

MATH 245 Homework 4

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Question 1: Find eigenvalues and eigenfunctions

(a)

$-\frac{d^2}{dx^2}X(x) = \lambda X(x)$ in $0 < x < l$ with boundary conditions $X'(0) = 0 = X(l)$

Case 1: Positive eigenvalues, $\lambda = \beta^2 > 0$

Re-writing as $X'' + \lambda X = 0$, we will get the characteristic equation: $r^2 + \beta^2 = 0$. Since our characteristic equation has complex roots $r = \pm i\beta$, our solutions take the form

$$X(x) = A \sin(\beta x) + B \cos(\beta x)$$

Differentiating, we find that

$$X'(x) = \beta A \cos(\beta x) - \beta B \sin(\beta x)$$

Now plugging in our initial condition $X'(0) = 0$, we get $X'(0) = \beta A = 0$. And since we are in a case where $\beta \neq 0$, this means $B = 0$ so $X(x) = A \sin(\beta x)$. Now we use our boundary condition $X(l) = 0$ to get $X(l) = A \sin(\beta l) = 0$. If $A = 0$, then $X(0) = 0$ and this contradicts the definition of an eigenfunction. Therefore, we must have $\sin(\beta l) = 0$, which can only occur if $\beta = \frac{n\pi}{l}$. Therefore, this case gives us eigenvalues

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad n = 1, 2, 3, \dots$$

with eigenfunctions

$$X_n(x) = \sin\left(\frac{n\pi x}{l}\right)$$

Case 2: Zero eigenvalues, $\lambda = 0$ $X'' = 0$ implies that $X(x)$ is of the form $Ax + B$, with derivative $X'(x) = A$. Now plugging in our initial condition $X'(0) = 0$, we get $A = 0$, which means $X(x) = B$. But from the boundary condition $X(l) = 0$, we get $B = 0$ and so $X(0) = 0$. Therefore, there are no eigenfunctions $X(x)$ that satisfy $-\frac{d^2}{dx^2}X(x) = \lambda X(x)$ when $\lambda = 0$ and hence no zero eigenvalues.

Case 3: Negative eigenvalues, $\lambda = -\beta^2 < 0$ This case gives us the characteristic equation: $r^2 - \beta^2 = 0$. Since our characteristic equation has distinct real roots $r = \pm\beta$, our solutions take the form

$$X(x) = Ae^{\beta x} + Be^{-\beta x}$$

Differentiating, we find that

$$X'(x) = \beta Ae^{\beta x} - \beta Be^{-\beta x}$$

$$X'(0) = \beta A - \beta B = 0$$

Since we are in a case where $\beta \neq 0$, this means $A - B = 0$, or $A = B$. Then the boundary condition gives

$$X(l) = Be^{\beta l} + Be^{-\beta l} = 0$$

Since $e^{\beta l}$ and $e^{-\beta l}$ are nonzero for all values of l , we must have $B = 0$, implying that again $X(x) = 0$.

Thus, this problem has only positive eigenvalues and their associated eigenfunctions as found in Case 1.

(b)

$x^2 X''(x) + xX'(x) + \lambda X(x) = 0$ in $1 < x < e$ with boundary conditions $X(1) = 0 = X(e)$.

We recognize this equation as having the same form as a second-order Cauchy-Euler equation, a linear homogeneous ODE of the form $ax^2y + bxy' + cy = 0$ with the auxiliary equation $ar(r-1) + br + c = 0$. Here, $a = b = 1$ and $c = \lambda$, so we have

$$r(r-1) + r + \lambda = 0$$

$$r^2 - r + r + \lambda = 0$$

$$r^2 + \lambda = 0$$

Case 1: Positive eigenvalues, $\lambda = \beta^2 > 0$

$$X(x) = A \sin(\beta \ln x) + B \cos(\beta \ln x)$$

$$X(1) = A \sin(0) + B \cos(0) = 0 \quad \longrightarrow \quad B = 0$$

$$X(e) = A \sin(\beta \ln e) = 0$$

$$A \sin(\beta) = 0$$

If $A = 0$, then $X(0) = 0$ and this contradicts the definition of an eigenfunction. Therefore, we must have $\sin(\beta) = 0$, which can only occur if $\beta = n\pi$. Therefore, this case gives us eigenvalues

