Eigenvectors

Practical Linear Algebra | Lecture 11

Eigenvectors and Eigenvalues

- For a given matrix A, an eigenvector is a vector v that satisfies

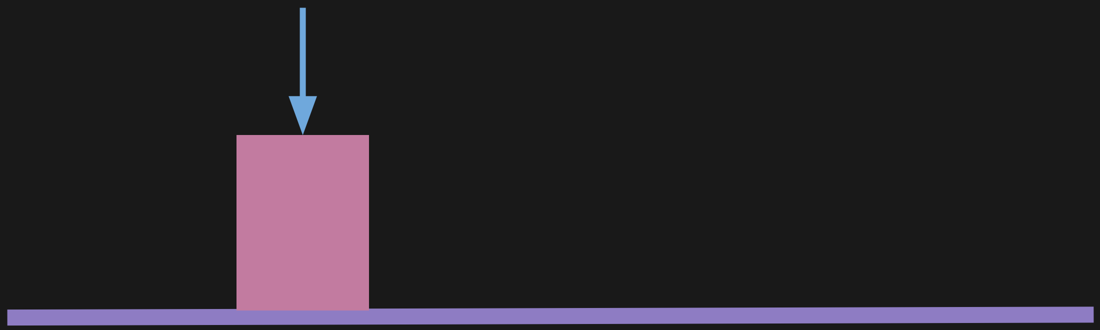
$$Av = \lambda v$$

- λ is a scalar called an eigenvalue
- Two or more different eigenvectors can have the same eigenvalue
- The zero vector isn't considered an eigenvector ($v \neq 0$)
- Definition of eigenvector implies A is square
- Eigenvectors and eigenvalues of a real matrix can be complex!

  $v \in \mathbb{C}^n, \ \lambda \in \mathbb{C}$

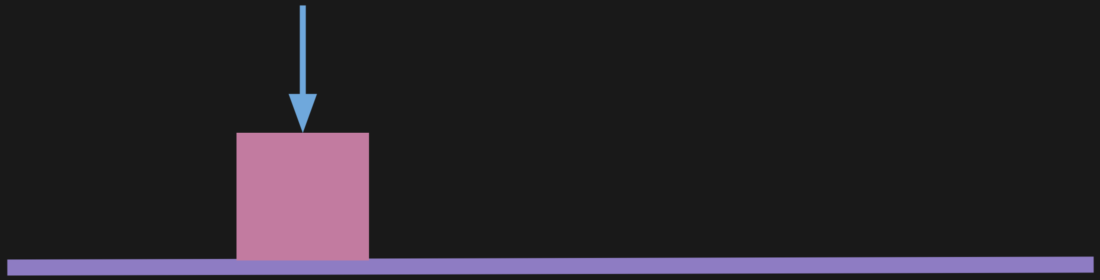
Why do we care?

- Eigenvectors are the “natural directions” of a system



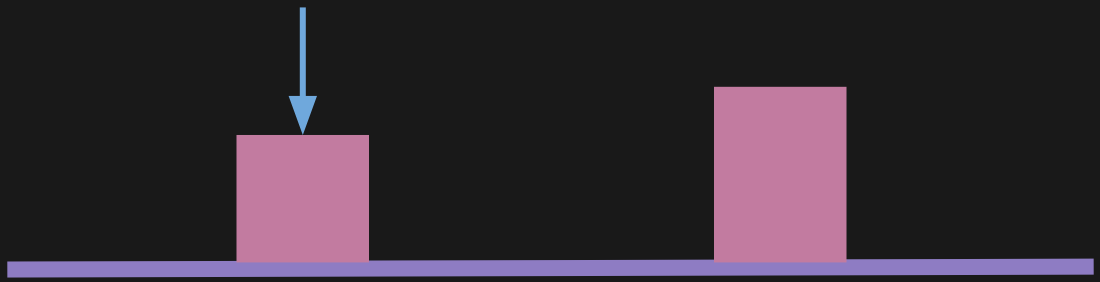
Why do we care?

- Eigenvectors are the “natural directions” of a system



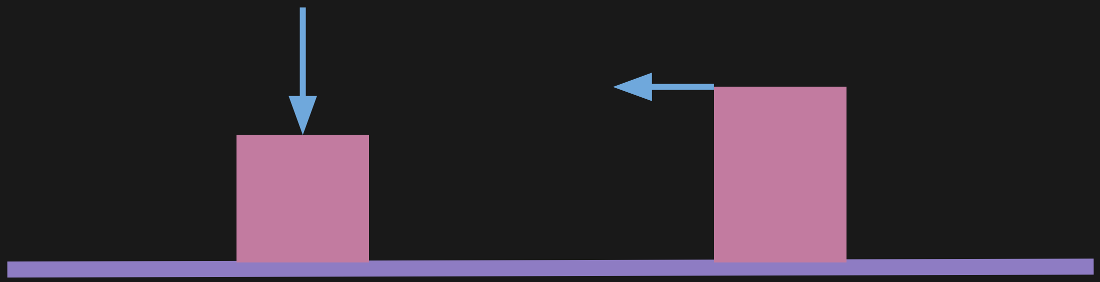
Why do we care?

