

# MATH 245 Homework 2

Ruby Krasnow and Tommy Thach

2024-02-18

## Question 1

Determine the region in which the given equation is hyperbolic, parabolic, elliptic, or singular.

a)

$$u_{xx} - y^2 u_{yy} + u_x - u + x^2 = 0$$

$a = 1, b = 0, c = -y^2$ , so we have  $b^2 - ac = 0 - (-y^2) = y^2$ . This will be positive everywhere except for  $y = 0$ , so the equation is hyperbolic where  $y \neq 0$  and parabolic for  $y = 0$ .

b)

$$u_{xx} - y u_{yy} + x u_x + y u_y + u = 0$$

$a = 1, b = 0, c = -y$ , so we have  $b^2 - ac = 0 - (-y) = y$ . Thus, the equation will be hyperbolic where  $y > 0$ , parabolic where  $y = 0$ , and elliptic where  $y < 0$ .

Both PDEs are nowhere singular, because the coefficient of  $u_{xx}$  is never zero.

## Question 2

Using a factorization similar to the wave equation, solve the following IVP:

$$\begin{cases} u_{xx} + 2u_{xy} - 3u_{yy} = 0 & x \in \mathbb{R}, y > 0 \\ u(0, x) = \sin x & x \in \mathbb{R} \\ u_y(0, x) = x & x \in \mathbb{R} \end{cases}$$

We are looking for a solution  $u(x, y)$  for the above IVP. First, we can factor the given PDE as follows:

$$\left( \frac{\partial}{\partial x} + 3 \frac{\partial}{\partial y} \right) \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) u = 0$$

or

$$(\partial_x + 3\partial_y)(\partial_x - \partial_y)u = 0$$

Then set  $\left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) u = v$ , giving us

$$\left( \frac{\partial}{\partial x} + 3 \frac{\partial}{\partial y} \right) v = v_x + 3v_y = 0$$

which we know has the solution  $v(x, y) = f(3x - y)$ , so

$$u_x - u_y = f(3x - y)$$

Now we can reorder our equation as

$$(\partial_x - \partial_y)(\partial_x + 3\partial_y)u = 0$$

and set  $w = (\partial_x + 3\partial_y)u$

Then

$$w_x - w_y = 0$$

Which we know has the solution  $w(x, y) = g(x + y)$ . So  $u_x + 3u_y = g(x + y)$ , which gives us a system of two equations:

$$\begin{cases} u_x - u_y = f(3x - y) \\ u_x + 3u_y = g(x + y) \end{cases}$$

Subtract the first equation from the second:

$$4u_y = -f(3x - y) + g(x + y)$$

Now we can integrate with respect to  $y$  to find that:

$$u(x, y) = F(3x - y) + G(x + y)$$

where  $F$  is the antiderivative of  $-f$  with respect to  $y$  and  $G$  is the antiderivative of  $g$  with respect to  $y$ .

Using the fact that  $u(0, x) = \sin x$ ,

$$u(0, x) = \sin x = F(3x) + G(x) \tag{1}$$

now replace  $x$  with a new neutral variable,  $\alpha$  and differentiate:

$$\sin \alpha = F(3\alpha) + G(\alpha)$$

$$\cos \alpha = 3F'(3\alpha) + G'(\alpha) \tag{2}$$

But we can also differentiate  $u(x, y) = F(3x - y) + G(x + y)$  with respect to  $y$  to get

$$u_y(x, y) = -F'(3x - y) + G'(x + y)$$

but from our initial conditions, we know

$$u_y(0, x) = -F'(3x - 0) + G'(x + 0) = x$$

Let's replace  $x$  by our neutral variable  $\alpha$  and solve for  $F'$ :

$$F'(\alpha) = G'(3\alpha) - \alpha$$

Now plug this into (2):

$$\cos \alpha = 3G'(\alpha) - 3\alpha + G'(\alpha)$$

$$G(\alpha) = \frac{1}{4} \int \cos \alpha + 3\alpha = \frac{\sin \alpha}{4} + \frac{3\alpha^2}{8}$$

So that means (1) becomes:

$$\sin \alpha = F(3\alpha) + \frac{\sin \alpha}{4} + \frac{3\alpha^2}{8}$$

$$F(\alpha) = \frac{3 \sin(\frac{\alpha}{3})}{4} - \frac{\alpha^2}{24}$$

Which means  $u(x, y) = F(3x - y) + G(x + y)$  becomes

Solution

$$u(x, y) = \frac{3}{4} \sin \left( x - \frac{y}{3} \right) - \frac{(3x - y)^2}{24} + \frac{\sin(x + y)}{4} + \frac{3(x + y)^2}{8}$$

Check solution

$$u_y = \frac{-1}{4} \cos \left( x - \frac{y}{3} \right) + \frac{\cos(x + y)}{4} + x + \frac{2y}{3}$$

$$u_{yy} = \frac{-1}{12} \sin \left( x - \frac{y}{3} \right) - \frac{\sin(x + y)}{4} + \frac{2}{3}$$

$$u_x = \frac{3}{4} \cos \left( x - \frac{y}{3} \right) + \frac{\cos(x + y)}{4} + y$$

$$u_{xx} = \frac{-3}{4} \sin \left( x - \frac{y}{3} \right) - \frac{\sin(x + y)}{4}$$

$$u_{xy} = \frac{1}{4} \sin \left( x - \frac{y}{3} \right) - \frac{\sin(x + y)}{4} + 1$$

Check that  $u_{xx} + 2u_{xy} - 3u_{yy} = 0$

$$\left( \frac{-3}{4} + \frac{2}{4} + \frac{1}{4} \right) \sin \left( x - \frac{y}{3} \right) + \left( \frac{-1}{4} + \frac{-2}{4} + \frac{3}{4} \right) \sin(x + y) + (0 + 2 - 3) = 0$$

### Question 3

Solve the Neumann boundary value problem for the wave equation on half line:

$$\begin{cases} u_{tt} = c^2 u_{xx} + f(t, x) & 0 < x < \infty \\ u(0, x) = \phi(x) & 0 < x < \infty \\ u_t(0, x) = \psi(x) & 0 < x < \infty \\ u_x(t, 0) = h(t) & t > 0 \end{cases}$$

The method we will use is to first convert to homogenous Neumann boundary conditions, and then extend the equation to the entire line using the reflection method.

First, define  $w(t, x) = u(t, x) - H(t)$ , where  $H'(t) = h(t)$ .

