MATH 245 Homework 3

Ruby Krasnow and Tommy Thach

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Question 1 (Parity of Solution)

(a) Prove that if initial conditions of the wave equation on the whole line are even(odd), the solution is even(odd).

Consider the IVP

$$\begin{cases} u_{tt} = c^2 u_{xx} & -\infty < x < \infty \\ u(0,x) = \varphi(x) & -\infty < x < \infty \\ u_t(0,x) = \psi(x) & -\infty < x < \infty \end{cases}$$

where $\varphi(x)$ and $\psi(x)$ are even functions, i.e., $\varphi(-x) = \varphi(x)$ and $\psi(-x) = \psi(x)$. Then using D'Alembert's formula, we know the solution u is

$$u(x,t) = \frac{\varphi(x+ct) + \varphi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds$$

Now plug in $-x$:

$$\begin{aligned} u(-x,t) &= \frac{\varphi(-x+ct) + \varphi(-x-ct)}{2} + \frac{1}{2c}\int_{-x-ct}^{-x+ct} \psi(s)ds \\ &= \frac{\varphi(x-ct) + \varphi(x+ct)}{2} + \frac{1}{2c}\int_{-x-ct}^{-x+ct} \psi(-s)ds \end{aligned}$$

After setting $y = -s, dy = -ds$ in the integral of ψ:

$$u(-x,t) = \frac{\varphi(x-ct) + \varphi(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy = u(x,t).$$

Therefore, even initial conditions imply that the solution of the wave equation is even.

Similarly, assume $\varphi(x)$ and $\psi(x)$ are odd functions, i.e., $\varphi(-x) = -\varphi(x)$ and $\psi(-x) = -\psi(x)$. Then using D'Alembert's formula, we know $u(-x,t)$ is

$$\begin{aligned} u(-x,t) &= \frac{\varphi(-x+ct) + \varphi(-x-ct)}{2} + \frac{1}{2c}\int_{-x-ct}^{-x+ct} \psi(s)ds \\ &= \frac{-\varphi(x-ct) - \varphi(x+ct)}{2} - \frac{1}{2c}\int_{-x-ct}^{-x+ct} \psi(-s)ds \end{aligned}$$

After setting $y = -s, dy = -ds$ in the integral of ψ:

$$u(-x,t) = \frac{-\varphi(x-ct) - \varphi(x+ct)}{2} - \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(-y)dy = -u(x,t).$$

Therefore, using the uniqueness of the wave equation, odd initial conditions imply that the solution of the wave equation is odd.

(b) Prove that if the initial condition of the heat equation on the whole line is even(odd), the solution is even(odd).

Consider the IVP

$$\begin{cases} u_t = ku_{xx} & -\infty < x < \infty \\ u(0,x) = f(x) \end{cases}$$

Then we know

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)\, dy$$

Which means, when $x = -x$:

$$u(-x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(-x-y)^2/4kt} f(y)\, dy$$

Now use the change of variable $y = -s, dy = -ds$:

$$u(-x,t) = \frac{-1}{\sqrt{4\pi kt}} \int_{\infty}^{-\infty} e^{-(-x+s)^2/4kt} f(-s)\, ds$$

$$= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-((-1)(x-s))^2/4kt} f(-s)\, ds$$

But $e^{-((-1)(x-s))^2/4kt}$ is equivalent to $e^{-(x-s)^2/4kt}$, so we have

$$u(-x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-s)^2/4kt} f(-s)\, ds$$

Now assume f is odd. By definition, we have $f(-s) = -f(s)$, so

$$u(-x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-s)^2/4kt} f(-s)\, ds = \frac{-1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-s)^2/4kt} f(s)\, ds = -u(x,t)$$

And therefore our solution is odd when our initial $f(x)$ is an odd function. Similarly, if we assume $f(x)$ is even, by definition we have $f(-s) = f(s)$, so

$$u(-x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-s)^2/4kt} f(-s)\, ds = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-s)^2/4kt} f(s)\, ds = u(x,t),$$

meaning (using the uniqueness of the heat equation), our solution is also even.

Question 2 (Speed of Heat vs Wave)

Consider the traveling wave $u(x,t) = f(x - at)$ where f is a given function of one variable.

