Exploring Vectors

Practical Linear Algebra | Lecture 2

Scalars

- A scalar is a single number

$$3$$

Scalars

- A scalar is a single number

$$\pi$$

Scalars

- A scalar is a single number

$$\alpha$$

Scalars

- A scalar is a single number

$$x$$

Scalars

- A scalar is a single number

$$x_1$$

Vectors

- A vector is an ordered sequence of numbers

$$\begin{bmatrix} -11 \\ 4 \\ 9 \end{bmatrix}$$

Vectors

- A vector is an ordered sequence of numbers

$$\begin{bmatrix} -1 \\ 0 \\ \pi \\ 98 \end{bmatrix}$$

Vectors

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$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Vectors

- A vector is an ordered sequence of numbers

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Vectors

$$x \in \mathbb{R}^n$$

$$\alpha \in \mathbb{R}$$

Column Vectors and Row Vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \begin{bmatrix} -11 \\ 4 \\ 9 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 0 \\ \pi \\ 98 \end{bmatrix}$$

$$\begin{bmatrix} 15 & 3 & -1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} y_1 & y_2 & y_3 & y_4 \end{bmatrix}$$

$$\begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}$$

Transpose

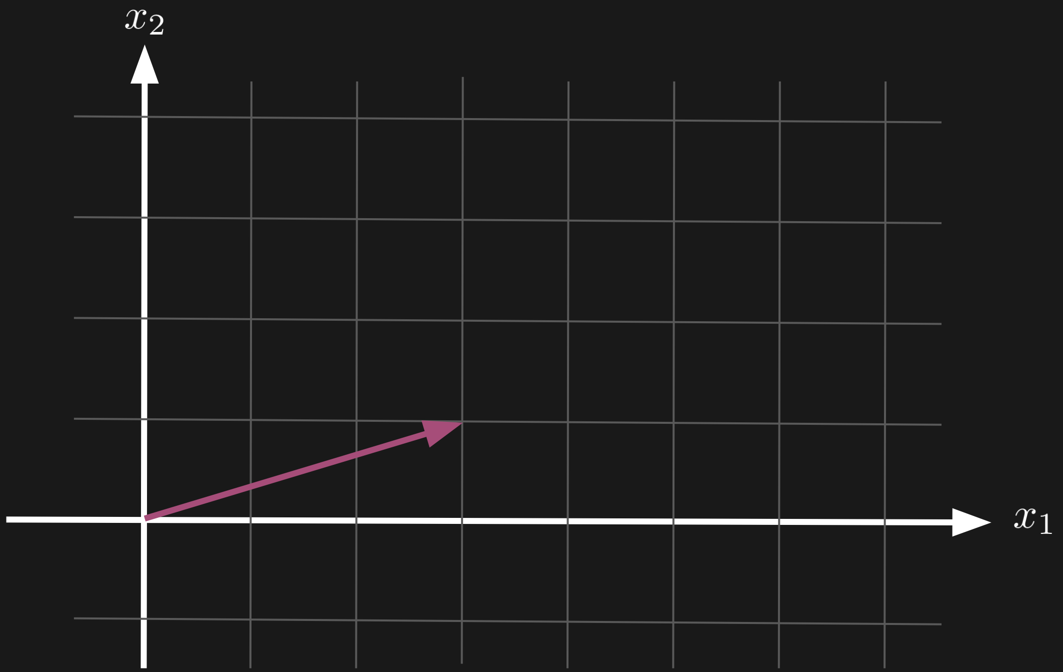
$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}^T = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}$$