$$\lambda_n = n^2 \pi^2 \quad n = 1, 2, 3, \dots$$

with eigenfunctions

$$X_n(x) = \sin(n\pi \ln x)$$

Case 2: Zero eigenvalues, $\lambda = 0$ $X'' = 0$ implies that $X(x)$ is of the form $Ax + B$. Now plugging in our initial condition $X(1) = 0$, we get $A + B = 0$, which means $X(x) = Ax - A$. But from the boundary condition $X(e) = 0$, we get $A(e-1) = 0$ which is only possible if $A = 0$ and accordingly $X(0) = 0$. Therefore, there are no eigenfunctions $X(x)$ that satisfy the boundary conditions when $X'' = 0$ and hence no zero eigenvalues.

Case 3: Negative eigenvalues, $\lambda = -\beta^2 < 0$ This case gives us the characteristic equation: $r^2 - \beta^2 = 0$. Since our characteristic equation has distinct real roots $r = \pm\beta$, our solutions take the form

$$X(x) = Ax^\beta + Bx^{-\beta}$$

$$X(1) = A + B = 0 \quad \longrightarrow \quad A = -B$$

$$X(x) = Ax^\beta - Ax^{-\beta}$$

Then the other boundary condition gives $X(e) = Ae^\beta - Ae^{-\beta} = 0$. Since e^β and $e^{-\beta}$ are always nonzero, we must have $A = 0$, implying that again $X(x) = 0$. Thus, this problem has only positive eigenvalues and their associated eigenfunctions as found in Case 1.

(c)

On the interval $0 \leq x \leq 1$ of length one, consider the eigenvalue problem

$$-X'' = \lambda X, \quad X'(0) + X(0) = 0, \quad X(1) = 0$$

(i) Find an eigenfunction with eigenvalue zero. Call it $X_0(x)$.

$X'' = 0$ implies that $X(x)$ is of the form $Ax + B$ and $X'(x) = A$. Then $X(1) = 0$ becomes $A + B = 0$, or $A = -B$. Now we look at our second condition,

$$X'(0) + X(0) = 0$$

$X'(0)$ is just A and $X(0)$ is just B , so again this gives us $A = -B$. Thus, we have $X(x) = Ax - A = A(x - 1)$, so we have found

$$X_0(x) = x - 1$$

(ii) Find an equation for the positive eigenvalues $\lambda = \beta^2$.

$$\begin{aligned} X(x) &= A \sin(\beta x) + B \cos(\beta x) \\ X'(x) &= \beta A \cos(\beta x) - \beta B \sin(\beta x) \end{aligned}$$

$$\begin{aligned} X(1) &= A \sin(\beta) + B \cos(\beta) = 0 \\ A \sin(\beta) &= -B \cos(\beta) \end{aligned}$$

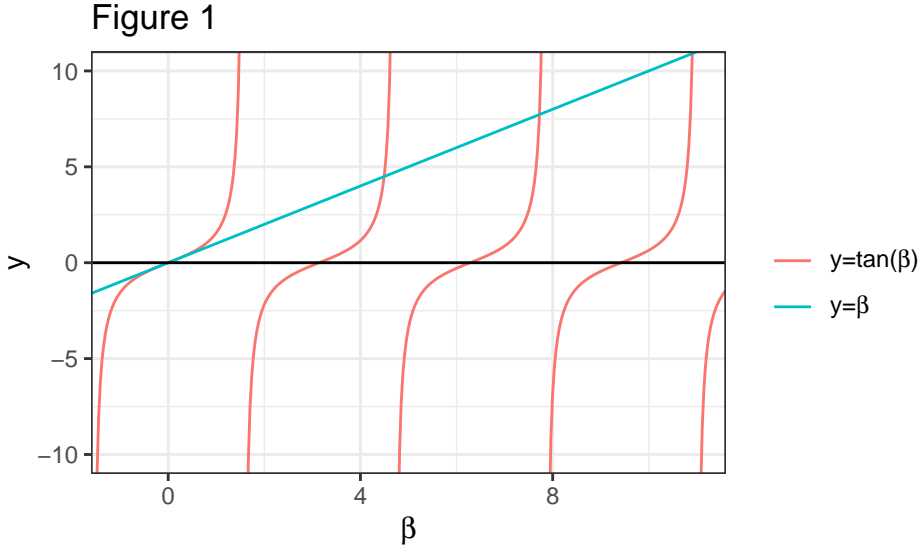
$$-\frac{B}{A} = \frac{\sin(\beta)}{\cos(\beta)} = \tan \beta \tag{1}$$

