BIOL 274 Homework 7

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Visualizing Solutions of Linear Systems with Constant Coefficients

For each of the following systems, state the general solution and sketch the phase portrait (with at least 4 different solutions / ICs shown):

Question 1

$$\dot{x} = x + 2y$$
$$\dot{y} = 3x + 2y$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

$$\det \begin{bmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)(2-\lambda) - 6 = 0$$

$$2 - \lambda - 2\lambda + \lambda^2 - 6 = 0$$

$$\lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1)$$

So we have found two distinct real eigenvalues, $\lambda_1 = 4,\ \lambda_2 = -1$. Now we need to solve $AX = \lambda X$, or $(A - \lambda I)X = 0$ with these two values of λ. Let's start with the first eigenvalue, 4.

$$A - 4I = \begin{bmatrix} 1-4 & 2 \\ 3 & 2-4 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} -3 & 2 \\ 0 & 0 \end{bmatrix}$$

$$-3x_1 + 2x_2 = 0 \quad \rightarrow \quad 2x_2 = 3x_1 \quad \rightarrow \quad v_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

And repeat for the second eigenvalue, -1:

$$A - (-I) = \begin{bmatrix} 1+1 & 2 \\ 3 & 2+1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$x_1 + x_2 = 0 \quad \rightarrow \quad x_2 = -x_1 \quad \rightarrow \quad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Solution

$$X = C_1 e^{4t} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

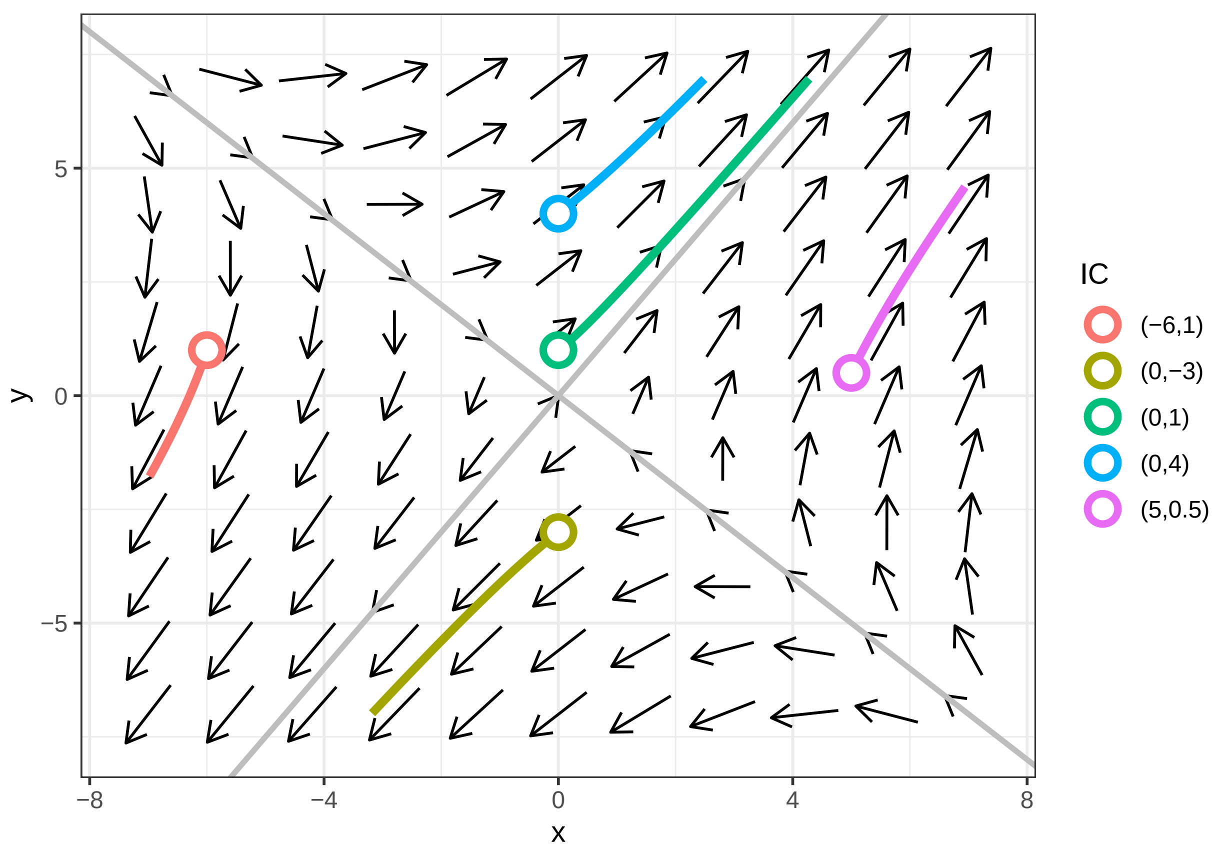
Check solution

$$X' = C_1 \begin{bmatrix} 8e^{4t} \\ 12e^{4t} \end{bmatrix} + C_2 \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix}$$

$$x' = 8C_1e^{4t} - C_2e^{-t} = 2C_1e^{4t} + C_2e^{-t} + 2(3C_1e^{4t} - C_2e^{-t}) = x + 2y$$

$$y' = 12C_1e^{4t} + C_2e^{-t} = 3(2C_1e^{4t} + C_2e^{-t}) + 2(3C_1e^{4t} - C_2e^{-t}) = 3x + 2y$$

One positive eigenvalue and one negative eigenvalue gives a saddle:



Question 2

$$\dot{x} = 4x + 2y$$
$$\dot{y} = -x + y$$

$$A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$$

$$\det \begin{bmatrix} 4-\lambda & 2 \\ -1 & 1-\lambda \end{bmatrix} = 0$$

$$(4-\lambda)(1-\lambda) + 2 = 0$$
$$\lambda^2 - 5\lambda + 6 = 0$$
$$(\lambda - 3)(\lambda - 2) = 0$$