Then our IBVP becomes:

$$\begin{cases} \square w = \square u - H''(t) = f(t, x) - H''(t) \equiv F(t, x) & x \geq 0, t > 0 \\ w(0, x) = u(0, x) - H(0) = \phi(x) - H(0) \equiv \Phi(x) & x \geq 0 \\ w_t(0, x) = u_t(0, x) - H'(0) = \psi(x) - H'(0) \equiv \Psi(x) & x \geq 0 \\ w_x(0, x) = u_x(t, 0) - H'(0) = u_x(t, 0) - h(t) \equiv 0 & t \geq 0 \end{cases}$$

Now, since we have homogeneous Neumann boundary conditions, we will use even extensions of the initial data to the whole line, namely:

$$\Phi_{\text{even}}(x) = \begin{cases} \Phi(x) & \text{for } x \geq 0 \\ \Phi(-x) & \text{for } x \leq 0 \end{cases}, \quad \Psi_{\text{even}}(x) = \begin{cases} \Psi(x) & \text{for } x \geq 0 \\ \Psi(-x) & \text{for } x \leq 0 \end{cases}, \quad F_{\text{even}}(t, x) = \begin{cases} F(t, x) & \text{for } x \geq 0 \\ F(t, -x) & \text{for } x \leq 0 \end{cases} \quad (3)$$

Then we assume  $U(t, x)$  is an extension of  $w(t, x)$  to the whole line. Since the initial condition and non-homogeneous parts are both even, the solution  $U$  is even, so we automatically have  $U_x(t, 0) = 0$ .

Consider

$$\begin{cases} \square U(t, x) = F_{\text{even}}(t, x) & \text{for } x \in \mathbb{R}, t > 0 \\ U(0, x) = \Phi_{\text{even}}(x) & \text{for } x \in \mathbb{R} \\ U_t(0, x) = \Psi_{\text{even}}(x) & \text{for } x \in \mathbb{R} \end{cases} \quad (4)$$

The solution at point  $(t_0, x_0)$  is

$$U(t_0, x_0) = \frac{\Phi_{\text{even}}(x_0 + ct_0) + \Phi_{\text{even}}(x_0 - ct_0)}{2} + \frac{1}{2c} \int_{x_0 - ct_0}^{x_0 + ct_0} \Psi_{\text{even}}(x) dx + \frac{1}{2c} \int_0^{t_0} \int_{x_0 - c(t_0 - t)}^{x_0 + c(t_0 - t)} F_{\text{even}}(t, x) dx dt$$

Assume  $t_0 > 0, x_0 > 0 \Rightarrow x_0 + ct_0 > 0$ . Now assume  $x_0 > ct_0$ , so the domain of dependence is entirely to the right of the line  $x = ct$ . This means  $w(t_0, x_0) = U(t_0, x_0)$  and our solution is

$$\begin{aligned} w(t_0, x_0) &= \frac{\Phi(x_0 + ct_0) + \Phi(x_0 - ct_0)}{2} + \frac{1}{2c} \int_{x_0 - ct_0}^{x_0 + ct_0} \Psi(x) dx + \frac{1}{2c} \int_0^{t_0} \int_{x_0 - c(t_0 - t)}^{x_0 + c(t_0 - t)} F(t, x) dx dt \\ &= \frac{\phi(x_0 + ct_0) - H(0) + \phi(x_0 - ct_0) - H(0)}{2} + \frac{1}{2c} \int_{x_0 - ct_0}^{x_0 + ct_0} \psi(x) - H'(0) dx + \frac{1}{2c} \int_0^{t_0} \int_{x_0 - c(t_0 - t)}^{x_0 + c(t_0 - t)} f(t, x) - H''(t) dx dt \\ w(t_0, x_0) &= \frac{\phi(x_0 + ct_0) + \phi(x_0 - ct_0)}{2} - H(0) + \frac{1}{2c} \int_{x_0 - ct_0}^{x_0 + ct_0} \psi(x) dx - \frac{H'(0)}{2c} [(x_0 + ct_0) - (x_0 - ct_0)] + \\ &\quad \frac{1}{2c} \int_0^{t_0} \int_{x_0 - c(t_0 - t)}^{x_0 + c(t_0 - t)} f(t, x) dx dt - \frac{1}{2c} \int_0^{t_0} H''(t) \int_{x_0 - c(t_0 - t)}^{x_0 + c(t_0 - t)} dx dt \end{aligned}$$

We can rewrite this as

$$\begin{aligned} w(t_0, x_0) &= \frac{\phi(x_0 + ct_0) + \phi(x_0 - ct_0)}{2} + \frac{1}{2c} \int_{x_0 - ct_0}^{x_0 + ct_0} \psi(x) dx + \frac{1}{2c} \int_0^{t_0} \int_{x_0 - c(t_0 - t)}^{x_0 + c(t_0 - t)} f(t, x) dx dt \\ &\quad - H(0) - H'(0)t_0 - \frac{1}{2c} \int_0^{t_0} H''(t) 2c(t_0 - t) dt. \end{aligned}$$

Call the section in blue “A”.

This means we have

$$w(t_0, x_0) = A - H(0) - H'(0)t_0 - \int_0^{t_0} H''(t)(t_0 - t) dt \quad (5)$$

Now we use integration by parts on the final integral in (5), setting

$$u = (t_0 - t), \quad du = -dt, \quad dv = H''(t)dt, \quad v = H'(t)$$

,

so that

$$\int_0^{t_0} H''(t)(t_0 - t) dt = H'(t)(t_0 - t)|_0^{t_0} + \int_0^{t_0} H'(t) dt = -H'(0)t_0 + H(t_0) - H(0)$$

So (5) becomes

$$\begin{aligned} w(t_0, x_0) &= A - H(0) - H'(0)t_0 + H'(0)t_0 - H(t_0) + H(0) \\ w(t_0, x_0) &= A - H(t_0) \end{aligned}$$

Since  $u(t, x) = u(t, x) + H(t)$ , we get  $u(t_0, x_0) = A$ , or

$$u(t_0, x_0) = \frac{\phi(x_0 + ct_0) + \phi(x_0 - ct_0)}{2} + \frac{1}{2c} \int_{x_0 - ct_0}^{x_0 + ct_0} \psi(x) dx + \frac{1}{2c} \int_0^{t_0} \int_{x_0 - c(t_0 - t)}^{x_0 + c(t_0 - t)} f(t, x) dx dt \quad (6)$$

when  $x_0 > ct_0$ .

Now we must consider what happens when  $x_0 < ct_0$ , such that  $x_0 - ct_0 < 0$ . Then we have

$$\begin{aligned} w(t_0, x_0) &= \frac{\phi(x_0 + ct_0) - H(0) + \phi(ct_0 - x_0) - H(0)}{2} + \\ &\frac{1}{2c} \int_{x_0 - ct_0}^0 \psi(-x) - H'(0) dx + \frac{1}{2c} \int_0^{x_0 + ct_0} \psi(x) - H'(0) dx + \\ &\frac{1}{2c} \iint_{\Delta} F_{\text{even}}(t, x) dx dt \end{aligned}$$

