MATH 245 Homework 1

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Question 1

Show the function is a solution of the PDE:
(a) $u_{xx} + u_{yy} = 0$
(i) $u(x, y) = e^x \sin(y)$

$$u_x = e^x \sin(y),\ u_{xx} = e^x \sin(y),\ u_y = e^x \cos(y),\ u_{yy} = -e^x \sin(y)$$

Which means that $u_{xx} + u_{yy} = e^x \sin(y) + (-e^x \sin(y)) = 0$ and so $u(x, y) = e^x \sin(y)$ is a solution to the PDE.
(ii) $u(x, y) = \log \sqrt{x^2 + y^2}$
Assuming this is meant to be $u(x, y) = \ln \sqrt{x^2 + y^2}$,

$$u_x = \left(x^2 + y^2\right)^{-1/2} \left(\frac{1}{2}\right) \left(x^2 + y^2\right)^{-1/2} (2x) = x \left(x^2 + y^2\right)^{-1}$$

$$u_{xx} = x(-1) \left(x^2 + y^2\right)^{-2} (2x) + \left(x^2 + y^2\right)^{-1}$$

$$u_y = \left(x^2 + y^2\right)^{-1/2} \left(\frac{1}{2}\right) \left(x^2 + y^2\right)^{-1/2} (2y) = y \left(x^2 + y^2\right)^{-1}$$

$$u_{yy} = y(-1) \left(x^2 + y^2\right)^{-2} (2y) + \left(x^2 + y^2\right)^{-1}$$

$$u_{xx} + u_{yy} = \frac{-2x^2}{\left(x^2 + y^2\right)^2} + \frac{-2y^2}{\left(x^2 + y^2\right)^2} + \frac{2}{\left(x^2 + y^2\right)} =$$

$$\frac{-2\left(x^2 + y^2\right)}{\left(x^2 + y^2\right)^2} + \frac{2}{\left(x^2 + y^2\right)} = 0$$

Which means that $u(x, y) = \log \sqrt{x^2 + y^2}$ is a solution to the PDE.

(b) $bu_x + au_y + u = 0$, $u(x, y) = \exp\left(\frac{-x}{b}\right) f(ax - by)$ for arbitrary differentiable function f.

$$u_x = e^{\frac{-x}{b}} \left(\frac{-1}{b}\right) f(ax - by) + f'(ax - by)(a) e^{\frac{-x}{b}}$$

$$u_y = e^{\frac{-x}{b}} f'(ax - by)(-b)$$

Then,

$$bu_x + au_y + u = -e^{\frac{-x}{b}} f(ax - by) + (ab)\left(e^{\frac{-x}{b}}\right) f'(ax - by) + e^{\frac{-x}{b}} f'(ax - by)(-ab) + e^{\frac{-x}{b}} f(ax - by) = 0$$

So $u(x,y) = \exp\left(\frac{-x}{b}\right) f(ax - by)$ is a solution to the PDE for an arbitrary differentiable function f.

(c) $u_{xx} - \frac{1}{x}u_x - x^2 u_{yy} = 0$, $u(x,y) = f\left(2y + x^2\right) + g\left(2y - x^2\right)$ for arbitrary twice-differentiable functions f and g.

$$u_x = \left(f'\left(2y + x^2\right)\right)(2x) + \left(g'\left(2y - x^2\right)\right)(-2x)$$
$$u_y = 2\left(f'\left(2y + x^2\right)\right) + 2\left(g'\left(2y - x^2\right)\right)$$
$$u_{xx} = 2f' + (f'')(4x^2) - 2g' + (g'')(4x^2)$$
$$u_{yy} = 4f'' + 4g''$$
$$u_{xx} - \frac{1}{x}u_x - x^2 u_{yy} =$$
$$2f' + 4x^2 f'' - 2g' + 4x^2 g'' - \frac{1}{x}\left(2xf' - 2xg'\right) - x^2\left(4f'' + 4g''\right) =$$
$$2f' - 2f' + 4x^2 f'' - 4x^2 f'' - 2g' + 2g' + 4x^2 g'' - 4x^2 g'' = 0$$

So $u(x,y) = f\left(2y + x^2\right) + g\left(2y - x^2\right)$ is a solution to the PDE for arbitrary twice-differentiable functions f and g.

Question 2

(a) 2nd-order linear homogeneous

(b) 4th-order linear inhomogeneous

(c) 2nd-order quasi-linear homogeneous

(d) We can rewrite this as $u_{xx} + u_{yy} + f(x,y)u - g(x,y)u^5 = 0$, making it clear that this is a 2nd-order semi-linear homogeneous PDE.

