Homework 1

Tuesday, January 30, 2024 4:10 AM

UNC COLLEGE OF ARTS & SCIENCES

STOR415: INTRODUCTION TO OPTIMIZATION

DEPARTMENT OF STATISTICS AND OPERATIONS RESEARCH

SPRING 2024

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HOMEWORK 1: LINEAR ALGEBRA REVIEW AND BASIC LP FORMULATIONS

Question 1. (**5 points**): Let $A = \begin{bmatrix} 4 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & -3 \end{bmatrix}$. Compute the product AB using the definition of matrix multiplication.

Question 2. (**15 points**): We are given matrices

$$A = \begin{bmatrix} 2 & b & a \\ z & 4 & u \\ g & c & f \end{bmatrix}, \quad B = \begin{bmatrix} 3 & v & k \\ k & \ell & w \\ v & t & s \end{bmatrix}.$$

The numbers a, b, c, etc. are unknown constants.

1. Suppose $C = AB$ and the elements of C are denoted by C_{ij}. What is C_{12}? What is C_{23}? What is C_{32}?
2. Suppose $C = A^T B$ and the elements of C are denoted by C_{ij}. What is C_{12}? What is C_{23}? What is C_{32}?
3. Suppose $C = A^T B^T$ and the elements of C are denoted by C_{ij}. What is C_{12}? What is C_{23}? What is C_{32}?

Question 3. (**10 points**): I have up to \$1000 to invest. I can invest my money in stocks and bonds. Each dollar invested in stocks yields 15 cents of profit, and each dollar invested in bonds yields 5 cents of profit. No more than 25% of all money invested can be in stocks, and at least \$450 must be in bonds. Formulate an LP to maximize total profit earned from my investment.

Question 4. (**25 points**): Kooky Candy Company makes 2 kinds of gourmet chocolate bars. The milk chocolate bar sells for \$1.50, and the dark chocolate bar sells for \$2.00. Both require cocoa butter and almonds, which are currently available in limited supply (other required ingredients are available in essentially unlimited quantities). Kooky Candy can obtain 50,000 oz. of cocoa butter, and 25,000 oz. of almonds. A milk chocolate bar requires 2 oz. of cocoa butter and 1 oz of almonds, while a dark chocolate bar requires 5 oz. of cocoa butter and 1 oz. of almonds.

a) Formulate an LP to maximize the total revenue of the Kooky Candy Company.

b) Convert your solution to standard form. Then, convert the standard form into the matrix form, i.e. clearly state the matrix A and vectors b and c in the standard form.

Question 5. (45 points): Consider the following table indicating the nutritional value of different food types:

Foods	Price ($) per Serving	Calories per Serving	Fat (g) per Serving	Protein (g) per Serving	Carbohydrate (g) per Serving
Raw carrots	0.14	23	0.1	0.6	6
Baked potatoes	0.12	171	0.2	3.7	30
Wheat bread	0.2	65	0	2.2	13
Cheddar cheese	0.75	112	9.3	7	0
Peanut butter	0.15	188	16	7.7	2

You need to decide how many servings of each food to buy each day so that you minimize the total cost of buying your food while satisfying the following daily nutritional requirements:

- Calories must be at least 2000,
- Fat must be at least 50g,
- Protein must be at least 100g,
- Carbohydrates must be at least 250g.

Formulate an LP to determine how many servings of each of the aforementioned foods meet all of the nutrional requirements, while minimizing the total cost of food (you may buy fractional numbers of servings).

The end

1. Let $A = \begin{bmatrix} 4 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & -3 \end{bmatrix}$

Compute AB using def. of matrix multiplication.

$$\begin{bmatrix} 4 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 10 & -7 \\ 10 & -9 \end{bmatrix}$$

2. we are given matrices

$$A = \begin{bmatrix} z & b & a \\ z & 4 & u \\ g & c & f \end{bmatrix} \qquad B = \begin{bmatrix} 3 & v & k \\ k & l & w \\ v & t & s \end{bmatrix}$$

Let $a, b, c, \ldots$ be unknown constants

a. Suppose $C = AB$ and the elements of C are denoted by c_{ij}. what is c_{12}, c_{23}, c_{32}?

(12: row 1, column 2)

$$\underset{A}{\begin{bmatrix} 2 & b & a \\ z & 4 & u \\ g & c & f \end{bmatrix}} \underset{B}{\begin{bmatrix} 3 & v & k \\ k & l & w \\ v & t & s \end{bmatrix}} = \underset{C}{\begin{bmatrix} 6+bk+av & 2v+bl+at & 2k+bw+as \\ 3z+4k+uv & zv+4l+ut & zk+4w+us \\ 3g+ck+fv & gv+cl+ft & gk+cw+fs \end{bmatrix}}$$

$$c_{12} = 2v + bl + at$$

$$c_{23} = zk + 4w + us$$

$$c_{32} = gv + cl + ft$$

b. Suppose $C = A^T B$ and the elements of C are denoted by c_{ij}. What is c_{12}? what is c_{23}? what is c_{32}?

