# Homework 4

Wednesday, February 21, 2024 8:11 PM

**STOR415: Introduction to Optimization**
**Department of Statistics and Operations Research**
———————— **Fall 2022** ————————

Instructor: Michael O'Neill

## Homework 4: MCNFP and Graphical Solutions

- For this homework, you will need to submit both handwritten work and the output and code from Jupyter notebooks. Gradescope only allows the submission of a single PDF, however, you can combine multiple PDFs into one using free tools online or Adobe Acrobat, which all UNC students should have access to. When submitting this assignment, please assign pages to specific questions to make it easier to grade.
- For each coding problem, create an ipynb with exactly the same name as is required in the problem. In the Julia code, declare variables with the name given in the problem. Then, after solving the problem, in the last cell of your notebook print (or use @show) all the values of all of the variables in your optimization problem as well as the value of the objective function.
- Please comment add comments to your code describing the variables, constraints and objective function of your model.
- Ensure that your notebook runs properly before submitting it. In the main bar, perform **Clear Outputs of All Cells** then **Run All** to ensure that there are no errors.
- To generate a PDF of your notebook:
  - In the main bar, click **Export** (may be hidden behind a 3 dots dropdown menu)
  - Choose Export as PDF (may require additional extensions).
  - If Export as PDF fails or does not give proper output, export the file as HTML, open this HTML file in a web browser and save the HTML file as a PDF.
  - If you have issues with this, first open the folder containing your notebooks in VSCode (instead of just the notebook itself), then try exporting to PDF. This often fixes issues that may otherwise occur.
  - If you cannot get exporting to work in VSCode, you may use a Jupyter Notebook to PDF con-

verter online.

**Question 1**. (**40 points**):

Oilco has oil fields in Los Angeles and San Diego. The Los Angeles field can produce 400,000 barrels per day and the San Diego field can produce 500,000 barrels per day.

Oil is sent from the fields to a refinery, in either Dallas or Houston (assume each refinery has unlimited capacity). To refine 100,000 barrels costs $700 at Dallas and $900 at Houston.

Refined oil is shipped to customers in New York and Chicago. New York customers require 300,000 barrels per day and Chicago customers require 400,000 barrels per day.

The costs of shipping 100,000 barrels of oil (refined or unrefined) between cities are shown below.



| From/To($) | Dallas | Houston | New York | Chicago |
|---|---|---|---|---|
| Los Angels | 300 | 110 | - | - |
| San Diego | 420 | 100 | - | - |
| Dallas | - | - | 450 | 550 |
| Houston | - | - | 470 | 530 |

a) Formulate a minimum-cost-network-flow-problem (MCNFP) to minimize the total cost of meeting all demands.

b) Model this problem in JuMP and solve it. Display the values of all variables. (The optimal value is $10,470.)

Graphically solve the following LPs in questions 2-5. Clearly show the isoprofit/isocost lines. Clearly state which case applies (unique optimal solution, multiple solutions, unbounded, infeasible). If there are multiple optimal solutions, please write down the set of optimal solutions. In each problem also solve the problem in which "max" is replaced by "min".

In the "min" case it may be possible to just give a quick logical argument to prove that it is the optimal solution, rather than drawing the isocost lines. If this is the case, then provide that argument.

**Question 2**. (**10 points**):

$$\begin{aligned} \max_{x} \quad & x_1 + x_2 \\ s.t. \quad & x_1 + 2x_2 \leq 6 \\ & -x_1 + x_2 \geq 6 \\ & x_1, x_2 \geq 0. \end{aligned}$$

**Question 3**. (**10 points**):

$$\begin{aligned} \max_{x} \quad & 4x_1 + x_2 \\ s.t. \quad & 8x_1 + 2x_2 \leq 16 \\ & x_1 + 2x_2 \leq 10 \\ & x_1, x_2 \geq 0. \end{aligned}$$

**Question 4**. (**10 points**):

$$\begin{aligned} \max_{x} \quad & -x_1 + 4x_2 \\ s.t. \quad & x_1 - x_2 \leq 6 \\ & 2x_1 + 3x_2 \geq 3 \\ & x_1, x_2 \geq 0. \end{aligned}$$





**Question 5**. (**10 points**):

$$\begin{aligned} \max_{x} \quad & 3x_1 + x_2 \\ s.t. \quad & 2x_1 + x_2 \leq 6 \\ & x_1 + 3x_2 \leq 9 \\ & x_1, x_2 \geq 0. \end{aligned}$$

**Question 6**. (**20 points**): Suppose we are given an LP. We say that the feasible region is **bounded** if there is some number $M > 0$ s.t. the norm of any feasible $x$ (i.e. $\sqrt{x_1^2 + \cdots + x_n^2}$) is at most $M$. It is unbounded otherwise. Which of the following statements is true? Give a rigorous proof, or a clear counterexample.

