Markov Logic Networks

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Statistical relational learning

Combining logic with probability

Motivation

- First-order logic is a powerful language to represent complex relational information
- Probability is the standard way to represent uncertainty in knowledge
- Combining the two would allow to model complex probabilistic relationships in the domain of interest

Combining logic with probability

logic graphical models

- Graphical models are a mean to represent joint probabilities highlighting the relational structure among variables
- A compressed representation of such models can be obtained using templates, cliques in the graphs sharing common parameters (e.g. as in HMM for BN or CRF for MN)
- Logic can be seen as a language to build templates for graphical models
- Logic based versions of HMM, BN and MN have been defined

First-order logic (in a nutshell)

Symbols

- *Constant* symbols representing objects in the domain (e.g. `Nick, Polly`)
- *Variable* symbols which take objects in the domain as values (e.g. `x,y`)
- *Function* symbols which mapping tuples of objects to objects (e.g. `BandOf`). Each function symbol has an arity (i.e. number of arguments)
- *Predicate* symbols representing relations among objects or object attributes (e.g. `Singer, SangTogether`). Each predicate symbol has an arity.

First-order logic

Terms

- A *term* is an expression representing an object in the domain. It can be:
  - A constant (e.g. `Niel`)
  - A variable (e.g. `x`)
  - A function applied to a tuple of objects. E.g.: `BandOf(Niel)`, `SonOf(f,m)`, `Age(MotherOf(John))`

First-order logic

Formulas

- A (well formed) *atomic formula* (or *atom*) is a predicate applied to a tuple of objects. E.g.:
  `Singer(Nick),SangTogether(Nick,Polly)`
  `Friends(X,BrotherOf(Emy))`
- Composite formulas are constructed from atomic formulas using logical connectives and quantifiers

First-order logic

Connectives

negation	$\neg F$: true iff formula F is false
conjunction	$F_1 \wedge F_2$: true iff both formulas F_1, F_2 are true
disjunction	$F_1 \vee F_2$: true iff at least one of the two formulas F_1, F_2 is true
implication	$F_1 \Rightarrow F_2$ true iff F_1 is false or F_2 is true (same as $F_2 \vee \neg F_1$)
equivalence	$F_1 \Leftrightarrow F_2$ true iff F_1 and F_2 are both true or both false (same as $(F_1 \Rightarrow F_2) \wedge (F_2 \Rightarrow F_1)$)

Literals

- A *positive literal* is an atomic formula
- A *negative literal* is a negated atomic formula

First-order logic

Quantifiers

existential quantifier $\exists x\ F_1$: true iff F_1 is true for at least one object x in the domain. E.g.:
$\exists x$ `Friends(x,BrotherOf(Emy))`

universal quantifier $\forall x\ F_1$: true iff F_1 is true for all objects x in the domain. E.g.:
$\forall x$ `Friends(x,BrotherOf(Emy))`

Scope

- The *scope* of a quantifier in a certain formula is the (sub)formula to which the quantifiers applies

First-order logic

Precedence

- Quantifiers have the highest precedence
- Negation has higher precedence than other connectives
- Conjunction has higher precedence than disjunction
- Disjunction have higher precedence than implication and equivalence
- Precedence rules can as usual be overruled using parentheses

Examples

- *Emy and her brother have no common friends*:

$$\neg\exists x\,(\texttt{Friends(x,Emy)} \land \texttt{Friends(x,BrotherOf(Emy))})$$

- *All birds fly*:

$$\forall x\,(\texttt{Bird(x)} \Rightarrow \texttt{Flies(x)})$$

First-order logic

Closed formulas

- A variable-occurrence within the scope of a quantifier is called *bound*. E.g. x in:

$$\forall x\ (\texttt{Bird(x)} \Rightarrow \texttt{Flies(x)})$$

- A variable-occurence outside the scope of any quantifier is called *free*. e.g. y in:

$$\neg\exists x\ (\texttt{Friends(x,Emy)} \land \texttt{Friends(x,y)})$$

- A *closed* formula is a formula which contains no free occurrence of variables

Note

- We will be interested in closed formulas only

First-order logic

Ground terms and formulas

- A *ground* term is a term containing no variables
- A *ground* formula is a formula made of only ground terms

