# MATH 245 Homework 4

Ruby Krasnow and Tommy Thach

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## Question 1: Find eigenvalues and eigenfunctions

### (a)

$-\frac{d^2}{dx^2}X(x) = \lambda X(x)$ in $0 < x < l$ with boundary conditions $X'(0) = 0 = X(l)$

Case 1: Positive eigenvalues, $\lambda = \beta^2 > 0$

Re-writing as $X'' + \lambda X = 0$, we will get the characteristic equation: $r^2 + \beta^2 = 0$. Since our characteristic equation has complex roots $r = \pm i\beta$, our solutions take the form

$$X(x) = A\sin(\beta x) + B\cos(\beta x)$$

Differentiating, we find that

$$X'(x) = \beta A\cos(\beta x) - \beta B\sin(\beta x)$$

Now plugging in our initial condition $X'(0) = 0$, we get $X'(0) = \beta A = 0$. And since we are in a case where $\beta \neq 0$, this means $B = 0$ so $X(x) = A\sin(\beta x)$. Now we use our boundary condition $X(l) = 0$ to get $X(l) = A\sin(\beta l) = 0$. If $A = 0$, then $X(0) = 0$ and this contradicts the definition of an eigenfunction. Therefore, we must have $\sin(\beta l) = 0$, which can only occur if $\beta = \frac{n\pi}{l}$. Therefore, this case gives us eigenvalues

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad n = 1, 2, 3...$$

with eigenfunctions

$$X_n(x) = \sin\left(\frac{n\pi x}{l}\right)$$

Case 2: Zero eigenvalues, $\lambda = 0$ $X'' = 0$ implies that $X(x)$ is of the form $Ax + B$, with derivative $X'(x) = A$. Now plugging in our initial condition $X'(0) = 0$, we get $A = 0$, which means $X(x) = B$. But from the boundary condition $X(l) = 0$, we get $B = 0$ and so $X(0) = 0$. Therefore, there are no eigenfunctions $X(x)$ that satisfy $-\frac{d^2}{dx^2}X(x) = \lambda X(x)$ when $\lambda = 0$ and hence no zero eigenvalues.

Case 3: Negative eigenvalues, $\lambda = -\beta^2 < 0$ This case gives us the characteristic equation: $r^2 - \beta^2 = 0$. Since our characteristic equation has distinct real roots $r = \pm\beta$, our solutions take the form

$$X(x) = Ae^{\beta x} + Be^{-\beta x}$$

Differentiating, we find that

$$X'(x) = \beta Ae^{\beta x} - \beta Be^{-\beta x}$$

$$X'(0) = \beta A - \beta B = 0$$

Since we are in a case where $\beta \neq 0$, this means $A - B = 0$, or $A = B$. Then the boundary condition gives

$$X(l) = Be^{\beta l} + Be^{-\beta l} = 0$$

Since $e^{\beta l}$ and $e^{-\beta l}$ are nonzero for all values of $l$, we must have $B = 0$, implying that again $X(x) = 0$.

Thus, this problem has only positive eigenvalues and their associated eigenfunctions as found in Case 1.

**(b)**

$x^2 X''(x) + xX'(x) + \lambda X(x) = 0$ in $1 < x < e$ with boundary conditions $X(1) = 0 = X(e)$.

We recognize this equation as having the same form as a second-order Cauchy-Euler equation, a linear homogeneous ODE of the form $ax^2y + bxy' + cy = 0$ with the auxiliary equation $ar(r-1) + br + c = 0$. Here, $a = b = 1$ and $c = \lambda$, so we have

$$r(r-1) + r + \lambda = 0$$
$$r^2 - r + r + \lambda = 0$$
$$r^2 + \lambda = 0$$

<u>Case 1:</u> Positive eigenvalues, $\lambda = \beta^2 > 0$

$$X(x) = A\sin(\beta \ln x) + B\cos(\beta \ln x)$$
$$X(1) = A\sin(0) + B\cos(0) = 0 \quad \longrightarrow \quad B = 0$$
$$X(e) = A\sin(\beta \ln e) = 0$$
$$A\sin(\beta) = 0$$

If $A = 0$, then $X(0) = 0$ and this contradicts the definition of an eigenfunction. Therefore, we must have $\sin(\beta) = 0$, which can only occur if $\beta = n\pi$. Therefore, this case gives us eigenvalues

$$\lambda_n = n^2\pi^2 \quad n = 1, 2, 3...$$

with eigenfunctions

$$X_n(x) = \sin(n\pi \ln x)$$

<u>Case 2:</u> Zero eigenvalues, $\lambda = 0$ $X'' = 0$ implies that $X(x)$ is of the form $Ax + B$. Now plugging in our initial condition $X(1) = 0$, we get $A + B = 0$, which means $X(x) = Ax - A$. But from the boundary condition $X(e) = 0$, we get $A(e-1) = 0$ which is only possible if $A = 0$ and accordingly $X(0) = 0$. Therefore, there are no eigenfunctions $X(x)$ that satisfy the boundary conditions when $X'' = 0$ and hence no zero eigenvalues.

