`https://github.com/rmhi/SU21Eisenstein`

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The code calculates invariants on certain arithmetic subgroups of $\mathrm{SU}(2,1)$. These arithmetic groups have finite index in a certain subgroup Υ defined below. There are three files:

1. `PSU(2,1)(OK,alpha) presentation.sage`

   The file computes a presentation of the group Υ defined below.

2. `SU(2,1)`

   The code calculates various invariants for a subgroup G of finite index in Υ. In particular, it will calculate the following:

   (a) A presentation of G.
   (b) A presentation of the pre-image $\tilde{G}$ of G in the universal cover of $\mathrm{SU}(2,1)$.
   (c) The abelianization of G.
   (d) The abelianization of $\tilde{G}$.
   (e) The "weight denominator" of G, i.e. the largest n such that G has a multiplier system of weight $\frac{1}{n}$.
   (f) The action on both abelianizations of a diamond operator corresponding to a matrix which normalizes G.

   The following subgroups are defined:

   - `Gamma(I)`

     The principal congruence subgroup of level I for a non-zero ideal I of $\mathcal{O}$.
   - `Gamma0(I)`

     The congruence subgroup of matrices which are upper triangular modulo I.
   - `Gamma1(I)`

     The congruence subgroup of matrices which are upper triangular modulo I, and where the diagonal entries are congruent to 1 modulo I.
   - `GammaNC(I)`

     The subgroup $\phi^{-1}(I)$, where $\phi : \Upsilon \to \mathcal{O}$ is a certain surjective homomorphism.

- `Index3congruence(a,b,c,d)`

  The index 3 subgroups of Υ. There are 40 of these subgroups, and they are all congruence subgroups. The variables `a,b,c,d` are integers. The subgroup depends only on the image of (a, b, c, d) in $\mathbb{P}^3(\mathbb{F}_3)$.

- `G & H`

  The intersection of two arithmetic groups `G` and `H`.

3. `SU(2,1)(OK,alpha) Sigma-sigma split.sage`

   Two homogeneous $\mathbb{Z}$-valued 2-cocycles are defined on Υ. One of these is the restriction of the standard cocycle σ on $\mathrm{SU}(2,1)$ corresponding to the universal cover of $\mathrm{SU}(2,1)$. The other cocycle Σ is the cup product of two 1-cocycles, which are defined on on Υ. More specifically, Σ is given by

   $$\Sigma(g,h) = \mathrm{Tr}\left(\frac{\phi(g)\bar{\phi}(h)}{\alpha}\right),$$

   where $\phi : \Upsilon \to \mathbb{Z}[\omega]$ is a certain surjective homomorphism.

   The code proves that $4 \cdot \sigma = \Sigma$ in $H^2(\mathrm{PSU}(\mathcal{O}, \alpha), \frac{1}{3}\mathbb{Z})$.

   More precisely, there is a central extension corresponding to the cocycle $4 \cdot \sigma - \Sigma$:

   $$1 \to \mathbb{Z} \to \tilde{\Upsilon} \to \Upsilon \to 1.$$

   The code calculates a presentation for $\tilde{\Upsilon}$ from the presentation of Υ. Using this presentation, it calculates the abelianization $\tilde{\Upsilon}^{\mathrm{ab}} = \tilde{\Upsilon}/[\tilde{\Upsilon}, \tilde{\Upsilon}]$.

The arithmetic group $\Upsilon = \mathrm{PSU}(2,1)(\mathcal{O}, \alpha)$.

Let ω be a primitive cube root of unity and let

$$k = \mathbb{Q}(\omega), \qquad \mathcal{O} = \mathbb{Z}[\omega].$$

The matrix $J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ induces a Hermitian form of on k^3 by

$$\langle v, w \rangle = \bar{v}^t \cdot J \cdot w.$$

Define the following groups of isometries of this form:

$$\mathrm{SU}(2,1) = \{g \in \mathrm{SL}_3(\mathbb{C}) : \bar{g}^t J g = J\} \qquad \mathrm{SU}(2,1)(\mathcal{O}) = \mathrm{SU}(2,1) \cap \mathrm{SL}_3(\mathcal{O})$$

The element $\alpha = 2\omega + 1$ is a square root of -3 in $\mathcal{O}$. Define the principle congruence subgroup:

$$\mathrm{SU}(2,1)(\mathcal{O}, \alpha) = \{g \in \mathrm{SU}(2,1)(\mathcal{O}) : g \equiv I_3 \bmod \alpha\}.$$

The group $\mathrm{SU}(2,1)(\mathcal{O}, \alpha)$ splits as a direct sum:

$$\mathrm{SU}(2,1)(\mathcal{O}, \alpha) = \mu_3 \times \Upsilon,$$

where μ_3 is the centre of $\mathrm{SU}(2,1)$ (i.e. the cyclic group generated by $\omega \cdot I_3$, and Υ is the following congruence subgroup:

$$\Upsilon = \{g \in \mathrm{SU}(2,1)(\mathcal{O}, \alpha) : g_{1,1} \equiv 1 \bmod 3\}.$$

The code investigates the group Υ and its subgroups of finite index. Often in the code, this group is called `PSU(2,1)(OK,alpha)`.

Instructions

The code was developed on sage version 9.0, but may run on other versions. Before running the code, create a subdirectory "`SU(2,1)(OK,alpha) subgroup data`". The code will save the results of calculation in this subdirectory, in order to avoid doing the same calculation more than once.

Example

Open sage in a terminal on the directory containing the code. Then type

```
load("SU21.sage")
```

Wait a few seconds or minutes while some tests are run (these tests may be turned off by editing the first line of `SU21.sage`). Define the level 3 principal congruence subgroup as follows:

```
G = Gamma(3)
```

To calculate the abelianization of this group, type

```
G.H_1()
```

You should get the result after a few seconds. To calculate the abelianization of the pre-image of this group in the universal cover of $\mathrm{SU}(2,1)$, type

```
G.H_1_met()
```

This might take a few minutes. If you'd like to know how the calculation is progression, then you could instead type

```
G.H_1_met(verbose=True)
```

The subdirectory SU(2,1)(OK,alpha) subgroup data will now contain a file `Gamma(3).sobj`. This file contains the results of the calculations. If you restart sage and repeat the commands above, then the result should arrive much more quickly, because it is being read from the file.

The speed of many of the calculations depends mainly on the index of the subgroup. Since $\mathrm{SU}(2,1)$ is an 8-dimensional Lie group, the index of the principal congruence subgroup `Gamma(I)` increases roughly like $\mathrm{N}(I)^8$.