BIOL 274 Homework 6

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Question 1

Write each of the following systems in matrix/vector notation:

(a)

$$\begin{aligned} x' &= 4x - 6y \\ y' &= 0.7x + y \end{aligned}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ 0.7 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{or}$$

$$\dot{X} = \begin{bmatrix} 4 & -6 \\ 0.7 & 1 \end{bmatrix} X$$

(b)

$$\begin{aligned} \frac{dx_1}{dt} - 16x_2 &= -2x_1 \\ \frac{dx_2}{dt} + 4x_1 &= 0 \end{aligned}$$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -2 & 16 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \text{or}$$

$$\dot{X} = \begin{bmatrix} -2 & 16 \\ -4 & 0 \end{bmatrix} X$$

(c)

$$\begin{aligned} \dot{v} &= v - u + \sin t \\ \dot{u} &= 2v + u - \cos t \end{aligned}$$

$$\begin{bmatrix} \dot{v} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix} + \begin{bmatrix} \sin t \\ -\cos t \end{bmatrix}, \quad \text{or}$$

$$\dot{V} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} V + \begin{bmatrix} \sin t \\ -\cos t \end{bmatrix}$$

Question 2

Find the general solution for each of the following systems:

(a)

$$\dot{x} = 9x - 8y$$
$$\dot{y} = 24x - 19y$$

$$A = \begin{bmatrix} 9 & -8 \\ 24 & -19 \end{bmatrix}$$

$$\det \begin{bmatrix} 9-\lambda & -8 \\ 24 & -19\lambda \end{bmatrix} = 0$$

$$(9-\lambda)(-19-\lambda) + 192 = 0$$
$$-171 - 9\lambda + 19\lambda + \lambda^2 + 192 = \lambda^2 + 10\lambda + 21 = (\lambda+7)(\lambda+3)$$

So we have found two distinct real eigenvalues, $\lambda_1 = -7,\ \lambda_2 = -3$. Now we need to solve $AX = \lambda X$, or $(A - \lambda I)X = 0$ with these two values of λ. Let's start with the first eigenvalue, -7.

$$A + 7I = \begin{bmatrix} 9+7 & -8 \\ 24 & -19+7 \end{bmatrix} = \begin{bmatrix} 16 & -8 \\ 24 & -12 \end{bmatrix}$$

Now we use Gauss-Jordan elimination:

$$\begin{bmatrix} 16 & -8 \\ 24 & -12 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}$$

$$-2x_1 + x_2 = 0 \quad \rightarrow \quad x_2 = 2x_1 \quad \rightarrow \quad v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

And repeat for the second eigenvalue, -3:

$$A + 3I = \begin{bmatrix} 9+3 & -8 \\ 24 & -19+3 \end{bmatrix} = \begin{bmatrix} 12 & -8 \\ 24 & -16 \end{bmatrix}$$

$$\begin{bmatrix} 12 & -8 \\ 24 & -16 \end{bmatrix} \longrightarrow \begin{bmatrix} -3 & 2 \\ 0 & 0 \end{bmatrix}$$

$$-3x_1 + 2x_2 = 0 \quad \rightarrow \quad 2x_2 = 3x_1 \quad \rightarrow \quad v_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

So we have found $\lambda_1 = -7$ with eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\lambda_2 = -3$ with eigenvector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. We can plug these values into our formula for the general solution of a system of linear first-order ODEs with constant coefficients:

$$X = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2$$

Solution

$$X = C_1 e^{-7t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{-3t} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

We can differentiate our solution and check that it satisfies the original system of differential equations:

Check solution

$$X' = C_1 \begin{bmatrix} -7e^{-7t} \\ -14e^{-7t} \end{bmatrix} + C_2 \begin{bmatrix} -6e^{-3t} \\ -9e^{-3t} \end{bmatrix}$$

$$x' = -7C_1e^{-7t} - 6C_2e^{-3t} = 9(C_1e^{-7t} + 2C_2e^{-3t}) - 8(2C_1e^{-7t} + 3C_2e^{-3t}) = 9x - 8y$$
$$y' = -14C_1e^{-7t} - 9C_2e^{-3t} = 24(C_1e^{-7t} + 2C_2e^{-3t}) - 19(2C_1e^{-7t} + 3C_2e^{-3t}) = 24x - 19y$$

(b)

$$\dot{x} = 2y - x$$
$$\dot{y} = -2x - y$$

$$B = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}$$

$$\det \begin{bmatrix} -1-\lambda & 2 \\ -2 & -1-\lambda \end{bmatrix} = 0$$

$$(-1-\lambda)(-1-\lambda) + 4 = 0$$
$$\lambda^2 + 2\lambda + 5 = 0$$
$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \longrightarrow \quad \frac{-2 \pm \sqrt{4 - 4(5)}}{2} = -1 \pm 2i$$

