MATH 245 Homework 4

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Question 1: Find eigenvalues and eigenfunctions

(a)

$-\frac{d^2}{dx^2}X(x) = \lambda X(x)$ in $0 < x < l$ with boundary conditions $X'(0) = 0 = X(l)$

<u>Case 1:</u> Positive eigenvalues, $\lambda = \beta^2 > 0$

Re-writing as $X'' + \lambda X = 0$, we will get the characteristic equation: $r^2 + \beta^2 = 0$. Since our characteristic equation has complex roots $r = \pm i\beta$, our solutions take the form

$$X(x) = A\sin(\beta x) + B\cos(\beta x)$$

Differentiating, we find that

$$X'(x) = \beta A\cos(\beta x) - \beta B\sin(\beta x)$$

Now plugging in our initial condition $X'(0) = 0$, we get $X'(0) = \beta A = 0$. And since we are in a case where $\beta \neq 0$, this means $B = 0$ so $X(x) = A\sin(\beta x)$. Now we use our boundary condition $X(l) = 0$ to get $X(l) = A\sin(\beta l) = 0$. If $A = 0$, then $X(0) = 0$ and this contradicts the definition of an eigenfunction. Therefore, we must have $\sin(\beta l) = 0$, which can only occur if $\beta = \frac{(2n+1)\pi}{l}$. Therefore, this case gives us eigenvalues

$$\lambda_n = \left(\frac{(2n+1)\pi}{l}\right)^2 \quad n = 1, 2, 3...$$

with eigenfunctions

$$X_n(x) = \sin\left(\frac{(2n+1)\pi x}{l}\right)$$

<u>Case 2:</u> Zero eigenvalues, $\lambda = 0$ $X'' = 0$ implies that $X(x)$ is of the form $Ax + B$, with derivative $X'(x) = A$. Now plugging in our initial condition $X'(0) = 0$, we get $A = 0$, which means $X(x) = B$. But from the boundary condition $X(l) = 0$, we get $B = 0$ and so $X(0) = 0$. Therefore, there are no eigenfunctions $X(x)$ that satisfy $-\frac{d^2}{dx^2}X(x) = \lambda X(x)$ when $\lambda = 0$ and hence no zero eigenvalues.

<u>Case 3:</u> Negative eigenvalues, $\lambda = -\beta^2 < 0$ This case gives us the characteristic equation: $r^2 - \beta^2 = 0$. Since our characteristic equation has distinct real roots $r = \pm\beta$, our solutions take the form

$$X(x) = Ae^{\beta x} + Be^{-\beta x}$$

Differentiating, we find that

$$X'(x) = \beta Ae^{\beta x} - \beta Be^{-\beta x}$$

$$X'(0) = \beta A - \beta B = 0$$

Since we are in a case where $\beta \neq 0$, this means $A - B = 0$, or $A = B$. Then the boundary condition gives

$$X(l) = Be^{\beta l} + Be^{-\beta l} = 0$$

Since $e^{\beta l}$ and $e^{-\beta l}$ are nonzero for all values of l, we must have $B = 0$, implying that again $X(x) = 0$.

Thus, this problem has only positive eigenvalues and their associated eigenfunctions as found in Case 1.

(b)

$x^2 X''(x) + xX'(x) + \lambda X(x) = 0$ in $1 < x < e$ with boundary conditions $X(1) = 0 = X(e)$.

We recognize this equation as having the same form as a second-order Cauchy-Euler equation, a linear homogeneous ODE of the form $ax^2y + bxy' + cy = 0$ with the auxiliary equation $ar(r-1) + br + c = 0$. Here, $a = b = 1$ and $c = \lambda$, so we have

$$r(r-1) + r + \lambda = 0$$
$$r^2 - r + r + \lambda = 0$$
$$r^2 + \lambda = 0$$

Case 1: Positive eigenvalues, $\lambda = \beta^2 > 0$

$$X(x) = A\sin(\beta \ln x) + B\cos(\beta \ln x)$$
$$X(1) = A\sin(0) + B\cos(0) = 0 \quad \longrightarrow \quad B = 0$$
$$X(e) = A\sin(\beta \ln e) = 0$$
$$A\sin(\beta) = 0$$