- Eigenvectors are the “natural directions” of a system



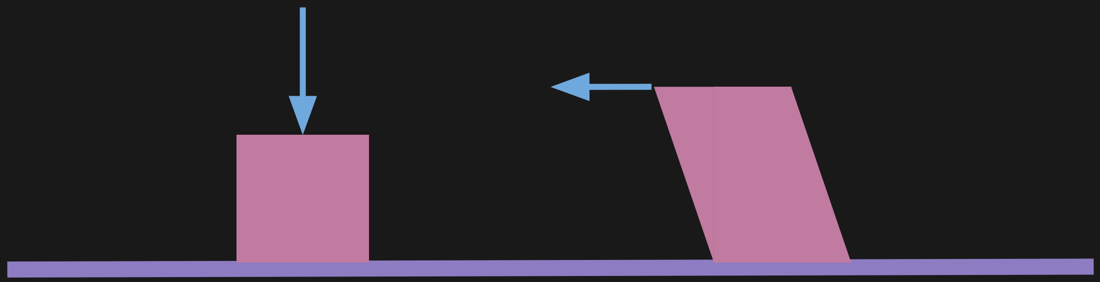
Why do we care?

- Eigenvectors are the “natural directions” of a system



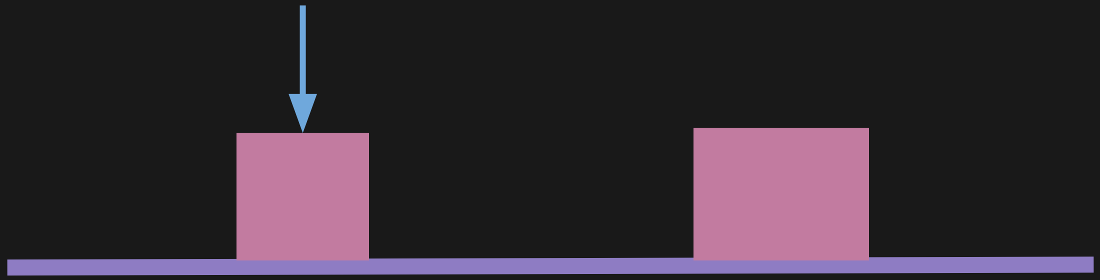
Why do we care?

- Eigenvectors are the “natural directions” of a system



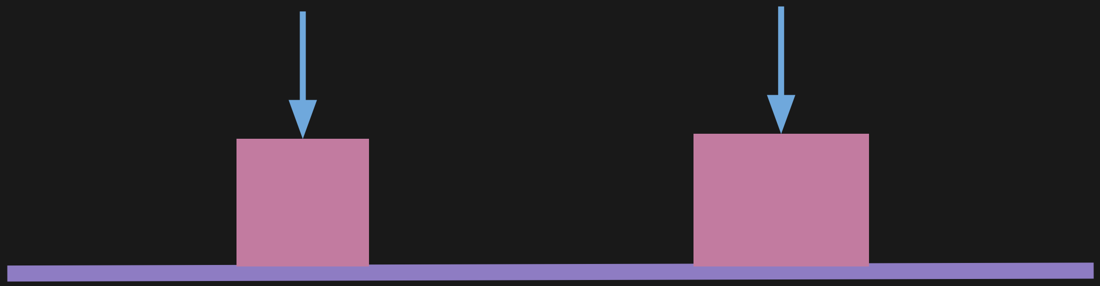
Why do we care?

- Eigenvectors are the “natural directions” of a system
- Eigenvalues quantify the system’s sensitivity to a given input direction



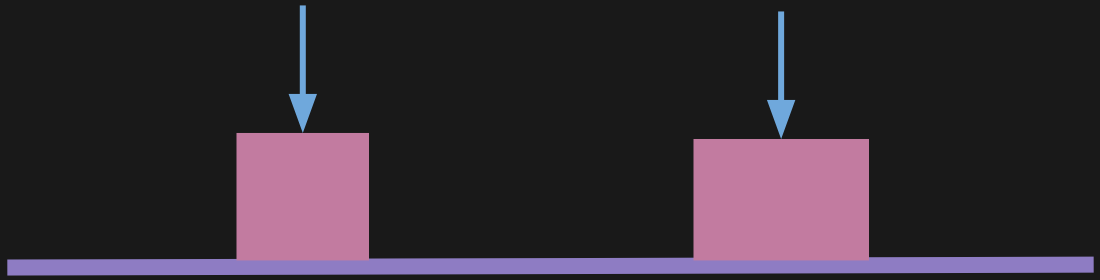
Why do we care?

- Eigenvectors are the “natural directions” of a system
- Eigenvalues quantify the system’s sensitivity to a given input direction



Why do we care?

- Eigenvectors are the “natural directions” of a system
- Eigenvalues quantify the system’s sensitivity to a given input direction



How to find eigenvectors

$$Av = \lambda v$$

$$Av - \lambda v = 0$$

$$Av - \lambda I v = 0$$

$$(A - \lambda I)v = 0$$

$$v \in \mathbf{null}(A - \lambda I)$$

How to find eigenvectors

$$(A - \lambda I)v = 0$$

- $A - \lambda I$ has linearly dependent columns
- $A - \lambda I$ must be **not** invertible – the matrix is singular
- Inverse of two-by-two matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{\mathbf{det}(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

How to find eigenvectors

- We want to find all values of λ that make $A - \lambda I$ singular

$$\mathbf{det}(A - \lambda I) = 0$$

- In other words, we want to find the roots of the characteristic polynomial

$$\mathbf{det}(A - \lambda I) = \mathbf{det}\left(\begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix}\right)$$

$$= (a - \lambda)(d - \lambda) - bc = 0$$

- Exactly n roots exist (possibly complex, possibly repeated)

