MATH 245 Homework 6

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(Wednesday morning correction to Problem 5)

Problem 1

The function $u : D \subset \mathbb{R}^2 \to \mathbb{R}$ is subharmonic if $\Delta u = u_{xx} + u_{yy} \geq 0$ and superharmonic if $\Delta u = u_{xx} + u_{yy} \leq 0$. Does the maximum and minimum principle hold for subharmonic functions or superharmonic on a connected bounded domain $D \subset \mathbb{R}^2$? If yes, state and prove it, otherwise give a counterexample.

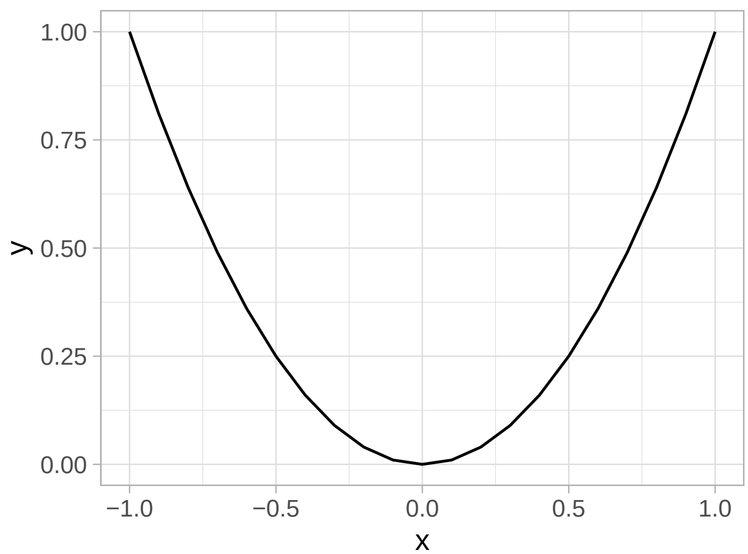
Subharmonic

Minimum - No
Take $u(x, y) = x^2$ on the connected bounded domain $D = \{(x, y) : -1 \leq x \leq 1, -1 \leq y \leq 1\}$. Then $\Delta u = 2 > 0$, so u is subharmonic. However, the minimum of u on D occurs at $x = 0$ where $u(0, y) = 0$ and not on the boundary of D, where $u = 1$.

Maximum - Yes

Let D be a bounded domain in $\mathbb{R}^2$ and $u : D \subset \mathbb{R}^2 \to \mathbb{R}$. Assume u is continuous on $\bar{D}$, where $\bar{D} = D \cup \partial D$, and is a continuous subharmonic function such that $\Delta u = u_{xx} + u_{yy} \geq 0$.

Then because D is bounded and u is continuous on $\bar{D}$, u must attain its maximum and minimum in $\bar{D}$. Define $M = \max_{\partial D} u = u(x_0)$ on ∂D. Our goal is to show that $\max_D u \leq \max_{\partial D} u = M$.



For every $\varepsilon > 0$, we define $v(x) = u(x) + \varepsilon|x|^2$. Then

$$\begin{aligned}
\Delta v &= \Delta u + \varepsilon \Delta |x|^2 \\
&= \Delta u + \varepsilon \sum_{i=1}^{n} \frac{\partial^2 \Delta |x|^2}{\partial x_i^2} \\
&= \Delta u + \varepsilon(2 + 2 + ... + 2) \\
&= \Delta u + 2n\varepsilon
\end{aligned}$$

