Matrix Properties

Practical Linear Algebra | Lecture 5

Transpose of a Matrix

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} \qquad A^T = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{m1} \\ A_{12} & A_{22} & \cdots & A_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{mn} \end{bmatrix}$$

- If $A \in \mathbb{R}^{m \times n}$, then $A^T \in \mathbb{R}^{n \times m}$
- The columns of A are the rows of A^T, and the rows of A are the columns of A^T
- Just swap the rows and columns of A to get A^T

Range and Rank

- The range of a matrix is the span of its columns

$$A \in \mathbb{R}^{m \times n}$$

$$\textbf{range}(A) = \{Ax \mid x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$$

- The rank of a matrix is the dimension of its range

$$\textbf{rank}(A) = \textbf{dim range}(A)$$

Range and Rank

- If $A \in \mathbb{R}^{m \times n}$ and $\textbf{range}(A) = \mathbb{R}^m$, then we say A is onto
- If A is onto, $\textbf{range}(A)$ contains all possible m-vectors
- If A is onto, the columns of A span $\mathbb{R}^m$
- If A is onto, the rows of A are independent (see proof)

Range and Rank

- **Rank facts**
  - $\textbf{rank}(A)$ is the number of independent columns
  - $\textbf{rank}(A) = \textbf{rank}(A^T)$
    - Row rank and column rank are always the same for any matrix
    - Number of independent row vectors is always equal to number of independent column vectors
    - We can't prove this now – we need *QR factorization*, which will come later
  - For any $A \in \mathbb{R}^{m \times n}$, we have $\textbf{rank}(A) \leq \textbf{min}(m, n)$
    - Implied by above fact
    - If $\textbf{rank}(A) = \textbf{min}(m, n)$, we say the matrix is full rank

Digression: A Proof

- Let's prove that if A is onto, the rows of A are independent
- Let $A \in \mathbb{R}^{m \times n}$
- Definition of onto: $\textbf{range}(A) = \mathbb{R}^m$
- By definition, $\textbf{range}(A) = \mathbb{R}^m$ means $\textbf{rank}(A) = m$
- Remember that $\textbf{rank}(A) = \textbf{rank}(A^T)$, so $\textbf{rank}(A^T) = m$
- A has m rows, but the row rank of A is m. This is only possible if the rows are independent
- **Implication:** if A is onto, it can't be a skinny matrix (visually obvious)

Nullspace

$$\textbf{null}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$$

- The nullspace of a matrix is the set of input vectors that get mapped to zero
- Equivalently, the nullspace is the set of vectors orthogonal to all rows of the matrix

Nullspace

- **Visual intuition**
  - Let $A \in \mathbb{R}^{m \times n}$. This means A has n column vectors, where each vector is m-dimensional
  - Draw the span of the columns of the matrix A. You'll end up with a subspace of m-dimensional space.
  - Look at the origin of this subspace (the point where all coordinates are zero). This is the zero vector in m-dimensional space
  - The nullspace is the set of all coefficients that make the n column vectors sum to zero

Zero Nullspace

- It's possible there's only one case where the linear combination of the columns of A sum to the zero vector: *all the coefficients are zero*
- This means the columns of A are linearly independent
- If $x = 0$ is the only element of $\mathbf{null}(A)$, we say that A is one-to-one. This term means that there's a one-to-one correspondence between the values of x and the output vector – if you know one, you can uniquely recover the other