How to find eigenvectors

- The set of all eigenvalues of a matrix A is called the spectrum of A
  - Denoted $\mathbf{spec}(A)$

Our Plan

- Eigenvectors reveal something deep about matrices…
- We will:
  - Review complex numbers
  - Introduce symmetric matrices
  - Prove symmetric matrices have real eigenvalues
  - Prove real eigenvalues mean real eigenvectors exist
  - Prove symmetric matrices have orthogonal eigenvectors
- *These facts build up to something amazing…*

Complex Numbers and Vectors

complex number $$z = a + bi \in \mathbb{C}$$

$$a, b \in \mathbb{R} \qquad i = \sqrt{-1}$$

complex conjugate $$\overline{z} = a - bi$$

Complex Numbers and Vectors

some properties

$$a + bi + a - bi = 2a$$

$$z + \overline{z} = 2\,\mathrm{Re}(z)$$

$$z\overline{z} = a^2 + b^2$$

$$\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$$

Complex Numbers and Vectors

complex vector $$v \in \mathbb{C}^n$$

$$v + \overline{v} = 2\,\mathrm{Re}(v)$$

$$\overline{v}^T v = \|v\|^2$$

Symmetric Matrices

- A matrix is symmetric if it satisfies

$$A^T = A$$

- Definition implies A is square
- Example: $\begin{bmatrix} 4 & 3 & 8 \\ 3 & 1 & 5 \\ 8 & 5 & 9 \end{bmatrix}$

Symmetric Matrices

- Symmetric matrices are worth studying
  - They show up naturally in many problems
  - We’ll see later they tell us a lot about arbitrary matrices
- We’ll stick to real symmetric matrices for the rest of this lecture

Symmetric matrices have real eigenvalues

(Assume $A^T = A,\ A \in \mathbb{R}^{n \times n}$)

$$Av = \lambda v$$

$$\overline{Av} = \overline{\lambda v}$$

$$\overline{A}\,\overline{v} = \overline{\lambda}\,\overline{v}$$

$$A\overline{v} = \overline{\lambda}\,\overline{v}$$

$$(A\overline{v})^T = (\overline{\lambda}\,\overline{v})^T$$

$$\overline{v}^T A^T = \overline{\lambda}\,\overline{v}^T$$

Symmetric matrices have real eigenvalues

(Assume $A^T = A,\ A \in \mathbb{R}^{n\times n}$)

$$\overline{v}^T A^T = \overline{\lambda}\,\overline{v}^T$$

$$\overline{v}^T A = \overline{\lambda}\,\overline{v}^T$$

$$\overline{v}^T A v = \overline{\lambda}\,\overline{v}^T v$$

$$\lambda \overline{v}^T v = \overline{\lambda}\,\overline{v}^T v$$

$$\lambda \|v\|^2 = \overline{\lambda} \|v\|^2$$

$$\lambda = \overline{\lambda} \Rightarrow \lambda \in \mathbb{R}$$

Real eigenvalues mean real eigenvectors exist

(Assume $\lambda \in \mathbb{R}$, $A \in \mathbb{R}^{n \times n}$)

$$Av = \lambda v$$

$$\overline{Av} = \overline{\lambda v}$$

$$\overline{A}\overline{v} = \overline{\lambda}\overline{v}$$

$$A\overline{v} = \lambda\overline{v}$$

$$A\overline{v} + Av = \lambda\overline{v} + Av$$

$$A\overline{v} + Av = \lambda\overline{v} + \lambda v$$

Real eigenvalues mean real eigenvectors exist

(Assume $\lambda \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$)

$$A\overline{v} + Av = \lambda\overline{v} + \lambda v$$

$$A(\overline{v} + v) = \lambda(\overline{v} + v)$$

$$A(2\,\mathrm{Re}(v)) = \lambda(2\,\mathrm{Re}(v))$$

$$A\,\mathrm{Re}(v) = \lambda\,\mathrm{Re}(v)$$

$\Rightarrow \mathrm{Re}(v)$ is an eigenvector of A

Orthogonal Eigenvectors

- For any two distinct eigenvalues of a symmetric matrix, the associated eigenvectors are orthogonal
  (Assume $\lambda_1, \lambda_2 \in \mathbb{C},\ \lambda_1 \neq \lambda_2,\ A^T = A$)

$$
\begin{aligned}
\lambda_1 v_1^T v_2 &= (\lambda_1 v_1)^T v_2 \\
&= (A v_1)^T v_2 \\
&= v_1^T A^T v_2 \\
&= v_1^T A v_2 \\
&= v_1^T \lambda_2 v_2
\end{aligned}
$$

Orthogonal Eigenvectors

- For any two distinct eigenvalues of a symmetric matrix, the associated eigenvectors are orthogonal
  (Assume $\lambda_1, \lambda_2 \in \mathbb{C},\ \lambda_1 \neq \lambda_2,\ A^T = A$)

$$\lambda_1 v_1^T v_2 = \lambda_2 v_1^T v_2$$

$$\lambda_1 v_1^T v_2 - \lambda_2 v_1^T v_2 = 0$$

$$(\lambda_1 - \lambda_2) v_1^T v_2 = 0$$

$$\lambda_1 - \lambda_2 \neq 0 \Rightarrow v_1^T v_2 = 0$$

Orthogonal Eigenvectors

- What about repeated eigenvalues?
- Are the associated eigenvectors still orthogonal?

