1 Calculus

1.1 Solution

1. We have

$$\lim_{x\to 2} \quad \frac{1}{x-2} - \frac{2(2x-3)}{x^3-3x^2+2x} \tag{1}$$

$$= \lim_{x\to 2} \quad \frac{1}{x-2} - \frac{2(2x-3)}{x(x-1)(x-2)} \tag{2}$$

$$= \lim_{x\to 2} \quad \frac{x(x-1)-2(2x-3)}{x(x-1)(x-2)} \tag{3}$$

$$= \lim_{x\to 2} \quad \frac{x^2-5x+6}{x(x-1)(x-2)} \tag{4}$$

$$= \lim_{x\to 2} \quad \frac{(x-2)(x-3)}{x(x-1)(x-2)}[x-2\neq 0] \tag{5}$$

$$= \lim_{x\to 2} \quad \frac{x-3}{x(x-1)} \tag{6}$$

$$= \frac{-1}{2} \tag{7}$$

$$\tag{8}$$

2. put $y = 2+x$ so that when $x \to 0, y \to 2$. Then

$$\lim_{x\to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x} = \lim_{y\to 2} \frac{y^{\frac{1}{2}}-2^{\frac{1}{2}}}{y-2} \tag{9}$$

$$= \frac{1}{2}(2)^{\frac{1}{2}-1} \tag{10}$$

$$= \frac{1}{2} \times 2^{-\frac{1}{2}} \tag{11}$$

$$= \frac{1}{2\sqrt{2}} \tag{12}$$

$$\tag{13}$$

3.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \tag{14}$$

$$= \lim_{h\to 0} \frac{a(x+h)^2+b(x+h)+c-ax^2-bx-c}{h} \tag{15}$$

$$= \lim_{h\to 0} \frac{bh+ah^2+2axh}{h} \tag{16}$$

$$= \lim_{h\to 0} ah+2ax+b \tag{17}$$

$$= 2ax+b \tag{18}$$

$$\tag{19}$$

4.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \tag{20}$$

$$= \lim_{h\to 0} \frac{1}{h}\frac{1}{x+h} - \frac{1}{x} \tag{21}$$

$$= \lim_{h\to 0} \frac{-h}{h(x+h)x} \tag{22}$$

$$= \frac{-1}{x^2} \tag{23}$$

$$\tag{24}$$

5.

$$\begin{aligned}
f(x) &= x^4 + x - 1 && (25)\\
f^{(1)}(x) &= 4x^3 + 1 && (26)\\
f^{(2)}(x) &= 12x^2 && (27)\\
f^{(3)}(x) &= 24x && (28)\\
f^{(4)}(x) &= 24 && (29)\\
& && (30)
\end{aligned}$$

And all other derivatives are zero, thus

$$\begin{aligned}
x^4 + x - 2 = 0 + (x-1) \times 5 + \frac{(x-1)^2}{2!} \times 12 + \frac{(x-1)^3}{3!} \times 24 + \frac{(x-1)^4}{4!} \times 24 && (31)\\
= 5(x-1) + 6(x-1)^2 + 4(x-1)^3 + (x-1)^4 && (32)\\
&& (33)
\end{aligned}$$

6.

$$\begin{aligned}
\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[(x^2-1)(y+2)\right] && (34)\\
= (y+2)\frac{\partial}{\partial x}\left[(x^2-1)\right] && (35)\\
= (y+2)2(x) && (36)\\
= 2(x)(y+2) && (37)\\
&& (38)
\end{aligned}$$

Similarly

$$\begin{aligned}
\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[(x^2-1)(y+2)\right] && (39)\\
= (x^2-1)\frac{\partial}{\partial y}\left[(y+2)\right] && (40)\\
= (x^2-1) \cdot 1 && (41)\\
= (x^2-1) && (42)\\
&& (43)
\end{aligned}$$

7. we observe that

$$\begin{aligned}
e^{x+y+1} = e^x e^y e && (44)\\
&& (45)
\end{aligned}$$

So

$$\begin{aligned}
\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(e e^x e^y) && (46)\\
= e e^y \frac{\partial}{\partial x}(e^x) && (47)\\
= e e^y (e^x) && (48)\\
= e^{x+y+1} && (49)\\
&& (50)
\end{aligned}$$

Similarly

$$\begin{aligned}
\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(e e^x e^y) && (51)\\
= e e^x \frac{\partial}{\partial y}(e^y) && (52)\\
= e e^x (e^y) && (53)\\
= e^{x+y+1} && (54)\\
&& (55)
\end{aligned}$$

8. Using the product rule

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left(e^{-x}\right)\sin(x+y) + e^{-x}\frac{\partial}{\partial x}\left(\sin(x+y)\right) \qquad (56)\\
&= -e^{-x}\sin(x+y) + e^{-x}(1\cdot\cos(x+y)) \qquad (57)\\
&= e^{-x}(\cos(x+y) - \sin(x+y)) \qquad (58)\\
& \qquad (59)
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(e^{-x}\right)\sin(x+y) + e^{-x}\frac{\partial}{\partial y}\left(\sin(x+y)\right) \qquad (60)\\
&= -0 + e^{-x}(1\cdot\cos(x+y)) \qquad (61)\\
&= e^{-x}(\cos(x+y)) \qquad (62)\\
& \qquad (63)
\end{aligned}
$$

9. The partial derivatives of the function $u(x,y) = e^x \sin y$ are

$$
\frac{\partial u}{\partial x} = e^x \sin y, \frac{\partial u}{\partial x} = e^x \cos y \qquad (64)
$$

$$
\qquad (65)
$$

Thus the Hessian is $\begin{pmatrix} e^x \sin y & e^x \cos y \\ e^x \cos y & -e^x \sin y \end{pmatrix}$

10. The gradient of Q is

$$
\Delta(x^2 + 5y^2 + 4xy - 2yz) = (2x + 4y, 10y + 4x - 2z, -2y) \qquad (66)
$$

$$
\qquad (67)
$$

The Hessian matrix of Q is $\begin{pmatrix} 2 & 4 & 0 \\ 4 & 10 & -2 \\ 0 & -2 & 0 \end{pmatrix}$