MATH 245 Homework 4

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Question 1: Find eigenvalues and eigenfunctions

(a)

$-\frac{d^2}{dx^2}X(x) = \lambda X(x)$ in $0 < x < l$ with boundary conditions $X'(0) = 0 = X(l)$

Case 1: Positive eigenvalues, $\lambda = \beta^2 > 0$

Re-writing as $X'' + \lambda X = 0$, we will get the characteristic equation: $r^2 + \beta^2 = 0$. Since our characteristic equation has complex roots $r = \pm i\beta$, our solutions take the form

$$X(x) = A\sin(\beta x) + B\cos(\beta x)$$

Differentiating, we find that

$$X'(x) = \beta A\cos(\beta x) - \beta B\sin(\beta x)$$

Now plugging in our initial condition $X'(0) = 0$, we get $X'(0) = \beta A = 0$. And since we are in a case where $\beta \neq 0$, this means $B = 0$ so $X(x) = A\sin(\beta x)$. Now we use our boundary condition $X(l) = 0$ to get $X(l) = A\sin(\beta l) = 0$. If $A = 0$, then $X(0) = 0$ and this contradicts the definition of an eigenfunction. Therefore, we must have $\sin(\beta l) = 0$, which can only occur if $\beta = \frac{(2n+1)\pi}{l}$. Therefore, this case gives us eigenvalues

$$\lambda_n = \left(\frac{(2n+1)\pi}{l}\right)^2 \quad n = 1, 2, 3...$$

with eigenfunctions

$$X_n(x) = \sin\left(\frac{(2n+1)\pi x}{l}\right)$$

Case 2: Zero eigenvalues, $\lambda = 0$ $X'' = 0$ implies that $X(x)$ is of the form $Ax + B$, with derivative $X'(x) = A$. Now plugging in our initial condition $X'(0) = 0$, we get $A = 0$, which means $X(x) = B$. But from the boundary condition $X(l) = 0$, we get $B = 0$ and so $X(0) = 0$. Therefore, there are no eigenfunctions $X(x)$ that satisfy $-\frac{d^2}{dx^2}X(x) = \lambda X(x)$ when $\lambda = 0$ and hence no zero eigenvalues.

Case 3: Negative eigenvalues, $\lambda = -\beta^2 < 0$ This case gives us the characteristic equation: $r^2 - \beta^2 = 0$. Since our characteristic equation has distinct real roots $r = \pm\beta$, our solutions take the form

$$X(x) = Ae^{\beta x} + Be^{-\beta x}$$

Differentiating, we find that

$$X'(x) = \beta Ae^{\beta x} - \beta Be^{-\beta x}$$

$$X'(0) = \beta A - \beta B = 0$$

Since we are in a case where $\beta \neq 0$, this means $A - B = 0$, or $A = B$. Then the boundary condition gives

$$X(l) = Be^{\beta l} + Be^{-\beta l} = 0$$

Since $e^{\beta l}$ and $e^{-\beta l}$ are nonzero for all values of l, we must have $B = 0$, implying that again $X(x) = 0$.

Thus, this problem has only positive eigenvalues and their associated eigenfunctions as found in Case 1.

(b)

$x^2X''(x) + xX'(x) + \lambda X(x) = 0$ in $1 < x < e$ with boundary conditions $X(1) = 0 = X(e)$.

We recognize this equation as having the same form as a second-order Cauchy-Euler equation, a linear homogeneous ODE of the form $ax^2y + bxy' + cy = 0$ with the auxiliary equation $ar(r-1) + br + c = 0$. Here, $a = b = 1$ and $c = \lambda$, so we have

$$r(r-1) + r + \lambda = 0$$
$$r^2 - r + r + \lambda = 0$$
$$r^2 + \lambda = 0$$

<u>Case 1:</u> Positive eigenvalues, $\lambda = \beta^2 > 0$

$$X(x) = A\sin(\beta \ln x) + B\cos(\beta \ln x)$$
$$X(1) = A\sin(0) + B\cos(0) = 0 \quad \longrightarrow \quad B = 0$$
$$X(e) = A\sin(\beta \ln e) = 0$$
$$A\sin(\beta) = 0$$

If $A = 0$, then $X(0) = 0$ and this contradicts the definition of an eigenfunction. Therefore, we must have $\sin(\beta) = 0$, which can only occur if $\beta = (2n+1)\pi$. Therefore, this case gives us eigenvalues

$$\lambda_n = (2n+1)^2\pi^2 \quad n = 1, 2, 3...$$

with eigenfunctions

$$X_n(x) = \sin((2n+1)\pi \ln x)$$

<u>Case 2:</u> Zero eigenvalues, $\lambda = 0$ $X'' = 0$ implies that $X(x)$ is of the form $Ax+B$. Now plugging in our initial condition $X(1) = 0$, we get $A + B = 0$, which means $X(x) = Ax - A$. But from the boundary condition $X(e) = 0$, we get $A(e-1) = 0$ which is only possible if $A = 0$ and accordingly $X(0) = 0$. Therefore, there are no eigenfunctions $X(x)$ that satisfy the boundary conditions when $X'' = 0$ and hence no zero eigenvalues.