Eigenspaces

- Set of all eigenvectors with the same eigenvalue is called the eigenspace associated with that eigenvalue
  - Includes zero vector to make it a subspace
- We can always find an orthonormal basis for any eigenspace
  - How?

$$Av_1 = \lambda_1 v_1, \; Av_2 = \lambda_1 v_2$$

$$A(v_1 + v_2) = \lambda_1 (v_1 + v_2)$$

$$A(c_1 v_1 + c_2 v_2) = \lambda_1 (c_1 v_1 + c_2 v_2)$$

- Just apply QR factorization to $\begin{bmatrix} v_1 & v_2 \end{bmatrix}$

Eigenspaces

- So what?
- All these facts together mean that every real symmetric matrix has n eigenvectors that form an orthonormal basis for $\mathbb{R}^n$
- Now for the payoff...

Spectral Theorem

$$\begin{aligned} Av_1 &= \lambda_1 v_1 \\ Av_2 &= \lambda_2 v_2 \\ &\vdots \\ Av_n &= \lambda_n v_n \end{aligned}$$

- Can we write this in matrix form?

$$A \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} \lambda_1 v_1 & \lambda_2 v_2 & \cdots & \lambda_n v_n \end{bmatrix}$$

$$A \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Spectral Theorem

- Define $Q \equiv \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$ and $\Lambda \equiv \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$

$$A \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$AQ = Q\Lambda$$

Spectral Theorem

- A little rearranging makes the magic happen...

$$
\begin{aligned}
AQ &= Q\Lambda \\
AQQ^{-1} &= Q\Lambda Q^{-1} \\
A &= Q\Lambda Q^{-1} \\
A &= Q\Lambda Q^{T}
\end{aligned}
$$

- Behold, the spectral theorem! *(look how short the proof is)*
- $Q\Lambda Q^T$ is called the spectral decomposition of A

Spectral Theorem

- Some intuition...

$$A = Q\Lambda Q^T$$

$$A = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix}$$

$$A = \begin{bmatrix} \lambda_1 v_1 & \lambda_2 v_2 & \cdots & \lambda_n v_n \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix}$$

Spectral Theorem

$$Ax = \begin{bmatrix} \lambda_1 v_1 & \lambda_2 v_2 & \cdots & \lambda_n v_n \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix} x$$

$$Ax = \begin{bmatrix} \lambda_1 v_1 & \lambda_2 v_2 & \cdots & \lambda_n v_n \end{bmatrix} \begin{bmatrix} v_1^T x \\ v_2^T x \\ \vdots \\ v_n^T x \end{bmatrix}$$

$$Ax = \lambda_1 v_1 v_1^T x + \lambda_2 v_2 v_2^T x + \cdots + \lambda_n v_n v_n^T x$$

$$Ax = \sum_{i=1}^{n} \lambda_i v_i v_i^T x$$

Spectral Theorem

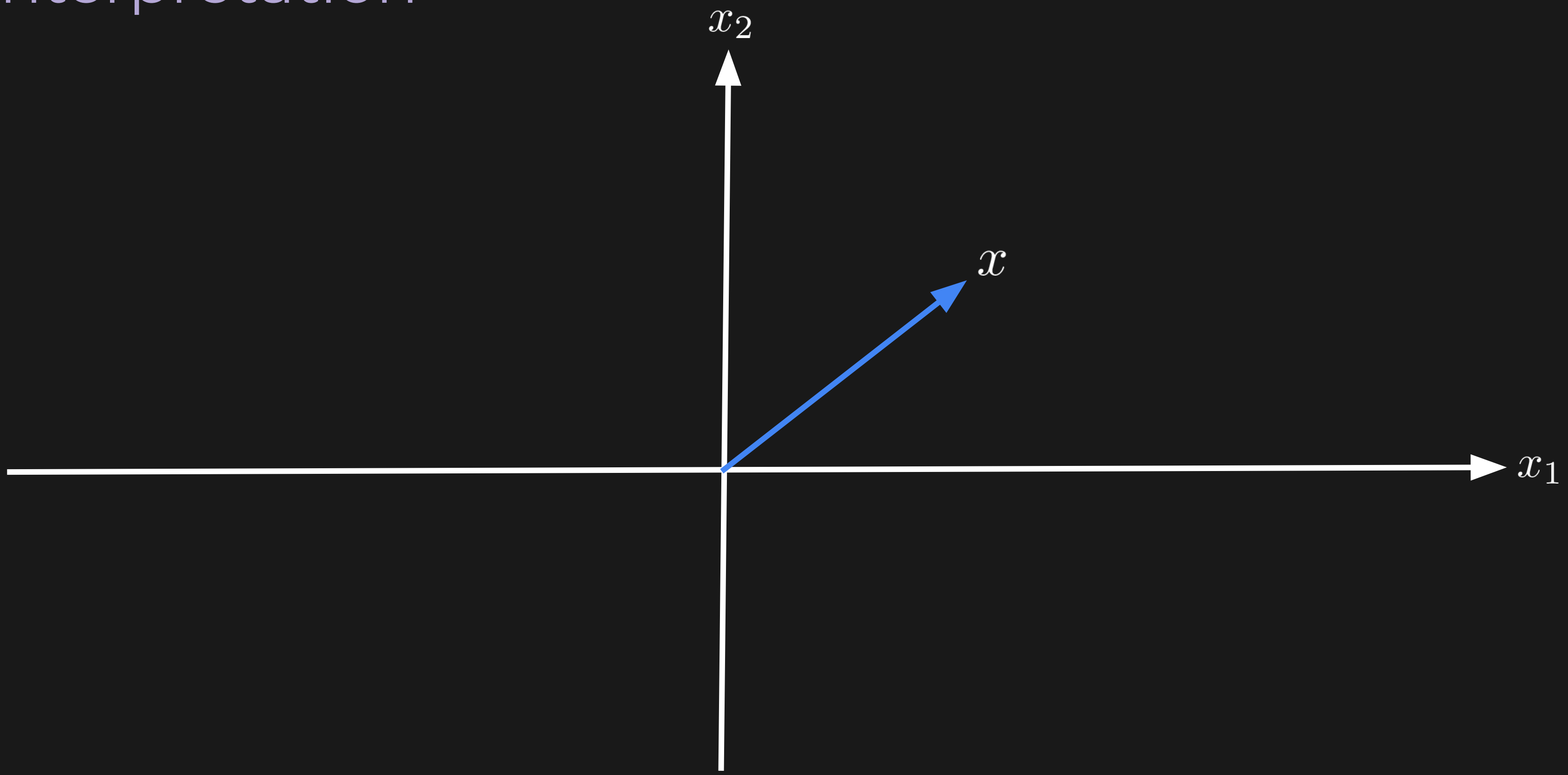
$$A = \sum_{i=1}^{n} \lambda_i v_i v_i^T$$

