MATH 245 Homework 5

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Problem 1: Inhomogeneous Heat Equation

Using the method of separation of variables, solve the inhomogeneous heat equation:

$$\begin{cases} u_t - ku_{xx} = x\cos t & 0 < x < \pi, \quad t > 0 \\ u(0,x) = \sin(\pi x) & 0 < x < \pi \\ u(t,0) = t^2, \ \ u(t,\pi) = 2t & t > 0 \end{cases} \tag{1}$$

Problem 2: Inhomogeneous Wave Equation

Using the method of separation of variables, solve the inhomogeneous wave equation:

$$\begin{cases} u_{tt} - c^2 u_{xx} = F(t,x) & 0 < x < L, \quad t > 0 \\ u(0,x) = \phi(x) & 0 < x < L \\ u_t(0,x) = \psi(x) & 0 < x < L \\ u_x(t,0) = h(t), \ \ u_x(t,L) = g(t) & t > 0 \end{cases} \tag{2}$$

for constants a and $k > 0$

Problem 3: Damped Heat Equation

Using the method of separation of variables, solve the damped heat equation:

$$\begin{cases} u_t + au = ku_{xx} & -\pi < x < \pi, \quad t > 0 \\ u(0,x) = \phi(x) & -\pi < x < \pi \\ u(t,\pi) = u(t,-\pi) & t > 0 \\ u_x(t,\pi) = u_x(t,-\pi) & t > 0 \end{cases} \tag{3}$$

for constants a and $k > 0$

Problem 4: Beam Equation

Using the method of separation of variables, solve the beam equation:

$$\begin{cases} u_{tt} = c^2 u_{xxxx} & 0 < x < L, \quad t > 0 \\ u(0,x) = \phi(x) & 0 < x < L \\ u_t(0,x) = \psi(x) & 0 < x < L \\ u(t,0) = u(t,L) = 0 & t > 0 \\ u_{xx}(t,0) = u_{xx}(t,L) = 0 & \end{cases} \tag{4}$$

Problem 5: Radioactive Decay Problem

Using the method of separation of variables, solve the radioactive decay problem, for constants $A, a > 0$.

$$\begin{cases} u_t - u_{xx} = Ae^{-ax} \\ u(0,x) = \sin x \\ u(t,0) = u(t,\pi) = 0 \end{cases} \tag{5}$$

We want to find a separated solution of the form $u(t,x) = X(x)T(t)$. Recall that for the analogous homogeneous PDE with homogeneous Dirichlet boundary conditions, we consider the following eigenvalue problem $X'' + \lambda X = 0, \quad X(0) = X(l) = 0$, which we have shown to have the eigenvalues and eigenvectors

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \sin\left(\frac{n\pi x}{l}\right), \quad n = 1, 2, 3...$$

Giving us the Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \tag{6}$$

where

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Now, we assume that our solution to (5) will take a similar form as (6), where $l = \pi$ and $f(t,x) = Ae^{-ax}$:

$$u(t,x) = b_0(t) + \sum_{n=1}^{\infty} b_n(t)\sin(nx)$$

We can differentiate this and plug it into (5) as follows:

$$u_t(t,x) = b_0'(t) + \sum_{n=1}^{\infty} b_n'(t)\sin(nx)$$

$$u_{xx}(t,x) = -\sum_{n=1}^{\infty} b_n(t)\; n^2 \sin(nx)$$

$$b_0'(t) + \sum_{n=1}^{\infty} b_n'(t)\sin(nx) + \sum_{n=1}^{\infty} b_n(t)\; n^2\sin(nx) = Ae^{-ax} \tag{7}$$

For each fixed t, we write Ae^{-ax} as a Fourier sine series:

$$Ae^{-ax} = q_0(t) + \sum_{n=1}^{\infty} q_n(t)\sin(nx) \tag{8}$$

where

$$\begin{aligned} q_0(t) &= \frac{1}{l}\int_0^l f(t,x)dx \\ &= \frac{A}{\pi}\int_0^{\pi} e^{-ax}dx \\ &= \frac{-A}{a\pi}\left[e^{-ax}\right]_0^{pi} \\ &= \frac{-A}{a\pi}\left(e^{-a\pi} - 1\right) \end{aligned}$$

$$\begin{aligned}
q_n(t) &= \frac{2}{l}\int_0^l f(t,x)\sin\left(\frac{n\pi x}{l}\right)dx \\
&= \frac{2A}{\pi}\int_0^\pi e^{-ax}\sin(nx)dx
\end{aligned}$$