So we have found two distinct real eigenvalues, $\lambda_1 = 3$, $\lambda_2 = 2$. Now we need to solve $AX = \lambda X$, or $(A - \lambda I)X = 0$ with these two values of λ. Let's start with the first eigenvalue, 3.

$$A - 3I = \begin{bmatrix} 4-3 & 2 \\ -1 & 1-3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$x_1 + 2x_2 = 0 \quad \rightarrow \quad 2x_2 = -x_1 \quad \rightarrow \quad v_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

And repeat for the second eigenvalue, 2:

$$A - 2I = \begin{bmatrix} 4-2 & 2 \\ -1 & 1-2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$x_1 + x_2 = 0 \quad \rightarrow \quad x_2 = -x_1 \quad \rightarrow \quad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Solution

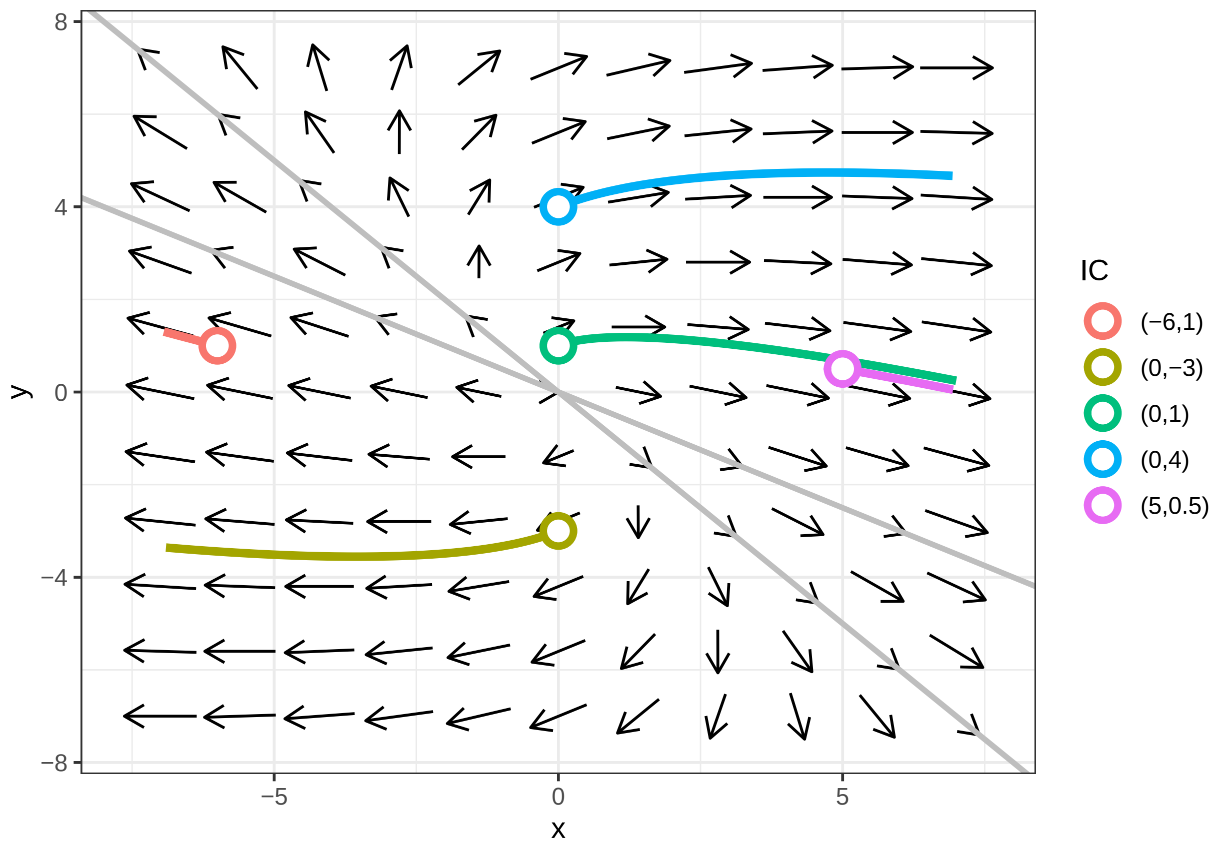
$$X = C_1 e^{3t} \begin{bmatrix} 2 \\ -1 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Check solution

$$x' = 6C_1 e^{3t} + 2C_2 e^{2t} = 4(2C_1 e^{3t} + C_2 e^{2t}) + 2(-C_1 e^{3t} - C_2 e^{2t}) = 4x + 2y$$

$$y' = -3C_1 e^{3t} - 2C_2 e^{2t} = -(2C_1 e^{3t} + C_2 e^{2t}) + (-C_1 e^{3t} - C_2 e^{2t}) = -x + y$$

Two positive eigenvalues gives a source, with solutions going towards the eigenvector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ with the eigenvalue $\lambda = 3$ of the larger magnitude.



Question 3

$$\dot{x} = 3x - 5y$$
$$\dot{y} = 5x + 3y$$

$$A = \begin{bmatrix} 3 & -5 \\ 5 & 3 \end{bmatrix}$$

$$\det \begin{bmatrix} 3-\lambda & -5 \\ 5 & 3-\lambda \end{bmatrix} = 0$$

$$(3-\lambda)(3-\lambda) + 25 = 0$$
$$\lambda^2 - 6\lambda + 34 = 0$$
$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \longrightarrow \quad \frac{6 \pm \sqrt{36 - 136}}{2} = \frac{6 \pm 10i}{2} = 3 \pm 5i$$

So we have complex eigenvalues, $\lambda_1 = 3+5i,\ \lambda_2 = 3-5i$. Now we need to solve $AX = \lambda X$, or $(A - \lambda I)X = 0$ with these two values of λ.

$$A - (3+5i)I = \begin{bmatrix} 3-3-5i & -5 \\ 5 & 3-3-5i \end{bmatrix} = \begin{bmatrix} -5i & -5 \\ 5 & -5i \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix}$$

$$ix_1 + x_2 = 0 \quad \to \quad ix_1 = -x_2 \quad \to \quad v_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Which means the eigenvector for the other eigenvalue is the conjugate of this vector, $v_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$.