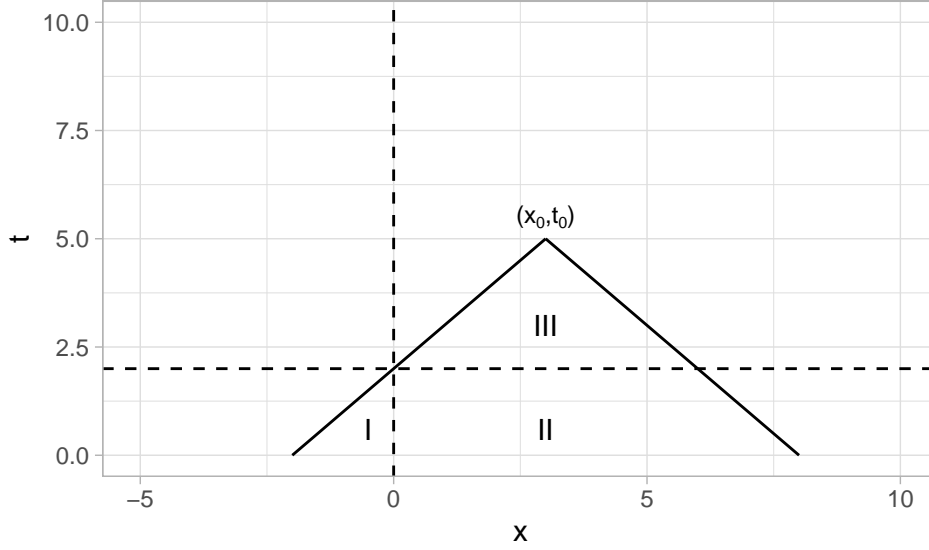
$$\begin{aligned} w(t_0, x_0) &= \frac{\phi(x_0 + ct_0) + \phi(ct_0 - x_0)}{2} - H(0) + \\ &\frac{1}{2c} \int_{x_0 - ct_0}^0 \psi(-x) - H'(0) dx + \frac{1}{2c} \int_0^{x_0 + ct_0} \psi(x) - H'(0) dx + \\ &\frac{1}{2c} \iint_{\Delta} F_{\text{even}}(t, x) dx dt \end{aligned}$$

Where  $\Delta$  is the 2D region in the diagram below.

Violet section:

$$\begin{aligned} &\frac{1}{2c} \int_{x_0 - ct_0}^0 \psi(-x) dx + \frac{1}{2c} \int_0^{x_0 + ct_0} \psi(x) dx - \frac{1}{2c} [H'(0)(0 - (x - ct_0)) + H'(0)(x + ct_0 - 0)] = \\ &\frac{1}{2c} \left[ \int_0^{ct_0 - x_0} \psi(x) dx + \int_0^{x_0 + ct_0} \psi(x) dx \right] - H'(0)t_0 \end{aligned}$$

Now we investigate  $\frac{1}{2c} \iint_{\Delta} F_{\text{even}}(t, x) dx dt$ .



$$\begin{aligned}
 \text{I} &= \frac{1}{2c} \int_0^{t_0 - \frac{x_0}{c}} \int_{x_0 - c(t_0 - t)}^0 F(t, -x) dx dt, & \text{II} &= \frac{1}{2c} \int_0^{t_0 - \frac{x_0}{c}} \int_0^{x_0 + c(t_0 - t)} F(t, x) dx dt, \\
 \text{III} &= \frac{1}{2c} \int_{t_0 - \frac{x_0}{c}}^{t_0} \int_{x_0 - c(t_0 - t)}^{x_0 + c(t_0 - t)} F(t, x) dx dt
 \end{aligned}$$

Now we will evaluate each integral separately:

$$\begin{aligned}
 \text{I} &= \frac{1}{2c} \int_0^{t_0 - \frac{x_0}{c}} \int_{x_0 - c(t_0 - t)}^0 f(t, -x) - H''(t) dx dt \\
 &= \frac{1}{2c} \int_0^{t_0 - \frac{x_0}{c}} \int_{x_0 - c(t_0 - t)}^0 f(t, -x) dx dt - \frac{1}{2c} \int_0^{t_0 - \frac{x_0}{c}} H''(t) \int_{x_0 - c(t_0 - t)}^0 dx dt \\
 &= -\frac{1}{2c} \int_0^{t_0 - \frac{x_0}{c}} \int_{-x_0 + c(t_0 - t)}^0 f(t, x) dx dt + \frac{1}{2c} \int_0^{t_0 - \frac{x_0}{c}} H''(t) (x_0 - c(t_0 - t)) dt \\
 &= \frac{1}{2c} \int_0^{t_0 - \frac{x_0}{c}} \int_0^{c(t_0 - t) - x_0} f(t, x) dx dt + \frac{1}{2c} \int_0^{t_0 - \frac{x_0}{c}} H''(t) (x_0 - c(t_0 - t)) dt \\
 \text{II} &= \frac{1}{2c} \int_0^{t_0 - \frac{x_0}{c}} \int_0^{x_0 + c(t_0 - t)} f(t, x) dx dt - \frac{1}{2c} \int_0^{t_0 - \frac{x_0}{c}} H''(t) \int_0^{x_0 + c(t_0 - t)} dx dt \\
 &= \frac{1}{2c} \int_0^{t_0 - \frac{x_0}{c}} \int_0^{x_0 + c(t_0 - t)} f(t, x) dx dt - \frac{1}{2c} \int_0^{t_0 - \frac{x_0}{c}} H''(t) (x_0 + c(t_0 - t)) dt \\
 \text{III} &= \frac{1}{2c} \int_{t_0 - \frac{x_0}{c}}^{t_0} \int_{x_0 - c(t_0 - t)}^{x_0 + c(t_0 - t)} f(t, x) dx dt - \frac{1}{2c} \int_{t_0 - \frac{x_0}{c}}^{t_0} H''(t) \int_{x_0 - c(t_0 - t)}^{x_0 + c(t_0 - t)} dx dt \\
 &= \frac{1}{2c} \int_{t_0 - \frac{x_0}{c}}^{t_0} \int_{x_0 - c(t_0 - t)}^{x_0 + c(t_0 - t)} f(t, x) dx dt - \frac{1}{2c} \int_{t_0 - \frac{x_0}{c}}^{t_0} H''(t) 2c(t_0 - t) dt \\
 &= \frac{1}{2c} \int_{t_0 - \frac{x_0}{c}}^{t_0} \int_{x_0 - c(t_0 - t)}^{x_0 + c(t_0 - t)} f(t, x) dx dt - \int_{t_0 - \frac{x_0}{c}}^{t_0} H''(t) (t_0 - t) dt
 \end{aligned}$$

Now we use integration by parts on the final integral in the above expression, setting

$$u = (t_0 - t), \quad du = -dt, \quad dv = H''(t)dt, \quad v = H'(t)$$

so that

$$\begin{aligned} \int_{t_0 - \frac{x_0}{c}}^{t_0} H''(t)(t_0 - t) dt &= H'(t)(t_0 - t) \Big|_{t_0 - \frac{x_0}{c}}^{t_0} + \int_{t_0 - \frac{x_0}{c}}^{t_0} H'(t) dt \\ &= -\frac{x_0}{c} H' \left( t_0 - \frac{x_0}{c} \right) + H(t_0) - H \left( t_0 - \frac{x_0}{c} \right) \end{aligned}$$

So III is

$$\frac{1}{2c} \int_{t_0 - \frac{x_0}{c}}^{t_0} \int_{x_0 - c(t_0 - t)}^{x_0 + c(t_0 - t)} f(t, x) dx dt + \frac{x_0}{c} H' \left( t_0 - \frac{x_0}{c} \right) - H(t_0) + H \left( t_0 - \frac{x_0}{c} \right)$$

