MATH 245 Homework 3

Ruby Krasnow and Tommy Thach

2024-02-23

##Done: 1 ##Not started: 2-6

Question 1 (Parity of Solution)

(a) Prove that if initial conditions of the wave equation on the whole line are even(odd), the solution is even(odd).

Consider the IVP

$$\begin{cases} u_{tt} = c^2 u_{xx} & -\infty < x < \infty \\ u(0,x) = \varphi(x) & -\infty < x < \infty \\ u_t(0,x) = \psi(x) & -\infty < x < \infty \end{cases}$$

where $\varphi(x)$ and $\psi(x)$ are even functions, i.e., $\varphi(-x) = \varphi(x)$ and $\psi(-x) = \psi(x)$. Then using D'Alembert's formula, we know the solution u is

$$u(x,t) = \frac{\varphi(x+ct) + \varphi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds$$

Now plug in $-x$:

$$\begin{aligned} u(-x,t) &= \frac{\varphi(-x+ct) + \varphi(-x-ct)}{2} + \frac{1}{2c}\int_{-x-ct}^{-x+ct} \psi(s)ds \\ &= \frac{\varphi(x-ct) + \varphi(x+ct)}{2} + \frac{1}{2c}\int_{-x-ct}^{-x+ct} \psi(-s)ds \end{aligned}$$

After setting $y = -s, dy = -ds$ in the integral of ψ:

$$u(-x,t) = \frac{\varphi(x-ct) + \varphi(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy = u(x,t).$$

Therefore, even initial conditions imply that the solution of the wave equation is even.

Similarly, assume $\varphi(x)$ and $\psi(x)$ are odd functions, i.e., $\varphi(-x) = -\varphi(x)$ and $\psi(-x) = -\psi(x)$. Then using D'Alembert's formula, we know $u(-x,t)$ is

$$\begin{aligned} u(-x,t) &= \frac{\varphi(-x+ct) + \varphi(-x-ct)}{2} + \frac{1}{2c}\int_{-x-ct}^{-x+ct} \psi(s)ds \\ &= \frac{-\varphi(x-ct) - \varphi(x+ct)}{2} - \frac{1}{2c}\int_{-x-ct}^{-x+ct} \psi(-s)ds \end{aligned}$$

After setting $y = -s, dy = -ds$ in the integral of ψ:

$$u(-x,t) = \frac{-\varphi(x-ct) - \varphi(x+ct)}{2} - \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(-y)dy = -u(x,t).$$

Therefore, odd initial conditions imply that the solution of the wave equation is odd.

(b) Prove that if the initial condition of the heat equation on the whole line is even(odd), the solution is even(odd).

Consider the IVP

$$\begin{cases} u_t = ku_{xx} & -\infty < x < \infty \\ u(0,x) = f(x) \end{cases}$$

Then we know

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)\ dy$$

Which means, when $x = -x$:

$$u(-x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(-x-y)^2/4kt} f(y)\ dy$$

Now use the change of variable $y = -s, dy = -ds$:

$$u(-x,t) = \frac{-1}{\sqrt{4\pi kt}}\int_{\infty}^{-\infty} e^{-(-x+s)^2/4kt} f(-s)\ ds$$

$$= \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-((-1)(x-s))^2/4kt} f(-s)\ ds$$

But $e^{-((-1)(x-s))^2/4kt}$ is equivalent to $e^{-(x-s)^2/4kt}$, so we have

$$u(-x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(x-s)^2/4kt} f(-s)\ ds$$

Now assume f is odd. By definition we have $f(-s) = -f(s)$, so

$$u(-x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(x-s)^2/4kt} f(-s)\ ds = \frac{-1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(x-s)^2/4kt} f(s)\ ds = -u(x,t)$$

And therefore our solution is odd when our initial $f(x)$ is an odd function. Similarly, if we assume $f(x)$ is even, by definition we have $f(-s) = f(s)$, so

$$u(-x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(x-s)^2/4kt} f(-s)\ ds = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(x-s)^2/4kt} f(s)\ ds = u(x,t),$$

meaning our solution is also even.

Question 2 (Speed of Heat vs Wave)

Consider the traveling wave $u(x,t) = f(x-at)$ where f is a given function of one variable.