$$\begin{aligned} X'(0) &= \beta A, \quad X(0) = B \\ X'(0) + X(0) &= \beta A + B = 0 \\ \beta &= -\frac{B}{A} \end{aligned} \tag{2}$$

Combining (1) and (2), we get the equation $\beta = \tan(\beta)$ for the positive eigenvalues.

(iii) Show graphically from part (b) that there are an infinite number of positive eigenvalues.

In part (b), we showed that β is a positive eigenvalue if $\beta = \tan(\beta)$. However, plotting the equations $y = \beta$ and $y = \tan \beta$ reveals that these two functions intersect an infinite number of times (Fig. 1), meaning there are an infinite number of positive eigenvalues.



(iv) Is there a negative eigenvalue?

This case gives us the characteristic equation: $r^2 - \beta^2 = 0$. Since our characteristic equation has distinct real roots $r = \pm\beta$, our solutions take the form

$$X(x) = Ae^{\beta x} + Be^{-\beta x}$$

Differentiating, we find that

$$X'(x) = \beta Ae^{\beta x} - \beta Be^{-\beta x}$$

$$X'(0) = \beta A - \beta B$$

$$X(0) = A + B$$

$$X'(0) + X(0) = \beta A - \beta B + A + B$$

$$0 = A(\beta + 1) + B(1 - \beta)$$

$$-A(\beta + 1) = B(1 - \beta)$$

$$\frac{\beta + 1}{1 - \beta} = -\frac{B}{A}$$

Now turning to our other boundary condition,

$$X'(1) = \beta Ae^{\beta} - \beta Be^{-\beta} = 0$$

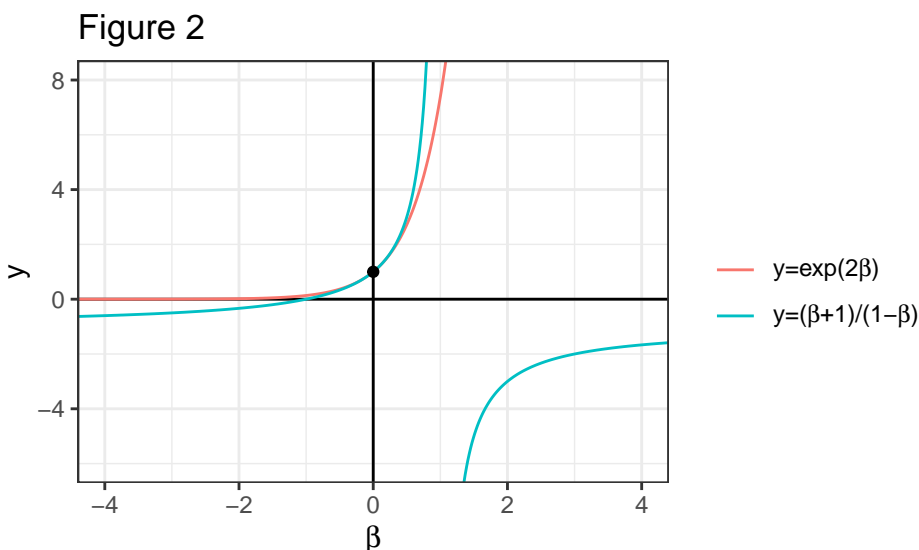
$$\beta Ae^{\beta} = \beta Be^{-\beta}$$

$$-\frac{B}{A} = -e^{2\beta}$$

This would give us the following equation:

$$\frac{\beta + 1}{1 - \beta} = -e^{2\beta}$$

But the only time that this equality holds true is when $\beta = 0$ (Fig. 2), but we are in the case when we defined β to be nonzero, meaning that we have reached a contradiction and we cannot have any negative eigenvalues.



