MATH 245 Homework 4

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Question 1: Find eigenvalues and eigenfunctions

(a)

$-\frac{d^2}{dx^2}X(x) = \lambda X(x)$ in $0 < x < l$ with boundary conditions $X'(0) = 0 = X(l)$

<u>Case 1:</u> Positive eigenvalues, $\lambda = \beta^2 > 0$

Re-writing as $X'' + \lambda X = 0$, we will get the characteristic equation: $r^2 + \beta^2 = 0$. Since our characteristic equation has complex roots $r = \pm i\beta$, our solutions take the form

$$X(x) = A\sin(\beta x) + B\cos(\beta x)$$

Differentiating, we find that

$$X'(x) = \beta A\cos(\beta x) - \beta B\sin(\beta x)$$

Now plugging in our initial condition $X'(0) = 0$, we get $X'(0) = \beta A = 0$. And since we are in a case where $\beta \neq 0$, this means $A = 0$ so $X(x) = B\cos(\beta x)$. Now we use our boundary condition $X(l) = 0$ to get $X(l) = B\cos(\beta l) = 0$. If $B = 0$, then $X(0) = 0$ and this contradicts the definition of an eigenfunction. Therefore, we must have $\cos(\beta l) = 0$, which can only occur if $\beta = \frac{(2n-1)\pi}{2l}$. Therefore, this case gives us eigenvalues

$$\lambda_n = \left(\frac{(2n-1)\pi}{2l}\right)^2 \quad n = 1, 2, 3...$$

with eigenfunctions

$$X_n(x) = \cos\left(\frac{(2n-1)\pi x}{2l}\right)$$

<u>Case 2:</u> Zero eigenvalues, $\lambda = 0$ $X'' = 0$ implies that $X(x)$ is of the form $Ax + B$, with derivative $X'(x) = A$. Now plugging in our initial condition $X'(0) = 0$, we get $A = 0$, which means $X(x) = B$. But from the boundary condition $X(l) = 0$, we get $B = 0$ and so $X(0) = 0$. Therefore, there are no eigenfunctions $X(x)$ that satisfy $-\frac{d^2}{dx^2}X(x) = \lambda X(x)$ when $\lambda = 0$ and hence no zero eigenvalues.

<u>Case 3:</u> Negative eigenvalues, $\lambda = -\beta^2 < 0$ This case gives us the characteristic equation: $r^2 - \beta^2 = 0$. Since our characteristic equation has distinct real roots $r = \pm\beta$, our solutions take the form

$$X(x) = Ae^{\beta x} + Be^{-\beta x}$$

Differentiating, we find that

$$X'(x) = \beta Ae^{\beta x} - \beta Be^{-\beta x}$$

$$X'(0) = \beta A - \beta B = 0$$

Since we are in a case where $\beta \neq 0$, this means $A - B = 0$, or $A = B$. Then the boundary condition gives

$$X(l) = Be^{\beta l} + Be^{-\beta l} = 0$$

Since $e^{\beta l}$ and $e^{-\beta l}$ are nonzero for all values of l, we must have $B = 0$, implying that again $X(x) = 0$.

Thus, this problem has only positive eigenvalues and their associated eigenfunctions as found in Case 1.

(b)

$x^2 X''(x) + xX'(x) + \lambda X(x) = 0$ in $1 < x < e$ with boundary conditions $X(1) = 0 = X(e)$.

We recognize this equation as having the same form as a second-order Cauchy-Euler equation, a linear homogeneous ODE of the form $ax^2 y + bxy' + cy = 0$ with the auxiliary equation $ar(r-1) + br + c = 0$. Here, $a = b = 1$ and $c = \lambda$, so we have

$$r(r-1) + r + \lambda = 0$$
$$r^2 - r + r + \lambda = 0$$
$$r^2 + \lambda = 0$$

Case 1: Positive eigenvalues, $\lambda = \beta^2 > 0$

$$X(x) = A\sin(\beta \ln x) + B\cos(\beta \ln x)$$
$$X(1) = A\sin(0) + B\cos(0) = 0 \quad \longrightarrow \quad B = 0$$
$$X(e) = A\sin(\beta \ln e) = 0$$
$$A\sin(\beta) = 0$$

If $A = 0$, then $X(0) = 0$ and this contradicts the definition of an eigenfunction. Therefore, we must have $\sin(\beta) = 0$, which can only occur if $\beta = n\pi$. Therefore, this case gives us eigenvalues

$$\lambda_n = n^2\pi^2 \quad n = 1, 2, 3...$$

with eigenfunctions

$$X_n(x) = \sin(n\pi \ln x)$$

Case 2: Zero eigenvalues, $\lambda = 0$ $X'' = 0$ implies that $X(x)$ is of the form $Ax + B$. Now plugging in our initial condition $X(1) = 0$, we get $A + B = 0$, which means $X(x) = Ax - A$. But from the boundary condition $X(e) = 0$, we get $A(e - 1) = 0$ which is only possible if $A = 0$ and accordingly $B = -A = 0$. Therefore, there are no eigenfunctions $X(x)$ that satisfy the boundary conditions when $X'' = 0$ and hence no zero eigenvalues.