And since u is subharmonic, we have that

$$\Delta v = \Delta u + 2n\varepsilon > 0 \tag{1}$$

Now assume the max of v is inside D, at some $x_1 \in D$. Then by (1), we have $\Delta v(x_1) > 0$. By generalization of the second derivative theorem, we know that at the maximum of a function $v : \mathbb{R}^n \to \mathbb{R}$, the Hessian of v, $\nabla^2 v(x_1)$, should be negative semi-definite, meaning that all eigenvalues λ_i are less than or equal to zero. But $\lambda_i \leq 0$ implies that

$$\Delta v(x_1) = \operatorname{tr}\left(\nabla^2 v(x_1)\right) = (\lambda_1 + \lambda_2 + ... + \lambda_n)(x_1) \leq 0$$

which is a contradiction of (1). Therefore, the maximum of v is at the boundary,

$$v(x) \leq \max_{\partial D} v \quad \forall\, x \in D$$

So, $\forall\, x \in D$,

$$\begin{aligned}
u(x) &= v(x) - \varepsilon|x|^2 \\
&\leq \max_{\partial D} v - \varepsilon|x|^2
\end{aligned}$$

But since we can write

$$\begin{aligned}\max_{\partial D} v - \varepsilon|x|^2 &= \max_{\partial D}\left(u(x) + \varepsilon|x|^2\right) - \varepsilon|x|^2 \\ &= \max_{\partial D} u + \varepsilon\left(R - |x|^2\right),\end{aligned}$$

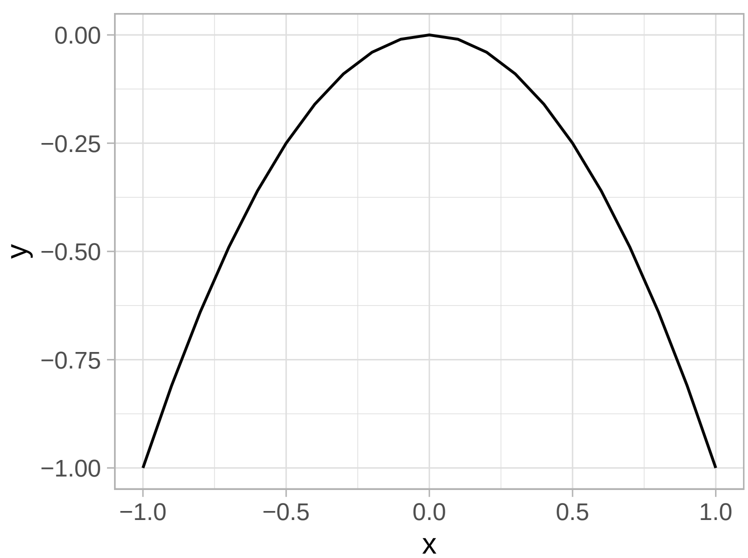
where R is the largest radius of the domain, we must have

$$u(x) \leq \max_{\partial D} u + \varepsilon\left(R - |x|^2\right).$$

Since this is true for any ε, we take $\varepsilon \to 0$ to find that $u(x) \leq \max_{\partial D} u$. $\square$

Superharmonic

Maximum - No
Take $u(x, y) = -x^2$ on the connected bounded domain $D = \{(x, y) : -1 \leq x \leq 1, -1 \leq y \leq 1\}$. Then $\Delta u = -2 < 0$, so u is superharmonic. However, the maximum of u on D occurs at $x = 0$ where $u(0, y) = 0$ and not on the boundary of D, where $u = -1$.



Minimum - Yes

Let D be a bounded domain in $\mathbb{R}^2$ and $u : D \subset \mathbb{R}^2 \to \mathbb{R}$. Assume u is continuous on $\bar{D}$, where $\bar{D} = D \cup \partial D$, and is a continuous superharmonic function such that $\Delta u = u_{xx} + u_{yy} \leq 0$.

