An Analysis of First-Order Logics of Probability*

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ABSTRACT

We consider two approaches to giving semantics to first-order logics of probability. The first approach puts a probability on the domain, and is appropriate for giving semantics to formulas involving statistical information such as "The probability that a randomly chosen bird flies is greater than 0.9." The second approach puts a probability on possible worlds, and is appropriate for giving semantics to formulas describing degrees of belief such as "The probability that Tweety (a particular bird) flies is greater than 0.9." We show that the two approaches can be easily combined, allowing us to reason in a straightforward way about statistical information and degrees of belief. We then consider axiomatizing these logics. In general, it can be shown that no complete axiomatization is possible. We provide axiom systems that are sound and complete in cases where a complete axiomatization is possible, showing that they do allow us to capture a great deal of interesting reasoning about probability.

1. Introduction

Consider the two statements "The probability that a randomly chosen bird will fly is greater than 0.9" and "The probability that Tweety (a particular bird) flies is greater than 0.9." It is quite straightforward to capture the second statement by using a possible world semantics along the lines of that used in [11, 13, 23]. Namely, we can imagine a number of possible worlds such that the predicate *Flies* has a different extension in each one. Thus, *Flies*(*Tweety*) would hold in some possible worlds, and not in others. We then put a probability distribution on this set of possible worlds, and check if the set of possible worlds where *Flies*(*Tweety*) holds has probability greater than 0.9.

However, as pointed out by Bacchus [2, 3], this particular possible worlds approach runs into difficulties when trying to represent the first statement, which we may believe as a result of statistical information of the form "More than 90% of all birds fly." What is the formula that should hold at a set of

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worlds whose probability is greater than 0.9? The most obvious candidate is perhaps $\forall x(Bird(x) \Rightarrow Flies(x))$. However, it might very well be the case that in each of the worlds we consider possible, there is at least one bird that doesn't fly. Hence, the statement $\forall x(Bird(x) \Rightarrow Flies(x))$ holds in none of the worlds (and so has probability 0). Thus it cannot be used to represent the statistical information. As Bacchus shows, other straightforward approaches do not work either.

There seems to be a fundamental difference between these two statements. The first can be viewed as a statement about what Hacking calls a *chance setup* [16], that is, about what one might expect as the result of performing some experiment or trial in a given situation. It can also be viewed as capturing statistical information about the world, since given some statistical information (say, that 90% of the individuals in a population have property P), then we can imagine a chance setup in which a randomly chosen individual has probability 0.9 of having property P. On the other hand, the second statement captures what has been called a *degree of belief* [3, 20]. The first statement seems to assume only one possible world (the "real" world), and in this world, some probability distribution over the set of birds. It is saying that if we consider a bird chosen at random, then with probability greater than 0.9 it will fly. The second statement implicitly assumes the existence of a number of possibilities (in some of which Tweety flies, while in others Tweety doesn't), with some probability over these possibilities.

Bacchus [3] provides a syntax and semantics for a first-order logic for reasoning about chance setups, where the probability is placed on the domain. This approach has difficulties dealing with degrees of belief. For example, if there is only one fixed world, in this world either Tweety flies or he doesn't, so *Flies*(*Tweety*) holds with either probability 1 or probability 0. In particular, a statement such as "The probability that Tweety flies is between 0.9 and 0.95" is guaranteed to be false! Recognizing this difficulty, Bacchus moves beyond the syntax of his logic to define the notion of a *belief function*, which lets us talk about the degree of belief in the formula α given a knowledge base β. However, it would clearly be useful to be able to capture reasoning about degrees of belief within a logic, rather than moving outside the logic to do so.

In this paper, we describe two first-order logics, one for capturing reasoning about chance setups (and hence statistical information) and another for reasoning about degrees of belief. We then show how the two can be easily combined in one framework, allowing us to simultaneously reason about statistical information and degrees of belief.

We go on to consider issues of axiomatizability. Bacchus is able to provide a complete axiomatization for his language because he allows probabilities to take on *nonstandard* values in arbitrary ordered fields. Results of a companion paper [1] show that if we require probabilities to be real-valued (as we do here), we cannot in general hope to have a complete axiomatization for our language. We give sound axiom systems here which we show are complete for certain restricted settings. This suggests that our axiom systems are sufficiently rich to capture a great deal of interesting reasoning about probability.

Although work relating first-order logic and probability goes back to Carnap [4], there has been relatively little work on providing formal first-order logics for reasoning about probability. Besides the work of Bacchus mentioned above, the approaches closest in spirit to that of the current paper are perhaps those of Feldman [8], Feldman and Harel [9], Fenstad [10], Gaifman [14], Keisler [19], Loś [21], and Scott and Krauss [24]. Gaifman [14] and Scott and Krauss [24] considered the problem of associating probabilities with classical first-order statements (which, as pointed out in [2], essentially corresponds to putting probabilities on possible worlds). Loś and Fenstad studied this problem as well, but allowed values for free variables to be chosen according to a probability on the domain [10, 21]. Keisler [19] investigated an infinitary logic with a measure on the domain, and obtained completeness and compactness results. Feldman and Harel [8, 9] considered a probabilistic dynamic logic, which extends first-order dynamic logic by adding probability. There are commonalities between the program-free fragment of Feldman and Harel's logic and our logics, but since their interest is in reasoning about probabilistic programs, their formalism is significantly more complex than ours, and they focus on proving that their logic is complete relative to its program-free fragment.

The rest of this paper is organized as follows. In the next section, we present a logic for reasoning about situations where we have probabilities on the domain. Our syntax here is essentially identical to that of Bacchus [3]; our semantics follows similar lines, with some subtle, yet important, technical differences. In Section 3 we present a logic for reasoning about situations where there are probabilities on possible worlds. In Section 4 we show that these approaches can be combined in a straightforward way. In Section 5 we consider the question of finding complete axiomatizations.

2. Probabilities on the Domain

We assume that we have a first-order language for reasoning about some domain. We take this language to consist of a collection Φ of predicate symbols and function symbols of various arities (as usual, we can identify constant symbols with functions symbols of arity 0). Given a formula φ in the logic, we also allow formulas of the form $w_x(\varphi) \leqslant \frac{1}{2}$, which can be interpreted as "the probability that a randomly chosen x in the domain satisfies φ is greater than or equal to $\frac{1}{2}$." We actually extend this to allow arbitrary sequences of distinct variables in the subscript. To understand the intuition behind this, suppose the formula $Son(x, y)$ says that x is the son of y. Now consider the three terms $w_x(Son(x, y))$, $w_y(Son(x, y))$, and $w_{\langle x, y\rangle}(Son(x, y))$. The first describes the probability that a randomly chosen x is the son of y; the second describes the probability that x is the son of a randomly chosen y; the third describes the probability that a randomly chosen pair (x, y) will have the property that x is the son of y.

We formalize these ideas by using a two-sorted language. The first sort consists of the function symbols and predicate symbols in Φ, together with a countable family of *object variables* $x^{\mathrm{o}}, y^{\mathrm{o}}, \ldots$. Terms of the first sort describe elements of the domain we want to reason about. Terms of the second sort represent real numbers, typically probabilities, which we want to be able to add and multiply. In order to accommodate this, the second sort consists of the binary function symbols $+$ and $\times$, which represent addition and multiplication, constant symbols $\mathbf{0}$ and $\mathbf{1}$, representing the real numbers 0 and 1, binary relation symbols $>$ and $=$, and a countable family of *field variables* $x^{\mathrm{f}}, y^{\mathrm{f}}, \ldots$, which are intended to range over the real numbers. (We drop the superscripts on the variables when it is clear from context what sort they are.)

We now define object terms, field terms, and formulas simultaneously by induction. We form object terms, which range over the domain of the first-order language, by starting with object variables and closing off under function application, so that if f is an n-ary function symbol in Φ and $t_1, \ldots, t_n$ are object terms, then $f(t_1, \ldots, t_n)$ is an object term. We form field terms, which range over the reals, by starting with $\mathbf{0}, \mathbf{1}$, and probability terms of the form $w_{\vec{x}}(\varphi)$, where φ is an arbitrary formula and $\vec{x}$ is a sequence $\langle x_1, \ldots, x_n \rangle$ of distinct object variables, and then closing off under $+$ and $\times$, so that $t_1 + t_2$ and $t_1 \times t_2$ are field terms if t_1 and t_2 are. We form formulas in the standard way. We start with *atomic formulas*: if P is an n-ary predicate symbols in Φ, and $t_1, \ldots, t_n$ are object terms, then $P(t_1, \ldots, t_n)$ is an atomic formula, while if t_1 and t_2 are field terms, then $t_1 = t_2$ and $t_1 > t_2$ are atomic formulas. We sometimes also consider the situation where there is an equality symbol for object terms; in this case, if t_1 and t_2 are object terms, then $t_1 = t_2$ is also an atomic formula. We then close off under conjunction, negation, and universal quantification, so that if φ_1 and φ_2 are formulas and x is a (field or object) variable, then $\varphi_1 \wedge \varphi_2, \neg\varphi_1$, and $\forall x \varphi_1$ are all formulas. We call the resulting language $\mathscr{L}_1(\Phi)$; if it includes equality between object terms, we call it $\mathscr{L}_1^{=}(\Phi)$.

We define $\vee$, $\Rightarrow$, and $\exists$, in terms of $\wedge$, $\neg$, and $\forall$ as usual. In addition, if t_1 and t_2 are two field terms, we use other standard abbreviations such as $t_1 < t_2$ for $t_2 > t_1$, $t_1 \geqslant t_2$ for $(t_1 > t_2) \vee (t_1 = t_2)$, $t_1 \geqslant \frac{1}{2}$ for $(\mathbf{1} + \mathbf{1}) \times t_1 \geqslant \mathbf{1}$, and so on.

The only differences between our syntax and that of Bacchus is that we write $w_{\vec{x}}(\varphi)$ rather than $[\varphi]_{\vec{x}}$, we do not consider what Bacchus calls *measuring functions* (functions which map object terms into field terms), and the only field functions we allow are $+$ and $\times$. The language is still quite rich, allowing us to express conditional probabilities, notions of independence, and statistical notions; we refer the reader to [3] for examples.

We define a *type-1 probability structure* to be a tuple (D, π, μ), where D is a domain, π assigns to the predicate and function symbols in Φ predicates and functions of the right arity over D (so that (D, π) is just a standard first-order structure), and μ is a discrete probability function on D. That is, we take μ to be a mapping from D to the real interval $[0, 1]$ such that $\Sigma_{d \in D} \mu(d) = 1$. For any $A \subseteq D$, we define $\mu(A) = \Sigma_{d \in A} \mu(d)$.[1] Given a probability function μ, we can then define a discrete probability function μ^n on the product domain D^n consisting of all n-tuples of elements of D by taking $\mu^n(d_1, \ldots, d_n) = \mu(d_1) \times \cdots \times \mu(d_n)$. Define a *valuation* to be a function mapping each object variable into an element of D and each field variable into an element of $\mathbb{R}$ (the reals). Given a type-1 probability structure M and valuation v, we proceed by induction to associate with every object (respectively field) term t an element $[t]_{(M,v)}$ of D (respectively $\mathbb{R}$), and with every formula φ a truth value, writing $(M, v) \models \varphi$ if the value **true** is associated with φ by (M, v). The definitions follow the lines of first-order logic, so we just give a few clauses of the definition here, leaving the remainder to the reader:

- $(M, v) \models (t_1 = t_2)$ iff $[t_1]_{(M,v)} = [t_2]_{(M,v)}$;
- $(M, v) \models \forall x^o \varphi$ iff $(M, v[x^o/d]) \models \varphi$ for all $d \in D$, where $v[x^o/d]$ is the valuation which is identical to v except that it maps x^o to d;
- $[w_{\langle x_1, \ldots, x_n \rangle}(\varphi)]_{(M,v)} = \mu^n(\{(d_1, \ldots, d_n) : (M, v[x_1/d_1, \ldots, x_n/d_n]) \models \varphi\})$.

The major difference between our semantics and that of Bacchus is that Bacchus allows *nonstandard* probability functions, which take values in arbitrary ordered fields, and are only finitely additive, not necessarily countably additive. Our probability functions are standard: they are real-valued and countably additive. (Bacchus allows such nonstandard probability functions in order to obtain a complete axiomatization for his language. We return to this point later.)

We write $M \models \varphi$ if $(M, v) \models \varphi$ for all valuations v, and write $\models_1 \varphi$, and say that *φ is valid with respect to type-1 structures*, if $M \models \varphi$ for all type-1 probability structures M.

[1] The restriction to discrete probability functions is made here for ease of exposition only. We discuss below how we can allow arbitrary probability functions on the domain. It might seem that for practical applications we should further restrict to *uniform* probability functions, i.e., ones that assign equal probability to all domain elements. Although we allow uniform probability functions, and the language is expressive enough to allow us to say that the probability on the domain is uniform (using the formula $\forall x \forall y (w_z(x = z) = w_z(y = z))$) we do not require them. There are a number of reasons for this. For one thing, there are no uniform probability functions in countable domains. (Such a probability function would have to assign probability 0 to each individual element in the domain, which means by countable additivity it would have to assign probability 0 to the whole domain.) And even if we restrict attention to finite domains, we can construct two-stage processes (where, for example, one of three urns is chosen at random, and then some ball in the chosen urn is chosen at random) where the most natural way to assign probabilities would not assign equal probability to every ball [5].

