} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{S}^{rs}\mathbf{L}^{ss} \\ \mathbf{S}^{sr}\mathbf{L}^{rr} & \mathbf{0} \end{bmatrix}$$

$$\tilde{\mathbf{M}}_3 = (\mathbf{M}_3 - \mathbf{I})\mathbf{M}_2\mathbf{M}_1 = \begin{bmatrix} \mathbf{F}^{rr}\mathbf{L}^{rr} - \mathbf{L}^{rr} & \mathbf{F}^{rr}\mathbf{S}^{rs}\mathbf{L}^{ss} - \mathbf{S}^{rs}\mathbf{L}^{ss} \\ \mathbf{F}^{ss}\mathbf{S}^{sr}\mathbf{L}^{rr} - \mathbf{S}^{sr}\mathbf{L}^{rr} & \mathbf{F}^{ss}\mathbf{L}^{ss} - \mathbf{L}^{ss} \end{bmatrix}$$

While these appear (and are) increasingly complex, they serve to disentangle the complex net of intraregional, spillover, and feedback effects.

6.7.4 A Note on Interregional Feedbacks

Interregional feedback effects in a two-region input–output model were explored in section 3.3.2. They were defined early on (Miller 1966, 1969) for the specific scenario of a change in final demand in region r only – so $\Delta\mathbf{f}^r \neq \mathbf{0}$ and $\Delta\mathbf{f}^s = \mathbf{0}$. Then a measure of the interregional feedback effect is found as the difference between the output change in region r that would be generated by the complete two-region model and the output change in region r that would be calculated from a single-region model. These outputs are

$$\Delta\mathbf{x}_T^r = [(\mathbf{I} - \mathbf{A}^{rr}) - \mathbf{A}^{rs}\mathbf{L}^{ss}\mathbf{A}^{sr}]^{-1}\Delta\mathbf{f}^r \text{ and } \Delta\mathbf{x}_S^r = (\mathbf{I} - \mathbf{A}^{rr})^{-1}\Delta\mathbf{f}^r$$

(with subscripts indicating "two-region" and "single-region" models, respectively). Consider the inverse matrix in $\Delta\mathbf{x}_T^r$, $[(\mathbf{I} - \mathbf{A}^{rr}) - \mathbf{A}^{rs}(\mathbf{I} - \mathbf{A}^{ss})^{-1}\mathbf{A}^{sr}]^{-1}$.

1. Factoring out $(\mathbf{I} - \mathbf{A}^{rr})$ gives

$$\{(\mathbf{I} - \mathbf{A}^{rr})[\mathbf{I} - (\mathbf{I} - \mathbf{A}^{rr})^{-1}\mathbf{A}^{rs}(\mathbf{I} - \mathbf{A}^{ss})^{-1}\mathbf{A}^{sr}]\}^{-1}$$

2. Using the rule that $(\mathbf{MN})^{-1} = \mathbf{N}^{-1}\mathbf{M}^{-1}$, we have

$$[\mathbf{I} - (\mathbf{I} - \mathbf{A}^{rr})^{-1}\mathbf{A}^{rs}(\mathbf{I} - \mathbf{A}^{ss})^{-1}\mathbf{A}^{sr}]^{-1}(\mathbf{I} - \mathbf{A}^{rr})^{-1}$$

Using $\mathbf{L}^{rr} = (\mathbf{I} - \mathbf{A}^{rr})^{-1}$ and $\mathbf{L}^{ss} = (\mathbf{I} - \mathbf{A}^{ss})^{-1}$, we have

$$\Delta\mathbf{x}_T^r = [\mathbf{I} - \mathbf{L}^{rr}\mathbf{A}^{rs}\mathbf{L}^{ss}\mathbf{A}^{sr}]^{-1}\mathbf{L}^{rr}\Delta\mathbf{f}^r \text{ and } \Delta\mathbf{x}_S^r = \mathbf{L}^{rr}\Delta\mathbf{f}^r$$

