# BIOL 274 Homework 4

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## Question 1

Solve each of the following IVPs:

(a)

$$\begin{cases} y'' + 6y' + 8y = 0 \\ y(0) = 3 \\ y'(0) = -16 \end{cases} \tag{1}$$

The characteristic equation is

$$\lambda^2 + 6\lambda + 8 = 0 \quad \longrightarrow \quad (\lambda + 4)(\lambda + 2) = 0$$

Which means we have two distinct real roots, $\lambda_1 = -4$ and $\lambda_1 = -2$.

Thus, our general solution is

$$y = C_1 e^{-4x} + C_2 e^{-2x}$$

We differentiate to find that

$$y' = -4C_1 e^{-4x} - 2C_2 e^{-2x}$$

Now we plug in our initial conditions:

$$3 = C_1 + C_2$$
$$-16 = -4C_1 - 2C_2$$

Multiply both sides of the first equation by 2 and add to the second equation:

$-10 = -2C_1 \quad \longrightarrow \quad C_1 = 5$, so $C_2 = 3 - 5 = -2$. Which means the solution to the IVP is

**Solution**

$$y(x) = 5e^{-4x} - 2e^{-2x}$$

**Check solution**

$$y'(x) = -20e^{-4x} + 4e^{-2x}$$

$$y''(x) = 80e^{-4x} - 8e^{-2x}$$

$$y'' + 6y' + 8y = (80 - 120 + 40)e^{-4x} + (-8 + 24 - 16)e^{-2x} = 0$$

(b)

$$\begin{cases} y'' + 6y' + 9y = 0 \\ y(0) = \frac{1}{3} \\ y'(0) = \frac{1}{2} \end{cases} \tag{2}$$

The characteristic equation is

$$\lambda^2 + 6\lambda + 9 = 0 \quad \longrightarrow \quad (\lambda + 3)(\lambda + 3) = 0$$

So we have a repeated real root, $\lambda_1 = \lambda_2 = -3$. Thus, our general solution is

$$y = C_1 e^{-3x} + C_2 x e^{-3x}$$

We differentiate to find that

$$y' = -3C_1 e^{-3x} + C_2 e^{-3x} - 3C_2 x e^{-3x}$$

Now we plug in our initial conditions:

$$\frac{1}{3} = C_1$$

$$\frac{1}{2} = -3C_1 + C_2 \quad \longrightarrow \quad C_2 = \frac{3}{2}$$

Which means the solution to the IVP is

**Solution**

$$y(x) = \frac{1}{3} e^{-3x} + \frac{3}{2} x e^{-3x}$$

**Check solution**

$$y'(x) = -e^{-3x} - \frac{9}{2} x e^{-3x} + \frac{3}{2} e^{-3x} = \frac{1}{2} e^{-3x} - \frac{9}{2} x e^{-3x}$$

$$y''(x) = -\frac{3}{2} e^{-3x} + \frac{27}{2} x e^{-3x} - \frac{9}{2} e^{-3x} = \frac{27}{2} x e^{-3x} - 6e^{-3x}$$

Plugging these expressions into our original ODE, we see that

$$y'' + 6y' + 9y = \left( \frac{27}{2} - \frac{54}{2} + \frac{27}{2} \right) x e^{-3x} + (-6 + 3 + 3)\, e^{-3x} = 0$$

## Question 2

Consider the equation

$$\frac{d^2 y}{dt^2} + \frac{k}{m} y = 0$$

for the motion of a single (undamped) harmonic oscillator.

(a) Consider the function $y(t) = \cos(\beta t)$. Under what conditions on $\beta$ is $y(t)$ a solution?

$$\lambda^2 + \frac{k}{m} = 0$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{0 \pm \sqrt{0 - 4\frac{k}{m}}}{2}$$

$$0 \pm \sqrt{\frac{k}{m}}\, i$$

$y(t) = \cos(\beta t)$ is only a solution if $\beta = \sqrt{\frac{k}{m}}$.

We can check this:

$$y(t) = \cos\left(\sqrt{\frac{k}{m}}t\right)$$

$$y'(t) = -\sqrt{\frac{k}{m}} \sin\left(\sqrt{\frac{k}{m}}t\right)$$

$$y''(t) = -\frac{k}{m} \cos\left(\sqrt{\frac{k}{m}}t\right)$$

$$y'' + \frac{k}{m}y = -\frac{k}{m} \cos\left(\sqrt{\frac{k}{m}}t\right) + \frac{k}{m} \cos\left(\sqrt{\frac{k}{m}}t\right) = 0$$

(b) What initial conditions ($t = 0$) correspond to this solution?

$y(0) = \cos(0) = 1$, so we must have the initial condition $y(0) = 1$.

(c) In terms of $k$ and $m$, what is the period of this solution?

