PDE pres

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2024-04-27

We represent the the population density of *B. violaceus* at time t and spatial coordinate x as $u(t, x)$.

Assume that the population occupies a domain of fixed length L, so that $0 < x < L$. The environment outside the domain is hostile, so there is no population there: if $x \leq 0$ or $x \geq L$, then $u(x, t) = 0$ for all t.

If we assumed simple linear population growth, our PDE would take the form

$$\frac{\partial u}{\partial t} = \mathrm{D}\frac{\partial u^2}{\partial x^2} + ru, \quad 0 < x < L \tag{1}$$

with the Dirichlet BCs $u(0, t) = u(0, L) = 0$.

This is the simplest version of the general reaction-diffusion model, for a homogeneous, unstructured population growing exponentially and dispersing in a uniform, one-dimensional environment (Andow et al. 1990; Skellam 1951).

We know how to solve this! The solution is

$$u(t, x) = \sum_{n=1}^{\infty} C_n e^{(r-\lambda_n)t} \sin\left(\frac{n\pi x}{L}\right),$$

where

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 D, \quad n = 1, 2, 3...$$

and the coefficients C_n are determined by the initial density $u(x, 0)$.

$$\lambda_1 = \left(\frac{\pi}{L}\right)^2 D \quad \Longrightarrow \quad r = \left(\frac{\pi}{L}\right)^2 D \quad \Longrightarrow \quad L_{cr} = \pi\sqrt{\frac{D}{r}}$$

A more realistic model considers the evolution of a compact initial population distribution in a large open space. Rather than imposing the restriction that the population always occupy a fixed region, we instead allow the population to grow freely. Only the initial condition is bounded, assumed for simplicity to be connected:

$$u(0, x) = \begin{cases} \phi(x) > 0 & 0 < x < L \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Here, L is the size of the initially occupied area and the function ϕ describes the initial population distribution. The equation describing the population dynamics, that is,

$$\frac{\partial u}{\partial t} = \mathrm{D}\frac{\partial u^2}{\partial x^2} + F(u) \tag{3}$$

should now be defined on a much larger domain than L; here, we assume $-\infty < x < \infty$. Note that $F(u)$ can now be a nonlinear function (like logistic growth), which is much more biologically realistic.

For biologically reasonable choices of $F(u)$, the growth function will be bounded by the linear function, such that $F(u) \leq ru$.

$$\mathrm{D}\frac{\partial u^2}{\partial x^2} + F(u) \leq \mathrm{D}\frac{\partial U^2}{\partial x^2} + rU \tag{4}$$

Now, note that for compact initial conditions like (2), there exist positive constants G and σ such that we bound the initial distribution from above by a Gaussian distribution:

$$U(0,x) = \frac{G}{\sqrt{4\pi\sigma^2}} \exp\left(-\frac{x^2}{4\sigma^2}\right) \tag{5}$$

$$\text{such that} \quad u(0,x) \leq U(0,x) \quad \text{for all } x \tag{6}$$

$$\frac{\partial U}{\partial t} = \mathrm{D}\frac{\partial U^2}{\partial x^2} + rU \tag{7}$$

The linear equation in (1) can be readily solved, since the change of variables $U(t,x) = W(t,x)e^{rt}$ reduces it to the standard diffusion equation for W. Using forward and inverse Fourier transforms, we find that

$$W(t,x) = \frac{G}{\sqrt{4\pi(\sigma^2 + Dt)}} \exp\left(-\frac{x^2}{4(\sigma^2 + Dt)}\right)$$

Converting back to our original variable, we conclude that the solution to (1) with the initial condition (5a) is a normal distribution, specifically

$$U(t,x) = \frac{G}{\sqrt{4\pi(\sigma^2 + Dt)}} \exp\left(-\frac{x^2}{4(\sigma^2 + Dt)} + rt\right)$$

Now, we can quantify the spatial spread of the population by considering how quickly a level set $U(x,t) = U_c$ moves in space, where U_c could be considered a threshold of detection. Rearranging the above equation, we can get the following expression

$$\left(\frac{x}{t}\right)^2 = 4rD + \frac{4r\sigma^2}{t} - \frac{4(\sigma^2 + Dt)}{t^2} \ln\left(\frac{U_c\sqrt{4\pi(\sigma^2 + Dt)}}{G}\right)$$

As t becomes large, this formula simplifies to $\frac{|x|}{t} \to \sqrt{4rD}$. Thus, the asymptotic rate of spread is $V = \sqrt{4rD}$, as we initially sought to show.

General reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \mathrm{D}\left(\frac{\partial u^2}{\partial x^2}\right) + f(u)$$

Linear growth

$$\frac{\partial u}{\partial t} = \mathrm{D}\left(\frac{\partial u^2}{\partial x^2}\right) + ru$$

Fisher's equation

$$\frac{\partial u}{\partial t} = \mathrm{D}\left(\frac{\partial u^2}{\partial x^2}\right) + ru\left(1 - \frac{u}{K}\right)$$

Harvest/fishing mortality

$$\frac{\partial u}{\partial t} = \mathrm{D}\left(\frac{\partial u^2}{\partial x^2}\right) + ru\left(1 - \frac{u}{K}\right) - qE(x)u$$

With advection

$$\frac{\partial u}{\partial t} = \mathrm{D}\left(\frac{\partial u^2}{\partial x^2}\right) - a\frac{\partial u}{\partial x} + f(u)$$

References

Andow, D. A., P. M. Kareiva, Simon A. Levin, and Akira Okubo. 1990. "Spread of Invading Organisms." *Landscape Ecology* 4 (2): 177–88. https://doi.org/10.1007/BF00132860.

Skellam, J. G. 1951. "RANDOM DISPERSAL IN THEORETICAL POPULATIONS." *Biometrika* 38 (1-2): 196–218. https://doi.org/10.1093/biomet/38.1-2.196.