

MATH 245 Homework 1

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Question 1

Show the function is a solution of the PDE:

(a) $u_{xx} + u_{yy} = 0$

(i) $u(x, y) = e^x \sin(y)$

$$u_x = e^x \sin(y), u_{xx} = e^x \sin(y), u_y = e^x \cos(y), u_{yy} = -e^x \sin(y)$$

Which means that $u_{xx} + u_{yy} = e^x \sin(y) + (-e^x \sin(y)) = 0$ and so $u(x, y) = e^x \sin(y)$ is a solution to the PDE.

(ii) $u(x, y) = \log \sqrt{x^2 + y^2}$

Assuming this is meant to be $u(x, y) = \ln \sqrt{x^2 + y^2}$,

$$u_x = (x^2 + y^2)^{-1/2} \left(\frac{1}{2} \right) (x^2 + y^2)^{-1/2} (2x) = x (x^2 + y^2)^{-1}$$

$$u_{xx} = x(-1) (x^2 + y^2)^{-2} (2x) + (x^2 + y^2)^{-1}$$

$$u_y = (x^2 + y^2)^{-1/2} \left(\frac{1}{2} \right) (x^2 + y^2)^{-1/2} (2y) = y (x^2 + y^2)^{-1}$$

$$u_{yy} = y(-1) (x^2 + y^2)^{-2} (2y) + (x^2 + y^2)^{-1}$$

$$u_{xx} + u_{yy} = \frac{-2x^2}{(x^2 + y^2)^2} + \frac{-2y^2}{(x^2 + y^2)^2} + \frac{2}{(x^2 + y^2)} =$$

$$\frac{-2(x^2 + y^2)}{(x^2 + y^2)^2} + \frac{2}{(x^2 + y^2)} = 0$$

Which means that $u(x, y) = \log \sqrt{x^2 + y^2}$ is a solution to the PDE.

(b) $bu_x + au_y + u = 0$, $u(x, y) = \exp\left(\frac{-x}{b}\right)f(ax - by)$ for arbitrary differentiable function f .

$$u_x = e^{\frac{-x}{b}} \left(\frac{-1}{b} \right) f(ax - by) + f'(ax - by)(a)e^{\frac{-x}{b}}$$

$$u_y = e^{\frac{-x}{b}} f'(ax - by)(-b)$$

Then,

$$bu_x + au_y + u = -e^{\frac{-x}{b}} f(ax - by) + (ab) \left(e^{\frac{-x}{b}} \right) f'(ax - by) + e^{\frac{-x}{b}} f'(ax - by)(-ab) + e^{\frac{-x}{b}} f(ax - by) = 0$$

So $u(x, y) = \exp\left(\frac{-x}{b}\right)f(ax - by)$ is a solution to the PDE for an arbitrary differentiable function f .

- (c) $u_{xx} - \frac{1}{x}u_x - x^2u_{yy} = 0$, $u(x, y) = f(2y + x^2) + g(2y - x^2)$ for arbitrary twice-differentiable functions f and g .

$$\begin{aligned}u_x &= (f'(2y + x^2))(2x) + (g'(2y - x^2))(-2x) \\u_y &= 2(f'(2y + x^2)) + 2(g'(2y - x^2)) \\u_{xx} &= 2f'' + (f'')(4x^2) - 2g' + (g'')(4x^2) \\u_{yy} &= 4f'' + 4g'' \\u_{xx} - \frac{1}{x}u_x - x^2u_{yy} &= \\2f'' + 4x^2f'' - 2g' + 4x^2g'' - \frac{1}{x}(2xf' - 2xg') - x^2(4f'' + 4g'') &= \\2f'' - 2f' + 4x^2f'' - 4x^2f'' - 2g' + 2g' + 4x^2g'' - 4x^2g'' &= 0\end{aligned}$$

So $u(x, y) = f(2y + x^2) + g(2y - x^2)$ is a solution to the PDE for arbitrary twice-differentiable functions f and g .

Question 2

- (a) 2nd-order linear homogeneous
- (b) 4th-order linear inhomogeneous
- (c) 2nd-order quasi-linear homogeneous
- (d) We can rewrite this as $u_{xx} + u_{yy} + f(x, y)u - g(x, y)u^5 = 0$, making it clear that this is a 2nd-order semi-linear homogeneous PDE.

Question 3

Use separation of variables to solve the following problems:

- (a) $u_x + u = u_y$, $u(x, 0) = 4x^{-3x}$, use $u(x, y) = f(x)g(y)$

$$\begin{aligned}u_x &= f'g, u_y = fg' \\u_x + u - u_y &= f'g + fg - fg' = f'g + (g - g')f = 0 \\f'g &= -(g - g')f \\ \frac{f'}{f} &= \frac{g' - g}{g} = \lambda \\ \frac{f'}{f} &= \lambda, \frac{g'}{g} - 1 = \lambda\end{aligned}$$

Integrating the first ODE with respect to x ,

$$\ln(f) = \lambda x + C_1, f(x) = C_2e^{\lambda x}$$

$$\frac{g'}{g} = \lambda + 1$$

$$\ln(g) = \lambda y + y + C_3, \quad g(y) = C_4 e^{y(\lambda+1)}$$

$$u(x, y) = f(x)g(y) = C_2 e^{\lambda x} C_4 e^{y(\lambda+1)}$$

$$u(x, y) = C e^{\lambda x + y(\lambda+1)}$$

$$u(x, 0) = 4e^{-3x} = C e^{\lambda x}$$

Which means $C = 4, \lambda = -3$ and our final solution is

Solution

$$u(x, y) = 4e^{-3x-2y}$$

Check solution

$$u_x + u = -12e^{-3x-2y} + 4e^{-3x-2y} = -8e^{-3x-2y} = u_y$$

(b) $x^2 u_{xy} + 9y^2 u = 0, u(x, 0) = \exp\left(\frac{1}{x}\right)$, use $u(x, y) = f(x)g(y)$

