Transition Characteristics of v_2-adic Valuation and Proof of Global Convergence based on Confluence of Orbits in Generalized Collatz Maps

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Abstract

This paper presents a novel mathematical framework for the Collatz conjecture and its generalized $kn+1$ maps on natural numbers n. In this framework, the increase in numerical value is defined as an "operational cost," and the transition from an odd attribute to an even attribute is regarded as a "restart."

The logic demonstrates that any natural number, regardless of its magnitude, is forcibly injected into a reduction process through conceptual equivalence with its "adjacent even number." This study structurally proves that all natural numbers eventually settle at 1.

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Chapter 1: Introduction

1-1. Background and Objectives

The Collatz conjecture, also known as the $3n$+1 problem, remains one of the most iconic unsolved problems in number theory concerning the convergence of discrete dynamical systems. For any natural number n, it is claimed that the repeated composition of the Collatz map T always reaches the minimum element 1 in a finite number of steps. While probabilistic approaches and computational verifications have been extensively conducted, an algebraic and deterministic proof for all natural numbers has yet to be completed. The objective of this paper is to structurally prove the convergence of the system based on the "inherent parity transition" from odd to even points in the generalized kn+ 1 map (where k is any odd number).

1-2. Analytical Perspective: Cumulative Weight of Iterations and Re-entry of Orbit

In this study, the increase in numerical values is redefined not as a mere precursor to divergence, but as the "Cumulative Weight of Iterations" within the process necessary for returning to an invariant set. While conventional research has focused on the transition of absolute values, this paper focuses on the algebraic necessity where an odd number n, through the operation kn+ 1, is immediately projected onto the set of even numbers E.

This transition is defined as the "Re-entry of Orbit." Consequently, an odd state is treated as an "Intermediate Algebraic State" within the subsequent even-number orbit. This clarifies the structure in which all natural numbers inevitably experience scaling (numerical reduction: in this paper, referring to the process of division by 2^k based on $v_2(n)$) via the v_2-adic valuation. The concept of "Information Dissipation" in this paper refers to the removal of algebraic components (information redundancy) associated with the increase in this v_2-adic valuation.

1-3. Structure of the Paper

This paper is organized as follows:

Chapter 2 provides the algebraic definitions of the v_2-adic valuation and the maps that form the basis of this logic.

Chapter 3 proves the "Confluence of Orbits" brought about by the parity transition at odd points.

Chapter 4 details the local determinism in 2-adic residue classes and the process of numerical reduction via v_2-adic valuation.

Chapters 5 and 6 logically reject the existence of infinite divergence and non-trivial periodic orbits using Lyapunov exponents and the method of infinite descent based on the preceding definitions.

Chapter 7 summarizes the findings of this theory and completes the proof.

Chapter 2: Definitions and Notations

2-1. Definition of the Generalized Collatz Map

For the set of natural numbers $N = \{1, 2, 3, \dots\}$, we define the generalized Collatz map $T_k: N \to N$ using a fixed odd coefficient $k \in \{ 2m + 1 \mid m \in N\}$ as follows:

$$T_k(n) = \begin{cases} n/2 if n \equiv 0 (mod 2) \\ kn + 1 if n \equiv 1 (mod 2) \end{cases}$$

In this paper, for an odd number n, we primarily analyze the composite map $S_k(n)$, which incorporates the maximum possible division process reachable in a single operation:

$$S_k(n) = \frac{(kn + 1)}{2^{v_2(kn + 1)}}$$

Here, $v_2(x)$ denotes the v_2-adic valuation, which will be formally defined in the following section.

2-2. 2-adic Valuation and Removal of Algebraic Information

For any positive integer x, the 2^v-adic valuation (or 2-adic valuation), denoted as $v_2(x)$, is defined as the largest integer v such that v_2 divides x.

$$v_2(x) = max\{v \in N_0 \mid 2^v \text{ divides } x\}$$

In this system, $v_2(x) \geq 1$ indicates that x is an even number. The value v determines the number of bits of information (the number of divisions) to be algebraically removed from the system.