And now we can find the sum of the three regions:

$$\begin{aligned} \text{I} + \text{II} + \text{III} &= \frac{1}{2c} \int_0^{t_0 - \frac{x_0}{c}} \int_0^{c(t_0 - t) - x_0} f(t, x) dx dt + \frac{1}{2c} \int_0^{t_0 - \frac{x_0}{c}} H''(t)(x_0 - c(t_0 - t)) dt + \\ &\quad \frac{1}{2c} \int_0^{t_0 - \frac{x_0}{c}} \int_0^{x_0 + c(t_0 - t)} f(t, x) dx dt - \frac{1}{2c} \int_0^{t_0 - \frac{x_0}{c}} H''(t)(x_0 + c(t_0 - t)) dt + \\ &\quad \frac{1}{2c} \int_{t_0 - \frac{x_0}{c}}^{t_0} \int_{x_0 - c(t_0 - t)}^{x_0 + c(t_0 - t)} f(t, x) dx dt + \\ &\quad \frac{x_0}{c} H' \left( t_0 - \frac{x_0}{c} \right) - H(t_0) + H \left( t_0 - \frac{x_0}{c} \right) \\ &= \frac{1}{2c} \int_0^{t_0 - \frac{x_0}{c}} \int_0^{c(t_0 - t) - x_0} f(t, x) dx dt + \frac{1}{2c} \int_0^{t_0 - \frac{x_0}{c}} \int_0^{x_0 + c(t_0 - t)} f(t, x) dx dt + \frac{1}{2c} \int_{t_0 - \frac{x_0}{c}}^{t_0} \int_{x_0 - c(t_0 - t)}^{x_0 + c(t_0 - t)} f(t, x) dx dt + \\ &\quad \frac{1}{2c} \int_0^{t_0 - \frac{x_0}{c}} H''(t)(x_0 - c(t_0 - t)) dt - \frac{1}{2c} \int_0^{t_0 - \frac{x_0}{c}} H''(t)(x_0 + c(t_0 - t)) dt + \\ &\quad \frac{x_0}{c} H' \left( t_0 - \frac{x_0}{c} \right) - H(t_0) + H \left( t_0 - \frac{x_0}{c} \right) \\ &\quad \frac{1}{2c} \int_0^{t_0 - \frac{x_0}{c}} H''(t)(x_0 - c(t_0 - t)) dt - \frac{1}{2c} \int_0^{t_0 - \frac{x_0}{c}} H''(t)(x_0 + c(t_0 - t)) dt = \\ &\quad \frac{1}{2c} \int_0^{t_0 - \frac{x_0}{c}} H''(t)(-c(t_0 - t)) dt - \frac{1}{2c} \int_0^{t_0 - \frac{x_0}{c}} H''(t)c(t_0 - t) dt = \\ &\quad -\frac{1}{2c} \int_0^{t_0 - \frac{x_0}{c}} H''(t)c(t_0 - t) dt - \frac{1}{2c} \int_0^{t_0 - \frac{x_0}{c}} H''(t)c(t_0 - t) dt = \\ &\quad - \int_0^{t_0 - \frac{x_0}{c}} H''(t)(t_0 - t) dt \end{aligned}$$

Now we use integration by parts, so that

$$\int_0^{t_0 - \frac{x_0}{c}} H''(t)(t_0 - t) dt = H'(t)(t_0 - t) \Big|_0^{t_0 - \frac{x_0}{c}} + \int_0^{t_0 - \frac{x_0}{c}} H'(t) dt = \frac{x_0}{c} H' \left( t_0 - \frac{x_0}{c} \right) + H \left( t_0 - \frac{x_0}{c} \right) - H(0)$$

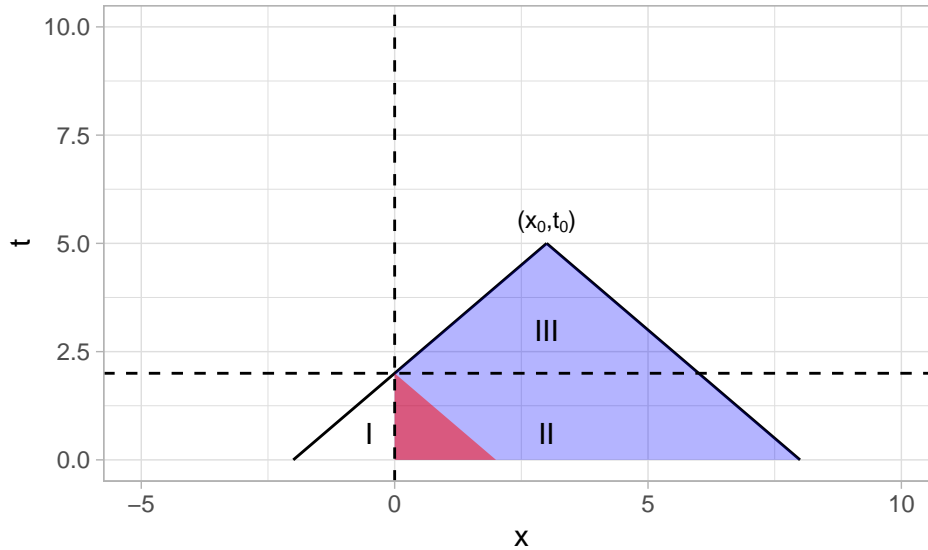
$$\text{I} + \text{II} + \text{III} =$$

$$\begin{aligned} & \frac{1}{2c} \int_0^{t_0 - \frac{x_0}{c}} \int_0^{c(t_0-t)-x_0} f(t, x) dx dt + \frac{1}{2c} \int_0^{t_0 - \frac{x_0}{c}} \int_0^{x_0+c(t_0-t)} f(t, x) dx dt + \frac{1}{2c} \int_{t_0 - \frac{x_0}{c}}^{t_0} \int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} f(t, x) dx dt \\ & - \frac{x_0}{c} H' \left( t_0 - \frac{x_0}{c} \right) - H \left( t_0 - \frac{x_0}{c} \right) + H(0) + \\ & \frac{x_0}{c} H' \left( t_0 - \frac{x_0}{c} \right) - H(t_0) + H \left( t_0 - \frac{x_0}{c} \right) \end{aligned}$$