Now we use Euler's formula:

$$e^{(3+5i)t} \begin{bmatrix} i \\ 1 \end{bmatrix} = e^{3t} e^{5it} \begin{bmatrix} i \\ 1 \end{bmatrix} = e^{3t} \left(\cos(5t) + i \sin(5t)\right) \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$e^{3t} \begin{bmatrix} i\cos(5t) - \sin(5t) \\ \cos(5t) + i\sin(5t) \end{bmatrix} = e^{3t} \left(\begin{bmatrix} -\sin(5t) \\ \cos(5t) \end{bmatrix} + i \begin{bmatrix} \cos(5t) \\ \sin(5t) \end{bmatrix} \right)$$

Having found our real and imaginary parts, we can now write our general solution as a linear combination of them:

Solution

$$X = C_1 e^{3t} \begin{bmatrix} -\sin 5t \\ \cos 5t \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} \cos 5t \\ \sin 5t \end{bmatrix}$$

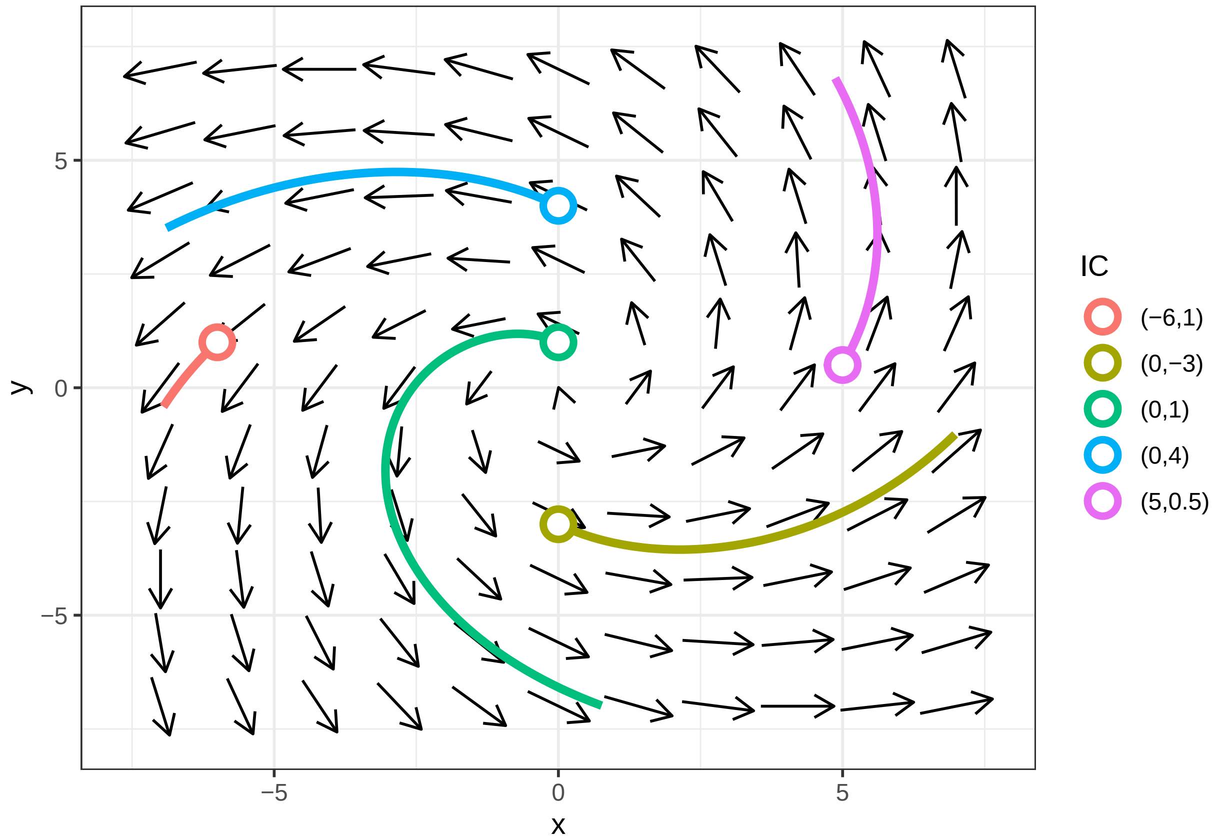
Check solution

$$x = -C_1 e^{3t} \sin 5t + C_2 e^{3t} \cos 5t$$
$$y = C_1 e^{3t} \cos 5t + C_2 e^{3t} \sin 5t$$

$$\begin{aligned} \dot{x} &= -5C_1 e^{3t} \cos 5t - 3C_1 e^{3t} \sin 5t - 5C_2 e^{3t} \sin 5t + 3C_2 e^{3t} \cos 5t \\ &= 3\left(-C_1 e^{3t} \sin 5t + C_2 e^{3t} \cos 5t\right) - 5\left(C_1 e^{3t} \cos 5t + C_2 e^{3t} \sin 5t\right) \\ &= 3x - 5y \end{aligned}$$

$$\begin{aligned} \dot{y} &= -5C_1 e^{3t} \sin 5t + 3C_1 e^{3t} \cos 5t + 5C_2 e^{3t} \cos 5t + 3C_2 e^{3t} \sin 5t \\ &= 5\left(-C_1 e^{3t} \sin 5t + C_2 e^{3t} \cos 5t\right) + 3\left(C_1 e^{3t} \cos 5t + C_2 e^{3t} \sin 5t\right) \\ &= 5x + 3y \end{aligned}$$

Complex eigenvalues with a positive real part gives a spiral source. Plugging in some test points (e.g., (1,1)) to our original matrix reveals that the spiral rotates counterclockwise.