First-order logic

First order language

- The set of symbols (constants, variables, functions, predicates, connectives, quantifiers) constitute a first-order *alphabet*
- A first order *language* given by the alphabet is the set of formulas which can be constructed from symbols in the alphabet

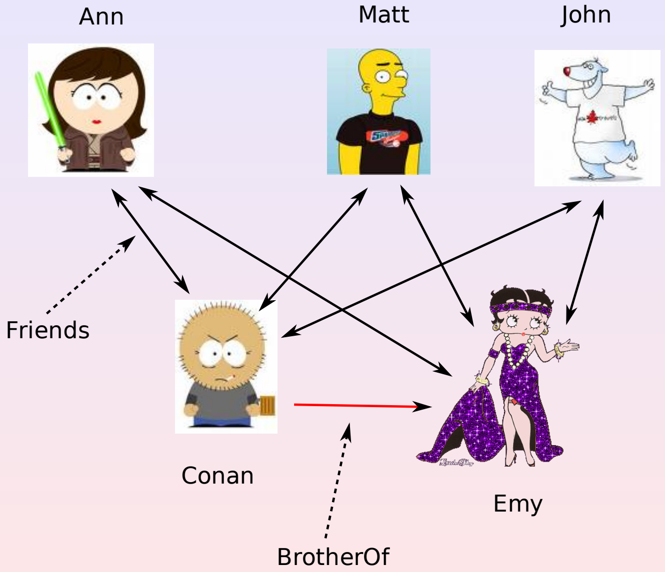
Knowledge base (KB)

- A first-order knowledge base is a set of formulas
- Formulas in the KB are implicitly conjoined
- A KB can thus be seen as a single large formula

Interpretation

- An *interpretation* provides *semantics* to a first order language by:
  1. defining a domain containing all possible objects
  2. mapping each ground term to an object in the domain
  3. assigning a truth value to each ground atomic formula (a *possible world*)
- The truth value of complex formulas can be obtained combining interpretation assignments with connective and quantifier rules

First-order logic: example

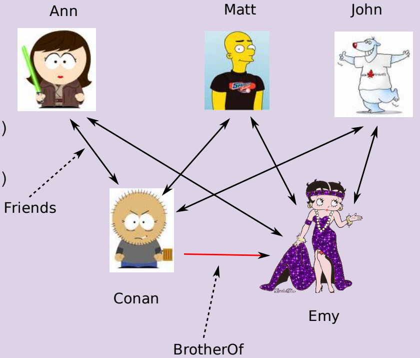


First-order logic: example

$\neg\exists x\,(\texttt{Friends(x,Emy)} \wedge \neg\texttt{Friends(x,BrotherOf(Emy))})$

- The formula is true under the interpretation as the following atomic formulas are true:

```
Friends(Ann, Emy)
Friends(Ann,BrotherOf(Emy))
Friends(Matt, Emy)
Friends(Matt,BrotherOf(Emy))
Friends(John, Emy)
Friends(John,BrotherOf(Emy))
```



First-order logic

Types

- Objects can by *typed* (e.g. people, cities, animals)
- A typed variable can only range over objects of the corresponding type
- a typed term can only take arguments from the corresponding type. E.g. `MotherOf(John),MotherOf(Amy)`

First-order logic

Inference in first-order logic

- A formula F is *satisfiable* iff there exists an intepretation under which the formula is true
- A formula F is *entailed* by a KB iff is is true for all interpretations for which the KB is true. We write it:

$$KB \models F$$

  the formula is a logical consequence of KB, not depending on the particular interpretation
- Logical entailment is usually done by *refutation*: proving that $KB \wedge \neg F$ is unsatisfiable

Note

- Logical entailment allows to *extend* a KB inferring new formulas which are true for the same interpretations for which the KB is true

First-order logic

Clausal form

- The *clausal* form or *conjunctive normal form* (CNF) is a regular form to represent formulas which is convenient for automated inference:
  - A *clause* is a disjunction of literals.
  - A KB in CNF is a conjunction of clauses.
- Variables in KB in CNF are always implicitly assumed to be universally quantified.
- Any KB can be converted in CNF by a mechanical sequence of steps
- Existential quantifiers are replaced by Skolem constants or functions