Nonzero Nullspace

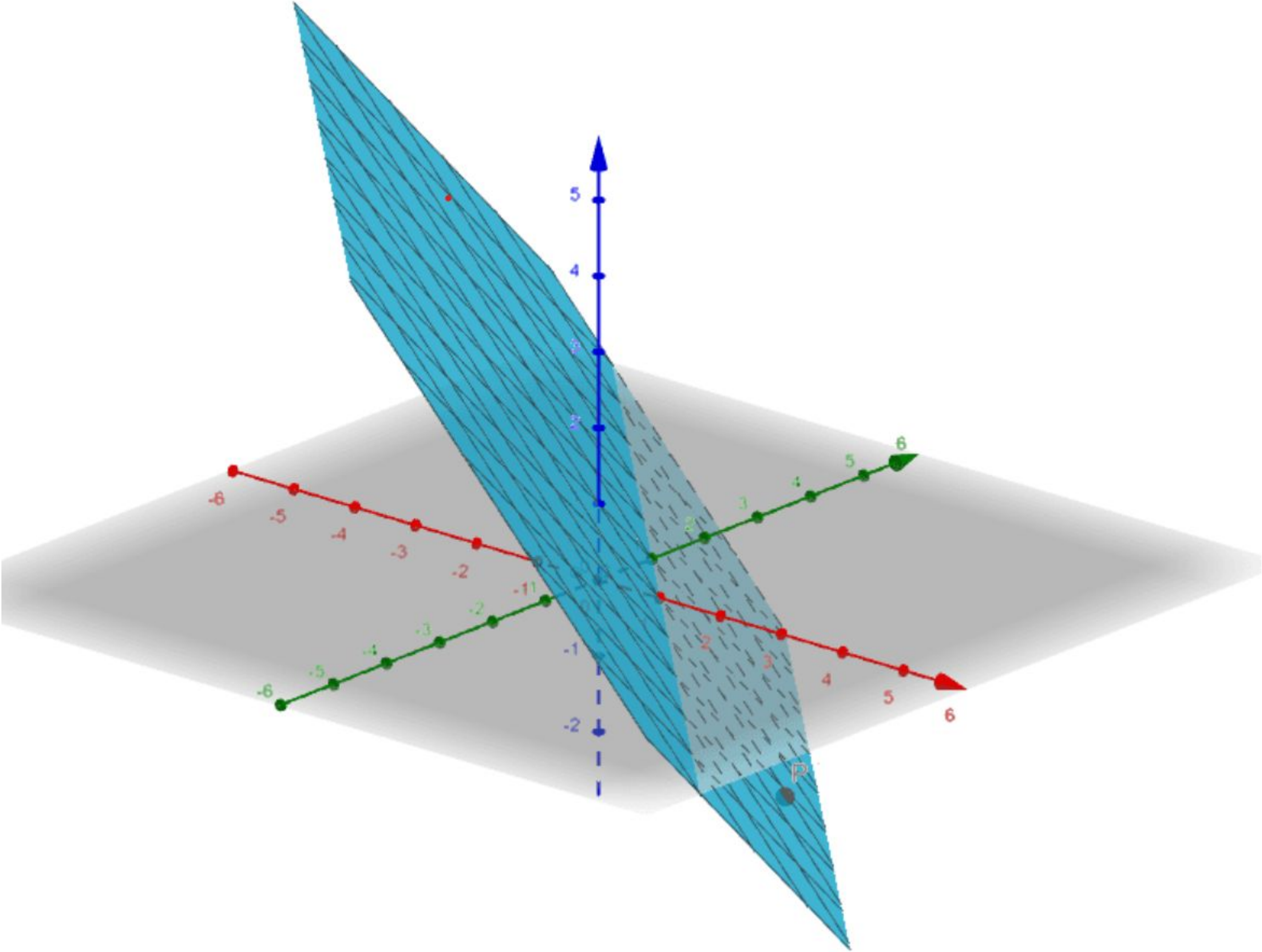
- If a matrix has nonzero vectors in its nullspace, there's an important practical implication
- Say we have a matrix A and two vectors x and y, where $Ax = 0$ but $Ay \neq 0$
- What happens to the vector $x + y$? Since matrix multiplication is linear, we have $A(x + y) = Ax + Ay = Ay$
- This means we can translate y by x (or any scalar times x), and our output vector will be exactly the same
- **Key point:** the matrix A cannot detect any translation in the x direction!

Nullity

- Nullspace is also sometimes called the kernel
- $\mathbf{null}(A) = \mathcal{N}(A) = \mathbf{ker}(A)$
- The nullity is the dimension of the nullspace
  - $\mathbf{nullity}(A) = \mathbf{dim}\ \mathbf{null}(A)$
- *Rank* is to *range* as *nullity* is to *nullspace*