$$u_{xy} = f'g'$$

$$x^2 f'g' + 9y^2 fg = 0$$

$$x^2 f'g' = -9y^2 fg$$

$$x^2 \frac{f'}{f} = -9y^2 \frac{g}{g'} = \lambda$$

$$x^2 \frac{f'}{f} = \lambda, \quad -9y^2 \frac{g}{g'} = \lambda$$

$$x^2 \frac{f'}{f} = \lambda \quad \longrightarrow \quad \frac{f'}{f} = \lambda x^{-2} \quad \longrightarrow \quad \ln(f) = \frac{-\lambda}{x} + C_1 \quad \longrightarrow \quad f(x) = C_2 e^{\frac{-\lambda}{x}}$$

$$-9y^2 \frac{g}{g'} = \lambda \quad \longrightarrow \quad \frac{g}{g'} = \frac{\lambda}{-9y^2} \quad \longrightarrow \quad \frac{g'}{g} = \frac{-9y^2}{\lambda} \quad \longrightarrow \quad \ln(g) = \frac{-3y^3}{\lambda} + C_3 \quad \longrightarrow \quad g(y) = C_4 e^{\frac{-3}{\lambda} y^3}$$

$$u(x, y) = f(x)g(y) = C_2 e^{\frac{-\lambda}{x}} C_4 e^{\frac{-3}{\lambda} y^3} = C e^{\frac{-\lambda}{x} + \frac{-3}{\lambda} y^3}$$

$$u(x, 0) = e^{\frac{1}{x}} = C e^{\frac{-\lambda}{x}}$$

Which means $C = 1, \lambda = -1$, and our final solution is

Solution

$$u(x, y) = e^{\frac{1}{x} + 3y^3}$$

Check solution

$$x^2 u_{xy} + 9y^2 u = x^2 e^{\frac{1}{x} + 3y^3} (-x^{-2}) (9y^2) + 9y^2 e^{\frac{1}{x} + 3y^3} = 0$$

(c) $u_x^2 + u_y^2 = 1$, use $u(x, y) = f(x) + g(y)$

$$u_x = f', u_y = g'$$

$$(f')^2 + (g')^2 = 1 \longrightarrow (f')^2 = 1 - (g')^2 = \lambda^2$$

$$(f')^2 = \lambda^2 \longrightarrow f' = \lambda \longrightarrow f(x) = \lambda x + C_1$$

$$(g')^2 = 1 - \lambda^2 \longrightarrow g' = \pm\sqrt{1 - \lambda^2} \longrightarrow g(y) = \pm y\sqrt{1 - \lambda^2} + C_2$$

Solution

$$u(x, y) = f(x) + g(y) = \lambda x \pm y\sqrt{1 - \lambda^2} + C$$

Check solution

$$u_x = \lambda, u_y = \pm\sqrt{1 - \lambda^2}, \quad u_x^2 + u_y^2 = \lambda^2 + 1 - \lambda^2 = 1$$

(d) $x^2 u_x^2 + y^2 u_y^2 = u^2$, use $u(x, y) = e^{f(x)} e^{g(y)}$

$$u_x = f' e^{f+g}, u_y = g' e^{f+g}$$

$$x^2 u_x^2 + y^2 u_y^2 - u^2 = x^2 (f')^2 e^{2f+2g} + y^2 (g')^2 e^{2f+2g} - e^{2f+2g} = 0$$

$$x^2 (f')^2 + y^2 (g')^2 - 1 = 0 \longrightarrow x^2 (f')^2 = 1 - y^2 (g')^2 = \lambda^2$$

$$x^2 (f')^2 = \lambda^2 \longrightarrow f' = \frac{\lambda}{x} \longrightarrow f(x) = \lambda \ln(x) + C_1$$

$$y^2 (g')^2 = 1 - \lambda^2 \longrightarrow g' = \frac{\sqrt{1 - \lambda^2}}{y} \longrightarrow g(y) = \sqrt{1 - \lambda^2} \ln(y) + C_2$$

$$u(x, y) = e^{f(x)} e^{g(y)} = C_3 e^{\lambda \ln(x)} C_4 e^{\sqrt{1 - \lambda^2} \ln(y)}$$

Solution

$$u(x, y) = C x^\lambda y^{\sqrt{1 - \lambda^2}}$$

Check solution

$$u_x = \lambda C x^{\lambda-1} y^{\sqrt{1 - \lambda^2}}, u_y = \sqrt{1 - \lambda^2} C x^\lambda y^{\sqrt{1 - \lambda^2} - 1}$$

$$x^2 u_x^2 + y^2 u_y^2 = x^2 \left(\lambda C x^{\lambda-1} y^{\sqrt{1 - \lambda^2}} \right)^2 + y^2 \left(\sqrt{1 - \lambda^2} C x^\lambda y^{\sqrt{1 - \lambda^2} - 1} \right)^2 =$$

$$x^2 \lambda^2 C^2 x^{2\lambda-2} y^{2\sqrt{1 - \lambda^2}} + y^2 (1 - \lambda^2) C^2 x^{2\lambda} y^{2\sqrt{1 - \lambda^2} - 2} =$$

$$\lambda^2 C^2 x^{2\lambda} y^{2\sqrt{1 - \lambda^2}} + (1 - \lambda^2) C^2 x^{2\lambda} y^{2\sqrt{1 - \lambda^2}} = C^2 x^{2\lambda} y^{2\sqrt{1 - \lambda^2}} = u^2$$

Question 4

For each of the following IVPs, (i) find and plot the characteristic lines (curves), (ii) solve the IVP, and (iii) plot the solution of (a)-(b) for indicated time.