Conversion to clausal form: example

First Order Logic	Clausal Form
"Every bird flies" $\forall x\ (\texttt{Bird(x)} \Rightarrow \texttt{Flies(x)})$	$\texttt{Flies(x)} \vee \neg\texttt{Bird(x)}$
"Every predator of a bird is a bird" $\forall x, y\ (\texttt{Predates(x,y)} \wedge \texttt{Bird(y)} \Rightarrow \texttt{Bird(x)})$	$\texttt{Bird(x)} \vee \neg\texttt{Bird(y)} \vee$ $\neg\texttt{Predates(x,y)}$
"Every prey has a predator" $\forall y\ (\texttt{Prey(y)} \Rightarrow \exists x\ \texttt{Predates(x,y)})$	$\texttt{Predates(PredatorOf(y),y)} \vee$ $\neg\texttt{Prey(y)}$

First-order logic

Problem of uncertainty

- In most real world scenarios, logic formulas are *typically* but not *always* true
- For instance:
  - “Every bird flies” : what about an ostrich (or Charlie Parker) ?
  - “Every predator of a bird is a bird”: what about lions with ostriches (or heroin with Parker) ?
  - “Every prey has a predator”: predators can be extinct
- A world failing to satisfy even a single formula would not be possible
- there could be no possible world satisfying all formulas

Handling uncertainty

- We can relax the hard constraint assumption on satisfying all formulas
- A possible world not satisfying a certain formula will simply be less likely
- The more formula a possible world satisfies, the more likely it is
- Each formula can have a weight indicating how strong a constraint it should be for possible worlds
- Higher weight indicates higher probability of a world satisfying the formula wrt one not satisfying it

Markov Logic networks

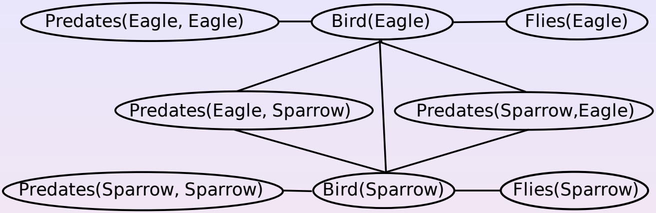
Definition

- A Markov Logic Network (MLN) L is a set of pairs (F_i, w_i) where:
  - F_i is a formula in first-order logic
  - w_i is a real number (the weight of the formula)
- Applied to a finite set of constants $C = \{c_1, \ldots, c_{|C|}\}$ it defines a Markov network $M_{L,C}$:
  - $M_{L,C}$ has one binary node for each possible grounding of each atom in L. The value of the node is 1 if the ground atom is true, 0 otherwise.
  - $M_{L,C}$ has one feature for each possible grounding of each formula F_i in L. The value of the feature is 1 if the ground formula is true, 0 otherwise. The weight of the feature is the weight w_i of the corresponding formula

Intuition

- A MLN is a *template* for Markov Networks, based on logical descriptions
- Single atoms in the template will generate nodes in the network
- Formulas in the template will be generate cliques in the network
- There is an edge between two nodes iff the corresponding ground atoms appear together in at least one grounding of a formula in L

Markov Logic networks: example



Ground network

- A MLN with two (weighted) formulas:

$$w_1 \quad \forall x \, (\texttt{Bird(x)} \Rightarrow \texttt{Flies(x)})$$

$$w_2 \quad \forall x, y \, (\texttt{Predates(x,y)} \land \texttt{Bird(y)} \Rightarrow \texttt{Bird(x)})$$

- applied to a set of two constants $\{\texttt{Sparrow}, \texttt{Eagle}\}$
- generates the Markov Network shown in figure

Markov Logic networks

Joint probability

- A ground MLN specifies a joint probability distribution over possible worlds (i.e. truth value assignments to all ground atoms)
- The probability of a possible world x is:

$$p(x) = \frac{1}{Z} \exp\left(\sum_{i=1}^{F} w_i n_i(x)\right)$$

where:

- the sum ranges over formulas in the MLN (i.e. clique templates in the Markov Network)
- $n_i(x)$ is the number of true groundings of formula F_i in x
- The partition function Z sums over all possible worlds (i.e. all possible combination of truth assignments to ground atoms)

Markov Logic networks

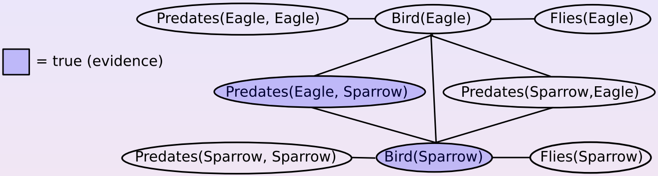
Adding evidence

- Evidence is usually available for a subset of the ground atoms (as their truth value assignment)
- The MLN can be used to compute the conditional probability of a possible world x (consistent with the evidence) given evidence e:

$$p(x|e) = \frac{1}{Z(e)} \exp\left(\sum_{i=1}^{F} w_i n_i(x)\right)$$

- where the partition function $Z(e)$ sums over all possible worlds consistent with the evidence.

Example: evidence

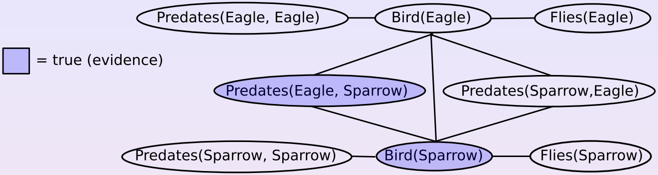


Including evidence

- Suppose that we have (true) evidence *e* given by these two facts:

  `Bird(Sparrow)`
  `predates(Eagle,Sparrow)`

Example: assignment 1



Computing probability

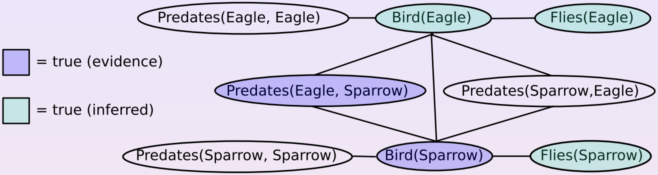
$$p(x) = \frac{1}{Z} \exp(w_1 + 3w_2)$$

- The probability of a world with only evidence atoms set as true violates two ground formulas:

$\texttt{Bird(Sparrow)} \Rightarrow \texttt{Flies(Sparrow)}$

$\texttt{Predates(Eagle,Sparrow)} \wedge \texttt{Bird(Sparrow)} \Rightarrow \texttt{Bird(Eagle)}$

Example: assignment 2

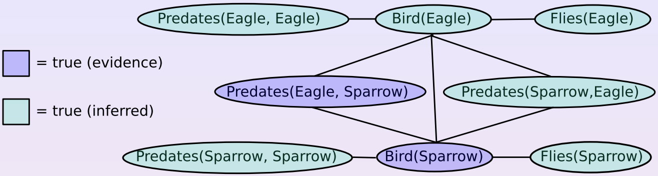


Computing probability

$$p(x) = \frac{1}{Z} \exp(2w_1 + 4w_2)$$

- This possible world is the most likely among all possible worlds as it satisfies all constraints.

Example: assignment 3



Computing probability

$$p(x) = \frac{1}{Z} \exp(2w_1 + 4w_2)$$

- This possible world has also highest probability.
- The problem is that we did not encode constraints saying that:
  - A bird is not likely to be predator of itself
  - A prey is not likely to be predator of its predator

Hard constraints

Impossible worlds

- It is always possible to make certain worlds impossible by adding constraints with infinite weight
- Infinite weight constraints behave like pure logic formulas: any possible world has to satisfy them, otherwise it receives zero probability

Example

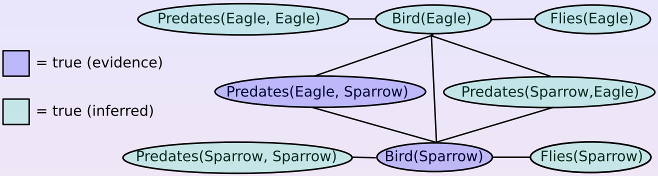
- Let's add the infinite weight constraint:

  "Nobody can be a self-predator"

  $w_3 \quad \forall x \; \neg \texttt{Predates(X,X)}$

  to the previous example

Hard constraint: assignment 3

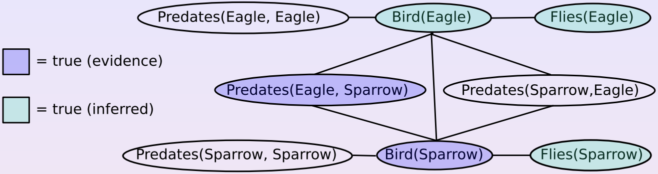


Computing joint probability

$$p(x) = \frac{1}{Z} \exp(2w_1 + 4w_2) = 0$$

- The numerator does not contain w_3, as the no-self-predator constraint is never satified
- However the partition function Z sums over all possible worlds, including those in which the constraint is satisfied.
- As $w_3 = \infty$, the partition function takes infinite value and the possible worlds gets zero probability.

Hard constraint: assignment 2



Computing joint probability

$$p(x) = \frac{1}{Z}\exp(2w_1 + 4w_2 + 2w_3) \neq 0$$

- The only non-zero probability possible worlds are those always satisying hard constraints
- Infinite weight features cancel out between numerator and possible worlds at denominator which also satisfy the constraints, while those which do not become zero

Inference

Assumptions

- For simplicity of presentation, we will consider MLN in form:
    - function-free (only predicates)
    - clausal
- However the methods can be applied to other forms as well
- We will use general first-order logic form when describing applications

Inference

MPE inference

- One of the basic tasks consists of predicting the most probable state of the world given some evidence (the *most probable explanation*)
- The problem is a special case of MAP inference (*maximum a posteriori* inference), in which we are interested in the state of a subset of variables which do not necessarily include all those without evidence.

Inference

MPE inference in MLN

- MPE inference in MLN reduces to finding the truth assignment for variables (i.e. nodes) without evidence maximizing the weighted sum of satisfied clauses (i.e. features)
- The problem can be addressed with any weighted satisfiability solver
- MaxWalkSAT has been successfully used for MPE inference in MLN.

MaxWalkSAT

Description

- Weighted version of WalkSAT
- Stochastic local search algorithm:
  1. Pick an unsatisfied clause at random
  2. Flip the truth value of an atom in the clause
- The atom to flip is chosen in one of two possible ways with a certain probability:
  - randomly
  - in order to maximize the weighted sum of the clauses satisfied with the flip
- The stochastic behaviour (hopefully) allows to escape local minima

MaxWalkSAT pseudocode

```
procedure MaxWalkSAT(weighted_clauses, max_flips, max_tries, target, p)
    vars ← variables in weighted_clauses
    for i ← 1 to max_tries do
        soln ← a random truth assignment to vars
        cost ← sum of weights of unsatisfied clauses in soln
        for j ← 1 to max_flips do
            if cost ≤ target then
                return "Success, solution is", soln
            end if
            c ← a randomly chosen unsatisfied clause
            if Uniform(0,1) < p then
                v_f ← a randomly chosen variable from c
            else
                for all variable v in c do
                    compute DeltaCost(v)
                end for
                v_f ← v with lowest DeltaCost(v)
            end if
            soln ← soln with v_f flipped
            cost ← cost + DeltaCost(v_f)
        end for
    end for
    return "Failure, best assignment is", best soln found
end procedure
```

MaxWalkSAT

Ingredients

- *target* is the maximum cost considered acceptable for a solution
- *max_tries* is the number of walk restarts
- *max_flips* is the number of flips in a single walk
- *p* is the probability of flipping a random variable
- Uniform(0,1) picks a number uniformly at random from [0,1]
- DeltaCost(*v*) computes the change in cost obtained by flipping variable *v* in the current solution

Marginal and conditional probabilities

- Another basic inference task is that of computing the marginal probability that a formula holds, possibly given some evidence on the truth value of other formulas
- Exact inference in generic MLN is intractable (as it is for the generic MN obtained by the grounding)
- MCMC sampling techniques have been used as an approximate alternative