Nullspace vs. Range

- **Remember**: Nullspace is about the *inputs*, range is about the *outputs*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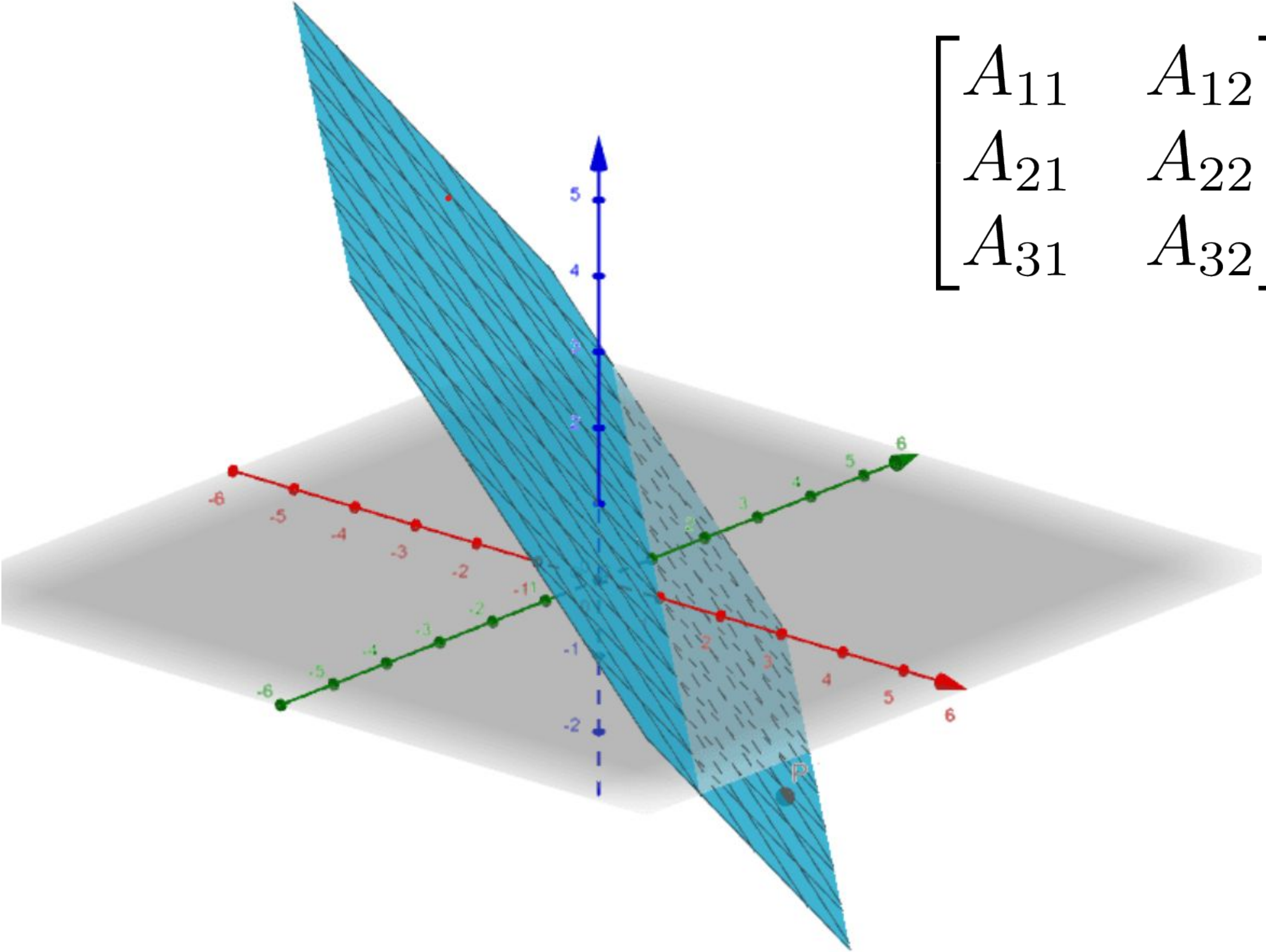
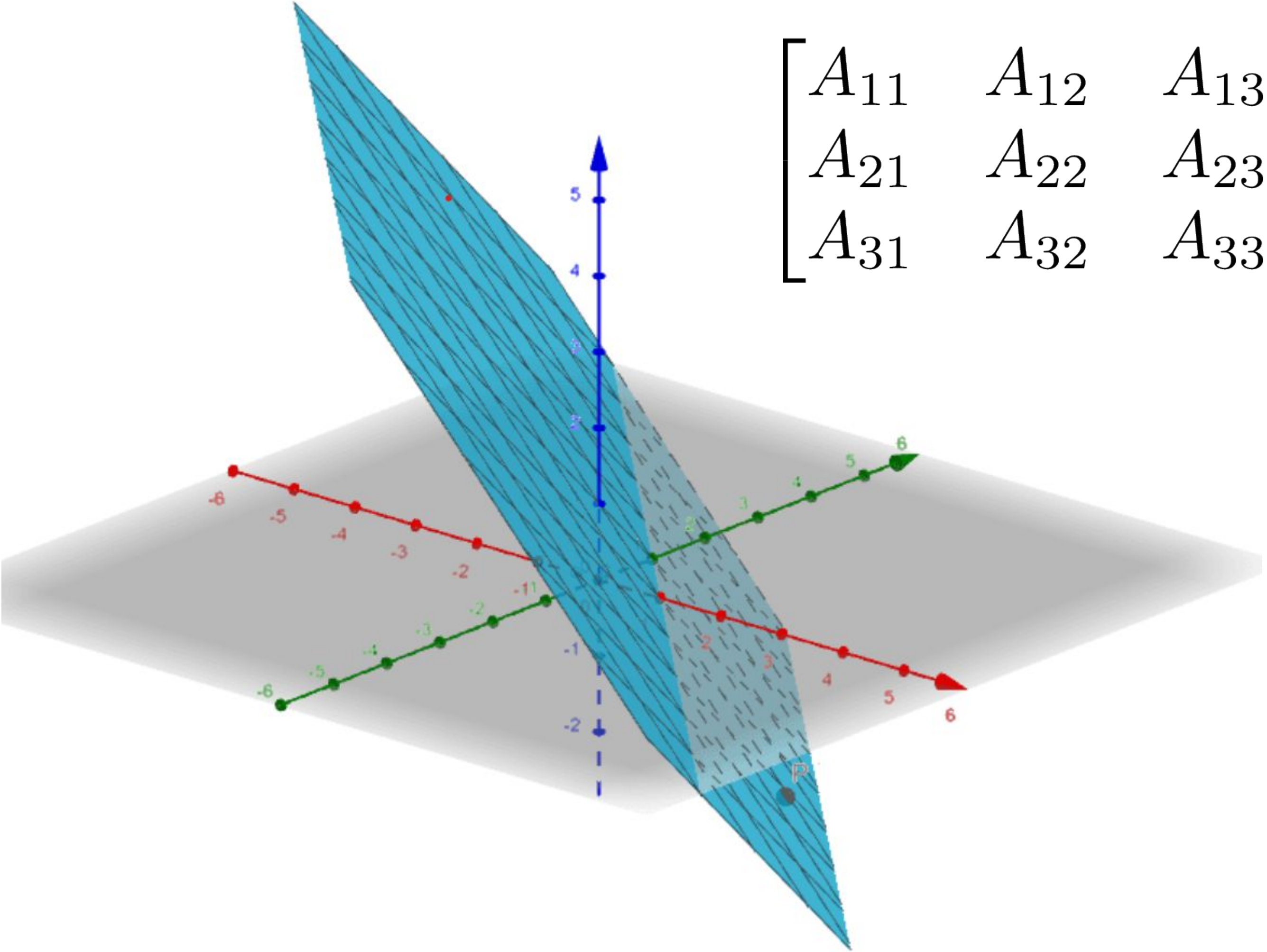