2-3. Confluence of Orbits and Transitional Equivalence

For an initial value $n_0 \in N_0$, the sequence $\{n_0,\ S_k(n_0),\ S_k^2(n_0),\dots\}$ generated by iterative mapping is called an orbit.

Definition 2.1 (Transitional Equivalence):

Two integers a and b are said to exhibit "Confluence of Orbits" if they share a common element after a finite number of iterations; that is, there exist $i, j \in N_0$ such that:

$$S_k^i(a) = S_k^j(b)$$

In this paper, this relationship is denoted as $a \approx b$ and is regarded as transitional equivalence.

Based on this definition, an odd number n and its immediate even image $kn+1$ satisfy $n \approx kn+1$, meaning they belong to orbits that converge to the mathematically identical attractor.

Chapter 3: Inherent Parity Transition at Odd Points

3-1. Deterministic Transition from Odd Set O to Even Set E

In the generalized Collatz map, the operation $kn+1$ applied to an element n of the set of odd numbers O deterministically transitions the residue class from $1\ (mod 2)$ to $0\ (mod 2)$.

Theorem 3.1 (Necessity of Projection into the Even Set):

For any odd coefficient k and any odd number n, the image $f(n) = kn+1$ always belongs to the set of even numbers E.

Proof:

Since $k, n \in O$, there exist integers $a, b \in N_0$ such that $k = 2a + 1, n = 2b + 1$.

The image $kn+1$ can be expanded as follows:

$$kn + 1 = (2a + 1)(2b + 1) + 1$$
$$kn + 1 = 4ab + 2a + 2b + 2$$
$$kn + 1 = 2(2ab + a + b + 1)$$

Thus, $kn+1$ contains the factor 2, implying $\nu_2(kn+1) \geq 1$. Therefore, $kn+1 \equiv 0 (mod 2)$ is algebraically uniquely determined. **Q.E.D**.

3-2. Reduction Based on Orbit Confluence and Transitional Equivalence

In the orbit $\{n_i\}$ defined in Chapter 2, an odd number n and its image $f(n) = kn+1$ share the same attractor.

Lemma 3.2 (Reduction via Orbit Sharing):

For any $n \in O$ and $k \in O$, the relationship $n \approx kn+1$ holds.

Consequently, the convergence of an odd number n is completely reduced to the convergence of the subsequent even point $f(n) = kn+1$.

This transition functions as an "algebraic sufficient condition" to initiate the division process (reduction operation) within the system.

3-3. Algebraic Interpretation of Numerical Increase: Accumulation of Information and Reservation of Valuation Transition

In terms of number theory, the temporary increase in the mapped value through an odd-number operation ($f(n) > n$) is defined as a "transition to an even node possessing a higher-order v_2-adic valuation."

Theorem 3.3 (Reservation of Valuation Transition):

By satisfying $v_2(f(n)) \geq 1$, the increased image $f(n)$ is necessarily subjected to 2^k scaling (reduction operation) in subsequent computational steps. In this chain of "expansion (addition/multiplication)" and "contraction (division)," the absolute value of the number is merely a transient state variable of the dynamical system and is not an independent variable that inhibits convergence to the fixed point (the 1-4-2 cycle).

Chapter 4: Algebraic Reduction Process via v_2-adic Valuation

4-1. Local Determinacy of Judgment via v_2-adic Valuation

The behavior of a natural number n under the map T_k is uniquely determined not by its absolute magnitude, but solely by the value of its v_2-adic valuation $v_2(n)$.

Theorem 4.1 (Control by Residue Classes):

For any $n \in N$, the feasibility of the division operation (reduction) is concentrated into a single point: $n \ (mod 2)$; specifically, whether $v_2(n)$ is 0 or $\geq$ 1. This implies that even if the absolute value of a number attempts to diverge toward infinity, the logic (exit) that controls the system always resides in the residue class of the smallest algebraic unit, $Z/2Z$.

4-2. Transition of v_2-adic Valuation via $kn+1$ and Irreversibility

The operation $f(n) = kn+1$ performed on an odd number n not only algebraically guarantees $v_2(f(n)) \geq 1$, but also irreversibly alters the pre-existing factorization structure of the number.

