Least Norm

Practical Linear Algebra | Lecture 9

Problem Setup

$$y = Ax$$

$$A \in \mathbb{R}^{m \times n}$$

- If A is fat, we have an underdetermined system of equations
- More variables than equations
- There are infinitely many solutions for x
- We can use this freedom in x to satisfy some other constraint

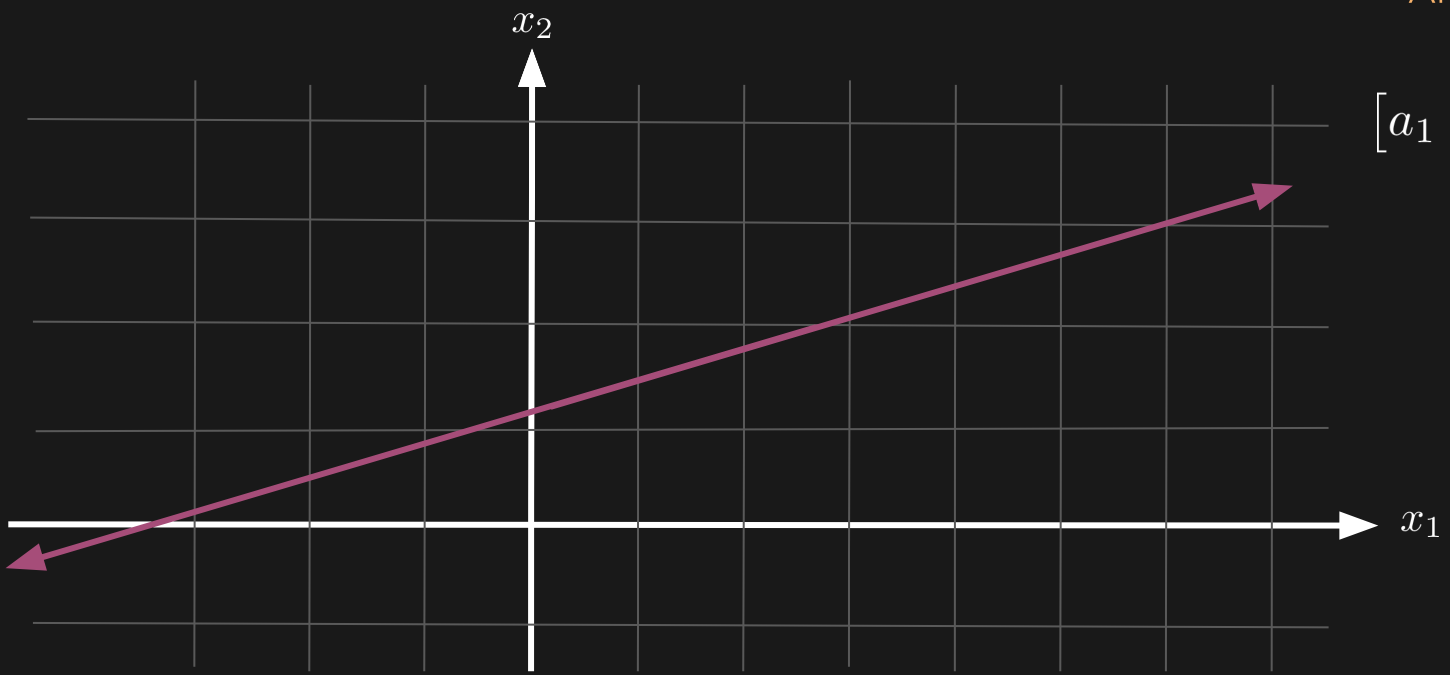
Least Norm

$$\begin{array}{ll} \text{minimize} & \|x\| \\ \text{subject to} & Ax = y \end{array}$$

$$\{x \mid Ax = y\} = \{x_p + x_n \mid x_n \in \mathbf{null}(A)\}$$

- One very natural constraint is to minimize the norm of x
- Any solution can be written as the sum of a particular solution and a vector in the nullspace of A
- This important fact provides insight into the problem