<u>Case 3:</u> Negative eigenvalues, $\lambda = -\beta^2 < 0$ This case gives us the characteristic equation: $r^2 - \beta^2 = 0$. Since our characteristic equation has distinct real roots $r = \pm\beta$, our solutions take the form

$$X(x) = Ax^{\beta} + Bx^{-\beta}$$

$$X(1) = A + B = 0 \quad \longrightarrow \quad A = -B$$
$$X(x) = Ax^{\beta} - Ax^{-\beta}$$

Then the other boundary condition gives $X(e) = Ae^{\beta} - Ae^{-\beta} = 0$. Since $e^{\beta}$ and $e^{-\beta}$ are always nonzero, we must have $A = 0$, implying that again $X(x) = 0$.Thus, this problem has only positive eigenvalues and their associated eigenfunctions as found in Case 1.

**(c)**

On the interval $0 \leq x \leq 1$ of length one, consider the eigenvalue problem

$$-X'' = \lambda X, \quad X'(0) + X(0) = 0, \quad X(1) = 0$$

(i) Find an eigenfunction with eigenvalue zero. Call it $X_0(x)$.

$X'' = 0$ implies that $X(x)$ is of the form $Ax + B$ and $X'(x) = A$. Then $X(1) = 0$ becomes $A + B = 0$, or $A = -B$. Now we look at our second condition,

$$X'(0) + X(0) = 0$$

$X'(0)$ is just $A$ and $X(0)$ is just $B$, so again this gives us $A = -B$. Thus, we have $X(x) = Ax - A = A(x-1)$, so we have found

$$X_0(x) = x - 1$$

(ii) Find an equation for the positive eigenvalues $\lambda = \beta^2$.

$$X(x) = A \sin(\beta x) + B \cos(\beta x)$$
$$X'(x) = \beta A \cos(\beta x) - \beta B \sin(\beta x)$$

$$X(1) = A \sin(\beta) + B \cos(\beta) = 0$$
$$A \sin(\beta) = -B \cos(\beta)$$

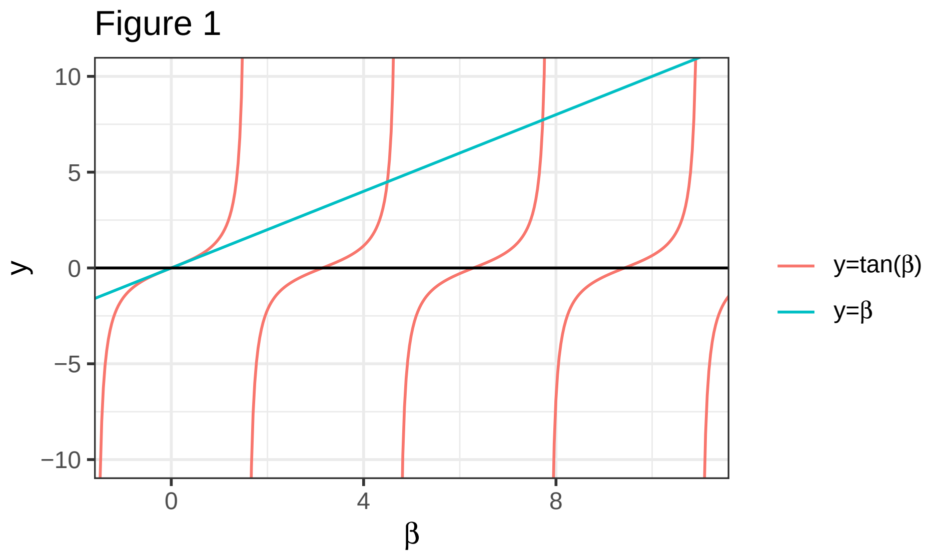
$$-\frac{B}{A} = \frac{\sin(\beta)}{\cos(\beta)} = \tan \beta \tag{1}$$

$$X'(0) = \beta A, \quad X(0) = B$$
$$X'(0) + X(0) = \beta A + B = 0$$
$$\beta = -\frac{B}{A} \tag{2}$$

Combining (1) and (2), we get the equation $\beta = \tan(\beta)$ for the positive eigenvalues.

(iii) Show graphically from part (b) that there are an infinite number of positive eigenvalues.