$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Conservation of Dimension

Conservation of Dimension

- *Sounds pretty cool*

Conservation of Dimension

- Rank-nullity theorem
  - $\mathbf{rank}(A) + \mathbf{nullity}(A) = n$ for any $A \in \mathbb{R}^{m \times n}$
- Each dimension in the input space either goes to the output or gets "crushed" into the zero vector
- Physical meaning: for any sensor matrix A, the input dimensions will either show up in the output measurements, or be ignored
- Proof requires QR factorization

Orthogonality

- A set of vectors is orthogonal if the dot product between any two of them is zero
- For short, instead of saying “set of orthogonal vectors”, we just say “orthogonal vectors”

Normalized vectors

- A normalized vector is a vector whose norm is one

Orthonormal vectors

- A set of orthogonal, normalized vectors is said to be orthonormal
- **Some properties**
  - An orthonormal set of vectors is independent
  - If we take an orthonormal set of k vectors and arrange them as the columns of a matrix U, then $U^TU = I_k$

Orthogonal matrices

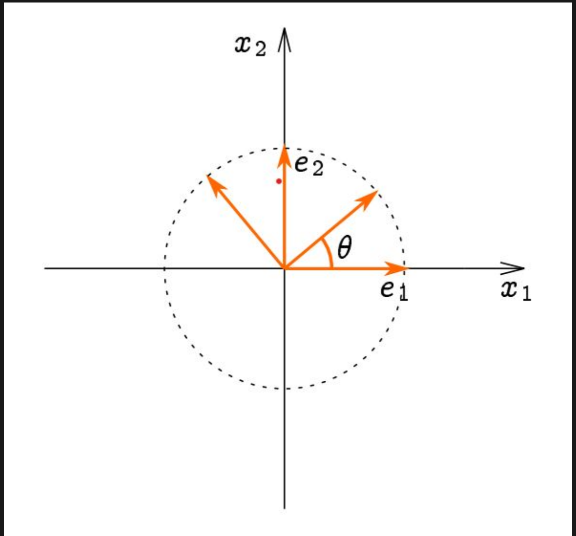
- An orthogonal matrix is a *square* matrix U that satisfies $U^T U = I$
- Careful: *Orthogonal* matrices have *orthonormal* columns! Don't make the mistake of calling them "orthonormal" matrices
  - *As far as I know there's no term for matrices whose column vectors are simply orthogonal, not orthonormal*

Orthogonal matrices

- Let $U \in \mathbb{R}^{n \times n}$. Since the n column vectors of U are independent, and each column vector has n elements, we know they span $\mathbb{R}^n$
- The columns of an orthogonal matrix $U \in \mathbb{R}^{n \times n}$ form a basis for $\mathbb{R}^n$
- Orthogonal matrices preserve lengths and angles, and so are said to be isometric (distance-preserving)

Rotation matrices

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



Rotation matrices

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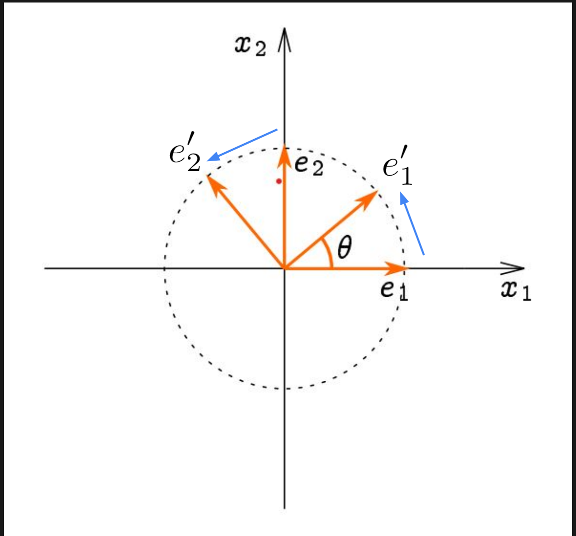
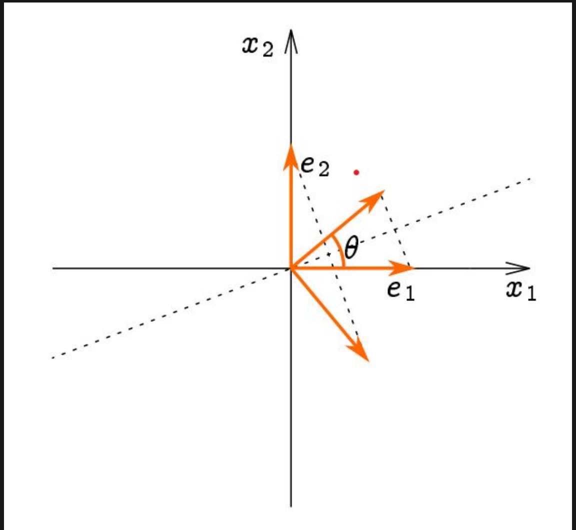


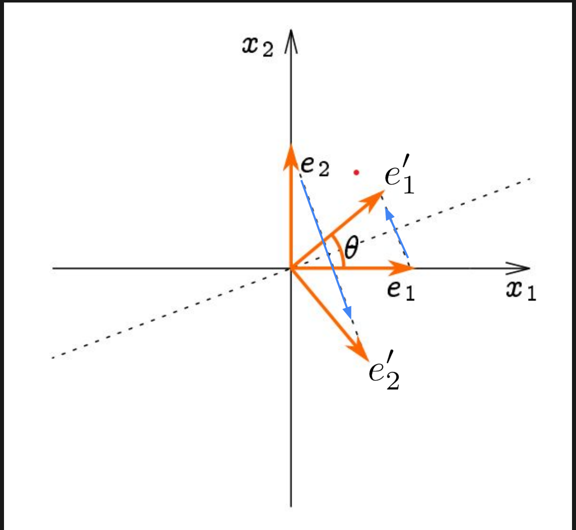
Reflection matrices

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$



Reflection matrices

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$



Next Time

- Matrices in Python
- Application: Robotics