Question 3

Use separation of variables to solve the following problems:

(a) $u_x + u = u_y$, $u(x,0) = 4x^{-3x}$, use $u(x,y) = f(x)g(y)$

$$u_x = f'g,\ u_y = fg'$$
$$u_x + u - u_y = f'g + fg - fg' = f'g + (g - g')f = 0$$
$$f'g = -(g - g')f$$
$$\frac{f'}{f} = \frac{g' - g}{g} = \lambda$$
$$\frac{f'}{f} = \lambda,\ \frac{g'}{g} - 1 = \lambda$$

Integrating the first ODE with respect to x,

$$\ln(f) = \lambda x + C_1,\ f(x) = C_2 e^{\lambda x}$$

$$\frac{g'}{g} = \lambda + 1$$
$$\ln(g) = \lambda y + y + C_3,\ g(y) = C_4 e^{y(\lambda+1)}$$
$$u(x,y) = f(x)g(y) = C_2 e^{\lambda x} C_4 e^{y(\lambda+1)}$$
$$u(x,y) = Ce^{\lambda x + y(\lambda+1)}$$
$$u(x,0) = 4e^{-3x} = Ce^{\lambda x}$$

Which means $C = 4, \lambda = -3$ and our final solution is

Solution

$$u(x,y) = 4e^{-3x-2y}$$

Check solution

$$u_x + u = -12e^{-3x-2y} + 4e^{-3x-2y} = -8e^{-3x-2y} = u_y$$

(b) $x^2 u_{xy} + 9y^2 u = 0,\ u(x,0) = \exp\left(\frac{1}{x}\right)$, use $u(x,y) = f(x)g(y)$

$$u_{xy} = f'g'$$
$$x^2 f'g' + 9y^2 fg = 0$$

$$x^2 f'g' = -9y^2 fg$$
$$x^2 \frac{f'}{f} = -9y^2 \frac{g}{g'} = \lambda$$
$$x^2 \frac{f'}{f} = \lambda,\ -9y^2 \frac{g}{g'} = \lambda$$

$$x^2 \frac{f'}{f} = \lambda \quad \longrightarrow \quad \frac{f'}{f} = \lambda x^{-2} \quad \longrightarrow \quad \ln(f) = \frac{-\lambda}{x} + C_1 \quad \longrightarrow \quad f(x) = C_2 e^{\frac{-\lambda}{x}}$$

$$-9y^2 \frac{g}{g'} = \lambda \quad \longrightarrow \quad \frac{g}{g'} = \frac{\lambda}{-9y^2} \quad \longrightarrow \quad \frac{g'}{g} = \frac{-9y^2}{\lambda} \quad \longrightarrow \quad \ln(g) = \frac{-3y^3}{\lambda} + C_3 \quad \longrightarrow \quad g(y) = C_4 e^{\frac{-3}{\lambda} y^3}$$

$$u(x,y) = f(x)g(y) = C_2 e^{\frac{-\lambda}{x}} C_4 e^{\frac{-3}{\lambda} y^3} = Ce^{\frac{-\lambda}{x} + \frac{-3}{\lambda} y^3}$$
$$u(x,0) = e^{\frac{1}{x}} = Ce^{\frac{-\lambda}{x}}$$

Which means $C = 1, \lambda = -1$, and our final solution is

Solution

$$u(x,y) = e^{\frac{1}{x} + 3y^3}$$

Check solution

$$x^2 u_{xy} + 9y^2 u = x^2 e^{\frac{1}{x} + 3y^3} \left(-x^{-2}\right)\left(9y^2\right) + 9y^2 e^{\frac{1}{x} + y^3} = 0$$

(c) $u_x^2 + u_y^2 = 1$, use $u(x,y) = f(x) + g(y)$

$$u_x = f', u_y = g'$$

$$(f')^2 + (g')^2 = 1 \quad \longrightarrow \quad (f')^2 = 1 - (g')^2 = \lambda^2$$

$$(f')^2 = \lambda^2 \quad \longrightarrow \quad f' = \lambda \quad \longrightarrow \quad f(x) = \lambda x + C_1$$

$$(g')^2 = 1 - \lambda^2 \quad \longrightarrow \quad g' = \pm\sqrt{1-\lambda^2} \quad \longrightarrow \quad g(y) = \pm y\sqrt{1-\lambda^2} + C_2$$

Solution

$$u(x,y) = f(x) + g(y) = \lambda x \pm y\sqrt{1-\lambda^2} + C$$

Check solution

$$u_x = \lambda, u_y = \pm\sqrt{1-\lambda^2}, \quad u_x^2 + u_y^2 = \lambda^2 + 1 - \lambda^2 = 1$$