If $A = 0$, then $X(0) = 0$ and this contradicts the definition of an eigenfunction. Therefore, we must have $\sin(\beta) = 0$, which can only occur if $\beta = (2n+1)\pi$. Therefore, this case gives us eigenvalues

$$\lambda_n = (2n+1)^2\pi^2 \quad n = 1, 2, 3...$$

with eigenfunctions

$$X_n(x) = \sin((2n+1)\pi \ln x)$$

Case 2: Zero eigenvalues, $\lambda = 0$ $X'' = 0$ implies that $X(x)$ is of the form $Ax + B$. Now plugging in our initial condition $X(1) = 0$, we get $A + B = 0$, which means $X(x) = Ax - A$. But from the boundary condition $X(e) = 0$, we get $A(e-1) = 0$ which is only possible if $A = 0$ and accordingly $X(0) = 0$. Therefore, there are no eigenfunctions $X(x)$ that satisfy the boundary conditions when $X'' = 0$ and hence no zero eigenvalues.

Case 3: Negative eigenvalues, $\lambda = -\beta^2 < 0$ This case gives us the characteristic equation: $r^2 - \beta^2 = 0$. Since our characteristic equation has distinct real roots $r = \pm\beta$, our solutions take the form

$$X(x) = Ax^{\beta} + Bx^{-\beta}$$

$$X(1) = A + B = 0 \quad \longrightarrow \quad A = -B$$
$$X(x) = Ax^{\beta} - Ax^{-\beta}$$

Then the other boundary condition gives $X(e) = Ae^{\beta} - Ae^{-\beta} = 0$. Since e^{β} and $e^{-\beta}$ are always nonzero, we must have $A = 0$, implying that again $X(x) = 0$.Thus, this problem has only positive eigenvalues and their associated eigenfunctions as found in Case 1.

(c)

On the interval $0 \le x \le 1$ of length one, consider the eigenvalue problem

$$-X'' = \lambda X, \quad X'(0) + X(0) = 0, \quad X(1) = 0$$

(i) Find an eigenfunction with eigenvalue zero. Call it $X_0(x)$.

$X'' = 0$ implies that $X(x)$ is of the form $Ax + B$ and $X'(x) = A$. Then $X(1) = 0$ becomes $A + B = 0$, or $A = -B$. Now we look at our second condition,

$$X'(0) + X(0) = 0$$

$X'(0)$ is just A and $X(0)$ is just B, so again this gives us $A = -B$. Thus, we have $X(x) = Ax - A = A(x-1)$, so we have found

$$X_0(x) = x - 1$$

(ii) Find an equation for the positive eigenvalues $\lambda = \beta^2$.

$$X(x) = A\sin(\beta x) + B\cos(\beta x)$$
$$X'(x) = \beta A\cos(\beta x) - \beta B\sin(\beta x)$$