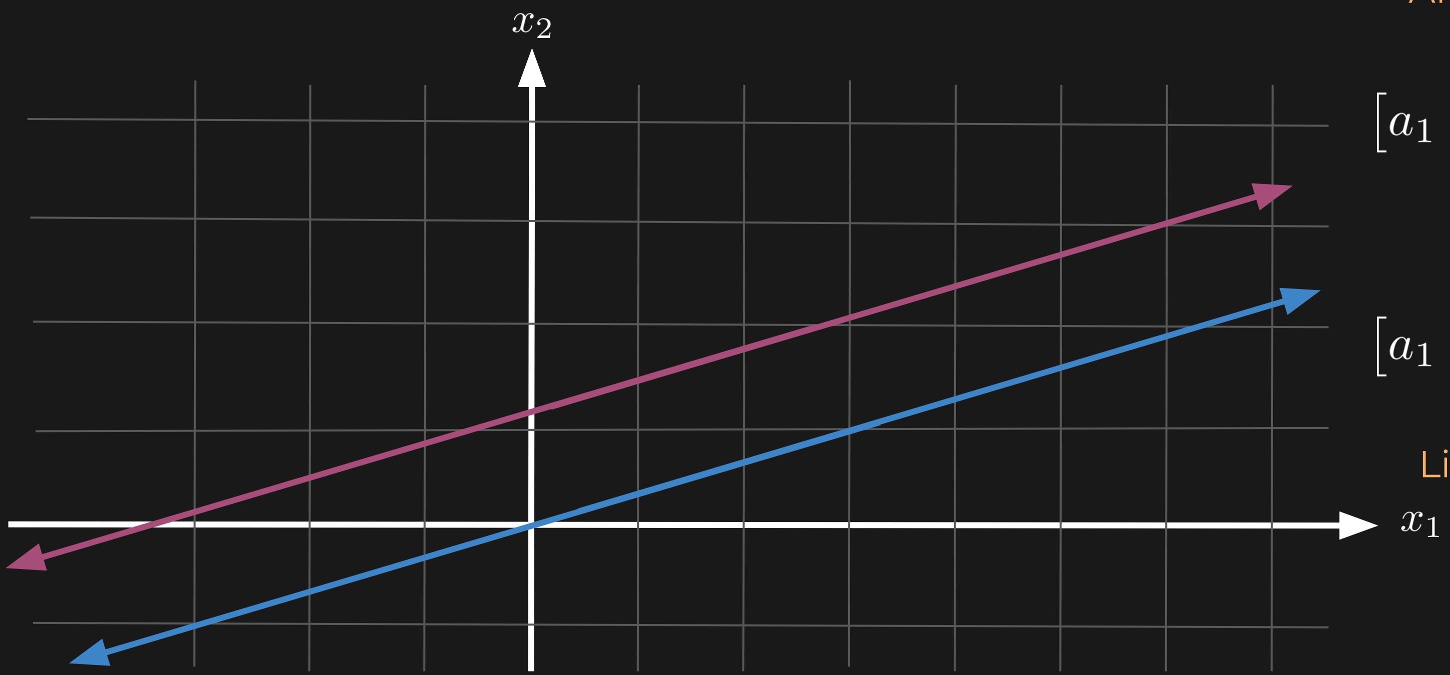
Geometric Interpretation



Affine subspace

$$\begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = y$$

Geometric Interpretation



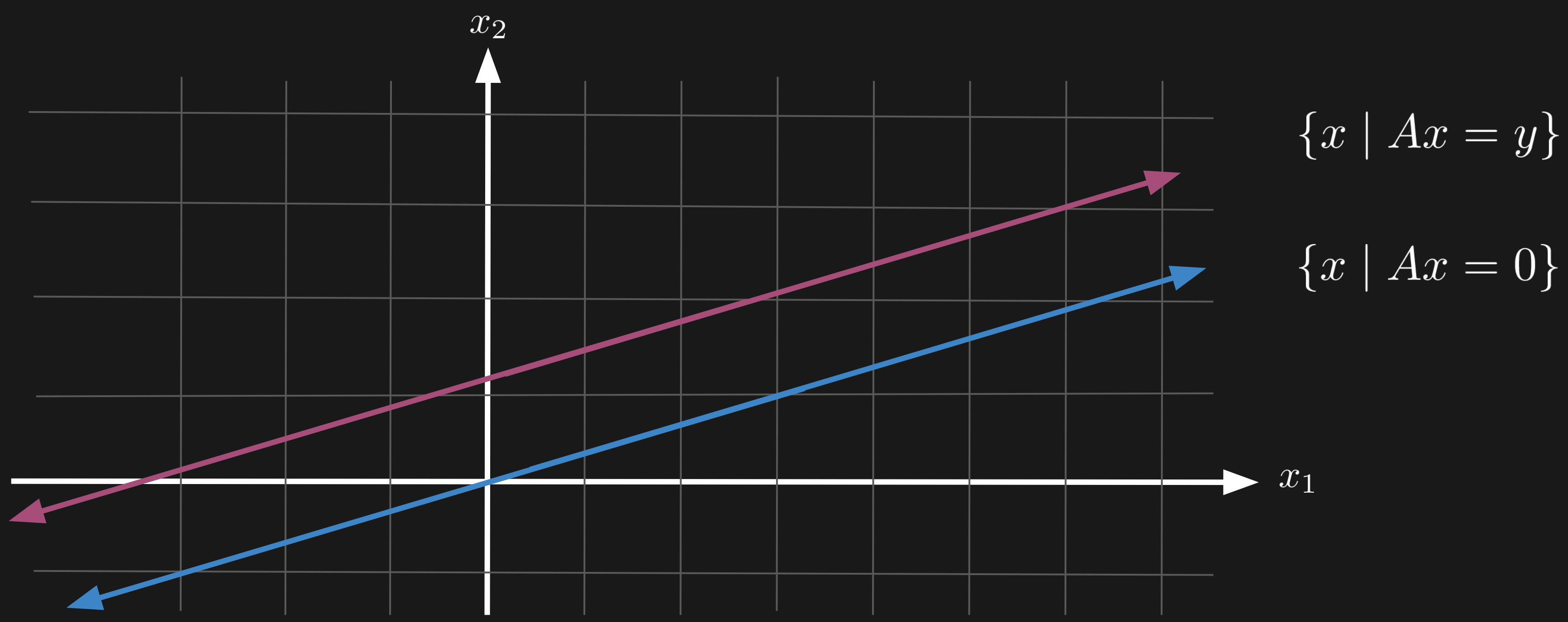