Question 4

$$\dot{x} = 6x - 8y$$
$$\dot{y} = -3x + 4y$$

$$A = \begin{bmatrix} 6 & -8 \\ -3 & 4 \end{bmatrix}$$

$$\det \begin{bmatrix} 6-\lambda & -8 \\ -3 & 4-\lambda \end{bmatrix} = 0$$

$$(6-\lambda)(4-\lambda) - 24 = 0$$
$$\lambda^2 - 10\lambda = \lambda(\lambda - 10) = 0$$

So we have found two distinct real eigenvalues, $\lambda_1 = 0$, $\lambda_2 = 10$. Now we need to solve $AX = \lambda X$, or $(A - \lambda I)X = 0$ with these two values of λ. Let's start with the first eigenvalue, 0.

$$A - 0I = \begin{bmatrix} 6 & -8 \\ -3 & 4 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 3 & -4 \\ 0 & 0 \end{bmatrix}$$

$$3x_1 - 4x_2 = 0 \quad \rightarrow \quad 3x_1 = 4x_2 \quad \rightarrow \quad v_1 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

And repeat for the second eigenvalue, 10:

$$A - 10I = \begin{bmatrix} 6-10 & -8 \\ -3 & 4-10 \end{bmatrix} = \begin{bmatrix} -4 & -8 \\ -3 & -6 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$x_1 + 2x_2 = 0 \quad \rightarrow \quad x_1 = -2x_2 \quad \rightarrow \quad v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

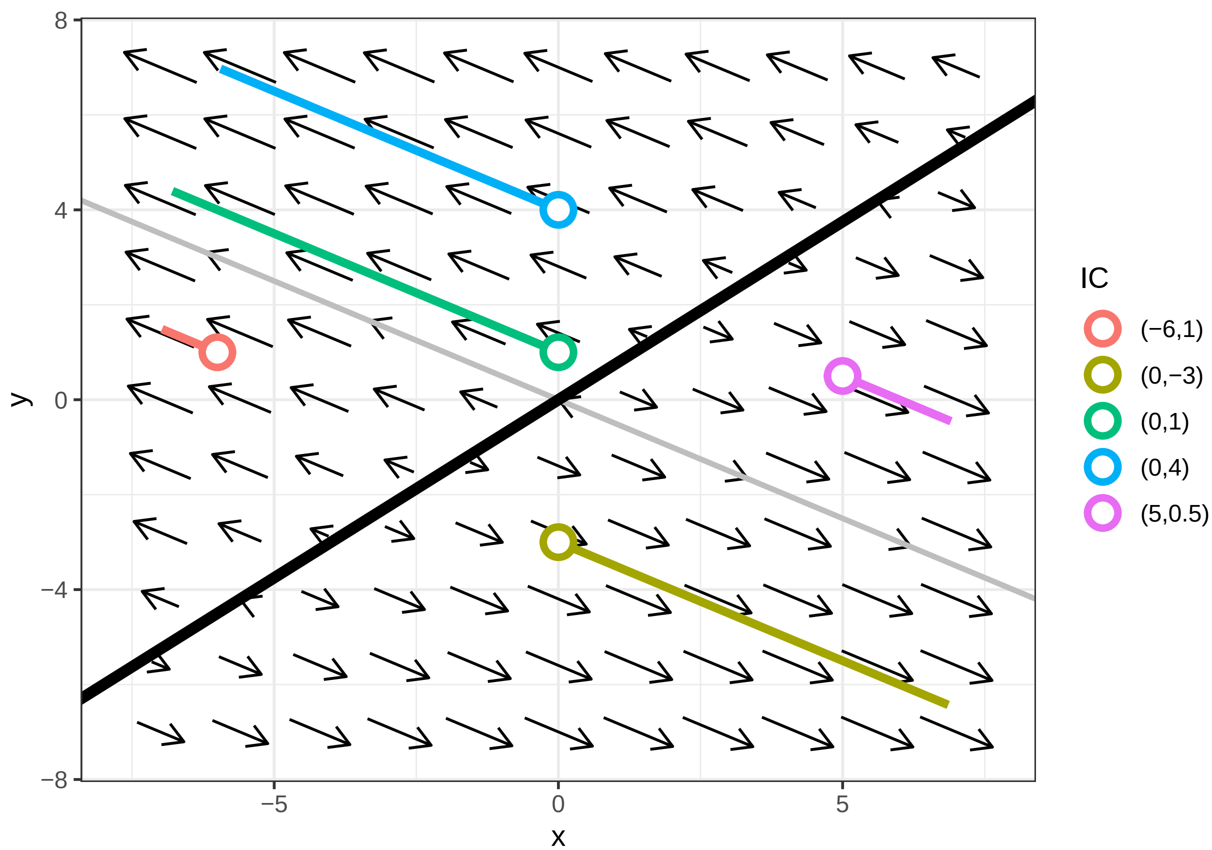
Solution

$$X = C_1 \begin{bmatrix} 4 \\ 3 \end{bmatrix} + C_2 e^{10t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Check solution

$$x' = -20C_2e^{10t} = 6(4C_1 - 2C_2e^{10t)} - 8(3C_1 + C_2e^{10t}) = 6x - 8y$$
$$y' = 10C_2e^{10t} = -3(4C_1 - 2C_2e^{10t)} + 4(3C_1 + C_2e^{10t}) = -3x + 4y$$

The zero eigenvalue gives a line of equilibria, emphasized with the black line, with solutions pointing outward (because of the positive eigenvalue 10) parallel to the other eigenvector.