Constructing the ground MN

- In order to perform a specific inference task, it is not necessary in general to ground the whole network, as parts of it could have no influence on the computation of the desired probability
- Grounding only the needed part of the network can allow significant savings both in memory and in time to run the inference

Partial grounding: intuition

- A standard inference task is that of computing the probability that F_1 holds given that F_2 does.
- We will focus on the common simple case in which F_1, F_2 are conjunctions of ground literals:
  1. All atoms in F_1 are added to the network one after the other
  2. If an atom is also in F_2 (has evidence), nothing more is needed for it
  3. Otherwise, its Markov blanket is added, and each atom in the blanket is checked in the same way

Partial grounding: pseudocode

procedure CONSTRUCTNETWORK(F_1,F_2,L,C)
inputs:
F_1 a set of query ground atoms
F_2 a set of evidence ground atoms
L a Markov Logic Network
C a set of constants
output: M a ground Markov Network
calls: $MB(q)$ the Markov blanket of q in $M_{L,C}$
  $G \leftarrow F_1$
  while $F_1 \neq \emptyset$ **do**
    for all $q \in F_1$ **do**
      if $q \notin F_2$ **then**
        $F_1 \leftarrow F_1 \cup (MB(q) \setminus G)$
        $G \leftarrow G \cup MB(q)$
      end if
      $F_1 \leftarrow F_1 \setminus \{q\}$
    end for
  end while
  return M the ground MN composed of all nodes in G and all arcs between them in $M_{L,C}$, with features and weights of the corresponding cliques
end procedure

Inference

Gibbs sampling

- Inference in the partial ground network is done by Gibbs sampling.
- The basic step consists of sampling a ground atom given its Markov blanket
- The probability of X_l given that its Markov blanket has state $\boldsymbol{B}_l = \boldsymbol{b}_l$ is $p(X_l = x_l | \boldsymbol{B}_l = \boldsymbol{b}_l) =$

$$\frac{\exp \sum_{f_i \in F_l} w_i f_i(X_l = x_l, \boldsymbol{B}_l = \boldsymbol{b}_l)}{\exp \sum_{f_i \in F_l} w_i f_i(X_l = 0, \boldsymbol{B}_l = \boldsymbol{b}_l) + \exp \sum_{f_i \in F_l} w_i f_i(X_l = 1, \boldsymbol{B}_l = \boldsymbol{b}_l)}$$

  where:
  - F_l is the set of ground formulas containing X_l
  - $f_i(X_l = x_l, \boldsymbol{B}_l = \boldsymbol{b}_l)$ is the truth value of the ith formula when $X_l = x_l$ and $\boldsymbol{B}_l = \boldsymbol{b}_l$
- The probability of the conjuction of literals is the fraction of samples (at chain convergence) in which all literals are true

Inference

Multimodal distributions

- As the distribution is likely to have many modes, multiple independently initialized chains are run
- Efficiency in modeling the multimodal distribution can be obtained starting each chain from a mode reached using MaxWalkSAT

Handling hard constraints

- Hard constraints break the space of possible worlds into separate regions
- This violate the MCMC assumption of reachability
- Very strong constraints create areas of very low probability difficult to traverse
- The problem can be addressed by *slice sampling* MCMC, a technique aimed at sampling from slices of the distribution with a frequency proportional to the probability of the slice

Learning

Maximum likelihood parameter estimation

- Parameter estimation amounts at learning weights of formulas
- We can learn weights from training examples as possible worlds.
- Let's consider a single possible world as training example, made of:
  - a set of constants $\mathcal{C}$ defining a specific MN from the MLN
  - a truth value for each ground atom in the resulting MN
- We usually make a *closed world* assumption, where we only specify the true ground atoms, while all others are assumed to be false.
- As all groundings of the same formula will share the same weight, learning can be also done on a single possible world

Learning

Maximum likelihood parameter estimation

- Weights of formulas can be learned maximizing the likelihood of the possible world:

$$w^{\max} = \mathrm{argmax}_w p_w(x) = \mathrm{argmax}_w \frac{1}{Z} \exp\left(\sum_{i=1}^{F} w_i n_i(x)\right)$$

