MATH 245 Homework 2

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Question 1

Determine the region in which the given equation is hyperbolic, parabolic, elliptic, or singular.

a)

$$u_{xx} + y^2 u_{yy} + u_x - u + x^2 = 0$$

$a = 1, b = 0, c = -y^2$, so we have $b^2 - ac = 0 - (-y^2) = y^2$. This will be positive everywhere except for $y = 0$, so the equation is hyperbolic where $y \neq 0$ and parabolic for $y = 0$.

b)

$$u_{xx} - y u_{yy} + x u_x + y u_y + u = 0$$

$a = 1, b = 0, c = -y$, so we have $b^2 - ac = 0 - (-y) = y$. Thus, the equation will be hyperbolic where $y > 0$, parabolic where $y = 0$, and elliptic where $y < 0$.

Question 2

Using a factorization similar to the wave equation, solve the following IVP:

$$\begin{cases} u_{xx} + 2u_{xy} - 3u_{yy} = 0 & x \subset \mathbb{R},\ y > 0 \\ u(0, x) = \sin x & x \subset \mathbb{R} \\ u_y(0, x) = x & x \subset \mathbb{R} \end{cases} \tag{1}$$

First, we can factor the equation as follows:

$$\left(\frac{\partial}{\partial x} + 3\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) u = 0$$

or

$$\left(\partial_x + 3\partial_y\right)\left(\partial_x - \partial_y\right) u = 0$$

Then set $\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) u = v$, giving us

$\left(\frac{\partial}{\partial x} + 3\frac{\partial}{\partial y}\right) v = v_x + 3v_t = 0$

which we know has the solution $v(x, y) = f(3x - y)$, so

$$u_x - u_y = f(3x - y)$$

On (characteristic) lines with the slope $y = -x + c$, or $y + x =$ constant, we must have $u_x - u_y = f(3x - y) = 0$. Set $\eta = x + y$ and $\xi = x$. Then by the chain rule,

$$u_x = u_\eta + u_\xi, \; u_y = u_\eta$$

And let's rewrite y as $y = \eta - x = \eta - \xi$.

So

$$u_x - u_y = f(3x - y) \quad \longrightarrow \quad u_\xi = f(3\xi - \eta + \xi)$$

$$u_\xi = f(4\xi - \eta)$$

Now integrate with respect to ξ:

$$u(\eta, \xi) = F(4\xi - \eta) + G(\eta)$$

where F is the antiderivative of f with respect to ξ.

Now convert back to our original variables:

$$u(x, y) = F(3x - y) + G(x + y)$$

Using the fact that $u(0, x) = \sin x$,

$$u(0, x) = \sin x = F(3x) + G(x) \tag{2}$$

now replace x with a new neutral variable, α and differentiate:

$$\sin \alpha = F(3\alpha) + G(\alpha)$$

$$\cos \alpha = 3F'(3\alpha) + G'(\alpha) \tag{3}$$

But we can also differentiate $u(x, y) = F(3x - y) + G(x + y)$ with respect to y to get

$$u_y(x, y) = -F'(3x - y) + G'(x + y)$$

but from our initial conditions, we know

$$u_y(0, x) = -F'(3x - 0) + G'(x + 0) = x$$

Let's replace x by our neutral variable α and solve for F':

$$F'(\alpha) = G'(3\alpha) - \alpha$$

Now plug this into 3:

$$\cos \alpha = 3G'(\alpha) - 3\alpha + G'(\alpha)$$

$$G(\alpha) = \frac{1}{4} \int \cos \alpha + 3\alpha = \frac{\sin \alpha}{4} + \frac{3\alpha^2}{8}$$

So that means 2 becomes:

$$\sin \alpha = F(3\alpha) + \frac{\sin \alpha}{4} + \frac{3\alpha^2}{8}$$

$$F(\alpha) = \frac{3 \sin\left(\frac{\alpha}{3}\right)}{4} - \frac{\alpha^2}{24}$$

Which means $u(x, y) = F(3x - y) + G(x + y)$ becomes

$$u(x,y) = \frac{3}{4}\sin\left(x - \frac{y}{3}\right) - \frac{(3x-y)^2}{24} + \frac{\sin(x+y)}{4} + \frac{3(x+y)^2}{8} =$$

