Matrix Basics

Practical Linear Algebra | Lecture 4

Matrices and their properties

- A matrix is a 2D grid of numbers

$$\begin{bmatrix} 4 & 5 & -12 & 0 \\ -10 & 98 & 0 & -1 \\ 2 & -44 & 1 & 9 \end{bmatrix}$$

Matrices and their properties

- Abstract form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m \times n}$$

Matrices and their properties

- A matrix can be thought of as a collection of column vectors

$$A = \begin{bmatrix} | & | & & | \\ c_1 & c_2 & \cdots & c_n \\ | & | & & | \end{bmatrix}$$

Matrices and their properties

- A matrix can also be thought of as a collection of row vectors

$$A = \begin{bmatrix} — & r_1 & — \\ — & r_2 & — \\ & \vdots & \\ — & r_m & — \end{bmatrix}$$

Matrices and their properties

- Matrix-vector multiplication

$$A \in \mathbb{R}^{m \times n} \qquad x \in \mathbb{R}^n$$

$$Ax$$

Matrices and their properties

- Matrix-vector multiplication

$$\begin{bmatrix} — & a_1^T & — \\ — & a_2^T & — \\ & \vdots & \\ — & a_m^T & — \end{bmatrix} x = \begin{bmatrix} a_1^T x \\ a_2^T x \\ \vdots \\ a_m^T x \end{bmatrix}$$

Matrices and their properties

- Matrix-matrix multiplication

$$A \in \mathbb{R}^{m \times n} \qquad B \in \mathbb{R}^{n \times k}$$

$$AB$$

Matrices and their properties

- Matrix-matrix multiplication

$$\begin{bmatrix} — & a_1^T & — \\ — & a_2^T & — \\ & \vdots & \\ — & a_m^T & — \end{bmatrix} \begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_k \\ | & | & & | \end{bmatrix} = \begin{bmatrix} a_1^T b_1 & a_1^T b_2 & \cdots & a_1^T b_k \\ a_2^T b_1 & a_2^T b_2 & \cdots & a_2^T b_k \\ \vdots & & & \\ a_m^T b_1 & a_m^T b_2 & \cdots & a_m^T b_k \end{bmatrix}$$

Matrix equations

- A matrix can be used to represent a linear system of equations

$$y = Ax$$

Matrix equations

- Matrix-vector multiplication: dot product of matrix rows with vector

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Matrix equations

- Matrix-vector multiplication: mixture of matrix columns (addition of vectors)

$$y = Ax = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

Matrix equations

- Skinny, fat, and square matrices

$$\mathbb{R}^{m \times n}$$

skinny	fat	square
$m > n$	$m < n$	$m = n$

Matrix equations

- Identity matrix

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Matrix equations

- Identity matrix

$$A \in \mathbb{R}^{n \times n}$$

$$AI_n = I_n A = A$$

Matrix equations

- Identity matrix

$$I_n = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

Matrix equations

- Inverse matrix

$$A \in \mathbb{R}^{n \times n}$$

$$A^{-1}A = AA^{-1} = I_n$$

Matrix equations

- Solving a linear system of equations with the inverse (assuming A is invertible)

$$A \in \mathbb{R}^{n \times n}$$

$$y = Ax$$

$$A^{-1}y = A^{-1}Ax$$

$$A^{-1}y = x$$

Next Time

- Matrix properties: nullspace, range, rank and orthogonality