Case 3: Negative eigenvalues, $\lambda = -\beta^2 < 0$ This case gives us the characteristic equation: $r^2 - \beta^2 = 0$. Since our characteristic equation has distinct real roots $r = \pm\beta$, our solutions take the form

$$X(x) = Ax^\beta + Bx^{-\beta}$$

$$X(1) = A + B = 0 \quad \longrightarrow \quad A = -B$$
$$X(x) = Ax^\beta - Ax^{-\beta}$$

Then the other boundary condition gives $X(e) = Ae^\beta - Ae^{-\beta} = 0 \quad \Rightarrow A(e^\beta - e^{-\beta}) = 0$. If we had $e^\beta - e^{-\beta} = 0$, that would imply $\beta = -\beta \quad \Rightarrow \beta = 0$, but we are in a case when we defined β to be nonzero. So, we must have $A = 0$, implying that again $X(x) = 0$. Thus, this problem has only positive eigenvalues and their associated eigenfunctions as found in Case 1.

(c)

On the interval $0 \le x \le 1$ of length one, consider the eigenvalue problem

$$-X'' = \lambda X, \quad X'(0) + X(0) = 0, \quad X(1) = 0$$

(i) Find an eigenfunction with eigenvalue zero. Call it $X_0(x)$.

$X'' = 0$ implies that $X(x)$ is of the form $Ax + B$ and $X'(x) = A$. Then $X(1) = 0$ becomes $A + B = 0$, or $A = -B$. Now we look at our second condition,

$$X'(0) + X(0) = 0$$

$X'(0)$ is just A and $X(0)$ is just B, so again this gives us $A = -B$. Thus, we have $X(x) = Ax - A = A(x-1)$, so we have found

$$X_0(x) = x - 1$$

(ii) Find an equation for the positive eigenvalues $\lambda = \beta^2$.

$$X(x) = A\sin(\beta x) + B\cos(\beta x)$$
$$X'(x) = \beta A\cos(\beta x) - \beta B\sin(\beta x)$$

$$X(1) = A\sin(\beta) + B\cos(\beta) = 0$$
$$A\sin(\beta) = -B\cos(\beta)$$

Before dividing by A, we need to verify that $A \ne 0$. In this case, if $A = 0$, then $X = B\cos\beta x$, so $X'(0) + X(0) = 0 + B = 0$ implies $B = 0$, a contradiction. Similarly, $\cos\beta \ne 0$, since if $\cos\beta = 0$, then $\sin\beta \ne 0$ (sin and cos are not zero at the same time), and so $A\sin\beta = -B\cos\beta = 0 \Rightarrow A = 0$, leading to the same contradiction as before.

$$-\frac{B}{A} = \frac{\sin(\beta)}{\cos(\beta)} = \tan\beta \tag{1}$$

$$X'(0) = \beta A, \quad X(0) = B$$
$$X'(0) + X(0) = \beta A + B = 0$$
$$\beta = -\frac{B}{A} \tag{2}$$

Combining (1) and (2), we get the equation $\beta = \tan(\beta)$ for the positive eigenvalues.

(iii) Show graphically from part (b) that there are an infinite number of positive eigenvalues.

In part (b), we showed that β is a positive eigenvalue if $\beta = \tan(\beta)$. However, plotting the equations $y = \beta$ and $y = \tan\beta$ reveals that these two functions intersect an infinite number of times (Fig. 1), meaning there are an infinite number of positive eigenvalues.

Figure 1
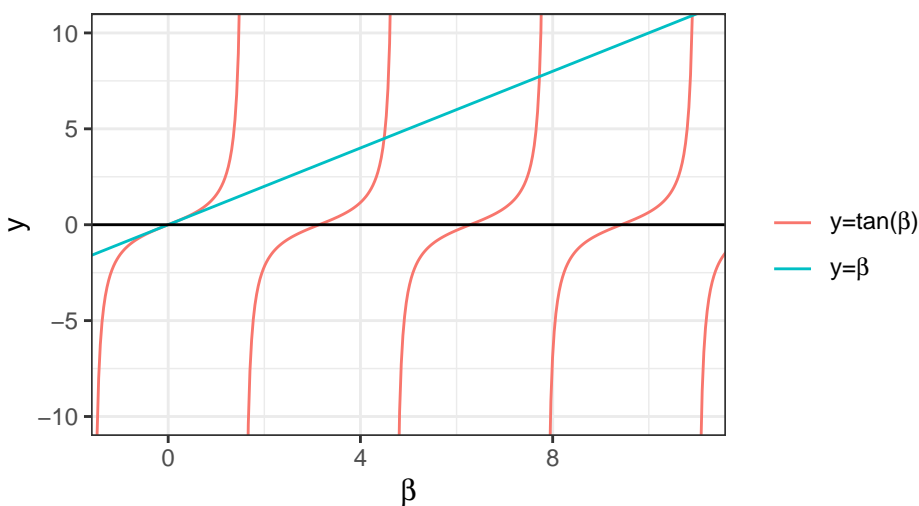
(iv) Is there a negative eigenvalue?