In part (b), we showed that $\beta$ is a positive eigenvalue if $\beta = \tan(\beta)$. However, plotting the equations $y = \beta$ and $y = \tan \beta$ reveals that these two functions intersect an infinite number of times (Fig. 1), meaning there are an infinite number of positive eigenvalues.




(iv) Is there a negative eigenvalue?

This case gives us the characteristic equation: $r^2 - \beta^2 = 0$. Since our characteristic equation has distinct real roots $r = \pm\beta$, our solutions take the form

$$X(x) = Ae^{\beta x} + Be^{-\beta x}$$

Differentiating, we find that

$$X'(x) = \beta Ae^{\beta x} - \beta Be^{-\beta x}$$

$$X'(0) = \beta A - \beta B$$
$$X(0) = A + B$$

$$X'(0) + X(0) = \beta A - \beta B + A + B$$
$$0 = A(\beta + 1) + B(1 - \beta)$$
$$-A(\beta + 1) = B(1 - \beta)$$
$$\frac{\beta + 1}{1 - \beta} = -\frac{B}{A}$$

Now turning to our other boundary condition,

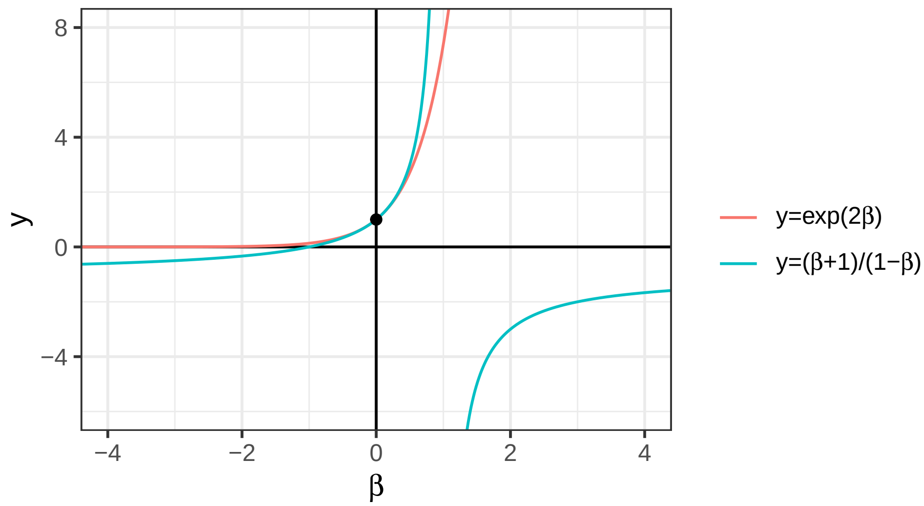
$$X'(1) = \beta Ae^{\beta} - \beta Be^{-\beta} = 0$$
$$\beta Ae^{\beta} = \beta Be^{-\beta}$$
$$-\frac{B}{A} = -e^{2\beta}$$

This would give us the following equation:

$$\frac{\beta + 1}{1 - \beta} = -e^{2\beta}$$

But the only time that this equality holds true is when $\beta = 0$ (Fig. 2), but we are in the case when we defined $\beta$ to be nonzero, meaning that we have reached a contradiction and we cannot have any negative eigenvalues.

Figure 2




## Question 2

Find the Fourier series of $f(x)$. Does the Fourier-series converge (i) pointwise, or (ii) uniformly?

### (a)

$$f(x) = \begin{cases} 1 - |x| & |x| \leq 1 \\ 1 & 1 < |x| \leq \pi \end{cases}$$

The Fourier series for a function $f(x)$ on an interval $-l < x < l$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right)$$

where

$$a_0 = \frac{1}{2l} \int_{-l}^{l} f(x) dx$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

Here, $l = \pi$. Let's start with $a_0$:

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \\
&= \frac{1}{2\pi}\left[\int_{-\pi}^{-1} 1\ dx + \int_{-1}^{1}(1-|x|)dx + \int_{1}^{\pi} 1\ dx\right] \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} 1\ dx - \int_{-1}^{1}|x|\,dx \\
&= 1 - 2\int_{0}^{1} x\ dx \\
&= 1 - 1 = 0
\end{aligned}$$

where we used the evenness of $|x|$ to rewrite $\int_{-1}^{1}|x|\,dx$ as $2\int_0^1 x\ dx$. Thus, $a_0 = 0$. Now we perform a similar procedure to find $a_n$:

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{-1}\cos(nx)\ dx + \int_{-1}^{1}(1-|x|)\cos(nx)\ dx + \int_{1}^{\pi}\cos(nx)\ dx\right] \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(nx)\ dx - \frac{1}{\pi}\int_{-1}^{1}|x|\cos(nx)\ dx
\end{aligned}$$

