MATH 245 Homework 2

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Question 1

Determine the region in which the given equation is hyperbolic, parabolic, elliptic, or singular.

a)

$$u_{xx} + y^2 u_{yy} + u_x - u + x^2 = 0$$

b)

$$u_{xx} - y u_{yy} + x u_x + y u_y + u = 0$$

Question 2

Using a factorization similar to the wave equation, solve the following IVP:

$$\begin{cases} u_{xx} + 2u_{xy} - 3u_{yy} = 0 & x \subset \mathbb{R},\ y > 0 \\ u(0, x) = \sin x & x \subset \mathbb{R} \\ u_y(0, x) = x & x \subset \mathbb{R} \end{cases} \tag{1}$$

First, we can factor the equation as follows:

$$\left(\frac{\partial}{\partial x} + 3\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) u = 0$$

or

$$\left(\partial_x + 3\partial_y\right)\left(\partial_x - \partial_y\right) u = 0$$

Then set $\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) u = v$, giving us

$\left(\frac{\partial}{\partial x} + 3\frac{\partial}{\partial y}\right) v = v_x + 3v_t = 0$

which we know has the solution $v(x, y) = f(3x - y)$, so

$$u_x - u_y = f(3x - y)$$

On (characteristic) lines with the slope $y = -x + c$, or $y + x =$ constant, we must have $u_x - u_y = f(3x - y) = 0$. Set $\eta = x + y$ and $\xi = x$. Then by the chain rule,

$$u_x = u_\eta + u_\xi,\ u_y = u_\eta$$

And let's rewrite y as $y = \eta - x = \eta - \xi$.

So

$$u_x - u_y = f(3x - y) \quad \longrightarrow \quad u_\xi = f(3\xi - \eta + \xi)$$

$$u_\xi = f(4\xi - \eta)$$

Now integrate with respect to ξ:

$$u(\eta, \xi) = F(4\xi - \eta) + G(\eta)$$

where F is the antiderivative of f with respect to ξ.

Now convert back to our original variables:

$$u(x, y) = F(3x - y) + G(x + y)$$

Using the fact that $u(0, x) = \sin x$,

$$u(0, x) = \sin x = F(3x) + G(x) \tag{2}$$

now replace x with a new neutral variable, α and differentiate:

$$\sin\alpha = F(3\alpha) + G(\alpha)$$

$$\cos\alpha = 3F'(3\alpha) + G'(\alpha) \tag{3}$$

But we can also differentiate $u(x, y) = F(3x - y) + G(x + y)$ with respect to y to get

$$u_y(x, y) = -F'(3x - y) + G'(x + y)$$

but from our initial conditions, we know

$$u_y(0, x) = -F'(3x - 0) + G'(x + 0) = x$$

Let's replace x by our neutral variable α and solve for F':

$$F'(\alpha) = G'(3\alpha) - \alpha$$

Now plug this into 3:

$$\cos\alpha = 3G'(\alpha) - 3\alpha + G'(\alpha)$$

$$G(\alpha) = \frac{1}{4}\int \cos\alpha + 3\alpha = \frac{\sin\alpha}{4} + \frac{3\alpha^2}{8}$$

So that means 2 becomes:

$$\sin\alpha = F(3\alpha) + \frac{\sin\alpha}{4} + \frac{3\alpha^2}{8}$$

$$F(\alpha) = \frac{3\sin\left(\frac{\alpha}{3}\right)}{4} - \frac{\alpha^2}{24}$$

Which means $u(x, y) = F(3x - y) + G(x + y)$ becomes

$$u(x, y) = \frac{3}{4}\sin\left(x - \frac{y}{3}\right) - \frac{(3x - y)^2}{24} + \frac{\sin(x + y)}{4} + \frac{3(x + y)^2}{8} =$$

Solution

$$u(x,y) = \frac{3}{4}\sin\left(x - \frac{y}{3}\right) + \frac{\sin(x+y)}{4} + xy + \frac{y^2}{3}$$

Check solution

$$u_y = \frac{-1}{4}\cos\left(x - \frac{y}{3}\right) + \frac{\cos(x+y)}{4} + x + \frac{2y}{3}$$

$$u_{yy} = \frac{-1}{12}\sin\left(x - \frac{y}{3}\right) - \frac{\sin(x+y)}{4} + \frac{2}{3}$$

$$u_x = \frac{3}{4}\cos\left(x - \frac{y}{3}\right) + \frac{\cos(x+y)}{4} + y$$

$$u_{xx} = \frac{-3}{4}\sin\left(x - \frac{y}{3}\right) - \frac{\sin(x+y)}{4}$$

$$u_{xy} = \frac{1}{4}\sin\left(x - \frac{y}{3}\right) - \frac{\sin(x+y)}{4} + 1$$

Check that $u_{xx} + 2u_{xy} - 3u_{yy} = 0$

$$\left(\frac{-3}{4} + \frac{2}{4} + \frac{1}{4}\right)\sin\left(x - \frac{y}{3}\right) + \left(\frac{-1}{4} + \frac{-2}{4} + \frac{3}{4}\right)\sin(x+y) + (0 + 2 - 3) = 0$$