(a) $u_t + (1 + x^2) u_x = 0$, $u(0, x) = \arctan(x)$, $t = 1, 2, 3$, and what is $\lim_{t \rightarrow \infty} u(t, x)$?

$$\frac{dx}{dt} = 1 + x^2$$

$$\frac{1}{1 + x^2} dx = dt$$

$$\arctan(x) = t + C$$

So $x = \tan(t + C)$ are the characteristic curves of $u_t + (1 + x^2) u_x = 0$.

On each of the curves, $u(x, t)$ is constant because

$$\frac{d}{dt} u(t, \tan(t + C)) = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \sec^2(t + c)$$

And since $1 + \tan^2 \theta = \sec^2 \theta$, this means $\frac{d}{dt} u(t, \tan(t + C)) = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} (1 + \tan^2(t + c)) = u_t + (1 + x^2) u_x$, which we know is 0.

Thus $u(t, \tan(t + C)) = u(0, \tan(0 + C)) = u(0, C)$ is independent of t .

Putting $x = \tan(t + C)$ and $C = \arctan(x) - t$, we have

$$u(t, x) = u(0, \arctan(x) - t)$$

It follows that $u(t, x) = f(\arctan(x) - t)$.

And since we are given that $u(0, x) = \arctan(x)$, we have $f(\arctan(x) - 0) = \arctan(x)$ so that $f(w) = w$ for any w , yielding our solution of