(a) If it is a solution of the wave equation, show that the speed must be $a = \pm c$ (unless f is a linear function).

$$u_t = -af'(x-at), \quad u_{tt} = a^2 f''(x-at) \tag{1}$$

$$u_x = f'(x-at), \quad u_{xx} = f''(x-at) \tag{2}$$

If $u(x,t) = f(x-at)$ is a solution to the wave equation, then we must have $u_{tt} = c^2 u_{xx}$. By substituting (1) and (2) into the PDE, we get

$$a^2 f'' = c^2 f'',$$

which implies at $a^2 = c^2$, or $a = \pm c$. However, if f is a linear function, then the second derivative $f'' = 0$ and we cannot conclude that $a^2 = c^2$.

(b) If it is a solution of the diffusion equation, find f and show that the speed a is arbitrary.

If $u(x,t) = f(x-at)$ is a solution to the wave equation, then we must have $u_t = ku_{xx}$. By substituting (1) and (2) into the PDE, we get

$$\begin{aligned} -af' &= kf'' \\ -af' &= kf'' \\ \frac{-a}{k} &= \frac{f''}{f'} \end{aligned}$$

Now if we set $\xi = x - at$,

$$\frac{-a}{k} = \frac{f''}{f'} = \frac{d}{d\xi}\ln(f')$$

So integrating with respect to ξ,

$$\frac{-a}{k}\xi + C = \ln(f')$$

$$f'(\xi) = e^{\frac{-a}{k}\xi + C_1} = C_2 e^{\frac{-a}{k}\xi}$$

Now if we integrate again with respect to ξ,

$$f(\xi) = \frac{-k}{a} C_2 e^{\frac{-a}{k}\xi} + C_3$$

Set a new constant $C_4 = \frac{-k}{a} C_3$,

$$f(\xi) = C_4 e^{\frac{-a}{k}\xi} + C_3$$

If we change back to the original variables,

$$f(x-at) = C_4 e^{\frac{-a(x-at)}{k}} + C_3$$

Thus, we found the solution

$$u(x,t) = f(x-at) = Ae^{\frac{-a}{k}(x-at)} + B, \tag{3}$$

where A and B are constants.

We can check that this satisfies the heat equation:

$$u_t = A\frac{a^2}{k} e^{\frac{-a}{k}(x-at)}$$

$$u_x = A\frac{-a}{k}e^{\frac{-a}{k}(x-at)}$$

$$u_{xx} = A\frac{a^2}{k^2}e^{\frac{-a}{k}(x-at)}$$

$$u_t - ku_{xx} = A\frac{a^2}{k}e^{\frac{-a}{k}(x-at)} - kA\frac{a^2}{k^2}e^{\frac{-a}{k}(x-at)} = 0$$

However, (3) will be a solution regardless of the value of a; i.e., a is arbitrary.

Question 3 (Heat Equation with Robin boundary)

Consider the following problem with a Robin boundary condition:

$$(*) \begin{cases} u_t = ku_{xx} & 0 < x < \infty, 0 < t < \infty \\ u(0,x) = x & t = 0, 0 < x < \infty \\ u_x(0,t) - 2u(0,t) = 0 & x = 0 \end{cases}$$

The purpose of this exercise is to verify the solution formula for $(*)$. Let $f(x) = x$ for $x > 0$ and let $f(x) = x + 1 - e^{2x}$ for $x < 0$, and let

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)\, dy.$$

(a) What PDE and initial conditions does $v(x,t)$ satisfy for $-\infty < x < \infty$?

$v(x,t)$ satisfies

$$\begin{cases} v_t = kv_{xx} & -\infty < x < \infty, 0 < t < \infty \\ v(0,x) = f(x) & -\infty < x < \infty \end{cases} \tag{4}$$

where

$$f(x) = \begin{cases} x & x > 0 \\ x + 1 - e^{2x} & x < 0 \end{cases} \tag{5}$$

(b) Let $w = v_x - 2v$. What PDE and initial condition does $w(x,y)$ satisfy for $-\infty < x < \infty$?