$$X(1) = A\sin(\beta) + B\cos(\beta) = 0$$
$$A\sin(\beta) = -B\cos(\beta)$$

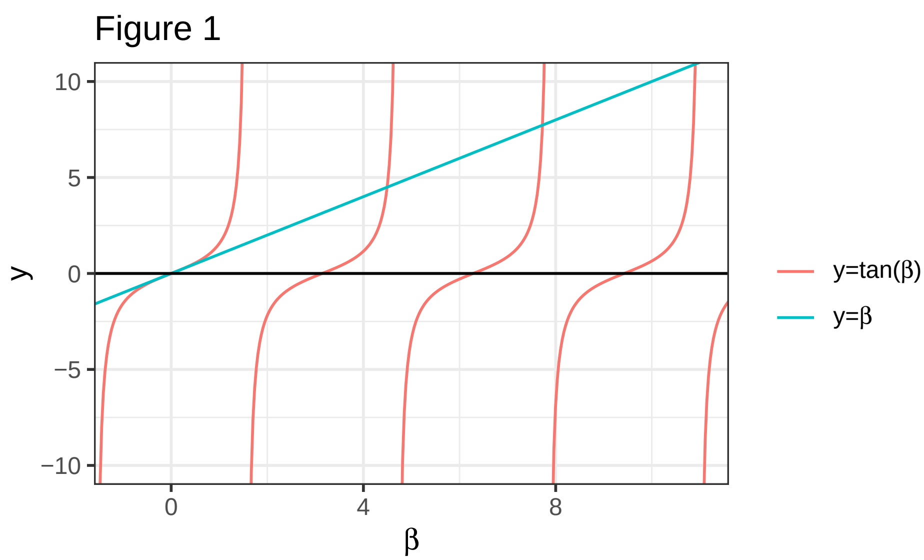
$$-\frac{B}{A} = \frac{\sin(\beta)}{\cos(\beta)} = \tan\beta \tag{1}$$

$$X'(0) = \beta A, \quad X(0) = B$$
$$X'(0) + X(0) = \beta A + B = 0$$
$$\beta = -\frac{B}{A} \tag{2}$$

Combining (1) and (2), we get the equation $\beta = \tan(\beta)$ for the positive eigenvalues.

(iii) Show graphically from part (b) that there are an infinite number of positive eigenvalues.

In part (b), we showed that β is a positive eigenvalue if $\beta = \tan(\beta)$. However, plotting the equations $y = \beta$ and $y = \tan\beta$ reveals that these two functions intersect an infinite number of times (Fig. 1), meaning there are an infinite number of positive eigenvalues.



(iv) Is there a negative eigenvalue?

This case gives us the characteristic equation: $r^2 - \beta^2 = 0$. Since our characteristic equation has distinct real roots $r = \pm\beta$, our solutions take the form

$$X(x) = Ae^{\beta x} + Be^{-\beta x}$$

Differentiating, we find that

$$X'(x) = \beta Ae^{\beta x} - \beta Be^{-\beta x}$$

$$X'(0) = \beta A - \beta B$$
$$X(0) = A + B$$

$$X'(0) + X(0) = \beta A - \beta B + A + B$$
$$0 = A(\beta + 1) + B(1 - \beta)$$
$$-A(\beta + 1) = B(1 - \beta)$$
$$\frac{\beta + 1}{1 - \beta} = -\frac{B}{A}$$

Now turning to our other boundary condition,

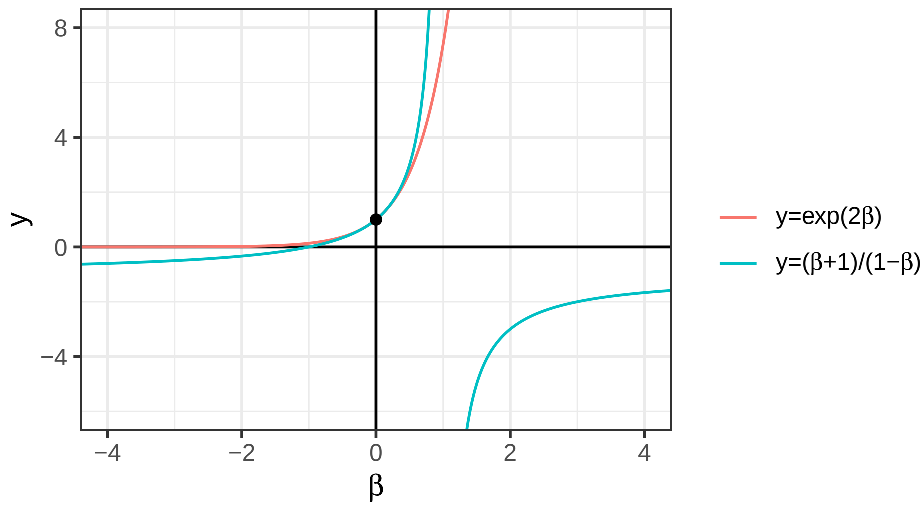
$$X'(1) = \beta Ae^{\beta} - \beta Be^{-\beta} = 0$$
$$\beta Ae^{\beta} = \beta Be^{-\beta}$$
$$-\frac{B}{A} = -e^{2\beta}$$