$$\text{I} + \text{II} + \text{III} =$$

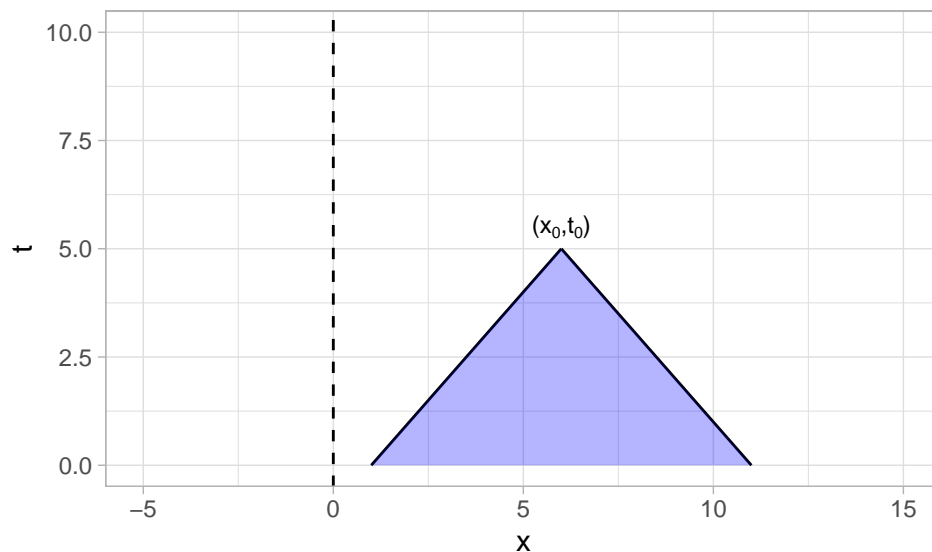
$$\begin{aligned} & \frac{1}{2c} \int_0^{t_0 - \frac{x_0}{c}} \int_0^{c(t_0-t)-x_0} f(t, x) dx dt + \frac{1}{2c} \int_0^{t_0 - \frac{x_0}{c}} \int_0^{x_0+c(t_0-t)} f(t, x) dx dt + \frac{1}{2c} \int_{t_0 - \frac{x_0}{c}}^{t_0} \int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} f(t, x) dx dt + \\ & H(0) - H(t_0) \end{aligned}$$

$$\begin{aligned} w(t_0, x_0) &= \frac{\phi(x_0 + ct_0) + \phi(ct_0 - x_0)}{2} - H(0) + \\ & \frac{1}{2c} \left[ \int_0^{ct_0-x_0} \psi(x) dx + \int_0^{x_0+ct_0} \psi(x) dx \right] - H'(0)t_0 + \\ & \frac{1}{2c} \int_0^{t_0 - \frac{x_0}{c}} \int_0^{c(t_0-t)-x_0} f(t, x) dx dt + \frac{1}{2c} \int_0^{t_0 - \frac{x_0}{c}} \int_0^{x_0+c(t_0-t)} f(t, x) dx dt + \frac{1}{2c} \int_{t_0 - \frac{x_0}{c}}^{t_0} \int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} f(t, x) dx dt + \\ & H(0) - H(t_0) \end{aligned}$$



When  $x_0 < ct_0$ , we're integrating  $f$  over the domain of dependence (the shaded blue region, including the red triangle) but then adding  $f$  integrated over the red triangle. When  $x_0 > ct_0$ , we just integrate  $f$  over the blue triangle.





### Solution

So for  $ct_0 > x_0$ ,  $u(t_0, x_0) = w(t_0, x_0) + H(t_0)$ , which means

$$\begin{aligned}
 u(t_0, x_0) &= \frac{\phi(x_0 + ct_0) + \phi(ct_0 - x_0)}{2} + \\
 &\frac{1}{2c} \left[ \int_0^{ct_0 - x_0} \psi(x) dx + \int_0^{x_0 + ct_0} \psi(x) dx \right] - H'(0)t_0 + \\
 &\frac{1}{2c} \int_0^{t_0 - \frac{x_0}{c}} \int_0^{c(t_0 - t) - x_0} f(t, x) dx dt + \frac{1}{2c} \int_0^{t_0 - \frac{x_0}{c}} \int_0^{x_0 + c(t_0 - t)} f(t, x) dx dt + \\
 &\frac{1}{2c} \int_{t_0 - \frac{x_0}{c}}^{t_0} \int_{x_0 - c(t_0 - t)}^{x_0 + c(t_0 - t)} f(t, x) dx dt
 \end{aligned}$$

While for  $ct_0 > x_0$ , from (6), we have

$$u(t_0, x_0) = \frac{\phi(x_0 + ct_0) + \phi(x_0 - ct_0)}{2} + \frac{1}{2c} \int_{x_0 - ct_0}^{x_0 + ct_0} \psi(x) dx + \frac{1}{2c} \int_0^{t_0} \int_{x_0 - c(t_0 - t)}^{x_0 + c(t_0 - t)} f(t, x) dx dt$$

### Question 4

Consider the 3D wave equation for  $u(t, x, y, z)$ :

$$u_{tt} = c^2 \Delta u \quad (x, y, z) \in \mathbb{R}^3, \quad t > 0$$

Change the coordinates to spherical coordinates. Assume the solution is spherically symmetric, so that  $u(t, x, y, z) = u(t, r)$  and does not depend on  $\theta$  and  $\phi$ . Find the solution for  $u(0, r) = 0$  and

$$u_t(0, r) = \begin{cases} 1 & |r| \leq 1 \\ 0 & |r| > 1 \end{cases}$$

Hint: use the substitution  $u(t, r) = \frac{1}{r}w(t, r)$ .

First, we need to derive the formula for the Laplacian in spherical coordinates.

We know the equation for the Laplacian in polar coordinates is:

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

Now let's convert to spherical coordinates:

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{s^2 + z^2}$$

$$\begin{aligned} x &= s \cos \phi, & y &= s \sin \phi, & z &= r \cos \theta \\ s &= r \sin \theta \end{aligned}$$

By the two-dimensional Laplacian, we have

$$u_{zz} + u_{ss} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta},$$

$$u_{xx} + u_{yy} = u_{ss} + \frac{1}{s}u_s + \frac{1}{s^2}u_{\phi\phi}$$

We add these two equations and cancel  $u_s$  to get

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \frac{1}{s}u_s + \frac{1}{s^2}u_{\phi\phi}$$

Now since  $u$  doesn't depend on  $\theta$  or  $\phi$ , we have

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{s}u_s = u_{rr} + \frac{1}{r}u_r + \frac{1}{r \sin \theta}u_s$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial s} + \frac{\partial u}{\partial \theta} \frac{\partial \theta}{\partial s} + \frac{\partial u}{\partial \phi} \frac{\partial \phi}{\partial s} = u_r \frac{1}{\sin \theta} + 0 + 0 = u_r \frac{s}{r}$$

So with our change of variables, we have

$$u_{tt} = c^2 \left( u_{rr} + \frac{2}{r}u_r \right)$$

Now set  $w = ru$ , or  $u = \frac{w}{r}$ . Then

$$w_t = ru_t, \quad w_{tt} = ru_{tt}, \quad u_{tt} = \frac{w_{tt}}{r}$$

$$w_t = ru_t, \quad w_{tt} = ru_{tt}, \quad u_{tt} = \frac{w_{tt}}{r}$$

$$u_r = \frac{w_r}{r} - \frac{w}{r^2}$$

$$u_{rr} = \frac{w_{rr}}{r} - \frac{2w_r}{r^2} + \frac{2w}{r^3}$$

So  $u_{tt} = c^2 (u_{rr} + \frac{2}{r}u_r)$  becomes

$$\frac{w_{tt}}{r} = c^2 \left( \frac{w_{rr}}{r} - \frac{2w_r}{r^2} + \frac{2w}{r^3} + \frac{2}{r} \left( \frac{w_r}{r} - \frac{w}{r^2} \right) \right),$$

which simplifies to

$$w_{tt} = c^2 w_{rr}.$$

But this is just the wave equation, giving us the following IVP:

$$\begin{cases} w_{tt} = c^2 w_{rr} & t > 0, r \geq 0 \\ u(0, r) = 0 & r \geq 0 \\ u_t(0, r) = \begin{cases} 1 & |r| \leq 1 \\ 0 & |r| > 1 \end{cases} & r \geq 0 \end{cases}$$