Let $v(x,t)$ be a solution of $v_x = v_{tt}$. Now, we know that any partial derivatives of a solution to the heat equation are also solutions, and any linear combination of solutions is also a solution. Thus, if $v(x,t)$ is a solution to $v_x = v_{tt}$, so is $w = v_x - 2v$. The initial conditions will be

$$w(x,0) = v_x(0,x) - 2v(0,x) = f'(x) - 2f(x) = \begin{cases} 1 - 2x & \text{if } x > 0 \\ -1 - 2x & \text{if } x < 0 \end{cases}$$

Thus, $w(x,t)$ satisfies the PDE $v_x = v_{tt}$ with initial conditions

$$w(x,0) = \begin{cases} 1 - 2x & \text{if } x > 0 \\ -1 - 2x & \text{if } x < 0 \end{cases}$$

(c) Show that $f'(x) - 2f(x)$ is an odd function for $x \neq 0$.

$$f(x) = \begin{cases} x & \text{if } x > 0 \\ x + 1 - e^{2x} & \text{if } x < 0 \end{cases}$$

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ 1 - 2e^{2x} & \text{if } x < 0 \end{cases}$$

$$f'(x) - 2f(x) = \begin{cases} 1 - 2x & \text{if } x > 0 \\ -1 - 2x & \text{if } x < 0 \end{cases}$$

Which means that

$$f'(-x) - 2f(-x) = \begin{cases} 1 - 2(-x) & \text{if } (-x) > 0 \\ -1 - 2(-x) & \text{if } (-x) < 0 \end{cases}$$

But this is equivalent to

$$f'(-x) - 2f(-x) = \begin{cases} 1 + 2x & \text{if } x < 0 \\ -1 + 2x & \text{if } x > 0 \end{cases}$$

which is exactly $-(f'(x) - 2f(x))$. Therefore, $f'(x) - 2f(x)$ is an odd function of x.

(d) Use Exercise 2.4.11 to show that w is an odd function of x.

By Problem 1 in this assignment, we know that given the diffusion equation on the whole line with the usual initial condition $u(x, 0) = \phi(x), \phi(x)$ odd, the solution $u(x, t)$ is also an odd function of x.

To summarize, we know that

- $f'(x) - 2f(x)$ is an odd function of x
- Given a diffusion equation with odd initial conditions, the solution is also odd
- $w(x, t)$ is a solution to the diffusion equation with initial conditions $f'(x) - 2f(x)$

Combining these three points, it is clear that $w(x, t)$ must be an odd function of x.

(e) Deduce that $v(x, t)$ satisfies $(*)$ for $x > 0$. Assuming uniqueness, deduce that the solution of $(*)$ is given by

$$v(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)\, dy.$$

Since $w(x, t)$ is an odd function, it must automatically satisfy the boundary conditions $w(0, t) = 0$. We can solve the problem on the half-line by restricting $x > 0$. We know that the solution to

$$w_t = kw_{xx}, \quad w(x, 0) = f'(x) - 2f(x), \quad -\infty < x < \infty$$

is given by

$$w(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \left[f'(y) - 2f(y)\right]\, dy.$$

Now we can use our solution for w to solve for v:

$$v_x - 2v = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \left[f'(y) - 2f(y)\right]\, dy$$

$$v_x - 2v = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f'(y)\, dy - \frac{2}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)\, dy$$

We can see that the first term on the right-hand side is the solution to $(vx)_t = k(vx)_{xx}, -\infty < x < \infty$ with the initial condition $v_x(x,0) = f'(x)$, i.e., it equals v_x. Thus, we can subtract v_x from both sides of the above equation and divide by -2 to find show that $(*)$ is indeed satisfied by

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)\, dy, \quad x > 0$$

Because any solution satisfying the heat equation with specified initial and boundary conditions must be the one and only solution, this must be the unique solution to $(*)$.

Question 4 (Maximum Principle)

Consider two solutions $u(t,x)$ and $v(t,x)$ of the diffusion equation in $\{0 \le x \le l, 0 \le t \le \infty\}$

(a) Let $M(T)$ = the maximum of $u(t,x)$ in the closed rectangle $\Omega_T = \{0 \le x \le l, 0 \le t \le T\}$. Does $M(T)$ increase or decrease as a function of T?

The maximum principle says that if $u(x,t)$ satisfies the diffusion equation in a rectangle $\{0 \le x \le x, 0 \le t \le T\}$, then the maximum value of $u(x,t)$ is assumed either initially ($t = 0$) or on the lateral sides ($x = 0$ or $x = l$).