A) If the LP has an unbounded objective value (that is, for any feasible $x$, there is always another feasible $\hat{x}$ with a larger (smaller) objective value when maximizing (minimizing)), then its feasible region must be unbounded.

B) If the feasible region of the LP is unbounded, then its objective value is unbounded.

—————— **The end** ——————



1. a) $X_{LA,D}$ : barrels from LA to Dallas

$X_{LA,H}$ : barrels from LA to Houston

$X_{SD,D}$ : barrels from SD to Dallas

$X_{SD,H}$ : barrels from SD to Houston

$X_{D,NY}$ :

$X_{D,C}$ : ⋮

$X_{H,NY}$ :

$X_{H,C}$ : barrels from Houston to Chicago

$\min z = 300X_{LA,D} + 110X_{LA,H} + 420_{SD,D} + 100_{SD,H} +$

$x$

$$(450+700)x_{D,NY} + (550+700)x_{D,C} + (470+900)x_{H,NY} + (530+900)x_{H,C}$$

s.t.

$$x_{LA,D} + x_{LA,H} \le 4$$

$$x_{SD,D} + x_{SD,H} \le 5$$

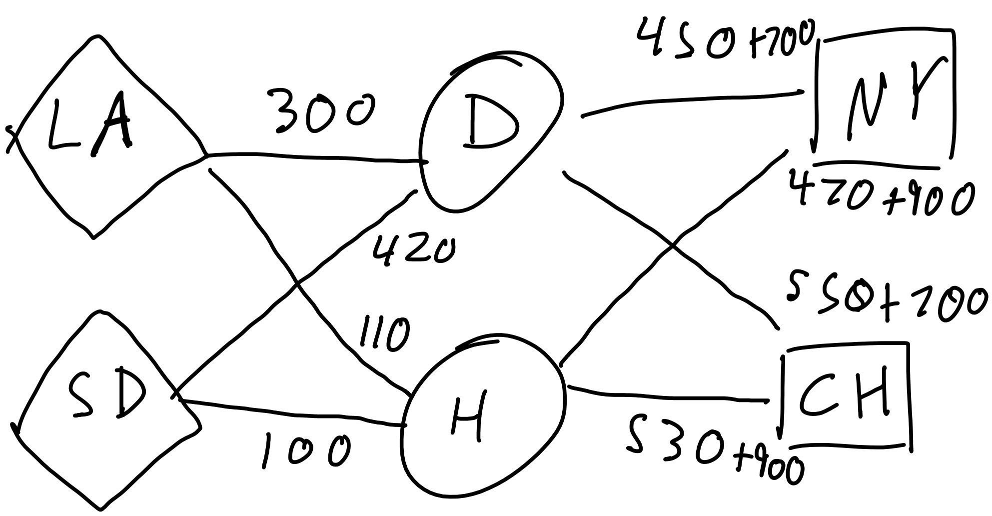
$$x_{D,NY} + x_{H,NY} = 3$$

$$x_{D,C} + x_{H,C} = 4$$

$$x_{D,NY} + x_{D,C} \le x_{LA,D} + x_{SD,D}$$

$$x_{H,NY} + x_{H,C} \le x_{LA,H} + x_{SD,H}$$

$$x \ge 0$$

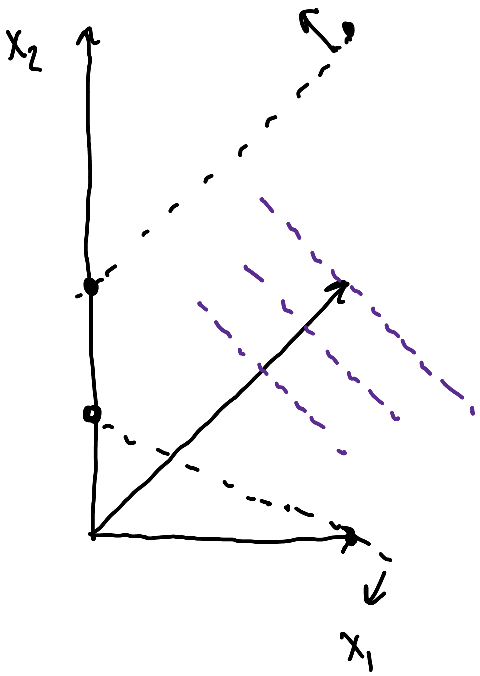



2) $\max_{x} \; x_1 + x_2$

s.t. $x_1 + 2x_2 \le 6 \rightarrow x_2 \le 3 - \frac{1}{2}x_1$

$-x_1 + x_2 \ge 6 \rightarrow x_2 \ge 6 + x_1$

$x_1, x_2 \ge \emptyset$



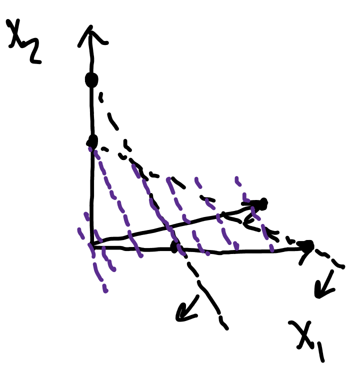

Clearly, the two constraints to the left have no feasible region. t.f. this LP is formulated s.t. it is an infeasible problem.