Question 5

$$\dot{x} = 4x - 2y$$
$$\dot{y} = 2x$$

$$A = \begin{bmatrix} 4 & -2 \\ 2 & 0 \end{bmatrix}$$

$$\det \begin{bmatrix} 4-\lambda & -2 \\ 2 & -\lambda \end{bmatrix} = 0$$

$$(4-\lambda)(-\lambda) + 4 = 0$$
$$\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0$$

So we have a repeated eigenvalue, $\lambda_1 = \lambda_2 = 2$.

If the eigenvalues of A are repeated (i.e. $\lambda_1 = \lambda_2$), then the general solution to the system $\dot{X} = AX$ is $X = e^{\lambda t}v_0 + te^{\lambda t}v_1$, where $v_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ and $v_1 = (A - \lambda I)v_0$.

$$\begin{aligned} v_1 &= (A - 2I)v_0 \\ &= \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \\ &= \begin{bmatrix} 2x_0 - 2y_0 \\ 2x_0 - 2y_0 \end{bmatrix} \end{aligned}$$

Which means the general solution is:

Solution

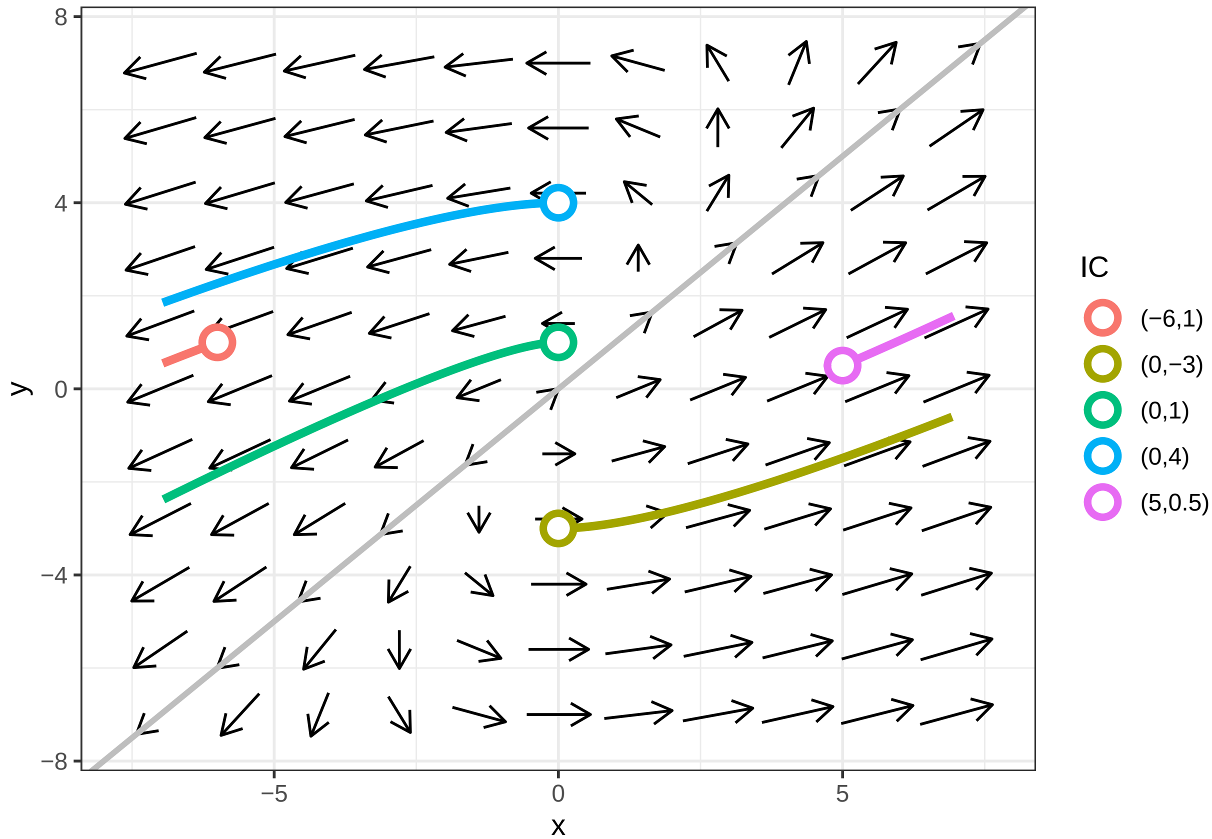
$$X = e^{2t} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + 2te^{2t} \begin{bmatrix} x_0 - y_0 \\ x_0 - y_0 \end{bmatrix}$$

Check solution

$$\begin{aligned} \dot{x} &= 2e^{2t}x_0 + 4te^{2t}(x_0 - y_0) + 2e^{2t}(x_0 - y_0) \\ &= 4e^{2t}x_0 + 4te^{2t}(x_0 - y_0) - 2e^{2t}y_0 \\ &= 4(e^{2t}x_0 + 2te^{2t}(x_0 - y_0)) - 2(e^{2t}y_0 + 2te^{2t}(x_0 - y_0)) \\ &= 4x - 2y \end{aligned}$$

$$\begin{aligned} \dot{y} &= 2e^{2t}y_0 + 4te^{2t}(x_0 - y_0) + 2e^{2t}(x_0 - y_0) \\ &= 2e^{2t}x_0 + 4te^{2t}(x_0 - y_0) \\ &= 2(e^{2t}x_0 + 2te^{2t}(x_0 - y_0)) \\ &= 2x \end{aligned}$$

The positive repeated eigenvalue makes this a source, with solutions going towards the straight-line solution with the same slope (1, because y=x) as v_1.



