MATH 245 Homework 1

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2024-01-29

Question 1

Show the function is a solution of the PDE:
(a) $u_{xx} + u_{yy} = 0$
(i) $u(x, y) = e^x \sin(y)$

$$u_x = e^x \sin(y),\ u_{xx} = e^x \sin(y),\ u_y = e^x \cos(y),\ u_{yy} = -e^x \sin(y)$$

Which means that $u_{xx} + u_{yy} = e^x \sin(y) + (-e^x \sin(y)) = 0$ and so $u(x, y) = e^x \sin(y)$ is a solution to the PDE.
(ii) $u(x, y) = \log \sqrt{x^2 + y^2}$
Assuming this is meant to be $u(x, y) = \ln \sqrt{x^2 + y^2}$,

$$u_x = \left(x^2 + y^2\right)^{-1/2} \left(\frac{1}{2}\right) \left(x^2 + y^2\right)^{-1/2} (2x) = x \left(x^2 + y^2\right)^{-1}$$

$$u_{xx} = x(-1) \left(x^2 + y^2\right)^{-2} (2x) + \left(x^2 + y^2\right)^{-1}$$

$$u_y = \left(x^2 + y^2\right)^{-1/2} \left(\frac{1}{2}\right) \left(x^2 + y^2\right)^{-1/2} (2y) = y \left(x^2 + y^2\right)^{-1}$$

$$u_{yy} = y(-1) \left(x^2 + y^2\right)^{-2} (2y) + \left(x^2 + y^2\right)^{-1}$$

$$u_{xx} + u_{yy} = \frac{-2x^2}{\left(x^2 + y^2\right)^2} + \frac{-2y^2}{\left(x^2 + y^2\right)^2} + \frac{2}{\left(x^2 + y^2\right)} =$$

$$\frac{-2\left(x^2 + y^2\right)}{\left(x^2 + y^2\right)^2} + \frac{2}{\left(x^2 + y^2\right)} = 0$$

Which means that $u(x, y) = \log \sqrt{x^2 + y^2}$ is a solution to the PDE.

(b) $bu_x + au_y + u = 0$, $u(x, y) = \exp\left(\frac{-x}{b}\right) f(ax - by)$ for arbitrary differentiable function f.

$$u_x = e^{\frac{-x}{b}} \left(\frac{-1}{b}\right) f(ax - by) + f'(ax - by)(a) e^{\frac{-x}{b}}$$

$$u_y = e^{\frac{-x}{b}} f'(ax - by)(-b)$$

Then,

$$bu_x + au_y + u = -e^{\frac{-x}{b}} f(ax - by) + (ab)\left(e^{\frac{-x}{b}}\right) f'(ax - by) + e^{\frac{-x}{b}} f'(ax - by)(-ab) + e^{\frac{-x}{b}} f(ax - by) = 0$$

So $u(x,y) = \exp\left(\frac{-x}{b}\right) f(ax - by)$ is a solution to the PDE for an arbitrary differentiable function f.

(c) $u_{xx} - \frac{1}{x}u_x - x^2 u_{yy} = 0$, $u(x,y) = f\left(2y + x^2\right) + g\left(2y - x^2\right)$ for arbitrary twice-differentiable functions f and g.

$$\begin{aligned}
u_x &= \left(f'\left(2y + x^2\right)\right)(2x) + \left(g'\left(2y - x^2\right)\right)(-2x)\\
u_y &= 2\left(f'\left(2y + x^2\right)\right) + 2\left(g'\left(2y - x^2\right)\right)\\
u_{xx} &= 2f' + (f'')(4x^2) - 2g' + (g'')(4x^2)\\
u_{yy} &= 4f'' + 4g''\\
u_{xx} - \frac{1}{x}u_x - x^2 u_{yy} &=\\
2f' + 4x^2 f'' - 2g' + 4x^2 g'' - \frac{1}{x}\left(2xf' - 2xg'\right) - x^2\left(4f'' + 4g''\right) &=\\
2f' - 2f' + 4x^2 f'' - 4x^2 f'' - 2g' + 2g' + 4x^2 g'' - 4x^2 g'' &= 0
\end{aligned}$$

So $u(x,y) = f\left(2y + x^2\right) + g\left(2y - x^2\right)$ is a solution to the PDE for arbitrary twice-differentiable functions f and g.