- $v_i v_i^T$ is a dyad
    - It's the outer product of each eigenvector with itself
- Any real symmetric matrix is the sum of dyads made from its eigenvectors, weighted by its eigenvalues!

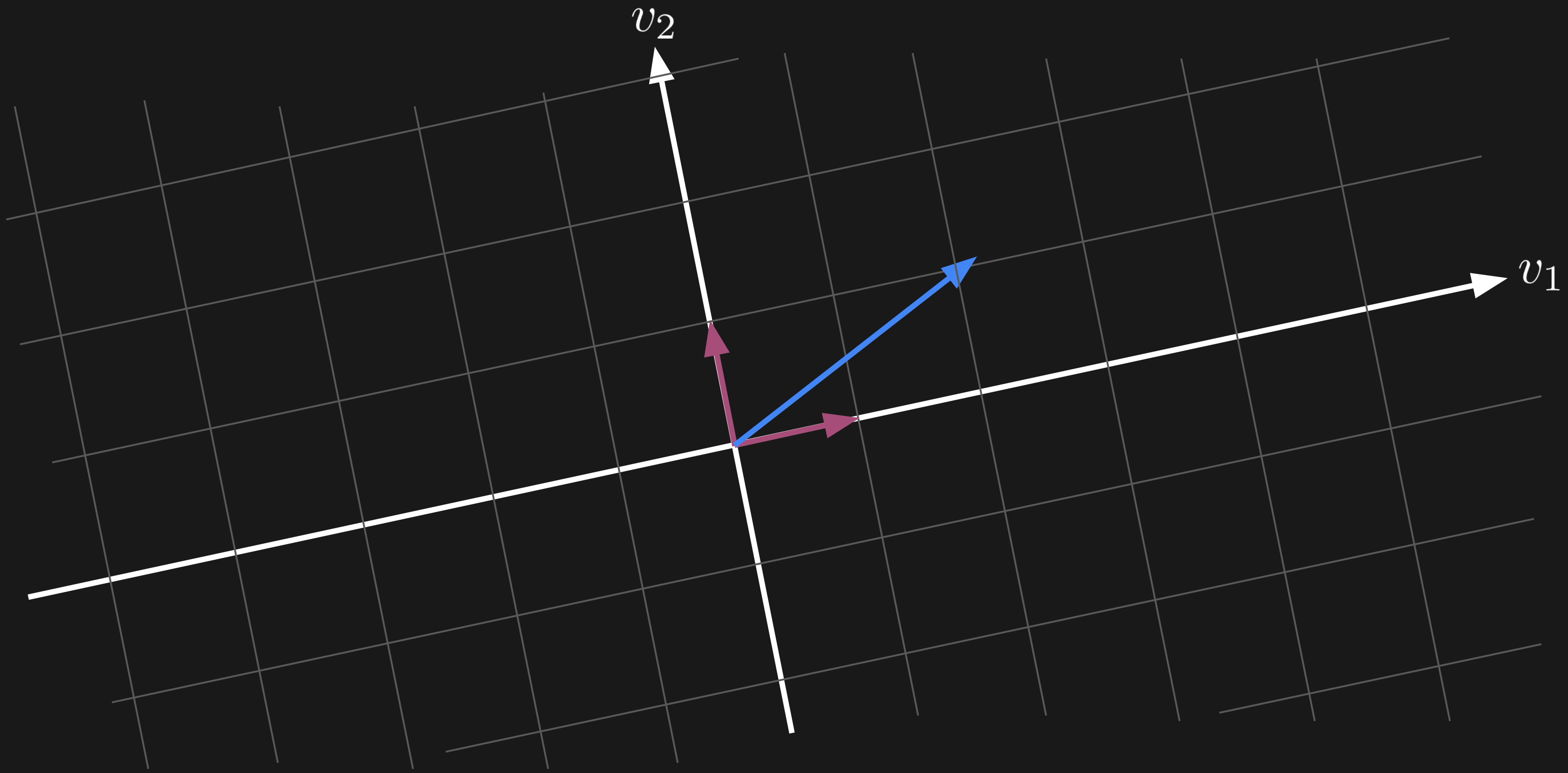
Interpretation

- Eigenvectors and eigenvalues tell you how sensitive the matrix is to different input directions
- Some input directions might be very important, while some might be unimportant
- Eigenvectors form a natural basis for the matrix