Then because D is bounded and u is continuous on $\bar{D}$, u must attain its maximum and minimum in $\bar{D}$. Define $m = \min_{\partial D} u = u(x_0)$ on ∂D. Our goal is to show that $\min_D u \geq$

$\min_{\partial D} u = m$. For every $\varepsilon > 0$, we define $v(x) = u(x) - \varepsilon|x|^2$. Then

$$\begin{aligned}
\Delta v &= \Delta u - \varepsilon \Delta |x|^2 \\
&= \Delta u - \varepsilon \sum_{i=1}^{n} \frac{\partial^2 \Delta |x|^2}{\partial x_i^2} \\
&= \Delta u - \varepsilon(2 + 2 + ... + 2) \\
&= \Delta u - 2n\varepsilon
\end{aligned}$$

And since u is superharmonic, we have that

$$\Delta v = \Delta u - 2n\varepsilon < 0 \tag{2}$$

Now assume the min of v is inside D, at some $x_1 \in D$. Then by (2), we have $\Delta v(x_1) < 0$. By generalization of the second derivative theorem, we know that at the minimum of a function $v : \mathbb{R}^n \to \mathbb{R}$, the Hessian of v, $\nabla^2 v(x_1)$, should be positive semi-definite, meaning that all eigenvalues λ_i are greater than or equal to zero. But $\lambda_i \geq 0$ implies that

$$\Delta v(x_1) = \operatorname{tr}\left(\nabla^2 v(x_1)\right) = (\lambda_1 + \lambda_2 + ... + \lambda_n)(x_1) \geq 0$$

which is a contradiction of (2). Therefore, the minimum of v is at the boundary,

$$v(x) \geq \min_{\partial D} v \quad \forall\, x \in D$$

So, $\forall\, x \in D$,

$$\begin{aligned}
u(x) &= v(x) + \varepsilon|x|^2 \\
&\leq \min_{\partial D} v + \varepsilon|x|^2
\end{aligned}$$

But since we can write

$$\begin{aligned}
\min_{\partial D} v + \varepsilon|x|^2 &= \min_{\partial D} \left(u(x) - \varepsilon|x|^2\right) + \varepsilon|x|^2 \\
&= \min_{\partial D} u + \varepsilon\left(|x|^2 - R\right),
\end{aligned}$$

where R is the largest radius of the domain, we must have

$$u(x) \leq \min_{\partial D} u + \varepsilon\left(|x|^2 - R\right).$$

Since this is true for any ε, we take $\varepsilon \to 0$ to find that $u(x) \geq \min_{\partial D} u$. $\square$

Problem 2

Suppose u is harmonic on the disk $D = \{(x,y) : x^2 + y^2 < 4\}$ and $u = \sin(\theta) + 1$ on ∂D. Without finding the solution u, find the maximum value of u in $D \cup \partial D$ and value of u at the origin.

By the maximum principle, u obtains its maximum in $D \cup \partial D$ on the boundary ∂D and not inside. The maximum of $u = \sin\theta + 1$ on ∂D is at $\theta = \frac{\pi}{2}$, where $\sin\theta$ is at its maximum. Therefore, the maximum value of u in $D \cup \partial D$ is $u = \sin\left(\frac{\pi}{2}\right) + 1 = 1 + 1 = 2$.

The mean value property says that for a harmonic function u in a disk D, continuous in its closure $\bar{D}$, the value of u at the center of D equals the average of u on its circumference. The average value of a continuous function $f(x)$ over the interval $[a,b]$ is given by

$$f_{avg}(x) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Here,

$$\begin{aligned}
u(0) &= \text{average of u over } \partial B(0,2) \\
&= \frac{1}{2\pi R}\int_{\partial B(0,R)} u\,ds \\
&= \frac{1}{4\pi}\int_{\partial B(0,R)} u\,ds \\
&= \frac{1}{4\pi}\int_{\theta=0}^{\theta=2\pi} u \cdot R\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} u\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} (\sin\theta + 1)\,d\theta \\
&= \frac{1}{2\pi}\Big[(\cos\theta + \theta)\Big]_0^{2\pi} \\
&= \frac{1}{2\pi}\Big[(\cos 2\pi + 2\pi) - \cos 0\Big] \\
&= \frac{1}{2\pi}(1 + 2\pi - 1) = 1
\end{aligned}$$

Therefore, the value of u at the origin is 1.