(d) $x^2u_x^2 + y^2u_y^2 = u^2$, use $u(x,y) = e^{f(x)}e^{g(y)}$

$$u_x = f'e^{f+g}, u_y = g'e^{f+g}$$

$$x^2u_x^2 + y^2u_y^2 - u^2 = x^2(f')^2e^{2f+2g} + y^2(g')^2e^{2f+2g} - e^{2f+2g} = 0$$

$$x^2(f')^2 + y^2(g')^2 - 1 = 0 \quad \longrightarrow \quad x^2(f')^2 = 1 - y^2(g')^2 = \lambda^2$$

$$x^2(f')^2 = \lambda^2 \quad \longrightarrow \quad f' = \frac{\lambda}{x} \quad \longrightarrow \quad f(x) = \lambda\ln(x) + C_1$$

$$y^2(g')^2 = 1 - \lambda^2 \quad \longrightarrow \quad g' = \frac{\sqrt{1-\lambda^2}}{y} \quad \longrightarrow \quad g(y) = \sqrt{1-\lambda^2}\ln(y) + C_2$$

$$u(x,y) = e^{f(x)}e^{g(y)} = C_3e^{\lambda\ln(x)}C_4e^{\sqrt{1-\lambda^2}\ln(y)}$$

Solution

$$u(x,y) = Cx^\lambda y^{\sqrt{1-\lambda^2}}$$

Check solution

$$u_x = \lambda Cx^{\lambda-1}y^{\sqrt{1-\lambda^2}}, u_y = \sqrt{1-\lambda^2}Cx^\lambda y^{\sqrt{1-\lambda^2}-1}$$

$$x^2u_x^2 + y^2u_y^2 = x^2\left(\lambda Cx^{\lambda-1}y^{\sqrt{1-\lambda^2}}\right)^2 + y^2\left(\sqrt{1-\lambda^2}Cx^\lambda y^{\sqrt{1-\lambda^2}-1}\right)^2 =$$

$$x^2\lambda^2C^2x^{2\lambda-2}y^{2\sqrt{1-\lambda^2}} + y^2(1-\lambda^2)C^2x^{2\lambda}y^{2\sqrt{1-\lambda^2}-2} =$$

$$\lambda^2C^2x^{2\lambda}y^{2\sqrt{1-\lambda^2}} + (1-\lambda^2)C^2x^{2\lambda}y^{2\sqrt{1-\lambda^2}} = C^2x^{2\lambda}y^{2\sqrt{1-\lambda^2}} = u^2$$

Question 4

For each of the following IVPs, (i) find and plot the characteristic lines (curves), (ii) solve the IVP, and (iii) plot the solution of (a)-(b) for indicated time.

(a) $u_t + \left(1+x^2\right) u_x = 0$, $u(0,x) = \arctan(x)$, $t = 1,2,3$, and what is $\lim_{t\to\infty} u(t,x)$?

$$\frac{dx}{dt} = 1 + x^2$$
$$\frac{1}{1+x^2} dx = dt$$

$$\arctan(x) = t + C$$

So $x = \tan(t+C)$ are the characteristic curves of $u_t + \left(1+x^2\right) u_x = 0$.
On each of the curves, $u(x,t)$ is constant because

$$\frac{d}{dt} u(t, \tan(t+C)) = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \sec^2(t+c)$$

And since $1 + \tan^2\theta = \sec^2\theta$, this means $\frac{d}{dt} u(t, \tan(t+C)) = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\left(1 + \tan^2(t+c)\right) = u_t + \left(1+x^2\right) u_x$, which we know is 0.

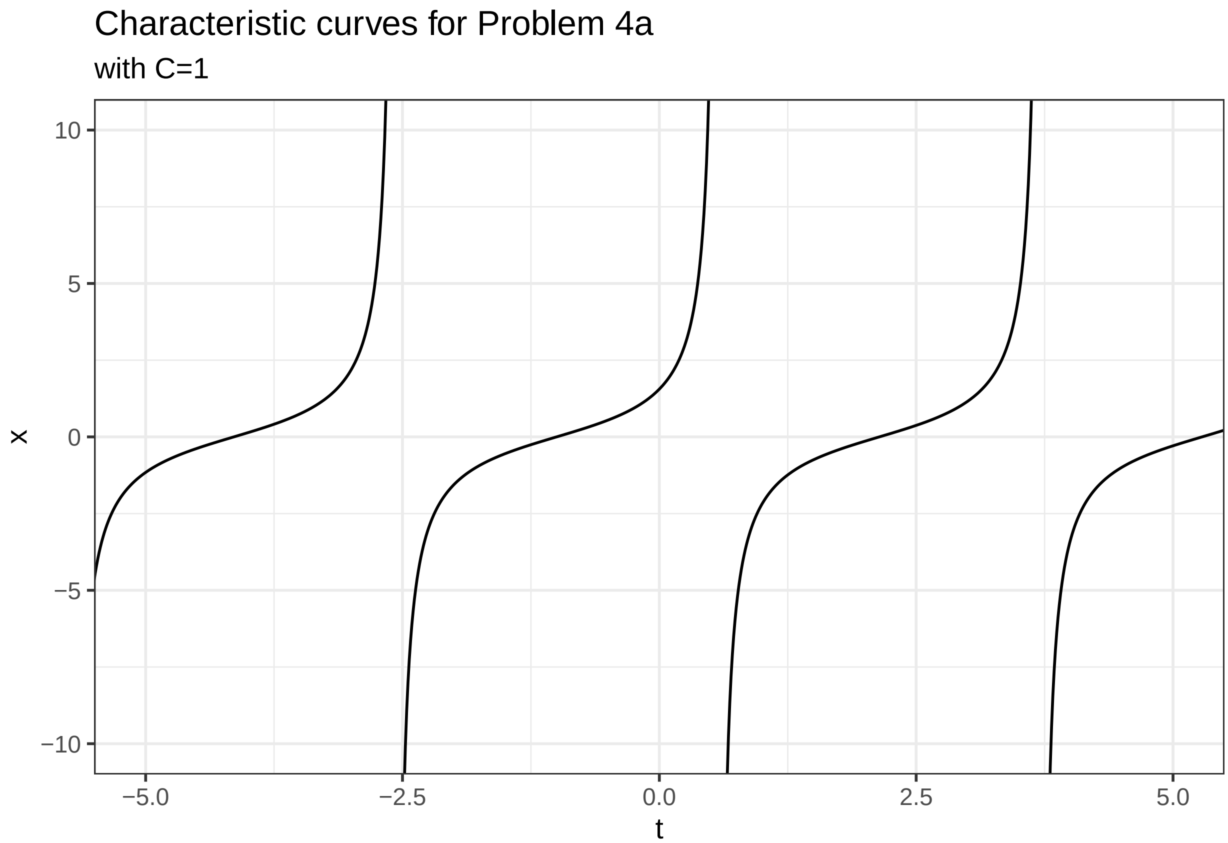
Thus $u(t, \tan(t+C)) = u(0, \tan(0+C)) = u(0,C)$ is independent of t.
Putting $x = \tan(t+C)$ and $C = \arctan(x) - t$, we have