As an example, suppose the language has only one predicate, the binary predicate *Son*, and we have a structure $M = (\{a, b, c\}, \pi, \mu)$ such that $\pi(Son)$ consists of only the pair (a, b), and

$$\mu(a) = \tfrac{1}{3}, \qquad \mu(b) = \tfrac{1}{2}, \qquad \mu(c) = \tfrac{1}{6}.$$

Thus, the structure M can be viewed as describing a chance setup—a particular experimental situation—where the probability of picking a is $\frac{1}{3}$, the probability of picking b is $\frac{1}{2}$, and the probability of picking c is $\frac{1}{6}$. Let v be a valuation such that $v(x) = a$ and $v(y) = c$. Then it is easy to check that we have

$$[w_x(Son(x, y))]_{(M,v)} = 0,$$
$$[w_y(Son(x, y))]_{(M,v)} = \tfrac{1}{2},$$
$$[w_{\langle x,y\rangle}(Son(x, y))]_{(M,v)} = \tfrac{1}{6}.$$

Thus, if we pick an x at random from the domain (according to the chance setup described by M) and fix y to be c, the probability that x is a son of y is 0: no member of the domain is a son of c. If we fix x to be a and pick a y at random from the domain, the probability that x is a son of y is $\frac{1}{2}$, which is exactly the probability that $y = b$. Finally, if we pick pairs at random (by choosing the first element of the pair, replacing it, and then choosing the second element) the probability of picking a pair (x, y) such that x is a son of y is $\frac{1}{6}$.

This example shows that the syntax and semantics of this logic are well suited for reasoning about chance setups. We can construct similar examples to show that it is appropriate for reasoning about statistical information in large domains. But, as discussed in the introduction, the logic is not well suited for making statements about degrees of belief about properties of particular individuals. For example, although in the logic it is consistent that the probability that a randomly chosen bird flies is between 0.9 and 0.95, it is inconsistent that the probability that Tweety flies is between 0.9 and 0.95. To make this more formal, note that in a formula such as $w_x(\varphi) \geqslant 0.9$, the w_x binds the free occurrences of x in φ just as the $\forall x$ binds all free occurrences of x in φ in the formula $\forall x \varphi$. We define a formula to be *closed* if no variables in the formula are free. Just as for first-order logic, we can show that the truth of a formula depends only on the values assigned by the valuation to the free variables. In particular, it follows that the truth of a closed formula is independent of the valuation.

Proposition 2.1. *Suppose φ is a formula in $\mathcal{L}_1(\Phi)$ all of whose free variables are contained in the set X. Let M be a type-1 probability structure and let v_1 and v_2*

be two valuations that agree on X (so that $v_1(y) = v_2(y)$ for all $y \in X$). Then $(M, v_1) \models \varphi$ iff $(M, v_2) \models \varphi$.

Proof. By a straightforward induction on the structure of φ, much as in the case of the corresponding result for first-order logic. We leave details to the reader. $\square$

If φ is a closed formula, then by definition it has no free variables. Notice that if we take X in the preceding proposition to be the empty set, then all valuations agree on X. It follows that the truth of a closed formula is independent of the valuation.

Corollary 2.2. *If φ is a closed formula, then for all valuations v_1 and v_2, we have $(M, v_1) \models \varphi$ iff $(M, v_2) \models \varphi$.*

It follows from Corollary 2.2 that if φ is a closed formula, that either $M \models \varphi$ or $M \models \neg\varphi$ for each type-1 probability structure M. This means that in a type-1 probability structure M, a closed formula is true for all choices of random variable x or for none of them. Thus we get:

Lemma 2.3 [3, Lemma 5.1]. *If φ is a closed formula, then for any vector $\vec{x}$ of distinct object variables, $\models_1 ((w_{\vec{x}}(\varphi) = \mathbf{0}) \vee (w_{\vec{x}}(\varphi) = \mathbf{1}))$.*

As we mentioned above, our restriction to discrete probability functions on the domain is not essential. We can allow arbitrary probability functions by associating with the probability function its domain, that is the σ-algebra of subsets of D to which the probability function assigns a probability. (A σ-algebra is a set of subsets that contains the empty set and is closed under complementation and countable union.) Thus, a type-1 probability structure would become a tuple of the form $(D, \pi, \mathcal{X}, \mu)$, where $\mathcal{X}$ is a σ-algebra of subsets of D and μ is a probability function on $\mathcal{X}$. We can define a σ-algebra $\mathcal{X}^n$ on D^n and a product measure μ^n on D^n in a straightforward way [17]. The only problem that arises is that we might need to take the probability of a *nonmeasurable* set, i.e., one not in the σ-algebra. For example, suppose we consider the structure $M = (D, \pi, \mathcal{X}, \mu)$. We earlier defined $[w_x(\varphi(x))]_{(M,v)}$ as $\mu(D_\varphi)$, where

$$D_\varphi = \{d \in D : (M, v[x/d]) \models \varphi\} .$$

However, there is now no reason to believe that $D_\varphi \in \mathcal{X}$, so that $\mu(D_\varphi)$ may not be well defined. We can get around this problem by requiring that all definable sets be measurable; this is the solution taken in [3]. Alternatively, we can interpret w_x as an *inner measure* rather than a probability; see [12, 13] for further details.

3. Probabilities on Possible Worlds

Lemma 2.3 shows that, in a precise sense, type-1 probability structures are inappropriate for reasoning about degrees of belief. In practice, it might well be the case that the way we derive our degrees of belief is from the statistical information at our disposal. Suppose we know that the probability that a randomly chosen bird flies is greater than 0.9. We can express this in $\mathscr{L}_1(\{Flies, Bird\})$ by the conditional probability statement

$$w_x(Flies(x) \mid Bird(x)) > 0.9 ,$$

which we view as an abbreviation for

$$w_x(Flies(x) \wedge Bird(x)) > 0.9 w_x(Bird(x)) .$$ [2]

If we know that Tweety is a bird, then we might conclude that the probability that Tweety flies is as least 0.9. Thus, if we take $w(Tweety)$ to represent the probability that Tweety flies, we might take as a default assumption a statement like

$$\begin{aligned} &Bird(Tweety) \wedge w_x(Flies(x) \mid Bird(x)) > 0.9 \Rightarrow \\ &w(Flies(Tweety)) > 0.9 . \end{aligned}$$

As pointed out by Bacchus and others, this type of reasoning is fraught with difficulties. It is quite clear that this default assumption is not sound in general. In particular, if we have more specific information about Tweety, such as the fact that Tweety is a penguin, then we no longer want to draw the conclusion that the probability that Tweety flies is at least 0.9. Bacchus provides some heuristics for deriving such degrees of belief [3]. While this is a very interesting topic to pursue, it seems useful to have a formal model that allows us to directly capture degrees of belief. Such a formal model can be constructed in a straightforward way using possible worlds, as we now show.

[2] This is a more appropriate way of formalizing the fact that most birds fly than $w_x(Bird(x) \Rightarrow Flies(x)) > 0.9$. The formula $Bird(x) \Rightarrow Flies(x)$ is equivalent to $\neg Bird(x) \vee Flies(x)$, so the implication would hold with high probability even if no bird in the domain flew, as long as less than 10% of the domain consisted of birds. (I'd like to thank Fahiem Bacchus for pointing this out to me.) Also note that the representation of conditional probability used here is somewhat nonstandard. The conditional probability of A given B is typically taken to be the probability of $A \cap B$ divided by the probability of B. We have cleared the denominator here to avoid having to deal with the difficulty of dividing by 0 should the probability of B be 0. This results in some anamolous interpretations of formulas. For example, $w(\alpha \mid \beta) = r$ is taken as an abbreviation for $w(\alpha \wedge \beta) = rw(\beta)$. If $w(\beta) = 0$, then $w(\alpha \wedge \beta) = 0$, so $w(\alpha \mid \beta) = r$ is true for all values of r. On the other hand, for similar reasons, $w(\alpha \mid \beta) < r$ and $w(\alpha \mid \beta) > r$ are both false for all values of r if $w(\beta) = 0$.

The syntax for a logic for reasoning about possible worlds is essentially the same as the syntax used in the previous section. Starting with a set Φ of function and predicate symbols, we form more complicated formulas and terms as before except that instead of allowing probability terms of the form $w_{\vec{x}}(\varphi)$, where $\vec{x}$ is some vector of distinct object variables, we only allow probability terms of the form $w(\varphi)$, interpreted as "the probability of φ." Since we are no longer going to put a probability distribution on the domain, it does not make sense to talk about the probability that a random choice for $\vec{x}$ will satisfy φ. For example, in the term $w(\mathit{Flies}(\mathit{Tweety}))$ considered above, it would not really make sense to consider the probability that a randomly chosen x satisfies the property that Tweety flies. It does make sense to talk about the probability of φ though: this will be the probability of the set of possible worlds where φ is true. We call the resulting language $\mathscr{L}_2(\Phi)$; if it includes equality between object terms, we call it $\mathscr{L}_2^=(\Phi)$.

More formally, a *type-2 probability structure* is a tuple (D, S, π, μ), where D is a domain, S is a set of *states* or *possible worlds*, for each state $s \in S$, $\pi(s)$ assigns to the predicate and function symbols in Φ predicates and functions of the right arity over D, and μ is a discrete probability function on S. Note the key difference between type-1 and type-2 probability structures: in type-1 probability structures, the probability is taken over the domain D, while in type-2 probability structures, the probability is taken over S, the set of states. Given a type-2 probability structure M, a state s, and valuation v, we can associate with every object (respectively field) term t an element $[t]_{(M,s,v)}$ of D (respectively $\mathbb{R}$), and with every formula φ a truth value, writing $(M, s, v) \models \varphi$ if the value **true** is associated with φ by (M, s, v). Note that we now need the state to provide meanings for the predicate and function symbols; they might have different meanings in each state. Again, we just give a few clauses of the definition here, which should suffice to indicate the similarities and differences between type-1 and type-2 probability structures:

- $(M, s, v) \models P(x)$ iff $v(x) \in \pi(s)(P)$;
- $(M, s, v) \models (t_1 = t_2)$ iff $[t_1]_{(M,s,v)} = [t_2]_{(M,s,v)}$;
- $(M, s, v) \models \forall x^{\circ}\varphi$ iff $(M, s, v[x^{\circ}/d]) \models \varphi$ for all $d \in D$;
- $[w(\varphi)]_{(M,s,v)} = \mu(\{s' \in S: (M, s', v) \models \varphi\})$.

We say $M \models \varphi$ if $(M, s, v) \models \varphi$ for all states s in M and all valuations v, and say φ is *valid with respect to type-2 structures*, and write $\models_2 \varphi$, if $M \models \varphi$ for all type-2 probability structures M.

As expected, in type-2 probability structures, it is completely consistent for the probability that Tweety flies to be between 0.9 and 0.95. Lemma 2.3 does not hold for type-2 probability structures. A sentence such as $0.9 \leq w(\mathit{Flies}(\mathit{Tweety})) \leq 0.95$ is true in a structure M (independent of the state s) precisely if the set of states where $\mathit{Flies}(\mathit{Tweety})$ is true has probability between

0.9 and 0.95. However, there is no straightforward way to capture statistical information using $\mathcal{L}_2$.[3]

Possible extensions

We have made a number of simplifying assumptions in our presentation of type-2 probability structures. We now briefly discuss how they might be dropped.

(1) As in the case of type-1 probability structures, we can allow arbitrary probability functions, not just discrete ones, by associating with the probability function the σ-algebra of subsets of S which forms its domain.
(2) We have assumed that all functions and predicates are *flexible*, i.e., they may take on different meanings at each state. We can easily designate some functions and predicates to be *rigid*, so that they take on the same meaning at all stages.
(3) We have assumed that there is only one domain. There are a number of ways to extend the model to allow each state to have associated with it a different domain. The situation is analogous to the problem of extending standard first-order modal logic by allowing different domains. In particular we have to explain the semantics of formulas such as $\exists x(w(\varphi(x) = \frac{1}{2}))$ (this is known as the problem of *quantifying in*). If we take this formula to be true at a state s if, roughly speaking, there is some d in the domain of s such that $w(\varphi(d) = \frac{1}{2})$, we may have a problem if d is not in the domain of all other states. The interested reader can consult [15] for a number of approaches to dealing with the problem; all these approaches can be modified to apply to our situation.
(4) We have assumed that there is only one probability measure μ on the set of states. We may want to allow uncertainty about the probability functions. We can achieve this by associating with each state a (possibly

[3] We remark that there is a sense in which we can translate back and forth between domain-based probability and possible-world-based probability. For example, there is an effective translation that maps a formula φ in $\mathcal{L}_1^{=}$ to a formula φ' in language $\mathcal{L}_2^{=}$, and a translation that maps a type-1 structure M to a type-2 structure M' such that $M \models \varphi$ iff $M' \models \varphi'$. Similar mappings exist in the other direction. The key step in the translation from φ to φ' is to replace a probability term such as $w_x(\psi(x))$ in φ by $w_x(\psi(\mathbf{a}))$, where $\mathbf{a}$ is a fresh constant symbol. Given a type-1 structure $M = (D, \pi, \mu)$ over a domain D, we construct a corresponding type-2 structure $M' = (D, S, \pi', \mu')$ over the same domain D, such that for each $d \in D$, there is a nonempty set of states $S_d = \{s' : \pi'(s)(\mathbf{a}) = d\}$ such that $\mu'(S_d) = \mu(d)$. For the translation in the other direction, we replace a predicate $P(x_1, \ldots, x_n)$ in an $\mathcal{L}_2$ formula by a predicate $P\{x_1, \ldots, x_n, s)$, where intuitively s ranges over states. Thus, the dependence of the predicate P on the state is explicitly encoded in P^*. Further details can be found in [1]. Despite the existence of these translations, we would still argue that $\mathcal{L}_1$ is not the right language for reasoning about probability over possible worlds, while $\mathcal{L}_2$ is not the right language for reasoning about probability over the domain.

different) probability function on the set of states (cf. [11, 18]). Thus a structure would now consist of a tuple $(D, S, \pi, \{\mu^s : s \in S\})$; in order to evaluate the value of the (field) term $w(\varphi)$ in a state s, we use the probability function μ^s.

4. Probabilities on the Domain and on Possible Worlds

In the previous sections we presented structures to capture two different modes of probabilistic reasoning. We do not want to say that one mode is more "right" than another; they both have their place. Clearly there might be situations where we want to do both modes of reasoning simultaneously. We consider three examples here.