In the equation for simple harmonic oscillation $y(t) = A\cos(\omega t + \varphi)$, $\omega = \sqrt{\frac{k}{m}}$ represents the angular frequency, measured in radians per second. $\omega$ is equivalent to $\frac{2\pi}{T}$, where $T$ is the period of the oscillation (the time it takes to complete one oscillation and return to the starting position). This means we can find T by taking $\frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{k}{m}}}$, or

$$T = 2\pi\sqrt{\frac{m}{k}}$$

## Question 3

Find the general solution for each of the following.
(a)

$$y'' + 3y' + 2y = 4t^2 - 2t \tag{3}$$

The characteristic equation is

$$\lambda^2 + 3\lambda + 2 = 0 \quad \longrightarrow \quad (\lambda + 2)(\lambda + 1) = 0$$

Which means we have two distinct real roots, $\lambda_1 = -2$ and $\lambda_1 = -1$.

Thus, our complementary function (solution) is

$$y_c = C_1 e^{-2t} + C_2 e^{-t}$$

Let's assume our solution takes the form $y = At^2 + Bt + C$. Then $y' = 2At + B$ and $y'' = 2A$, so the left-hand side of 3 becomes

$$2A + 6At + 3B + 2At^2 + 2Bt + 2C = 4t^2 - 2t$$

We can split this up by coefficients of $t$ into

$$2At^2 = 4t^2$$

$$6At + 2Bt = -2t$$

$$2A + 3B + 2C = 0$$

Which means that $A = 2,\ B = -7,\ C = \frac{17}{2}$ and so

$$y_p = 2t^2 - 7t + \frac{17}{2}$$

Combine $y_p$ with $y_c$ to find our general solution:

**Solution**

$$y(t) = C_1 e^{-2t} + C_2 e^{-t} + 2t^2 - 7t + \frac{17}{2}$$

**Check solution**

$$y'(t) = -2C_1 e^{-2t} - C_2 e^{-t} + 4t - 7$$

$$y''(t) = 4C_1 e^{-2t} + C_2 e^{-t} + 4$$

$$y'' + 3y' + 2y = (4C_1 - 6C_1 + 2C_1)\, e^{-2t} + (C_2 - 3C_1 + 2C_1)\, e^{-t} + 4 + 12t - 21 + 4t^2 - 14t + 17 = 4t^2 - 2t$$

(b)

$$y'' - y' - 2y = 15e^2 t \tag{4}$$

The characteristic equation is

$$\lambda^2 - \lambda - 2 = 0 \quad \longrightarrow \quad (\lambda - 2)(\lambda + 1) = 0$$

Which means we have two distinct real roots, $\lambda_1 = 2$ and $\lambda_1 = -1$.

Thus, our complementary function (solution) is

$$y_c = C_1 e^{2t} + C_2 e^{-t}$$

Let's first assume our solution takes the form $y = Ae^{2t}$. Then $y' = 2Ae^{2t}$ and $y'' = 4Ae^{2t}$, so the left-hand side of 4 becomes

$$4Ae^{2t} - 2Ae^{2t} - 2Ae^{2t} = 0 \neq 15e^2t$$

Oh no! What happened? Well, $e^{2t}$ satisfies the associated homogenous DE and so is part of the complementary solution. Instead, we will try $y = Ate^{2t}$. Then $y' = 2Ate^{2t} + Ae^{2t}$ and $y'' = 4Ate^{2t} + 4Ae^{2t}$, so the left-hand side of 4 becomes

$$y'' - y' - 2y = (4A - 2A - 2A)\, te^{2t} + (4A - A)\, e^{2t}$$

Which gives us $3Ae^{2t} = 15e^{2t}$, meaning $A = 5$ and we have

$$y_p = 5te^{2t}$$

Combine $y_p$ with $y_c$ to find our general solution:

**Solution**

$$y(t) = C_1e^{2t} + C_2e^{-t} + 5te^{2t}$$

or

$$y(t) = (C_1 + 5t)e^{2t} + C_2e^{-t}$$

**Check solution**

$$y'(t) = (2C_1 + 10t + 5)e^{2t} - C_2e^{-t}$$

$$y''(t) = (4C_1 + 20t + 20)e^{2t} + C_2e^{-t}$$

$$y'' - y' - 2y = (4C_1 + 20t + 20 - 2C_1 - 10t - 5 - 2C_1 - 10t)e^{2t} + (C_2 + C_2 - 2C_2)e^{-t} = 15e^{2t}$$

(c)

$$y'' - 2y' - 3y = \cos(2t) \tag{5}$$

The characteristic equation is

$$\lambda^2 - 2\lambda - 3 = 0 \quad \longrightarrow \quad (\lambda - 3)(\lambda + 1) = 0$$

Which means we have two distinct real roots, $\lambda_1 = 3$ and $\lambda_1 = -1$.