Question 6

$$\dot{x} = -4x + y$$
$$\dot{y} = 3x - 2y$$

$$A = \begin{bmatrix} -4 & 1 \\ 3 & -2 \end{bmatrix}$$

$$\det \begin{bmatrix} -4-\lambda & 1 \\ 3 & -2-\lambda \end{bmatrix} = 0$$

$$(-4-\lambda)(-2-\lambda) - 3 = 0$$

$$\lambda^2 + 6\lambda + 5 = (\lambda+5)(\lambda+1) = 0$$

So we have found two distinct real eigenvalues, $\lambda_1 = -5$, $\lambda_2 = -1$. Now we need to solve $AX = \lambda X$, or $(A - \lambda I)X = 0$ with these two values of λ. Let's start with the first eigenvalue, 0.

$$A + 5I = \begin{bmatrix} -4+5 & 1 \\ 3 & -2+5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$x_1 + x_2 = 0 \quad \rightarrow \quad x_1 = -x_2 \quad \rightarrow \quad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A + I = \begin{bmatrix} -4+1 & 1 \\ 3 & -2+1 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}$$

$$3x_1 - x_2 = 0 \quad \rightarrow \quad 3x_1 = x_2 \quad \rightarrow \quad v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

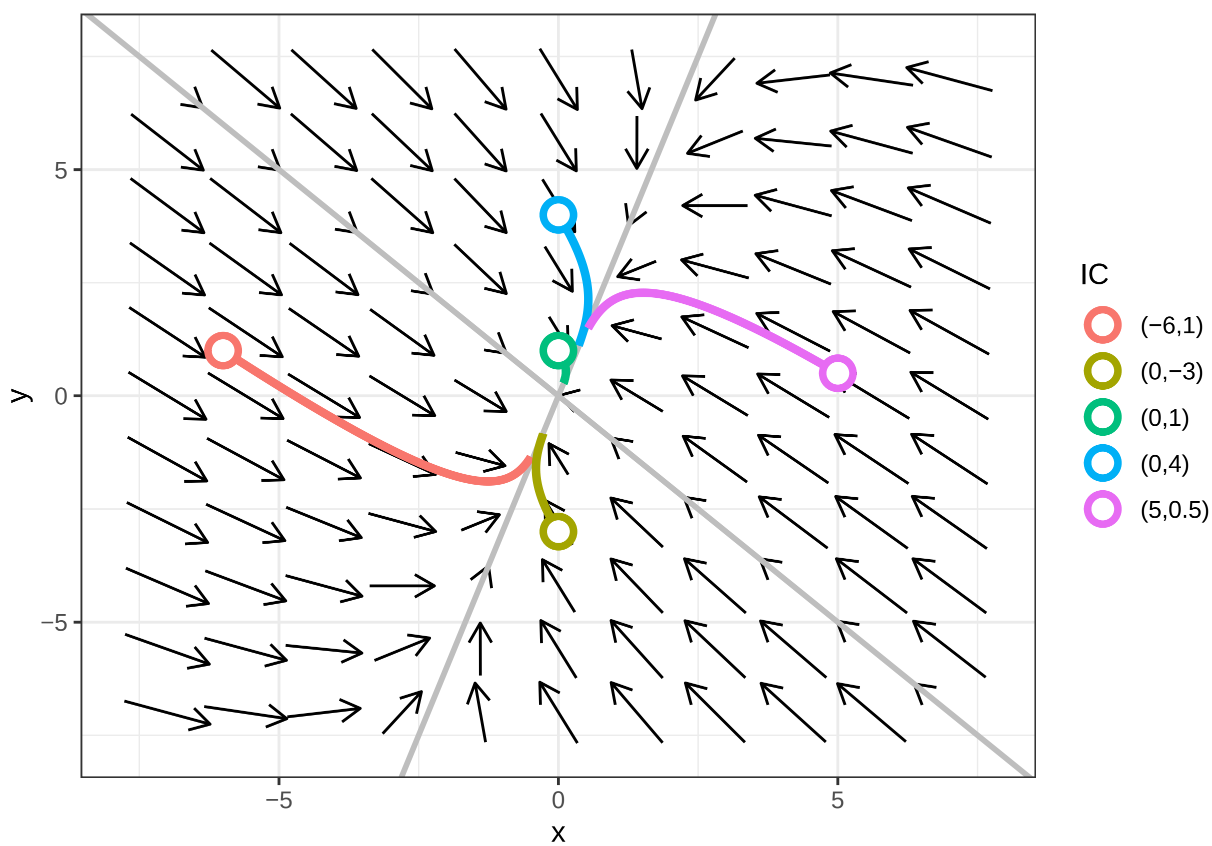
Solution

$$X = C_1 e^{-5t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Check solution

$$x' = -5C_1 e^{-5t} - C_2 e^{-t} = -4(C_1 e^{-5t} + C_2 e^{-t}) + (-C_1 e^{-5t} + 3C_2 e^{-t}) = -4x + y$$
$$y' = 5C_1 e^{-5t} - 3C_2 e^{-t} = 3(C_1 e^{-5t} + C_2 e^{-t}) - 2(-C_1 e^{-5t} + 3C_2 e^{-t}) = 3x - 2y$$

Two negative eigenvalues make this a sink, with solutions going towards the eigenvector $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ with the larger (less negative) eigenvalue.