Example 4.1. Consider the statement "The probability that Tweety flies is greater than the probability that a randomly chosen bird flies." This can be captured by the formula

$$w(\mathit{Flies}(\mathit{Tweety})) > w_x(\mathit{Flies}(x)) .$$

Example 4.2. For a more complicated example, consider two statements like "The probability that a randomly chosen bird flies is greater than 0.99" and "The probability that a randomly chosen bird flies is greater than 0.9." An agent might consider the first statement rather unlikely to be true, and so take it to hold with probability less than 0.2, while he might consider the second statement exceedingly likely to be true, and so take it to hold with probability greater than 0.95. We can capture this by combining the syntax of the previous two sections to get:

$$\begin{aligned} &w(w_x(\mathit{Flies}(x) \mid \mathit{Bird}(x)) > 0.99) < 0.2 \wedge \\ &w(w_x(\mathit{Flies}(x) \mid \mathit{Bird}(x)) > 0.90) > 0.95 . \end{aligned} \tag{1}$$

Example 4.3. The connection between probabilities on the domain and degrees of belief is an important one, that needs further investigation. Perhaps the most obvious connection we can expect to hold between an agent's degree of belief in $\varphi(\boldsymbol{a})$, for a particular constant $\mathbf{a}$, and the probability that $\varphi(x)$ holds for a randomly chosen individual x is equality, as characterized by the following equation:

$$w(\varphi(\mathbf{a})) = w_x(\varphi(x)) . \tag{2}$$

Another connection, provided by what has been called *Miller's principle* (see [22, 27]), can be viewed as saying that for any real number r_0 the conditional probability of $\varphi(\mathbf{a})$, given that the probability that a randomly chosen x satisfies

φ is r_0, is itself r_0. Assuming that the real variable r does not appear free in φ, we can express (this instance of) Miller's principle in our notation as

$$\forall r[w(\varphi(\mathbf{a})|(w_x(\varphi(x)) = r)) = r] .$$

We examine the connection between Miller's principle and (2) after we define our formal semantics.

Given a set Φ of function and predicate symbols, let $\mathscr{L}_3(\Phi)$ be the language that results by allowing probability terms both of the form $w_{\vec{x}}(\varphi)$, where $\vec{x}$ is a vector of distinct object variables, and of the form $w(\varphi)$; we take $\mathscr{L}_3^=(\Phi)$ to be the extension of $\mathscr{L}_3(\Phi)$ that includes equality between object terms. To give semantics to formulas in $\mathscr{L}_3(\Phi)$ (respectively $\mathscr{L}_3^=(\Phi)$), we will clearly need probability functions over both the set of states and over the domain. Let a *type-3 probability structure* be a tuple of the form $(D, S, \pi, \mu_D, \mu_S)$, where D, S, and π are as for type-2 probability structures, μ_D is a discrete probability function on D and μ_S is a discrete probability function on S. Intuitively, type-3 structures are obtained by combining type-1 and type-2 structures.

Given a type-3 probability structure M, a state s, and valuation v, we can give semantics to terms and formulas along much the same lines as in type-1 and type-2 structures. For example, we have:

- $[w_{\langle x_1,\ldots,x_n\rangle}(\varphi)]_{(M,s,v)}$
  $= \mu_D^n(\{(d_1, \ldots, d_n): (M, s, v[x_1/d_1, \ldots, x_n/d_n]) \models \varphi\})$;
- $[w(\varphi)]_{(M,s,v)} = \mu_S(\{s' \in S: (M, s', v) \models \varphi\})$.

It is now easy to construct a structure M where formula (1) is satisfied. We can take *Bird* to be a rigid designator in M, so that the same domain elements are birds in all the states of M. On the other hand, *Flies* will not be rigid. In most of the states in M (i.e., in a set of states of probability greater than 0.95), the extension of *Flies* will be such that more than 90% of the domain elements that satisfy *Bird* also satisfy *Flies*. However, there will only be a few states (i.e., a set of states of probability less than 0.2) where it will be the case that more than 99% of birds fly.

The assumption that *Flies* is not rigid is crucial here. Since we have assumed that in a given type-3 probability structure we have one fixed probability function on the domain, it is easy to see that if *all* the predicate and function symbols that appear in φ are rigid, then the truth of a formula such as $w_x(\varphi(x)) = r$ is independent of the state; it is either true in all states or false in all states.

Lemma 4.4. *If M is a type-3 structure such that all the predicate and function symbols appearing in φ are rigid, then*

(a) *for all* r *with* $0 \leq r \leq 1$, *if* $(M, s, v) \models (w_x(\varphi(x)) = r)$ *for some state* s *in* M, *then* $(M, s', v) \models (w_x(\varphi(x)) = r)$ *for all states* s' *in* M;
(b) *for all* r *with* $0 \leq r \leq 1$, *if* $(M, s, v) \models (w_x(\varphi(x)) = r)$ *for some state* s *in* M, *then* $(M, s', v) \not\models (w_x(\varphi(x)) = r)$ *for all states* s' *in* M;
(c) $M \models \forall r[(w(w_x(\varphi(x)) = r) = \mathbf{1}) \vee (w(w_x(\varphi(x)) = r) = \mathbf{0})]$.

Note the analogy between this result and Lemma 2.3.

Of course, we can easily extend type-3 structures to allow the probability function on the domain to be a function on the state. Thus at each state s we would have a (possibly different) probability function μ_D^s on the domain. When computing the value of a field term such as $w_x(\varphi(x))$ at state s, we use the function μ_D^s. Other extensions of type-3 structures, along the lines discussed for type-1 and type-2 structures, are possible as well.

As we discussed above, it is not clear how to go from statistical information to degrees of belief. One connection is suggested by Miller's principle, and another is suggested by (2). As the following theorem shows, in type-3 structures, as we have defined them, there is a close connection between Miller's principle and (2).

Theorem 4.5. *If* M *is a type-3 structure such that all the predicate and function symbols in* φ *are rigid except for the constant symbol* **a**, *then*

$$M \models ([w(\varphi(\mathbf{a})) = w_x(\varphi(x))] \equiv \forall r[w(\varphi(\mathbf{a}) | (w_x(\varphi(x)) = r)) = r]) .$$

Proof. Suppose that $M = (D, S, \pi, \mu_D, \mu_D)$ and all predicate and functions symbols in φ are rigid in M except for **a**.

For the $\Rightarrow$ direction, suppose that

$$(M, s, v) \models (w(\varphi(\mathbf{a})) = w_x(\varphi(x)))$$

for some state $s \in S$ and valuation v. We want to show that

$$(M, s, v) \models (\forall r[w(\varphi(\mathbf{a}) | (w_x(\varphi(x)) = r)) = r]) .$$

Choose a real number r_0. There are now two cases to consider. If

$$(M, s, v[r/r_0]) \models (w_x(\varphi(x)) = r) ,$$

then by assumption

$$(M, s, v[r/r_0]) \models (w(\varphi(\mathbf{a})) = r) .$$

By Lemma 4.4, we have $(M, s', v[r/r_0]) \models (w_x(\varphi(x)) = r)$ for all $s' \in S$. This has two consequences:

$$(M, s', v[r/r_0]) \models (\varphi(\mathbf{a}) \vee (w_x(\varphi(x)) = r))$$
$$\text{iff } (M, s', v[r/r_0]) \models \varphi(\mathbf{a}) , \tag{3}$$

$$(M, s', v[r/r_0]) \models (w(w_x(\varphi(x)) = r) = \mathbf{1}) . \tag{4}$$

From (3) and the fact that $(M, s, v[r/r_0]) \models (w(\varphi(\mathbf{a})) = r)$, we get

$$(M, s, v[r/r_0]) \models (w(\varphi(\mathbf{a}) \wedge (w_x(\varphi(x)) = r)) = r) .$$

Unwinding the definition of conditional probability, it easily follows that

$$(M, s, v[r/r_0]) \models (w(\varphi(\mathbf{a}) | (w_x(\varphi(x)) = r)) = r) .$$

For the second case, suppose that $(M, s, v[r/r_0]) \not\models (w_x(\varphi(x)) = r)$. By Lemma 4.4, we have $(M, s', v[r/r_0]) \not\models (w_x(\varphi(x)) = r)$ for all $s' \in S$. Thus

$$\mu_S(\{s' \in S\colon (M, s', v[r/r_0]) \not\models (w_x(\varphi(x)) = r) \wedge \varphi(\mathbf{a}))\}) = 0 .$$

It again easily follow that

$$(M, s, v[r/r_0]) \models (w(\varphi(\mathbf{a}) | (w_x(\varphi(x)) = r)) = r) .$$

Thus we get

$$(M, s, v) \models \forall r[w(\varphi(\mathbf{a}) | (w_x(\varphi(x)) = r)) = r] .$$

For the converse, suppose

$$(M, s, v) \models \forall r[w(\varphi(\mathbf{a}) | (w_x(\varphi(x)) = r)) = r] .$$

Choose r_0 such that $(M, s, v[r/r_0]) \models (w_x(\varphi(x)) = r)$. By Miller's principle we have

$$(M, s, v[r/r_0]) \models (w(\varphi(\mathbf{a}) \wedge (w_x(\varphi(x)) = r)) = r \times w(w_x(\varphi(x)) = r)) .$$

By Lemma 4.4, we have $(M, s, v[r/r_0]) \models (w(w_x(\varphi(x)) = r) = \mathbf{1})$. Thus, $(M, s, v[r/r_0]) \models (w(\varphi(\mathbf{a})) = r)$. It follows that

$$(M, s, v[r/r_0]) \models (w(\varphi(\mathbf{a})) = w_x(\varphi(x))) .$$

Since r does not appear free in φ (by assumption), we get $(M, s, v) \models (w(\varphi(\mathbf{a}) = w_x(\varphi(x)))$.

We have just shown that

$$(M, s, v) \models ([w(\varphi(\mathbf{a})) = w_x(\varphi(x))] \equiv \forall r[w(\varphi(\mathbf{a}) | (w_x(\varphi(x)) = r)) = r]) .$$

Since we chose s and v arbitrarily, the theorem follows. □

While this result does not begin to settle the issue of how to connect statistical information with degrees of belief, it does show that type-3 structures provide a useful framework in which to discuss the issue.

We remark that the idea of there being two types of probability has arisen in the literature before. The most prominent example is perhaps the work of Carnap [4], who talks about probability$_1$ and probability$_2$. Probability$_2$ corresponds to relative frequence or statistical information and thus to our type 1. Probability$_1$ corresponds to what Carnap calls *degree* of *confirmation*. This is not quite the same as our type 2; degree of confirmation considers to what extent a body of evidence supports or confirms a belief. However, there is some commonality in spirit. Skyrms [26] talks about first- and second-order probabilities, where first-order probabilities represent propensities or frequency—essentially statistical information—while second-order probabilities represent degrees of belief. These are called first- and second-order probabilities since typically one has a degree of belief about statistical information (this is the case in our second example above). Although $\mathscr{L}_3(\Phi)$ allows arbitrary alternation of the two types of probability, the semantics does support the intuition that these really are two fundamentally *different* types of probability.

5. On Obtaining Complete Axiomatizations

In order to guide (and perhaps help us automate) our reasoning about probabilities, it would be nice to have a complete deductive system. Unfortunately, results of [1] show that in general we will not be able to obtain such a system. We briefly review the relevant results here, and then show that we can obtain complete axiomatizations for important special cases.

5.1. Decidability and undecidability results

All the results in the subsection are taken from [1]. The first result is positive:

Theorem 5.1. *If Φ consists only of unary predicates, then the validity problem for $\mathscr{L}_1(\Phi)$ with respect to type-1 probability structures is decidable.*

The restrictions made in the previous result (to a language with only unary predicates, without equality between object terms) are both necessary. Once we allow equality in the language, the validity problem is no longer decidable, even if Φ is empty. In fact, the set of valid formulas is not even recursively enumerable (r.e.). And a binary predicate in Φ is enough to guarantee that the set of valid formulas is not r.e., even without equality between object terms.

Theorem 5.2.

(1) *For all* Φ, *the set of* $\mathscr{L}_1^{=}(\Phi)$ *formulas valid with respect to type-1 structures is not r.e.*

(2) *If* Φ *contains at least one predicate of arity greater than or equal to two, then the set of* $\mathscr{L}_1(\Phi)$ *formulas valid with respect to type-2 probability structures is not r.e.*

Once we move to $\mathscr{L}_2$, the situation is even worse. Even with only one unary predicates in Φ, the set of valid $\mathscr{L}_2(\Phi)$ formulas is not r.e. If we have equality, then the set of valid formulas is not r.e. as long as Φ has at least one constant symbol. (Note that $\varphi \Rightarrow (w(\varphi) = \mathbf{1})$ is valid if φ contains no nonlogical symbols—that is, φ does not contain any function or predicate symbols, other than equality—so we cannot make any nontrivial probability statements if Φ is empty.)

Theorem 5.3.

(1) *If* Φ *contains at least one predicate of arity greater than or equal to one, then the set of* $\mathscr{L}_2(\Phi)$ *formulas valid with respect to type-2 probability structures is not r.e.*

(2) *If* Φ *is nonempty, then the set of* $\mathscr{L}_2^{=}(\Phi)$ *formulas valid with respect to type-2 probability structures is not r.e.*

These results paint a rather discouraging picture as far as complete axiomatizations go. If a logic is to have a complete recursive axiomatization, then the set of valid formulas must be r.e. (we can enumerate the valid formulas by just carrying out all possible proofs). Thus, for all the cases cited in the previous theorems for which the set of valid formulas is not r.e., there can be no complete axiomatization.[4]

There is some good news in this bleak picture. In many applications it

[4] We remark that in [1], the exact degree of undecidability of the difficulty of the validity problem for all these logics is completely characterized. It turns out to be widely undecidable, much harder than the validity problem for the first-order theory of arithmetic. In fact, with just one binary predicate in the language, the validity problem is harder than that for the first-order theory of real analysis, where we allow quantification over real numbers as well as over natural numbers! As a consequence, our logics of probability are not even decidable relative to the full theory of real analysis. In retrospect, this is perhaps not surprising. A probability function is a higher-order function on sets, so reasoning about probability causes extra complications over and above reasoning about real numbers and natural numbers. We refer the reader to [1] for details.

suffices to restrict attention to structures of size at most N (i.e., structures whose domain has at most N elements), some fixed N. In this case, we get decidability.