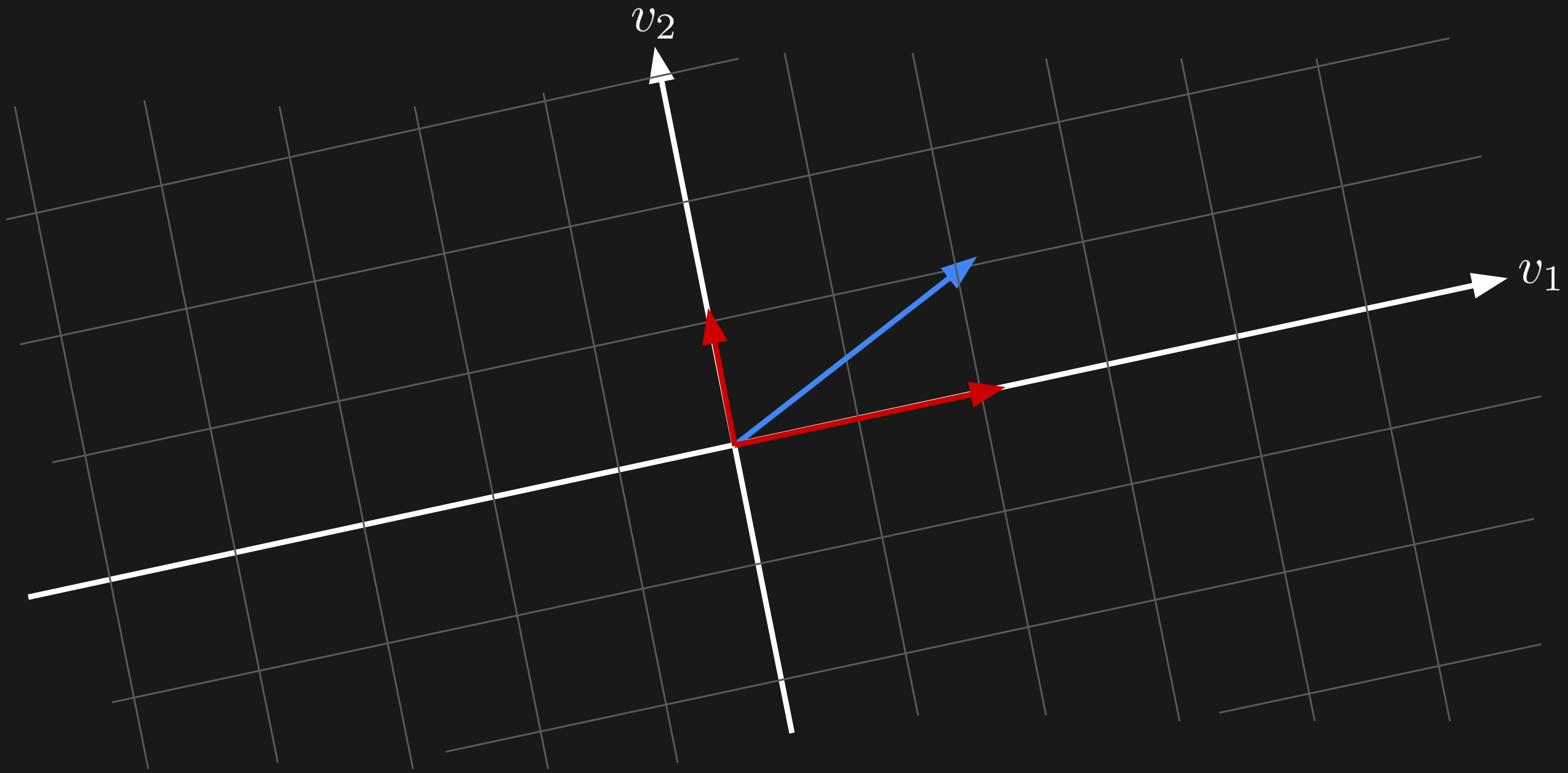
Interpretation



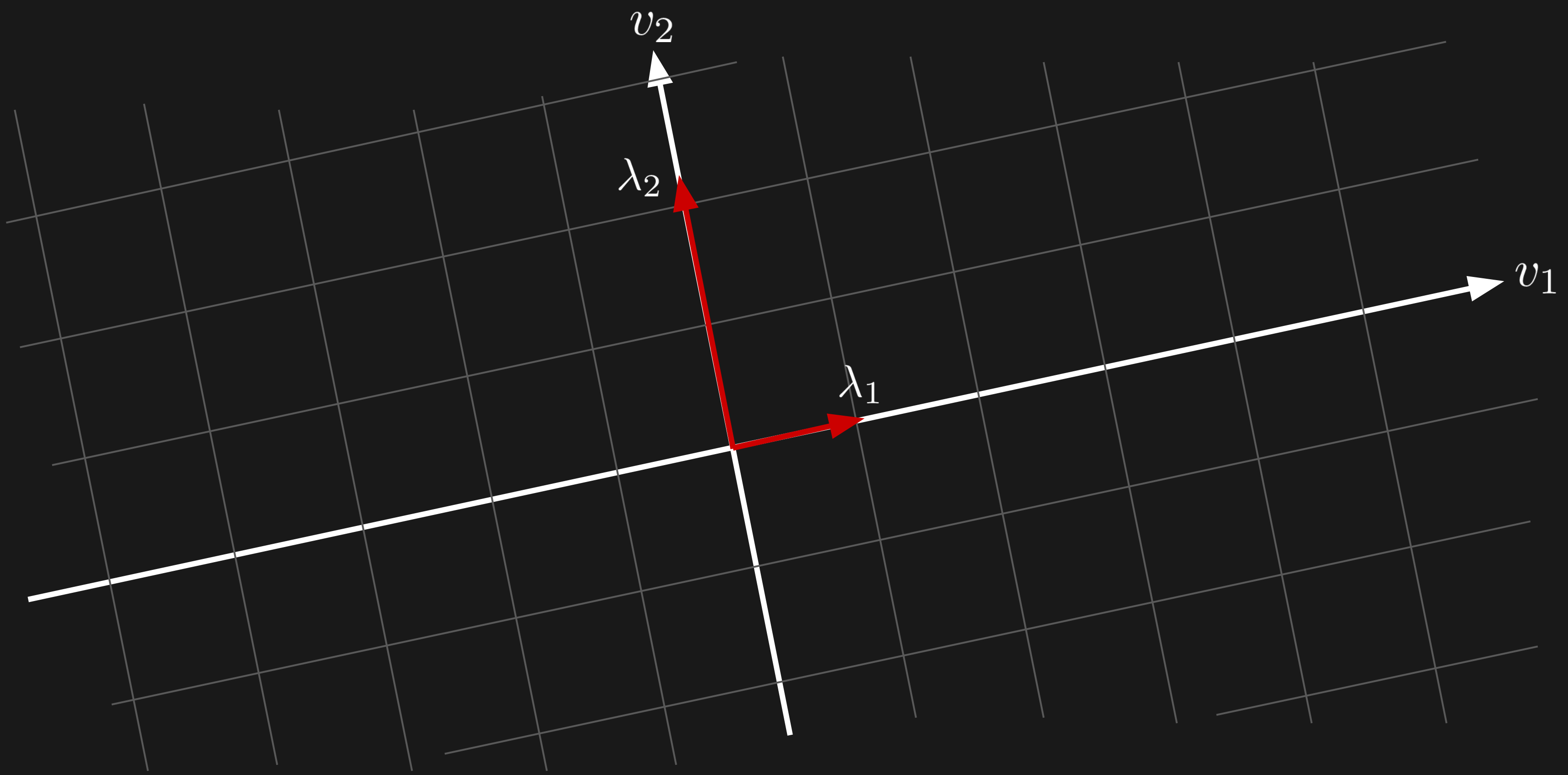
Interpretation



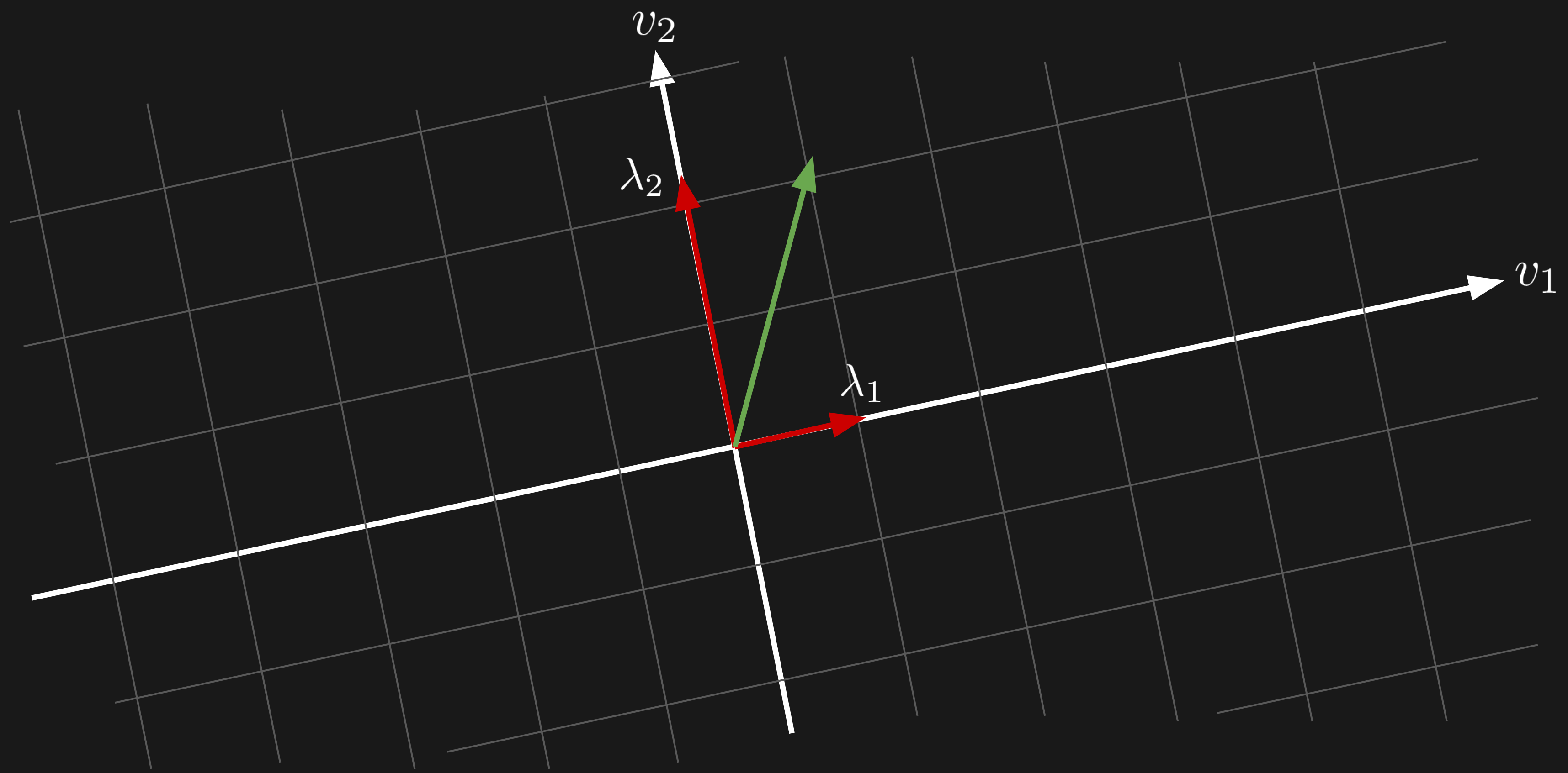
Interpretation