Although  $r$  is nonnegative, since  $\varphi = 0$  and there are absolute value bars in  $\phi$ , we can safely use d'Alembert's formula to find the solution:

$$w(t, r) = \frac{\varphi(r+ct) + \varphi(r-ct)}{2} + \frac{1}{2c} \int_{r-ct}^{r+ct} \psi(s) ds$$

Since  $\varphi = 0$ ,

$$w(t, r) = \frac{1}{2c} \int_{r-ct}^{r+ct} \psi(s) ds$$

Now we have 4 cases:

Case 1:  $r - ct \geq -1, r + ct \leq 1$

$$\begin{aligned} w(t, r) &= \frac{1}{2c} \int_{r-ct}^{r+ct} s ds \\ &= \frac{1}{4c} ((r+ct)^2 - (r-ct)^2) \\ &= \frac{1}{4c} (r^2 + 2crt + c^2t^2 - r^2 + 2crt - c^2t^2) = \frac{4crt}{4c} = rt \end{aligned}$$

Case 2:  $r - ct < -1, r + ct > 1$

$$\begin{aligned} w(t, r) &= \frac{1}{2c} \int_{-1}^1 s ds = \\ &= \frac{1}{4c} (1 - 1) = 0 \end{aligned}$$

Case 3:  $r - ct < -1, r + ct \leq 1$

$$\begin{aligned} w(t, r) &= \frac{1}{2c} \int_{-1}^{r+ct} s ds \\ &= \frac{1}{4c} ((r+ct)^2 - 1) \end{aligned}$$

Case 4:  $r - ct \geq -1, r + ct > 1$

$$\begin{aligned} w(t, r) &= \frac{1}{2c} \int_{r-ct}^1 s ds \\ &= \frac{1}{4c} (1 - (r-ct)^2) \end{aligned}$$

Since  $u = \frac{w}{r}$ , this means we have

Solution

$$u(t, r) = \begin{cases} t & \text{if } r - ct \geq -1, r + ct \leq 1 \\ 0 & \text{if } r - ct < -1, r + ct > 1 \\ \frac{1}{4rc}((r + ct)^2 - 1) & \text{if } r - ct < -1, r + ct \leq 1 \\ \frac{1}{4rc}(1 - (r - ct)^2) & \text{if } r - ct \geq -1, r + ct > 1 \end{cases}$$

## Question 5

Consider the following Dirichlet boundary value problem:

$$\begin{cases} u_{tt} + x(t, x)u_t = u_{xx} & 0 < x < 1 \\ u(0, x) = \phi(x) & 0 < x < 1 \\ u_t(0, x) = \psi(x) & 0 < x < 1 \\ u(t, 0) = u(t, 1) = 0 & t \geq 0 \end{cases}$$

Assume that  $|a(t, x)| \leq m$  for some constant  $m$  and all  $0 < x < 1$  and  $t \geq 0$ . Let

$$E(t) = \frac{1}{2} \int_0^1 (u_t(t, x)^2 + u_x(t, x)^2) dx$$

(1) Show that

$$\frac{dE(t)}{dt} \leq 2mE(t) \quad (7)$$

for  $t \geq 0$ .

First differentiate  $E(t)$ :

$$\begin{aligned} \frac{dE}{dt} &= \frac{d}{dt} \left[ \frac{1}{2} \int_0^1 (u_t^2 + u_x^2) dx \right] \\ &= \frac{1}{2} \int_0^1 \frac{\partial}{\partial t} (u_t^2 + u_x^2) dx \end{aligned} \quad (8)$$

$$= \frac{1}{2} \int_0^1 2u_t u_{tt} + 2u_x u_{xt} dx \quad (9)$$

$$= \int_0^1 u_t u_{tt} dx + \int_0^1 u_x u_{xt} dx =: I + J. \quad (10)$$

The equality (8) follows from differentiation under the integral sign, while (9) follows from the chain rule for partial derivatives.

Consider the integral  $J$  in (10). By integrating by parts, we can move one of the partials  $\frac{\partial}{\partial x}$  to the other factor, at the cost of introducing a minus sign and a boundary term. Hence  $J$  becomes

$$J = \int_0^1 u_x u_{xt} dx = - \int_0^1 u_{xx} u_t dx + [u_x u_t]_{x=0}^{x=1}. \quad (11)$$

The boundary term vanishes, since  $u(t, 0) \equiv u(t, 1) \equiv 0$  for  $t > 0$  implies that  $u_t$  is identically zero at  $x = 0, 1$ . So, substituting (11) for  $J$ , equation (10) becomes

$$\begin{aligned} I + J &= \int_0^1 u_t u_{tt} dx - \int_0^1 u_{xx} u_t dx \\ &= \int_0^1 u_t (u_{tt} - u_{xx}) dx \\ &= \int_0^1 u_t (-a u_t) dx \end{aligned} \tag{12}$$

$$= \int_0^1 (-a) u_t^2 dx. \tag{13}$$

Here, equality (12) just uses the PDE. Since  $u_t^2 \geq 0$  and  $-a \leq |-a| \leq m$ , we see that  $(-a)u_t^2 \leq mu_t^2$ . Hence, the expression in (13) satisfies the inequality

$$I + J = \int_0^1 (-a) u_t^2 dx \leq \int_0^1 m u_t^2 dx \leq m \int_0^1 u_t^2 + u_x^2 dx = 2mE. \tag{14}$$

Where we also used the fact that  $u_x^2 \geq 0$  means that  $m \int_0^1 u_t^2 dx \leq m \int_0^1 (u_t^2 + u_x^2) dx$ .

The desired inequality (7) follows from (14) and the fact that  $\frac{dE}{dt} = I + J$ .

- (2) Use part (1) and show that  $\frac{d}{dt} (e^{-2mE(t)}) \leq 0$  for all  $t \geq 0$ . Hence, by integration from  $[0, t]$ , we get that

$$E(t) \leq e^{2mt} E(0) \text{ for all } t \geq 0. \tag{15}$$

By the product rule,

$$\begin{aligned} \frac{d}{dt} (e^{-2mt} E) &= -2me^{-2mt} E + e^{-2mt} \frac{dE}{dt} \\ &= e^{-2mt} \left( \frac{dE}{dt} - 2mE \right) \\ &\leq e^{-2mt} \cdot 0 = 0. \end{aligned}$$

- (3) If  $\phi(x) = \psi(x) = 0$  for all  $0 < x < 1$ , what does this say about  $E(t)$  for  $t \geq 0$  and hence about  $u(t, x)$  for  $t \geq 0$ ?