3) $\max\limits_{x} \; 4x_1 + x_2$

s.t. $8x_1 + 2x_2 \le 16 \rightarrow x_2 \le 8 - 4x_1$

$x_1 + 2x_2 \le 10 \rightarrow x_2 \le 5 - \frac{1}{2}x_1$

$x_1, x_2 \ge \emptyset$




$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$4(2) + (0) = 8$ is a max solution

$x_2 = 8 - 4x_1$

$x_1 + 2(8 - 4x_1) = 10$

For the min problem,

$\Rightarrow x_1 + 16 - 8x_1 = 10 \Rightarrow -7x_1 = -6 \Rightarrow x_1 = \frac{6}{7}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$

$4(0) + (5) = 5$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$4(0) + (0) = 0$ is the min optimal solution

$x_2 = 8 - 4\left(\frac{6}{7}\right) = \frac{32}{7}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6/7 \\ 32/7 \end{bmatrix}$

$4(6/7) + (32/7) = 56/7 = 8$ is also a max solution

Since these are both optimal solutions we know that there are multiple optimal solutions. More precisely it is true that,

$\{(x_1, x_2) \mid 8x_1 + 2x_2 = 16, \ 6/7 \leq x_1 \leq 2, \ 0 \leq x_2 \leq 32/7\}$

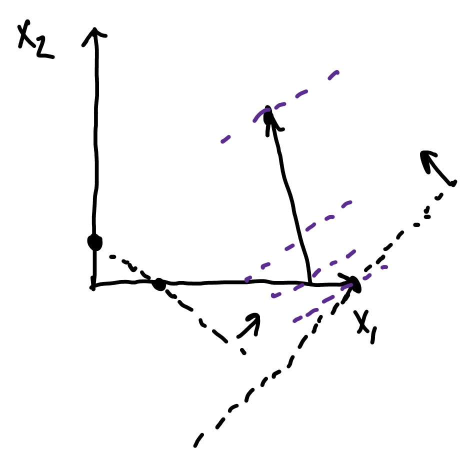
is the set of optimal solutions for the optimization problem.

4) $\max\limits_{x} \ -x_1 + 4x_2$

s.t. $x_1 - x_2 \leq 6 \rightarrow x_2 \geq -6 + x_1$

$2x_1 + 3x_2 \geq 3 \rightarrow x_2 \geq 1 - \frac{2}{3}x_1$

$x_1, x_2 \geq 0$




This problem is unbounded.