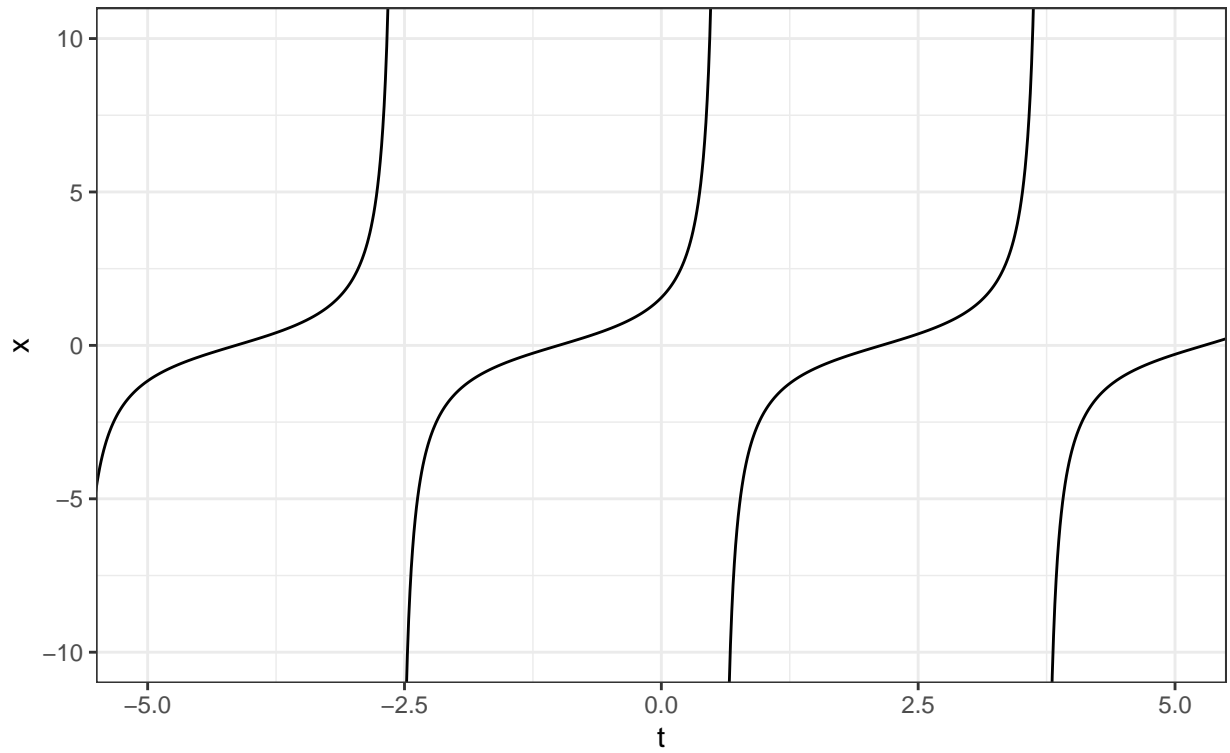
Solution

$$u(t, x) = \arctan(x) - t$$

Check solution

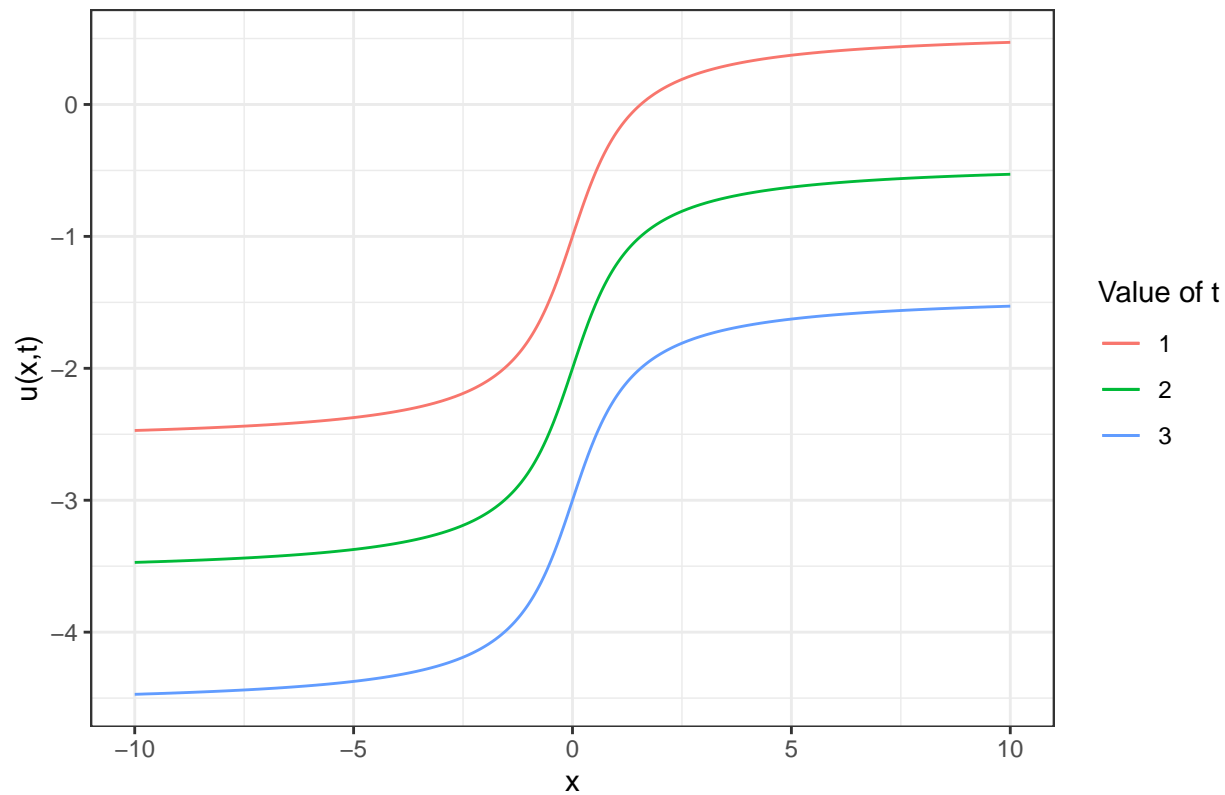
Since $u_t = -1$, $u_x = \frac{1}{1+x^2}$, that means $u_t + (1 + x^2) u_x = -1 + 1 = 0$

Characteristic curves for Problem 4a
with C=1



Since the range of \arctan is limited to $(-\frac{\pi}{2}, \frac{\pi}{2})$, as $\lim_{t \rightarrow \infty}$ that means $u(t, x) = \arctan(x) - t$ will go to $-\infty$.

Solution curves for Problem 4a



(b) $u_t - xu_x = 0$, $u(0, x) = \frac{1}{1+x^2}$, $t = 1, 2, 3$, and what is $\lim_{t \rightarrow \infty} u(t, x)$?

The directional derivative of u in the direction of the vector $(1, -x)$ is zero. The curves in the tx plane with $(1, -x)$ as a tangent vector have slopes $-x$:

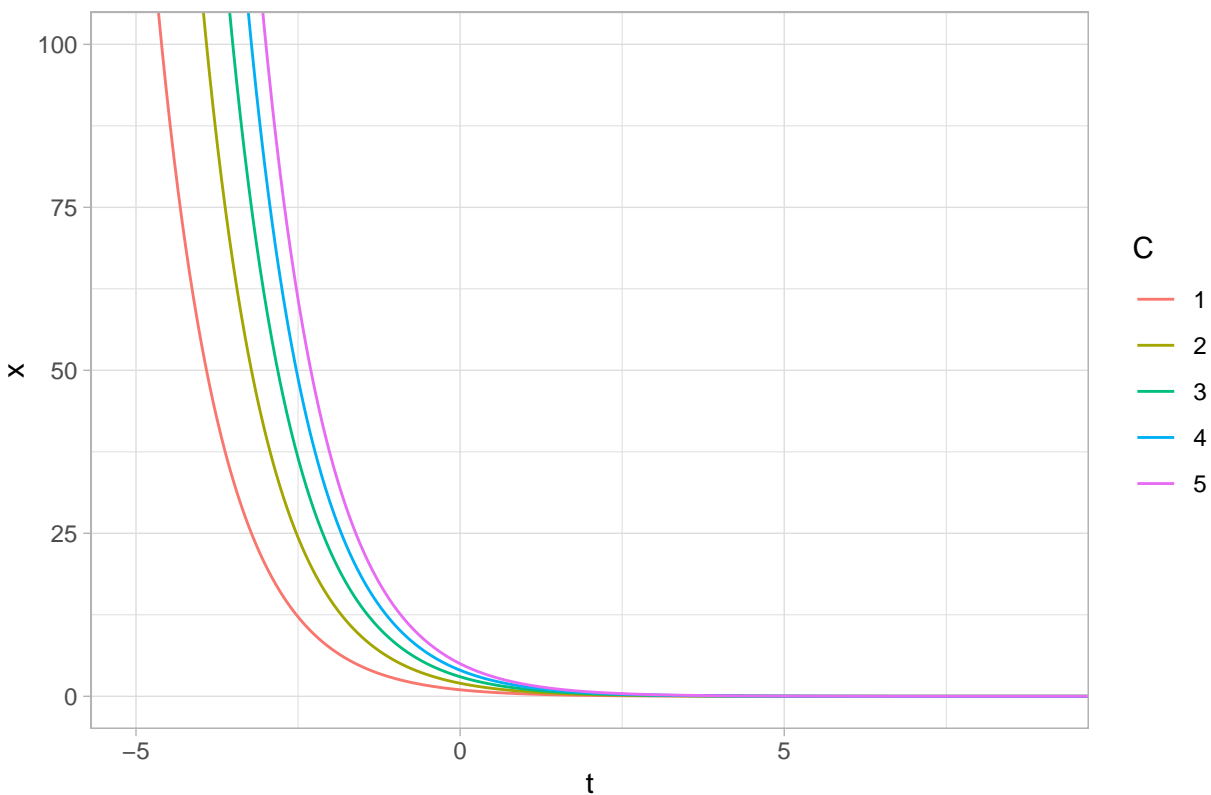
$$\frac{dx}{dt} = -x$$

$$\frac{1}{x} dx = -dt$$

Solving this ODE gives the equations for the characteristic lines:

$$x = Ce^{-t}$$

Characteristic curves for Problem 4b



On each of the curves, $u(x, t)$ is constant because

$$\frac{d}{dt}u(t, Ce^{-t}) = \frac{\partial u}{\partial t} - Ce^{-t} \frac{\partial u}{\partial x} = u_t - xu_x$$

and we know that $u_t - xu_x = 0$.