$$u(t,x) = u(0, \arctan(x) - t)$$

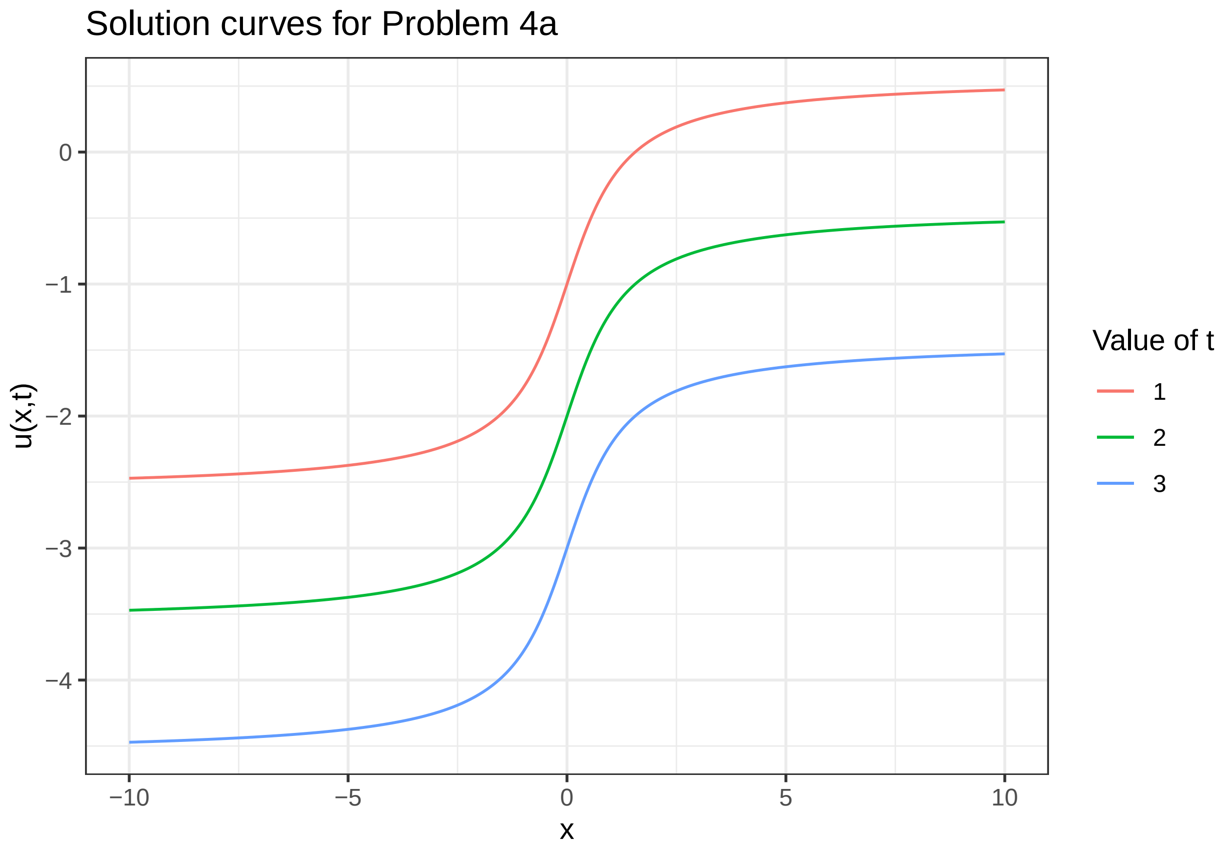
It follows that $u(t,x) = f(\arctan(x) - t)$.
And since we are given that $u(0,x) = \arctan(x)$, we have $f(\arctan(x) - 0) = \arctan(x)$ so that $f(w) = w$ for any w, yielding our solution of

Solution

$$u(t,x) = \arctan(x) - t$$



Since the range of arctan is limited to $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, as $\lim_{t\to\infty}$ that means $u(t,x) = \arctan(x) - t$ will go to $-\infty$.