Question 7

$$\begin{aligned}\dot{x} &= -9x - 3y \\ \dot{y} &= 3x + y\end{aligned}$$

$$A = \begin{bmatrix} -9 & -3 \\ 3 & 1 \end{bmatrix}$$

$$\det \begin{bmatrix} -9-\lambda & -3 \\ 3 & 1-\lambda \end{bmatrix} = 0$$

$$(-9-\lambda)(1-\lambda) + 9 = 0$$

$$\lambda^2 + 8\lambda = \lambda(\lambda+8) = 0$$

So we have found two distinct real eigenvalues, $\lambda_1 = 0$, $\lambda_2 = -8$. Now we need to solve $AX = \lambda X$, or $(A - \lambda I)X = 0$ with these two values of λ. Let's start with the first eigenvalue, 0.

$$A - 0I = \begin{bmatrix} -9 & -3 \\ 3 & 1 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$$

$$3x_1 + x_2 = 0 \quad \rightarrow \quad 3x_1 = -x_2 \quad \rightarrow \quad v_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

And repeat for the second eigenvalue, -8:

$$A + 8I = \begin{bmatrix} -9+8 & -3 \\ 3 & 1+8 \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ 3 & 9 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$$

$$x_1 + 3x_2 = 0 \quad \rightarrow \quad x_1 = -3x_2 \quad \rightarrow \quad v_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

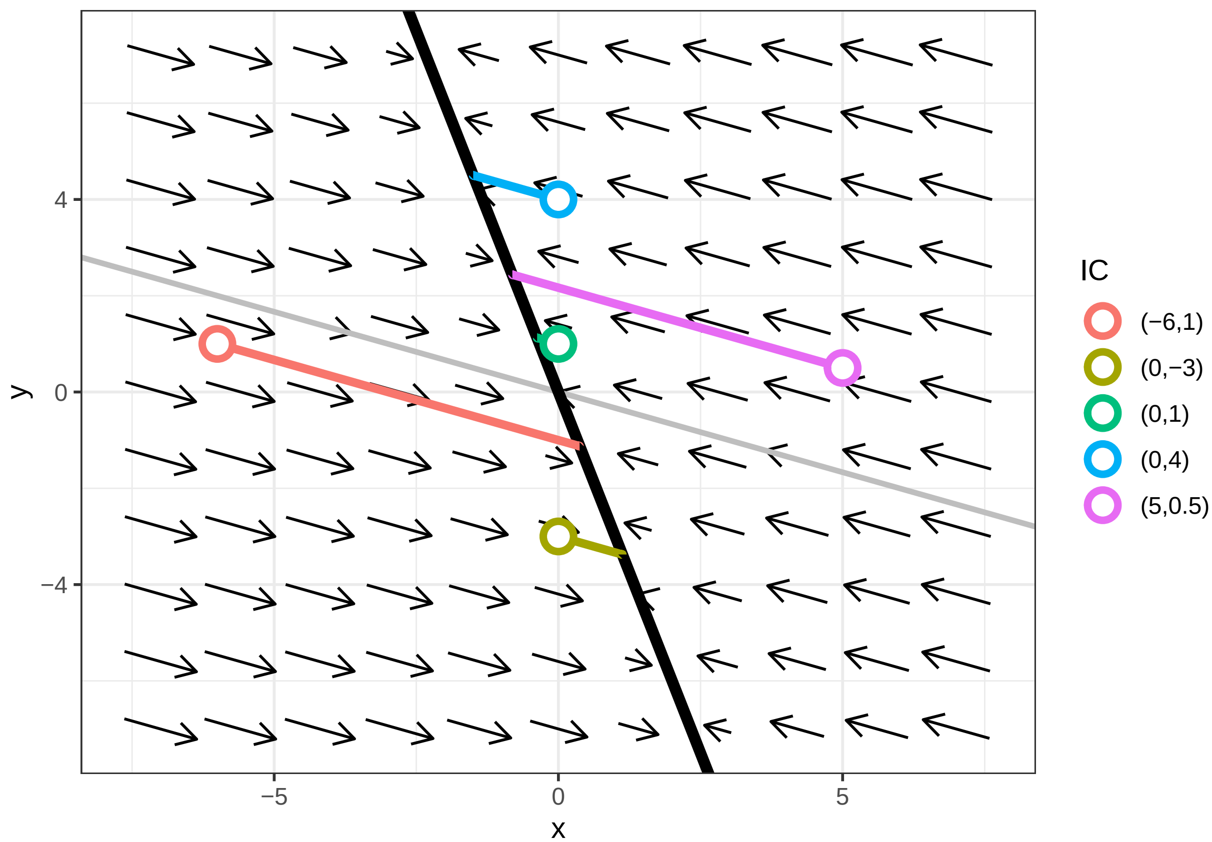
Solution

$$X = C_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} + C_2 e^{-8t} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Check solution

$$\begin{aligned}x' &= -24C_2e^{-8t} = -9(C_1 + 3C_2e^{-8t}) - 3(-3C_1 - C_2e^{-8t}) = -9x - 3y \\ y' &= 8C_2e^{-8t} = 3(C_1 + 3C_2e^{-8t}) + (-3C_1 - C_2e^{-8t}) = 3x + y\end{aligned}$$

The zero eigenvalue gives a line of equilibria, with solutions drawn towards the line (because the other eigenvalue is negative) parallel with the other eigenvector.