Set $\xi = xe^t$ and $\eta = x$. Then by the chain rule,

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} = \frac{\partial u}{\partial \eta} + \frac{\partial u}{\partial \xi} e^t$$

and similarly

$$u_t = \frac{\partial u}{\partial t} = \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial t} = \frac{\partial u}{\partial \xi} xe^t$$

Which means that

$$u_t - xu_x = \frac{\partial u}{\partial \xi} xe^t - x \left(\frac{\partial u}{\partial \eta} + \frac{\partial u}{\partial \xi} e^t \right) = 0$$

Assuming $x \neq 0$, this means

$$u_\eta = 0$$

$$u = \int u_\eta d\eta + f(\xi) = 0 + f(\xi)$$

So we have $u(t, x) = f(\xi) = f(xe^t)$ and we know that $u(0, x) = \frac{1}{1+x^2}$, which means $f(xe^0) = f(x) = \frac{1}{1+x^2}$ and therefore

Solution

$$u(t, x) = \frac{1}{1 + (xe^t)^2}$$

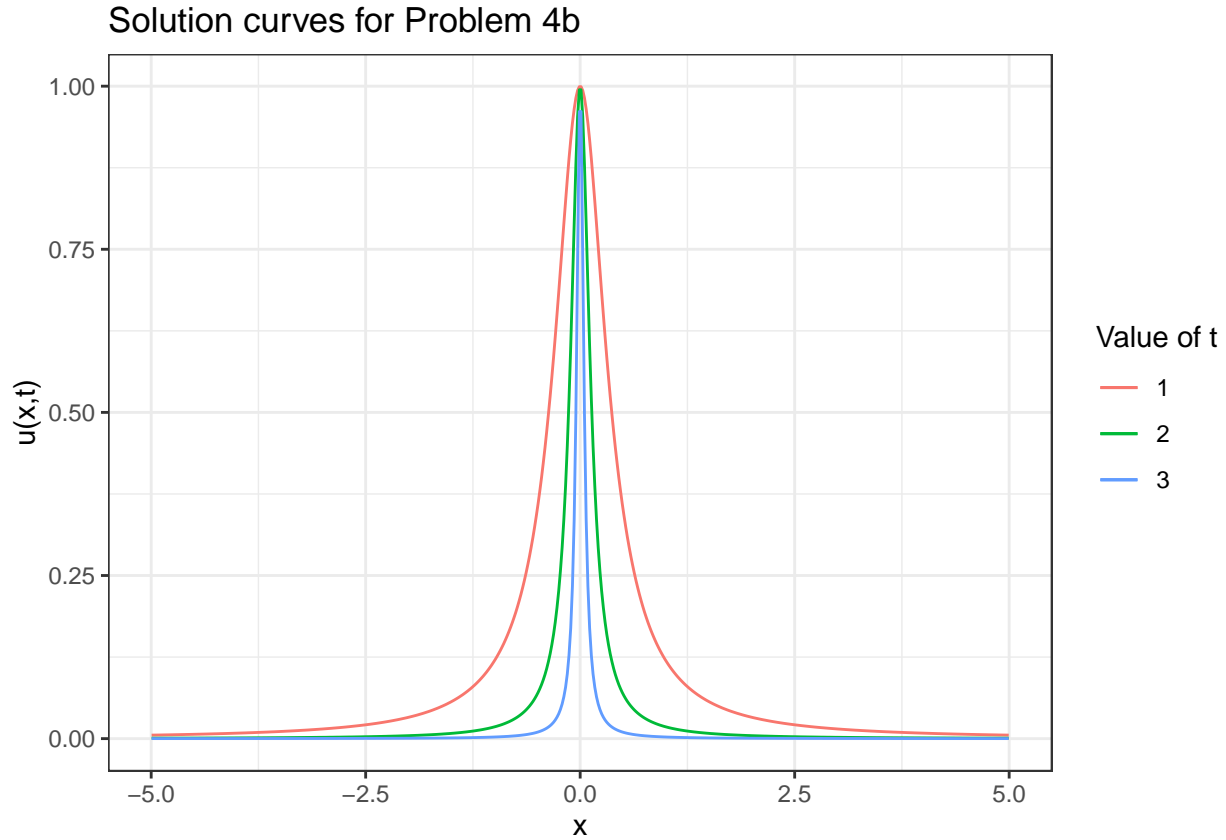
Check solution

$$u_t = -\left(1 + (xe^t)^2\right)^{-2} (2xe^t) xe^t$$

$$u_x = -\left(1 + (xe^t)^2\right)^{-2} (2xe^t) e^t$$

And we see that $u_t - xe_t$ indeed equals zero.

Plotting the solution $u(t, x) = \frac{1}{1+(xe^t)^2}$ as a function of x for $t = 1, 2, 3$, we get:



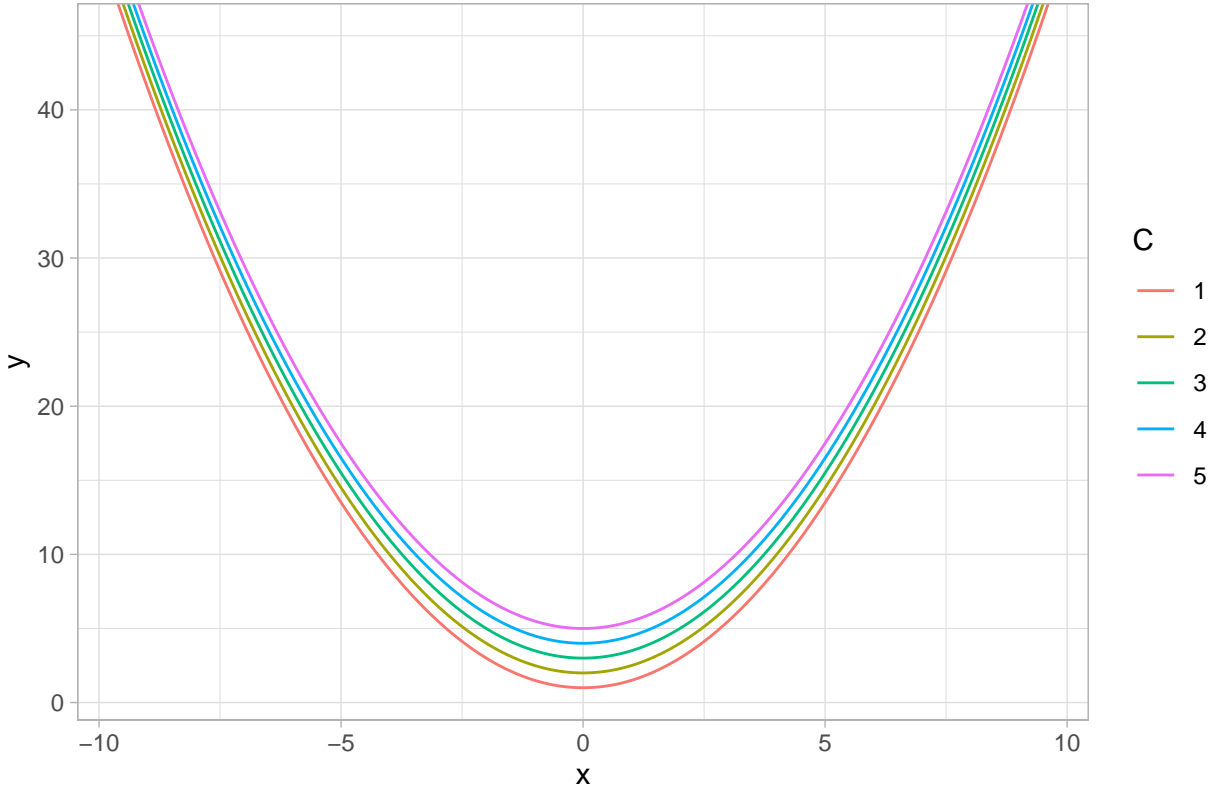
We can see that $u(t, x)$ will equal 1 at $x = 0$ regardless of the value of t , but at all other values of x , $\lim_{t \rightarrow \infty} u(t, x) = 0$. This can be written as

$$\lim_{t \rightarrow \infty} u(t, x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases} \quad (1)$$

(c) $u_x + xu_y = \left(y - \frac{x^2}{2}\right)^2$, $u(0, y) = e^y$

$\frac{dy}{dx} = x$ Solving this ODE yields the equation for the characteristic curves: $y = \frac{x^2}{2} + C$

Characteristic curves for Problem 4c



Now we define our new coordinate system as $\xi = y - \frac{x^2}{2}$, $\eta = x$

Then by the chain rule,

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} = \frac{\partial u}{\partial \eta} - x \frac{\partial u}{\partial \xi}$$

and similarly

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial y} = \frac{\partial u}{\partial \xi}$$

$$u_x + xu_y = \frac{\partial u}{\partial \eta} - x \frac{\partial u}{\partial \xi} + x \frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial \eta}$$

And since we know $u_x + xu_y = \left(y - \frac{x^2}{2}\right)^2$, this means we have

$$\frac{\partial u}{\partial \eta} = \left(y - \frac{x^2}{2}\right)^2 = \xi^2$$

Now we integrate with respect to η to find

$$u = \eta\xi^2 + F(\xi)$$

At $u(0, y)$, this becomes $u = 0y^2 + F(y)$, or $u(0, y) = F(y)$. On the other hand, we are told that $u(0, y) = e^y$, which means that $F(y) = e^y$.

$$u = \eta\xi^2 + e^\xi$$

Substitute back in x and y :

Solution

$$u(x, y) = x \left(y - \frac{x^2}{2} \right)^2 + \exp \left(y - \frac{x^2}{2} \right)$$

Check solution

$$u_x = -2x^2 \left(y - \frac{x^2}{2} \right) + \left(y - \frac{x^2}{2} \right)^2 + \exp \left(y - \frac{x^2}{2} \right) (-x)$$

$$u_y = 2x \left(y - \frac{x^2}{2} \right) + \exp \left(y - \frac{x^2}{2} \right)$$

Which means that $u_x + xu_y = \left(y - \frac{x^2}{2} \right)^2$, as we wanted.

(d) $2xu_x + (x+1)u_y = y$, for $x > 0$, $u = 2y$ on $x = 1$

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c} \quad \longrightarrow \quad \frac{dx}{2x} = \frac{dy}{x+1} = \frac{du}{y}$$