Theorem 5.4. *If we restrict to structures of size at most N then, for all Φ the validity problem for $\mathscr{L}_1^=(\Phi)$ (respectively, $\mathscr{L}_2^=(\Phi)$, $\mathscr{L}_3^=(\Phi)$) with respect to type-1 (respectively, type-2, type-3) probability structures is decidable.*

A fortiori, the same result holds if equality is not in the language. We also get decidability if we restrict to structures of size exactly N.

The restriction to bounded structures is necessary though.

Theorem 5.5. *For all Φ (respectively, for all nonempty Φ, for all Φ) the set of $\mathscr{L}_1^=(\Phi)$ (respectively, $\mathscr{L}_2^=(\Phi)$, $\mathscr{L}_3^=(\Phi)$) formulas valid with respect to type-1 (respectively, type-2, type-3) probability structures of finite size is not r.e.*

5.2. An axiom system for probability on the domain

Although the previous results tell us that we cannot in general get a complete axiomatization for reasoning about probability, it is still useful to obtain a collection of sound axioms that lets us carry out a great deal of probabilistic reasoning.

In order to carry out our reasoning, we will clearly need axioms for doing first-order reasoning. In order to reason about probabilities, which we take to be real numbers, we need the theory of real closed fields. An *ordered field* is a field with a linear ordering $<$. A *real closed field* is an ordered field where every positive element has a square root and every polynomial of odd degree has a root. Tarski [28] showed (see also [25]) that the theory of real closed fields coincides with the theory of the reals (for the first-order language with equality and nonlogical symbols $+$, $\times$, $\leqslant$, $\mathbf{0}$, $\mathbf{1}$). That is, a first-order formula involving these symbols is true of the reals if and only if it is true in every real closed field. He also showed that the theory of real closed fields is decidable and has an elegant complete axiomazation. We incorporate this into our axiomatization too, since the language of real closed fields is a sublanguage of $\mathscr{L}_1(\Phi)$.

Consider the following collection of axioms, which we call AX_1.

First-order reasoning

PC. All instances of a standard complete axiomatization for first-order predicate calculus, including axioms for equality if equality is in the language (see, for example, [7]).

MP. From φ and $\varphi \Rightarrow \psi$ infer ψ (modus ponens).

Gen. From φ infer $\forall x \varphi$ (universal generalization)

Reasoning about real closed fields

RCF. All instances of a standard complete axiomatization for real closed fields (see, for example, [25]). The axioms of RCF consist of the standard axioms for fields (saying that addition and multiplication are commutative and associative, multiplication distributes over addition, **1** is the identity element for multiplication, and so on), axioms that say $\leq$ is a total linear order, an axiom that says that every positive number has a square root, and an axiom schema that says that every odd-degree polynomial has a root.

Reasoning about probabilities over the domain

PD1. $\forall x_1 \cdots \forall x_n \varphi \Rightarrow w_{\langle x_1, \ldots, x_n \rangle}(\varphi) = \mathbf{1}$, where $\langle x_1, \ldots, x_n \rangle$ is a sequence of distinct object variables.

PD2. $w_{\vec{x}}(\varphi) \geq \mathbf{0}$.

PD3. $w_{\vec{x}}(\varphi \wedge \psi) + w_{\vec{x}}(\varphi \wedge \neg \psi) = w_{\vec{x}}(\varphi)$.

PD4. $w_{\vec{x}}(\varphi) = w_{\vec{x}[x_i/z]}(\varphi[x_i/z])$, where z is an object variable which does not appear in $\vec{x}$ or φ.

PD5. $w_{\vec{x},\vec{y}}(\varphi \wedge \psi) = w_{\vec{x}}(\varphi) \times w_{\vec{y}}(\psi)$, if none of the free variables of φ is contained in $\vec{y}$, none of the free variables of ψ is contained in $\vec{x}$, and $\vec{x}$ and $\vec{y}$ are disjoint.

RPD1. From $\varphi \equiv \psi$ infer $w_{\vec{x}}(\varphi) = w_{\vec{x}}(\psi)$.

Note that PD4 allows us to rename bound variables, while PD5 lets us do reasoning based on the independence of the random variables. AX_1 is a straightforward extension of the axiom system used in [13] for reasoning about the propositional case. Not surprisingly, it is also quite similar to the collection of axioms given in [3]. Bacchus does not use the axioms for real closed fields; instead he uses the axioms for ordered fields, since he allows his probability functions to take values in arbitrary ordered fields. His axioms for reasoning about probabilities are essentially the same as ours (indeed, axioms PD1, PD2, and PD4 are also used by Bacchus, while PD5 is a weaker version of one of his axioms).

It is easy to check that these axioms are *sound* with respect to type-1 probability structures: if $AX_1 \vdash \varphi$ then $M \models \varphi$ for every axiom φ.

Theorem 5.6. *AX_1 is sound with respect to type-1 probability structures.*

Proof. It suffices to show that every instance of each axiom is valid and that the inference rules preserve validity. The only nontrivial case is axiom PD5.

Suppose that none of the free variables in φ is contained in $\vec{y}$ and none of the free variables in ψ is contained in $\vec{x}$, and $\vec{x}$, $\vec{y}$ are disjoint sequences of variables. We can assume without loss of generality that $\vec{x} = \langle x_1, \ldots, x_k \rangle$ and $\vec{y} = \langle x_{k+1}, \ldots, x_n \rangle$. Let

$$A = \{(d_1, \ldots, d_n)\colon (M, v[x_1/d_1, \ldots, x_n/d_n]) \models (\varphi \wedge \psi)\} ,$$

let

$$B = \{(d_1, \ldots, d_k)\colon (M, v[x_1/d_1, \ldots, x_k/d_k]) \models \varphi\} ,$$

and let

$$C = \{(d_{k+1}, \ldots, d_n)\colon (M, v[x_{k+1}/d_{k+1}, \ldots, x_n/d_n]) \models \psi\} .$$

By definition we have

$$[w_{\langle x_1, \ldots, x_n \rangle}(\varphi \wedge \psi)]_{(M,v)} = \mu^n(A) ,$$
$$[w_{\langle x_1, \ldots, x_k \rangle}(\varphi)]_{(M,v)} = \mu^k(B) ,$$
$$[w_{\langle x_{k+1}, \ldots, x_n \rangle}(\psi)]_{(M,v)} = \mu^{n-k}(C) .$$

From Proposition 2.1, it follows that

$$(M, v[x_1/d_1, \ldots, x_n/d_n]) \models (\varphi \wedge \psi)$$
$$\text{iff } (M, v[x_1/d_1, \ldots, x_k/d_k]) \models \varphi \quad \text{and}$$
$$(M, v[x_{k+1}/d_{k+1}, \ldots, x_n/d_n]) \models \psi .$$

Thus, $A = B \times C$. By the definition of product measure, it follows that $\mu^n(A) = \mu^k(B) \times \mu^{n-k}(C)$, and hence that $w_{\vec{x},\vec{y}}(\varphi \wedge \psi) = w_{\vec{x}}(\varphi) \times w_{\vec{y}}(\psi)$, as desired. Thus every instance of PD5 is valid. □

By the results of Section 5.1, we cannot hope that AX_1 (or any other axiom system!) will be complete for $\mathscr{L}_1(\Phi)$ once Φ has a predicate of arity at least two, nor can it be complete for $\mathscr{L}_1^=$. However, if we restrict Φ to consist only of unary predicates and do not have equality between object terms in the language, then it is complete.

Theorem 5.7. *If Φ consists only of unary predicates, then AX_1 is a sound and complete axiomatization for the language $\mathscr{L}_1(\Phi)$ with respect to type-1 probability structures.*

Proof. Soundness follows from Theorem 5.6. For completeness, suppose φ is valid. We show in Appendix A that we can effectively find a formula $\varphi_1 \wedge \varphi_2$ such that

(1) $AX_1 \vdash (\varphi_1 \wedge \varphi_2) \Rightarrow \varphi$,
(2) φ_1 is a pure first-order formula over Φ (and so is formed from the function and predicate symbols in Φ and object variables, using first-order quantification), φ_2 is a formula in the language of real closed fields (and so is formed from $\mathbf{0}$, $\mathbf{1}$, $+$, $\times$, $>$, $=$, and field variables, using first-order quantification over field variables), and
(3) both φ_1 and φ_2 are valid.

Since φ_1 is a valid pure first-order formula, we have $\{\mathrm{PC}, \mathrm{MP}\} \vdash \varphi_1$; since φ_2 is a valid formula in the language of real closed fields, $\{\mathrm{RCF}, \mathrm{MP}\} \vdash \varphi_2$. From (1), it follows that $AX_1 \vdash \varphi$. The details of the proof can be found in Appendix A. We remark that this proof gives us an immediate proof of Theorem 5.1, since, as we mentioned above, the theory of real closed fields is known to be decidable, as is first-order logic with only unary predicates [6]. □

Although the restriction to only unary predicates is clearly a severe one, a great deal of interesting probabilistic reasoning can be done in this language. In particular, our examples with flying birds can be carried out in this language. This result suggests that, although it is not complete, AX_1 is rich enough to let us carry out a great deal of probabilistic reasoning. The next result reinforces this impression.

Let AX_1^N be AX_1 together with the following axiom, which says that the domain has size at most N:

FIN$_N$. $\exists x_1 \cdots x_N \forall y((y = x_1) \vee \cdots \vee (y = x_N))$.

Theorem 5.8. *AX_1^N is a sound and complete axiomatization for $\mathscr{L}_1^=(\Phi)$ with respect to type-1 probability structures of size at most N.*

Proof. See Appendix A.

We can of course modify axiom FIN$_N$ to say that the domain has exactly N elements, and get a complete axiomatization with respect to structures of size exactly N.

5.3. An axiom system for probability on possible worlds

In order to reason about type-2 structures, we must replace the axioms for reasoning about probabilities over the domain with axioms for reasoning about probabilities over possible worlds. Consider the following axioms:

Reasoning about probabilities over possible worlds

PW1. $\varphi \Rightarrow (w(\varphi) = \mathbf{1})$, if no function and predicate symbols in Φ appear in φ except in the argument ψ of a probability term of the form $w(\psi)$.

PW2. $w(\varphi) \geqslant \mathbf{0}$.

PW3. $w(\varphi \wedge \psi) + w(\varphi \wedge \neg\psi) = w(\varphi)$.

RPW1. From $\varphi \equiv \psi$ infer $w(\varphi) = w(\psi)$.

PW2, PW3, and RPW1 are the result of replacing $w_{\bar{x}}$ in PD2, PD3, and RPD1, respectively, by w. PW1 is the analogue of PD1. Note that we cannot get a sound axiom simply by replacing the $w_{\bar{x}}$ in PD1 by w. For example, it might very well be the case that $\forall x P(x)$ holds at some possible worlds and not at others, so that, for example, we may have $\forall x P(x) \wedge (w(P(x)) = \frac{1}{2})$ holding at some possible world. On the other hand, since we use the same probability function to evaluate probability terms at all possible worlds, it is clear that if φ is a formula all of whose function and predicate symbols appear only in the arguments of probability terms (for example, φ might be a formula such as $(x = y) \Rightarrow (w(P(x) \wedge Q(y)) = \frac{1}{2}))$, then the truth of φ is independent of the possible world. Thus, if φ is true at some possible world, then it must be true at all of them. The validity of all instances of PW1 in type-2 structures follows.

Let AX_2 be the system that results by combining these axioms for reasoning about probabilities in possible worlds together with the axioms and rules of inference for first-order reasoning and for reasoning about real closed fields, with one small caveat. The standard axiomatization for first-order logic (see, for example [7]) has the substitution axiom $\forall x\varphi \Rightarrow \varphi[x/t]$, where t is a term that is *substitutable* for x. We do not give a careful definition for *substitutable* here (one can be found in [7]); intuitively, we do not want to substitute t if t contains a variable y which will end up in the scope of a quantifier. Here we have to extend the definition of substitutable even further so as not to allow the substitution of terms which contain non-rigid function and constant symbols into the scope of the w. To understand why, suppose we have a type-2 structure M consisting of two states, say s_1 and s_2, each of which has probability $\frac{1}{2}$, and exactly two domain elements, say d_1 and d_2. Suppose $(M,s_1) \models (P(d_1) \wedge \neg P(d_2))$ while $(M,s_2) \models (P(d_2) \wedge \neg P(d_1))$. Finally, let **a** be a constant symbol such that in s_1, the interpretation of $\mathbf{a}$ is d_2 (i.e., $\pi(s_1)(\mathbf{a}) = d_2$) and in s_1, the interpretation of $\mathbf{a}$ is d_1. Now it is easy to see that $(M,s_1) \models \forall x(w(P(x)) = \frac{1}{2})$ (informally, this is because both $P(d_1)$ and $P(d_2)$ hold at $\frac{1}{2}$ of the states), while $(M,s_1) \models (w(P(\mathbf{a})) = \mathbf{0})$. Thus,

$$\forall x(w(P(x)) = \tfrac{1}{2} \Rightarrow (w(P(\mathbf{a})) = \tfrac{1}{2})$$

is not valid in M. The problem here is that $\mathbf{a}$ is not a rigid designator. Once we restrict substitution appropriately, as described above, the problem disappears.

With this restriction, it is easy to show:

Theorem 5.9. *AX_2 is sound with respect to type-2 probability structures.*

While AX_2 is sound with respect to type-2 probability structures, the results of Section 5.1 tell us that it cannot be complete with respect to $\mathcal{L}_2(\Phi)$ (respectively $\mathcal{L}_2^=(\Phi)$) for any nontrivial Φ. However, we can get an analogue to Theorem 5.8. Let AX_2^N be AX_2 together with the axiom FIN_N.

Theorem 5.10. *AX_2^N is a sound and complete axiomatization for $\mathcal{L}_2^=(\Phi)$ with respect to type-2 probability structures of size at most N.*

Proof. See Appendix A. $\square$

5.4. A combined axiom system

Of course, we can combine AX_1 and AX_2 to get AX_3, which is a sound axiomatization for $\mathcal{L}_3$ with respect to type-3 structures. Again, we cannot hope to prove completeness in general, but, as before, we can prove that AX_3^N is complete with respect to type-3 structures of size at most N. We omit further details here.