So we have complex eigenvalues, $\lambda_1 = -1 + 2i$, $\lambda_2 = -1 - 2i$. Now we need to solve $AX = \lambda X$, or $(A - \lambda I)X = 0$ with these two values of λ.

$$B - (-1+2i)I = \begin{bmatrix} -1+1-2i & 2 \\ -2 & -1+1-i \end{bmatrix} = \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} i & -1 \\ 0 & 0 \end{bmatrix}$$

$$ix_1 - x_2 = 0 \quad \rightarrow \quad ix_1 = x_2 \quad \rightarrow \quad v_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Which means the eigenvector for the other eigenvalue is the conjugate of this vector, $v_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$.

Now we use Euler's formula:

$$e^{(-1+2i)t} \begin{bmatrix} -i \\ 1 \end{bmatrix} = e^{-t}e^{2it} \begin{bmatrix} -i \\ 1 \end{bmatrix} = e^{-t} \left(\cos(2t) + i \sin(2t)\right) \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$e^{-t} \begin{bmatrix} -i\cos(2t) + \sin(2t) \\ \cos(2t) + i\sin(2t) \end{bmatrix} = e^{-t} \left(\begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix} + i \begin{bmatrix} -\cos(2t) \\ \sin(2t) \end{bmatrix} \right)$$

Having found our real and imaginary parts, we can now write our general solution as a linear combination of them:

Solution

$$X = C_1 e^{-t} \begin{bmatrix} \sin 2t \\ \cos 2t \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} -\cos 2t \\ \sin 2t \end{bmatrix}$$

Check solution

$$\begin{aligned} x &= C_1 e^{-t} \sin 2t - C_2 e^{-t} \cos 2t \\ y &= C_1 e^{-t} \cos 2t + C_2 e^{-t} \sin 2t \end{aligned}$$

$$X' = 2C_1 e^{-t} \begin{bmatrix} \cos 2t \\ -\sin 2t \end{bmatrix} - C_1 e^{-t} \begin{bmatrix} \sin 2t \\ \cos 2t \end{bmatrix} + 2C_2 e^{-t} \begin{bmatrix} \sin 2t \\ \cos 2t \end{bmatrix} - C_2 e^{-t} \begin{bmatrix} -\cos 2t \\ \sin 2t \end{bmatrix}$$

$$\begin{aligned} \dot{x} &= 2C_1 e^{-t} \cos 2t - C_1 e^{-t} \sin 2t + 2C_2 e^{-t} \sin 2t + C_2 e^{-t} \cos 2t \\ &= 2\left(C_1 e^{-t} \cos 2t + C_2 e^{-t} \sin 2t\right) - \left(C_1 e^{-t} \sin 2t - C_2 e^{-t} \cos 2t\right) \\ &= 2y - x \end{aligned}$$

$$\begin{aligned} \dot{y} &= -2C_1 e^{-t} \sin 2t - C_1 e^{-t} \cos 2t + 2C_2 e^{-t} \cos 2t - C_2 e^{-t} \sin 2t \\ &= -2\left(C_1 e^{-t} \sin 2t - C_2 e^{-t} \cos 2t\right) - \left(C_1 e^{-t} \cos 2t + C_2 e^{-t} \sin 2t\right) \\ &= -2x - y \end{aligned}$$

(c)

$$\begin{aligned} \dot{x} &= x + y \\ \dot{y} &= y \end{aligned}$$

$$C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\det \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)(1-\lambda) = 0$$

So we have a repeated eigenvalue, $\lambda_1 = \lambda_2 = 1$.

If the eigenvalues of A are repeated (i.e. $\lambda_1 = \lambda_2$), then the general solution to the system $\dot{X} = AX$ is $X = e^{\lambda t} v_0 + t e^{\lambda t} v_1$, where $v_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ and $v_1 = (A - \lambda I) v_0$.

$$
\begin{aligned}
v_1 &= (A - I)v_0 \\
&= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \\
&= \begin{bmatrix} y_0 \\ 0 \end{bmatrix}
\end{aligned}
$$

Which means the general solution is:

Solution

$$X = e^t \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + te^t \begin{bmatrix} y_0 \\ 0 \end{bmatrix}$$