Solution

$$u(x,y) = \frac{3}{4}\sin\left(x - \frac{y}{3}\right) + \frac{\sin(x+y)}{4} + xy + \frac{y^2}{3}$$

Check solution

$$u_y = \frac{-1}{4}\cos\left(x - \frac{y}{3}\right) + \frac{\cos(x+y)}{4} + x + \frac{2y}{3}$$

$$u_{yy} = \frac{-1}{12}\sin\left(x - \frac{y}{3}\right) - \frac{\sin(x+y)}{4} + \frac{2}{3}$$

$$u_x = \frac{3}{4}\cos\left(x - \frac{y}{3}\right) + \frac{\cos(x+y)}{4} + y$$

$$u_{xx} = \frac{-3}{4}\sin\left(x - \frac{y}{3}\right) - \frac{\sin(x+y)}{4}$$

$$u_{xy} = \frac{1}{4}\sin\left(x - \frac{y}{3}\right) - \frac{\sin(x+y)}{4} + 1$$

Check that $u_{xx} + 2u_{xy} - 3u_{yy} = 0$

$$\left(\frac{-3}{4} + \frac{2}{4} + \frac{1}{4}\right)\sin\left(x - \frac{y}{3}\right) + \left(\frac{-1}{4} + \frac{-2}{4} + \frac{3}{4}\right)\sin(x+y) + (0 + 2 - 3) = 0$$

Question 3

Solve the Neumann boundary value problem for the wave equation on half line:

$$\begin{cases} u_{tt} = c^2 u_{xx} + f(t,x) & 0 < x < \infty \\ u(0,x) = \phi x & 0 < x < \infty \\ u_t(0,x) = \psi x & 0 < x < \infty \\ u_x(t,0) = h(t) & t > 0 \end{cases} \tag{4}$$

Question 4

Consider the 3D wave equation for $u(t,x,y,z)$:

$$u_{tt} = c^2 \Delta u \qquad (x,y,z) \in \mathbb{R}^3, \quad t > 0$$

Change the coordinates to spherical coordinates. Assume the solution is spherically symmetric, so that $u(t,x,y,z) = u(t,r)$ and does not depend on θ and ϕ. Find the solution for $u(0,r) = 0$ and

$$u_t(0,r) = \begin{cases} 1 & |r| \le 1 \\ 0 & |r| > 1 \end{cases} \tag{5}$$

Hint: use the substitution $u(t,r) = \frac{1}{r}w(t,r)$.

First, we need to derive the formula for the Laplacian in spherical coordinates.

We know the equation for the Laplacian in polar coordinates is:

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Now let's convert to spherical coordinates:

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{s^2 + z^2}$$

$$\begin{aligned} x &= s\cos\phi \\ y &= s\sin\phi \\ z &= r\cos\theta \\ s &= r\sin\theta \end{aligned}$$

By the two-dimensional Laplacian, we have

$$u_{zz} + u_{ss} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$

and

$$u_{xx} + u_{yy} = u_{ss} + \frac{1}{s}u_s + \frac{1}{s^2}u_{\phi\phi}$$

We add these two equations and cancel u_s to get

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \frac{1}{s}u_s + \frac{1}{s^2}u_{\phi\phi}$$

Now since u doesn't depend on θ or ϕ, we have

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{s}u_s = u_{rr} + \frac{1}{r}u_r + \frac{1}{r\sin\theta}u_s$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial s} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial s} + \frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial s} = u_r\frac{1}{\sin\theta} + 0 + 0 = u_r\frac{s}{r}$$

So with our change of variables, we have

$$u_{tt} = c^2\left(u_{rr} + \frac{2}{r}u_r\right)$$

Now set $w = ru$, or $u = \frac{w}{r}$. Then

$$w_t = ru_t, \quad w_{tt} = ru_{tt}, \quad u_{tt} = \frac{w_{tt}}{r}$$

$$w_t = ru_t, \quad w_{tt} = ru_{tt}, \quad u_{tt} = \frac{w_{tt}}{r}$$

$$u_r = \frac{w_r}{r} - \frac{w}{r^2}$$

$$u_{rr} = \frac{w_{rr}}{r} - \frac{2w_r}{r^2} + \frac{2w}{r^3}$$

So

$$u_{tt} = c^2 \left(u_{rr} + \frac{2}{r} u_r \right)$$

becomes

$$\frac{w_{tt}}{r} = c^2 \left(\frac{w_{rr}}{r} - \frac{2w_r}{r^2} + \frac{2w}{r^3} + \frac{2}{r} \left(\frac{w_r}{r} - \frac{w}{r^2} \right) \right)$$