Question 2

Find the Fourier series of $f(x)$. Does the Fourier-series converge (i) pointwise, or (ii) uniformly?

(a)

$$f(x) = \begin{cases} 1 - |x| & |x| \leq 1 \\ 1 & 1 < |x| \leq \pi \end{cases}$$

The Fourier series for a function $f(x)$ on an interval $-l < x < l$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \right)$$

where

$$a_0 = \frac{1}{2l} \int_{-l}^l f(x) dx$$

$$a_n = \frac{1}{l} \int_{-l}^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{1}{l} \int_{-l}^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

Here, $l = \pi$. Let's start with a_0 :

$$\begin{aligned}
a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \\
&= \frac{1}{2\pi} \left[\int_{-\pi}^{-1} 1 dx + \int_{-1}^1 (1 - |x|) dx + \int_1^{\pi} 1 dx \right] \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} 1 dx - \int_{-1}^1 |x| dx \\
&= 1 - 2 \int_0^1 x dx \\
&= 1 - 1 = 0
\end{aligned}$$

where we used the evenness of $|x|$ to rewrite $\int_{-1}^1 |x| dx$ as $2 \int_0^1 x dx$. Thus, $a_0 = 0$. Now we perform a similar procedure to find a_n :

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \\
&= \frac{1}{\pi} \left[\int_{-\pi}^{-1} \cos(nx) dx + \int_{-1}^1 (1 - |x|) \cos(nx) dx + \int_1^{\pi} \cos(nx) dx \right] \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(nx) dx - \frac{1}{\pi} \int_{-1}^1 |x| \cos(nx) dx
\end{aligned}$$

Since $\sin(\pi n)$ and $\sin(-\pi n)$ are zero for any integer n , the first integral vanishes. We use the symmetry of $|x| \cos(nx)$ (both functions are even, and the product of two even functions is even) and rewrite the second integral as

$$-\frac{2}{\pi} \int_0^1 x \cos(nx) dx$$

We then use integration by parts:

$$\begin{aligned}
&-\frac{2}{\pi} \int_0^1 x \cos(nx) dx \\
&= -\frac{2}{n\pi} \left[x \sin(nx) \Big|_0^1 - \int_0^1 \sin(nx) dx \right] \\
&= -\frac{2}{n\pi} \left[\sin(n) + \frac{1}{n} \cos(nx) \Big|_0^1 \right] \\
&= -\frac{2}{n\pi} \left[\sin(n) + \frac{1}{n} \cos(n) - \frac{1}{n} \right] \\
a_n &= -\frac{2 \sin n}{n\pi} + \frac{2(\cos n - 1)}{n^2 \pi} \\
b_n &= \frac{1}{\pi} \left[\int_{-\pi}^{-1} \sin(nx) dx + \int_{-1}^1 (1 - |x|) \sin(nx) dx + \int_1^{\pi} \sin(nx) dx \right] \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(nx) dx - \frac{1}{\pi} \int_{-1}^1 |x| \sin(nx) dx
\end{aligned}$$

But $\sin x$ is an odd function and so is $|x| \sin(nx)$, so both of these integrals vanish and we find $b_n = 0$.

Now inserting the coefficients a_n we found above, we get the Fourier series:

$$f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\frac{\sin n}{n} + \frac{\cos n + 1}{n^2} \right) \cos(nx)$$

(b)

$$f(x) = |x| = \begin{cases} -x & -\pi \leq x \leq 0 \\ x & 0 < x \leq \pi \end{cases}$$

Let's start with a_0 :

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| dx \\ &= \frac{1}{\pi} \int_0^{\pi} x dx \\ &= \frac{1}{2\pi} x^2 \Big|_0^{\pi} = \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \\ &= \frac{2}{\pi} \int_0^{\pi} x \cos(nx) dx \\ &= \frac{2}{n\pi} x \sin(nx) \Big|_0^{\pi} - \frac{2}{n\pi} \int_0^{\pi} \sin(nx) dx \\ &= 0 + \frac{2}{n^2\pi} \cos(nx) \Big|_0^{\pi} \\ &= \frac{2}{n^2\pi} (-1)^n - \frac{2}{n^2\pi} \\ &= \frac{2}{n^2\pi} (-1)^{n+1} \end{aligned}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

But since $|x|$ is even and $\sin(nx)$ is odd, the product of the two functions is odd and so the integral vanishes over the symmetric interval.