This case gives us the characteristic equation: $r^2 - \beta^2 = 0$. Since our characteristic equation has distinct real roots $r = \pm\beta$, our solutions take the form

$$X(x) = Ae^{\beta x} + Be^{-\beta x}$$

Differentiating, we find that

$$X'(x) = \beta Ae^{\beta x} - \beta Be^{-\beta x}$$

$$X'(0) = \beta A - \beta B$$
$$X(0) = A + B$$

$$X'(0) + X(0) = \beta A - \beta B + A + B$$
$$0 = A(\beta + 1) + B(1 - \beta)$$
$$-A(\beta + 1) = B(1 - \beta)$$
$$\frac{\beta + 1}{1 - \beta} = -\frac{B}{A}$$

Now turning to our other boundary condition,

$$X(1) = Ae^{\beta} + Be^{-\beta} = 0$$

$$Ae^{\beta} = -Be^{-\beta}$$

$$-\frac{B}{A} = e^{2\beta}$$

We know $A \neq 0$, since otherwise $X(1) = Be^{-\beta} = 0$ implies $B = 0$, a contradiction. Now you also can't have $1 - \beta = 0$, since $\beta = 1$ leads to $-A(1 + 1) = -2A = 0$. This justifies all the division so far.

This would give us the following equation:

$$\frac{\beta + 1}{1 - \beta} = e^{2\beta}$$

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But the only time that this equality holds true is when $\beta = 0$ (Fig. 2), but we are in the case when we defined β to be nonzero, meaning that we have reached a contradiction and we cannot have any negative eigenvalues.

Figure 2
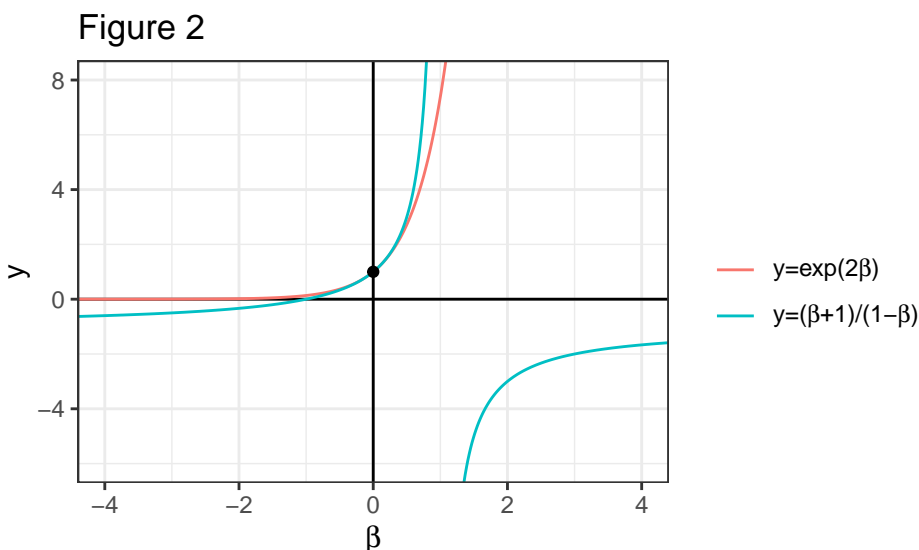
Question 2

Find the Fourier series of $f(x)$. Does the Fourier-series converge (i) pointwise, or (ii) uniformly?

(a)

$$f(x) = \begin{cases} 1 - |x| & |x| \leq 1 \\ 1 & 1 < |x| \leq \pi \end{cases}$$

The Fourier series for a function $f(x)$ on an interval $-l < x < l$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \right)$$

where

$$a_0 = \frac{1}{2l} \int_{-l}^{l} f(x) dx$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