6. Conclusions

We have provided natural semantics to capture two different kinds of probabilistic reasoning: in one, the probability is on the domain, and in the other, the probability is on a set of possible worlds. We also showed how these two modes of reasoning could be combined in a straightforward way.

We then considered the problem of providing sound and complete axioms to characterize first-order reasoning about probability. While complexity results of [1] show that in general there cannot be a complete axiomatization, we did provide sound axiom systems that we showed were rich enough to enable us to carry out a great deal of interesting probabilistic reasoning. In particular, together with an axiom guaranteeing finiteness, our axiom systems were shown to be complete for domains of bounded size.

Our results form an interesting contrast to those of Bacchus [3]. Bacchus gives a complete axiomatization for his language (which, as we remarked above, is essentially the same as our language $\mathcal{L}_1(\Phi)$ for reasoning about probabilities on the domain), thus showing that the set of formulas in his language that are valid with respect to the class of domains he considers is r.e. The reason for this difference is that Bacchus allows nonstandard probability functions, which are only required to be finitely additive and can take values in arbitrary ordered fields. Facts about the real numbers (such as the statement that 2 has a square root), are not valid in all the domains considered by Bacchus. It is not clear how much we lose by moving from the real numbers to arbitrary ordered fields. Our technical results, as well as the examples of Bacchus, suggest that the loss may not be too serious. It is worth noting that the move to nonstandard probability functions is the key reason that a complete axiomatization is obtainable. In [1] it is shown that all the undecidability results mentioned above can be proved even if we only require the probability function to be finitely additive, and restrict probabilities to taking only rational values.[5]

The situation here is somewhat analogous to that of axiomatizing arithmetic. Gödel's famous incompleteness result shows that the first-order theory of arithmetic (for the language with equality and nonlogical symbols $+$, $\times$, $\mathbf{0}$, $\mathbf{1}$, where the domain is the natural numbers) does not have a complete axiomatization. The axioms of Peano arithmetic are sound for arithmetic, but not complete. They are complete with respect to a larger class of domains (including so-called *nonstandard models*). Our results show that reasoning about probabilities is even *harder* than reasoning about arithmetic, and so cannot have a complete axiomatization. However, Bacchus' axioms are complete with respect to a larger class of structures, where probabilities can assume nonstandard values. And just as the axioms of Peano arithmetic are sufficiently rich to let us carry out a great deal of interesting arithmetic reasoning, so the axioms that we have provided (or the axioms of [3]) are sufficiently rich to enable us to carry out a great deal of interesting probabilistic reasoning.

Appendix A. Proofs of Theorems 5.7, 5.8 and 5.10

Before proving the theorems, we first show that a number of facts about probability—facts that we use repeatedly in our proofs—are provable in AX_1.

[5] Bacchus claims [3] that it is impossible to have a complete proof theory for countably additive probability functions. Although, as our results show, his claim is essentially correct (at least, as long as the language contains one binary predicate symbol or equality), the reason that he gives for this claim, namely, that the corresponding logic is not compact, is not correct. For example, even if $\Phi = \{P\}$, where P is a unary predicate, the logic is not compact. (Consider the set $\{w_x(P(x)) \neq \mathbf{0}, w_x(P(x)) < \frac{1}{2}, w_x(P(x)) < \frac{1}{3}, w_x(P(x)) \leq \frac{1}{4}, \ldots\}$. Any finite subset of these formulas is satisfiable, but the full set is not.) However, by Theorem 5.7, the logic in this case has a complete axiomatization.

We say two formulas φ and ψ are *mutually exclusive* if $\mathrm{PC} \vdash (\varphi \Rightarrow \neg\psi)$. A set $\varphi_1, \ldots, \varphi_k$ of formulas is mutually exclusive if each pair φ_i, φ_j, for $i \neq j$, is mutually exclusive.

Lemma A.1.

(1) *If* $\varphi_1, \ldots, \varphi_k$ *are mutually exclusive, then*

$$AX_1 \vdash (w_{\vec{x}}(\varphi_1 \vee \cdots \vee \varphi_k) = w_{\vec{x}}(\varphi_1) + \cdots + w_{\vec{x}}(\varphi_k)) .$$

(2) *If* $AX_1 \vdash \varphi$, *then* $AX_1 \vdash (w_{\vec{x}}(\varphi) = \mathbf{1})$.
(3) $AX_1 \vdash (w_{\vec{x}}(\varphi) + w_{\vec{x}}(\neg\varphi) = \mathbf{1})$.
(4) $AX_1 \vdash (w_{\vec{x}}(\varphi \wedge \psi) \leq w_{\vec{x}}(\varphi))$.
(5) $AX_1 \vdash ((w_{\vec{x}}(\psi) = \mathbf{1}) \Rightarrow (w_{\vec{x}}(\varphi \wedge \psi) = w_{\vec{x}}(\varphi)))$.
(6) $AX_1 \vdash ((w_{\vec{x}}(\psi) = \mathbf{1}) \Rightarrow (w_{\vec{x}}(\varphi \wedge \neg\psi) = \mathbf{0}))$.
(7) $AX_1 \vdash ((w_{\vec{x}}(\varphi \equiv \psi) = \mathbf{1}) \Rightarrow (w_{\vec{x}}(\varphi) = w_{\vec{x}}(\psi)))$.
(8) *If none of the variables free in* φ *are contained in* $\vec{y}$, *and the variables in* $\vec{x}$ *and* $\vec{y}$ *are distinct, then*

$$AX_1 \vdash (w_{\vec{x},\vec{y}}(\varphi) = w_{\vec{x}}(\varphi)) .$$

Proof. For part (1), let $\psi = \varphi_1 \vee \cdots \vee \varphi_k$. We proceed by induction on k, the number of disjuncts. First observe that using PD3 we get that

$$AX_1 \vdash (w_{\vec{x}}(\psi) = w_{\vec{x}}(\psi \wedge \varphi_1) + w_{\vec{x}}(\psi \wedge \neg\varphi_1)) .$$

Since the φ_i are mutually exclusive, we get that both

$$\mathrm{PC} \vdash ((\psi \wedge \varphi_1) \equiv \varphi_1)$$

and

$$\mathrm{PC} \vdash ((\psi \wedge \neg\varphi_1) \equiv (\varphi_2 \vee \cdots \vee \varphi_k)) .$$

Now using RPD1 and RCF, we get that

$$AX_1 \vdash (w_{\vec{x}}(\psi) = w_{\vec{x}}(\varphi_1) + w_{\vec{x}}(\varphi_2 \vee \cdots \vee \varphi_k)) .$$

We now continue by induction.

For part (2), suppose $\vec{x} = \langle x_1, \ldots, x_n \rangle$. By applying universal generalization (the rule Gen), we have that $AX_1 \vdash \forall x_1 \cdots x_n \varphi$. The result now follows from PD1.

For part (3), since $\mathrm{PC} \vdash (\varphi \vee \neg\varphi)$, from part (1) we get

$$AX_1 \vdash (w_{\vec{x}}(\varphi \vee \neg\varphi) = \mathbf{1}) .$$

Since the formulas φ and $\neg\varphi$ are mutually exclusive, the result now follows using part (1) and straightforward reasoning about equalities.

For part (4), observe that by PD3, we have

$$AX_1 \vdash (w_{\bar{x}}(\varphi) = w_{\bar{x}}(\varphi \wedge \psi) + w_{\bar{x}}(\varphi \wedge \neg\psi)) .$$

By PD2, we have

$$AX_1 \vdash (w_{\bar{x}}(\varphi \wedge \neg\psi) \geqslant \mathbf{0}) .$$

The result follows using straightforward reasoning about inequalities (which can be done using the axioms of RCF).

We prove parts (5) and (6) simultaneously. Observe that from part (3) we have

$$AX_1 \vdash ((w_{\bar{x}}(\psi) = \mathbf{1}) \Rightarrow (w_{\bar{x}}(\neg\psi) = \mathbf{0})) . \tag{A.1}$$

From part (4), we have

$$AX_1 \vdash ((w_{\bar{x}}(\neg\psi) = \mathbf{0}) \Rightarrow (w_{\bar{x}}(\varphi \wedge \neg\psi) = \mathbf{0})) . \tag{A.2}$$

Part (6) now follows from (A.1) and (A.2). For part (5), we need only put this together with the following instance of PD3:

$$AX_1 \vdash (w_{\bar{x}}(\varphi) = w_{\bar{x}}(\varphi \wedge \psi) + w_{\bar{x}}(\varphi \wedge \neg\psi)) .$$

In order to prove part (7), first observe that, by part (5), we get

$$AX_1 \vdash ((w_{\bar{x}}(\varphi \equiv \psi) = \mathbf{1}) \Rightarrow (w_{\bar{x}}((\varphi \equiv \psi) \wedge \varphi) = w_{\bar{x}}(\varphi)))$$

and

$$AX_1 \vdash ((w_{\bar{x}}(\varphi \equiv \psi) = \mathbf{1}) \Rightarrow (w_{\bar{x}}((\varphi \equiv \psi) \wedge \psi) = w_{\bar{x}}(\psi))) .$$

From the definition of $\equiv$, the formula $(\varphi \equiv \psi)$ is an abbreviation for $(\varphi \wedge \psi) \vee (\neg\varphi \wedge \neg\psi)$. Thus, we get

$$\mathrm{PC} \vdash (((\varphi \equiv \psi) \wedge \varphi) \equiv (\varphi \wedge \psi))$$

and

$$\mathrm{PC} \vdash (((\varphi \equiv \psi) \wedge \psi) \equiv (\varphi \wedge \psi)) .$$

By applying RPD1, we get

$$AX_1 \vdash ((w_{\bar{x}}(\varphi \equiv \psi) = \mathbf{1}) \Rightarrow (w_{\bar{x}}(\varphi \wedge \psi) = w_{\bar{x}}(\varphi)))$$

and

$$AX_1 \vdash ((w_{\vec{x}}(\varphi \equiv \psi) = \mathbf{1}) \Rightarrow (w_{\vec{x}}(\varphi \wedge \psi) = w_{\vec{x}}(\psi))) .$$

Part (7) now follows.

For part (8), given φ, let ψ be any sentence (formula with no free variables) such that $AX_1 \vdash \psi$. (For example, if the free variables of φ are contained in $\vec{x}$, we can take ψ to be $w_{\vec{x}}(\varphi) \geq \mathbf{0}$.) Observe that $AX_1 \vdash (\varphi \equiv (\varphi \wedge \psi))$. Thus, by RPD1, we get

$$AX_1 \vdash (w_{\vec{x},\vec{y}}(\varphi) = w_{\vec{x},\vec{y}}(\varphi \wedge \psi)) .$$

Applying PD5, we get

$$AX_1 \vdash (w_{\vec{x},\vec{y}}(\varphi \wedge \psi) = w_{\vec{x}}(\varphi) \times w_{\vec{y}}(\psi)) .$$

By part (2), we know $AX_1 \vdash (w_{\vec{y}}(\psi) = \mathbf{1})$, so part (8) follows. □

We are now ready to prove Theorem 5.7. Recall it says that AX_1 is sound and complete for the language $\mathscr{L}_1(\Phi)$, if Φ contains only unary predicates.

Proof of Theorem 5.7. We have already dealt with soundness. In order to prove completeness, suppose $\models_1 \varphi$. We want to show that φ is provable in AX_1. The proof is somewhat technical; we just sketch the highlights here, leaving details to the reader.

We first need to develop some machinery. Given a finite set of formulas $\psi_1, \ldots, \psi_k$, define an *atom over* $\psi_1, \ldots, \psi_k$ to be a formula of the form $\psi_1' \wedge \cdots \wedge \psi_k'$, where each ψ_i' is either ψ_i or $\neg\psi_i$. Note that the atoms are mutually exclusive. Moreover, note that ψ_i is provably equivalent to the disjunction of the 2^{k-1} atoms which have ψ_i as one of their conjuncts. Thus, given any formula φ of the form $(\varphi_1 \wedge \psi_1) \vee \cdots \vee (\varphi_k \wedge \psi_k)$, by using propositional reasoning (in particular, by using only axioms of the form $(p \wedge (q \vee r)) \equiv ((p \wedge q) \vee (p \wedge r)))$, we can rewrite φ to a probably equivalent formula of the form $(\tau_1 \wedge \sigma_1) \vee \cdots \vee (\tau_m \wedge \sigma_m)$, where the σ_j are atoms over $\psi_1, \ldots, \psi_k$ (since there are 2^k distinct atoms, we must have $m \leq 2^k$) and the τ_j are disjunctions of some subset of φ_i.

Define a *pure first-order formula over* Φ to be one formed from the function and predicate symbols in Φ and object variables, using first-order quantification over object variables; define a *formula in the language of real closed fields* to be one formed from $\mathbf{0}$, $\mathbf{1}$, $+$, $\times$, $>$, $=$, and field variables, using first-order quantification over field variables; finally, a *formula in the language of real closed fields augmented with probability terms* is a formula in the language of real closed fields where we allow in addition probability terms of the form $w_{\vec{x}}(\psi)$.

Ultimately, we want to reduce φ to a conjunction of a pure first-order formula and a formula in the language of real closed fields. We need to first get φ into a certain canonical form in order to accomplish this goal.

Claim 5.7.1. *We can effectively find a formula φ^* provably equivalent to φ such that φ^* is in the following canonical form*:

$$(\varphi_1 \wedge \psi_1) \vee \cdots \vee (\varphi_k \wedge \psi_k),$$

where

(1) *φ_i, $i = 1, \ldots, k$, is a pure first-order formula over Φ,*
(2) *ψ_i, $i = 1, \ldots, k$, is a formula in the language of real closed fields augmented by probability formulas,*
(3) *there is a fixed object variable x_0 such that for every probability term $w_{\vec{x}}(\psi)$ that occurs in φ^*, we have that $\vec{x} = \langle x_0 \rangle$ and that ψ is a conjunction of the form $Q_1(x_0) \wedge \cdots \wedge Q_n(x_0)$, where each Q_i is either P_i or $\neg P_i$ for some unary predicate P_i in Φ,*
(4) *the formulas $\psi_1, \ldots, \psi_k$ are mutually exclusive,*
(5) *for every pure first-order subformula of φ^* of the form $\forall x^{\mathrm{o}} \varphi'$, the formula φ' is a Boolean combination of atomic formulas of the form $P(x^{\mathrm{o}})$ (so that, in particular, $\forall x^{\mathrm{o}} \varphi'$ is a closed formula).*

Moreover, the same variables are free in φ and φ^*.