Question 2

(a) 2nd-order linear homogeneous

(b) 4th-order linear inhomogeneous

(c) 2nd-order quasi-linear homogeneous

(d) We can rewrite this as $u_{xx} + u_{yy} + f(x,y)u - g(x,y)u^5 = 0$, making it clear that this is a 2nd-order semi-linear homogeneous PDE.

Question 3

Use separation of variables to solve the following problems:

(a) $u_x + u = u_y$, $u(x,0) = 4x^{-3x}$, use $u(x,y) = f(x)g(y)$

(b) $x^2 u_{xy} + 9y^2 u = 0$, $u(x,0) = \exp\left(\frac{1}{x}\right)$, use $u(x,y) = f(x)g(y)$

(c) $u_x^2 + u_y^2 = 1$, use $u(x,y) = f(x) + g(y)$

(d) $x^2 u_x^2 + y^2 u_y^2 = u^2$, use $u(x,y) = e^{f(x)} e^{g(y)}$

Question 4

For each of the following IVPs, (i) find and plot the characteristic lines (curves), (ii) solve the IVP, and (iii) plot the solution of (a)-(b) for indicated time.

(a) $u_t + \left(1+x^2\right) u_x = 0$, $u(0,x) = \arctan(x)$, $t = 1,2,3$, and what is $\lim_{t\to\infty} u(t,x)$?

$$\frac{dx}{dt} = 1 + x^2$$
$$\frac{1}{1+x^2} dx = dt$$

$$\arctan(x) = t + C$$

So $x = \tan(t+C)$ are the characteristic curves of $u_t + \left(1+x^2\right) u_x = 0$.
On each of the curves, $u(x,t)$ is constant because

$$\frac{d}{dt} u(t, \tan(t+C)) = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \sec^2(t+c)$$

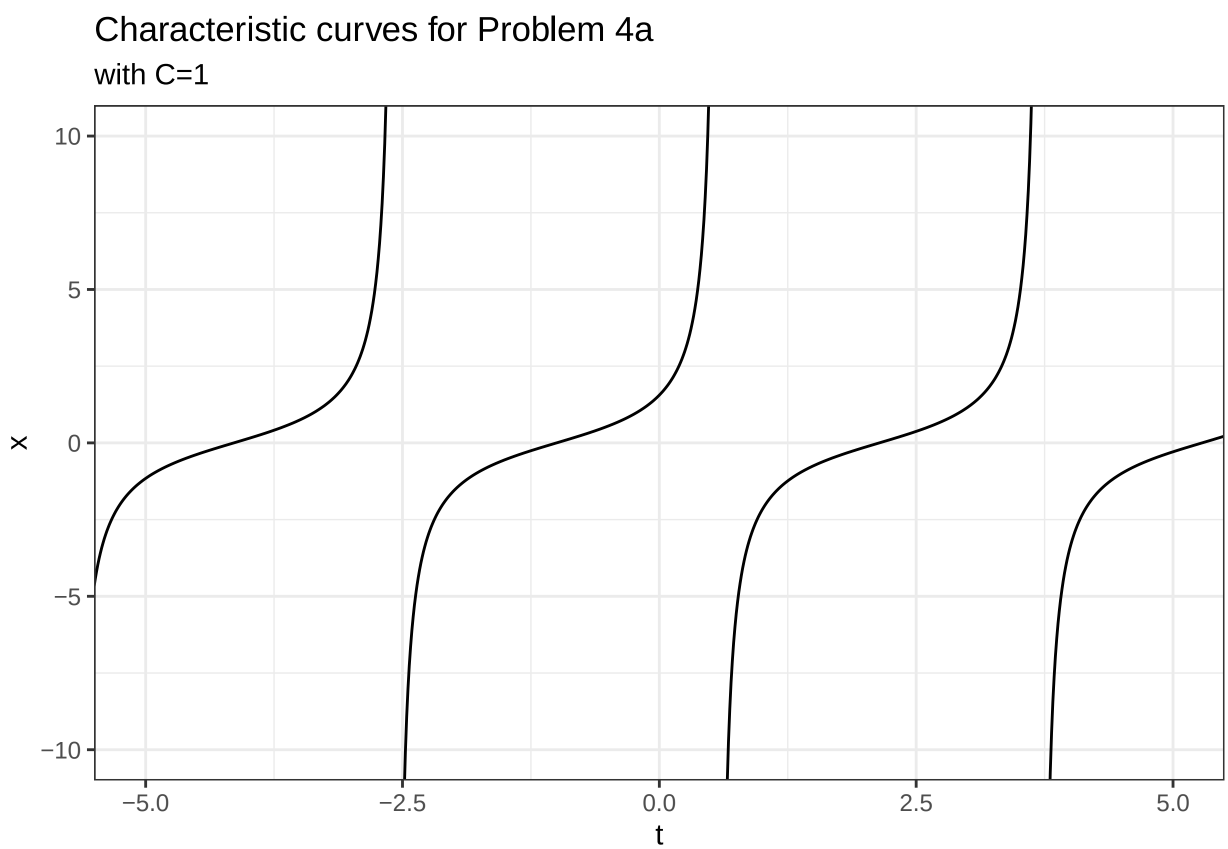
And since $1 + \tan^2\theta = \sec^2\theta$, this means $\frac{d}{dt} u(t, \tan(t+C)) = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\left(1 + \tan^2(t+c)\right) = u_t + \left(1+x^2\right) u_x$, which we know is 0.