Finally, using $\mathbf{F}^{rr} = [\mathbf{I} - \mathbf{L}^{rr}\mathbf{A}^{rs}\mathbf{L}^{ss}\mathbf{A}^{sr}]^{-1}$ from $\mathbf{M}_3$, above,

$$\Delta\mathbf{x}_T^r - \Delta\mathbf{x}_S^r = \mathbf{F}^{rr}\mathbf{L}^{rr}\Delta\mathbf{f}^r - \mathbf{L}^{rr}\Delta\mathbf{f}^r = (\mathbf{F}^{rr}\mathbf{L}^{rr} - \mathbf{L}^{rr})\Delta\mathbf{f}^r = (\mathbf{F}^{rr} - \mathbf{I})\mathbf{L}^{rr}\Delta\mathbf{f}$$

The $\mathbf{F}^{rr}\mathbf{L}^{rr}$ term is exactly the upper left element in the multiplier matrix from the multiplicative decomposition in (6.63), and the $(\mathbf{F}^{rr} - \mathbf{I})\mathbf{L}^{rr}$ term (for the difference in gross outputs in the two models) is exactly the upper left element in $\tilde{\mathbf{M}}_3$ from the additive decomposition of net effects.

6.7.5 Numerical Illustration

We reconsider the two-region example from Chapter 3, in light of these decomposition possibilities. In that example we had

$$\mathbf{Z} = \begin{bmatrix} \mathbf{Z}^{rr} & \mathbf{Z}^{rs} \\ \mathbf{Z}^{sr} & \mathbf{Z}^{ss} \end{bmatrix} = \begin{bmatrix} 150 & 500 & 50 & 25 & 75 \\ 200 & 100 & 400 & 200 & 100 \\ 300 & 500 & 50 & 60 & 40 \\ 75 & 100 & 60 & 200 & 250 \\ 50 & 25 & 25 & 150 & 100 \end{bmatrix}$$

and

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}^r \\ \mathbf{x}^s \end{bmatrix} = \begin{bmatrix} 1000 \\ 2000 \\ 1000 \\ 1200 \\ 800 \end{bmatrix}$$

with associated direct and total requirements matrices of

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}^{rr} & \mathbf{A}^{rs} \\ \mathbf{A}^{sr} & \mathbf{A}^{ss} \end{bmatrix} = \begin{bmatrix} 0.1500 & 0.2500 & 0.0500 & 0.0208 & 0.0938 \\ 0.2000 & 0.0500 & 0.4000 & 0.1667 & 0.1250 \\ 0.3000 & 0.2500 & 0.0500 & 0.0500 & 0.0500 \\ 0.0750 & 0.0500 & 0.0600 & 0.1667 & 0.3125 \\ 0.0500 & 0.0125 & 0.0250 & 0.1250 & 0.1250 \end{bmatrix}$$

and

$$\mathbf{L} = \begin{bmatrix} 1.4234 & 0.4652 & 0.2909 & 0.1917 & 0.3041 \\ 0.6346 & 1.4237 & 0.6707 & 0.4092 & 0.4558 \\ 0.6383 & 0.5369 & 1.3363 & 0.2501 & 0.3108 \\ 0.2672 & 0.2000 & 0.1973 & 1.3406 & 0.5473 \\ 0.1468 & 0.0908 & 0.0926 & 0.2155 & 1.2538 \end{bmatrix}$$

In addition,[25]

$$\mathbf{L}^{rr} = (\mathbf{I} - \mathbf{A}^{rr})^{-1} = \begin{bmatrix} 1.3651 & 0.4253 & 0.2509 \\ 0.5273 & 1.3481 & 0.5954 \\ 0.5698 & 0.4890 & 1.2885 \end{bmatrix}$$

and

$$\mathbf{L}^{ss} = (\mathbf{I} - \mathbf{A}^{ss})^{-1} = \begin{bmatrix} 1.2679 & 0.4528 \\ 0.1811 & 1.2075 \end{bmatrix}$$