This would give us the following equation:

$$\frac{\beta + 1}{1 - \beta} = -e^{2\beta}$$

But the only time that this equality holds true is when $\beta = 0$ (Fig. 2), but we are in the case when we defined β to be nonzero, meaning that we have reached a contradiction and we cannot have any negative eigenvalues.

Figure 2



Question 2

Find the Fourier series of $f(x)$. Does the Fourier-series converge (i) pointwise, or (ii) uniformly?

(a)

$$f(x) = \begin{cases} 1 - |x| & |x| \leq 1 \\ 1 & 1 < |x| \leq \pi \end{cases}$$

The Fourier series for a function $f(x)$ on an interval $-l < x < l$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right)$$

where

$$a_0 = \frac{1}{2l} \int_{-l}^{l} f(x) dx$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

Let's start with a_0:

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \\
&= \frac{1}{2\pi}\left[\int_{-\pi}^{-1} 1\ dx + \int_{-1}^{1}(1-|x|)dx + \int_{1}^{\pi} 1\ dx\right] \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} 1\ dx - \int_{-1}^{1}|x|\ dx \\
&= 1 - 2\int_{0}^{1} x\ dx \\
&= 1 - 1 = 0
\end{aligned}$$

where we used the evenness of $|x|$ to rewrite $\int_{-1}^{1}|x|\,dx$ as $2\int_0^1 x\ dx$. Thus, $a_0 = 0$. Now we perform a similar procedure to find a_n:

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{-1}\cos(nx)\ dx + \int_{-1}^{1}(1-|x|)\cos(nx)\ dx + \int_{1}^{\pi}\cos(nx)\ dx\right] \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(nx)\ dx - \frac{1}{\pi}\int_{-1}^{1}|x|\cos(nx)\ dx
\end{aligned}$$

Since $\sin(\pi n)$ and $\sin(-\pi n)$ are zero for any integer n, the first integral vanishes. We use the symmetry of $|x|\cos(nx)$ (both functions are even, and the product of two even functions is even) and rewrite the second integral as

$$-\frac{2}{\pi}\int_0^1 x\cos(nx)\ dx$$

We then use integration by parts:

$$\begin{aligned}
&-\frac{2}{\pi}\int_0^1 x\cos(nx)\ dx \\
&= -\frac{2}{n\pi}\left[x\sin(nx)\Big|_0^1 - \int_0^1\sin(nx)\ dx\right] \\
&= -\frac{2}{n\pi}\left[\sin(n) + \frac{1}{n}\cos(nx)\Big|_0^1\right] \\
&= -\frac{2}{n\pi}\left[\sin(n) + \frac{1}{n}\cos(n) - \frac{1}{n}\right]
\end{aligned}$$

$$a_n = -\frac{2\sin n}{n\pi} + \frac{2(\cos n - 1)}{n^2\pi}$$

$$\begin{aligned}
b_n &= \frac{1}{\pi}\left[\int_{-\pi}^{-1}\sin(nx)\ dx + \int_{-1}^{1}(1-|x|)\sin(nx)\ dx + \int_{1}^{\pi}\sin(nx)\ dx\right] \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\sin(nx)\ dx - \frac{1}{\pi}\int_{-1}^{1}|x|\sin(nx)\ dx
\end{aligned}$$

But $\sin x$ is an odd function and so is $|x|\sin(nx)$, so both of these integrals vanish and we find $b_n = 0$.