Check solution

Since $u_t = -1, u_x = \frac{1}{1+x^2}$, that means $u_t + \left(1+x^2\right) u_x = -1+1 = 0$

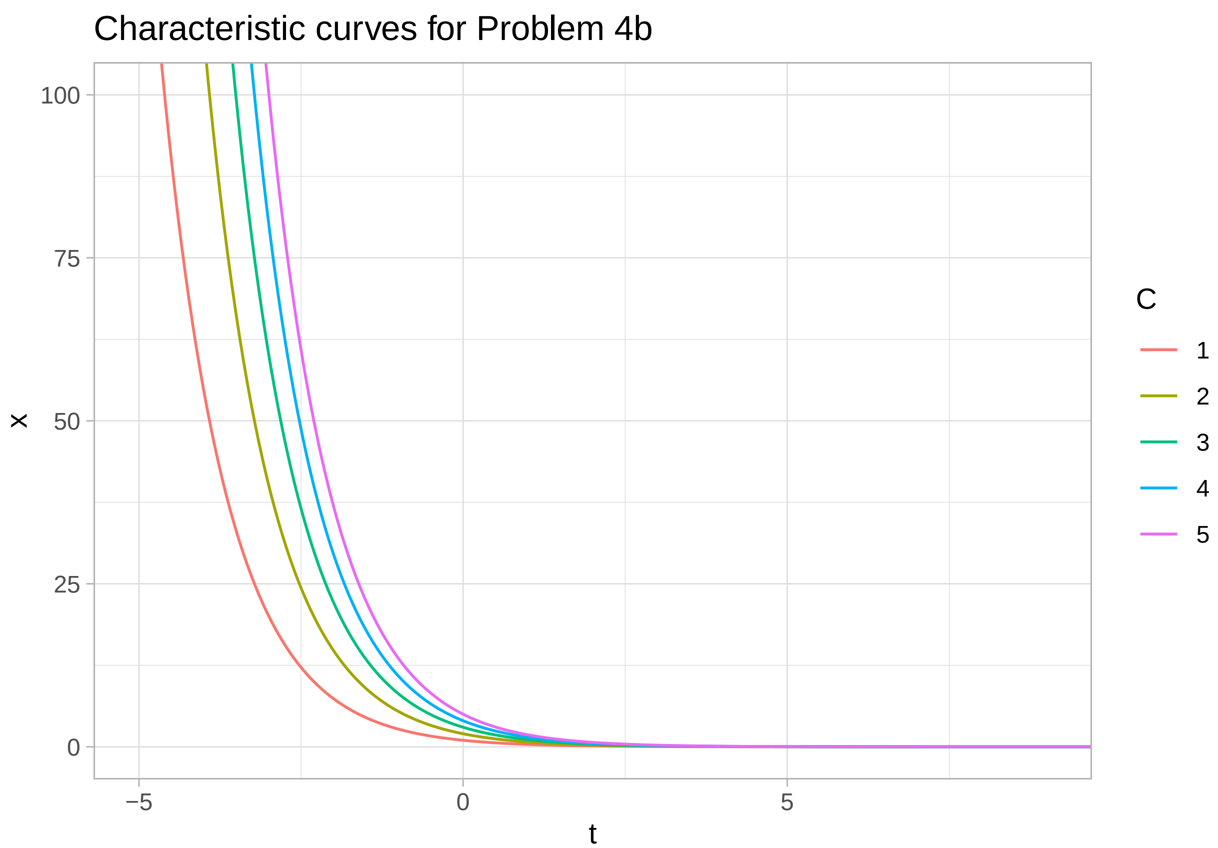
(b) $u_t - x u_x = 0$, $u(0,x) = \frac{1}{1+x^2}$, $t = 1,2,3$, and what is $\lim_{t\to\infty} u(t,x)$?

The directional derivative of u in the direction of the vector $(1,-x)$ is zero. The curves in the tx plane with $(1,-x)$ as a tangent vector have slopes $-x$:

$$\frac{dx}{dt} = -x$$
$$\frac{1}{x} dx = -dt$$

Solving this ODE gives the equations for the characteristic lines:

$$x = Ce^{-t}$$



On each of the curves, $u(x, t)$ is constant because

$$\frac{d}{dt}u(t, Ce^{-t}) = \frac{\partial u}{\partial t} - Ce^{-t}\frac{\partial u}{\partial x} = u_t - xu_x$$

and we know that $u_t - xu_x = 0$.