Since  $u(0, x) \equiv \phi(x) \equiv 0$  for  $0 < x < 1$ , we see that  $u_x$  is identically zero at time  $t = 0$ . Similarly,  $u_t(0, x) \equiv \psi(x) \equiv 0$  for  $0 < x < 1$ . Thus at  $t = 0$ ,

$$E(0) = \frac{1}{2} \int_0^1 (u_t(t, 0)^2 + u_x(t, 0)^2) dx = \frac{1}{2} \int_0^1 0^2 + 0^2 dx = 0. \tag{16}$$

But (16) together with the inequality (15) implies that

$$0 \leq E(t) \leq e^{2mt} E(0) \leq 0$$

for all  $t \geq 0$ . Hence the energy  $E$  is identically zero. Since the integrand  $u_t^2 + u_x^2$  is nonnegative, this is only possible if  $u_t \equiv u_x \equiv 0$  for  $t > 0, 0 < x < 1$ , meaning that  $u$  varies with neither time nor position. But this implies that  $u$  must be identically zero everywhere.

(4) Use the previous part to prove uniqueness of the following problem:

$$\begin{cases} u_{tt} + a(t, x)u_t = u_{xx} & 0 < x < 1, t > 0 \\ u(0, x) = \phi(x) & 0 < x < 1 \\ u_t(0, x) = \psi(x) & 0 < x < 1 \\ u(t, 0) = f(t) & t \geq 0 \\ u(t, 1) = g(t) & t \geq 0 \end{cases}$$

Let  $u$  and  $v$  be two solutions, and define  $w := u - v$ . Observe that  $w$  satisfies the original boundary value problem along with the conditions specified in part (3), implying that  $w \equiv 0$ . Hence  $u - v \equiv 0$  and so any solution  $u$  must be unique.  $\square$

## Problem 6

Does the D'Alembert method work for the wave equation  $u_{tt} = c(x)^2 u_{xx}$ ? What about  $u_{tt} = c(t)^2 u_{xx}$ ? Why?

Let's assume that we can factor this wave equation with  $c = c(x)$  as we could with constant  $c$ :

$$(\partial_t + c(x)\partial_x)(\partial_t - c(x)\partial_x)u = 0$$

Now, let's distribute:

$$\begin{aligned} (\partial_t + c(x)\partial_x)(u_t - c(x)u_x) &= 0 \\ u_{tt} - c(x)u_{xt} + c(x)u_{xt} - c(x)(c'(x)u_x + c(x)u_{xx}) &= 0 \\ u_{tt} - c(x)^2 u_{xx} - c(x)c'(x)u_x &= 0 \end{aligned}$$

But this is only equivalent to our original equation if  $c'(x) = 0$  for all  $x$ , i.e., if  $c$  is constant.

We can repeat the process for  $c = c(t)$ :

$$\begin{aligned} (\partial_t + c(t)\partial_x)(\partial_t - c(t)\partial_x)u &= 0 \\ (\partial_t + c(t)\partial_x)(u_t - c(t)u_x) &= 0 \\ u_{tt} - (c'(t)u_x + c(t)u_{xt}) + c(t)u_{xt} - c(t)^2 u_{xx} &= 0 \\ u_{tt} - c(t)^2 u_{xx} - c'(t)u_x &= 0 \end{aligned}$$

Which, as before, is only equivalent to our original equation if  $c'(t) = 0$  for all  $t$ , i.e., if  $c$  is constant.

## Problem 7 (The Poisson-Darboux Equation)

Solve the initial value problem

$$\begin{cases} u_{tt} - u_{xx} - \frac{2}{x}u_x = 0 & -\infty < x < \infty, t > 0 \\ u(0, x) = 0 & -\infty < x < \infty \\ u_t(0, x) = g(x) & -\infty < x < \infty \end{cases}$$

where  $g(x) = g(-x)$  is an even function. Hint: set  $w = xu$ .

Using the results from when we set  $w = ru$  in Problem 4,

$$u_t = \frac{w_t}{x}, u_{tt} = \frac{w_{tt}}{x}$$

$$u_x = \frac{w_x}{x} - \frac{w}{x^2}$$

$$u_{xx} = \frac{w_{xx}}{x} - \frac{2w_x}{x^2} + \frac{2w}{x^3}$$

So  $u_{tt} - u_{xx} - \frac{2}{x}u_x = 0$  becomes

$$\frac{w_{tt}}{x} - \frac{w_{xx}}{x} + \frac{2w_x}{x^2} - \frac{2w}{x^3} - \frac{2}{x} \left( \frac{w_x}{x} - \frac{w}{x^2} \right) = 0$$

Which simplifies to  $w_{tt} - w_{xx} = 0$ , and so  $w$  solves the wave equation when  $-\infty < x < \infty, t > 0$ .

Now observe that  $w(0, x) = xu(0, x) = 0$  and  $w_t(0, x) = xu_t(0, x) = xg(x)$ . Hence,  $w$  solves the initial value problem:

$$\begin{cases} w_{tt} - w_{xx} & -\infty < x < \infty, t > 0 \\ w(0, x) = 0 & -\infty < x < \infty, t > 0 \\ w_t(0, x) = xg(x) & -\infty < x < \infty, t > 0 \end{cases} \quad (17)$$

where  $g(x)$  is an even function.

By d'Alembert's formula, the solution to (17) is given by:

$$w(t, x) = \frac{1}{2} \int_{x-t}^{x+t} sg(s) ds$$

Hence since  $w_t = xu_t$ ,

$$w(t, x) = \frac{1}{2x} \int_{x-t}^{x+t} sg(s) ds, \quad (18)$$

assuming  $x \neq 0$ .

To handle the case when  $x = 0$ , assume that  $u$  is continuous in  $x$ , so that  $u(t, 0) = \lim_{x \rightarrow 0} u(t, x)$ . Define

$$I(y) = \int_0^y sg(s) ds.$$

This allows us to rewrite (18) as

$$u(t, x) = \frac{I(x+t) - I(x-t)}{2x} = \frac{I(x+t) - I(t-x)}{2x},$$

Noting that the evenness of  $g(s)$  means that  $sg(s)$  is odd, and so  $I(y)$  is even (one can check by substitution that  $I(y) = I(-y)$ ). Hence, letting  $x \rightarrow 0$ , we see that

$$u(t, 0) = \lim_{x \rightarrow 0} \frac{I(t+x) - I(t-x)}{2x}. \quad (19)$$

But this is just the form of a derivative, namely the symmetric derivative of  $I$  at  $t$ . By the Fundamental Theorem of Calculus, assuming  $g$  is continuous, we know the limit (19) exists and is necessarily equal to  $I'(t) = tg(t)$ .

Thus  $u(t, x)$  is given for any  $t \geq 0, x \in \mathbb{R}$  by

$$u(t, x) = \begin{cases} \frac{1}{2x} \int_{x-t}^{x+t} sg(s) ds, & x \neq 0, \\ tg(t), & x = 0 \end{cases} \quad (20)$$

## Problem 8

Solve the following characteristic initial value problem:

$$\begin{cases} y^3 u_{xx} - y u_{yy} + u_y = 0 & 0 < x < 4, \quad |y| \leq 2\sqrt{2} \\ u(x, y) = f(x) & x + \frac{y^2}{2} = 4 \text{ for } 2 \leq x \leq 4 \\ u(x, y) = g(x) & x - \frac{y^2}{2} = 0 \text{ for } 0 \leq x \leq 2 \end{cases}$$

where  $f(2) = g(2)$ . Hint: Use the transformation  $\eta = x - \frac{y^2}{2}$  and  $\xi = x + \frac{y^2}{2}$  and express the PDE in the coordinates  $(\xi, \eta)$ .