Since $\sin(\pi n)$ and $\sin(-\pi n)$ are zero for any integer $n$, the first integral vanishes. We use the symmetry of $|x|\cos(nx)$ (both functions are even, and the product of two even functions is even) and rewrite the second integral as

$$-\frac{2}{\pi}\int_0^1 x\cos(nx)\ dx$$

We then use integration by parts:

$$\begin{aligned}
&-\frac{2}{\pi}\int_0^1 x\cos(nx)\ dx \\
&= -\frac{2}{n\pi}\left[x\sin(nx)\Big|_0^1 - \int_0^1 \sin(nx)\ dx\right] \\
&= -\frac{2}{n\pi}\left[\sin(n) + \frac{1}{n}\cos(nx)\Big|_0^1\right] \\
&= -\frac{2}{n\pi}\left[\sin(n) + \frac{1}{n}\cos(n) - \frac{1}{n}\right]
\end{aligned}$$

$$a_n = -\frac{2\sin n}{n\pi} + \frac{2(\cos n - 1)}{n^2\pi}$$

$$\begin{aligned}
b_n &= \frac{1}{\pi}\left[\int_{-\pi}^{-1}\sin(nx)\ dx + \int_{-1}^{1}(1-|x|)\sin(nx)\ dx + \int_{1}^{\pi}\sin(nx)\ dx\right] \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\sin(nx)\ dx - \frac{1}{\pi}\int_{-1}^{1}|x|\sin(nx)\ dx
\end{aligned}$$

But $\sin x$ is an odd function and so is $|x|\sin(nx)$, so both of these integrals vanish and we find $b_n = 0$.

Now inserting the coefficients $a_n$ we found above, we get the Fourier series:

$$f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \left( \frac{\sin n}{n} + \frac{\cos n + 1}{n^2} \right) \cos(nx)$$

Since $f(x)$ and $f'(x)$ are both piecewise continuous, the Fourier series converges pointwise. However, since $f(x)$ is not continuous (only piecewise), it does not converge uniformly.

**(b)**

$$f(x) = |x| = \begin{cases} -x & -\pi \leq x \leq 0 \\ x & 0 < x \leq \pi \end{cases}$$

Let's start with $a_0$:

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| \, dx \\ &= \frac{1}{\pi} \int_0^{\pi} x \; dx \\ &= \frac{1}{2\pi} x^2 \Big|_0^{\pi} = \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \\ &= \frac{2}{\pi} \int_0^{\pi} x \cos(nx) dx \\ &= \frac{2}{n\pi} x \sin(nx) \Big|_0^{\pi} - \frac{2}{n\pi} \int_0^{\pi} \sin(nx) dx \\ &= 0 + \frac{2}{n^2 \pi} \cos(nx) \Big|_0^{\pi} \\ &= \frac{2}{n^2\pi} (-1)^n - \frac{2}{n^2\pi} \\ &= \frac{2}{n^2\pi} (-1)^{n+1} \end{aligned}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

But since $|x|$ is even and $\sin(nx)$ is odd, the product of the two functions is odd and so the integral vanishes over the symmetric interval.

Then we plug in our formulas for $a_0$ and $a_n$:

$$f(x) = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \left[ \frac{1}{n^2} (-1)^{n+1} \cos(nx) \right]$$

Since $f(x)$ and $f'(x)$ are both piecewise continuous, the Fourier series converges pointwise. Also, $f(x)$ is continuous and $f'(x)$ is piecewise continuous. Finally, the periodic boundary conditions are satisfied because $f(-\pi) = \pi = f(\pi)$. Therefore, the Fourier series also converges uniformly.

**(c)**

$$f(x) = x + x^2, \quad -\pi \le x \le \pi$$

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x + x^2 dx \\
&= \frac{1}{2\pi}\left(\frac{x^2}{2} + \frac{x^3}{3}\right)\Big|_{-\pi}^{\pi} \\
&= \frac{1}{2\pi}\left(\frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} + \frac{\pi^3}{3}\right) \\
&= \frac{1}{2\pi}\left(\frac{2\pi^3}{3}\right) = \frac{\pi^2}{3}
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)\,dx
\end{aligned}$$

Applying integration by parts to the first integral,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)\,dx = \frac{1}{n\pi}x\sin(nx)\Big|_{-\pi}^{\pi} - \frac{1}{n\pi}\int_{-\pi}^{\pi}\sin(nx)\,dx$$

$$= \frac{1}{n\pi}x\sin(nx)\Big|_{-\pi}^{\pi} - \frac{-1}{n^2\pi}\cos(nx)\Big|_{-\pi}^{\pi} = 0 - 0 = 0$$