Problem 3

Use the maximum principle to show that the solution of the Dirichlet problem for $\Delta u = 0$ depends continuously on the boundary data.

Define u_1 and v to be two solutions to the Dirichlet problems

$$\begin{cases} \Delta u_1 = 0 & \text{in } D \\ u_1 = h(x) & \text{on } \partial D \end{cases}$$

$$\begin{cases} \Delta u_2 = 0 & \text{in } D \\ u_2 = h(x) + \varepsilon(x) & \text{on } \partial D \end{cases}$$

where $\varepsilon(x)$ is small $\forall x$ on ∂D. Now let $w = u_1 - u_2$. Then w solves the following Dirichlet problem:

$$\begin{cases} \Delta w = 0 & \text{in } D \\ w = \varepsilon(x) & \text{on } \partial D \end{cases}$$

By the maximum and minimum principles,

$$\min_{\partial D} \varepsilon(x) \leq w \leq \max_{\partial D} \varepsilon(x)$$

If $\varepsilon \to 0$ for all x on ∂D, then $w = u_1 - u_2$ approaches zero for all $x \in D$.

Problem 4

Solve the following problem on semi-annulus domain:

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 & 0 < a < r < b, 0 < \theta < \pi \\ u(r,0) = u(r,\pi) = 0 & 0 < a < r < b \\ u_r(a,\theta) = \theta & 0 < \theta < \pi \\ u_r(b,\theta) = 0 & 0 < \theta < \pi \end{cases} \tag{3}$$

We look for a separated solution of the form $u = \Theta(\theta)R(r)$. First, we rewrite $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ as

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

Separating variables, we get

$$\frac{R''}{R}r^2 + \frac{R'}{R}r = -\frac{\Theta''}{\Theta}$$

Since the left-hand side is a function of only R and the right-hand side is a function of only Θ, we can set both sides equal to a constant λ and separate this into two problems, one in Θ and one in R. Start with θ:

$$\Theta'' + \Theta\lambda = 0$$

From our boundary conditions,

$$R(r)\Theta(0) = R(r)\Theta(\pi) = 0 \quad \Longrightarrow \quad \Theta(\pi) = \Theta(0) = 0$$

So we have an eigenvalue problem with homogenous Dirichlet BCs, which we have shown to have the eigenvalues and eigenvectors

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \sin\left(\frac{n\pi x}{l}\right), \quad n = 1, 2, 3...$$

Here, that means

$$\lambda_n = n^2, \quad X_n(x) = \sin(n\theta), \quad n = 1, 2, 3...$$

Now onto the R problem:

$$\frac{R''}{R}r^2 + \frac{R'}{R}r = \lambda$$

$$R''r^2 + R'r - \lambda R = 0$$

This is an Euler ODE of the form $ax^2y'' + bxy' + cy = 0$, which has the characteristic equation $am(m-1) + bm + c$ and, if m_1, m_2 are distinct real roots, the general solution $y = C_1x^{m_1} + C_2x^{m_2}$. (Note that I have used m instead of the standard r to avoid confusion with the variable r from (3).

Here, $a = 1, b = 1, c = -\lambda$, so the characteristic equation becomes $m(m-1) + m - \lambda$, which simplifies to $m^2 - \lambda$. Thus, our roots are $\pm\sqrt{\lambda} = \pm n$. Thus, $R(r) = Cr^n + Dr^{-n}$.