Set  $\eta = x - \frac{y^2}{2}$  and  $\xi = x + \frac{y^2}{2}$ . Then

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \\ \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial \xi} \right) + \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial \eta} \right) \\ &= \frac{\partial^2 u}{\partial \xi^2} \frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \eta \partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \eta \partial \xi} \frac{\partial \eta}{\partial x} + \frac{\partial^2 u}{\partial \eta^2} \frac{\partial \eta}{\partial x} \end{aligned}$$

Which we can simplify to

$$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

Now we do the same for  $y$ :

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial y} = y \frac{\partial u}{\partial \xi} - y \frac{\partial u}{\partial \eta}$$

But since  $\eta = x - \frac{y^2}{2}$  and  $\xi = x + \frac{y^2}{2}$ , we can rewrite  $y$  as  $\sqrt{\xi - \eta}$ .

$$\frac{\partial u}{\partial y} = \sqrt{\xi - \eta} \left( \frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta} \right)$$

Now using the product rule,

$$\frac{\partial^2 u}{\partial y^2} = \sqrt{\xi - \eta} \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta} \right) + \left( \frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta} \right)$$

Where the second term has been simplified because  $\frac{\partial}{\partial y}$  of  $\sqrt{\xi - \eta}$  is simply  $\frac{\partial}{\partial y} y = 1$ .

Now we use the chain rule again for the first term:

Now using the product rule,

$$\frac{\partial^2 u}{\partial y^2} = \sqrt{\xi - \eta} \left( \frac{\partial^2 u}{\partial \xi^2} \frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \eta \partial \xi} \frac{\partial \eta}{\partial y} - \frac{\partial^2 u}{\partial \eta^2} \frac{\partial \eta}{\partial y} - \frac{\partial^2 u}{\partial \eta \partial \xi} \frac{\partial \xi}{\partial y} \right) + \left( \frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta} \right)$$

Which simplifies to

$$u_{yy} = (\xi - \eta) (u_{\xi\xi} + u_{\eta\eta} - 2u_{\xi\eta}) + u_{\xi} - u_{\eta}$$



This means  $y^3 u_{xx} - y u_{yy} + u_y = 0$  becomes

$$(\xi - \eta)^{\frac{3}{2}} (u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) - (\xi - \eta)^{\frac{1}{2}} [(\xi - \eta) (u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) + u_{\xi} - u_{\eta}] + (\xi - \eta)^{\frac{1}{2}} (u_{\xi} - u_{\eta}) = 0$$

$$(\xi - \eta)^{\frac{3}{2}} (u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) - (\xi - \eta)^{\frac{3}{2}} (u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) - (\xi - \eta)^{\frac{1}{2}} (u_{\xi} - u_{\eta}) + (\xi - \eta)^{\frac{1}{2}} (u_{\xi} - u_{\eta}) = 0$$

$$(\xi - \eta)^{\frac{3}{2}} (4 u_{\xi\eta}) = 0$$

But by integrating with respect to both variables, we recognize  $u_{\xi\eta} = 0$  as having the solution

$$u(\xi, \eta) = h_1(\xi) + h_2(\eta),$$

or in our original coordinates,

$$u(x, y) = h_1 \left( x + \frac{y^2}{2} \right) + h_2 \left( x - \frac{y^2}{2} \right)$$

Now we use our given conditions. When  $2 \leq x \leq 4$ , we have  $g(x) = h_1(2x) + h_2(0)$ , which means  $h_1(x) = g\left(\frac{x}{2}\right) - h_2(0)$ .

Similarly, when  $0 \leq x \leq 2$ , we have  $f(x) = h_1(4) + h_2(2x - 4)$ , which means  $h_2(x) = f\left(\frac{x}{2} + 2\right) - h_1(4)$ .

Thus,

$$h_1(x) + h_2(x) = g\left(\frac{x}{2}\right) - h_2(0) + f\left(\frac{x}{2} + 2\right) - h_1(4)$$

But  $h_2(0) + h_1(4)$  is just  $g(x)$  when  $x = 2$  (or equivalently,  $f(x)$  when  $x = 2$ ),

so after replacing the  $x$  in  $h_1(x)$  by  $x + \frac{y^2}{2}$  and the  $x$  in  $h_2(x)$  by  $x - \frac{y^2}{2}$ , we get

Solution

$$u(x, y) = g\left(\frac{x}{2} + \frac{y^2}{4}\right) + f\left(\frac{x}{2} + \frac{y^2}{4} + 2\right) - f(2)$$

Check solution

$$u_y(x, y) = \frac{y}{2}g' \left( \frac{x}{2} + \frac{y^2}{4} \right) + \frac{y}{2}f' \left( \frac{x}{2} + \frac{y^2}{4} + 2 \right)$$

$$u_{yy}(x, y) = \frac{y^2}{4}g'' \left( \frac{x}{2} + \frac{y^2}{4} \right) + \frac{1}{2}g' \left( \frac{x}{2} + \frac{y^2}{4} \right) + \frac{y^2}{4}f'' \left( \frac{x}{2} + \frac{y^2}{4} + 2 \right) + \frac{1}{2}f' \left( \frac{x}{2} + \frac{y^2}{4} + 2 \right)$$

$$u_x(x, y) = \frac{1}{2}g' \left( \frac{x}{2} + \frac{y^2}{4} \right) + \frac{1}{2}f' \left( \frac{x}{2} + \frac{y^2}{4} + 2 \right)$$

$$u_{xx}(x, y) = \frac{1}{4}g'' \left( \frac{x}{2} + \frac{y^2}{4} \right) + \frac{1}{4}f'' \left( \frac{x}{2} + \frac{y^2}{4} + 2 \right)$$

This means  $y^3u_{xx} - yu_{yy} + u_y = 0$  becomes

$$\begin{aligned} & \left[ \frac{y^3}{4}g'' \left( \frac{x}{2} + \frac{y^2}{4} \right) + \frac{y^3}{4}f'' \left( \frac{x}{2} + \frac{y^2}{4} + 2 \right) \right] - \\ & \left[ \frac{y^3}{4}g'' \left( \frac{x}{2} + \frac{y^2}{4} \right) + \frac{y}{2}g' \left( \frac{x}{2} + \frac{y^2}{4} \right) + \frac{y^3}{4}f'' \left( \frac{x}{2} + \frac{y^2}{4} + 2 \right) + \frac{y}{2}f' \left( \frac{x}{2} + \frac{y^2}{4} + 2 \right) \right] + \\ & \left[ \frac{y}{2}g' \left( \frac{x}{2} + \frac{y^2}{4} \right) + \frac{y}{2}f' \left( \frac{x}{2} + \frac{y^2}{4} + 2 \right) \right] = 0 \end{aligned}$$