Thus, our complementary function (solution) is

$$y_c = C_1e^{3t} + C_2e^{-t}$$

Let's assume our solution takes the form $y = A\sin(2t) + B\cos(2t)$. Then

$$y' = 2A\cos(2t) - 2B\sin(2t)$$

$$y'' = -4A\sin(2t) - 4B\cos(2t)$$

So the left-hand side of 5 becomes

$$y'' - 2y' - 3y = (-4A + 4B - 3A)\sin(2t) + (-4B - 4A - 3B)\cos(2t)$$

Compare the coefficients of the sin and cos terms to the original DE:

$$-7A + 4B = 0$$

$$-4A - 7B = 1$$

$$A = \frac{4B}{7}$$

$$-\frac{16B}{7} - \frac{49B}{7} = 1$$

Which gives us that $B = \frac{-7}{65}$, $A = \frac{-4}{65}$. Thus, our particular solution is

$$y_p = -\frac{4}{65}\sin(2t) - \frac{7}{65}\cos(2t)$$

Combine $y_p$ with $y_c$ to find our general solution:

**Solution**

$$y(t) = C_1 e^{3t} + C_2 e^{-t} - \frac{4}{65}\sin(2t) - \frac{7}{65}\cos(2t)$$

**Check solution**

$$y'(t) = 3C_1 e^{3t} - C_2 e^{-t} + \frac{14}{65}\sin(2t) - \frac{8}{65}\cos(2t)$$

$$y''(t) = 9C_1 e^{3t} + C_2 e^{-t} + \frac{16}{65}\sin(2t) + \frac{28}{65}\cos(2t)$$

$$y''-2y'-3y = (9-6-3)C_1 e^{3t} + (1+2-3)C_2 e^{-t} + \left(\frac{16}{65} - \frac{28}{65} + \frac{12}{65}\right)\sin(2t) + \left(\frac{28}{65} + \frac{16}{65} + \frac{21}{65}\right)\cos(2t) = \cos(2t)$$

(d)

$$y'' + 4y = e^t + \sin(2t) \tag{6}$$

The characteristic equation is

$$\lambda^2 + 4 = 0 \quad \longrightarrow \quad \lambda = 0 \pm 2i$$

Thus, our complementary function (solution) is

$$y_c = C_1 \sin(2t) + C_2 \cos(2t)$$

Let's break up our right hand side to find $y_{p_1}$ and $y_{p_2}$.

For $y_{p_1}$, we assume our solution takes the form $y = Ae^t$. Then $Ae^t + 4Ae^t = e^t$, so $A = \frac{1}{5}$ and we have

$$y_{p_1} = \frac{e^t}{5}$$

For $y_{p_2}$, we cannot assume our solution takes the form $y = A\sin(2t) + B\cos(2t)$ because we had complex roots $\pm 2i$ for our characteristic equation. Instead, we try

$$y = At\sin(2t) + Bt\cos(2t)$$

$$y' = 2At\cos(2t) + A\sin(2t) + B\cos(2t) - 2Bt\sin(2t) =$$
$$(2At + B)\cos(2t) + (-2Bt + A)\sin(2t)$$

$$y'' = (-4At - 2B)\sin(2t) + 2A\cos(2t) + (-4Bt + 2A)\cos(2t) - 2B\sin(2t) =$$

$$(-4At - 4B)\sin(2t) + (-4Bt + 4A)\cos(2t)$$

$$(-4At - 4B + 4At)\sin(2t) + (-4Bt + 4A + 4Bt)\cos(2t) = \sin(2t)$$

$$-4B\sin(2t) + 4A\cos(2t) = \sin(2t)$$
$$A = 0, B = -\frac{1}{4}$$
$$y_{p_2} = -t\frac{\cos(2t)}{4}$$
$$y_p = \frac{e^t}{5} - \frac{t\cos(2t)}{4}$$

Now adding this to our complementary function, we get

**Solution**

$$y(t) = C_1\sin(2t) + C_2\cos(2t) + \frac{e^t}{5} - \frac{t\cos(2t)}{4}$$

**Check solution**

$$y'(t) = 2C_1\cos(2t) - 2C_2\sin(2t) + \frac{e^t}{5} - \frac{\cos(2t)}{4} + \frac{t\sin(2t)}{2}$$

$$y''(t) = -4C_1\sin(2t) - 4C_2\cos(2t) + \frac{e^t}{5} + \frac{\sin(2t)}{2} + \frac{\sin(2t)}{2} + t\cos(2t)$$

$$y'' + 4y = (-4C_1 + 1 + 4C_1)\sin(2t) + (-4C_2 + t + 4C_2 - t)\cos(2t) + \frac{e^t}{5} + + \frac{4e^t}{5} =$$
$$e^t + \sin(2t)$$