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Here, $l = \pi$. Let's start with a_0:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{1}{\pi} \int_0^{\pi} f(x)dx$$

$$= \frac{1}{\pi} \left(\int_0^1 (1 - |x|)\ dx + \int_1^{\pi} dx \right)$$

$$= \frac{1}{\pi} \left(\int_0^1 (1 - x)\ dx + (\pi - 1) \right)$$

$$= \frac{1}{\pi} \left(\int_0^1 (1 - x)\ dx + (\pi - 1) \right)$$

$$= \frac{1}{\pi} \left((x - x^2/2)\Big|_0^1 + (\pi - 1) \right)$$

$$= \frac{1}{\pi} \left(\frac{1}{2} + (\pi - 1) \right)$$

$$= \frac{1}{\pi} \left(\pi - \frac{1}{2} \right)$$

where we used the evenness of $|x|$ to rewrite $\int_{-\pi}^{\pi} f(x)dx\ dx$ as $2 \int_0^{\pi} f(x)dx\ dx$. Thus, $a_0 = \frac{1}{\pi} \left(\pi - \frac{1}{2} \right)$. Now we perform a similar procedure to find a_n:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{-1} \cos(nx)\ dx + \int_{-1}^{1} (1 - |x|) \cos(nx)\ dx + \int_1^{\pi} \cos(nx)\ dx \right]$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(nx)\ dx - \frac{1}{\pi} \int_{-1}^{1} |x| \cos(nx)\ dx$$

Since $\sin(\pi n)$ and $\sin(-\pi n)$ are zero for any integer n, the first integral vanishes. We use the symmetry of $|x| \cos(nx)$ (both functions are even, and the product of two even functions is even) and rewrite the second integral as

$$-\frac{2}{\pi} \int_0^1 x \cos(nx)\ dx$$

We then use integration by parts:

$$-\frac{2}{\pi} \int_0^1 x \cos(nx)\ dx$$

$$= -\frac{2}{n\pi} \left[x \sin(nx)\Big|_0^1 - \int_0^1 \sin(nx)\ dx \right]$$

$$= -\frac{2}{n\pi} \left[\sin(n) + \frac{1}{n} \cos(nx)\Big|_0^1 \right]$$

$$= -\frac{2}{n\pi} \left[\sin(n) + \frac{1}{n} \cos(n) - \frac{1}{n} \right]$$

$$a_n = -\frac{2 \sin n}{n\pi} + \frac{2(\cos n - 1)}{n^2 \pi}$$

$$b_n = \frac{1}{\pi} \left[\int_{-\pi}^{-1} \sin(nx) \, dx + \int_{-1}^{1} (1 - |x|) \sin(nx) \, dx + \int_{1}^{\pi} \sin(nx) \, dx \right]$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(nx) \, dx - \frac{1}{\pi} \int_{-1}^{1} |x| \sin(nx) \, dx$$

But $\sin x$ is an odd function and so is $|x| \sin(nx)$, so both of these integrals vanish and we find $b_n = 0$.

Now inserting the coefficients we found above, we get the Fourier series:

$$f(x) = \frac{1}{\pi} \left(\pi - \frac{1}{2} \right) + \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\frac{\sin n}{n} + \frac{\cos n + 1}{n^2} \right) \cos(nx)$$

Since $f(x)$ and $f'(x)$ are both piecewise continuous, the Fourier series converges pointwise. However, since $f(x)$ is not continuous (only piecewise), it does not converge uniformly.

(b)

$$f(x) = |x| = \begin{cases} -x & -\pi \le x \le 0 \\ x & 0 < x \le \pi \end{cases}$$

Let's start with a_0:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| \, dx$$

$$= \frac{1}{\pi} \int_{0}^{\pi} x \, dx$$

$$= \frac{1}{2\pi} x^2 \Big|_{0}^{\pi} = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} x \cos(nx) dx$$

$$= \frac{2}{n\pi} x \sin(nx) \Big|_{0}^{\pi} - \frac{2}{n\pi} \int_{0}^{\pi} \sin(nx) dx$$

$$= 0 + \frac{2}{n^2\pi} \cos(nx) \Big|_{0}^{\pi}$$

$$= \frac{2}{n^2\pi} (-1)^n - \frac{2}{n^2\pi}$$

$$= \frac{2}{n^2\pi} (-1)^{n+1}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

But since $|x|$ is even and $\sin(nx)$ is odd, the product of the two functions is odd and so the integral vanishes over the symmetric interval.

Then we plug in our formulas for a_0 and a_n:

$$f(x) = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \left[\frac{1}{n^2} (-1)^{n+1} \cos(nx) \right]$$

Since $f(x)$ and $f'(x)$ are both piecewise continuous, the Fourier series converges pointwise. Also, $f(x)$ is continuous and $f'(x)$ is piecewise continuous. Finally, the periodic boundary conditions are satisfied because $f(-\pi) = \pi = f(\pi)$. Therefore, the Fourier series also converges uniformly.