Our separated solutions thus take the form

$$u_n(r,\theta) = (Cr^n + Dr^{-n})\sin(n\theta), \quad n = 1, 2, 3...$$

$$u(r,\theta) = \sum_{n=1}^{\infty}(Cr^n + Dr^{-n})\sin(n\theta)$$

Now we use our boundary conditions $u_r(a,\theta) = \theta$ and $u_r(b,\theta) = 0$:

$$u_r(r,\theta) = \sum_{n=1}^{\infty}\left(nCr^{n-1} - nDr^{-n-1}\right)\sin(n\theta)$$

$$u_r(b,\theta) = 0 = \sum_{n=1}^{\infty}\left(nCb^{n-1} - nDb^{-n-1}\right)\sin(n\theta)$$

$$Cb^n = Db^{-n} \to C = Db^{-2n}$$

$$u_r(a,\theta) = \theta = \sum_{n=1}^{\infty}\left(nDb^{-2n}a^{n-1} - nDa^{-n-1}\right)\sin(n\theta)$$

$$\frac{nD}{a}\left(b^{-2n}a^n - a^{-n}\right) = \int_0^{\pi}\theta\sin(n\theta)d\theta$$

$$\int_0^\pi \theta \sin(n\theta) d\theta = \theta(-1/n \cos n\theta)\Big|_0^\pi = \pi(-1/n)(-1)^n = \frac{\pi(-1)^{n+1}}{n}$$

$$\frac{nD}{a}\left(b^{-2n}a^n - a^{-n}\right) = \frac{\pi(-1)^{n+1}}{n}$$

Therefore, our solution to (3) is

Solution

$$u(r,\theta) = \sum_{n=1}^{\infty} (Cr^n + Dr^{-n}) \sin(n\theta)$$

where

$$D = \frac{a\pi(-1)^{n+1}}{n^2\left(b^{-2n}a^n - a^{-n}\right)}$$

$$C = Db^{-2n}$$

Problem 5

Consider the Neumann problem with periodic BC:

$$\begin{cases} \Delta u = 0 & r^2 > a^2 \\ u_r(a,\theta) = g(\theta) & r^2 = a^2 \end{cases} \tag{4}$$

Find the necessary condition for the existence of a solution and then solve the problem. Hint: Use Green's identity

$$\iint\limits_D (u\Delta v - v\Delta u)\, dA = \int\limits_{\partial D} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds \tag{5}$$

where D is a domain and n is a unit normal vector. Note that for a 2-dimensional domain D, $dA = dx\, dy$ and ds is arc length.

Since (5) must hold for any (u, v), take $v = 1$. Then $\Delta v = 0 = \frac{\partial v}{\partial n}$, so

$$\iint\limits_{D} (u\Delta v - v\Delta u)\, dA = \int\limits_{\partial D} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) ds \quad \text{becomes}$$

$$\iint\limits_{D} (0 - \Delta u)\, dA = \int\limits_{\partial D} \left(0 - \frac{\partial u}{\partial n} \right) ds$$

$$0 = -\int\limits_{\partial D} g(\theta)\; d\theta$$

Where we used the fact that by (4), $\Delta u = 0$ and $\frac{\partial u}{\partial n} = u_r = g(\theta)$. Thus, the necessary condition for the existence of a solution is

$$\int\limits_{\partial D} g(\theta)\; d\theta = 0$$

To solve the problem, we now look for a separated solution of the form $u = \Theta(\theta)R(r)$. In polar coordinates, $\Delta u = 0$ becomes $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$, giving us $R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$. From the previous problem, we know that this yields two problems, one in Θ and one in R.

For Θ, we have $\Theta'' + \Theta\lambda = 0$ with periodic BCs. Using the table provided and setting $l = \pi$, this means we have eigenvalues

$$\begin{cases} \lambda_0 = 0 \\ \lambda_n = \left(\frac{n\pi}{l}\right)^2 = n^2 \quad n = 1, 2, 3, ... \end{cases}$$

with eigenfunctions

$$\begin{cases} \Theta_0(\theta) = C_0 \\ \Theta_n(\theta) = \Big\{ \cos(n\theta), \sin(n\theta) \Big\} \end{cases}$$

Now we return to the R problem. As before, we get an Euler equation with $a = 1, b = 1, c = -\lambda$, so the characteristic equation becomes $m(m-1) + m - \lambda$, which simplifies to $m^2 - \lambda$.