Thus $u(t, \tan(t+C)) = u(0, \tan(0+C)) = u(0,C)$ is independent of t.
Putting $x = \tan(t+C)$ and $C = \arctan(x) - t$, we have

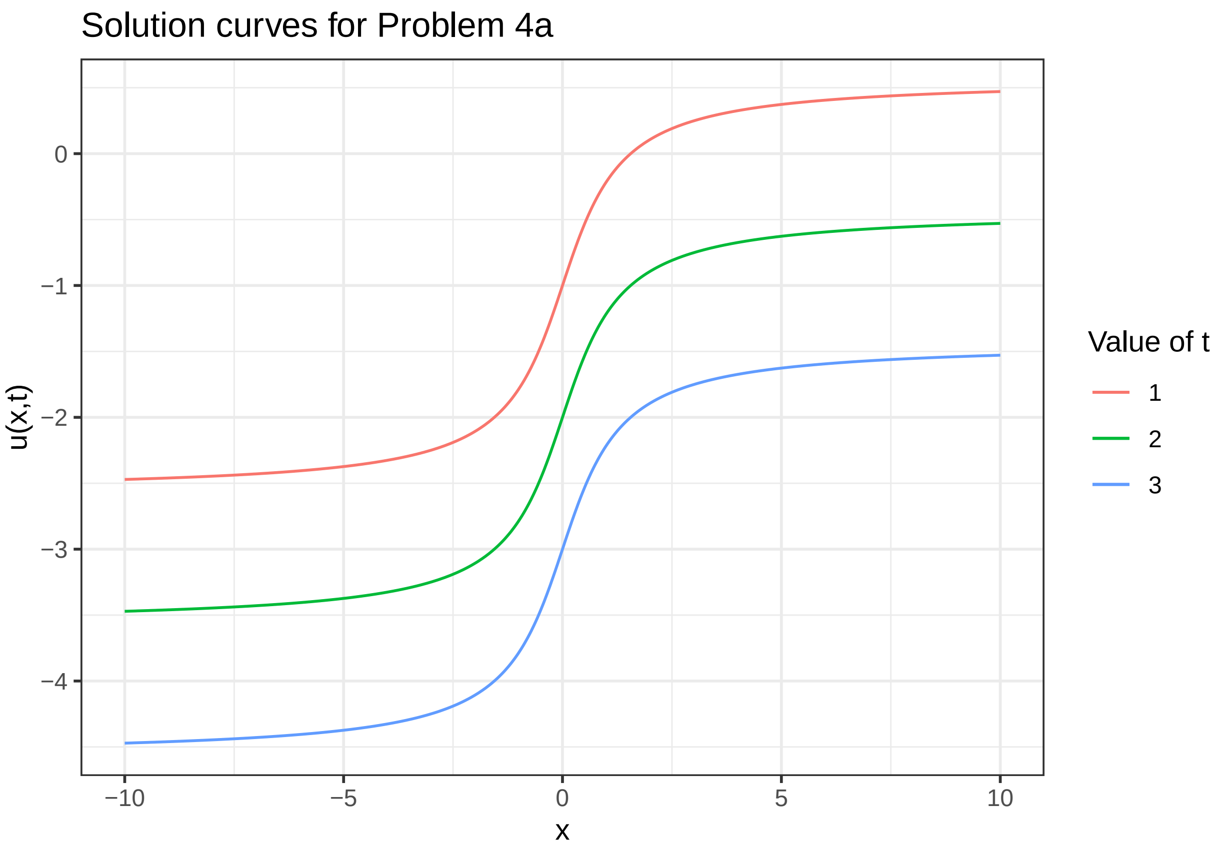
$$u(t,x) = u(0, \arctan(x) - t)$$

It follows that $u(t,x) = f(\arctan(x) - t)$.
And since we are given that $u(0,x) = \arctan(x)$, we have $f(\arctan(x) - 0) = \arctan(x)$ so that $f(w) = w$ for any w, yielding our solution of

$$u(t,x) = \arctan(x) - t$$