Set $\xi = xe^t$ and $\eta = x$. Then by the chain rule,

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} =$$

$$\frac{\partial u}{\partial \eta} + \frac{\partial u}{\partial \xi}e^t$$

and similarly

$$u_t = \frac{\partial u}{\partial t} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} = \frac{\partial u}{\partial \xi}xe^t$$

Which means that

$$u_t - xu_x = \frac{\partial u}{\partial \xi}xe^t - x\left(\frac{\partial u}{\partial \eta} + \frac{\partial u}{\partial \xi}e^t\right) = 0$$

Assuming $x \neq 0$, this means

$$u_\eta = 0$$

$$u = \int u_\eta \, d\eta \, + f(\xi) = 0 + f(\xi)$$

So we have $u(t,x) = f(\xi) = f(xe^t)$ and we know that $u(0,x) = \frac{1}{1+x^2}$, which means $f(xe^0) = f(x) = \frac{1}{1+x^2}$ and therefore

Solution

$$u(t,x) = \frac{1}{1+(xe^t)^2}$$

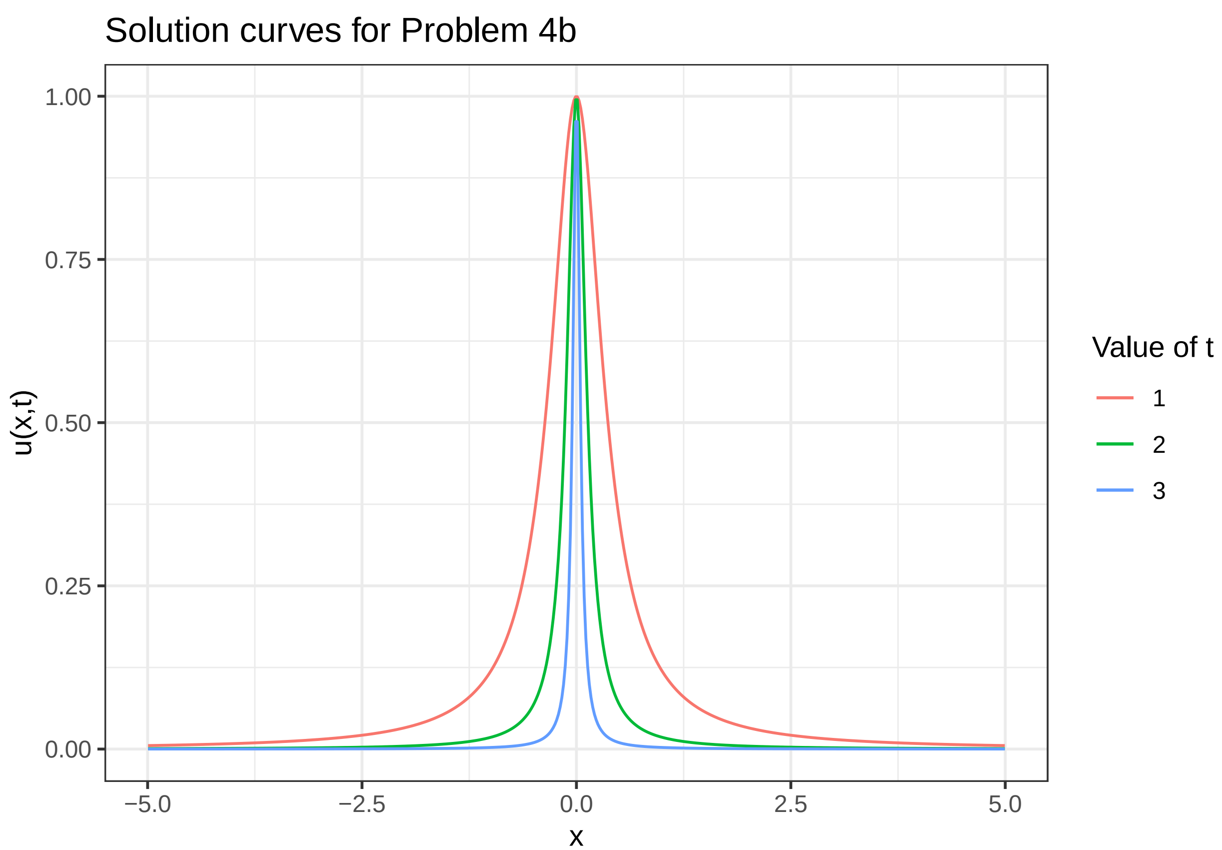
Check solution

$$u_t = -\left(1+\left(xe^t\right)^2\right)^{-2}\left(2xe^t\right)xe^t$$

$$u_x = -\left(1+\left(xe^t\right)^2\right)^{-2}\left(2xe^t\right)e^t$$

And we see that $u_t - xe_t$ indeed equals zero.