Then we plug in our formulas for a_0 and a_n :

$$f(x) = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \left[\frac{1}{n^2} (-1)^{n+1} \cos(nx) \right]$$

(c)

$$f(x) = x + x^2, \quad -\pi \leq x \leq \pi$$

$$\begin{aligned}
a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x + x^2 dx \\
&= \frac{1}{2\pi} \left(\frac{x^2}{2} + \frac{x^3}{3} \right) \Big|_{-\pi}^{\pi} \\
&= \frac{1}{2\pi} \left(\frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} + \frac{\pi^3}{3} \right) \\
&= \frac{1}{2\pi} \left(\frac{2\pi^3}{3} \right) = \frac{\pi^2}{3}
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos(nx) dx + \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(nx) dx
\end{aligned}$$

Applying integration by parts to the first integral,

$$\begin{aligned}
\frac{1}{\pi} \int_{-\pi}^{\pi} x \cos(nx) dx &= \frac{1}{n\pi} x \sin(nx) \Big|_{-\pi}^{\pi} - \frac{1}{n\pi} \int_{-\pi}^{\pi} \sin(nx) dx \\
&= \frac{1}{n\pi} x \sin(nx) \Big|_{-\pi}^{\pi} - \frac{-1}{n^2\pi} \cos(nx) \Big|_{-\pi}^{\pi} = 0 - 0 = 0
\end{aligned}$$

For the second integral, we can use the integration by parts table method demonstrated in class to find that

$$\frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(nx) dx = \frac{x^2}{n\pi} \sin(nx) \Big|_{-\pi}^{\pi} + \frac{2x}{n^2\pi} \cos(nx) \Big|_{-\pi}^{\pi} + \frac{2}{n^3\pi} \sin(nx) \Big|_{-\pi}^{\pi}$$

The first and last terms (with $\sin(nx)$) vanish. Evaluating the middle term, we get:

$$\frac{2\pi \cos(n\pi)}{n^2\pi} - \frac{-2\pi \cos(n\pi)}{n^2\pi} = \frac{4 \cos(n\pi)}{n^2} = \frac{4}{n^2} (-1)^n$$

Thus, we have found our coefficients a_n :

$$a_n = \frac{4}{n^2} (-1)^n$$

Now for b_n :

$$\begin{aligned}
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx) + \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \sin(nx) dx
\end{aligned}$$

But since x^2 is even and \sin is odd, the product under the second integral is odd functions and thus evaluates to zero over the symmetric interval $-\pi < x < \pi$. Applying integration by parts to the first integral,

$$b_n = \frac{-1}{n\pi} x \cos(nx) \Big|_{-\pi}^{\pi} + \frac{1}{n\pi} \int_{-\pi}^{\pi} \cos(nx) dx$$

But since \cos is even and $\cos \pi = \cos -\pi$, both terms are zero. Thus, $b_n = 0$ and

$$\begin{aligned}
f(x) &= a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \\
f(x) &= \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n \cos(nx)}{n^2}
\end{aligned}$$

Question 3

a)

Find the Fourier sine series of

$$f(x) = \begin{cases} 1 & 0 < x < \pi/2 \\ 2 & \pi/2 < x < \pi \end{cases}$$

The Fourier sine series of a function $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right)$$

where

$$A_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$A_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx$$

$$\begin{aligned} A_n &= \frac{2}{\pi} \left[\int_0^{\pi/2} \sin(nx) dx + 2 \int_{\pi/2}^{\pi} \sin(nx) dx \right] \\ &= \frac{2}{\pi} \left[\left. -\frac{1}{n} \cos(nx) \right|_0^{\pi/2} - \frac{2}{n} \cos(nx) \right]_{\pi/2}^{\pi} \\ &= \frac{2}{n\pi} \left[-\cos\left(\frac{n\pi}{2}\right) + 1 - 2\cos(n\pi) + 2\cos\left(\frac{n\pi}{2}\right) \right] \\ &= \frac{2}{n\pi} \left[1 + \cos\left(\frac{n\pi}{2}\right) - 2\cos(n\pi) \right] \end{aligned}$$

$$f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[1 + \cos\left(\frac{n\pi}{2}\right) - 2\cos(n\pi) \right] \sin(nx)$$

b)