From these we can generate the additional components needed for these decompositions, namely

$$\mathbf{S}^{rs} = \mathbf{L}^{rr}\mathbf{A}^{rs} = \begin{bmatrix} 0.1119 & 0.1937 \\ 0.2654 & 0.2477 \\ 0.1578 & 0.1790 \end{bmatrix} \text{ and } \mathbf{S}^{sr} = \mathbf{L}^{ss}\mathbf{A}^{sr} = \begin{bmatrix} 0.1177 & 0.0691 & 0.0874 \\ 0.0740 & 0.0242 & 0.0411 \end{bmatrix}$$

$$\mathbf{F}^{rr} = [\mathbf{I} - \mathbf{S}^{rs}\mathbf{S}^{sr}]^{-1} = \begin{bmatrix} 1.0296 & 0.0134 & 0.0191 \\ 0.0535 & 1.0262 & 0.0359 \\ 0.0343 & 0.0164 & 1.0228 \end{bmatrix}$$

and

$$\mathbf{F}^{ss} = [\mathbf{I} - \mathbf{S}^{sr}\mathbf{S}^{rs}]^{-1} = \begin{bmatrix} 1.0488 & 0.0599 \\ 0.0228 & 1.0297 \end{bmatrix}$$

The **M** matrices for the multiplicative decomposition are easily found to be

$$\mathbf{M}_1 = \begin{bmatrix} 1.3651 & 0.4253 & 0.2509 & 0 & 0 \\ 0.5273 & 1.3481 & 0.5954 & 0 & 0 \\ 0.5698 & 0.4890 & 1.2885 & 0 & 0 \\ 0 & 0 & 0 & 1.2679 & 0.4528 \\ 0 & 0 & 0 & 0.1811 & 1.2075 \end{bmatrix}$$

for *intraregional transfer* effects, as is expected, since only $\mathbf{L}^{rr}$ and $\mathbf{L}^{ss}$ appear in this matrix. Next

$$\mathbf{M}_2 = \begin{bmatrix} 1 & 0 & 0 & 0.1119 & 0.1937 \\ 0 & 1 & 0 & 0.2654 & 0.2477 \\ 0 & 0 & 1 & 0.1578 & 0.1790 \\ 0.1177 & 0.0691 & 0.0874 & 1 & 0 \\ 0.0740 & 0.0242 & 0.0411 & 0 & 1 \end{bmatrix}$$

[25] Remember that $\mathbf{L}^{rr}$ does not designate the 3×3 submatrix in the upper left of $\mathbf{L}$, and similarly $\mathbf{L}^{ss}$ is not the 2×2 submatrix in the lower right of $\mathbf{L}$.

contains *interregional spillover* ("open-loop") effects only, transmitted from r to s (upper right) and from s to r (lower left). Finally

$$\mathbf{M}_3 = \begin{bmatrix} 1.0296 & 0.0134 & 0.0191 & 0 & 0 \\ 0.0535 & 1.0262 & 0.0359 & 0 & 0 \\ 0.0343 & 0.0164 & 1.0228 & 0 & 0 \\ 0 & 0 & 0 & 1.0488 & 0.0599 \\ 0 & 0 & 0 & 0.0228 & 1.0297 \end{bmatrix}$$

identifies *interregional feedback* ("closed-loop") effects.

We first use the multiplicative decomposition to find $\mathbf{x}^{new} = \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1\mathbf{f}^{new}$ for our example (Chapter 3) with $(\mathbf{f}^{new})' = \begin{bmatrix} 100 & 0 & 0 & 0 & 0 \end{bmatrix}$. This will generate $\mathbf{x}^{new} = \begin{bmatrix} 142.34 \\ 63.46 \\ 63.83 \\ 26.72 \\ 14.68 \end{bmatrix}$, as we found in that chapter. Now, however, the effects can be disentangled. Specifically,