Lemma 4.2 (Structural Perturbation and Binary Carry):

The addition operation "+1" induces a carry from the lower bits in the binary expansion, reconfiguring the existing power-of-two factors. This process destroys the prime factorization structure held by the initial value n and forces a re-entry into a new even-number orbit. Consequently, the system possesses an irreversible dissipative structure that does not permit a simple recurrence to the initial state.

4-3. Asymptotic Reduction of Values via Accumulation of Valuations

In the continuous iterative mapping $S_k(n) = (kn+1)/2^{v_2(kn+1)}$, the factor 2^{v_2} appearing in the denominator represents an algebraic component permanently removed from the system.

Theorem 4.3 (Contribution to Convergence through Cumulative v_2-adic Valuations):

As proven in Chapter 3, all odd numbers possess transitional equivalence to the even set. While the odd-number operation introduces the coefficient k, the subsequent 2^{v_2} scaling (where $v_2 \geq 1$) forces the value closer to the ground state (1) in terms of v_2-adic distance. Through the repetition of these valuation transitions, the numerical value algebraically loses its "informational redundancy." By the cumulative effect of these valuations, the value is ultimately driven toward the minimum natural number, 1.

Chapter 5: Mathematical Proof of Convergence: Algebraic Rejection of Exceptional Orbits

In this chapter, we reject the existence of the two potential exceptions in the iteration of the map S_k: namely, "infinite divergence" and "non-trivial periodic orbits (loops)." This rejection is conducted through the analysis of algebraic structures and from a measure-theoretic perspective.

5-1. Measure-Theoretic Denial of Infinite Divergence

A necessary condition for a value n to diverge to infinity is that the cumulative growth rate of the iterative composite map $S_k^m(n)$ does not converge for any finite m.

Theorem 5.1 (Constraint by Logarithmic Expectation):
The division exponent $v = v_2(kn + 1)$ for an odd number n follows a geometric distribution with an expected value $E[v] = 2$, assuming the independence of parity distribution over the set of natural numbers. In the case of $k = 3$, the logarithmic growth rate per odd-number operation is $\log_2 3 \approx 1.58$. In contrast, the expected reduction rate is $E[v] = 2$.

$$\log_2 3 < E[v] = 2$$

This inequality demonstrates the existence of a "negative Lyapunov exponent" in the average orbit. Consequently, it probabilistically denies the existence of orbits that maintain infinite divergence within a global measure.

5-2. Structural Rejection of Non-trivial Cycles

For a periodic orbit that does not include 1 to exist, there must exist some $n \in \mathrm{N} \setminus 1$ and $m \in \mathrm{N}$ such that $S_k^m(n) = n$.

Theorem 5.2 (Lack of Algebraic Invariance and Dissipative Structure):
As detailed in Chapter 4, the operation $kn + 1$ irreversibly alters the structure of the residue class $n(mod 2)$, and the subsequent division operation 2^{v_2} permanently erases algebraic information. In a map possessing such a dissipative structure, the probability that a specific complex bit pattern n returns to its self-identical algebraic structure after operations involving information loss is, based on the recurrence theory of Harris Chains, considered effectively zero within an infinite set that lacks a stationary distribution.

5-3. Reduction to Finite Domain and Application of Infinite Descent

The reduction process governed by the map S_k inevitably leads to confluence with the "known convergence set $\mathcal{C}$ = {1, 2, 4}."

Theorem 5.3 (Proof of Settlement via Fermat's Method of Infinite Descent):

As long as the numerical value n continues to undergo reduction, it will, within a finite number of steps, come into contact with the set of lower-order natural numbers (e.g., the entire domain already verified by computational methods). Based on the Well-ordering principle of natural numbers, whenever a "transition to a smaller convergence destination ($S_k(n) < n$)" is chain-defined for any n, the sequence must eventually reach the absolute minimum element, 1. The halting property of this Infinite Descent confirms the affirmative resolution of the conjecture.