Since the range of arctan is limited to $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, as $\lim_{t\to\infty}$ that means $u(t, x) = \arctan(x) - t$ will go to $-\infty$.



We can check our solution of $u(t, x) = \arctan(x) - t$ by differentiating, since $u_t = -1, u_x = \frac{1}{1+x^2}$, that means $u_t + \left(1 + x^2\right) u_x = -1 + 1 = 0$.

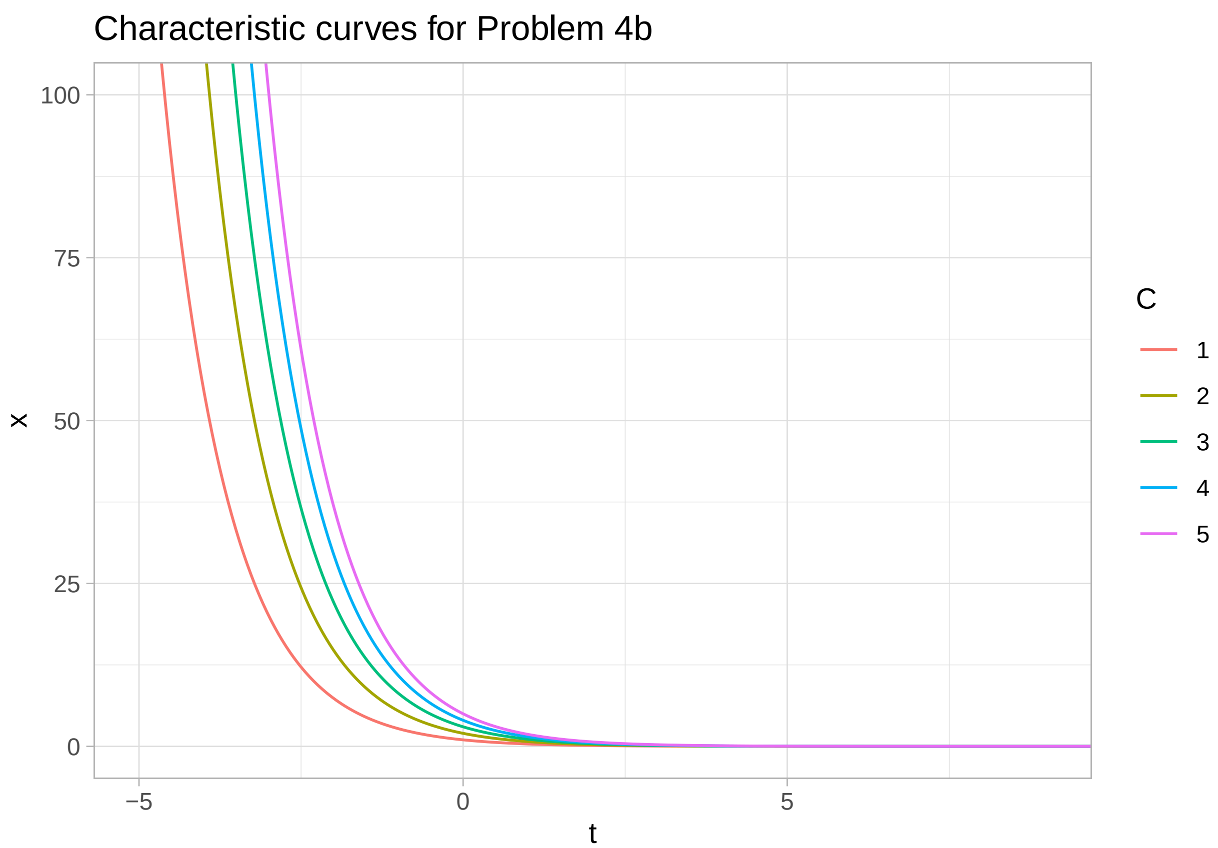
(b) $u_t - x u_x = 0$, $u(0, x) = \frac{1}{1+x^2}$, $t = 1, 2, 3$, and what is $\lim_{t\to\infty} u(t, x)$?