For the second integral, we can use the integration by parts table method demonstrated in class to find that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)\,dx = \frac{x^2}{n\pi}\sin(nx)\Big|_{-\pi}^{\pi} + \frac{2x}{n^2\pi}\cos(nx)\Big|_{-\pi}^{\pi} + \frac{2}{n^3\pi}\sin(nx)\Big|_{-\pi}^{\pi}$$

The first and last terms (with $\sin(nx)$) vanish. Evaluating the middle term, we get:

$$\frac{2\pi\cos(n\pi)}{n^2\pi} - \frac{-2\pi\cos(n\pi)}{n^2\pi} = \frac{4\cos(n\pi)}{n^2} = \frac{4}{n^2}(-1)^n$$

Thus, we have found our coefficients $a_n$:

$$a_n = \frac{4}{n^2}(-1)^n$$

Now for $b_n$:

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx) + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin(nx)dx
\end{aligned}$$

But since $x^2$ is even and $\sin$ is odd, the product under the second integral is odd functions and thus evaluates to zero over the symmetric interval $-\pi < x < \pi$. Applying integration by parts to the first integral,

$$b_n = \frac{-1}{n\pi}x\cos(nx)\Big|_{-\pi}^{\pi} + \frac{1}{n\pi}\int_{-\pi}^{\pi}\cos(nx)$$

But since cos is even and $\cos\pi = \cos -\pi$, both terms are zero. Thus, $b_n = 0$ and

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos(nx)}{n^2}$$

Here, $f(x)$ and $f'(x)$ are both continuous, so the Fourier series converges piecewise. However, $f(-\pi) = \pi^2 - \pi$ while $f(\pi) = \pi^2 + \pi$, so the periodic boundary conditions are not satisfied and the Fourier series does not converge uniformly.

## Question 3

### a)

Find the Fourier sine series of

$$f(x) = \begin{cases} 1 & 0 < x < \pi/2 \\ 2 & \pi/2 < x < \pi \end{cases}$$

The Fourier sine series of a function $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right)$$

where

$$B_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$B_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx$$

$$\begin{aligned}
B_n &= \frac{2}{\pi}\left[\int_0^{\pi/2} \sin(nx)dx + 2\int_{\pi/2}^{\pi} \sin(nx)dx\right] \\
&= \frac{2}{\pi}\left[\frac{-1}{n}\cos(nx)\Big|_0^{\pi/2} - \frac{2}{n}\cos(nx)\Big|_{\pi/2}^{\pi}\right] \\
&= \frac{2}{n\pi}\left[-\cos\left(\frac{n\pi}{2}\right) + 1 - 2\cos(n\pi) + 2\cos\left(\frac{n\pi}{2}\right)\right] \\
&= \frac{2}{n\pi}\left[1 + \cos\left(\frac{n\pi}{2}\right) - 2\cos(n\pi)\right]
\end{aligned}$$

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\left[1 + \cos\left(\frac{n\pi}{2}\right) - 2\cos(n\pi)\right]\sin(nx)$$

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\left[1 + \cos\left(\frac{n\pi}{2}\right) + 2(-1)^{n+1}\right]\sin(nx)$$

$$f(x) = \sum_{n_{odd}} + \sum_{n_{even}}$$

$$\begin{aligned}\sum_{n_{even}} &= \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k}\left[1+\cos\left(\frac{2k\pi}{2}\right)+2(-1)^{2k+1}\right]\sin(2kx)\\ &= \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k}\left[1+(-1)^n-2\right]\sin(2kx)\\ &= \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{n+1}}{2k}\sin(2kx)\end{aligned}$$

$$\begin{aligned}\sum_{n_{odd}} &= \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\left[1+\cos\left(\frac{(2k-1)\pi}{2}\right)+2(-1)^{2k-1+1}\right]\sin((2k-1)x)\\ &= \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{3}{2k-1}\sin((2k-1)x)\end{aligned}$$

$$f(x) = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{n+1}}{2k}\sin(2kx)+\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{3}{2k-1}\sin((2k-1)x)$$