which simplifies to

$$w_{tt} = c^2 w_{rr}$$

but this is just the wave equation, which we know has the solution

$$w(t,r) = \frac{\varphi(r+ct) + \varphi(r-ct)}{2} + \frac{1}{2c} \int_{r-ct}^{r+ct} \psi(s) ds$$

Since $\varphi = 0$,

$$w(t,r) = \frac{1}{2c} \int_{r-ct}^{r+ct} \psi(s) ds$$

Now we have 4 cases:

Case 1: $r - ct \geq -1, r + ct \leq 1$

$$w(t,r) = \frac{1}{2c} \int_{r-ct}^{r+ct} s \; ds$$

Case 2:$r - ct < -1, r + ct > 1$

$$w(t,r) = \frac{1}{2c} \int_{-1}^{1} s \; ds$$

Case 3:$r - ct < -1, r + ct \leq 1$

$$w(t,r) = \frac{1}{2c} \int_{-1}^{r+ct} s \; ds$$

Case 4:$r - ct \geq -1, r + ct > 1$

$$w(t,r) = \frac{1}{2c} \int_{r-ct}^{1} s \; ds$$

Since $u = \frac{w}{r}$, this means we have

$$u(t,r) = \begin{cases} \frac{1}{2c} \int_{r-ct}^{r+ct} s \; ds & r - ct \geq -1, r + ct \leq 1 \\ \frac{1}{2c} \int_{-1}^{1} s \; ds & r - ct < -1, r + ct > 1 \\ \frac{1}{2c} \int_{-1}^{r+ct} s \; ds & r - ct < -1, r + ct \leq 1 \\ \frac{1}{2c} \int_{r-ct}^{1} s \; ds & r - ct \geq -1, r + ct > 1 \end{cases} \tag{6}$$

Question 5

Consider the following Dirichlet boundary value problem:

$$\begin{cases} u_{tt} + x(t,x)u_t = u_{xx} & 0 < x < 1 \\ u(0,x) = \phi(x) & 0 < x < 1 \\ u_t(0,x) = \psi(x) & 0 < x < 1 \\ u(t,0) = u(t,1) = 0 & t \geq 0 \end{cases} \tag{7}$$

Assume that $|a(t,x)| \leq m$ for some constant m and all $0 < x < 1$ and $t \geq 0$. Let

$$E(t) = \frac{1}{2} \int_0^1 \left(u_t(t,x)^2 + u_x(t,x)^2 \right) dx$$

(1) Show that $\frac{dE(t)}{dt} \leq 2mE(t)$ for $t \geq 0$.
(2) Use part (a) and show that $\frac{d}{dt}\left(e^{-2mE(t)}\right) \leq 0$ for all $t \geq 0$. Hence, by integration from $[0,t]$, we get that

$$E(t) \leq e^{2mt}E(0) \quad \text{for all } t \geq 0$$

(3) If $\phi(x) = \psi(x) = 0$ for all $0 < x < 1$, what does this say about $E(t)$ for $t \geq 0$ and hence about $u(t,x)$ for $t \geq 0$?

(4) Use the previous part to prove uniqueness of the following problem:

$$\begin{cases} u_{tt} + a(t,x)u_t = u_{xx} & 0 < x < 1, t > 0 \\ u(0,x) = \phi(x) & 0 < x < 1 \\ u_t(0,x) = \psi(x) & 0 < x < 1 \\ u(t,0) = f(t) & t \geq 0 \\ u(t,1) = g(t) & t \geq 0 \end{cases} \tag{8}$$

Problem 6

Does the D'Alembert method work for the wave equation $u_{tt} = c(x)^2 u_{xx}$? What about $u_{tt} = c(t)^2 u_{xx}$? Why?

Let's try the factorization method if $c = c(x)$:

$$\left(\partial_t + c(x)\partial_x\right)\left(\partial_t - c(x)\partial_x\right) u = 0$$

Set $\xi = x + c(x)t$ and $\eta = x - c(x)t$.