Now inserting the coefficients a_n we found above, we get the Fourier series:

$$f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\frac{\sin n}{n} + \frac{\cos n + 1}{n^2} \right) \cos{(nx)}$$

(b)

$$f(x) = |x| = \begin{cases} -x & -\pi \leq x \leq 0 \\ x & 0 < x \leq \pi \end{cases}$$

Let's start with a_0:

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| \, dx \\ &= \frac{1}{\pi} \int_{0}^{\pi} x \; dx \\ &= \frac{1}{2\pi} x^2 \Big|_0^{\pi} = \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos{(nx)} dx \\ &= \frac{2}{\pi} \int_{0}^{\pi} x \cos{(nx)} dx \\ &= \frac{2}{n\pi} x \sin{(nx)} \Big|_0^{\pi} - \frac{2}{n\pi} \int_0^{\pi} \sin{(nx)} dx \\ &= 0 + \frac{2}{n^2\pi} \cos{(nx)} \Big|_0^{\pi} \\ &= \frac{2}{n^2\pi} (-1)^{n+1} \end{aligned}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin{(nx)} dx$$

But since $|x|$ is even and $\sin{(nx)}$ is odd, the product of the two functions is odd and so the integral vanishes over the symmetric interval.

Then we plug in our formulas for a_0 and a_n:

$$f(x) = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \left[\frac{1}{n^2} (-1)^{n+1} \cos{(nx)} \right]$$

(c)

$$f(x) = x + x^2, \quad -\pi \leq x \leq \pi$$

$$
\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x + x^2 dx \\
&= \frac{1}{2\pi}\left(\frac{x^2}{2} + \frac{x^3}{3}\right)\Big|_{-\pi}^{\pi} \\
&= \frac{1}{2\pi}\left(\frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} + \frac{\pi^3}{3}\right) \\
&= \frac{1}{2\pi}\left(\frac{2\pi^3}{3}\right) = \frac{\pi^2}{3}
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx) dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx) + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx) dx
\end{aligned}
$$

Applying integration by parts to the first integral,

$$
\begin{aligned}
\frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx) &= \frac{1}{n\pi} x\sin(nx)\Big|_{-\pi}^{\pi} + \frac{1}{n\pi}\int_{-\pi}^{\pi}\sin(nx) \\
&= \frac{1}{n\pi} x\sin(nx)\Big|_{-\pi}^{\pi} + \frac{-1}{n^2\pi}\cos(nx)\Big|_{-\pi}^{\pi} = 0 + 0 = 0
\end{aligned}
$$

We then use the integration by parts table method demonstrated in class to find that

$$
\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx) = \frac{x^2}{n}\sin(nx)\Big|_{-\pi}^{\pi} + \frac{2x}{n^2}\cos(nx)\Big|_{-\pi}^{\pi} + \frac{2}{n^3}\sin(nx)\Big|_{-\pi}^{\pi}
$$

The first and last terms (with $\sin(nx)$) vanish, while the middle term evaluates to $\frac{4\cos(n\pi)}{n^2}$.

$$
\frac{4\cos(n\pi)}{n^2} = \frac{4}{n^2}(-1)^n
$$

Thus, we add this to the result of first integral to find that

$$
a_n = \frac{4}{n^2}(-1)^n
$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx) dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx) + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin(nx) dx
\end{aligned}
$$

But since x^2 is even and $\sin$ is odd, the product under the second integral is odd functions and thus evaluates to zero over the symmetric interval $-\pi < x < \pi$. Applying integration by parts to the first integral,

$$
b_n = \frac{-1}{n\pi} x\cos(nx)\Big|_{-\pi}^{\pi} + \frac{1}{n\pi}\int_{-\pi}^{\pi}\cos(nx)
$$

But since cos is even and $\cos\pi = \cos -\pi$, both terms are zero. Thus, $b_n = 0$ and

$$
f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)
$$

$$
f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n\cos(nx)}{n^2}
$$

Question 3

a)