The directional derivative of u in the direction of the vector $(1, -x)$ is zero. The curves in the tx plane with $(1, -x)$ as a tangent vector have slopes $-x$:

$$\frac{dx}{dt} = -x$$
$$\frac{1}{x} dx = -dt$$

Solving this ODE gives the equations for the characteristic lines:

$$x = Ce^{-t}$$



On each of the curves, $u(x,t)$ is constant because

$$\frac{d}{dt}u(t, Ce^{-t}) = \frac{\partial u}{\partial t} - Ce^{-t}\frac{\partial u}{\partial x} = u_t - xu_x$$

and we know that $u_t - xu_x = 0$.

Set $\xi = xe^t$ and $\eta = x$. Then by the chain rule,

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} =$$

$$\frac{\partial u}{\partial \eta} + \frac{\partial u}{\partial \xi}e^t$$

and similarly

$$u_t = \frac{\partial u}{\partial t} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} = \frac{\partial u}{\partial \xi}xe^t$$

Which means that

$$u_t - xu_x = \frac{\partial u}{\partial \xi}xe^t - x\left(\frac{\partial u}{\partial \eta} + \frac{\partial u}{\partial \xi}e^t\right) = 0$$

Assuming $x \neq 0$, this means

$$u_\eta = 0$$

$$u = \int u_\eta \, d\eta \, + f(\xi) = 0 + f(\xi)$$

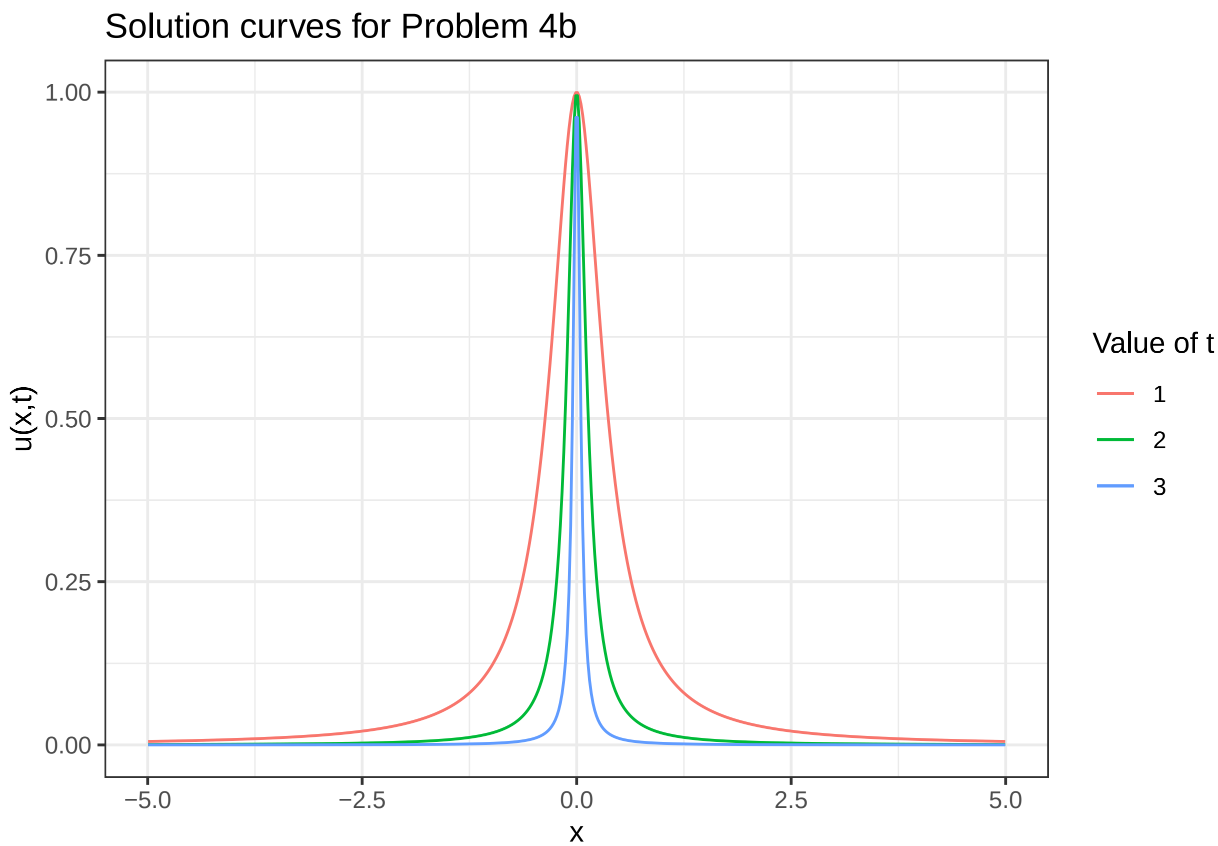
So we have $u(t,x) = f(\xi) = f(xe^t)$ and we know that $u(0,x) = \frac{1}{1+x^2}$, which means $f(xe^0) = f(x) = \frac{1}{1+x^2}$ and therefore $u(t,x) = \frac{1}{1+(xe^t)^2}$.

We check by differentiating:

$$u_t = -\left(1 + \left(xe^t\right)^2\right)^{-2} \left(2xe^t\right) xe^t$$

$$u_x = -\left(1 + \left(xe^t\right)^2\right)^{-2} \left(2xe^t\right) e^t$$

And we see that $u_t - xe_t$ indeed equals zero. Plotting the solution $u(t,x) = \frac{1}{1+(xe^t)^2}$ as a function of x for $t = 1, 2, 3$, we get:

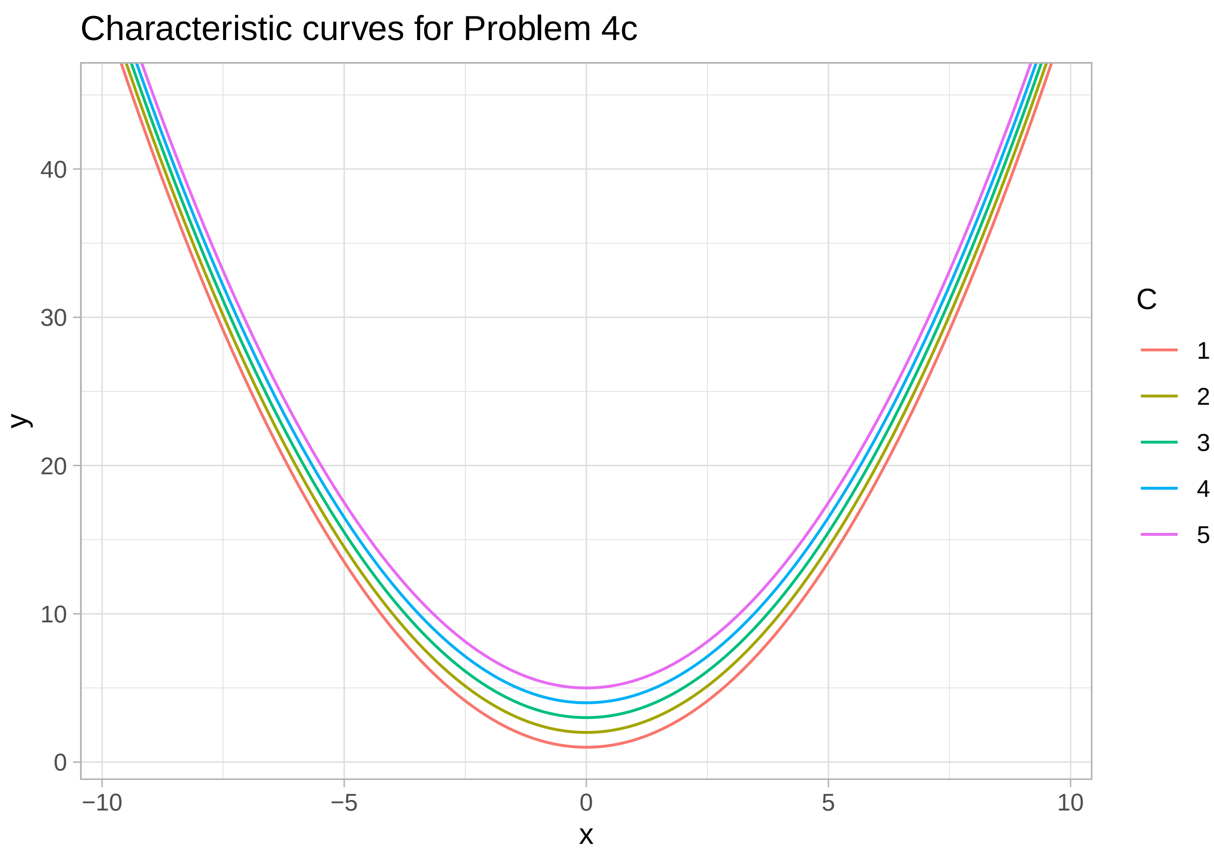


We can see that $u(t,x)$ will equal 1 at $x = 0$ regardless of the value of t, but at all other values of x, $\lim_{t\to\infty} u(t,x) = 0$. This can be written as

$$\lim_{t\to\infty} u(t,x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases} \tag{1}$$

(c) $u_x + xu_y = \left(y - \frac{x^2}{2}\right)^2$, $u(0,y) = e^y$

$\frac{dy}{dx} = x$ Solving this ODE yields the equation for the characteristic curves: $y = \frac{x^2}{2} + C$



Now we define our new coordinate system as $\xi = y - \frac{x^2}{2}$, $\eta = x$

Then by the chain rule,

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} =$$

$$\frac{\partial u}{\partial \eta} - x\frac{\partial u}{\partial \xi}$$

and similarly

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} =$$

$$\frac{\partial u}{\partial \xi}$$

$$u_x + xu_y = \frac{\partial u}{\partial \eta} - x\frac{\partial u}{\partial \xi} + x\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial \eta}$$

And since we know $u_x + xu_y = \left(y - \frac{x^2}{2}\right)^2$, this means we have

$$\frac{\partial u}{\partial \eta} = \left(y - \frac{x^2}{2}\right)^2 = \xi^2$$

Now we integrate with respect to η to find

$$u = \eta\xi^2 + F(\xi)$$

At $u(0,y)$, this becomes $u = 0y^2 + F(y)$, or $u(0,y) = F(y)$. On the other hand, we are told that $u(0,y) = e^y$, which means that $F(y) = e^y$.

$$u = \eta\xi^2 + e^\xi$$

Substitute back in x and y:

$$u(x,y) = x\left(y - \frac{x^2}{2}\right)^2 + \exp\left(y - \frac{x^2}{2}\right)$$

We can check by differentiating:

$$u_x = -2x^2\left(y - \frac{x^2}{2}\right) + \left(y - \frac{x^2}{2}\right)^2 + \exp\left(y - \frac{x^2}{2}\right)(-x)$$

$$u_y = 2x\left(y - \frac{x^2}{2}\right) + \exp\left(y - \frac{x^2}{2}\right)$$

Which means that $u_x + xu_y = \left(y - \frac{x^2}{2}\right)^2$, as we wanted.