**b)**

Find the Fourier-cosine-series of $f(x) = |\sin x|$. Then find

$$\sum_{k=1}^{\infty}\frac{1}{4k^2-1}$$

The Fourier cosine series of a function $f(x)$ is

$$f(x) = \frac{1}{2}A_0+\sum_{n=1}^{\infty}A_n\cos\left(\frac{n\pi x}{l}\right)$$

where

$$A_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

Assuming $l=\pi$, we have

$$A_n = \frac{2}{\pi}\int_0^{\pi}|\sin x|\cos(nx)dx \tag{3}$$

But since sin is non-negative on the interval $0 \leq x \leq \pi$, we can drop the absolute value bars and use the following trigonometric identity to rewrite the integral: $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$

$$\begin{aligned}A_n &= \frac{1}{\pi}\int_0^{\pi}\sin((1+n)x)+\sin((1-n)x)dx\\ &= \frac{1}{\pi}\left[\frac{-1}{1+n}\cos((1+n)x)\Big|_0^{\pi}-\frac{1}{1-n}\cos((1-n)x)\Big|_0^{\pi}\right]\\ &= \frac{1}{\pi}\left[\frac{-1}{1+n}[(-1)^{1+n}-1]-\frac{1}{1-n}[(-1)^{1-n}-1]\right]\\ &= \frac{1}{\pi}\left[(-1)^{1+n}-1)\right]\left(\frac{-1}{1+n}-\frac{1}{1-n}\right)\end{aligned}$$

Thus, we have found a formula for $A_n$,

$$A_n = \frac{-2}{\pi}\frac{(-1)^n + 1}{n^2 - 1}$$

This formula works for all $n \neq 1$ For $n = 0$,

$$A_0 = \frac{-2}{\pi}\frac{(-1)^0 + 1}{0 - 1} = \frac{4}{\pi}$$

For $n = 1$, we can go back to (3) and plug in the appropriate value of $n$:

$$A_1 = \frac{1}{\pi}\int_0^{\pi} \sin(2x)\, dx = \frac{1}{2\pi}\cos(2x)\Big|_0^{\pi} dx = 0$$

The Fourier cosine series is therefore

$$f(x) = \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n + 1}{n^2 - 1}\cos(nx)$$

Now, $(-1)^n + 1$ will be zero for all odd $n$ and two for all even $n$. So substituting $n = 2k$, we gwt

$$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{2k=2}^{\infty}\frac{1}{4k^2 - 1}\cos(2kx)$$

Starting the index at $2k = 2$ is equivalent to starting it at $k = 1$, so after making that change and evaluating both sides at $x = 0$, we can solve for the sum given in the question:

$$0 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2 - 1}$$

$$\sum_{k=1}^{\infty}\frac{1}{4k^2 - 1} = \frac{1}{2}$$

**c)**

The Riemann Zeta function is defined for $s > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$$

By computing the Fourier series of $x^2$ over $-\pi < x < \pi$ and using Parseval's identity, compute $\zeta(4)$.

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx \\
&= \frac{1}{2\pi}\left(\frac{x^3}{3}\right)\Big|_{-\pi}^{\pi} \\
&= \frac{1}{2\pi}\left(\frac{\pi^3}{3} - \frac{-\pi^3}{3}\right) \\
&= \frac{1}{2\pi}\left(\frac{2\pi^3}{3}\right) = \frac{\pi^2}{3}
\end{aligned}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos{(nx)} dx$$

But in part (2c), we found that this is equivalent to $\frac{4}{n^2}(-1)^n$, so

$$a_n = \frac{4}{n^2}(-1)^n$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin{(nx)} dx$$

Since $x^2$ is an even function, $b_n = 0$.

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos{(nx)}}{n^2}$$

Parseval's Identity says that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} (x^2)^2 \; dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} (x^2)^2 \; dx = \left(\frac{\pi^2}{3}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{4(-1)^n}{n^2}\right)^2$$

$$\frac{1}{\pi}\int_{0}^{\pi} x^4 \; dx = \frac{\pi^4}{9} + \sum_{n=1}^{\infty}\frac{8}{n^4}$$

$$\frac{1}{\pi}\frac{x^5}{5}\bigg|_0^\pi = \frac{\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{\pi^4}{5} = \frac{\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{4\pi^4}{45} = 8\sum_{n=1}^{\infty}\frac{1}{n^4}$$

Thus,

$$\zeta(4) = \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

**(d)** Use the Fourier series in 2c and the pointwise convergence theorem to find $\zeta(2)$. Then find

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

Take the following Fourier series and set $x = \pi$.

$$f(x) = x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos{(nx)}}{n^2}$$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Thus,

$$\zeta(2) = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Now, we want to find

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{1}{1} + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \ ...$$

Let's also write out some terms in the series representation of $\zeta(2)$:

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \ ...$$

We can see that the top series is simply the bottom series without the terms where the denominator is the square of an even integer; which we can write as:

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} &= \sum_{n=1}^{\infty}\frac{1}{n^2} - \sum_{n=1}^{\infty}\frac{1}{(2n)^2} \\
&= \sum_{n=1}^{\infty}\frac{1}{n^2} - \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2} \\
&= \frac{\pi^2}{6} - \frac{1}{4}\frac{\pi^2}{6} \\
&= \frac{\pi^2}{8}
\end{aligned}$$