Affine subspace

$$\begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = y$$

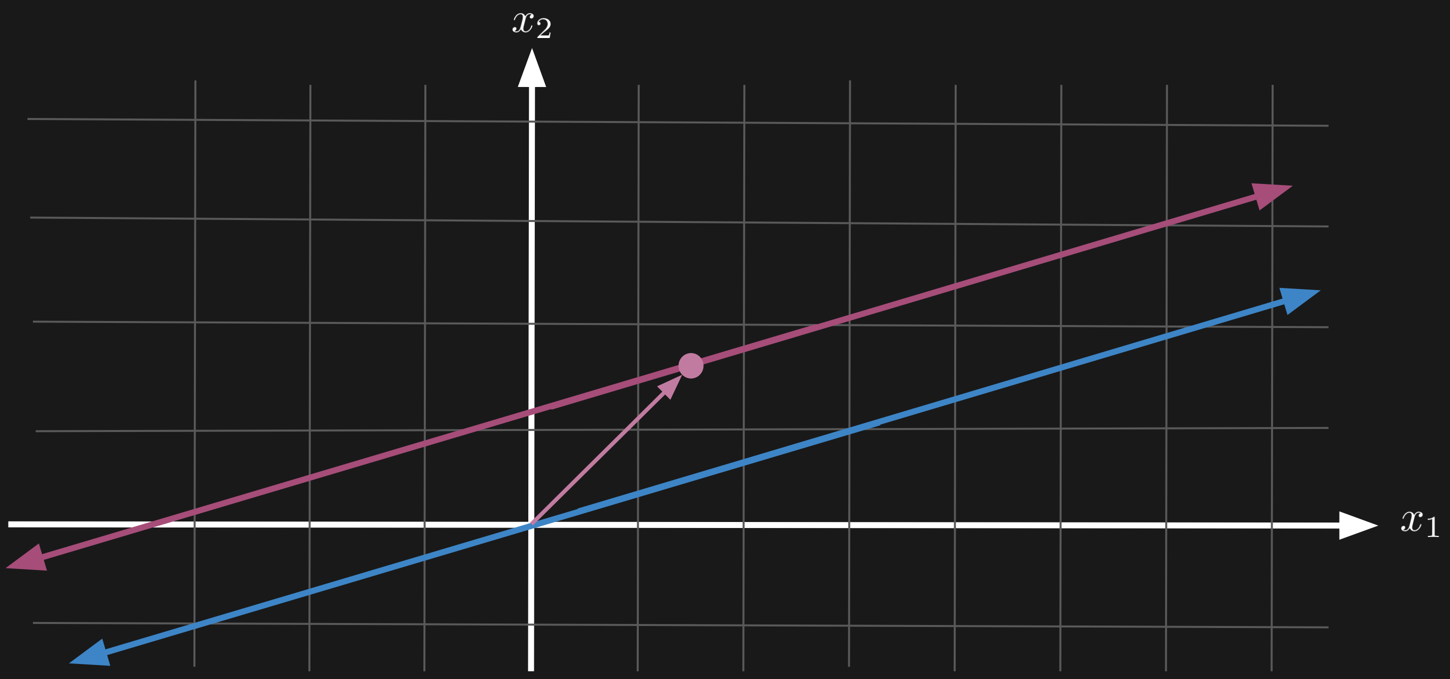
$$\begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Linear subspace