Now we do integration by parts twice on $\int_0^\pi e^{-ax}\sin(nx)dx$

First with $u=\sin(nx)$, $du = n\cos(nx)\,dx$, $dv = e^{-ax}\,dx$, $v=\frac{-1}{a}e^{-ax}$,

and the second time with $u = n\cos(nx)$, $du = -n^2\sin(nx)\,dx$, $dv=\frac{1}{a}e^{-ax}\,dx$, $v=\frac{-1}{a^2}e^{-ax}$

$$\begin{aligned}
\int_0^\pi e^{-ax}\sin(nx)dx &= \left[\frac{-\sin(nx)}{a}e^{-ax}\right]_0^\pi + \frac{n}{a}\int_0^\pi e^{-ax}\cos\left(\frac{n\pi x}{l}\right)dx \\
&= 0 - 0 + \frac{n}{a}\int_0^\pi e^{-ax}\cos(nx)dx \\
&= \left[\frac{-n\cos(nx)}{a^2}e^{-ax}\right]_0^\pi - \frac{n^2}{a^2}\int_0^\pi e^{-ax}\sin(nx)dx
\end{aligned}$$

Then moving the integrals to the same side,

$$\begin{aligned}
\left(1+\frac{n^2}{a^2}\right)\int_0^\pi e^{-ax}\sin(nx)dx &= \left[\frac{-n\cos(nx)}{a^2}e^{-ax}\right]_0^\pi \\
&= \frac{-n\cos(n\pi)}{a^2}e^{-a\pi} + \frac{n}{a^2} \\
(n^2+a^2)\int_0^\pi e^{-ax}\sin(nx)dx &= n(-1)^{n+1}e^{-a\pi}+n
\end{aligned}$$

Thus,

$$\int_0^\pi e^{-ax}\sin(nx)dx = \frac{n(-1)^{n+1}e^{-a\pi}+n}{n^2+a^2}$$

Which means that

$$q_n = \frac{2A}{\pi}\frac{n(-1)^{n+1}e^{-a\pi}+n}{n^2+a^2}$$

Now, by (7) and (8), we get the following equations:

$$\begin{cases} b_0'(t) = q_0(t) \\ b_n'(t) + b_n(t)n^2 = q_n(t) \end{cases}$$

From $b_0'(t) = q_0(t)$ we get

$$b_0(t) = \int_0^t q_0(s)\,ds$$

Since $q_0 = \frac{-A}{a\pi}\left(e^{-a\pi}-1\right)$, this means

$$b_0(t) = \frac{-At}{a\pi}\left(e^{-a\pi}-1\right) + b_0(0)$$

On the other hand, we have $b_n'(t)+b_n(t)n^2 = q_n(t)$, which we solve as follows:

$$\mu(t) = \exp\left(\int_0^t n^2\ ds\right) = \exp\left(n^2 t\right)$$

$$b_n(t) = \frac{1}{\mu(t)}\left[\int_0^t \mu(s) q_n(s)\ ds + b_n(0)\right]$$

$$b_n(t) = b_n(0)\,\mu(t)^{-1} + \int_0^t \frac{\mu(s)}{\mu(t)} q_n(s)\ ds$$

$$b_n(t) = e^{-n^2} b_n(0) + \int_0^t \frac{\exp\left(n^2 s\right)}{\exp\left(n^2 t\right)} q_n(s)\ ds$$

$$b_n(t) = e^{-n^2} b_n(0) + \int_0^t e^{n^2(s-t)} q_n(s)\ ds$$

$$u(0,x) = \sin x = b_0(0) + \sum_{n=1}^{\infty} b_n(0) \sin(nx)$$

so using our equations to find the coefficients of a Fourier sine series,

$$b_0(0) = \frac{2}{\pi}\int_0^{\pi} \sin(nx) dx$$

$$= \frac{-2}{n\pi}\cos(nx)\Big|_0^{\pi}\ dx = \frac{2}{n\pi}(1+(-1)^{n+1})$$

$$b_n(0) = \frac{1}{\pi}\int_0^{\pi} \sin(x)\sin(nx) dx$$

We evaluate this integral using the following trigonometric identity:

$$\sin\alpha\sin\beta = \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{2} \quad \implies \quad \sin(x)\sin(nx) = \frac{\cos(-x) - \cos(3x)}{2}$$

$$b_n(0) = \frac{1}{2\pi}\int_0^{\pi} \cos(x) - \cos(3x) dx = 0$$

Because $\sin 0, \sin \pi$, and $\sin 3\pi$ all equal zero.