Therefore, we conclude that

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

## Question 4

Compute the complex Fourier series of the following functions:

### (a)

Compute the complex Fourier series of $f(x) = e^x$ and show that

$$\coth\pi = \frac{1}{\pi} + \frac{2}{\pi}\left(\frac{1}{1+1^2} + \frac{1}{1+2^2} + \frac{1}{1+3^2} + \ ...\right)$$

The complex Fourier series of a function $f(x)$ is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}$$

where

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}\, dx$$

$$
\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-inx}\;dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x(1-in)}\;dx \\
&= \frac{1}{2\pi}\frac{1}{1-in}e^{x(1-in)}\Big|_{-\pi}^{\pi} \\
&= \frac{1}{2\pi}\frac{1}{1-in}\left[e^{\pi}e^{-in\pi} - e^{-\pi}e^{in\pi}\right]
\end{aligned}
$$

and since $e^{in\pi} = \cos(n\pi) + i\sin(n\pi) = (-1)^n = \cos(-n\pi) + i\sin(-n\pi) = e^{-in\pi}$,

$$
\begin{aligned}
c_n &= \frac{1}{2\pi}\frac{(-1)^n}{1-in}\left[e^{\pi} - e^{-\pi}\right] \\
&= \frac{1}{\pi}\frac{(-1)^n}{1-in}\sinh\pi
\end{aligned}
$$

$$
f(x) = e^x = \sum_{n=-\infty}^{\infty}\frac{1}{\pi}\frac{(-1)^n}{1-in}(\sinh\pi)\,e^{inx}
$$

Set $x = \pi$ and $x = -\pi$ and then add the two expressions, noting that $e^{in\pi} = e^{-in\pi} = (-1)^n$ and $(-1)^n(-1)^n = (-1)^{2n} = 1$.

$$
e^{\pi} = \sum_{n=-\infty}^{\infty}\frac{1}{\pi}\frac{1}{1-in}(\sinh\pi)
$$

$$
e^{-\pi} = \sum_{n=-\infty}^{\infty}\frac{1}{\pi}\frac{1}{1-in}(\sinh\pi)
$$

$$
\frac{e^{\pi}+e^{-\pi}}{2} = \sum_{n=-\infty}^{\infty}\frac{1}{\pi}\frac{1}{1-in}(\sinh\pi)
$$

But note that $\frac{e^{\pi}+e^{-\pi}}{2} = \cosh\pi$.

$$
\cosh\pi = \sum_{n=-\infty}^{\infty}\frac{1}{\pi}\frac{1}{1-in}(\sinh\pi)
$$

Divide both sides by $\sinh\pi$:

$$
\coth\pi = \sum_{n=-\infty}^{\infty}\frac{1}{\pi}\frac{1}{1-in}
$$

Multiply the right-hand side by $\frac{1+in}{1+in}$:

$$
\coth\pi = \sum_{n=-\infty}^{\infty}\frac{1}{\pi}\frac{1+n}{1+n^2}
$$

$$
= \sum_{n=-\infty}^{\infty}\frac{1}{\pi}\frac{1}{1+n^2} + \sum_{n=-\infty}^{\infty}\frac{1}{\pi}\frac{n}{1+n^2}
$$

The second sum will disappear because the $n$ term in the numerator means that all negative terms in the series will cancel with the corresponding positive $n$ terms, while the expression evaluates to zero at $n = 0$. Thus,

$$\begin{aligned}\coth\pi &= \sum_{n=-\infty}^{\infty} \frac{1}{\pi}\frac{1}{1+n^2} \\ &= \frac{1}{\pi} + \frac{2}{\pi}\left(\frac{1}{1+1^2} + \frac{1}{1+2^2} + \frac{1}{1+3^2} + \ ...\right)\end{aligned}$$

which is what we initially sought to show.

### (b)

Find the complex Fourier series of $xe^{ix}$. Then use your result to find the Fourier series of $x\cos x$ and $x\sin x$.

