MATH 245 Homework 4

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Question 1: Find eigenvalues and eigenfunctions

(a)

$-\frac{d^2}{dx^2}X(x) = \lambda X(x)$ in $0 < x < l$ with boundary conditions $X'(0) = 0 = X(l)$.

(b)

$x^2 X''(x) + xX'(x) + \lambda X(x) = 0$ in $1 < x < e$ with boundary conditions $X(1) = 0 = X(e)$.

(c)

On the interval $0 \leq x \leq 1$ of length one, consider the eigenvalue problem

$$-X'' = \lambda X, \quad X'(0) + X(0) = 0, \quad X(1) = 0$$

(i) Find an eigenfunction with eigenvalue zero. Call it $X_0(x)$.

(ii) Find an equation for the positive eigenvalues $\lambda = \beta^2$.

(iii) Show graphically from part (b) that there are an infinite number of positive eigenvalues.

(iv) Is there a negative eigenvalue?

Question 2

Find the Fourier-series of $f(x)$. Does the Fourier-series converge (i) pointwise, or (ii) uniformly?

(a)

$$f(x) = \begin{cases} 1 - |x| & |x| \leq 1 \\ 1 & 1 < |x| \leq \pi \end{cases}$$

(b)

$$f(x) = |x| = \begin{cases} -x & -\pi \leq x \leq 0 \\ x & 0 < x \leq \pi \end{cases}$$

(c)

$$f(x) = x + x^2, \quad -\pi \leq x \leq \pi$$

Question 3

(a) Find the Fourier-sine-series of

$$f(x) = \begin{cases} 1 & 0 < x < \pi/2 \\ 2 & \pi/2 < x < \pi \end{cases}$$

(b) Find the Fourier-cosine-series of $f(x) = |\sin x|$. Then find

$$\sum_{k=1}^{\infty} \frac{1}{4k^2 - 1}$$

(c) The Riemann Zeta function is defined for $s > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

By computing the Fourier series of x^2 over $-\pi < x < \pi$ and using Parseval's identity, compute $\zeta(4)$.

(d) Use the Fourier series in 2c and the pointwise convergence theorem to find $\zeta(2)$. Then find

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

Question 4

Compute the complex Fourier series of the following functions:

(a) Compute the complex Fourier series of $f(x) = e^x$ and show that

$$\coth \pi = \frac{1}{\pi} + \frac{2}{\pi} \left(\frac{1}{1+1^2} + \frac{1}{1+2^2} + \frac{1}{1+3^2} + \ ... \right)$$

(b) Find the complex Fourier series of xe^{ix}. Then use your result to find the Fourier series of $x \cos x$ and $x \sin x$.

Question 5.

Find the function represented by the new series which is obtained by termwise integration of the following series from 0 to x.

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{\cos kx}{k} = \log \left(2 \cos \left(\frac{x}{2} \right) \right), \quad -\pi < x < \pi$$