Proof. We prove that φ can be simplified in this way by induction on the structure of φ. If φ is an atomic formula of the form $P(t_1, \ldots, t_n)$, then the result is immediate. The result is also immediate if φ is an atomic formula of the form $(t_1 > t_2)$ or $(t_1 = t_2)$, where t_1 and t_2 are field terms, neither of which contain probability terms. If φ is of the form $(\varphi' \wedge \varphi'')$ or $\neg \varphi'$, we can get the result by straightforward propositional reasoning, forming the appropriate atoms to get mutual exclusion among the ψ_i's. Thus, there remain only three cases: (a) φ is of the form $\forall x^{\mathrm{o}} \varphi'$, (b) φ is of the form $\forall x^{\mathrm{f}} \varphi'$, (c) φ contains a probability term of the form $w_{\vec{x}}(\varphi')$.

In the first case, we can assume without loss of generality that φ' is in canonical form, and so is of the form $(\varphi_1 \wedge \psi_1) \vee \cdots \vee (\varphi_k \wedge \psi_k)$. Since the variable x^{o} does not occur free in any of the formulas $\psi_1, \ldots, \psi_k$, by straightforward first-order reasoning (using the fact that the ψ_i's are mutually exclusive) we can show that

$$\mathrm{PC} \vdash \left(\forall x^{\mathrm{o}} \varphi' \equiv \left(\bigvee_{i=1}^{k} (\psi_i \wedge \forall x^{\mathrm{o}} \varphi_i) \right) \right).$$

Now we want to rewrite $\forall x^{\mathrm{o}} \varphi_i$ so that Claim 5.7.1(5) holds, namely, so that all that remains in the scope of $\forall x^{\mathrm{o}}$ is a Boolean combination of atomic formulas of the form $P(x^{o})$. By the induction hypothesis, we can assume that φ_i is a Boolean combination of atomic formulas of the form $P(x^{o})$ and formulas where x^{o} does not appear free. Using the same ideas as discussed above in the context of atoms, we can show that φ_i is provably equivalent to a formula of the form $(\alpha_1 \vee \beta_1) \wedge \cdots \wedge (\alpha_m \vee \beta_m)$, where each α_i is a Boolean combination of formulas of the form $P(x^{o})$, the variable x^{o} does not appear free in any of the β_i, and the β_i are mutually exclusive. We can now proceed just as above to pull the β_i out of the scope of the $\forall x^{o}$. Namely, we can show that

$$\mathrm{PC} \vdash \left(\forall x^{o}\varphi_i \equiv \left(\bigvee_{i=1}^{m} (\beta_i \wedge \forall x^{o}\alpha_i)\right)\right).$$

This completes the proof of case (a).

The proof of case (b) is similar (but easier), and is left to the reader.

Now consider case (c), where we have a term of the form $w_{\vec{x}}(\varphi')$. By the induction hypothesis and rule RPD1, we can again assume without loss of generality that φ' is in canonical form; i.e., that φ' is in the form $(\varphi_1 \wedge \psi_1) \vee \cdots \vee (\varphi_k \wedge \psi_k)$, where the ψ_i are mutually exclusive. By Lemma A.1(1) we have

$$AX_1 \vdash (w_{\vec{x}}(\varphi) = w_{\vec{x}}(\varphi_1 \wedge \psi_1) + \cdots + w_{\vec{x}}(\varphi_k \wedge \psi_k)). \tag{A.3}$$

By (A.3), we can restrict attention to terms of the form $w_{\vec{x}}(\varphi^{\mathrm{fo}} \wedge \varphi^{\mathrm{rcf}})$, where φ^{fo} is a pure first-order formula and φ^{rcf} is a formula in the language of real closed fields augmented by probability terms.

We now proceed very much along the lines of case (a). Suppose $\vec{x} = \langle x_1, \ldots, x_n \rangle$. By the induction hypothesis, the only variables free in φ^{rcf} are field variables (since there are no free object variables in the probability terms, by Claim 5.7.1(3)), so we get that $\mathrm{PC} \vdash (\varphi^{\mathrm{rcf}} \Rightarrow \forall x_1 \cdots x_n \varphi^{\mathrm{rcf}})$. Using PD1, we get $AX_1 \vdash (\varphi^{\mathrm{rcf}} \Rightarrow (w_{\vec{x}}(\varphi^{\mathrm{rcf}}) = \mathbf{1}))$. By applying parts (5) and (6) of Lemma A.1, we get

$$AX_1 \vdash (\varphi^{\mathrm{rcf}} \Rightarrow (w_{\vec{x}}(\varphi^{\mathrm{fo}} \wedge \varphi^{\mathrm{rcf}}) = w_{\vec{x}}(\varphi^{\mathrm{fo}}))), \tag{A.4}$$

and

$$AX_1 \vdash (\neg\varphi^{\mathrm{rcf}} \Rightarrow (w_{\vec{x}}(\varphi^{\mathrm{fo}} \wedge \varphi^{\mathrm{rcf}}) = \mathbf{0})). \tag{A.5}$$

By ordinary propositional reasoning we can show that

$$\mathrm{PC} \vdash (\varphi \equiv ((\varphi \wedge \varphi^{\mathrm{rcf}}) \vee (\varphi \wedge \neg\varphi^{\mathrm{rcf}}))).$$

By standard first-order reasoning about equalities, thanks to (A.4), we can replace all occurrences of $w_{\vec{x}}(\varphi^{\mathrm{fo}} \wedge \varphi^{\mathrm{rcf}})$ in $(\varphi \wedge \varphi^{\mathrm{rcf}})$ by $w_{\vec{x}}(\varphi^{\mathrm{fo}})$, and thanks to (A.5), we can replace all occurrences of $w_{\vec{x}}(\varphi^{\mathrm{fo}} \wedge \varphi^{\mathrm{rcf}})$ in $(\varphi \wedge \neg\varphi^{\mathrm{rcf}})$ by $\mathbf{0}$.

Thus we have transformed φ to a provably equivalent formula where the argument in a probability term is a pure first-order formula; i.e., we can restrict attention to terms of the form $w_{\vec{x}}(\varphi^{\mathrm{fo}})$ where φ^{fo} is a pure first-order formula. We are still not done with this case; we must prove Claim 5.7.1(3). Now, using Claim 5.7.1(1) and standard first-order reasoning, φ^{fo} is provably equivalent to a formula of the form $(\alpha_1 \wedge \beta_1) \vee \cdots \vee (\alpha_k \wedge \beta_k)$, where each α_i is the conjunction of atomic formulas of the form $P(y)$ or $\neg P(y)$, where y is one of the variables appearing in $\vec{x}$, none of the variables in $\vec{x}$ appears free in β_i, and the β_i are mutually exclusive. Using Lemma A.1(1) again and the fact that the β_i are mutually exclusive, we can show that

$$AX_1 \vdash (w_{\vec{x}}(\varphi^{\mathrm{fo}}) = w_{\vec{x}}(\alpha_1 \wedge \beta_1) + \cdots + w_{\vec{x}}(\alpha_k \wedge \beta_k)) .$$

Thus, we can restrict attention to a term of the form $w_{\vec{x}}(\alpha_i \wedge \beta_i)$, where none of the variables in $\vec{x}$ appears free in β_i. Using analogues to (A.4) and (A.5), we pull the β_i out of the scope of $w_{\vec{x}}$, just as we pulled φ^{rcf} out of the scope of $w_{\vec{x}}(\varphi^{\mathrm{fo}} \wedge \varphi^{\mathrm{rcf}})$. This means we can reduce to considering terms of the form $w_{\vec{x}}(\alpha_i)$, where α_i is a conjunction of atomic formulas of the form $P(y)$ or $\neg P(y)$, and y is one of the variables in $\vec{x}$. We can then apply PD5 (and Lemma A.1(8)) repeatedly to reduce to the case where the sequence $\vec{x}$ in the subscript consists of a single variable. For example, using PD5, we can show

$$AX_1 \vdash (w_{\langle x,y\rangle}(P(x) \wedge \neg Q(y)) = w_x(P(x)) \times w_y(\neg Q(y))) .$$

Finally, by applying PD4, we can reduce to the case that the variable is the same for all probability terms. This proves Claim 5.7.1(3).

In order to complete the proof of Claim 5.7.1, we need only observe that the transformations required to get a formula φ into the canonical form required by Claim 5.7.1 are all effective. Moreover, they do not introduce any new variables, so that the same variables are free in φ and φ^*. $\square$

Claim 5.7.2. *Given φ, we can effectively find a formula $(\varphi' \wedge \psi')$ such that*

(1) *φ' is a pure first-order formula,*

(2) *ψ' is a formula in the language of real closed fields,*

(3) $AX_1 \vdash ((\varphi' \wedge \psi') \Rightarrow \varphi)$,

(4) *φ is valid iff $(\varphi' \wedge \psi')$ is valid.*

Proof. We can assume without loss of generality that φ is in the canonical form described in Claim 5.7.1. Let $P_1, \ldots, P_n$ be the atomic formulas that appear in the arguments of probability terms in φ, and let x_0 be the fixed object variable that appears in the probability terms. Consider the 2^n atoms over $P_1(x_0), \ldots, P_n(x_0)$; call them $\alpha_1, \ldots, \alpha_{2^n}$. As we have already observed, we can replace a probability term whose argument is a Boolean combination of

$P_1(x_0), \ldots, P_n(x_0)$ by a sum of probability terms whose arguments are (disjoint) atoms. Thus, φ is provably equivalent to a formula where all the probability terms are of the form $w_{x_0}(\alpha_i)$. Without loss of generality, we will assume that φ is in this form to start with. Since the α_i are mutually exclusive and their disjunction is provable, using Lemma A.1(1) and (2), we can show

$$AX_1 \vdash (w_{x_0}(\alpha_1) + \cdots + w_{x_0}(\alpha_{2^n}) = \mathbf{1}).$$

We now show that we can replace these probability terms by variables, thus completely getting rid of probability terms from the formula. Let $y_1, \ldots, y_{2^n}$ be fresh field variables, not appearing in φ; we think of y_i as representing $w_{x_0}(\alpha_i)$. Let $\varphi_{\bar{y}}$ be the result of replacing each probability term $w_{x_0}(\alpha_i)$ that appears in φ by y_i. Let φ'' be the universal closure[6] of the formula

$$\forall y_1 \cdots y_{2^n}\left(\left((y_1 + \cdots + y_{2^n} = \mathbf{1}) \wedge \left(\bigwedge_{i=1}^{2^n} y_i \geqslant \mathbf{0}\right)\right) \Rightarrow \varphi_{\bar{y}}\right).$$

Intuitively, φ'' says that φ holds for all ways of assigning probability to the 2^n atoms $\alpha_1, \ldots, \alpha_{2^n}$ (as long as the probabilities are positive and sum to 1). Clearly $AX_1 \vdash (\varphi'' \Rightarrow \varphi)$, since if we instantiate the y_i in φ'' with $w_x(\alpha_i)$, as we observed above, $w_{x_0}(\alpha_1) + \cdots + w_{x_0}(\alpha_{2^n}) = \mathbf{1}$ is provable, as is (by PD2) $w_x(\alpha_i) \geqslant \mathbf{0}$. Moreover, if φ is valid, then φ'' is valid. This follows from the observation that for every choice of values of the y_i with $y_1 + \cdots + y_{2^n} = \mathbf{1}$ and $y_i \geqslant \mathbf{0}$, $i = 1, \ldots, 2^n$, it is possible to define a probability function μ on the domain such that $w_x(\alpha_i) = y_i$. Clearly it is also the case that if φ'' is valid, then so is φ, since $\varphi'' \Rightarrow \varphi$ is provable.

Observe that the formula φ'' has no occurrences of probability terms. By using Claim 5.7.1, we can effectively find a formula φ''' provably equivalent to φ'' such that φ''' is of the form $(\varphi_1 \wedge \psi_1) \vee \cdots \vee (\varphi_k \wedge \psi_k)$, where each φ_i is a pure first-order formula and each ψ_i is a formula in the language of real closed fields (there are no probability terms in the ψ_i since there were none in φ'') and the ψ_i's are mutually exclusive. Moreover, each φ_i and ψ_i is a closed formula since φ'' is. By the arguments above, we know that $AX_1 \vdash (\varphi''' \Rightarrow \varphi)$. It immediately follows that $AX_1 \vdash ((\varphi_i \wedge \psi_i) \Rightarrow \varphi)$ for each disjunct $(\varphi_i \wedge \psi_i)$ of φ.