By the chain rule,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$= \partial_\xi(1 + c') + \partial_\eta\,(1 - c')$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial t}$$

$$= c\partial_\xi - c\partial_\eta$$

When we plug these back into

$$(\partial_t + c\partial_x)(\partial_t - c\partial_x)u = 0$$

we get

$$(c\partial_\xi - c\partial_\eta + c\partial_\xi(1+c') + c\partial_\eta(1-c'))(c\partial_\xi - c\partial_\eta - c\partial_\xi(1+c') - c\partial_\eta(1-c'))u = 0$$

When c is constant, $c' = 0$ and this simplifies to $-4c^2 u_{\xi\eta} = 0$, allowing us to integrate to find $u(x,t) = F(x+ct) + G(x-ct)$. But we cannot do the same simplification when we have the c' terms, meaning we cannot solve the wave equation by the same method when the wave speed is not constant.

Similarly,

Let's try the factorization method if $c = c(t)$:

$$(\partial_t + c(t)\partial_x)(\partial_t - c(t)\partial_x)u = 0$$

Set $\xi = x + c(t)t$ and $\eta = x - c(t)t$.

By the chain rule,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x} = \partial_\xi + \partial_\eta$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial t} = \partial_\xi(c't + c) - \partial_\eta(c't + c)$$

When we plug these back into

$$(\partial_t + c\partial_x)(\partial_t - c\partial_x)u = 0$$

we get

$$(\partial_\xi(c't+c) - \partial_\eta(c't+c) + c\partial_\xi + c\partial_\eta)(\partial_\xi(c't+c) - \partial_\eta(c't+c) - c\partial_\xi - c\partial_\eta)u = 0$$

Where again we cannot cancel terms to simplify our equation as we could with constant c.

Problem 7 (The Poisson-Darboux Equation)

Solve the initial value problem

$$\begin{cases} u_{tt} - u_{xx} - \frac{2}{x}u_x = 0 & -\infty < x < \infty, t > 0 \\ u(0,x) = 0 & -\infty < x < \infty \\ u_t(0,x) = g(x) & -\infty < x < \infty \end{cases} \tag{9}$$

where $g(x) = g(-x)$ is an even function. Hint: set $w = xu$.

Using the results from when we set $w = ru$ in Problem 4,

$$u_t = \frac{w_t}{x}, u_{tt} = \frac{w_{tt}}{x}$$

$$u_x = \frac{w_x}{x} - \frac{w}{x^2}$$

$$u_{xx} = \frac{w_{xx}}{x} - \frac{2w_x}{x^2} + \frac{2w}{x^3}$$

So $u_{tt} - u_{xx} - \frac{2}{x} u_x = 0$ becomes

$$\frac{w_{tt}}{x} - \frac{w_{xx}}{x} + \frac{2w_x}{x^2} - \frac{2w}{x^3} - \frac{2}{x}\left(\frac{w_x}{x} - \frac{w}{x^2}\right) = 0$$

Which simplifies to $w_{tt} - w_{xx} = 0$, the wave equation with $c = 1$. By the D'Alembert formula, since we have $\varphi = 0$,

$$w(t,x) = \frac{1}{2c}\int_{x-t}^{x+t} \psi(s) ds$$

$w_t = xu_t$ Since $y = x$ is an odd function and $u_t(0,x) = g(x)$ is an even function, $w_t = xg(x) = h(x)$ must be an odd function.

Problem 8

Solve the following characteristic initial value problem:

$$\begin{cases} y^3 u_{xx} - y u_{yy} + u_y = 0 & 0 < x < 4, \quad |y| \leq 2\sqrt{2} \\ u(x,y) = f(x) & x + \frac{y^2}{2} = 4 \text{ for } 2 \leq x \leq 4 \\ u(x,y) = g(x) & x - \frac{y^2}{2} = 0 \text{ for } 0 \leq x \leq 2 \end{cases} \tag{10}$$

where $f(2) = g(2)$. Hint: Use the transformation $\eta = x - \frac{y^2}{2}$ and $\xi = x + \frac{y^2}{2}$ and express the PDE in the coordinates (ξ, η).