

MATH 245 Homework 1

Ruby Krasnow

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Question 1

Show the function is a solution of the PDE:

(a) $u_{xx} + u_{yy} = 0$

(i) $u(x, y) = e^x \sin(y)$

$$u_x = e^x \sin(y), u_{xx} = e^x \sin(y), u_y = e^x \cos(y), u_{yy} = -e^x \sin(y)$$

Which means that $u_{xx} + u_{yy} = e^x \sin(y) + (-e^x \sin(y)) = 0$ and so $u(x, y) = e^x \sin(y)$ is a solution to the PDE.

(ii) $u(x, y) = \log \sqrt{x^2 + y^2}$

Assuming this is meant to be $u(x, y) = \ln \sqrt{x^2 + y^2}$,

$$u_x = (x^2 + y^2)^{-1/2} \left(\frac{1}{2} \right) (x^2 + y^2)^{-1/2} (2x) = x (x^2 + y^2)^{-1}$$

$$u_{xx} = x(-1) (x^2 + y^2)^{-2} (2x) + (x^2 + y^2)^{-1}$$

$$u_y = (x^2 + y^2)^{-1/2} \left(\frac{1}{2} \right) (x^2 + y^2)^{-1/2} (2y) = y (x^2 + y^2)^{-1}$$

$$u_{yy} = y(-1) (x^2 + y^2)^{-2} (2y) + (x^2 + y^2)^{-1}$$

$$u_{xx} + u_{yy} = \frac{-2x^2}{(x^2 + y^2)^2} + \frac{-2y^2}{(x^2 + y^2)^2} + \frac{2}{(x^2 + y^2)} =$$

$$\frac{-2(x^2 + y^2)}{(x^2 + y^2)^2} + \frac{2}{(x^2 + y^2)} = 0$$

Which means that $u(x, y) = \log \sqrt{x^2 + y^2}$ is a solution to the PDE.

(b) $bu_x + au_y + u = 0$, $u(x, y) = \exp\left(\frac{-x}{b}\right)f(ax - by)$ for arbitrary differentiable function f .

$$u_x = e^{\frac{-x}{b}} \left(\frac{-1}{b} \right) f(ax - by) + f'(ax - by)(a)e^{\frac{-x}{b}}$$

$$u_y = e^{\frac{-x}{b}} f'(ax - by)(-b)$$

Then,

$$bu_x + au_y + u = -e^{\frac{-x}{b}} f(ax - by) + (ab) \left(e^{\frac{-x}{b}} \right) f'(ax - by) + e^{\frac{-x}{b}} f'(ax - by)(-ab) + e^{\frac{-x}{b}} f(ax - by) = 0$$

So $u(x, y) = \exp\left(\frac{-x}{b}\right)f(ax - by)$ is a solution to the PDE for an arbitrary differentiable function f .

- (c) $u_{xx} - \frac{1}{x}u_x - x^2u_{yy} = 0$, $u(x, y) = f(2y + x^2) + g(2y - x^2)$ for arbitrary twice-differentiable functions f and g .

$$\begin{aligned}u_x &= (f'(2y + x^2))(2x) + (g'(2y - x^2))(-2x) \\u_y &= 2(f'(2y + x^2)) + 2(g'(2y - x^2)) \\u_{xx} &= 2f'' + (f'')(4x^2) - 2g' + (g'')(4x^2) \\u_{yy} &= 4f'' + 4g'' \\u_{xx} - \frac{1}{x}u_x - x^2u_{yy} &= \\2f'' + 4x^2f'' - 2g' + 4x^2g'' - \frac{1}{x}(2xf' - 2xg') - x^2(4f'' + 4g'') &= \\2f'' - 2f' + 4x^2f'' - 4x^2f'' - 2g' + 2g' + 4x^2g'' - 4x^2g'' &= 0\end{aligned}$$

So $u(x, y) = f(2y + x^2) + g(2y - x^2)$ is a solution to the PDE for arbitrary twice-differentiable functions f and g .

Question 2

- (a) 2nd-order linear homogenous
(b) 4th-order linear inhomogenous
(c) 2nd-order quasi-linear homogeneous
(d) We can rewrite this as $u^{-5}u_{xx} + u^{-5}u_{yy} + u^{-4}f(x, y) = g(x, y)$ to see that it is a 2nd-order quasi-linear inhomogeneous PDE.

Question 3

Use separation of variables to solve the following problems:

(a) $u_x + u = u_y$, $u(x, 0) = 4x^{-3x}$, use $u(x, y) = f(x)g(y)$

(b) $x^2u_{xy} + 9y^2u = 0$, $u(x, 0) = \exp\left(\frac{1}{x}\right)$, use $u(x, y) = f(x)g(y)$

(c) $u_x^2 + u_y^2 = 1$, use $u(x, y) = f(x) + g(y)$

(d) $x^2u_x^2 + y^2u_y^2 = u^2$, use $u(x, y) = e^{f(x)}e^{g(y)}$

Question 4

For each of the following IVPs, (i) find and plot the characteristic lines (curves), (ii) solve the IVP, and (iii) plot the solution of (a)-(b) for indicated time.