Since φ'' is equivalent to φ''', and we have already shown that φ is valid iff φ'' is valid, it follows that φ is valid iff φ''' is valid. We now show that if φ''' is valid iff $(\varphi_i \wedge \psi_i)$ is valid for some $i \in \{1, \ldots, k\}$. Clearly if $(\varphi_i \wedge \psi_i)$ is valid, then so is φ'''. For the converse, suppose φ''' is valid. By the result of Tarski mentioned above, we know that a formula in the language of real closed fields is valid iff it is true of the reals. Since the ψ_i are mutually exclusive, at most one

[6] Recall that the universal closure of a formula ξ is the result of universally quantifying the free variables in ξ. Thus, if the free variables in ξ are $z_1, \ldots, z_k$, then the universal closure of ξ is $\forall z_1 \ldots z_k \xi$. Note that the universal closure of a formula is guaranteed to be a closed formula.

can be true of the reals. We cannot have all the ψ_i being false of the reals, for then φ''' could not be valid. Thus, exactly one of the ψ_i must be true of the reals, say ψ_{i_0}. It is now easy to see that φ_{i_0} must be valid (since if there is some first-order structure where $\neg\varphi_{i_0}$ is not satisfiable in some first-order structure, then $\neg\varphi'''$ is also satisfiable in that structure augmented by the reals). We can now take the φ' and ψ' required to prove the claim to be φ_{i_0} and ψ_{i_0}. From the decidability of the theory of real closed fields, it follows that we can effectively find the required φ_{i_0} and ψ_{i_0}. □

The theorem now follows quickly from Claim 5.7.2. Given a valid formula φ, we simply construct the φ' and ψ' guaranteed to exist by Claim 5.7.2. Since φ' is valid, we have $\mathrm{PC} \vdash \varphi'$; since ψ' is valid, we have $\mathrm{RCF} \vdash \psi'$. Thus $AX_1 \vdash (\varphi' \wedge \psi')$. From Claim 5.7.2, we now get $AX_1 \vdash \varphi$. □

We next want to prove Theorem 5.8; recall that this theorem says that AX_1^N is sound and complete for $\mathcal{L}_1^=(\Phi)$ with respect to the domains of size at most N. As we shall see, many of the ideas in the proof of Theorem 5.7 will reappear in the proof of this theorem. For simplicity, we do this proof (and the following proof of Theorem 5.10) under the assumption that Φ contains no function symbols, although it may contain arbitrary predicate symbols. (Since we can always replace a k-ary function symbol with a $(k+1)$-ary predicate symbol, this assumption really entails no loss of generality.) In particular, this assumption implies that in an atomic formula of the form $(t_1 = t_2)$, t_1 and t_2 are either both field terms or both object variables.

Proof of Theorem 5.8. Clearly AX_1^N is sound. To prove completeness, suppose φ is valid with respect to type-1 structures of size at most N. Let *Exactly*(M) be the formula that says that there are exactly M elements in the domain. More formally, let *Exactly*$'(z_1, \ldots, z_M)$ be the formula

$$\Big(\bigwedge_{i,j=1,\ldots,M,\ i\neq j} (z_i \neq z_j)\Big) \wedge \forall y((y = z_1) \vee \cdots \vee (y = z_M)),$$

which says that the z_i represent the M different domain elements, and let *Exactly*(M) be the formula $\exists z_1 \cdots z_M\ \mathit{Exactly}'(z_1, \ldots, z_M)$. It is easy to see that

$$\{\mathrm{PC,MP,FIN}_N\} \vdash \Big(\varphi \equiv \Big(\bigwedge_{M=1}^{N} (\mathit{Exactly}(M) \Rightarrow \varphi)\Big)\Big).$$

Thus, each of the formulas *Exactly*$(M) \Rightarrow \varphi$ is valid, and in order to show that $AX_1^N \vdash \varphi$, it suffices to show, for $M = 1, \ldots, N$, that

$$AX_1^N \vdash (\mathit{Exactly}(M) \Rightarrow \varphi). \tag{A.6}$$

Note that we can assume without loss of generality that the variables $z_1, \ldots, z_M$ in $Exactly'(z_1, \ldots, z_M)$ do not appear free in φ. Now using standard first-order reasoning and the fact that $z_1, \ldots, z_M$ do not appear free in φ, we get

$$\mathrm{PC} \vdash (\forall z_1 \cdots z_M(Exactly'(z_1, \ldots, z_M) \Rightarrow \varphi) \equiv \\ (\exists z_1 \cdots z_M Exactly'(z_1, \ldots, z_M) \Rightarrow \varphi)) .$$

Since, by definition, $\exists z_1 \cdots z_M Exactly'(z_1, \ldots, z_M)$ is just $Exactly(M)$, the validity of $Exactly(M) \Rightarrow \varphi$ implies the validity of $Exactly'(z_1, \ldots, z_M) \Rightarrow \varphi$, and (given the rule Gen), in order to prove (A.6) it suffices to prove

$$AX_1 \vdash (Exactly'(z_1, \ldots, z_M) \Rightarrow \varphi) . \tag{A.7}$$

We prove (A.7) using techniques similar to those used in Theorem 5.7. Again, the first step is to reduce φ to a certain canonical form. The following claim is in fact almost identical to Claim 5.7.1, the major difference coming in the details of the third clause and the fact that we no longer require an analogue to Claim 5.7.1(5).

Claim 5.8.1. *We can effectively find a formula φ^* such that*

$$AX_1 \vdash (Exactly'(z_1, \ldots, z_M) \Rightarrow (\varphi \equiv \varphi^*)) ,$$

and φ^* is in the following canonical form:

$$(\varphi_1 \wedge \psi_1) \vee \cdots \vee (\varphi_k \wedge \psi_k) ,$$

where

(1) *φ_i, $i = 1, \ldots, k$, is a pure first-order formula over Φ,*
(2) *ψ_i, $i = 1, \ldots, k$, is a formula in the language of real closed fields augmented by probability formulas,*
(3) *there is a fixed object variable x_0 such that for every probability term $w_{\vec{x}}(\psi)$ that occurs in φ^*, we have that $\vec{x} = \langle x_0 \rangle$ and that ψ is a formula of the form $(x_0 = z_j)$, where z_j is one of the M free variables in $Exactly'(z_1, \ldots, z_M)$,*
(4) *the formulas $\psi_1, \ldots, \psi_k$ are mutually exclusive.*

Moreover, a variable is free in φ^* iff it is free in φ or it is one of $z_1, \ldots, z_M$.

Proof. Again, we proceed by induction on the structure of φ, and again, there are three nontrivial cases: (a) φ is of the form $\forall x^{\mathrm{o}}\varphi'$, (b) φ is of the form $\forall x^{\mathrm{f}}\varphi'$, (c) φ contains a term of the form $w_{\vec{x}}(\varphi')$.

We can deal with a formula of the form $\forall x^{\mathrm{o}}\varphi'$ just as in the corresponding part of the proof of Claim 5.7.1; indeed, since we no longer have to deal with an analogue of Claim 5.7.1(5), we don't have to work so hard. Dealing with a formula of the form $\forall x^{\mathrm{f}}\varphi'$ is similarly straightforward.

Now consider case (c), where φ contains a term of the form $w_{\vec{x}}(\varphi')$. By the induction hypothesis and rule RPD1, we can assume without loss of generality that φ' is in canonical form; i.e., that φ' is in the form $(\varphi_1 \wedge \psi_1) \vee \cdots \vee (\varphi_k \wedge \psi_k)$, where the ψ_i are mutually exclusive. Moreover, none of the variables that appear free in the ψ_i appear free in $\vec{x}$ (since, by Claim 5.8.1(3), it follows that the only free object variables that can appear in probability terms in ψ_i are in $\{z_1, \ldots, z_M\}$). Thus, just as in the proof of Claim 5.7.1, we can reduce to the case that the argument in the probability term is a pure first-order formula; i.e., we can restrict attention to terms of the form $w_{\vec{x}}(\varphi')$ where φ' is a pure first-order formula.

To get the idea of what we are going to do next, suppose φ' is the atomic formula $P(y_1, y_2)$. Further suppose that $P(z_1, z_1)$ and $P(z_1, z_3)$ hold, and that these are the only domain values for which P holds. Thus $P(y_1, y_2)$ is true iff

$$((y_1 = z_1) \wedge (y_2 = z_1)) \vee ((y_1 = z_1) \wedge (y_2 = z_3)).$$

It then follows that

$$w_{\vec{x}}(\varphi') \equiv w_{\vec{x}}((y_1 = z_1) \wedge (y_2 = z_1)) + w_{\vec{x}}((y_1 = z_1) \wedge (y_2 = z_3)) .$$

Thus, we have replaced a probability term by one whose arguments are of the form $(y_i = z_j)$. This can be done in general.

Suppose that the free variables in φ' are $y_1, \ldots, y_m$. Define an (M, m)-*sequence* to be one of the form $\langle i_1, \ldots, i_m \rangle$, where $1 \leq i_j \leq M$ (note that the i_j are not necessarily distinct). There are clearly M^m such (M, m)-sequences. If J is the (M, m)-sequence $\langle i_1, \ldots, i_m \rangle$, define $Eq(\vec{y}, J)$ to be an abbreviation for the formula

$$(y_1 = z_{i_1}) \wedge \cdots \wedge (y_m = z_{i_m}) .$$

Finally, if $\mathcal{J}$ is a set of (M, m)-sequences, let $Eq(\vec{y}, \mathcal{J})$ be an abbreviation for the formula $\vee_{J \in \mathcal{J}} Eq(\vec{y}, J)$. We can think of the z_j in $Exactly'(z_1, \ldots, z_M)$ as describing the elements of the domain. Then the formula $Eq(\vec{y}, \mathcal{J})$ holds exactly if the variables in $\vec{y}$ take on one of the values specified by a sequence in $\mathcal{J}$.

Now in every first-order structure, there is some set of domain values for which the formula φ' holds. For each (M, m)-sequence $J = \langle i_1, \ldots, i_m \rangle$, let φ'_J be an abbreviation for the formula $\varphi'[y_1/z_{i_1}, \ldots, y_m/z_{i_m}]$. Let $SEQ(M, m)$ be the set of all subsets of (M, m)-sequences. For each $\mathcal{J} \in SEQ(M, m)$, let $\varphi'_{\mathcal{J}}$

be an abbreviation for

$$\Big(\bigwedge_{J\in\mathcal{J}} \varphi'_J\Big) \wedge \Big(\bigwedge_{J\notin\mathcal{J}} \neg\varphi'_J\Big) .$$

Thus, $\varphi'_{\mathcal{J}}$ holds if φ' is true precisely of the domain elements described by $\mathcal{J}$. It is easy to see that

$$\mathrm{PC} \vdash (\varphi'_{\mathcal{J}} \equiv \forall x_1 \cdots x_n(\varphi' \equiv Eq(\vec{y}, \mathcal{J}))) . \tag{A.8}$$

Now in every first-order structure, there is some set of domain values for which the formula φ' holds. Thus, it is easy to see that

$$\mathrm{PC} \vdash \Big(\mathit{Exactly}'(z_1, \ldots, z_M) \Rightarrow \Big(\bigvee_{\mathcal{J}\in SEQ(M,m)} \varphi'_{\mathcal{J}} \Big)\Big) .$$

Thus we get

$$\mathrm{PC} \vdash \Big(\mathit{Exactly}'(z_1, \ldots, z_M) \Rightarrow \Big(\varphi \equiv \Big(\bigvee_{\mathcal{J}\in SEQ(M,m)} (\varphi \wedge \varphi'_{\mathcal{J}}) \Big)\Big)\Big) . \tag{A.9}$$

Suppose that $\vec{x}$ (the subscript in the probability term $w_{\vec{x}}(\varphi')$) is the sequence $\langle x_1, \ldots, x_n \rangle$. (Note that some of the x_i and y_j that appear in $\varphi'_{\mathcal{J}}$ may be identical.) From PD1 and (A.8), we get

$$AX_1 \vdash (\varphi'_{\mathcal{J}} \Rightarrow (w_{\vec{x}}(\varphi' \equiv Eq(\vec{y}, \mathcal{J})) = \mathbf{1})) . \tag{A.10}$$

Using Lemma A.1(7), we get

$$AX_1 \vdash ((w_{\vec{x}}(\varphi' \equiv Eq(\vec{y}, \mathcal{J})) = \mathbf{1}) \Rightarrow (w_{\vec{x}}(\varphi') = w_{\vec{x}}(Eq(\vec{y}, \mathcal{J})))) . \tag{A.11}$$

Let $\varphi_{\mathcal{J}}$ be the result of replacing all terms of the form $w_{\vec{x}}(\varphi')$ in φ by $w_{\vec{x}}(Eq(\vec{y}, \mathcal{J}))$. From (A.8), (A.10), and (A.11), it easily follows that

$$AX_1 \vdash ((\varphi \wedge \varphi'_{\mathcal{J}}) \equiv (\varphi_{\mathcal{J}} \wedge \varphi'_{\mathcal{J}})) . \tag{A.12}$$

Thus, from (A.9) and (A.12), we get

$$\mathrm{PC} \vdash \Big(\mathit{Exactly}'(z_1, \ldots, z_M) \Rightarrow \Big(\varphi \equiv \Big(\bigvee_{\mathcal{J}\in SEQ(M,m)} (\varphi_{\mathcal{J}} \wedge \varphi'_{\mathcal{J}}) \Big)\Big)\Big) .$$

Now we are almost done. The argument above says that we can replace all terms $w_{\vec{x}}(\varphi')$ in φ by probability terms whose argument is of the form $Eq(\vec{y}, \mathcal{J})$. Now $Eq(\vec{y}, \mathcal{J})$ is an abbreviation for $\vee_{J \in \mathcal{J}} Eq(\vec{y}, J)$; moreover, the disjuncts are mutually exclusive, since the z_i's represent distinct domain elements. Thus, by Lemma A.1(1), we have

$$AX_1 \vdash \Big(w_{\vec{x}}(Eq(\vec{y}, \mathcal{J})) = \sum_{J \in \mathcal{J}} w_{\vec{x}}(Eq(\vec{y}, J)) \Big) .$$

Thus, we can reduce consideration to probability terms whose arguments are of the form $(y_1 = z_{i_1}) \wedge \cdots \wedge (y_m = z_{i_m})$. Since none of the z_j appear in $\vec{x}$, by repeated applications of PD5 (and Lemma A.1(8)), we can reduce to the case where the sequence $\vec{x}$ in the subscript consists of a single variable, which by PD4 we can rename to x_0, and the argument of the probability term is a single conjunct of the form $(y = z_j)$.

To summarize, our arguments show that φ is equivalent to a formula φ' where all the probability terms are of the form $w_{x_0}(y = z_j)$. If y is the variable x_0, we are done. If not, then

$$AX_1 \vdash (\varphi' \equiv ((\varphi_0 \wedge (y \neq z_j)) \vee (\varphi_1 \wedge (y = z_j))) ,$$

where φ_0 (respectively φ_1) is the result of replacing all occurrences of the term $w_{x_0}(y = z_j)$ in φ' by $\mathbf{0}$ (respectively $\mathbf{1}$). (This follows since using PD1 and Lemma A.1 and the fact that x_0 does not appear free in the formula $(y = z_j)$, we can easily show that $AX_1 \vdash ((y = z_j) \Rightarrow (w_{x_0}(y = z_j) = \mathbf{1}))$ and $AX_1 \vdash ((y \neq z_j) \Rightarrow (w_{x_0}(y = z_j) = \mathbf{0}))$.) Thus, we can transform φ to a formula where all the probability terms are of the form $w_{x_0}(x_0 = z_j)$.