1. First, $\mathbf{M}_1\mathbf{f}^{new} = \begin{bmatrix} 136.51 \\ 52.73 \\ 56.98 \\ 0 \\ 0 \end{bmatrix}$ indicates the initial impact in region r, the origin of the final demand change.
2. Next, $\mathbf{M}_2\mathbf{M}_1\mathbf{f}^{new} = \begin{bmatrix} 136.51 \\ 52.73 \\ 56.98 \\ 24.69 \\ 13.71 \end{bmatrix}$ adds to (1) the increases in the two sectors of region s because of the spillovers from r. Note that outputs in r are unchanged from (1), since this calculation is concerned with spillovers only. Clearly the difference between the results in (2) and (1), $\mathbf{M}_2\mathbf{M}_1\mathbf{f}^{new} - \mathbf{M}_1\mathbf{f}^{new}$, will be the vector of changes in s only.
3. Finally, $\mathbf{M}_3\mathbf{M}_2\mathbf{M}_1\mathbf{f}^{new} = \begin{bmatrix} 142.34 \\ 63.46 \\ 63.83 \\ 26.72 \\ 14.68 \end{bmatrix} = \mathbf{L}\mathbf{f}^{new}$ then adds in the feedback effects in the two regions – in r where the stimulus originated and in s because of the stimulus from the spillovers. In this case, the difference between the results in (3) and (2),

$\begin{bmatrix} 5.83 \\ 10.73 \\ 6.84 \\ \hline 2.03 \\ 0.97 \end{bmatrix}$, nets out the feedback effects by themselves. The first three elements, $\begin{bmatrix} 5.83 \\ 10.73 \\ 6.84 \end{bmatrix}$, are exactly the interregional feedback amounts that we found for region r in Chapter 3.

Consider now the components of the additive decomposition

$$\mathbf{x}^{new} = \mathbf{M}\mathbf{f}^{new} = \mathbf{I}\mathbf{f}^{new} + \underbrace{(\mathbf{M}_1 - \mathbf{I})}_{\tilde{\mathbf{M}}_1}\mathbf{f}^{new} + \underbrace{(\mathbf{M}_2 - \mathbf{I})\mathbf{M}_1}_{\tilde{\mathbf{M}}_2}\mathbf{f}^{new} + \underbrace{(\mathbf{M}_3 - \mathbf{I})\mathbf{M}_2\mathbf{M}_1}_{\tilde{\mathbf{M}}_3}\mathbf{f}^{new}$$

These provide the net effects. For this example, these multiplier matrices are

$$\tilde{\mathbf{M}}_1 = \left[\begin{array}{ccc|cc} 0.3651 & 0.4253 & 0.2509 & 0 & 0 \\ 0.5273 & 0.3481 & 0.5954 & 0 & 0 \\ 0.5698 & 0.4890 & 0.2885 & 0 & 0 \\ \hline 0 & 0 & 0 & 0.2679 & 0.4528 \\ 0 & 0 & 0 & 0.1811 & 0.2075 \end{array}\right]$$

$$\tilde{\mathbf{M}}_2 = \left[\begin{array}{ccc|cc} 0 & 0 & 0 & 0.1769 & 0.2845 \\ 0 & 0 & 0 & 0.3814 & 0.4193 \\ 0 & 0 & 0 & 0.2325 & 0.2876 \\ \hline 0.2469 & 0.1859 & 0.1833 & 0 & 0 \\ 0.1371 & 0.0841 & 0.0858 & 0 & 0 \end{array}\right]$$

$$\tilde{\mathbf{M}}_3 = \left[\begin{array}{ccc|cc} 0.0583 & 0.0400 & 0.0400 & 0.0148 & 0.0195 \\ 0.1073 & 0.0756 & 0.0753 & 0.0278 & 0.0365 \\ 0.0684 & 0.0478 & 0.0477 & 0.0176 & 0.0232 \\ \hline 0.0203 & 0.0141 & 0.0141 & 0.0727 & 0.0944 \\ 0.0097 & 0.0067 & 0.0067 & 0.0343 & 0.0463 \end{array}\right]$$

The pieces of the decomposition in (6.64) are:

1. $\mathbf{If}^{new} = \begin{bmatrix} 100 \\ 0 \\ 0 \\ \\ 0 \\ 0 \end{bmatrix}$ is just the initial "shock."
2. Then $\tilde{\mathbf{M}}_1\mathbf{f}^{new} = \begin{bmatrix} 36.51 \\ 52.73 \\ 56.98 \\ \\ 0 \\ 0 \end{bmatrix}$ accounts for the indirect effects in r; the sum of (1) and (2) is just $\mathbf{M}_1\mathbf{f}^{new}$, by definition.
3. Next, $\tilde{\mathbf{M}}_2\mathbf{f}^{new} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \\ 24.69 \\ 13.71 \end{bmatrix}$ captures the spillovers; this is $\mathbf{M}_2\mathbf{M}_1\mathbf{f}^{new} - \mathbf{M}_1\mathbf{f}^{new}$, also by definition.
4. Finally, $\tilde{\mathbf{M}}_3\mathbf{f}^{new} = \begin{bmatrix} 5.83 \\ 10.73 \\ 6.84 \\ \\ 2.03 \\ 0.97 \end{bmatrix}$ isolates the contribution from the interregional feedbacks; by definition this is $\mathbf{M}_3\mathbf{M}_2\mathbf{M}_1\mathbf{f}^{new} - \mathbf{M}_2\mathbf{M}_1\mathbf{f}^{new}$.

The matrix components of these decompositions, $\mathbf{M}$ and $\tilde{\mathbf{M}}$, are multiplier matrices, and so various multipliers can be calculated in the same way as was done earlier in this chapter for $\mathbf{L}$ – for example, simple column sums, or weighted sums if employment, value added or other economic impacts are of interest.

An empirical example applying these kinds of decompositions can be found in Zhang and Zhao (2005). They present a detailed set of decompositions of initial, spillover, and feedback effects derived from the 17-sector version of the 2000 Chinese multiregional (CMRIO) model that has been aggregated spatially into two mega-regions – Coastal and Non-coastal regions.

6.8 Summary

In this chapter we have introduced the reader to a wide variety of multipliers that are frequently calculated and used in real-world applications of the input–output framework. While the array may seem bewildering at first glance, it is, in fact, incomplete. For example, instead of using household input coefficients, as in (6.11), to generate a household income multiplier, one can weight the elements of a column of **L** by the parallel concept of "government input" coefficients, representing dollar's worth of government payments by a sector per dollar's worth of that sector's output. These would be the elements needed in the added row of an **A** matrix that was being closed with respect to government operations, not households. In this way, we would generate government multipliers. And similarly, other multipliers associated with exogenous sectors can be calculated – for example, foreign trade multipliers.

The use of the input–output framework for impact analysis, due to changing final demands, using multipliers, constitutes one of the most frequent uses of the model. In subsequent chapters we will explore extensions to deal specifically with energy (Chapter 9) and environmental problems (Chapter 10), and alternative uses of the model, in which the data are transformed into alternative summary measures of economic activity such as decomposition of changes over time and linkage analysis, in which the relative "importance" of sectors is assessed.

We explored the added richness of the Miyazawa formulation of a "closed" model in which various income-consumption-output impacts can be isolated. And we also examined some of the many variations on early multiplier formulations – for example, when the approach is changed from (direct + indirect effects)/(direct effects) to (indirect effects)/(direct effects) – which essentially means subtracting one from a traditional multiplier. We also examined the conversion of (multiplier) effects into elasticity terms, giving percentage changes due to a one percent increase in an industry's final demand or output. Finally, we examined two approaches to the decomposition of multiplier effects; these provide mechanisms that explicitly identify the routes of transmission of the initial exogenous stimulus. (Additional approaches to disentangling economic structure are explored briefly in Chapter 14.) We illustrated these in the spatial case, with interregional spillovers and feedbacks. The approach is equally insightful for extended input–output models, as illustrated by the Miyazawa structure. This is a feature of many studies employing social accounting matrices (SAMs) and will be discussed further in Chapter 11.

Appendix 6.1 The Equivalence of Total Household Income Multipliers and the Elements in the Bottom Row of $(\mathbf{I} - \bar{\mathbf{A}})^{-1}$

Consider the general representation of our 3×3 model closed with respect to households (sector 3), and its inverse, similarly partitioned.