Therefore, our solution to (5) is:

Solution

$$u(t,x) = b_0(t) + \sum_{n=1}^{\infty} b_n(t)\sin(nx), \quad \text{where}$$

$$b_n(t) = \int_0^t e^{n^2(s-t)} q_n\ ds,$$

$$q_n = \frac{2A}{\pi}\frac{n(-1)^{n+1}e^{-a\pi} + n}{n^2 + a^2},$$

$$b_0(t) = \frac{-At}{a\pi}\left(e^{-a\pi} - 1\right) + b_0(0),$$

$$b_0(0) = \frac{2}{n\pi}(1+(-1)^{n+1})$$

Problem 6: Telegraph Equation

Using the method of separation of variables, solve the telegraph equation:

$$\begin{cases} u_{tt} + au_t + bu = c^2 u_{xx} & 0 < x < l, \quad t > 0 \\ u(0,x) = \phi(x) & 0 \le x \le l \\ u_t(0,x) = \psi(x) & 0 \le x \le l \\ u(t,0) = u(t,l) = 0 & t > 0 \end{cases} \tag{9}$$

for constants $a, b > 0$. Only find the solution when the characteristic equation of the time problem has real roots. Define the following energy:

$$E(t) = \frac{1}{2}\int_0^l (u_t^2 + c^2 u_x^2 + bu^2)\, dx$$

Show that $E(t) \le E(0)$ for all $t > 0$. Then prove that the telegraph equation has a unique solution.

Suppose we have a separated solution of the form $u(t,x) = X(x)T(t)$.

If we plug this into the homogeneous Dirichlet BCs, we find $u(t,0) = T(t)X(0) = 0, u(t,l) = T(t)X(l) = 0$. In order for both of these to be true, we must have either $T(t) = 0$ or $X(0) = X(l) = 0$.

But if $T(t) = 0$, then $u(t,x) = 0$ for all x, which contradicts our ICs. Thus, we must have $X(0) = X(l) = 0$.

Plugging our expression $u(t,x) = X(x)T(t)$ into the PDE in (9), it becomes

$$T''(t)X(x) + aT'(t)X(x) + bT(t)X(x) = c^2 T(t)X''(x)$$

$$-\frac{T''(t) + aT'(t) + bT(t)}{c^2 T(t)} = -\frac{X''(x)}{X(x)} = \lambda$$

Based on our assumptions thus far and the IBVP (9), we have three problems:

Spatial problem

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(l) = 0 \end{cases}$$

Time problem

$$T'' + aT' + (b + \lambda c^2)T = 0 \tag{10}$$

IVP

$$u(0,x) = T(0)X(x) = \phi(x)$$

The spatial problem is an eigenvalue problem with homogenous Dirichlet BCs, which we already know has the eigenvalues and eigenvectors

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \sin\left(\frac{n\pi x}{l}\right), \quad n = 1, 2, 3...$$

Now moving to the time problem, the equation (10) becomes the characteristic equation $r^2 + ar + (b + \lambda c^2) = 0$. From the Pythagorean Theorem, the roots of this equation are

$$\frac{-a \pm \sqrt{a^2 - 4(b + \lambda c^2)}}{2}$$

In order for the characteristic equation of the time problem to have only real roots, we must have $a^2 - 4(b + \lambda c^2) \geq 0$, or

$$\lambda \leq \frac{a^2 - 4b}{4c^2}$$

For each λ_n associated with the spatial problem, we get a solution to the time problem:

$$T_n(t) = C_n \exp\left[\frac{t}{2}(-a - a^2 - 4b - 4\lambda_n c^2)\right] + D_n \exp\left[\frac{t}{2}(-a + a^2 - 4b - 4\lambda_n c^2)\right]$$

where $C_n, D_n \in \mathbb{R}$

Therefore, the following are solutions of the PDE in (9): $u_n(t,x) = X_n(x)\, T_n(t), \quad n = 1, 2, 3...$