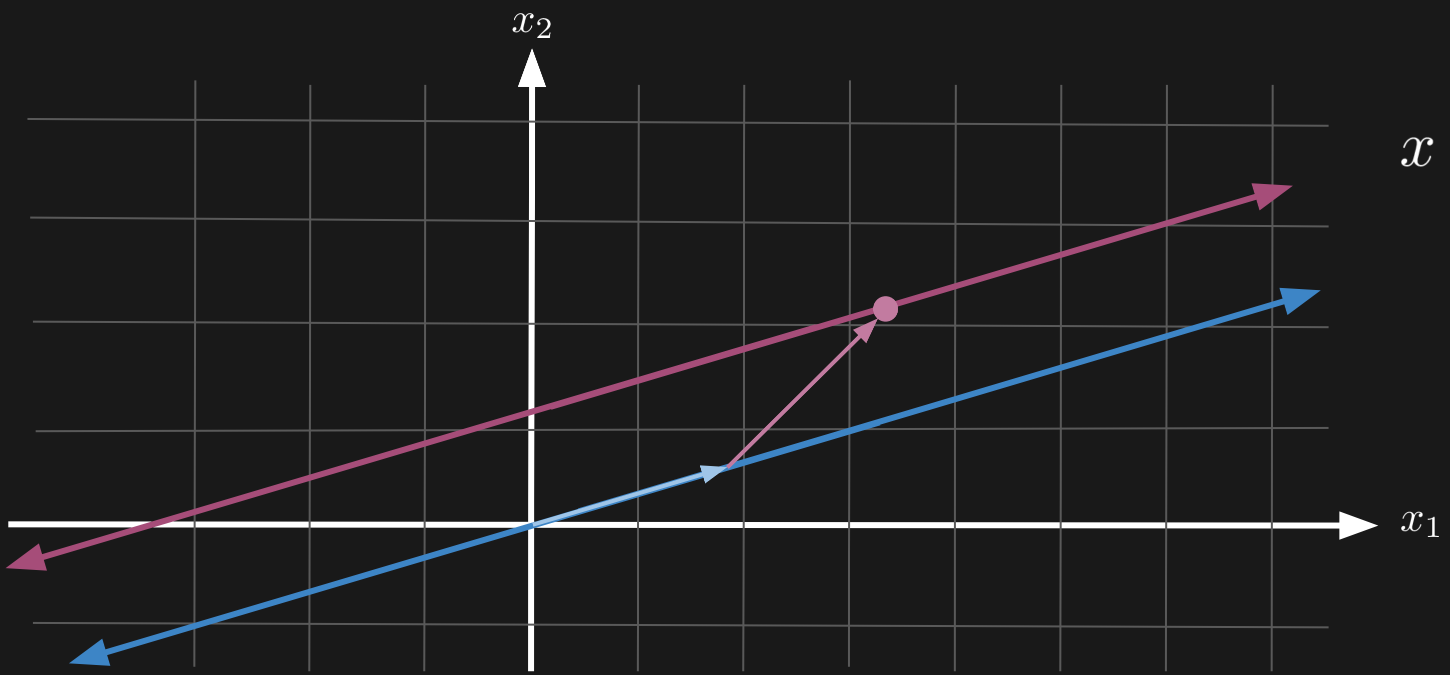
Geometric Interpretation



Geometric Interpretation

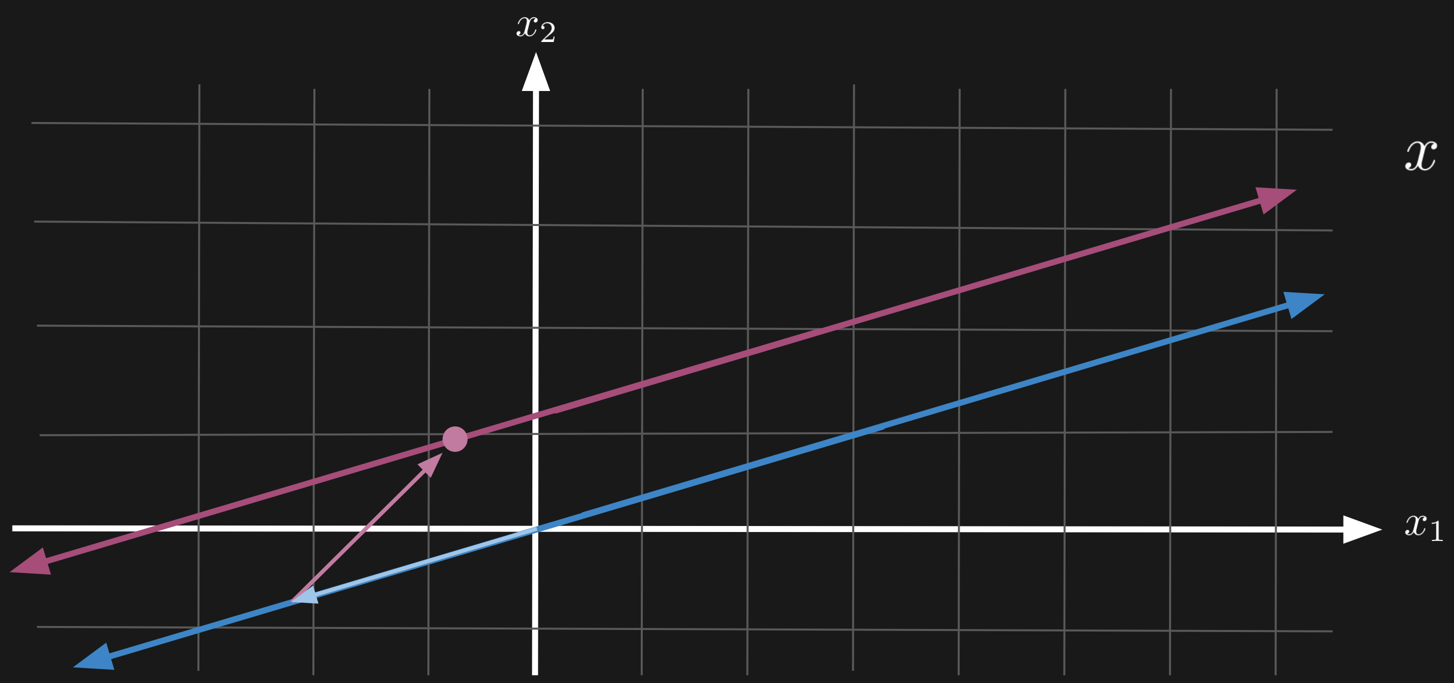


Geometric Interpretation



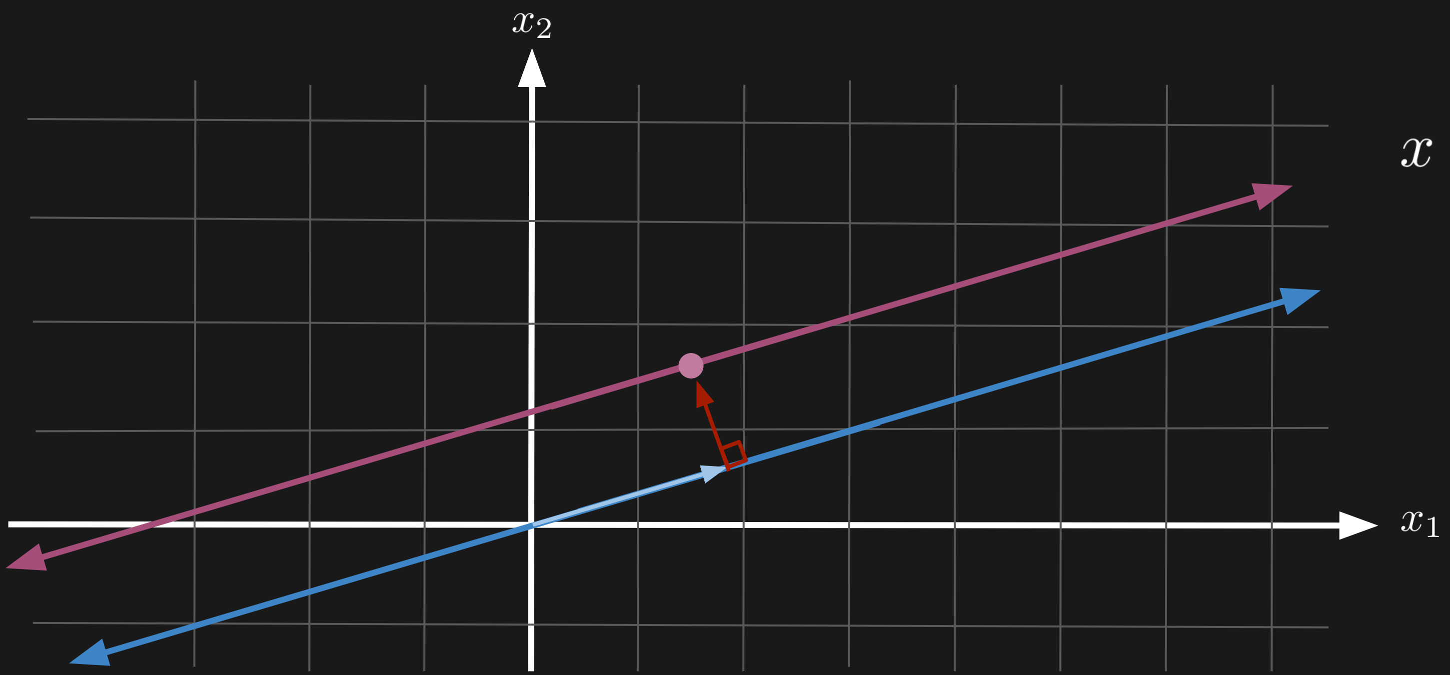
$$x = x_p + x_n$$

Geometric Interpretation



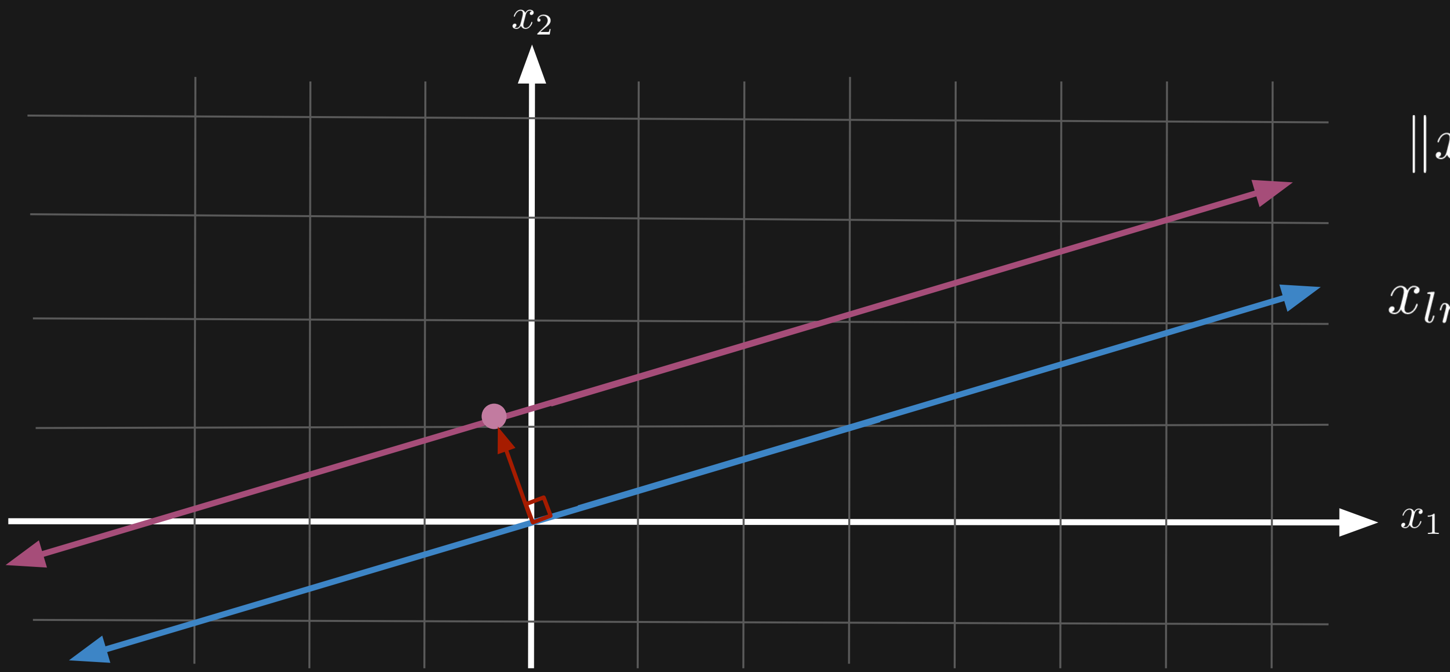
$$x = x_p + x_n$$

Geometric Interpretation



$$x = x_{ln} + x_n$$

Geometric Interpretation



$$\|x_{ln}\| \leq \|x\|$$

$$x_{ln} \perp \mathbf{null}(A)$$

Lagrange Multipliers

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g(x) = 0 \end{aligned}$$

Objective function

Equality constraints

$$L(x, \lambda) = f(x) + \lambda^T g(x)$$

Lagrangian function

Lagrange multiplier

Compute $\nabla_x L$ and $\nabla_\lambda L$ and solve for x and λ

Closed-Form Solution

$$\begin{aligned} \text{minimize} \quad & x^T x \\ \text{subject to} \quad & Ax = y \end{aligned}$$

$$L(x, \lambda) = x^T x + \lambda^T (Ax - y)$$

$$\nabla_x L = 2x + A^T \lambda = 0$$

$$\nabla_\lambda L = Ax - y = 0$$

Closed-Form Solution

$$\nabla_x L = 2x + A^T\lambda = 0 \qquad \Rightarrow x = -\frac{A^T\lambda}{2}$$

$$\nabla_\lambda L = Ax - y = 0 \qquad \Rightarrow -\frac{AA^T\lambda}{2} - y = 0$$