Check solution

$$x = e^t x_0 + te^t y_0, \quad y = e^t y_0$$

$$\dot{x} = e^t x_0 + te^t y_0 + e^t y_0 = x + y$$

$$\dot{y} = e^t y_0 = y$$

(d)

$$
\begin{aligned}
\dot{x} &= 7y - 2x \\
\dot{y} &= 3y
\end{aligned}
$$

$$D = \begin{bmatrix} -2 & 7 \\ 0 & 3 \end{bmatrix}$$

$$\det \begin{bmatrix} -2 - \lambda & 7 \\ 0 & 3 - \lambda \end{bmatrix} = 0$$

$$(-2 - \lambda)(3 - \lambda) = 0$$

So we have found two distinct real eigenvalues, $\lambda_1 = -2$, $\lambda_2 = 3$. Now we need to solve $DX = \lambda X$, or $(D - \lambda I)X = 0$ with these two values of λ. Let's start with the first eigenvalue, -2.

$$D + 2I = \begin{bmatrix} -2 + 2 & 7 \\ 0 & 3 + 2 \end{bmatrix} = \begin{bmatrix} 0 & 7 \\ 0 & 5 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$0x_1 + x_2 = 0 \quad \rightarrow \quad x_2 = 0 \quad \rightarrow \quad v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

And repeat for the second eigenvalue, 3:

$$D - 3I = \begin{bmatrix} -2 - 3 & 7 \\ 0 & 3 - 3 \end{bmatrix} = \begin{bmatrix} -5 & 7 \\ 0 & 0 \end{bmatrix}$$

$$-5x_1 + 7x_2 = 0 \quad \rightarrow \quad 7x_2 = 5x_1 \quad \rightarrow \quad v_2 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

So we have found $\lambda_1 = -2$ with eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\lambda_2 = 3$ with eigenvector $\begin{bmatrix} 7 \\ 5 \end{bmatrix}$. We can plug these values into our formula for the general solution of a system of linear first-order ODEs with constant coefficients:

Solution

$$X = C_1 e^{-2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

Check solution

$$\begin{aligned} x &= C_1 e^{-2t} + 7C_2 e^{3t} \\ y &= 5C_2 e^{3t} \end{aligned}$$

$$\begin{aligned} \dot{x} &= -2C_1 e^{-2t} + 21C_2 e^{3t} \\ &= 7\left(5C_2 e^{3t}\right) - 2\left(C_1 e^{-2t} + 7C_2 e^{3t}\right) \\ &= 7y - 2x \end{aligned}$$

$$\dot{y} = 15C_2 e^{3t} = 3y$$

(e)

$$\begin{aligned} \dot{x} &= x - 2y \\ \dot{y} &= 3x - 6y \end{aligned}$$

$$E = \begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix}$$

$$\det \begin{bmatrix} 1-\lambda & -2 \\ 3 & -6-\lambda \end{bmatrix} = 0$$

$$\begin{gathered} (1-\lambda)(-6-\lambda) + 6 = 0 \\ -6 - \lambda + 6\lambda + \lambda^2 + 6 = 0 \\ \lambda^2 + 5\lambda = 0 \end{gathered}$$

Or $\lambda(\lambda + 5) = 0$, which means $\lambda_1 = 0$ and $\lambda_2 = -5$.

$$E - 0I = \begin{bmatrix} 1-0 & -2 \\ 3 & -6-0 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

$$x_1 - 2x_2 = 0 \quad \rightarrow \quad 2x_2 = x_1 \quad \rightarrow \quad v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$E+5I=\begin{bmatrix}1+5 & -2\\ 3 & -6+5\end{bmatrix}=\begin{bmatrix}6 & -2\\ 3 & -1\end{bmatrix}$$

$$\begin{bmatrix}3 & -1\\ 0 & 0\end{bmatrix}$$

$$3x_1-x_2=0 \quad\rightarrow\quad x_2=3x_1 \quad\rightarrow\quad v_2=\begin{bmatrix}1\\ 3\end{bmatrix}$$

So we have found $\lambda_1=0$ with eigenvector $\begin{bmatrix}2\\ 1\end{bmatrix}$ and $\lambda_2=-5$ with eigenvector $\begin{bmatrix}1\\ 3\end{bmatrix}$. We can plug these values into our formula for the general solution of a system of linear first-order ODEs with constant coefficients:

Solution

$$X=C_1\begin{bmatrix}2\\ 1\end{bmatrix}+C_2e^{-5t}\begin{bmatrix}1\\ 3\end{bmatrix}$$

Check solution

$$x=2C_1+C_2e^{-5t}$$
$$y=C_1+3C_2e^{-5t}$$
$$\dot{x}=-5C_2e^{-5t}=\left(2C_1+C_2e^{-5t}\right)-2\left(C_1+3C_2e^{-5t}\right)=x-2y$$
$$\dot{y}=-15C_2e^{-5t}=3\left(2C_1+C_2e^{-5t}\right)-6\left(C_1+3C_2e^{-5t}\right)=3x-6y$$