Transpose

$$\begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Visualizing Vectors

- A vector is a line or a direction



$$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Visualizing Vectors

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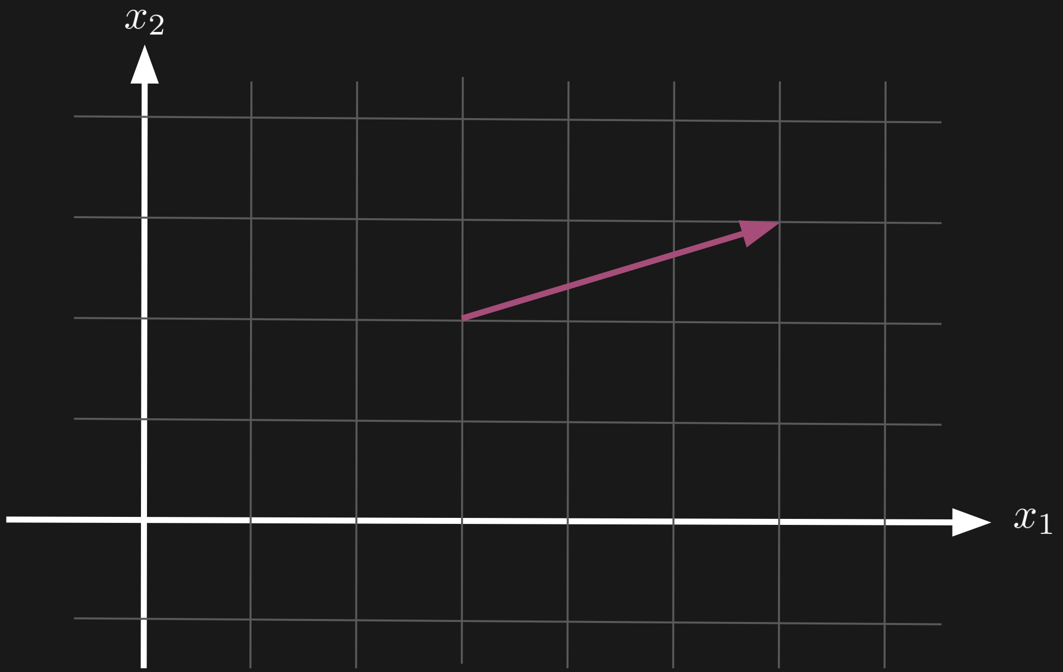