$$(\mathbf{I} - \bar{\mathbf{A}}) = \begin{bmatrix} (1-a_{11}) & -a_{12} & -a_{13} \\ -a_{21} & (1-a_{22}) & -a_{23} \\ -a_{31} & -a_{32} & (1-a_{33}) \end{bmatrix} = \begin{bmatrix} \mathbf{E} & \mathbf{F} \\ \mathbf{G} & \mathbf{H} \end{bmatrix}$$

$$(\mathbf{I}-\bar{\mathbf{A}})^{-1}=\bar{\mathbf{L}}=\begin{bmatrix}\bar{l}_{11} & \bar{l}_{12} & \bar{l}_{13}\\ \bar{l}_{21} & \bar{l}_{22} & \bar{l}_{23}\\ \\ \bar{l}_{31} & \bar{l}_{32} & \bar{l}_{33}\end{bmatrix}=\begin{bmatrix}\mathbf{S} & \mathbf{T}\\ \\ \mathbf{U} & \mathbf{V}\end{bmatrix}$$

From results on inverses of partitioned matrices in Appendix A, particularly (2) in (A.4), $\mathbf{GS}+\mathbf{HU}=\mathbf{0}$. Here, since $\mathbf{H}=1-a_{33}$, we can write $\mathbf{U}=a_{33}\mathbf{U}-\mathbf{GS}$, or

$$\begin{bmatrix}\bar{l}_{31} & \bar{l}_{32}\end{bmatrix}=a_{33}\begin{bmatrix}\bar{l}_{31} & \bar{l}_{32}\end{bmatrix}+\begin{bmatrix}a_{31} & a_{32}\end{bmatrix}\begin{bmatrix}\bar{l}_{11} & \bar{l}_{12}\\ \bar{l}_{21} & \bar{l}_{22}\end{bmatrix}$$

Written out and rearranged, this is

$$\bar{l}_{31}=a_{31}\bar{l}_{11}+a_{32}\bar{l}_{21}+a_{33}\bar{l}_{31}$$
$$\bar{l}_{32}=a_{31}\bar{l}_{12}+a_{32}\bar{l}_{22}+a_{33}\bar{l}_{32}$$

The three terms on the right-hand sides are exactly the terms in (6.12) – $\bar{m}(h)_j=\sum_{i=1}^{n+1}a_{n+1,i}\bar{l}_{ij}$ – for $j=1$ and $j=2$, where the $(n+1)=3$ and $i=3$ terms are those in the household row (or column). Thus, $\bar{m}(h)_1=\bar{l}_{31}$ and $\bar{m}(h)_2=\bar{l}_{32}$, and this is always true, for any $\bar{m}(h)_j$, for a model of any size with households endogenous. This is (6.13), namely $\bar{m}(h)_j=\bar{l}_{n+1,j}$.

Appendix 6.2 Relationship Between Type I and Type II Income Multipliers

To examine the value of the ratio between type II and type I income multipliers, we again use results on the inverse of a partitioned matrix. To begin we note, for any sector j, that both multipliers – in (6.14) and (6.15) – have the same denominator, $a_{n+1,j}$, and thus the ratio of the two multipliers for sector j is

$$R_j=\frac{m(h)_j^{II}}{m(h)_j^{I}}=\frac{\bar{l}_{n+1,j}}{\sum_{i=1}^{n}a_{n+1,i}l_{ij}} \tag{A6.2.1}$$

In matrix terms, with $\bar{\mathbf{L}}=\begin{bmatrix}\bar{\mathbf{L}}_{11} & \bar{\mathbf{L}}_{12}\\ \\ \bar{\mathbf{L}}_{21} & \bar{\mathbf{L}}_{22}\end{bmatrix}$, the numerator of the ratio in (A6.2.1) is the jth element of $\bar{\mathbf{L}}_{21}$ and the denominator is the corresponding element of $\mathbf{h}'_c\mathbf{L}$. Thus the n-element row vector of these ratios is

$$\mathbf{R}=[R_1,\ldots,R_n]=\bar{\mathbf{L}}_{21}[\langle\mathbf{h}'_c\mathbf{L}\rangle]^{-1} \tag{A6.2.2}$$

The reader should be clear that this matrix operation divides each $\bar{l}_{n+1,1},\ldots,\bar{l}_{n+1,n}$ by the corresponding $\sum_{i=1}^{n}a_{n+1,i}l_{ij}$. (Recall also that the notation $\langle\mathbf{x}\rangle$ is used instead of $\hat{\mathbf{x}}$

when the vector being diagonalized is represented by a matrix expression containing several elements, so that the hat does not fit easily.)