We can also express the minimization problem with,

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$

$-(6) + 4(0) = -6$ is the min optimal solution

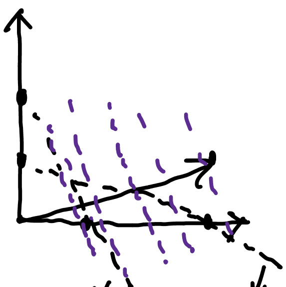
5) $\max \ 3x_1 + x_2$

$x$

s.t. $2x_1 + x_2 \le 6 \rightarrow x_2 \le 6 - 2x_1$

$x_1 + 3x_2 \le 9 \rightarrow x_2 \le 3 - \frac{1}{3}x_1$

$x_1, x_2 \ge \emptyset$



$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ \emptyset \end{bmatrix}$

$3(3) + (0) = 9$ is the max optimal solution

$x_2 = 6 - 2x_1$

$x_1 + 3(6 - 2x_1) = 9 \Rightarrow x_1 + 18 - 6x_1 = 9$

$\Rightarrow -5x_1 = -9 \Rightarrow x_1 = 9/5$

$x_2 = 6 - 2(9/5) = 12/5$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9/5 \\ 12/5 \end{bmatrix}$

$3(9/5) + 12/5 = 39/5 = 7.8$

For min problem

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

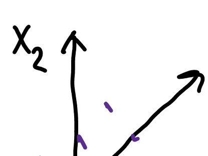
$3(0) + (0) = 0$ is the min optimal solution

6) a) Proof by Contradiction:

Assume for contradiction that the LP has an unbounded objective value but a bounded feasible region. As our unbounded objective value attempts to approach $\infty$ for a maximization problem (or $-\infty$ for a minimization problem) it will eventually be stopped the finite nature of our decision variables which are "fenced in" by the bounded feasible region. Therefore, If an LP has an unbounded objective value then its feasible region must be unbounded. $\boxtimes$

b) counter example:

$\min_{x} \; x_1 + x_2$



s.t. $x_1, x_2 \geq 0$


$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$0 + 0 = 0$$

Clearly, there exists a minimum solution to the problem above. Therefore, if the feasible region of the LP is unbounded it is not neccessary that its objective value be unbounded. $\square$

```julia
using JuMP, HiGHS

println("HOMEWORK 4:\n")

###Question 1###
println("Question 1\n")
println("b.\n")

# Model
model = Model(HiGHS.Optimizer)

# Decision Variables
@variable(model, x_LA_D >= 0)  # LA to Dallas
@variable(model, x_LA_H >= 0)  # LA to Houston
@variable(model, x_SD_D >= 0)  # SD to Dallas
@variable(model, x_SD_H >= 0)  # SD to Houston
@variable(model, x_D_NY >= 0)  # Dallas to New York
@variable(model, x_D_CH >= 0)  # Dallas to Chicago
@variable(model, x_H_NY >= 0)  # Houston to New York
@variable(model, x_H_CH >= 0)  # Houston to Chicago

# Objective
@objective(model, Min,
    300 * x_LA_D + 110 * x_LA_H +
    420 * x_SD_D + 100 * x_SD_H +
    (450 + 700) * x_D_NY + (550 + 700) * x_D_CH +
    (470 + 900) * x_H_NY + (530 + 900) * x_H_CH)

# Constraints
# Supply constraints
@constraint(model, x_LA_D + x_LA_H <= 4)  # Los Angeles can supply up
to 400,000 barrels
@constraint(model, x_SD_D + x_SD_H <= 5)  # San Diego can supply up to
500,000 barrels

# Demand constraints
@constraint(model, x_D_NY + x_H_NY >= 3)  # New York requires at least
300,000 barrels
@constraint(model, x_D_CH + x_H_CH >= 4)  # Chicago requires at least
400,000 barrels

# Refinery output constraints
@constraint(model, x_D_NY + x_D_CH <= x_LA_D + x_SD_D)  # Dallas
output
@constraint(model, x_H_NY + x_H_CH <= x_LA_H + x_SD_H)  # Houston
output

# Solve
optimize!(model)
```

```
# Results
println("Objective value: ", objective_value(model))
println("x_LA_D = ", value(x_LA_D))
println("x_LA_H = ", value(x_LA_H))
println("x_SD_D = ", value(x_SD_D))
println("x_SD_H = ", value(x_SD_H))
println("x_D_NY = ", value(x_D_NY))
println("x_D_CH = ", value(x_D_CH))
println("x_H_NY = ", value(x_H_NY))
println("x_H_CH = ", value(x_H_CH))

###Question 2###
println("Question 2\n")

# Model
model = Model(HiGHS.Optimizer)

# Decision Variables
@variable(model, x_1 >= 0)
@variable(model, x_2 >= 0)

# Objective
@objective(model, Max, x_1 + x_2)

# Constraints
@constraint(model, x_1 + 2x_2 <= 6)
@constraint(model, -x_1 + x_2 >= 6)

# Solve the model
#optimize!(model)

# Output the maximization results
#println("Objective value for the maximization is: ",
objective_value(model))
#println(" x_1 is ", value(x_1))
#println(" x_2 is ", value(x_2))

# Objective
@objective(model, Min, x_1 + x_2)

# Constraints
#optimize!(model)

# Output the  minimization results
#println("Objective value for the minimization is: ",
objective_value(model))
#println(" x_1 is ", value(x_1))
#println(" x_2 is ", value(x_2))

println("There is no feasible region so the problem is infeasible.")
```

```julia
###Question 3###
println("Question 3\n")

# Model
model = Model(HiGHS.Optimizer)

# Decision Variables
@variable(model, x_1 >= 0)
@variable(model, x_2 >= 0)

# Objective
@objective(model, Max, 4x_1 + x_2)

# Constraints
@constraint(model, 8x_1 + 2x_2 <= 16)
@constraint(model, x_1 + 2x_2 <= 10)

# Solve the model
optimize!(model)

# Output the maximization results
println("Objective value for the maximization is: ",
objective_value(model))
println(" x_1 is ", value(x_1))
println(" x_2 is ", value(x_2))

# Objective
@objective(model, Min, 4x_1 + x_2)

# Constraints
optimize!(model)

# Output the  minimization results
println("Objective value for the minimization is: ",
objective_value(model))
println(" x_1 is ", value(x_1))
println(" x_2 is ", value(x_2))

###Question 4###
println("Question 4\n")

# Model
model = Model(HiGHS.Optimizer)

# Decision Variables
@variable(model, x_1 >= 0)
@variable(model, x_2 >= 0)

# Objective
@objective(model, Max, -x_1 + 4x_2)
```

```julia
# Constraints
@constraint(model, x_1 - x_2 <= 6)
@constraint(model, 2x_1 + 3x_2 >= 3)

# Solve the model
#optimize!(model)

# Output the maximization results
#println("Objective value for the maximization is: ",
objective_value(model))
#println(" x_1 is ", value(x_1))
#println(" x_2 is ", value(x_2))
println("The maximization problem is unbounded.")

# Objective
@objective(model, Min, -x_1 + 4x_2)

# Constraints
optimize!(model)

# Output the  minimization results
println("Objective value for the minimization is: ",
objective_value(model))