Question 8

$$\dot{x} = -3x + 10y$$
$$\dot{y} = -x + 3y$$

$$A = \begin{bmatrix} -3 & 10 \\ -1 & 3 \end{bmatrix}$$

$$\det \begin{bmatrix} -3-\lambda & 10 \\ -1 & 3-\lambda \end{bmatrix} = 0$$

$$(-3-\lambda)(3-\lambda) + 10 = 0$$

$$\lambda^2 + 1 = 0$$

So we have complex eigenvalues, $\lambda_1 = i,\ \lambda_2 = -i$. Now we need to solve $AX = \lambda X$, or $(A - \lambda I)X = 0$ with these two values of λ.

$$A - iI = \begin{bmatrix} -3-i & 10 \\ -1 & 3-i \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} -1 & 3-i \\ 0 & 0 \end{bmatrix}$$

$$-x_1 + (3-i)x_2 = 0 \quad \rightarrow \quad x_1 = (3-i)x_2 \quad \rightarrow \quad v_1 = \begin{bmatrix} 3-i \\ 1 \end{bmatrix}$$

Which means the eigenvector for the other eigenvalue is the conjugate of this vector, $v_2 = \begin{bmatrix} 3+i \\ 1 \end{bmatrix}$.

Now we use Euler's formula:

$$e^{it} \begin{bmatrix} 3-i \\ 1 \end{bmatrix} = (\cos(t) + i\sin(t)) \begin{bmatrix} 3-i \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} (3-i)\cos t + (3i+1)\sin t \\ \cos t + i\sin t \end{bmatrix} = \begin{bmatrix} 3\cos t + \sin t \\ \cos t \end{bmatrix} + i \begin{bmatrix} 3\sin t - \cos t \\ \sin t \end{bmatrix}$$

Having found our real and imaginary parts, we can now write our general solution as a linear combination of them:

Solution

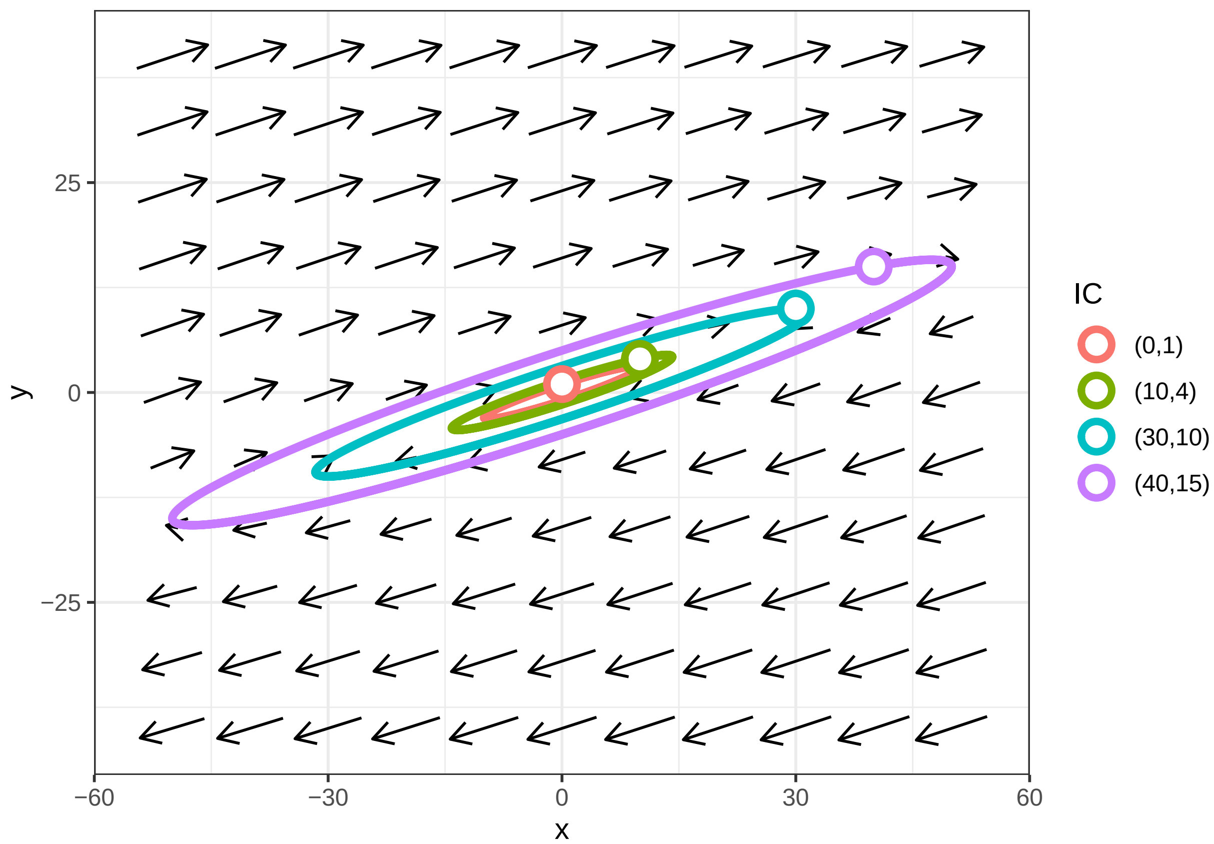
$$X = C_1 \begin{bmatrix} 3\cos t + \sin t \\ \cos t \end{bmatrix} + C_2 \begin{bmatrix} 3\sin t - \cos t \\ \sin t \end{bmatrix}$$

Check solution

$$\begin{aligned} x' &= -3C_1\sin t + C_1\cos t + 3C_2\cos t + C_2\sin t \\ &= -3(3C_1\cos t + C_1\sin t + 3C_2\sin t - C_2\cos t) + 10(C_1\cos t + C_2\sin t) \\ &= -3x + 10y \end{aligned}$$

$$\begin{aligned} y' &= -C_1\sin t + C_2\cos t \\ &= -(3C_1\cos t + C_1\sin t + 3C_2\sin t - C_2\cos t) + 3(C_1\cos t + C_2\sin t) \\ &= -x + 3y \end{aligned}$$

Purely imaginary complex eigenvalues give a center. Testing points with our original matrix reveals a clockwise direction of rotation.



25
0
−25
y
−60
−30
0
30
60
x
IC
(0,1)
(10,4)
(30,10)
(40,15)