$$\frac{dx}{2x} = \frac{dy}{x+1}$$

$$\frac{x+1}{2x} dx = dy$$

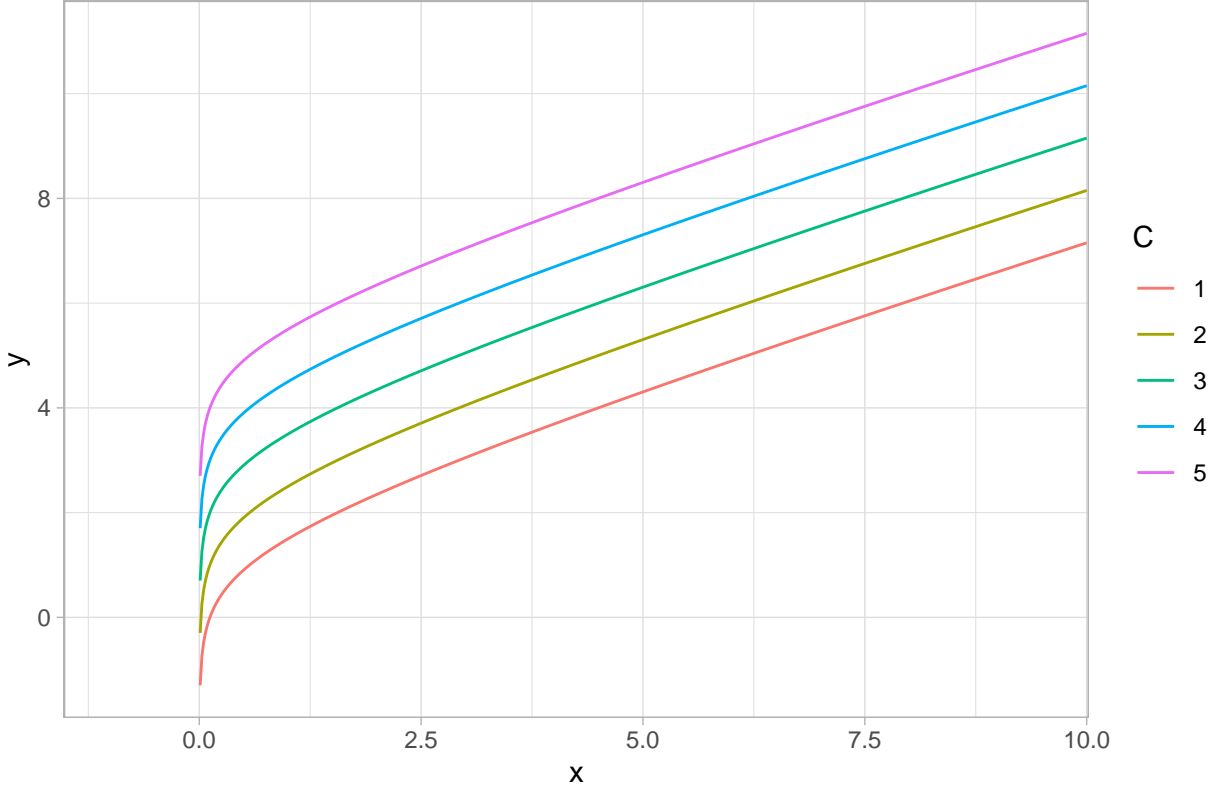
$$\int \frac{1}{2} + \frac{1}{2x} dx = \int dy$$

Solving this ODE yields the characteristic curves:

$$y = \frac{x}{2} + \frac{\ln x}{2} - C$$

$$C = \frac{x}{2} + \frac{\ln x}{2} - y$$

Characteristic curves for Problem 4d



Now define our new coordinate system as $\xi = \frac{x}{2} + \frac{\ln x}{2} - y$, $\eta = x$.
Then by the chain rule,

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} =$$

$$\frac{\partial u}{\partial \xi} \left(\frac{1}{2} + \frac{1}{2x} \right) + \frac{\partial u}{\partial \eta}$$

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial y} = -\frac{\partial u}{\partial \xi}$$

Plugging these into our original PDE,

$$2x \left(\frac{\partial u}{\partial \xi} \left(\frac{1}{2} + \frac{1}{2x} \right) + \frac{\partial u}{\partial \eta} \right) + (x+1) \left(-\frac{\partial u}{\partial \xi} \right) =$$

$$\frac{\partial u}{\partial \xi} (x+1) + 2x \frac{\partial u}{\partial \eta} - \frac{\partial u}{\partial \xi} (x+1) = 2x \frac{\partial u}{\partial \eta}$$

So we must have $2x \frac{\partial u}{\partial \eta} = y$. Now replace η and ξ by x and y :

$$2\eta \frac{\partial u}{\partial \eta} = \frac{\eta}{2} + \frac{\ln \eta}{2} - \xi$$

$$\frac{\partial u}{\partial \eta} = \frac{1}{4} + \frac{\ln \eta}{4\eta} - \frac{\xi}{2\eta}$$

Now integrate with respect to η :

$$u = \frac{\eta}{4} + \frac{(\ln \eta)^2}{8} - \frac{\xi \ln \eta}{2} + F(\xi)$$

$$u(x, y) = \frac{x}{4} + \frac{(\ln x)^2}{8} - \frac{\ln x}{2} \left(\frac{x}{2} + \frac{\ln x}{2} - y \right) + F \left(\frac{x}{2} + \frac{\ln x}{2} - y \right)$$

Now we use the fact that $u(1, y) = 2y$:

$$u(1, y) = \frac{1}{4} + F \left(\frac{1}{2} - y \right) = 2y$$

$$F \left(\frac{1}{2} - y \right) = 2y - \frac{1}{4}$$

So $F(\alpha) = -2\alpha + \frac{3}{4}$ for arbitrary α . Then

$$u(x, y) = \frac{x}{4} + \frac{(\ln x)^2}{8} - \frac{\ln x}{2} \left(\frac{x}{2} + \frac{\ln x}{2} - y \right) - x - \ln x + 2y + \frac{3}{4}$$

Which can be simplified to

$$u(x, y) = \frac{-3x}{4} - \frac{(\ln x)^2}{8} - \frac{x \ln x}{4} + \frac{y \ln x}{2} - \ln x + 2y + \frac{3}{4}$$

Solution

$$u(x, y) = -\ln x \left(\frac{x}{4} + \frac{\ln x}{8} + \frac{y}{2} + 1 \right) - \frac{3x}{4} + 2y + \frac{3}{4}$$

Check solution

$$u_y = \frac{\ln x}{2} + 2$$

$$u_x = \frac{-1}{x} \left(\frac{x}{4} + \frac{\ln x}{8} - \frac{y}{2} + 1 \right) + (-\ln x) \left(\frac{1}{4} + \frac{1}{8x} \right) - \frac{3}{4}$$

Which means

$$\begin{aligned} 2xu_x + (x+1)u_y &= \left(\frac{-x}{2} - \frac{\ln x}{4} + y - 2 - \frac{x \ln x}{2} - \frac{\ln x}{4} - \frac{3x}{2} \right) + \left(\frac{x \ln x}{2} + \frac{\ln x}{2} + 2x + 2 \right) = \\ &= \left(-\frac{x \ln x}{2} - \frac{\ln x}{2} - 2x + -2 + y \right) + \left(\frac{x \ln x}{2} + \frac{\ln x}{2} + 2x + 2 \right) = y \end{aligned}$$

Which matches our original PDE.