(c)

$$f(x) = x + x^2, \quad -\pi \leq x \leq \pi$$

$$
\begin{aligned}
a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x + x^2 dx \\
&= \frac{1}{2\pi} \left(\frac{x^2}{2} + \frac{x^3}{3} \right) \Big|_{-\pi}^{\pi} \\
&= \frac{1}{2\pi} \left(\frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} + \frac{\pi^3}{3} \right) \\
&= \frac{1}{2\pi} \left(\frac{2\pi^3}{3} \right) = \frac{\pi^2}{3}
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos{(nx)} \, dx \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos{(nx)} \, dx + \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos{(nx)} \, dx
\end{aligned}
$$

Applying integration by parts to the first integral,

$$
\begin{aligned}
\frac{1}{\pi} \int_{-\pi}^{\pi} x \cos{(nx)} \, dx &= \frac{1}{n\pi} x \sin{(nx)} \Big|_{-\pi}^{\pi} - \frac{1}{n\pi} \int_{-\pi}^{\pi} \sin{(nx)} \, dx \\
&= \frac{1}{n\pi} x \sin{(nx)} \Big|_{-\pi}^{\pi} - \frac{-1}{n^2\pi} \cos{(nx)} \Big|_{-\pi}^{\pi} = 0 - 0 = 0
\end{aligned}
$$

For the second integral, we can use the integration by parts table method demonstrated in class to find that

$$
\frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos{(nx)} \, dx = \frac{x^2}{n\pi} \sin{(nx)} \Big|_{-\pi}^{\pi} + \frac{2x}{n^2\pi} \cos{(nx)} \Big|_{-\pi}^{\pi} + \frac{2}{n^3\pi} \sin{(nx)} \Big|_{-\pi}^{\pi}
$$

The first and last terms (with $\sin{(nx)}$) vanish. Evaluating the middle term, we get:

$$
\frac{2\pi \cos{(n\pi)}}{n^2\pi} - \frac{-2\pi \cos{(n\pi)}}{n^2\pi} = \frac{4 \cos{(n\pi)}}{n^2} = \frac{4}{n^2}(-1)^n
$$

Thus, we have found our coefficients a_n:

$$
a_n = \frac{4}{n^2}(-1)^n
$$

Now for b_n:

$$
\begin{aligned}
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin{(nx)} dx \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin{(nx)} + \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \sin{(nx)} dx
\end{aligned}
$$

But since x^2 is even and sin is odd, the product under the second integral is odd functions and thus evaluates to zero over the symmetric interval $-\pi < x < \pi$. Applying integration by parts to the first integral,

$$b_n = \frac{-1}{n\pi} x \cos(nx) \Big|_{-\pi}^{\pi} + \frac{1}{n\pi} \int_{-\pi}^{\pi} \cos(nx)$$

But since cos is even and $\cos\pi = \cos-\pi$, both terms are zero. Thus, $b_n = 0$ and

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos(nx)}{n^2}$$

Here, $f(x)$ and $f'(x)$ are both continuous, so the Fourier series converges piecewise. However, $f(-\pi) = \pi^2 - \pi$ while $f(\pi) = \pi^2 + \pi$, so the periodic boundary conditions are not satisfied and the Fourier series does not converge uniformly.

Question 3

a)

Find the Fourier sine series of

$$f(x) = \begin{cases} 1 & 0 < x < \pi/2 \\ 2 & \pi/2 < x < \pi \end{cases}$$

The Fourier sine series of a function $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right)$$

where

$$B_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$B_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx$$

$$B_n = \frac{2}{\pi} \left[\int_0^{\pi/2} \sin(nx) dx + 2 \int_{\pi/2}^{\pi} \sin(nx) dx \right]$$

$$= \frac{2}{\pi} \left[\frac{-1}{n} \cos(nx) \Big|_0^{\pi/2} - \frac{2}{n} \cos(nx) \Big|_{\pi/2}^{\pi} \right]$$

$$= \frac{2}{n\pi} \left[-\cos\left(\frac{n\pi}{2}\right) + 1 - 2\cos(n\pi) + 2\cos\left(\frac{n\pi}{2}\right) \right]$$

$$= \frac{2}{n\pi} \left[1 + \cos\left(\frac{n\pi}{2}\right) - 2\cos(n\pi) \right]$$

$$f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[1 + \cos\left(\frac{n\pi}{2}\right) - 2\cos(n\pi) \right] \sin(nx)$$

$$f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[1 + \cos\left(\frac{n\pi}{2}\right) + 2(-1)^{n+1} \right] \sin(nx)$$

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$$f(x) = \sum_{n_{odd}} + \sum_{n_{even}}$$

$$\sum_{n_{even}} = \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k} \left[1 + \cos\left(\frac{2k\pi}{2}\right) + 2(-1)^{2k+1} \right] \sin(2kx)$$

$$= \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k} \left[1 + (-1)^n - 2 \right] \sin(2kx)$$

$$= \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{n+1}}{2k} \sin(2kx)$$

$$\sum_{n_{odd}} = \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k-1} \left[1 + \cos\left(\frac{(2k-1)\pi}{2}\right) + 2(-1)^{2k-1+1} \right] \sin((2k-1)x)$$

$$= \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{3}{2k-1} \sin((2k-1)x)$$

$$f(x) = \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{n+1}}{2k} \sin(2kx) + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{3}{2k-1} \sin((2k-1)x)$$

b)

