CSE 5015 (Fall 2025) Assignment 2

Due Wednesday, 8 October 2025

1. We are going to consider a Bayesian hypothesis test that a coin is exactly fair. Let θ be the "heads probability" of a coin
$$\Pr(\mathrm{H}) = \theta.$$
Let us place a somewhat atypical prior on θ that includes nontrivial prior probability that the coin is exactly fair (that is, $\theta = 1/2$):
$$\Pr(\theta = 1/2) = 0.5 \qquad p(\theta \mid \theta \neq 1/2) = \mathcal{U}[0, 1].$$
This can be interpreted as a prior that is a mixture between a uniform prior on the interval $[0, 1]$ with weight 0.5 (this is also a special of the beta distribution with $\alpha = \beta = 1$) and a "point mass" (Dirac delta function) at $\theta = 1/2$ with weight 0.5.

   Suppose we flip a coin $n = 10$ times and observe $x = 6$ "heads." What is the posterior distribution $p(\theta \mid x, n)$? What is the posterior probability that the coin is exactly fair?

   Note there is no probability density function corresponding to the prior. It will help to work in cases.

2. (Effect of weird priors.) Let us consider the following set of observations. We flip a coin independently $n = 1\,000$ times and observe $x = 900$ successes. Call the unknown bias of the coin $\theta \in (0, 1)$.

   For each of the prior distributions $p(\theta)$ below, please:

   - plot the prior distribution $p(\theta)$ over the range $0 < \theta < 1$
   - plot the posterior distribution given the above data, $p(\theta \mid \mathcal{D})$, over the range $0 < \theta < 1$
   - report the posterior mean, $\mathbb{E}[\theta \mid \mathcal{D}] = \int \theta \, p(\theta \mid \mathcal{D}) \, \mathrm{d}\theta$. (Here you may want to look up the posterior mean of a beta distribution.)

   (a) A uniform prior on θ, which can be realized by selecting the beta distribution with $\alpha = \beta = 1$:
$$p(\theta) = \mathcal{B}(\theta; \alpha = 1, \beta = 1).$$
   (b) A prior with extreme bias toward small values of θ:
$$p(\theta) = \mathcal{B}(\theta; \alpha = 1, \beta = 100).$$
   (c) A prior that has no support on values greater than $\theta = 1/2$:
$$p(\theta) = \begin{cases} 2 & \theta < 1/2; \\ 0 & \theta \geq 1/2. \end{cases}$$

3. (Optimal *Price is Right* bidding.) Suppose you have a standard normal belief about an unknown parameter θ, $p(\theta) = \mathcal{N}(\theta; 0, 1^2)$. You are asked to give a point estimate $\hat{\theta}$ of θ, but are told that there is a heavy penalty for guessing too high. The loss function is
$$\ell(\hat{\theta}, \theta; c) = \begin{cases} (\theta - \hat{\theta})^2 & \hat{\theta} < \theta; \\ c & \hat{\theta} \geq \theta \end{cases},$$

where $c > 0$ is a constant cost for overestimating. What is the Bayesian estimator in this case? How does it change as a function of c? Plot the optimal action as a function of c for $0 < c < 10$.

Hint: you should minimize certain expressions you encounter numerically as an analytic solution may not be available / advisable.