Again using results from Appendix A on the inverse of a partitioned matrix (specifically A.5), and with $(\mathbf{I} - \bar{\mathbf{A}}) = \begin{bmatrix} (1-a_{11}) & -a_{12} & -a_{13} \\ -a_{21} & (1-a_{22}) & -a_{23} \\ -a_{31} & -a_{32} & (1-a_{33}) \end{bmatrix} = \begin{bmatrix} \mathbf{E} & \mathbf{F} \\ \mathbf{G} & \mathbf{H} \end{bmatrix}$, we see that the components in (A6.2.2) are

$$\bar{\mathbf{L}}_{21} = -\bar{\mathbf{L}}_{22}(\mathbf{G}\mathbf{E}^{-1}) = -\bar{\mathbf{L}}_{22}(\mathbf{G}\mathbf{L}) \text{ and } \mathbf{h}'_c\mathbf{L} = -\mathbf{G}\mathbf{L}$$

Thus $\mathbf{R} = -\underset{(1\times 1)}{\bar{\mathbf{L}}_{22}}\,\underset{(1\times n)}{(\mathbf{G}\mathbf{L})}\underset{(n\times n)}{[\langle -\mathbf{G}\mathbf{L}\rangle]^{-1}} = \underset{(1\times 1)}{\bar{\mathbf{L}}_{22}}\,\underset{(1\times n)}{[1,\ldots,1]} = \bar{\mathbf{L}}_{22}\mathbf{i}'$; that is, the ratios are all the same and are equal to the element in the lower-right of the closed model inverse.

For the numerical example in section 6.2.2, we found that the ratio of these multipliers, which we designated k, was 1.29. Recall the inverse for our small example, in (6.6), namely $\bar{\mathbf{L}} = \begin{bmatrix} 1.365 & 0.425 & 0.251 \\ 0.527 & 1.348 & 0.595 \\ 0.570 & 0.489 & 1.289 \end{bmatrix}$, where, in particular (to two decimals), $\bar{\mathbf{L}}_{22} = 1.29$. (Differences are due to rounding and the detailed precision of the inversion process.)

This constancy of the ratios of the two types of multipliers was apparently first demonstrated by Sandoval (1967), in an article in which he showed that the ratio is equal to $\left|(\mathbf{I} - \bar{\mathbf{A}})\right| / |(\mathbf{I} - \mathbf{A})|$, the ratio of the determinants of the Leontief matrices (not inverses) of the closed and open models. [The reader familiar with determinants can easily verify this for the numerical example in this chapter – $|(\mathbf{I} - \mathbf{A})| = 0.7575$, $\left|(\mathbf{I} - \bar{\mathbf{A}})\right| = 0.587875$ and (to two decimal places) $|(\mathbf{I} - \mathbf{A})|/|(\mathbf{I} - \bar{\mathbf{A}})| = 1.29$.] In producing his result, Sandoval did not use results from the inverses of partitioned matrices but rather from the general definitions of inverses in terms of determinants and cofactors. (Other discussions of these topics can be found in Bradley and Gander, 1969, Katz, 1980, and ten Raa and Chakraborty, 1983.)

Problems

6.1 Rank sectors in terms of their importance as measured by output multipliers in each of the economies represented by the data in problems 2.1, 2.2, and 2.4–2.9 (include problem 2.10 if you did it.)

6.2 Consider one (or more) of the problems in Chapter 2. Using output multipliers, from problem 6.1, in conjunction with the new final demands in the problem in Chapter 2, derive the total value of output (across all sectors) associated with the new final demands. Compare your results with the total output obtained by summing the elements in the gross output vector which you found as the solution to the problem in Chapter 2. [In matrix notation, this is comparing $\mathbf{m}(o)\Delta\mathbf{f}$ with $\mathbf{i}'\Delta\mathbf{x} = \mathbf{i}'\mathbf{L}\Delta\mathbf{f}$; we know that they must be equal, since output multipliers are the column sums of the Leontief inverse – $\mathbf{m}(o) = \mathbf{i}'\mathbf{L}$.]