$$\lambda = -2(AA^T)^{-1}y$$

$$x_{ln} = A^T(AA^T)^{-1}y$$

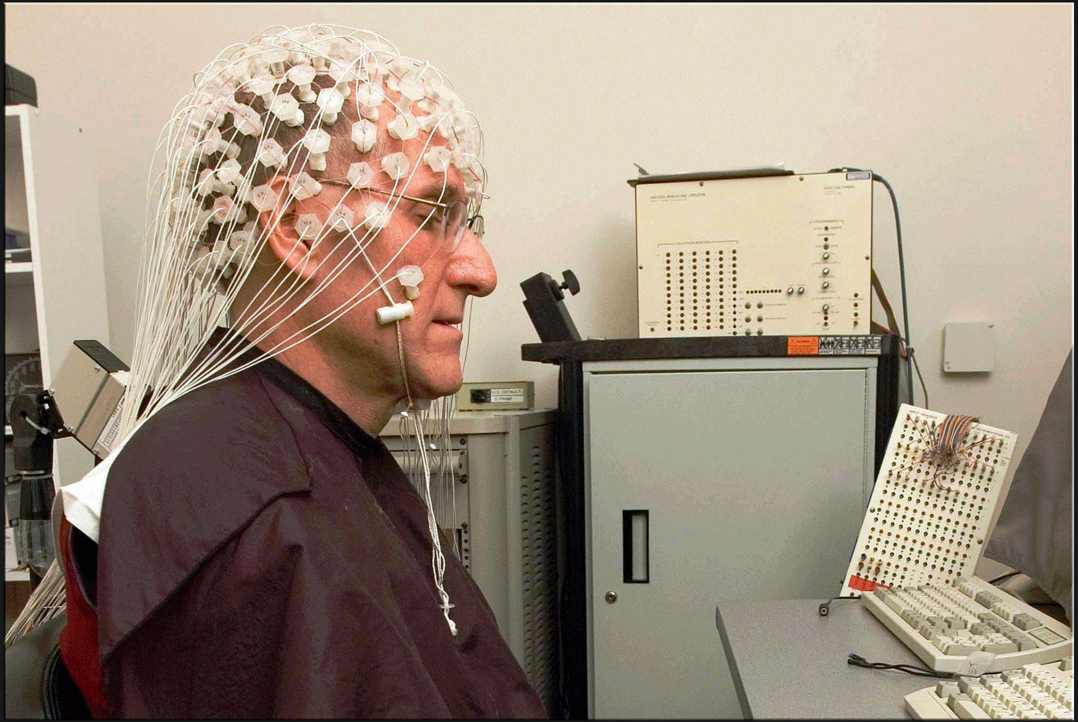
Pseudoinverse

A is skinny	Least squares	$x_{ls} = (A^T A)^{-1} A^T y$
A is fat	Least norm	$x_{ln} = A^T (A A^T)^{-1} y$

- Both matrices are called the Moore-Penrose inverse (pseudoinverse)
- Formula for pseudoinverse depends on whether A is skinny or fat
- In either case, A must be full rank

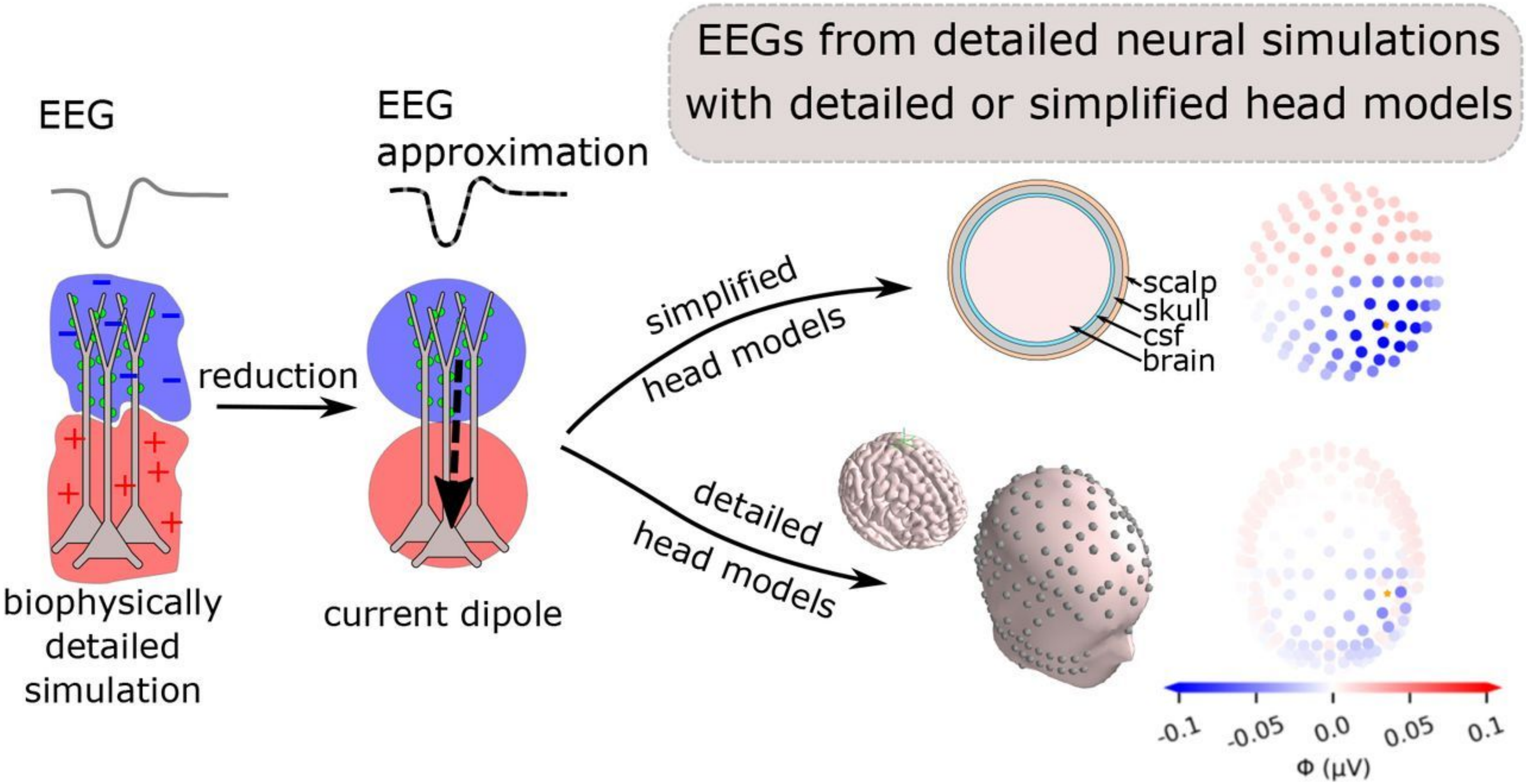
Application: Bioelectromagnetics

Electroencephalography
(EEG)