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Visualizing Vectors

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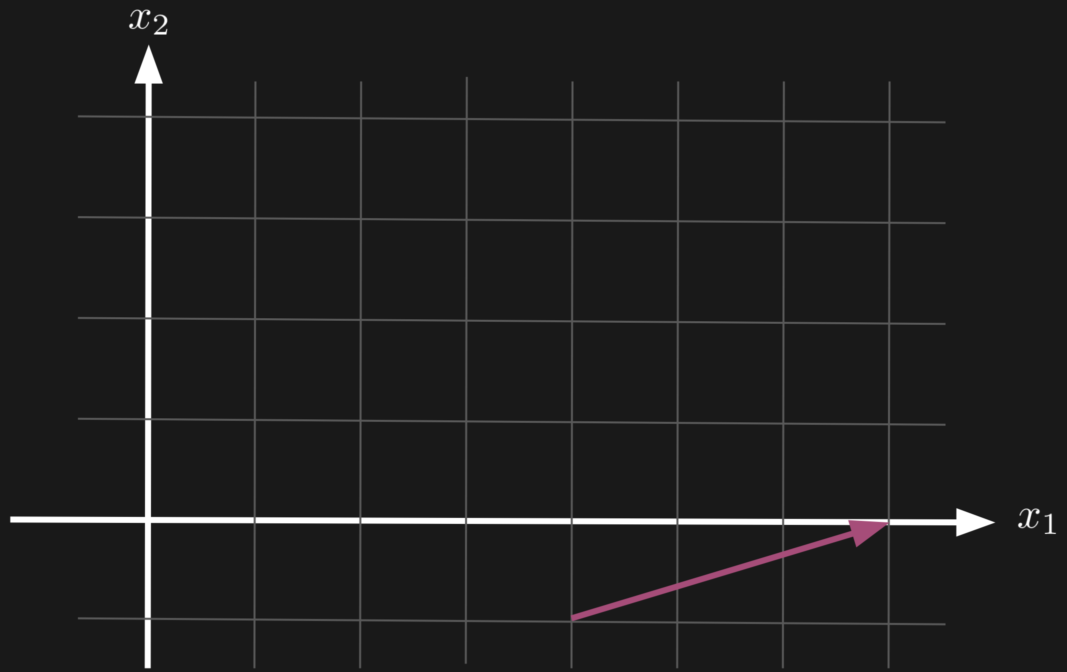
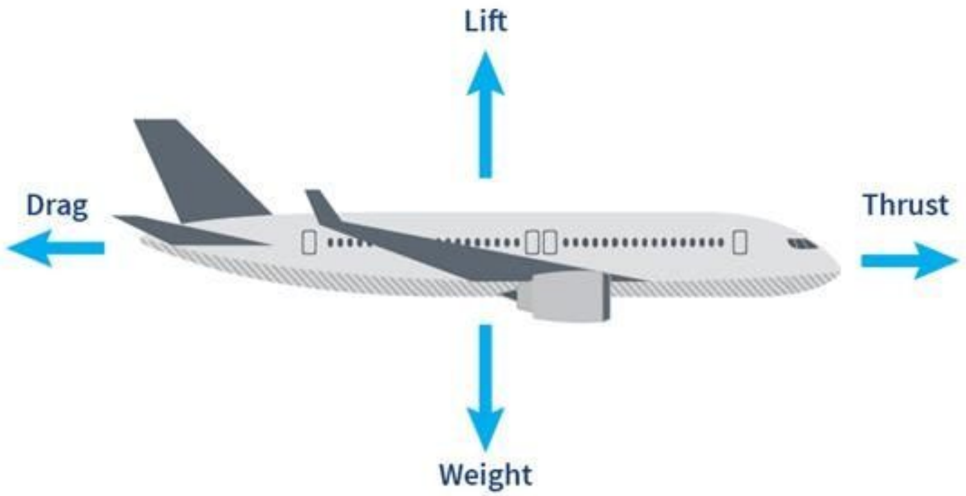


$$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Examples of Vectors

- Forces on an aircraft



Examples of Vectors

- Movie ratings

Ratings

Users

	Movie A	Movie B	Movie C	Movie D	Movie E	Movie F	Movie G		Movie ZZ
User A	5	5	-	-	4	3	4	…	-
User B	1	3	3	5		-	3	…	-
User C	-	-	-	1	5	-	-	…	-
User D	4	-			5	3	4	…	-
…									
User ZZZ	2	-	-	-	1	-	3	…	-

Examples of Vectors

- Web Search

$$
\begin{array}{l}
\text{Site 1} \\
\text{Site 2} \\
\text{Site 3} \\
\\
\text{Site } n-2 \\
\text{Site } n-1 \\
\text{Site } n
\end{array}
\quad
\begin{bmatrix} 1 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}
$$

Scalar-Vector Multiplication

$$\alpha v = \alpha \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} \alpha v_1 \\ \alpha v_2 \\ \vdots \\ \alpha v_n \end{bmatrix}$$

Vector Addition

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

Product of Two Vectors

- Element-wise multiplication
    - Also known as the Hadamard product or Schur product

$$x \circ y = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \circ \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 y_1 \\ x_2 y_2 \\ \vdots \\ x_n y_n \end{bmatrix}$$

Product of Two Vectors

- Element-wise multiplication
  - Written using either $\bigcirc$ or $\odot$

$$x \odot y = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \odot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 y_1 \\ x_2 y_2 \\ \vdots \\ x_n y_n \end{bmatrix}$$

Product of Two Vectors

- Dot product or inner product

$$x \cdot y = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

Product of Two Vectors

- Dot product or inner product

$$x \cdot y = x^T y = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Vector Norm