The complex Fourier series of a function $f(x)$ is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}$$

where

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}\ dx$$

$$\begin{aligned}c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{ix}e^{-inx}\ dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{ix(1-n)}\ dx\end{aligned}$$

Using integration by parts,

$$u = x, \quad du = dx, \quad dv = e^{ix(1-n)}\ dx, \quad v = \frac{1}{i(1-n)}e^{ix(1-n)}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{ix(1-n)}\ dx = \frac{1}{2\pi}\left[\frac{x}{i(1-n)}e^{ix(1-n)}\right]_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{i(1-n)}e^{ix(1-n)}\ dx$$

First, let's evaluate the first term on the right-hand side:

$$\begin{aligned}\frac{1}{2\pi}\left[\frac{x}{i(1-n)}e^{ix(1-n)}\right]_{-\pi}^{\pi} &= \frac{1}{2\pi}\left[\frac{\pi}{i(1-n)}e^{i\pi(1-n)} - \frac{-\pi}{i(1-n)}e^{-i\pi(1-n)}\right] \\ &= \frac{1}{2\pi}\left[\frac{\pi}{i(1-n)}e^{i\pi}e^{-i\pi n} - \frac{-\pi}{i(1-n)}e^{-i\pi}e^{i\pi n}\right] \\ &= \frac{1}{2\pi}\left[\frac{\pi}{i(1-n)}(-1)(-1)^n - \frac{-\pi}{i(1-n)}(-1)(-1)^n\right] \\ &= \frac{1}{2\pi}\left[\frac{2\pi}{i(1-n)}(-1)^{n+1}\right] \\ &= \frac{(-1)^{n+1}}{i(1-n)}\end{aligned}$$

Now we turn to the integral on the right-hand side:

$$\begin{aligned}
-\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{i(1-n)}e^{ix(1-n)}\,dx &= \frac{1}{2\pi}\left[\frac{e^{ix(1-n)}}{(1-n)^2}\right]_{-\pi}^{\pi}\\
&= \frac{1}{2\pi}\left[\frac{e^{i\pi(1-n)}}{(1-n)^2}-\frac{e^{-i\pi(1-n)}}{(1-n)^2}\right]\\
&= \frac{1}{2\pi}\left[\frac{(-1)^{n+1}}{(1-n)^2}-\frac{(-1)^{n+1}}{(1-n)^2}\right]\\
&= 0
\end{aligned}$$

Thus, $c_n = \frac{(-1)^{n+1}}{i(1-n)}$. Plugging this into the equation for a complex Fourier series, we find that

$$xe^{ix} = \sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}}{i(1-n)}e^{inx}$$

Now, we use Euler's identity

$$e^{ix} = \cos x + i\sin x$$
$$\cos x = \frac{e^{ix}+e^{-ix}}{2}$$

$$\begin{aligned}
x\cos x &= \frac{x\left(e^{ix}+e^{-ix}\right)}{2}\\
&= \frac{x}{2}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}}{i(1-n)}e^{inx}+\frac{x}{2}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}}{i(1-n)}e^{-inx}\\
&= \frac{x}{2}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}}{i(1-n)}(\cos nx+i\sin nx)+\frac{x}{2}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}}{i(1-n)}(\cos(-nx)+i\sin(-nx))\\
&= \frac{x}{2}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}}{i(1-n)}(\cos nx+i\sin nx)+\frac{x}{2}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}}{i(1-n)}(\cos(nx)-i\sin(nx))
\end{aligned}$$

The sin terms cancel and the cos terms add, yielding

$$x\cos x = x\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}}{i(1-n)}\cos nx$$

Similarly,

$$\begin{aligned}
\sin x &= \frac{e^{ix}-e^{-ix}}{2i}\\
&= \frac{x}{2i}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}}{i(1-n)}e^{inx}-\frac{x}{2i}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}}{i(1-n)}e^{-inx}\\
&= \frac{x}{2i}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}}{i(1-n)}(\cos nx+i\sin nx)-\frac{x}{2i}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}}{i(1-n)}(\cos(-nx)+i\sin(-nx))\\
&= \frac{x}{2i}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}}{i(1-n)}(\cos nx+i\sin nx)-\frac{x}{2i}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}}{i(1-n)}(\cos(nx)-i\sin(nx))
\end{aligned}$$

The cos terms cancel and the sin terms add, yielding

$$\frac{x}{i}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}}{i(1-n)}(-\sin nx)$$

Multiplying the two $i$s together cancels the negative sign, meaning the Fourier series becomes:

$$x\sin x = x\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}}{(1-n)}\sin nx$$

## Question 5

Find the function represented by the new series which is obtained by term-wise integration of the following series from 0 to $x$.

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\cos kx}{k} = \log\left(2\cos\left(\frac{x}{2}\right)\right), \quad -\pi < x < \pi$$

$$\int_0^x \sum_{k=1}^{\infty}(-1)^{k+1}\frac{\cos kx}{k} = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sin kx}{k^2}$$

$$\left[\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sin kx}{k^2}\right]\Bigg|_0^x = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sin kx}{k^2}$$

Therefore,

$$\int_0^x \log\left(2\cos\left(\frac{x}{2}\right)\right) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sin kx}{k^2}$$