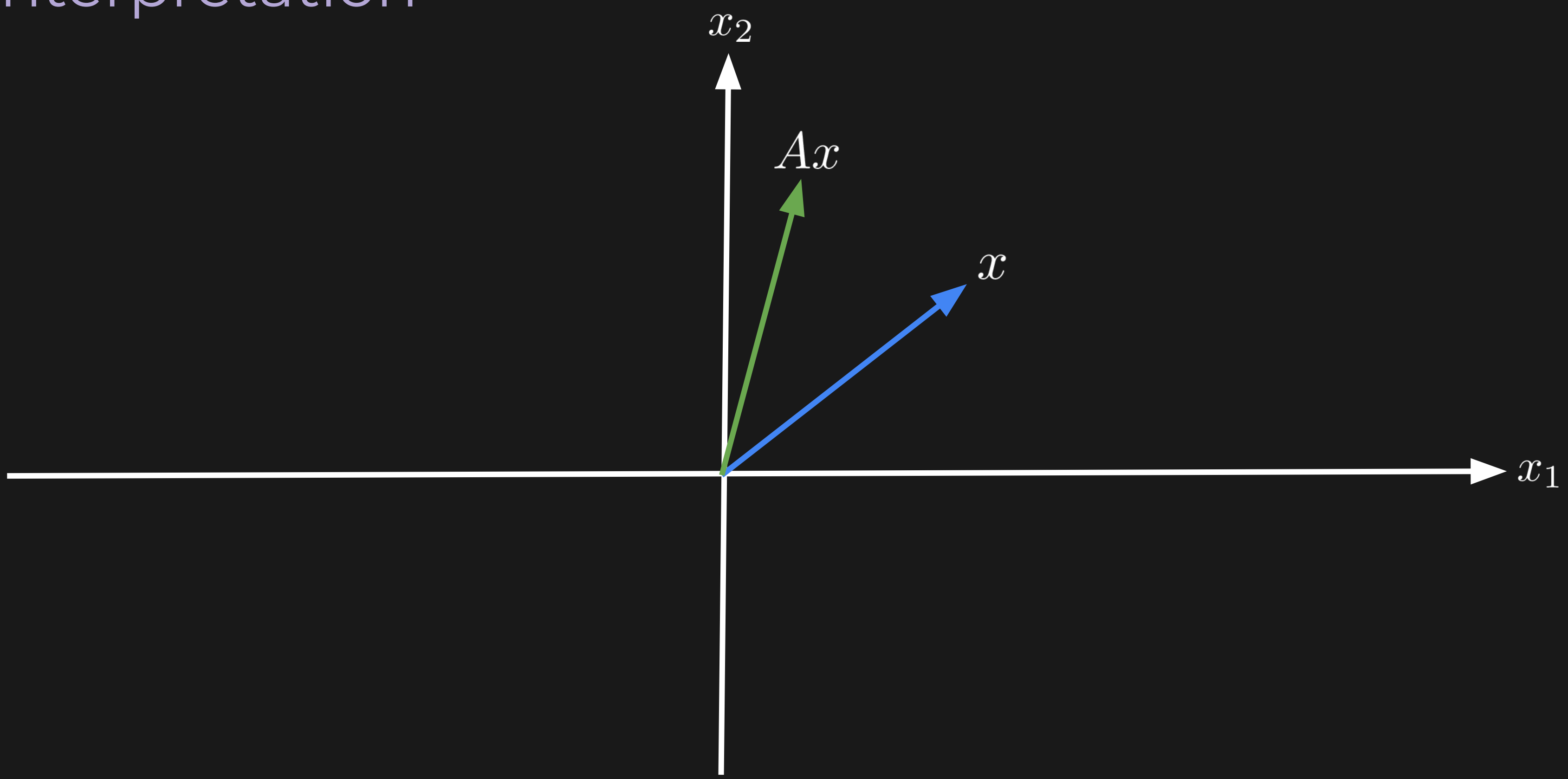
Interpretation



Interpretation



Interpretation



Beyond Symmetric Matrices

- Wouldn't it be nice if the spectral theorem could be generalized to any matrix?
    - It can! *(spoiler: that's the singular value decomposition)*

Next Time

- Applications of eigenvectors
  - Mechanical engineering, finance, red blood cells...
  - Markov processes
  - Ellipsoids