By the definition of maximum, if u assumes a higher value on the lateral sides $x = 0$ or $x = l$ when T is extended from some T_1 to a new T, $T_2 > T_1$, then the maximum of u will increase to that new value; otherwise, the maximum of u will remain the same. In other words, M is non-decreasing (increases, but not strictly) as a function of T.

(b) Let $m(T)$ = the minimum of $u(t,x)$ in the closed rectangle $\Omega_T = \{0 \le x \le l, 0 \le t \le T\}$. Does $m(T)$ increase or decrease as a function of T?

By the definition of minimum, if u assumes a lower value on the lateral sides $x = 0$ or $x = l$ when T is extended from some T_1 to a new T, $T_2 > T_1$, then the minimum of u will decrease to that new value; otherwise, the minimum of u will remain the same. In other words, M is non-increasing (decreases, but not strictly) as a function of T.

(c) Comparison Principle: Show that if $u \le v$ for $t = 0, x = 0$, and $x = l$, then $u \le v$ for $0 \le t \le \infty$ and $0 \le x \le l$.

Define $w(t,x) = u(t,x) - v(t,x)$ and note that w solves the heat equation in $\{0 \le x \le l, 0 \le t \le T\}$:

$$w_t - kw_{xx} = u_t - v_t - k(u_{xx} - v_{xx}) = (u_t - ku_{xx}) - (v_t - kv_{xx}) = 0 - 0 = 0$$

Since we are assuming $u \le v$ for $t = 0, x = 0$, and $x = l$, then $w \le 0$ for $t = 0, x = 0$, and $x = l$. But by the Maximum Principle, w assumes its maximum value on one of those three lines. This means we can take T to be arbitrarily large and still have $w \le 0$, i.e., $w \le 0$ on $\{0 \le x \le l, 0 \le t \le \infty\}$ and therefore $u \le v$ for $0 \le t \le \infty$ and $0 \le x \le l$.

Question 5 (Diffusion Equation with Dissipation)

Solve the following IVP for constant dissipation $b > 0$.

$$\begin{cases} u_t - ku_{xx} + bu = 0 & -\infty \le x \le \infty,\ t > 0 \\ u(0,x) = \psi(x), & -\infty \le x \le \infty \end{cases} \tag{6}$$

We will use the change of variables $u(t,x) = e^{-bt}v(t,x)$, and differentiate to find that:

$$u_t = e^{-bt}v_t - be^{-bt}v$$
$$u_x = e^{-bt}v_x$$
$$u_{xx} = e^{-bt}v_{xx}$$

So $u_t - ku_{xx} + bu = 0$ becomes

$$e^{-bt}v_t - be^{-bt}v - k\left(e^{-bt}v_{xx}\right) + be^{-bt}v = 0$$

After canceling the $be^{-bt}v$ terms and dividing by e^{-bt} (which we can do because it cannot equal zero), we are left with

$$v_t - kv_{xx} = 0, \quad v(0,x) = u(0,x) = \psi(x)$$

Hence, the general solution of v is

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)\, dy,$$

and the general solution of u is therefore

$$u(x,t) = \frac{e^{-bt}}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)\, dy.$$

Question 6

Suppose that there is a maximum principle for the wave equation and that $u(x,t)$ is a solution to the wave equation in the closed rectangle $\Omega_T = \{0 \le x \le \pi,\ 0 \le t \le \pi/c\}$. Then the maximum of u is assumed either initially (at $t = 0$) or on the lines $x = 0$ or $x = \pi$.

Consider the particular solution $u(x,t) = \sin x \sin ct$. This clearly satisfies the wave equation:

$$u_x = \cos x \sin ct$$
$$u_{xx} = -\sin x \sin ct$$
$$u_t = \sin x(c\cos ct)$$
$$u_{tt} = -\sin x(c^2 \sin ct) = c^2 u_{xx}$$

According to our presumed maximum principle, the maximum of $u(x,t)$ would have to be one of the following:

$$u(x,0) = \sin x \sin 0 = 0$$
$$u(0,t) = \sin 0 \sin ct = 0$$
$$u(\pi,t) = \sin \pi \sin ct = 0$$

But $u(x,t)$ has a higher value when $x = \pi/2$ and $t = \pi/2c$:

$$u\left(\frac{\pi}{2}, \frac{\pi}{2c}\right) = \sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) = 1$$

This contradicts the definition of maximum, so we have found a counterexample and therefore there is no maximum principle for the wave equation.