(a) $u_t + (1 + x^2) u_x = 0$, $u(0, x) = \arctan(x)$, $t = 1, 2, 3$, and what is $\lim_{x \rightarrow \infty} u(t, x)$?

$$\frac{dx}{dt} = 1 + x^2$$

$$\frac{1}{1 + x^2} dx = dt$$

$$\arctan(x) = t + C$$

So $x = \tan(t + C)$ are the characteristic curves of $u_t + (1 + x^2) u_x = 0$.

On each of the curves, $u(x, t)$ is constant because $\frac{d}{dt} u(t, \tan(t + C)) = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \sec^2(t + c)$.

And since $1 + \tan^2 \theta = \sec^2 \theta$, this means $\frac{d}{dt} u(t, \tan(t + C)) = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} (1 + \tan^2(t + c)) = u_t + (1 + x^2) u_x$, which we know is 0.

Thus $u(t, \tan(t + C)) = u(0, \tan(0 + C)) = u(0, C)$ is independent of t . Putting $x = \tan(t + C)$ and $C = \arctan(x) - t$, we have

$$u(t, x) = u(0, \arctan(x) - t)$$

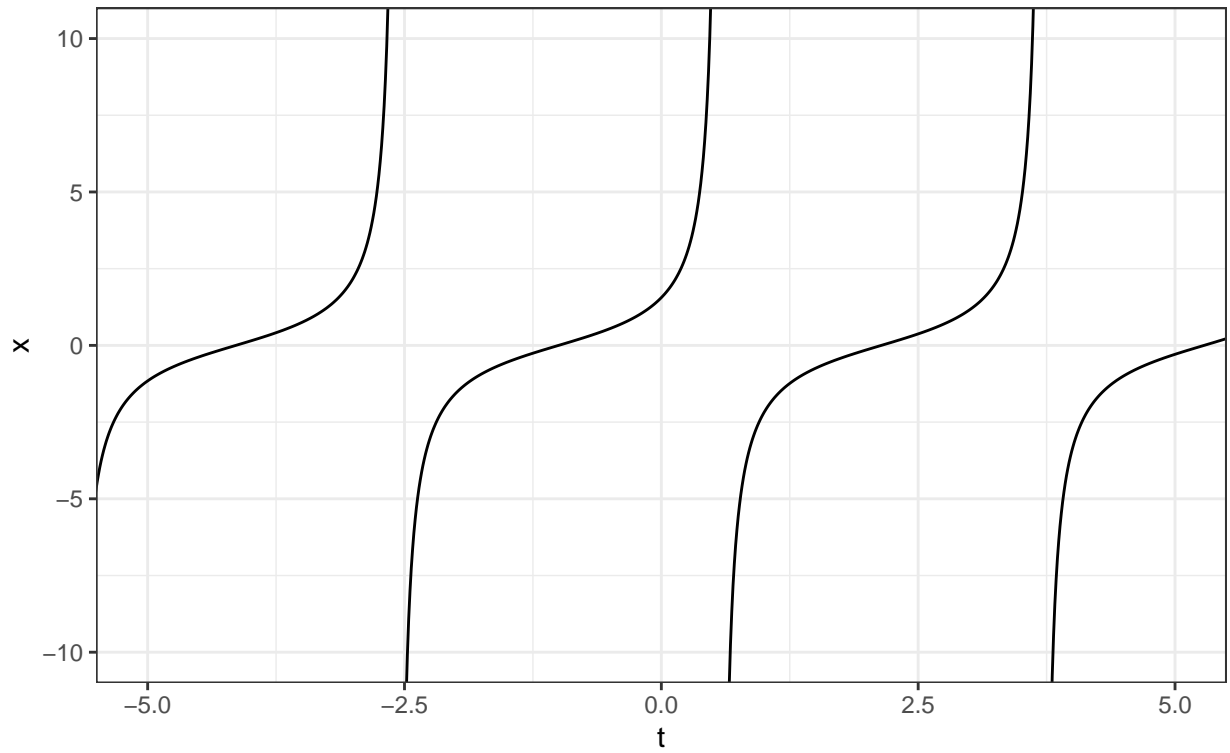
It follows that $u(t, x) = f(\arctan(x) - t)$. And since we are given that $u(0, x) = \arctan(x)$, we have $f(\arctan(x) - 0) = \arctan(x)$ so that $f(w) = w$ for any w , yielding our solution of

$$u(t, x) = \arctan(x) - t$$

```
C=1
df_4a <- tibble(x = seq(-2*pi, 2*pi, length.out = 1000),
               y=tan(x+C)) %>%
  mutate(y = ifelse(abs(y) < 20, y, NA))

ggplot(df_4a, aes(x=x, y=y)) +
  geom_line()+
  coord_cartesian(ylim=c(-10,10), xlim = c(-5,5))+
  theme_bw()+
  labs(x="t", y="x")+
  ggtitle("Characteristic curves for Problem 4a", subtitle = "with C=1")
```

Characteristic curves for Problem 4a
with C=1



Since the range of \arctan is limited to $(-\frac{\pi}{2}, \frac{\pi}{2})$, as $\lim_{t \rightarrow \infty}$ that means $u(t, x) = \arctan(x) - t$ will go to $-\infty$.