Find the Fourier-cosine-series of $f(x) = |\sin x|$. Then find

$$\sum_{k=1}^{\infty} \frac{1}{4k^2 - 1}$$

The Fourier cosine series of a function $f(x)$ is

$$f(x) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right)$$

where

$$A_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

Assuming $l = \pi$, we have

$$A_n = \frac{2}{\pi} \int_0^\pi |\sin x| \cos(nx) dx \tag{3}$$

But since sin is non-negative on the interval $0 \le x \le \pi$, we can drop the absolute value bars and use the following trigonometric identity to rewrite the integral: $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$

$$A_n = \frac{1}{\pi} \int_0^\pi \sin((1+n)x) + \sin((1-n)x) dx$$

$$= \frac{1}{\pi} \left[\frac{-1}{1+n} \cos((1+n)x) \Big|_0^\pi - \frac{1}{1-n} \cos((1-n)x) \Big|_0^\pi \right]$$

$$= \frac{1}{\pi} \left[\frac{-1}{1+n} [(-1)^{1+n} - 1] - \frac{1}{1-n} [(-1)^{1-n} - 1] \right]$$

$$= \frac{1}{\pi} [(-1)^{1+n} - 1)] \left(\frac{-1}{1+n} - \frac{1}{1-n} \right)$$

Thus, we have found a formula for A_n,

$$A_n = \frac{-2}{\pi} \frac{(-1)^n + 1}{n^2 - 1}$$

This formula works for all $n \neq 1$ For $n = 0$,

$$A_0 = \frac{-2}{\pi} \frac{(-1)^0 + 1}{0 - 1} = \frac{4}{\pi}$$

For $n = 1$, we can go back to (3) and plug in the appropriate value of n:

$$A_1 = \frac{1}{\pi} \int_0^{\pi} \sin(2x)\, dx = \frac{1}{2\pi} \cos(2x)\Big|_0^{\pi} dx = 0$$

The Fourier cosine series is therefore

$$f(x) = \frac{2}{\pi} - \frac{2}{\pi} \sum_{n=2}^{\infty} \frac{(-1)^n + 1}{n^2 - 1} \cos(nx)$$

Now, $(-1)^n + 1$ will be zero for all odd n and two for all even n. So substituting $n = 2k$, we gwt

$$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{2k=2}^{\infty} \frac{1}{4k^2 - 1} \cos(2kx)$$

Starting the index at $2k = 2$ is equivalent to starting it at $k = 1$, so after making that change and evaluating both sides at $x = 0$, we can solve for the sum given in the question:

$$0 = \frac{2}{\pi} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{4k^2 - 1}$$

$$\sum_{k=1}^{\infty} \frac{1}{4k^2 - 1} = \frac{1}{2}$$

c)

The Riemann Zeta function is defined for $s > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

By computing the Fourier series of x^2 over $-\pi < x < \pi$ and using Parseval's identity, compute $\zeta(4)$.

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 dx$$

$$= \frac{1}{2\pi} \left(\frac{x^3}{3} \right)\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi} \left(\frac{\pi^3}{3} - \frac{-\pi^3}{3} \right)$$

$$= \frac{1}{2\pi} \left(\frac{2\pi^3}{3} \right) = \frac{\pi^2}{3}$$

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$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos{(nx)} dx$$

But in part (2c), we found that this is equivalent to $\frac{4}{n^2}(-1)^n$, so

$$a_n = \frac{4}{n^2}(-1)^n$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \sin{(nx)} dx$$

Since x^2 is an even function, $b_n = 0$.

$$f(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n \cos{(nx)}}{n^2}$$

Parseval's Identity says that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} (x^2)^2 \, dx = a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} (x^2)^2 \, dx = \left(\frac{\pi^2}{3} \right)^2 + \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{4(-1)^n}{n^2} \right)^2$$

$$\frac{1}{\pi} \int_0^{\pi} x^4 \, dx = \frac{\pi^4}{9} + \sum_{n=1}^{\infty} \frac{8}{n^4}$$

$$\frac{1}{\pi} \frac{x^5}{5} \Big|_0^{\pi} = \frac{\pi^4}{9} + 8 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\frac{\pi^4}{5} = \frac{\pi^4}{9} + 8 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\frac{4\pi^4}{45} = 8 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

Thus,

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

(d) Use the Fourier series in 2c and the pointwise convergence theorem to find $\zeta(2)$. Then find

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

Take the following Fourier series and set $x = \pi$.

$$f(x) = x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n \cos{(nx)}}{n^2}$$

$$\pi^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

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$$\frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Thus,

$$\zeta(2) = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Now, we want to find

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{1}{1} + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \dots$$

Let's also write out some terms in the series representation of $\zeta(2)$:

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \dots$$

We can see that the top series is simply the bottom series without the terms where the denominator is the square of an even integer; which we can write as:

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \sum_{n=1}^{\infty}\frac{1}{n^2} - \sum_{n=1}^{\infty}\frac{1}{(2n)^2}$$

$$= \sum_{n=1}^{\infty}\frac{1}{n^2} - \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$= \frac{\pi^2}{6} - \frac{1}{4}\frac{\pi^2}{6}$$

$$= \frac{\pi^2}{8}$$

Therefore, we conclude that

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

Question 4

Compute the complex Fourier series of the following functions:

(a)

Compute the complex Fourier series of $f(x) = e^x$ and show that

$$\coth \pi = \frac{1}{\pi} + \frac{2}{\pi}\left(\frac{1}{1+1^2} + \frac{1}{1+2^2} + \frac{1}{1+3^2} + \dots\right)$$

The complex Fourier series of a function $f(x)$ is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}$$

where

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}\,dx$$

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$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^x e^{-inx} \, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{x(1-in)} \, dx$$

$$= \frac{1}{2\pi} \frac{1}{1-in} e^{x(1-in)} \Big|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi} \frac{1}{1-in} \left[e^\pi e^{-in\pi} - e^{-\pi} e^{in\pi} \right]$$

and since $e^{in\pi} = \cos(n\pi) + i \sin(n\pi) = (-1)^n = \cos(-n\pi) + i \sin(-n\pi) = e^{-in\pi}$,

$$c_n = \frac{1}{2\pi} \frac{(-1)^n}{1-in} \left[e^\pi - e^{-\pi} \right]$$

$$= \frac{1}{\pi} \frac{(-1)^n}{1-in} \sinh \pi$$

$$f(x) = e^x = \sum_{n=-\infty}^{\infty} \frac{1}{\pi} \frac{(-1)^n}{1-in} (\sinh \pi) e^{inx}$$

Set $x = \pi$ and $x = -\pi$ and then add the two expressions, noting that $e^{in\pi} = e^{-in\pi} = (-1)^n$ and $(-1)^n(-1)^n = (-1)^{2n} = 1$.

$$e^\pi = \sum_{n=-\infty}^{\infty} \frac{1}{\pi} \frac{1}{1-in} (\sinh \pi)$$

$$e^{-\pi} = \sum_{n=-\infty}^{\infty} \frac{1}{\pi} \frac{1}{1-in} (\sinh \pi)$$

$$\frac{e^\pi + e^{-\pi}}{2} = \sum_{n=-\infty}^{\infty} \frac{1}{\pi} \frac{1}{1-in} (\sinh \pi)$$

But note that $\frac{e^\pi + e^{-\pi}}{2} = \cosh \pi$.

$$\cosh \pi = \sum_{n=-\infty}^{\infty} \frac{1}{\pi} \frac{1}{1-in} (\sinh \pi)$$

Divide both sides by $\sinh \pi$:

$$\coth \pi = \sum_{n=-\infty}^{\infty} \frac{1}{\pi} \frac{1}{1-in}$$

Multiply the right-hand side by $\frac{1+in}{1+in}$:

$$\coth \pi = \sum_{n=-\infty}^{\infty} \frac{1}{\pi} \frac{1+n}{1+n^2}$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{\pi} \frac{1}{1+n^2} + \sum_{n=-\infty}^{\infty} \frac{1}{\pi} \frac{n}{1+n^2}$$

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The second sum will disappear because the n term in the numerator means that all negative terms in the series will cancel with the corresponding positive n terms, while the expression evaluates to zero at $n = 0$. Thus,

$$\coth \pi = \sum_{n=-\infty}^{\infty} \frac{1}{\pi} \frac{1}{1+n^2}$$

$$= \frac{1}{\pi} + \frac{2}{\pi} \left(\frac{1}{1+1^2} + \frac{1}{1+2^2} + \frac{1}{1+3^2} + \cdots \right)$$

which is what we initially sought to show.

(b)

Find the complex Fourier series of xe^{ix}. Then use your result to find the Fourier series of $x \cos x$ and $x \sin x$.

The complex Fourier series of a function $f(x)$ is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}$$

where

$$c_n = \frac{1}{2l} \int_{-l}^{l} f(x) e^{-in\pi x/l} \, dx$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} xe^{ix} e^{-inx} \, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} xe^{ix(1-n)} \, dx$$

Using integration by parts,

$$u = x, \quad du = dx, \quad dv = e^{ix(1-n)} \, dx, \quad v = \frac{1}{i(1-n)} e^{ix(1-n)}$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} xe^{ix(1-n)} \, dx = \frac{1}{2\pi} \left[\frac{x}{i(1-n)} e^{ix(1-n)} \right]_{-\pi}^{\pi} - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{i(1-n)} e^{ix(1-n)} \, dx$$