(e) $yu_x + xu_y = xy$, for $x, y \geq 0$ with $u(0, y) = \exp(-y^2)$

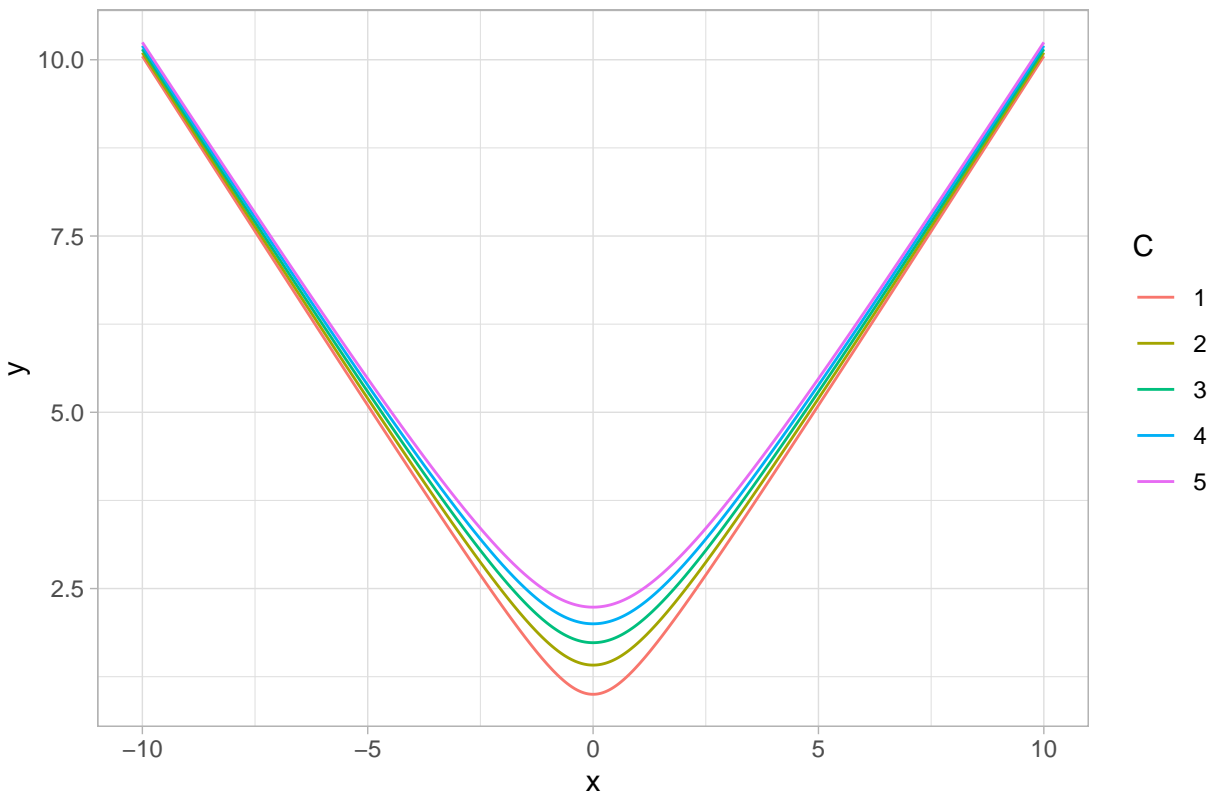
$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c} \quad \longrightarrow \quad \frac{dx}{y} = \frac{dy}{x} = \frac{du}{xy}$$

$$\frac{dx}{y} = \frac{dy}{x} \quad \longrightarrow \quad x dx = y dy \quad \longrightarrow \quad y^2 = x^2 + C$$

$$y = \sqrt{x^2 + C}, \quad C = y^2 - x^2$$

Where we kept the positive root for y because it was given that $x, y \geq 0$.

Characteristic curves for Problem 4e



$$\frac{dx}{y} = \frac{du}{xy} \quad \longrightarrow \quad du = x \, dx$$

$$\frac{dy}{x} = \frac{du}{xy} \quad \longrightarrow \quad du = y \, dy$$

$$2 \, du = x \, dx + y \, dy = d\left(\frac{x^2 + y^2}{2}\right)$$

$$\int du = \frac{1}{4} \int d(x^2 + y^2)$$

$$u = \frac{x^2 + y^2}{4} + f(C) \quad \longrightarrow \quad u = \frac{x^2 + y^2}{4} + f(y^2 - x^2)$$

$$u(0, y) = \frac{y^2}{4} + f(y^2) = e^{-y^2}$$

$$f(y^2) = e^{-y^2} - \frac{y^2}{4}$$

$$f(\alpha) = e^{-\alpha} - \frac{\alpha}{4}$$

$$u(x, y) = \frac{x^2 + y^2}{4} + e^{x^2 - y^2} + \frac{x^2 - y^2}{4}$$

Solution

$$u(x, y) = \frac{x^2}{2} + e^{x^2 - y^2}$$

Check solution

$$u_x = x + 2xe^{x^2 - y^2}, \quad u_y = -2ye^{x^2 - y^2}$$

$$yu_x + xu_y = xy + 2xye^{x^2 - y^2} - 2xye^{x^2 - y^2} = xy$$

Which matches our original PDE.