By the principle of superposition,

$$\begin{aligned}
u(t,x) &= \sum_{n=1}^{\infty} c'_n\, u_n(t,x) \\
&= \sum_{n=1}^{\infty} c'_n\, T_n(t)\, X_n(x) \\
&= \sum_{n=1}^{\infty} A_n\, \exp\left[\frac{t}{2}(-a - a^2 - 4b - 4\lambda_n c^2)\right]\, \sin\left(\frac{n\pi x}{l}\right) + \\
&\sum_{n=1}^{\infty} B_n\, \exp\left[\frac{t}{2}(-a + a^2 - 4b - 4\lambda_n c^2)\right]\, \sin\left(\frac{n\pi x}{l}\right)
\end{aligned}$$

where $A_n = c'_n C_n$ and $B_n = c'_n D_n$, with $A_n, B_n \in \mathbb{R}$.

We can get rid of these arbitrary constants by using our initial conditions, and . First, let us simplify the notation by defining $\gamma = \frac{t}{2}(-a - a^2 - 4b - 4\lambda_n c^2)$ and $\zeta = \frac{t}{2}(-a - a^2 - 4b - 4\lambda_n c^2)$.

Using our first initial condition, $u(0,x) = \phi(x)$:

$$u(t,x) = \sum_{n=1}^{\infty} A_n\, e^{\gamma t} \sin\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} B_n\, e^{\zeta t} \sin\left(\frac{n\pi x}{l}\right)$$

$$u(0,x) = \phi(x) = \sum_{n=1}^{\infty} (A_n + B_n) \sin\left(\frac{n\pi x}{l}\right)$$

We can use the equation for the coefficients inside a Fourier sine series to find $A_n + B_n$

$$A_n + B_n = \frac{2}{l} \int_0^l \phi(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

Now repeating for our second initial condition, $u_t(0,x) = \psi(x)$:

$$u_t(t,x) = \sum_{n=1}^{\infty} \gamma\, A_n\, e^{\gamma t}\, \sin\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} \zeta\, B_n\, e^{\zeta t}\, \sin\left(\frac{n\pi x}{l}\right)$$

$$u_t(0,x) = \psi(x) = \sum_{n=1}^{\infty} (\gamma A_n + \zeta B_n) \sin\left(\frac{n\pi x}{l}\right)$$

$$(\gamma A_n + \zeta B_n) = \frac{2}{l}\int_0^l \psi(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Which can be rearranged as

$$A_n = \frac{1}{\gamma}\left[-\zeta B_n + \frac{2}{l}\int_0^l \psi(x)\sin\left(\frac{n\pi x}{l}\right)dx\right]$$

Now we can solve for B_n:

$$\frac{-\zeta B_n}{\gamma} + \frac{2}{\gamma l}\int_0^l \psi(x)\sin\left(\frac{n\pi x}{l}\right)dx = -B_n + \frac{2}{l}\int_0^l \phi(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$B_n - \frac{\zeta B_n}{\gamma} = \left(1 - \frac{\zeta}{\gamma}\right)B_n = \frac{2}{l}\int_0^l \phi(x)\sin\left(\frac{n\pi x}{l}\right)dx - \frac{2}{\gamma l}\int_0^l \psi(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Therefore, our solution is:

Solution

$$u(t,x) = \sum_{n=1}^{\infty} A_n\, e^{\gamma t}\sin\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} B_n\, e^{\zeta t}\sin\left(\frac{n\pi x}{l}\right), \quad \text{where}$$

$$A_n = \frac{2}{l}\int_0^l \phi(x)\sin\left(\frac{n\pi x}{l}\right)dx - B_n$$

$$B_n = \frac{2\gamma}{(\gamma-\zeta)l}\int_0^l \phi(x)\sin\left(\frac{n\pi x}{l}\right)dx - \frac{2(\gamma-\zeta)}{l}\int_0^l \psi(x)\sin\left(\frac{n\pi x}{l}\right)dx,$$

$$\gamma = \frac{t}{2}(-a - a^2 - 4b - 4\lambda_n c^2)$$

$$\zeta = \frac{t}{2}(-a - a^2 - 4b - 4\lambda_n c^2)$$

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad n = 1, 2, 3...$$