- As usual we will equivalenty maximize the log-likelihood:

$$\log(p_w(x)) = \sum_{i=1}^{F} w_i n_i(x) - \log(Z)$$

Priors

- In order to combat overfitting Gaussian priors can be added to the weights as usual (see CRF)

Maximum likelihood parameter estimation

- The gradient of the log-likelihood wrt weights becomes:

$$\frac{\partial}{\partial w_i} \log p_w(x) = n_i(x) - \sum_{x'} p_w(x') n_i(x')$$

  where the sum is over all possible worlds x', i.e. all possible truth assignments for ground atoms in the MN
- Note that $p_w(x')$ is computed using the current parameter values w
- The i-th component of the gradient is the difference between number of true grounding of the i-th formula, and its expectation according to the current model

Applications

Entity resolution

- Determine which observations (e.g. noun phrases in texts) correspond to the same real-world object
- Typically addressed creating feature vectors for pairs of occurrences, and training a classifier to predict whether they match
- The pairwise approach doesn't model information on multiple related objects (e.g. if two bibliographic entries correspond to the same paper, the authors are also the same)
- Some implications hold only with a certain probability (e.g. if two authors in two bibliographic entries are the same, the entries are more likely to refer to the same paper)

Applications

MLN for entity resolution

- MLN can be used to address entity resolution tasks by:
  - not assuming that distinct names correspond to distinct objects
  - adding an equality predicate and its axioms: reflexivity, symmetry, transitivity
- Implications related to the equality predicate can be:
  - grounding of a predicate with equal constants have same truth value
  - constants appearing in a ground predicate with equal constants are equal (i.e. the “same paper $\rightarrow$ same author” implication, which holds only probabilistically in general)

Applications

MLN for entity resolution

- Weights for different instances of such axioms can be learned from data
- Inference is performed adding evidence on entity properties and relations, and querying for equality atoms
- The network performs *collective* entity resolution, as the most probable resolution for all entities is jointly produced

Entity resolution

Entity resolution in citation databases

- Each citation has: author, title, venue fields.
- Citation to field relations:

  `Author(bib,author)` `Title(bib,title)`
  `Venue(bib,venue)`

- field content relations:

  `HasWord(author,word)` `HasWord(title,word)`
  `HasWord(venue,word)`

- equivalence relations:

  `SameAuthor(author1,author2)`
  `SameTitle(title1,title2)`
  `SameVenue(venue1,venue2)`
  `SameBib(bib1,bib2)`

Entity resolution

Same words imply same entity

- E.g.:

$$\texttt{Title(b1,t1)} \wedge \texttt{Title(b2,t2)} \wedge \texttt{HasWord(t1,+w)} \wedge \texttt{HasWord(t2,+w)} \Rightarrow \texttt{SameBib(b1,b2)}$$

- here the '+' operator is a template: a rule is generated for each constant of the appropriate type (i.e. words)
- a separate weight is learned for separate words (e.g. stopwords like articles or prepositions are probably less informative than other words)

transitivity

- E.g.:

$$\texttt{SameBib(b1,b2)} \wedge \texttt{SameBib(b2,b3)} \Rightarrow \texttt{SameBib(b1,b3)}$$

Entity resolution

transitivity across entities

- E.g.:

```
Author(b1,a1) ∧ Author(b2,a2) ∧ SameBib(b1,b2)
   ⇒ SameAuthor(a1,a2)

Author(b1,a1) ∧ Author(b2,a2) ∧
   SameAuthor(a1,a2) ⇒ Samebib(b1,b2)
```

- The second rule is not a valid logic rule, but holds probabilistically (citations with same authors are more likely to be the same)

Resources

References

- Domingos, Pedro and Kok, Stanley and Lowd, Daniel and Poon, Hoifung and Richardson, Matthew and Singla, Parag (2007). *Markov Logic*. In L. De Raedt, P. Frasconi, K. Kersting and S. Muggleton (eds.), Probabilistic Inductive Logic Programming. New York: Springer.

Software

- The open source Alchemy system provides an implementation of MLN, with example networks for a number of tasks:
  - http://alchemy.cs.washington.edu/