(a) If it is a solution of the wave equation, show that the speed must be $a = \pm c$ (unless f is a linear function).

(b) If it is a solution of the diffusion equation, find f and show that the speed a is arbitrary.

Question 3 (Heat Equation with Robin boundary)

Consider the following problem with a Robin boundary condition:

$$\begin{cases} u_t = ku_{xx} & 0 < x < \infty, 0 < t < \infty \\ u(0, x) = x & t = 0, 0 < x < \infty \\ u_x(0, t) - 2u(0, t) = 0 & x = 0 \end{cases}$$

Question 4 (Maximum Principle)

Consider two solutions $u(t, x)$ and $v(t, x)$ of the diffusion equation in $\{0 \leq x \leq l, 0 \leq t \leq \infty\}$

(a) Let $M(T) =$ the maximum of $u(t, x)$ in the closed rectangle $\Omega_T = \{0 \leq x \leq l, 0 \leq t \leq T\}$. Does $M(T)$ increase or decrease as a function of T?

The maximum principle says that if $u(x, t)$ satisfies the diffusion equation in a rectangle $\{0 \leq x \leq x, 0 \leq t \leq T\}$, then the maximum value of $u(x, t)$ is assumed either initially ($t = 0$) or on the lateral sides ($x = 0$ or $x = l$).

By the definition of maximum, if u assumes a higher value on the lateral sides $x = 0$ or $x = l$ when T is extended from some T_1 to a new T, $T_2 > T_1$, then the maximum of u will increase to that new value; otherwise, the maximum of u will remain the same. In other words, M is non-decreasing (increases, but not strictly) as a function of T.

(b) Let $m(T) =$ the minimum of $u(t, x)$ in the closed rectangle $\Omega_T = \{0 \leq x \leq l, 0 \leq t \leq T\}$. Does $m(T)$ increase or decrease as a function of T?

By the definition of minimum, if u assumes a lower value on the lateral sides $x = 0$ or $x = l$ when T is extended from some T_1 to a new T, $T_2 > T_1$, then the minimum of u will decrease to that new value; otherwise, the minimum of u will remain the same. In other words, M is non-increasing (decreases, but not strictly) as a function of T.

(c) Comparison Principle
If $u \leq v$ for $t = 0, x = 0$, and $x = l$, then $u \leq v$ for $0 \leq t \leq \infty$ and $0 \leq x \leq l$.

Question 5 (Diffusion Equation with Dissipation).

Solve the following IVP for constant dissipation $b > 0$.

$$\begin{cases} u_t - ku_{xx} + bu = 0 & -\infty \leq x \leq \infty,\ t > 0 \\ u(0, x) = \psi(x), & -\infty \leq x \leq \infty \end{cases} \tag{1}$$

We will use the change of variables $u(t, x) = e^{-bt}v(t, x)$

Question 6

Suppose that there is a maximum principle for the wave equation and that $u(x,t)$ is a solution to the wave equation in the closed rectangle $\Omega_T = \{0 \le x \le \pi,\ 0 \le t \le \pi/c\}$. Then the maximum of u is assumed either initially (at $t = 0$) or on the lines $x = 0$ or $x = \pi$.

Consider the particular solution $u(x,t) = \sin x \sin ct$. This clearly satisfies the wave equation:

$$u_x = \cos x \sin ct$$

$$u_{xx} = -\sin x \sin ct$$
$$u_t = \sin x (c \cos ct)$$
$$u_{tt} = -\sin x (c^2 \sin ct) = c^2 u_{xx}$$

According to our presumed maximum principle, the maximum of $u(x,t)$ would have to be one of the following:

$$u(x,0) = \sin x \sin 0 = 0$$

$$u(0,t) = \sin 0 \sin ct = 0$$
$$u(\pi,t) = \sin \pi \sin ct = 0$$

But $u(x,t)$ has a higher value when $x = \pi/2$ and $t = \pi/2c$:

$$u\left(\frac{\pi}{2}, \frac{\pi}{2c}\right) = \sin\left(\frac{\pi}{2}\right) \sin\left(\frac{\pi}{2}\right) = 1$$

This contradicts the definition of maximum, so we have found a counterexample and therefore there is no maximum principle for the wave equation.