$$\underset{A^T}{\begin{bmatrix} 2 & z & g \\ b & 4 & c \\ a & u & f \end{bmatrix}} \underset{B}{\begin{bmatrix} 3 & v & k \\ k & l & w \\ v & t & s \end{bmatrix}} = \underset{C}{\begin{bmatrix} 6+zk+gv & 2v+zl+gt & 2k+zw+gs \\ 3b+4k+cv & bv+4l+ct & bk+4w+cs \\ 3a+uk+fv & av+ul+ft & ak+uw+fs \end{bmatrix}}$$

$$c_{12} = 2v + zl + gt$$

$$C_{23} = bk + 4u + cs$$

$$C_{32} = av + ue + ft$$

C. Suppose $C = A^T B^T$ and the elements of C are denoted C_{ij}. What is C_{12}? What is C_{23}? What is C_{32}?

$$\underbrace{\begin{bmatrix} 2 & z & g \\ b & 4 & c \\ a & u & f \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} 3 & k & v \\ v & e & t \\ k & w & s \end{bmatrix}}_{B^T} = \begin{bmatrix} 6 + 2v + gk & 2k + ze + gw & 2v + zt + gs \\ 3b + 4v + ck & bk + 4e + cw & bv + 4t + cs \\ 3a + uv + fk & ak + ue + fw & av + ut + fs \end{bmatrix}$$

$C_{12} = 2k + ze + gw$ $C_{23} = bv + 4t + cs$ $C_{32} = ak + ue + fw$

3. I have up to \$1,000 to invest. I can invest my money in stocks and bonds. Each dollar invested in stocks yields 15 cents of profit, and each dollar invested in bonds yields 5 cents of profit. No more than 25% of all money invested can be in stocks, and at least \$450 must be in bonds. Formulate an LP to maximize total profit earned from my investment.

$$\max z = 0.15x_1 + 0.05x_2$$

$$\text{S.t.} \quad x_1 + x_2 \leq 1{,}000$$

$$x_1 \leq 0.25(x_1 + x_2)$$

$$x_2 \geq 450$$

$$x_1 \geq 0$$

4. Kooky Candy Company makes 2 kinds of gourmet chocolate bars. The milk chocolate bar sells for $1.50; and the dark chocolate bar sells for $2.00. Both require cocoa butter and almonds, which are currently available in limited supply (other ingredients have unlimited quantity). Kooky Candy can obtain 50,000 oz of cocoa butter, and 25,000 oz. of almonds. A milk chocolate bar requires 2 oz. of cocoa butter and 1 oz of almonds, while a dark chocolate bar requires 5 oz of cocoa butter and 1 oz. of almonds.

a. Formulate an LP to maximize the total revenue of the Kooky Candy Company.

x_1 = milk chocolate bar x_2 = dark chocolate bar

$$\max \quad Z = 1.5x_1 + 2x_2$$

$$\text{S.t.} \quad 2x_1 + 5x_2 \leq 50{,}000$$

$$x_1 + x_2 \leq 25{,}000$$

$$x_1, x_2 \geq 0$$

6) Convert your solution to standard form.

3.3.1 The standard form

An LP is said to be in the **standard form** if it satisfies the following conditions simultaneously:

1. Each variable x_i in the LP is subject to the nonnegativity sign restriction (i.e., $x_i \geq 0$ for $i = 1, \cdots, n$).
2. Each other constraint in addition to the nonnegativity constraints is an equality constraint of the form $\sum_{j=1}^{n} a_{ij}x_j = b_i$ for some $i = 1, \cdots, m$.

→ convert to minimization problem

→ multiply objective by -1

$$\min Z = -1(1.5X_1 + 2X_2) = -1.5X_1 - 2X_2$$

→ Introduce slack variables to convert inequalities to equalities

$$\text{S.t. } 2X_1 + 5X_2 + S_1 = 50{,}000$$

$$X_1 + X_2 + S_2 = 25{,}000$$

$$X_1, X_2, S_1, S_2 \geq 0$$

5. Consider the following table indicating the nutritional value of different food types:

Food	Price ($) per serving	Cal. per serving	Fat (g) per serving	Protein (g) per serving	Carb (g) per serving
X_1 = Raw Carrots	0.14	23	0.1	0.6	6
X_2 = Baked Potatoes	0.12	171	0.2	3.7	30
X_3 = Wheat Bread	0.2	65	0	2.2	13
X_4 = Cheddar Cheese	0.75	112	9.3	7	0
X_5 = Peanut Butter	0.15	188	16	7.7	2

$$\min Z = 0.14X_1 + 0.12X_2 + 0.2X_3 + 0.75X_4 + 0.15X_5$$

$$\text{s.t. } 23X_1 + 171X_2 + 65X_3 + 112X_4 + 188X_5 \geq 2$$

$$23x_1 + 171x_2 + 65x_3 + 112x_4 + 188x_5 \geq 2{,}000$$

$$0.1x_1 + 0.2x_2 + \quad 9.3x_4 + 16x_5 \geq 50$$

$$0.6x_1 + 3.7x_2 + 2.2x_3 + 7x_4 + 2.7x_5 \geq 100$$

$$6x_1 + 30x_2 + 13x_3 + \quad 2x_5 \geq 250$$

$$x_1, \cdots, x_5 \geq 0$$