Plotting the solution $u(t,x) = \frac{1}{1+(xe^t)^2}$ as a function of x for $t = 1, 2, 3$, we get:

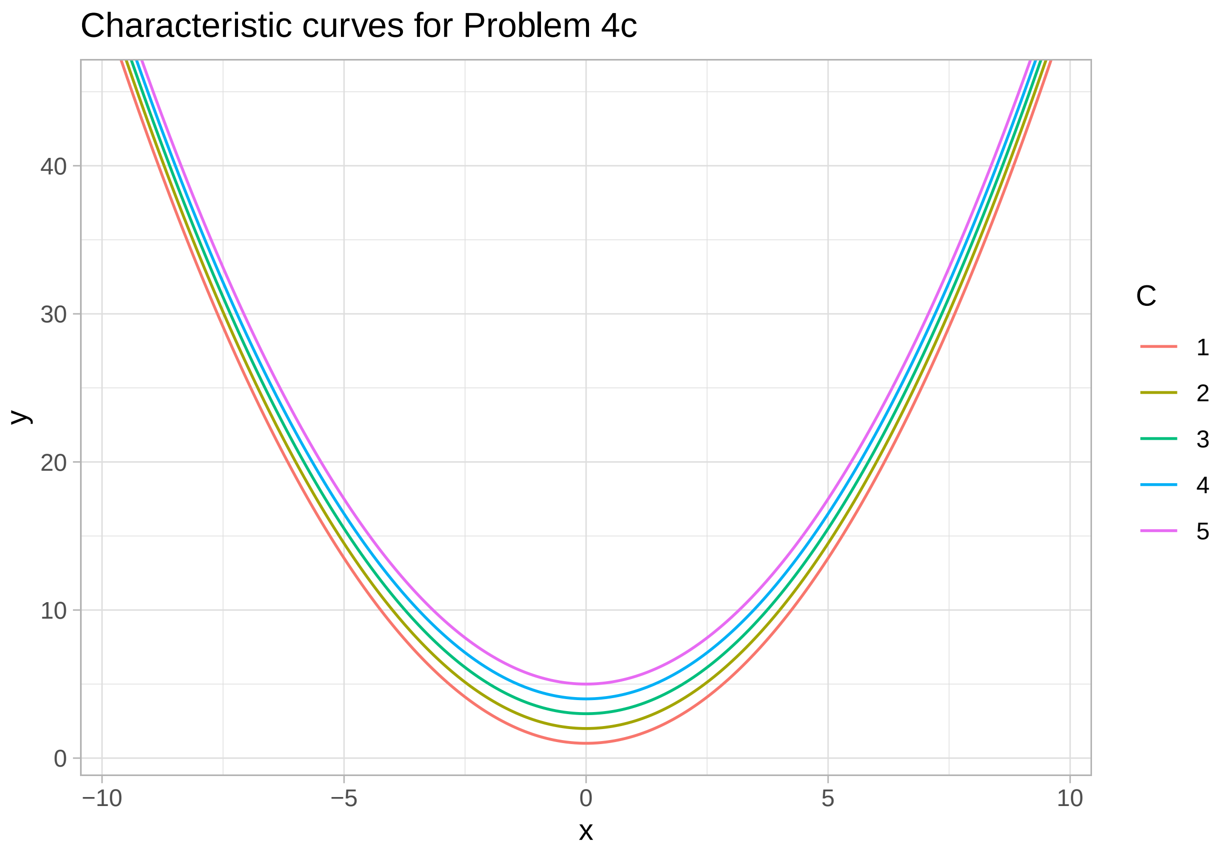


We can see that $u(t,x)$ will equal 1 at $x = 0$ regardless of the value of t, but at all other values of x, $\lim_{t\to\infty} u(t,x) = 0$. This can be written as

$$\lim_{t\to\infty} u(t,x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases} \tag{1}$$

(c) $u_x + xu_y = \left(y - \frac{x^2}{2}\right)^2$, $u(0, y) = e^y$

$\frac{dy}{dx} = x$ Solving this ODE yields the equation for the characteristic curves: $y = \frac{x^2}{2} + C$



Now we define our new coordinate system as $\xi = y - \frac{x^2}{2}$, $\eta = x$

Then by the chain rule,

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} =$$

$$\frac{\partial u}{\partial \eta} - x\frac{\partial u}{\partial \xi}$$

and similarly

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} =$$

$$\frac{\partial u}{\partial \xi}$$

$$u_x + xu_y = \frac{\partial u}{\partial \eta} - x\frac{\partial u}{\partial \xi} + x\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial \eta}$$

And since we know $u_x + xu_y = \left(y - \frac{x^2}{2}\right)^2$, this means we have

$$\frac{\partial u}{\partial \eta} = \left(y - \frac{x^2}{2}\right)^2 = \xi^2$$

Now we integrate with respect to η to find

$$u = \eta\xi^2 + F(\xi)$$

At $u(0,y)$, this becomes $u = 0y^2 + F(y)$, or $u(0,y) = F(y)$. On the other hand, we are told that $u(0,y) = e^y$, which means that $F(y) = e^y$.