<u>Case 3:</u> Negative eigenvalues, $\lambda = -\beta^2 < 0$ This case gives us the characteristic equation: $r^2 - \beta^2 = 0$. Since our characteristic equation has distinct real roots $r = \pm\beta$, our solutions take the form

$$X(x) = Ax^{\beta} + Bx^{-\beta}$$

$$X(1) = A + B = 0 \quad \longrightarrow \quad A = -B$$
$$X(x) = Ax^{\beta} - Ax^{-\beta}$$

Then the other boundary condition gives $X(e) = Ae^{\beta} - Ae^{-\beta} = 0$. Since e^{β} and $e^{-\beta}$ are always nonzero, we must have $A = 0$, implying that again $X(x) = 0$.Thus, this problem has only positive eigenvalues and their associated eigenfunctions as found in Case 1.

(c)

On the interval $0 \leq x \leq 1$ of length one, consider the eigenvalue problem

$$-X'' = \lambda X, \quad X'(0) + X(0) = 0, \quad X(1) = 0$$

(i) Find an eigenfunction with eigenvalue zero. Call it $X_0(x)$.

$X'' = 0$ implies that $X(x)$ is of the form $Ax + B$ and $X'(x) = A$. Then $X(1) = 0$ becomes $A + B = 0$, or $A = -B$. Now we look at our second condition,

$$X'(0) + X(0) = 0$$

$X'(0)$ is just A and $X(0)$ is just B, so again this gives us $A = -B$. Thus, we have $X(x) = Ax - A = A(x-1)$, so we have found

$$X_0(x) = x - 1$$

(ii) Find an equation for the positive eigenvalues $\lambda = \beta^2$.

$$X(x) = A\sin(\beta x) + B\cos(\beta x)$$
$$X'(x) = \beta A\cos(\beta x) - \beta B\sin(\beta x)$$

$$X(1) = A\sin(\beta) + B\cos(\beta) = 0$$
$$A\sin(\beta) = -B\cos(\beta)$$

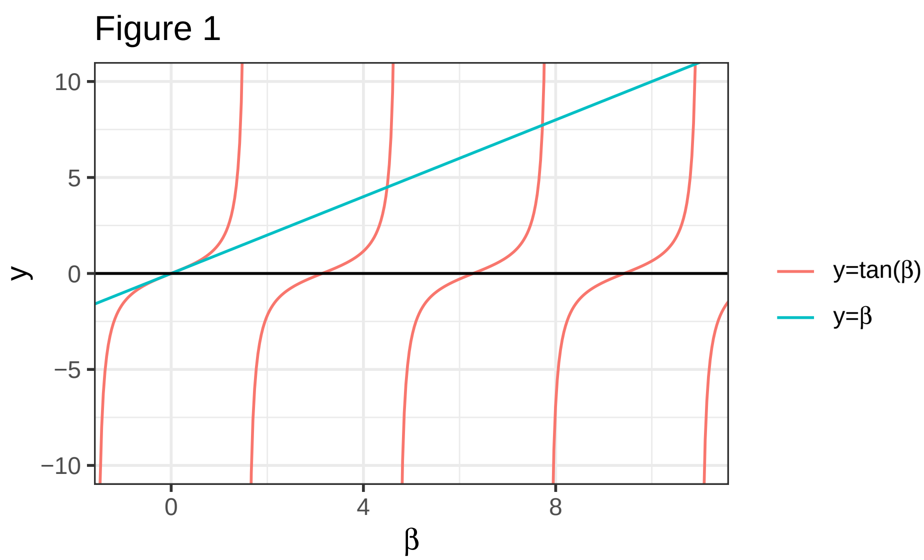
$$-\frac{B}{A} = \frac{\sin(\beta)}{\cos(\beta)} = \tan\beta \tag{1}$$

$$X'(0) = \beta A, \quad X(0) = B$$
$$X'(0) + X(0) = \beta A + B = 0$$
$$\beta = -\frac{B}{A} \tag{2}$$

Combining (1) and (2), we get the equation $\beta = \tan(\beta)$ for the positive eigenvalues.

(iii) Show graphically from part (b) that there are an infinite number of positive eigenvalues.

In part (b), we showed that β is a positive eigenvalue if $\beta = \tan(\beta)$. However, plotting the equations $y = \beta$ and $y = \tan\beta$ reveals that these two functions intersect an infinite number of times (Fig. 1), meaning there are an infinite number of positive eigenvalues.



(iv) Is there a negative eigenvalue?