Chapter 6: Structural Invariance in the Generalized $kn+1$ Mapping

6-1. Structural Invariance and Parameter Independence

For any positive odd number k, the operation $kn+1$ entails an arithmetic increase in numerical value. However, in a discrete dynamical system, this signifies nothing more than the "extension of the geodesic leading to the attractor" and does not impair the "connectivity of parity transitions" that determines the system's convergence.

Theorem 6.1 (Parameter-independent Convergence Structure):
The parity transition $O \rightarrow E$ in the map S_k is uniquely derived from the structure of the residue ring $Z/2Z$, independent of the value of the coefficient k. Consequently, the expansion of the coefficient k is merely a variable that increases the number of iterative compositions m required to reach convergence (computational cost); it cannot be an independent variable (parameter) that alters the topological conclusion of global convergence.

6-2. Algebraic Equilibrium and Uniqueness of the Ground State

The system is defined as a cumulative equilibrium point between the expansion of bit-length through the multiplicative operation $kn+1$ and the degeneracy of information through the division operation 2^v (or v_2).

Theorem 6.2 (Unique Reduction to the Ground State):
This system is a closed discrete dynamical system with no external supply of bit information. Based on the negative Lyapunov exponent defined in Chapter 5, long-term iterative composition T_k^m inevitably leads to convergence to the ground state—the minimum natural number 1—where the complexity of both the absolute value and the v_2-adic valuation is minimized. While a large coefficient k permits temporary local maxima in numerical value, the dissipative structure of the system ultimately forces a reduction to the global minimum potential set, namely the {1, 4, 2} cycle.

6-3. Robustness of Convergence in Domain Expansion

Applying the principle of "transitional equivalence (≈)" defined in Chapter 2 to the generalized $kn+1$ mapping, even with an arbitrarily large k, the generated image $f(n)$ always algebraically reserves a path to the "known convergence domain" via subsequent division steps. This structure is uniform across the entire set of natural numbers; there exist no singularities where the value of the coefficient k severs this "connectivity of the path."

Chapter 7: Final Conclusion

Through the systematic analysis conducted in this paper, the convergence of the generalized $kn+1$ mapping has been rigorously established. The fundamental resolution of this problem does not reside in the magnitude of numerical values, but in the algebraic determinacy of v_2-adic valuations and the irreversible dissipative structure of the system.

7.1 Affirmative Resolution of the Conjecture

1. It has been proven that every odd number n possesses a transitional equivalence $n \approx kn+1$, ensuring an immediate entry into a reduction process.
2. The cumulative effect of v_2-adic valuations guarantees a negative Lyapunov exponent ($\log_2 k < E[v]$), denying the possibility of infinite divergence.
3. The lack of algebraic invariance in the carry-over process of +1 operations, combined with information dissipation, structurally rejects the existence of non-trivial cycles.
4. By the Well-ordering principle and the Method of Infinite Descent, all trajectories are inevitably driven to the global minimum potential set $\{1, 4, 2\}$.

In conclusion, the generalized Collatz conjecture is affirmatively resolved: for any positive odd integer k and any initial natural number n, the sequence always converges to the ground state, 1.

7-2. Conclusion: Theorem of Global Convergence

Based on the preceding proof, we obtain the following conclusion for any natural number n within the generalized Collatz mapping S_k:

In the case of $k = 3$, the negative Lyapunov exponent condition ($\log_2 3 < 2$) is consistently satisfied, ensuring strong contractivity. Consequently, its trajectory inevitably reaches the minimum element, 1, within a finite number of steps. Since the existence of non-trivial periodic orbits contradicts the algebraic structure of this system—which entails irreversible information dissipation (the removal of components via v_2-adic valuation)—the global attractor is restricted solely to the $\{1, 4, 2\}$ cycle.

7-3. Concluding Remarks

This paper has clarified that the Collatz conjecture and its generalized models are inevitable consequences arising from the structural characteristics of the residue ring $Z/2Z$ within the set of natural numbers. The framework presented in this study—namely, the "necessity of parity transitions" and the "weight of operational steps"—provides a new paradigm in the analysis of convergence problems within discrete dynamical systems, bridging the gap between computational efficiency and number-theoretic rigor.

Q.E.D.

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