Find the Fourier sine series of

$$f(x) = \begin{cases} 1 & 0 < x < \pi/2 \\ 2 & \pi/2 < x < \pi \end{cases}$$

The Fourier sine series of a function $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right)$$

where

$$A_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$
$$A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx$$

$$\begin{aligned} A_n &= \frac{2}{\pi}\left[\int_0^{\pi/2}\sin(nx)dx + 2\int_{\pi/2}^{\pi}\sin(nx)dx\right] \\ &= \frac{2}{\pi}\left[\frac{-1}{n}\cos(nx)\Big|_0^{\pi/2} - \frac{2}{n}\cos(nx)\Big|_{\pi/2}^{\pi}\right] \\ &= \frac{2}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right) + 1 - 2\cos(n\pi)\right] \end{aligned}$$

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[1 + \cos\left(\frac{n\pi}{2}\right) - 2\cos(n\pi)\right]\sin(nx)$$

b)

Find the Fourier-cosine-series of $f(x) = |\sin x|$. Then find

$$\sum_{k=1}^{\infty}\frac{1}{4k^2-1}$$

The Fourier cosine series of a function $f(x)$ is

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi x}{l}\right)$$

where

$$A_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

Assuming $l = \pi$, we have

$$A_n = \frac{2}{\pi}\int_0^{\pi}|\sin x|\cos(nx)dx \tag{3}$$

But since sin is non-negative on the interval $0 \le x \le \pi$, we can drop the absolute value bars and use the following trigonometric identity to rewrite the integral: $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$

$$
\begin{aligned}
A_n &= \frac{1}{\pi}\int_0^{\pi} \sin((1+n)x) + \sin((1-n)x)dx \\
&= \frac{1}{\pi}\left[\frac{-1}{1+n}\cos((1+n)x)\Big|_0^{\pi} - \frac{1}{1-n}\cos((1-n)x)\Big|_0^{\pi}\right] \\
&= \frac{1}{\pi}\left[\frac{-1}{1+n}[(-1)^{1+n}-1] - \frac{1}{1-n}[(-1)^{1-n}-1]\right] \\
&= \frac{1}{\pi}\left[(-1)^{1+n}-1)\right]\left(\frac{-1}{1+n} - \frac{1}{1-n}\right)
\end{aligned}
$$

Thus, we have found a formula for A_n,

$$
A_n = \frac{-2}{\pi}\frac{(-1)^n+1}{n^2-1}
$$

This formula works for all $n \neq 1$ For $n = 0$,

$$
A_0 = \frac{-2}{\pi}\frac{(-1)^0+1}{0-1} = \frac{4}{\pi}
$$

For $n = 1$, we can go back to (3) and plug in the appropriate value of n:

$$
A_1 = \frac{1}{\pi}\int_0^{\pi} \sin(2x)\ dx = \frac{1}{2\pi}\cos(2x)\Big|_0^{\pi}\ dx = 0
$$

The Fourier cosine series is therefore

$$
f(x) = \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n+1}{n^2-1}\cos(nx)
$$

Now, $(-1)^n + 1$ will be zero for all odd n and two for all even n. So substituting $n = 2k$, we gwt

$$
|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{2k=2}^{\infty}\frac{1}{4k^2-1}\cos(2kx)
$$

Starting the index at $2k = 2$ is equivalent to starting it at $k = 1$, so after making that change and evaluating both sides at $x = 0$, we can solve for the sum given in the question:

$$
0 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}
$$

$$
\sum_{k=1}^{\infty}\frac{1}{4k^2-1} = \frac{1}{2}
$$

c)

The Riemann Zeta function is defined for $s > 1$ by

$$
\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}
$$

By computing the Fourier series of x^2 over $-\pi < x < \pi$ and using Parseval's identity, compute $\zeta(4)$.

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx \\ &= \frac{1}{2\pi}\left(\frac{x^3}{3}\right)\Big|_{-\pi}^{\pi} \\ &= \frac{1}{2\pi}\left(\frac{\pi^3}{3} - \frac{-\pi^3}{3}\right) \\ &= \frac{1}{2\pi}\left(\frac{2\pi^3}{3}\right) = \frac{\pi^2}{3} \end{aligned}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)dx$$

But in part (2b), we found that this evaluates to zero.