Again, in order to complete the proof of Claim 5.8.1, we need only observe that the transformations required to get a formula φ into the appropriate canonical form are all effective, and that no extra free variables are introduced in φ^* other than possibly some of the z_i. □

We can now prove an analogue of Claim 5.7.2.

Claim 5.8.2. *Given φ in the canonical form described in Claim 5.8.1, we can effectively find a formula $\varphi' \wedge \psi'$ such that*

(1) *φ' is a pure first-order formula,*
(2) *ψ' is a formula in the language of real closed fields,*
(3) $AX_1 \vdash ((\varphi' \wedge \psi') \Rightarrow \varphi)$,
(4) *$(Exactly'(z_1, \ldots, z_M) \Rightarrow \varphi)$ is valid iff $(Exactly'(z_1, \ldots, z_M) \Rightarrow (\varphi' \wedge \psi'))$ is valid.*

Proof. The proof is almost identical to that of Claim 5.7.2. Let $y_1, \ldots, y_M$ be fresh field variables, not appearing in φ; we now think of y_i as representing

$w_{x_0}(x_0 = z_i)$. Let $\varphi_{\bar{y}}$ be the result of replacing each probability term $w_{x_0}(x_0 = z_i)$ that appears in φ by y_i. Let φ'' be the result of universally quantifying all the variables other than $z_1, \ldots, z_M$ that appear free in the formula

$$\forall y_1 \cdots y_M\Big(\Big((y_1 + \cdots + y_M = \mathbf{1}) \wedge \Big(\bigwedge_{i=1}^{M} (y_i \geq \mathbf{0})\Big)\Big) \Rightarrow \varphi_{\bar{y}}\Big).$$

As in Claim 5.7.2, we can show that $\mathrm{PC} \vdash (\varphi'' \Rightarrow \varphi)$. Moreover, if $(\mathit{Exactly}(z_1, \ldots, z_M) \Rightarrow \varphi)$ is valid, then $(\mathit{Exactly}'(z_1, \ldots, z_M) \Rightarrow \varphi'')$ is valid.

Observe that the formula φ'' has no occurrences of probability terms. By using Claim 5.8.1, we can effectively find a formula φ''' provably equivalent to φ'' such that φ''' is of the form $(\varphi_1 \wedge \psi_1) \vee \cdots \vee (\varphi_k \wedge \psi_k)$, where each φ_i is a pure first-order formula and each ψ_i is a formula in the language of real closed fields, and the ψ_i are mutually exclusive. Since φ''' has no free field variables, each of the ψ_i must be a closed formula. Clearly $AX_1 \vdash ((\varphi_i \wedge \psi_i) \Rightarrow \varphi)$ for each disjunct $(\varphi_i \wedge \psi_i)$ of φ; moreover, using the same arguments as in Claim 5.7.2, we can show that for some i_0, we must have that $(\mathit{Exactly}'(z_1, \ldots, z_M) \Rightarrow \varphi)$ is valid iff $(\mathit{Exactly}'(z_1, \ldots, z_M) \Rightarrow (\varphi_{i_0} \wedge \psi_{i_0}))$ is valid, and that we can find this i_0 effectively. We now take φ' to be φ_{i_0} and ψ' to be ψ_{i_0}. $\square$

We can now easily prove (A.7) (and hence the theorem). Suppose $(\mathit{Exactly}'(z_1, \ldots, z_M) \Rightarrow \varphi)$ is valid. We simply construct the φ' and ψ' guaranteed to exist by Claim 5.8.2. It is now easy to see that $(\mathit{Exactly}'(z_1, \ldots, z_M) \Rightarrow (\varphi' \wedge \psi'))$ is valid iff ψ' is valid in real closed fields (or, equivalently, ψ' is true of the reals) and $(\mathit{Exactly}'(z_1, \ldots, z_m) \Rightarrow \varphi')$ is valid. Thus, $(\mathit{Exactly}'(z_1, \ldots, z_M) \Rightarrow \varphi)$ is valid iff

$$\mathrm{PC} \vdash (\mathit{Exactly}'(z_1, \ldots, z_M) \Rightarrow \varphi') \quad \text{and} \quad \mathrm{RCF} \vdash \psi'.$$

Thus, if $(\mathit{Exactly}'(z_1, \ldots, z_M) \Rightarrow \varphi)$ is valid, then

$$AX_1 \vdash ((\mathit{Exactly}'(z_1, \ldots, z_M) \Rightarrow \varphi') \wedge \psi'),$$

and hence

$$AX_1 \vdash (\mathit{Exactly}'(z_1, \ldots, z_M) \Rightarrow \varphi). \qquad \square$$

Finally we prove Theorem 5.10; recall that this theorem says that AX_2^N is sound and complete for $\mathcal{L}_2^=(\Phi)$ with respect to the domains of size at most N. Again, the proof follows the same basic pattern as the previous proofs. The key observation here is that the analogue of all but Lemma A.1(8) also holds for AX_2 (where we replace $w_{\bar{x}}$ by w). The proofs are essentially identical to those in Lemma A.1, except for part (2). In order to prove Lemma A.1(2), suppose that $AX_2 \vdash \varphi$. We want to show $AX_2 \vdash (w(\varphi) = \mathbf{1})$. By PW2, we have $AX_2 \vdash (w(\varphi) > \mathbf{0})$. We can now apply PW1 to the formula $(w(\varphi) > \mathbf{0})$ to get $AX_2 \vdash (w(w(\varphi) > \mathbf{0}) = \mathbf{1})$. By straightforward propositional reasoning, we also have $AX_2 \vdash (\varphi \equiv (w(\varphi) > \mathbf{0}))$. The result now follows using RPW1.

Proof of Theorem 5.10. Suppose that φ is valid with respect to type-2 structures of size at most N. We want to show that $AX_2^N \vdash \varphi$. Just as in the proof of Theorem 5.8, it suffices to prove

$$AX_2 \vdash (\mathit{Exactly}'(z_1, \ldots, z_M) \Rightarrow \varphi) . \tag{A.13}$$

In order to prove (A.13), we find an appropriate canonical form for formulas in $\mathscr{L}_2^{=}(\Phi)$.

Claim 5.10.1. *We can effectively find a formula φ^* such that*

$$AX_1 \vdash (\mathit{Exactly}'(z_1, \ldots, z_M) \Rightarrow (\varphi \equiv \varphi^*)) ,$$

and φ^* is in the following canonical form:

$$(\varphi_1 \wedge \psi_1) \vee \cdots \vee (\varphi_k \wedge \psi_k) ,$$

where

(1) *φ_i, $i = 1, \ldots, k$, is a pure first-order formula over Φ,*
(2) *ψ_i, $i = 1, \ldots, k$, is a formula in the language of real closed fields augmented by probability formulas,*
(3) *the argument ψ in every probability term $w(\psi)$ that occurs in φ^* is a Boolean combination of atomic formulas of the form $P(z_{j_1}, \ldots, z_{j_m})$ where P is an m-ary predicate symbol in Φ (thus, ψ is a quantifier-free formula, the only variables that can appear free in ψ are $z_1, \ldots, z_M$, and there are no equality terms of the form $(t_1 = t_2)$ in ψ),*
(4) *the formulas $\psi_1, \ldots, \psi_k$ are mutually exclusive.*

Moreover, a variable is free in φ^* iff it is free in φ or it is one of $z_1, \ldots, z_M$.

Proof. Again, we proceed by induction on the structure of φ. We discuss only the case where φ contains a term of the form $w(\varphi')$. By the induction hypothesis and rule RPW1, we can assume without loss of generality that φ' is in canonical form; i.e., that φ' is in the form $(\varphi_1 \wedge \psi_1) \vee \cdots \vee (\varphi_k \wedge \psi_k)$, where the ψ_i are mutually exclusive. Thus, using the appropriate analogue of Lemma A.1 (and using PW1 in place of PD1 to prove analogues of equations (A.4) and (A.5)), we can reduce just as in the previous proofs to the case that the argument in the probability term is a pure first-order formula; i.e., we can restrict attention to terms of the form $w(\varphi')$ where φ' is a pure first-order formula. By using equivalences of the form $\forall x(\psi \equiv \wedge_{i=1}^{M} \psi[x/z_i])$, we can easily find a quantifier-free formula φ'' such that

$$\mathrm{PC} \vdash (\mathit{Exactly}'(z_1, \ldots, z_M) \Rightarrow (\varphi' \equiv \varphi'')) .$$

Similar arguments to those used in Claim 5.8.1 now allow us to replace each occurrence of $w(\varphi')$ in φ by $w(\varphi'')$. We omit details here.

We now want to replace all free variables that occur in φ'' by $z_1, \ldots, z_M$. Suppose the free variables of φ'' are $y_1, \ldots, y_m$. Let $Eq(\vec{y}, J)$ be defined just as in the proof of Claim 5.8.1, where J is an (M, m)-sequence. Clearly we have $\mathrm{PC} \vdash (\vee_{J \in \mathcal{J}} Eq(\vec{y}, J))$. Thus,

$$\mathrm{PC} \vdash \Big(\varphi \equiv \Big(\bigvee_{J \in \mathcal{J}} (\varphi \wedge Eq(\vec{y}, J))\Big)\Big) . \tag{A.14}$$

Given $J = \langle i_1, \ldots, i_m \rangle$, let φ''_J be the result replacing all atomic formulas in $\varphi''[y_1/z_{i_1}, \ldots, y_m/z_{i_m}]$ of the form $(z_i = z_i)$ by **true**, and all atomic formulas of the form $(z_i = z_j)$, $i \neq j$, by **false**. Clearly

$$\mathrm{PC} \vdash (Eq(\vec{y}, J) \Rightarrow (\varphi'' \equiv \varphi''_J)) .$$

Thus

$$\mathrm{PC} \vdash (Eq(\vec{y}, J) \equiv (Eq(\vec{y}, J) \wedge (\varphi'' \equiv \varphi''_J))) .$$

Using RPW1 and (the analogue of) Lemma A.1(3), we can now show that

$$AX_2 \vdash (w(Eq(\vec{y}, J)) \leq w(\varphi'' \equiv \varphi''_J)) . \tag{A.15}$$

By PW1, we have $AX_2 \vdash (Eq(\vec{y}, J) \Rightarrow (w(Eq(\vec{y}, J)) = \mathbf{1}))$. Thus, from (A.15), we get

$$AX_2 \vdash (Eq(\vec{y}, J) \Rightarrow (w(\varphi \equiv \varphi''_J) = \mathbf{1})) .$$

Let φ_J be the result of replacing occurrences of $w(\varphi'')$ in φ by $w(\varphi''_J)$. Similar arguments to those used in Claim 5.8.1 now show

$$AX_2 \vdash ((\varphi \wedge Eq(\vec{y}, J)) \equiv (\varphi_J \wedge Eq(\vec{y}, J))) . \tag{A.16}$$

By combining (A.16) with (A.14), we can see that Claim 5.10.1 follows. □

Claim 5.10.2. *Given φ in the canonical form described in Claim* 5.10.1, *we can effectively find a formula $(\varphi' \wedge \psi')$ such that*

(1) *φ' is a pure first-order formula,*

(2) *ψ' is a formula in the language of real closed fields,*

(3) $AX_1 \vdash ((\varphi' \wedge \psi') \Rightarrow \varphi)$,
(4) $(Exactly'(z_1, \ldots, z_M) \Rightarrow \varphi)$ *is valid* iff $(Exactly'(z_1, \ldots, z_M) \Rightarrow (\varphi' \wedge \psi'))$ *is valid.*

Proof. Let $\beta_1, \ldots, \beta_n$ be all the atomic formulas that appear in probability terms in φ. Let $\alpha_1, \ldots, \alpha_{2^n}$ be the atoms over $\beta_1, \ldots, \beta_n$. We now proceed as in Claim 5.7.2. We can write each β_i as a disjunction of atoms. Thus, by Lemma A.1, we can replace all probability terms that appear in φ by a sum of probability terms whose arguments are (disjoint) atoms. Thus, we can assume without loss of generality that the probability terms that appear in φ are all of the form $w(\alpha_i)$. Let $\varphi_{\bar{y}}$ be the result of replacing each probability term $w(\alpha_i)$ that appears in φ by y_i. Let φ'' be the result of universally quantifying all the variables other than $z_1, \ldots, z_M$ that appear free in the formula

$$\forall y_1 \cdots y_{2^n}\Big(\Big(\big(y_1 + \cdots + y_{2^n} = \mathbf{1}\big) \wedge \Big(\bigwedge_{i=1}^{2^n} (y_i \geq \mathbf{0})\Big)\Big) \Rightarrow \varphi_{\bar{y}}\Big).$$

As in Claim 5.7.2, we can show that $\mathrm{PC} \vdash (\varphi'' \Rightarrow \varphi)$. Moreover, if $(Exactly'(z_1, \ldots, z_M) \Rightarrow \varphi)$ is valid, then $(Exactly'(z_1, \ldots, z_M) \Rightarrow \varphi'')$ is valid. (We remark that the validity of $(Exactly'(z_1, \ldots, z_M) \Rightarrow \varphi'')$ depends crucially on the fact that the predicates that appear as the conjuncts in the α_i only have z_i as their arguments, since this allows us to treat the atomic formulas as independent propositions. For example, if we had allowed arbitrary variables as arguments, and the only atomic formulas appearing in probability terms were $P(x)$ and $P(x')$, then we would have an atom of the form $(P(x) \wedge \neg P(x'))$. If φ included a conjunct of the form $(x = x')$, then this atom could not have positive probability, and we could not just replace it by a fresh variable y. Similar difficulties arise if we allow equalities of the form $(t_1 = t_2)$ in probability terms.) The rest of the proof now proceeds just as in Claims 5.7.2 and 5.8.2, so we omit details here. □

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