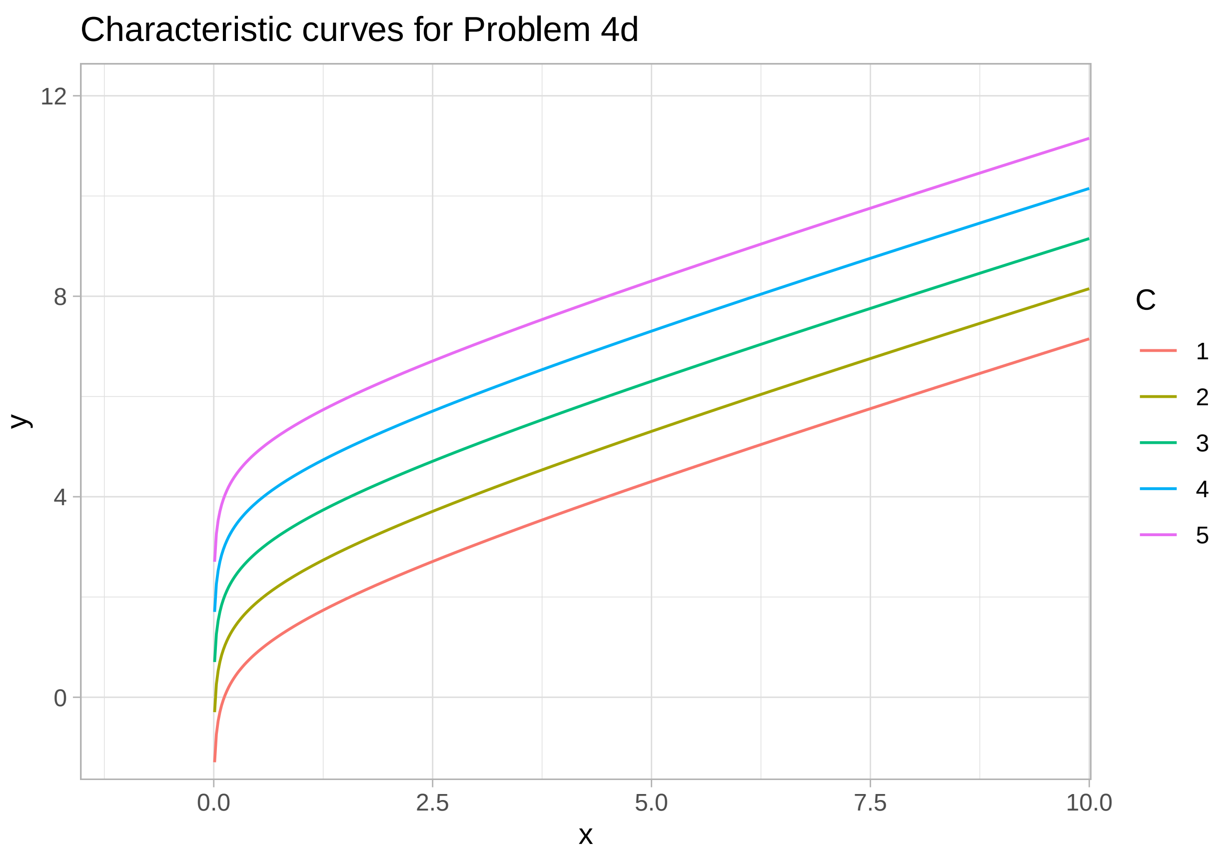
(d) $2xu_x + (x+1)u_y = y$, for $x > 0, u = 2y$ on $x = 1$

$$\frac{dx}{dy} = \frac{x+1}{2x} = \frac{1}{2} + \frac{1}{2x}$$

Solving this ODE yields the characteristic curves:

$$y = \frac{x}{2} + \frac{\ln(x)}{2} + C$$

$$C = 2y - x - \ln(x)$$



Now we define our new coordinate system as $2y - x - \ln(x)$, $\eta = x$

Then by the chain rule,

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} =$$

$$\frac{\partial u}{\partial \eta} + \frac{\partial u}{\partial \xi}\left(-1 - \frac{1}{x}\right)$$

Similarly,

$$u_y = \frac{\partial u}{\partial y} = 2\frac{\partial u}{\partial \xi}$$

Plugging into our initial PDE,

$$2xu_x + (x+1)u_y = 2x\frac{\partial u}{\partial \eta} + \frac{\partial u}{\partial \xi}2x\left(-1 - \frac{1}{x}\right) + 2(x+1)\frac{\partial u}{\partial \xi}$$