- The (Euclidean) norm of a vector is its length

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

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- The (Euclidean) norm of a vector is its length

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{x^T x}$$

Angle

- Formula for the angle between two vectors
  - Angle will lie between 0 and 180 degrees

$$\theta = \cos^{-1} \frac{x^T y}{\|x\| \|y\|}$$

Angle

- Formula is more intuitive if we rearrange it
    - Dot product tells you how “aligned” two vectors are

$$x^T y = \|x\| \|y\| \cos \theta$$

Angle

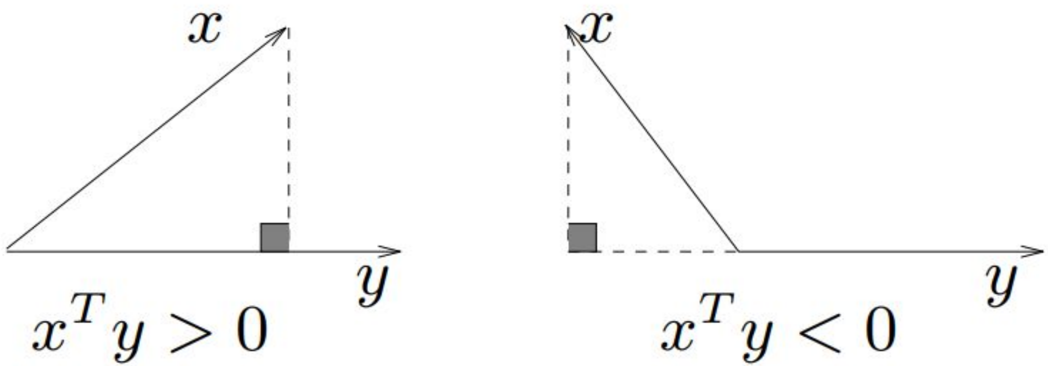

Image source: http://ee263.stanford.edu/lectures/linalg.pdf (page 5)

Subspaces

- A subspace is a set $S \subset \mathbb{R}^n$ that satisfies these two constraints:
  - $x + y \in S$ for all $x, y \in S$
  - $\alpha x \in S$ for all $\alpha \in \mathbb{R}$ and $x \in S$

Subspaces

- A subspace is a set $S \subset \mathbb{R}^n$ that satisfies these two constraints:
  - S is closed under addition
  - S is closed under scalar multiplication

Span

- The span of a set of vectors is the set of all linear combinations of those vectors
  - Spans are also subspaces

$$\mathbf{span}(v_1, v_2, \ldots, v_k) = \{\alpha_1 v_1 + \cdots + \alpha_k v_k \mid \alpha_i \in \mathbb{R}\}$$

Linear Independence

- A set of vectors $\{v_1, v_2, \ldots, v_k\}$ is linearly independent (or just independent) if none of the vectors in the set can be expressed as a linear combination of the other vectors

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- Equivalent condition:

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0 \Rightarrow \alpha_1 = \alpha_2 = \cdots = 0$$

Linear Independence

- Intuitive definition: A set of vectors is independent if removing any of the vectors would make the span smaller
- Said another way, if you have a set of linearly dependent vectors, then removing one vector won’t change the span – you won’t lose any information
    - You can remove one vector at a time until you get a set of independent vectors

Basis and Dimension

- A set of vectors $\{v_1, v_2, \ldots, v_k\}$ is a basis for a subspace S if these two constraints are satisfied:
  - $S = \mathbf{span}(v_1, v_2, \ldots, v_k)$
  - $\{v_1, v_2, \ldots, v_k\}$ is independent

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- The number of vectors in any basis of a given subspace S is always the same. This number is called the dimension of S.
    - Notation: $d = \mathbf{dim}\, S$

Next Time

- Vectors in NumPy
- Application: k-Nearest Neighbors algorithm in Scikit-learn