First, let's evaluate the first term on the right-hand side:

$$\frac{1}{2\pi} \left[\frac{x}{i(1-n)} e^{ix(1-n)} \right]_{-\pi}^{\pi} = \frac{1}{2\pi} \left[\frac{\pi}{i(1-n)} e^{i\pi(1-n)} - \frac{-\pi}{i(1-n)} e^{-i\pi(1-n)} \right]$$

$$= \frac{1}{2\pi} \left[\frac{\pi}{i(1-n)} e^{i\pi} e^{-i\pi n} - \frac{-\pi}{i(1-n)} e^{-i\pi} e^{i\pi n} \right]$$

$$= \frac{1}{2\pi} \left[\frac{\pi}{i(1-n)} (-1)(-1)^n - \frac{-\pi}{i(1-n)} (-1)(-1)^n \right]$$

$$= \frac{1}{2\pi} \left[\frac{2\pi}{i(1-n)} (-1)^{n+1} \right]$$

$$= \frac{(-1)^{n+1}}{i(1-n)}$$

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Now we turn to the integral on the right-hand side:

$$-\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{i(1-n)}e^{ix(1-n)}\,dx = \frac{1}{2\pi}\left[\frac{e^{ix(1-n)}}{(1-n)^2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{i\pi(1-n)}}{(1-n)^2} - \frac{e^{-i\pi(1-n)}}{(1-n)^2}\right]$$

$$= \frac{1}{2\pi}\left[\frac{(-1)^{n+1}}{(1-n)^2} - \frac{(-1)^{n+1}}{(1-n)^2}\right]$$

$$= 0$$

Thus, $c_n = \frac{(-1)^{n+1}}{i(1-n)}$. Plugging this into the equation for a complex Fourier series, we find that

$$xe^{ix} = \sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}}{i(1-n)}e^{inx}$$

Now, we use Euler's identity

$$e^{ix} = \cos x + i\sin x$$

$$\cos x = \frac{e^{ix}+e^{-ix}}{2}$$

$$x\cos x = \frac{x\left(e^{ix}+e^{-ix}\right)}{2}$$

$$= \frac{x}{2}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}}{i(1-n)}e^{inx} + \frac{x}{2}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}}{i(1-n)}e^{-inx}$$

$$= \frac{x}{2}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}}{i(1-n)}(\cos nx + i\sin nx) + \frac{x}{2}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}}{i(1-n)}(\cos(-nx) + i\sin(-nx))$$

$$= \frac{x}{2}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}}{i(1-n)}(\cos nx + i\sin nx) + \frac{x}{2}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}}{i(1-n)}(\cos(nx) - i\sin(nx))$$

The sin terms cancel and the cos terms add, yielding

$$x\cos x = x\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}}{i(1-n)}\cos nx$$

Similarly,

$$\sin x = \frac{e^{ix}-e^{-ix}}{2i}$$

$$= \frac{x}{2i}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}}{i(1-n)}e^{inx} - \frac{x}{2i}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}}{i(1-n)}e^{-inx}$$

$$= \frac{x}{2i}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}}{i(1-n)}(\cos nx + i\sin nx) - \frac{x}{2i}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}}{i(1-n)}(\cos(-nx) + i\sin(-nx))$$

$$= \frac{x}{2i}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}}{i(1-n)}(\cos nx + i\sin nx) - \frac{x}{2i}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}}{i(1-n)}(\cos(nx) - i\sin(nx))$$

The cos terms cancel and the sin terms add, yielding

$$\frac{x}{i} \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1}}{i(1-n)}(-\sin nx)$$

Multiplying the two is together cancels the negative sign, meaning the Fourier series becomes:

$$x \sin x = x \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1}}{(1-n)} \sin nx$$

**Note that if $n = 1$, then $c_n = 0$, since you end up with the integral of x over $(-\pi, \pi)$, which is zero. Hence the sum is properly over all terms with $n \neq 1$.

Question 5

Find the function represented by the new series which is obtained by term-wise integration of the following series from 0 to x.

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{\cos kx}{k} = \log\left(2 \cos\left(\frac{x}{2}\right)\right), \quad -\pi < x < \pi$$

$$\int_0^x \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\cos kx}{k} = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sin kx}{k^2}$$

$$\left[\sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sin kx}{k^2}\right]\Big|_0^x = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sin kx}{k^2}$$

Therefore,

$$\int_0^x \log\left(2 \cos\left(\frac{x}{2}\right)\right) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sin kx}{k^2}$$

If you wanted us to differentiate rather than integrate:

$$\frac{d}{dx} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\cos kx}{k} = \frac{d}{dx} \log\left(2 \cos\left(\frac{x}{2}\right)\right)$$

$$\sum_{k=1}^{\infty} (-1)^k \sin kx = \frac{-\sin \frac{x}{2}}{2 \cos \frac{x}{2}}$$

$$= -\frac{1}{2} \tan \frac{x}{2}$$

Thus,

$$\sum_{k=1}^{\infty} (-1)^k \sin kx = -\frac{1}{2} \tan \frac{x}{2}$$

Note that this sum is actually divergent, since the terms don't go to zero for any x apart from a few special points (like $x = 0$).