$$u = \eta\xi^2 + e^\xi$$

Substitute back in x and y:

Solution

$$u(x,y) = x\left(y - \frac{x^2}{2}\right)^2 + \exp\left(y - \frac{x^2}{2}\right)$$

Check solution

$$u_x = -2x^2\left(y - \frac{x^2}{2}\right) + \left(y - \frac{x^2}{2}\right)^2 + \exp\left(y - \frac{x^2}{2}\right)(-x)$$

$$u_y = 2x\left(y - \frac{x^2}{2}\right) + \exp\left(y - \frac{x^2}{2}\right)$$

Which means that $u_x + xu_y = \left(y - \frac{x^2}{2}\right)^2$, as we wanted.

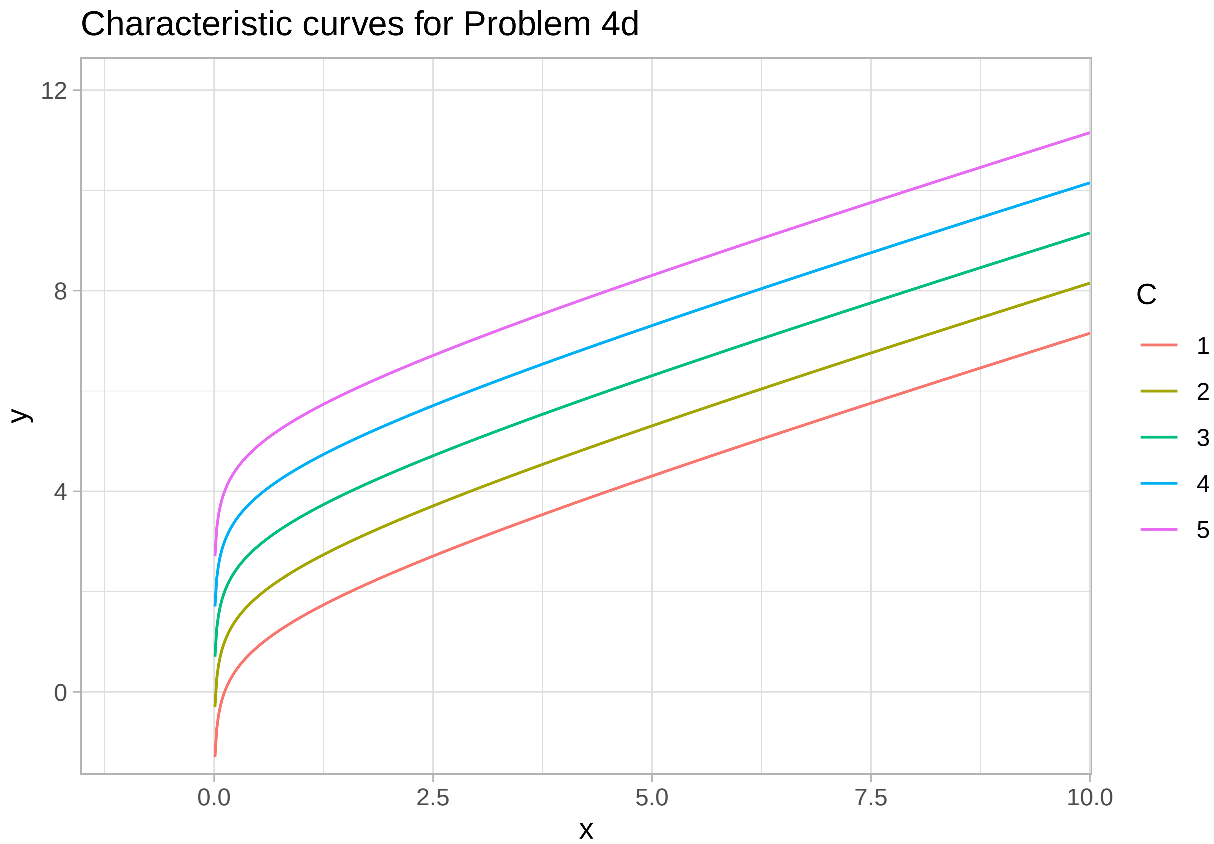
(d) $2xu_x + (x+1)u_y = y$, for $x > 0, u = 2y$ on $x = 1$

$$\frac{dx}{dy} = \frac{x+1}{2x} = \frac{1}{2} + \frac{1}{2x}$$

Solving this ODE yields the characteristic curves:

$$y = \frac{x}{2} + \frac{\ln(x)}{2} + C$$

$$C = 2y - x - \ln(x)$$


Characteristic curves for Problem 4d
12
8
4
0
y
0.0
2.5
5.0
7.5
10.0
x
C
1
2
3
4
5