6.3 Using the data in problem 2.3, find output multipliers and also both type I and type II income multipliers for the two sectors. Check that the ratio of the type II to the type I income multiplier is the same for both sectors.

6.4 You have assembled the following facts about the two sectors that make up the economy of a small country that you want to study (data pertain to the most recent quarter). Total interindustry inputs were \$50 and \$100, respectively, for Sectors 1 and 2. Sector 1's sales to final demand were \$60 and Sector 1's total output was \$100. Sector 2's sales to Sector 1 were \$30 and this represented 10 percent of Sector 2's total output. After national elections are held, it may turn out that different government policy will be forthcoming during the first quarter of the coming year.

   a. In which of the two sectors does an increase of \$100 in government purchases have the larger effect?
   b. How much larger is it than if the \$100 were spent on purchases of the other sector?

6.5 Consider an input output economy defined by $\mathbf{Z} = \begin{bmatrix} 140 & 350 \\ 800 & 50 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 1000 \\ 1000 \end{bmatrix}$.

   a. In the situation depicted in that question, if you were asked to design an advertising campaign to stimulate export sales of one of the goods produced in the country, would you concentrate your efforts on the product of sector 1 or of sector 2 or on some combination of the two? Why?
   b. If labor input coefficients for the two sectors in the region were found to be $a_{31} = 0.1$ and $a_{32} = 0.18$, how might your answer to part (a) of this question be changed, if at all?

6.6 Using the elements in the full two-region interregional Leontief inverse from problem 3.2, find:

   a. Simple intraregional output multipliers for sectors 1 and 2 [the vectors $\mathbf{m}(o)^{rr}$ and $\mathbf{m}(o)^{ss}$, as in (6.25) and (6.26)];
   b. Simple national (total) output multipliers for sectors 1 and 2 (vectors $\mathbf{m}(o)^{r}$ and $\mathbf{m}(o)^{s}$, as was done in (6.30) in the text];
   c. Sector-specific simple national output multipliers for sectors 1 and 2 in regions r and s. (This means finding the four multipliers in $\mathbf{m}(o)^{\cdot r} = [m(o)_{11}^{\cdot r}\ m(o)_{21}^{\cdot r}\ m(o)_{12}^{\cdot r}\ m(o)_{22}^{\cdot r}]$ and $\mathbf{m}(o)^{\cdot s}$, defined similarly.)

6.7 On the basis of the results in problem 6.6, above:

   a. For which sector's output does new final demand produce the largest total intraregional output stimulus in region r? In region s?
   b. For which sector in which region does an increase in final demand have the largest national (two-region) impact?
   c. To increase the output of sector 1 nationally (i.e., in both regions), would it be better to institute policies that would increase household demand in region r or in region s?
   d. Answer question (c) if the objective is now to increase sector 2 output nationally.

6.8 Answer problems 6.6 and 6.7, above, for the multiregional case, using the elements in $(\mathbf{I}-\mathbf{CA})^{-1}\mathbf{C}$ from problem 3.3.

6.9 The government in problem 3.4 is interested in starting an overseas advertising and promotion campaign in an attempt to increase export sales of the products of the country. There is specialization of production in the regions of the country; in particular, the products are shown in the table below:

	Region *A*	Region *B*	Region *C*
Manufacturing	Scissors	Cloth	Pottery
Agriculture	Oranges	Walnuts	None

For which product (or products) would increased export sales cause the greatest stimulation of the national economy?

6.10 If you have software (or patience), find $\left|(\mathbf{I}-\bar{\mathbf{A}})\right| / \left|(\mathbf{I}-\mathbf{A})\right|$ for our numerical example in which $\mathbf{A}=\begin{bmatrix} .15 & .25 \\ .20 & .05 \end{bmatrix}$ and $\bar{\mathbf{A}}=\begin{bmatrix} .15 & .25 & .05 \\ .20 & .05 & .40 \\ .30 & .25 & .05 \end{bmatrix}$, demonstrating that it is equal to $(1/g)=1.29$, as in Appendix 6.2.

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