This case gives us the characteristic equation: $r^2 - \beta^2 = 0$. Since our characteristic equation has distinct real roots $r = \pm\beta$, our solutions take the form

$$X(x) = Ae^{\beta x} + Be^{-\beta x}$$

Differentiating, we find that

$$X'(x) = \beta Ae^{\beta x} - \beta Be^{-\beta x}$$

$$X'(0) = \beta A - \beta B$$
$$X(0) = A + B$$

$$X'(0) + X(0) = \beta A - \beta B + A + B$$
$$0 = A(\beta + 1) + B(1 - \beta)$$
$$-A(\beta + 1) = B(1 - \beta)$$
$$\frac{\beta + 1}{1 - \beta} = -\frac{B}{A}$$

Now turning to our other boundary condition,

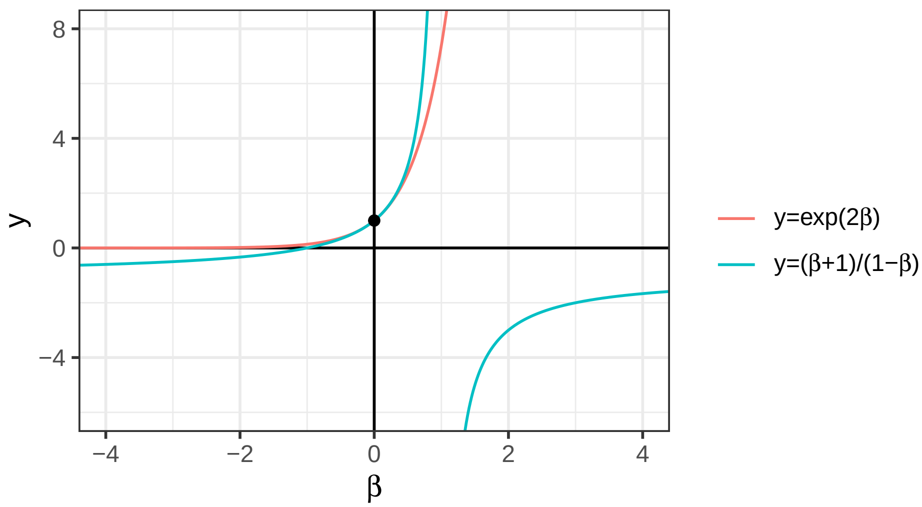
$$X'(1) = \beta Ae^{\beta} - \beta Be^{-\beta} = 0$$
$$\beta Ae^{\beta} = \beta Be^{-\beta}$$
$$-\frac{B}{A} = -e^{2\beta}$$

This would give us the following equation:

$$\frac{\beta + 1}{1 - \beta} = -e^{2\beta}$$

But the only time that this equality holds true is when $\beta = 0$ (Fig. 2), but we are in the case when we defined β to be nonzero, meaning that we have reached a contradiction and we cannot have any negative eigenvalues.

Figure 2



Question 2

Find the Fourier-series of $f(x)$. Does the Fourier-series converge (i) pointwise, or (ii) uniformly?

(a)

$$f(x) = \begin{cases} 1 - |x| & |x| \leq 1 \\ 1 & 1 < |x| \leq \pi \end{cases}$$

(b)

$$f(x) = |x| = \begin{cases} -x & -\pi \leq x \leq 0 \\ x & 0 < x \leq \pi \end{cases}$$

(c)

$$f(x) = x + x^2, \quad -\pi \leq x \leq \pi$$

Question 3

(a) Find the Fourier-sine-series of

$$f(x) = \begin{cases} 1 & 0 < x < \pi/2 \\ 2 & \pi/2 < x < \pi \end{cases}$$

(b) Find the Fourier-cosine-series of $f(x) = |\sin x|$. Then find

$$\sum_{k=1}^{\infty} \frac{1}{4k^2 - 1}$$

(c) The Riemann Zeta function is defined for $s > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

By computing the Fourier series of x^2 over $-\pi < x < \pi$ and using Parseval's identity, compute $\zeta(4)$.

(d) Use the Fourier series in 2c and the pointwise convergence theorem to find $\zeta(2)$. Then find

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

Question 4

Compute the complex Fourier series of the following functions:

(a) Compute the complex Fourier series of $f(x) = e^x$ and show that

$$\coth \pi = \frac{1}{\pi} + \frac{2}{\pi}\left(\frac{1}{1+1^2} + \frac{1}{1+2^2} + \frac{1}{1+3^2} + \ ...\right)$$

(b) Find the complex Fourier series of xe^{ix}. Then use your result to find the Fourier series of $x\cos x$ and $x \sin x$.

Question 5.

Find the function represented by the new series which is obtained by termwise integration of the following series from 0 to x.

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\cos kx}{k} = \log\left(2\cos\left(\frac{x}{2}\right)\right), \quad -\pi < x < \pi$$