Find the Fourier-cosine-series of $f(x) = |\sin x|$. Then find

$$\sum_{k=1}^{\infty} \frac{1}{4k^2 - 1}$$

The Fourier cosine series of a function $f(x)$ is

$$f(x) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right)$$

where

$$A_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

Assuming $l = \pi$, we have

$$A_n = \frac{2}{\pi} \int_0^{\pi} |\sin x| \cos(nx) dx \tag{3}$$

But since \sin is non-negative on the interval $0 \leq x \leq \pi$, we can drop the absolute value bars and use the following trigonometric identity to rewrite the integral: $\sin \alpha \cos \beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$

$$\begin{aligned}
A_n &= \frac{1}{\pi} \int_0^\pi \sin((1+n)x) + \sin((1-n)x) dx \\
&= \frac{1}{\pi} \left[\frac{-1}{1+n} \cos((1+n)x) \Big|_0^\pi - \frac{1}{1-n} \cos((1-n)x) \Big|_0^\pi \right] \\
&= \frac{1}{\pi} \left[\frac{-1}{1+n} [(-1)^{1+n} - 1] - \frac{1}{1-n} [(-1)^{1-n} - 1] \right] \\
&= \frac{1}{\pi} [(-1)^{1+n} - 1] \left(\frac{-1}{1+n} - \frac{1}{1-n} \right)
\end{aligned}$$

Thus, we have found a formula for A_n ,

$$A_n = \frac{-2(-1)^n + 1}{\pi(n^2 - 1)}$$

This formula works for all $n \neq 1$ For $n = 0$,

$$A_0 = \frac{-2(-1)^0 + 1}{\pi(0 - 1)} = \frac{4}{\pi}$$

For $n = 1$, we can go back to (3) and plug in the appropriate value of n :

$$A_1 = \frac{1}{\pi} \int_0^\pi \sin(2x) dx = \frac{1}{2\pi} \cos(2x) \Big|_0^\pi dx = 0$$

The Fourier cosine series is therefore

$$f(x) = \frac{2}{\pi} - \frac{2}{\pi} \sum_{n=2}^{\infty} \frac{(-1)^n + 1}{n^2 - 1} \cos(nx)$$

Now, $(-1)^n + 1$ will be zero for all odd n and two for all even n . So substituting $n = 2k$, we gwt

$$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{k=2}^{\infty} \frac{1}{4k^2 - 1} \cos(2kx)$$

Starting the index at $2k = 2$ is equivalent to starting it at $k = 1$, so after making that change and evaluating both sides at $x = 0$, we can solve for the sum given in the question:

$$\begin{aligned}
0 &= \frac{2}{\pi} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{4k^2 - 1} \\
\sum_{k=1}^{\infty} \frac{1}{4k^2 - 1} &= \frac{1}{2}
\end{aligned}$$

c)

The Riemann Zeta function is defined for $s > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

By computing the Fourier series of x^2 over $-\pi < x < \pi$ and using Parseval's identity, compute $\zeta(4)$.

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 dx \\ &= \frac{1}{2\pi} \left(\frac{x^3}{3} \right) \Big|_{-\pi}^{\pi} \\ &= \frac{1}{2\pi} \left(\frac{\pi^3}{3} - \frac{-\pi^3}{3} \right) \\ &= \frac{1}{2\pi} \left(\frac{2\pi^3}{3} \right) = \frac{\pi^2}{3} \end{aligned}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(nx) dx$$

But in part (2c), we found that this is equivalent to $\frac{4}{n^2}(-1)^n$, so

$$a_n = \frac{4}{n^2}(-1)^n$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \sin(nx) dx$$

Since x^2 is an even function, $b_n = 0$.

$$f(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n \cos(nx)}{n^2}$$