For λ_0, we get the repeated root $m_1 = m_2 = 0$, giving us the solution $R_0(r) = C_1 + C_2 \ln r$. For $\lambda_n = n^2$, our roots are $m = \pm n$. Thus, $R_n(r) = C_3 r^n + C_4 r^{-n}$.

We are only explicitly provided one BC, so in order to determine the unknown coefficients, we will need to assume that we should disregard the solutions that go to infinity as r approaches infinity, since our domain is the region *outside* the disk with radius a.

Thus, we exclude $\ln r$ and r^n, which will both go to $+\infty$ as $r \to +\infty$, and sum the remaining solutions to find that

$$
\begin{aligned}
u(r,\theta) &= A_0 + \sum_{n=1}^{\infty} r^{-n} \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right) \\
u_r(r,\theta) &= \sum_{n=1}^{\infty} -nr^{-n-1} \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right)
\end{aligned}
$$

Finally, we use the inhomogeneous BC at $r = a$. Setting $r = a$ in the series above, we require that

$$
g(\theta) = \sum_{n=1}^{\infty} -na^{-n-1} \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right)
$$

Since this is a Fourier series for $g(\theta)$, we can now solve for the unknown coefficients:

$$
\begin{aligned}
-nA_n a^{-n-1} &= \frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) \cos(n\theta)\, d\theta \\
-nB_n a^{-n-1} &= \frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) \sin(n\theta)\, d\theta
\end{aligned}
$$

Therefore, our solution to (4) is

Solution

$$
u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^{-n} \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right)
$$

where

$$
\begin{aligned}
A_n &= -\frac{a^{n+1}}{n\pi} \int_{-\pi}^{\pi} g(\theta) \cos(n\theta)\, d\theta \\
B_n &= -\frac{a^{n+1}}{n\pi} \int_{-\pi}^{\pi} g(\theta) \sin(n\theta)\, d\theta
\end{aligned}
$$

Problem 6.

Prove the uniqueness of the Neumann problem on $D \subset \mathbb{R}^2$

$$
\begin{cases} \Delta u = f & \text{in } D \\ \frac{\partial u}{\partial n} = g & \text{on } \partial D \end{cases} \tag{6}
$$

up to a constant. Use the following energy and Green's identity:

$$
E(u) = \frac{1}{2} \int_D |\nabla u|^2 \, dA
$$

Suppose we have two solutions u and v to (6) and define $w = u - v$. Then we have a new Neumann problem

$$\begin{cases} \Delta w = 0 & \text{in } D \\ \frac{\partial w}{\partial n} = 0 & \text{on } \partial D \end{cases} \tag{7}$$

Using the version of Green's first identity given in the textbook but in two dimensions, we have

$$\int\limits_{\partial D} v \frac{\partial u}{\partial n}\, ds = \iint\limits_{D} \nabla v \cdot \nabla u \; d\mathbf{x} + \iint\limits_{D} v \Delta u \; d\mathbf{x}$$

Now replace u and v in this formula with w:

$$\int\limits_{\partial D} w \frac{\partial w}{\partial n}\, ds = \iint\limits_{D} |\nabla w|^2 \; d\mathbf{x} + \iint\limits_{D} w \Delta w \; d\mathbf{x}$$

But from (7), $\Delta w = 0$ in D and $\frac{\partial w}{\partial n} = 0$ on ∂D, so

$$0 = \iint\limits_{D} |\nabla w|^2 \; d\mathbf{x}$$

By the vanishing theorem, it follows that $|\nabla w|^2 \equiv 0$ in D, so $\nabla w = 0$. But since a function with a vanishing gradient must be constant, provided that D is connected, this means that $w = c$. Thus, $c = u - v$ and the two solutions to the Neumann problem are unique up to a constant. $\square$