println(" x_1 is ", value(x_1))
println(" x_2 is ", value(x_2))

###Question 5###
println("Question 5\n")

# Model
model = Model(HiGHS.Optimizer)

# Decision Variables
@variable(model, x_1 >= 0)
@variable(model, x_2 >= 0)

# Objective
@objective(model, Max, 3x_1 + x_2)

# Constraints
@constraint(model, 2x_1 + x_2 <= 6)
@constraint(model, x_1 + 3x_2 <= 9)

# Solve the model
optimize!(model)

# Output the maximization results
println("Objective value for the maximization is: ",
objective_value(model))
```

```julia
println(" x_1 is ", value(x_1))
println(" x_2 is ", value(x_2))

# Objective
@objective(model, Min, 3x_1 + x_2)

# Constraints
optimize!(model)

# Output the  minimization results
println("Objective value for the minimization is: ",
objective_value(model))
println(" x_1 is ", value(x_1))
println(" x_2 is ", value(x_2))
```

```
HOMEWORK 4:

Question 1

b.

Running HiGHS 1.6.0: Copyright (c) 2023 HiGHS under MIT licence terms
Presolving model
6 rows, 8 cols, 16 nonzeros
6 rows, 8 cols, 16 nonzeros
Presolve : Reductions: rows 6(-0); columns 8(-0); elements 16(-0) -
Not reduced
Problem not reduced by presolve: solving the LP
Using EKK dual simplex solver - serial
  Iteration        Objective     Infeasibilities num(sum)
          0     0.0000000000e+00 Pr: 2(7) 0s
          5     1.0470000000e+04 Pr: 0(0) 0s
Model   status      : Optimal
Simplex   iterations: 5
Objective value     :  1.0470000000e+04
HiGHS run time      :          0.00
Objective value: 10470.0
x_LA_D = 3.0
x_LA_H = 0.0
x_SD_D = 0.0
x_SD_H = 4.0
x_D_NY = 3.0
x_D_CH = 0.0
x_H_NY = 0.0
x_H_CH = 4.0
Question 2

There is no feasible region so the problem is infeasible.
Question 3
```

```
Running HiGHS 1.6.0: Copyright (c) 2023 HiGHS under MIT licence terms
Presolving model
2 rows, 2 cols, 4 nonzeros
2 rows, 2 cols, 4 nonzeros
Presolve : Reductions: rows 2(-0); columns 2(-0); elements 4(-0) - Not
reduced
Problem not reduced by presolve: solving the LP
Using EKK dual simplex solver - serial
  Iteration        Objective     Infeasibilities num(sum)
          0    -1.4999972737e+00 Ph1: 2(4); Du: 2(1.5) 0s
          2     8.0000000000e+00 Pr: 0(0) 0s
Model   status      : Optimal
Simplex   iterations: 2
Objective value     :  8.0000000000e+00
HiGHS run time      :          0.00
Objective value for the maximization is: 8.0
 x_1 is 2.0
 x_2 is 0.0
Solving LP without presolve or with basis
Using EKK dual simplex solver - serial
  Iteration        Objective     Infeasibilities num(sum)
          0    -1.0000013946e+00 Ph1: 1(1); Du: 1(1) 0s
          1     0.0000000000e+00 Pr: 0(0) 0s
Model   status      : Optimal
Simplex   iterations: 1
Objective value     :  0.0000000000e+00
HiGHS run time      :          0.00
Objective value for the minimization is: 0.0
 x_1 is 0.0
 x_2 is 0.0
Question 4

The maximization problem is unbounded.
Running HiGHS 1.6.0: Copyright (c) 2023 HiGHS under MIT licence terms
Presolving model
2 rows, 2 cols, 4 nonzeros
0 rows, 0 cols, 0 nonzeros
Presolve : Reductions: rows 0(-2); columns 0(-2); elements 0(-4) -
Reduced to empty
Solving the original LP from the solution after postsolve
Model   status      : Optimal
Objective value     : -6.0000000000e+00
HiGHS run time      :          0.00
Objective value for the minimization is: -6.0
 x_1 is 6.0
 x_2 is 0.0
Question 5

Running HiGHS 1.6.0: Copyright (c) 2023 HiGHS under MIT licence terms
```

```
Presolving model
2 rows, 2 cols, 4 nonzeros
2 rows, 2 cols, 4 nonzeros
Presolve : Reductions: rows 2(-0); columns 2(-0); elements 4(-0) - Not
reduced
Problem not reduced by presolve: solving the LP
Using EKK dual simplex solver - serial
  Iteration        Objective     Infeasibilities num(sum)
          0    -3.9999952417e+00 Ph1: 2(7); Du: 2(4) 0s
          2     9.0000000000e+00 Pr: 0(0) 0s
Model   status      : Optimal
Simplex   iterations: 2
Objective value     :  9.0000000000e+00
HiGHS run time      :          0.00
Objective value for the maximization is: 9.0
 x_1 is 3.0
 x_2 is 0.0
Solving LP without presolve or with basis
Using EKK dual simplex solver - serial
  Iteration        Objective     Infeasibilities num(sum)
          0    -2.0000014690e+00 Ph1: 2(3); Du: 2(2) 0s
          1     0.0000000000e+00 Pr: 0(0) 0s
Model   status      : Optimal
Simplex   iterations: 1
Objective value     :  0.0000000000e+00
HiGHS run time      :          0.00
Objective value for the minimization is: 0.0
 x_1 is 0.0
 x_2 is 0.0
```