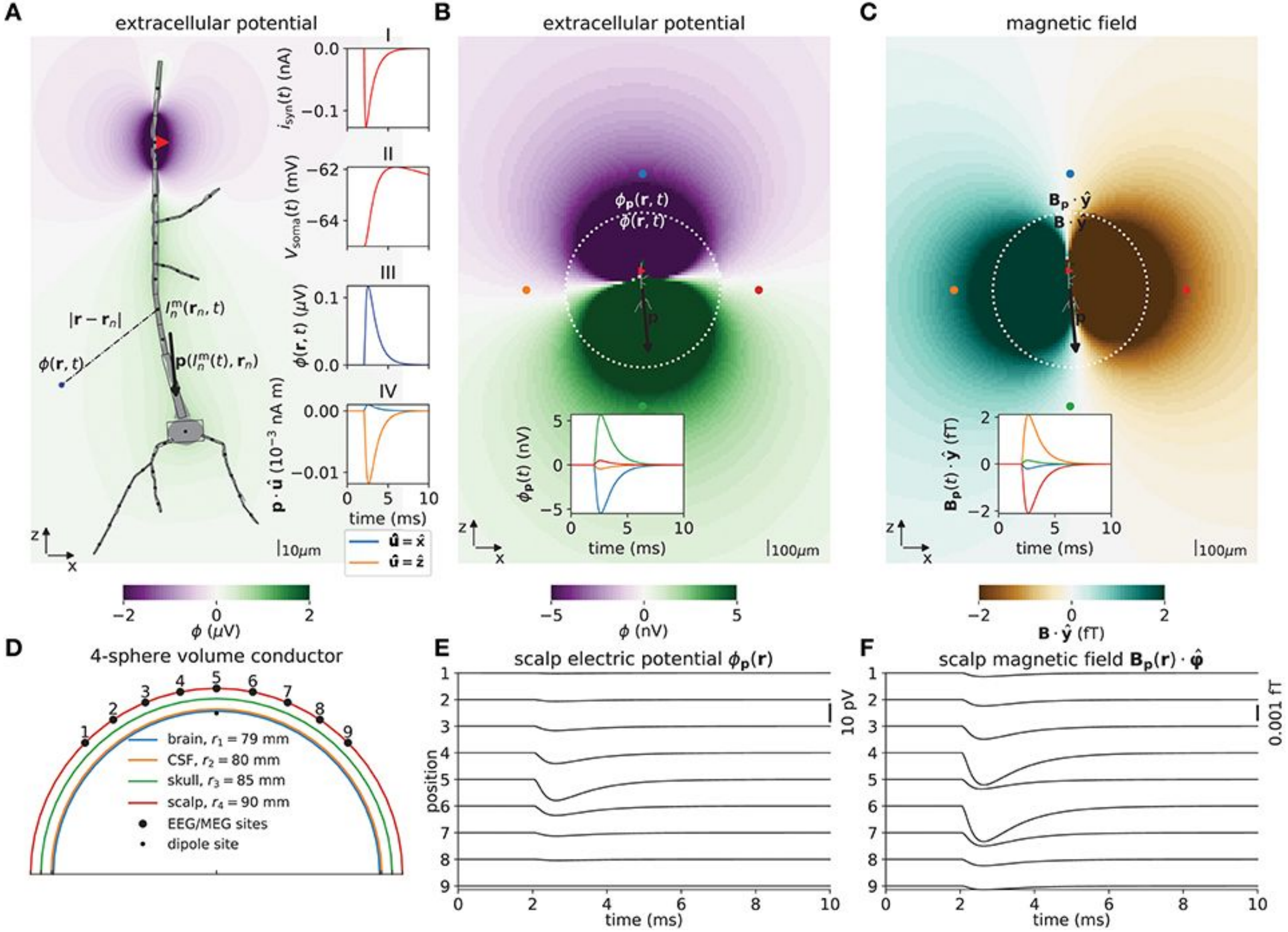


Application: Bioelectromagnetics

Magnetoencephalography
(MEG)



From Naess et. al., "Biophysically detailed forward modeling of the neural origin of EEG and MEG signals"



From Hagen et. al., "Multimodal Modeling of Neural Network Activity: Computing LFP, ECoG, EEG, and MEG Signals With LFPy 2.0"

Application: Bioelectromagnetics

$$d = Ls$$

Voltages measured by m electrodes

Leadfield matrix

Amplitudes of n current sources (neurons)

$$\hat{s} = L^T(LL^T)^{-1}d$$

Minimum norm estimate (MNE)

Application: Bioelectromagnetics

$$P = IV$$

Power in a circuit

$$V = IR$$

Ohm's law

$$\Rightarrow P = I^2 R$$

$$\Rightarrow P_1 + P_2 + \cdots + P_n = (I_1^2 + I_2^2 + \ldots I_n^2)R$$

Total power in many circuits

$$\sum_i P_i = I^T I R$$

Least norm objective function

- Least norm solution corresponds to minimum power exerted by brain
- Makes sense biologically – brain has evolved to become very efficient

Next Time

- QR factorization