MATH 245 Homework 5

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Problem 1: Inhomogeneous Heat Equation

Using the method of separation of variables, solve the inhomogeneous heat equation:

$$\begin{cases} u_t - ku_{xx} = x\cos t & 0 < x < \pi, \quad t > 0 \\ u(0,x) = \sin(\pi x) & 0 < x < \pi \\ u(t,0) = t^2, \ \ u(t,\pi) = 2t & t > 0 \end{cases} \tag{1}$$

Problem 2: Inhomogeneous Wave Equation

Using the method of separation of variables, solve the inhomogeneous wave equation:

$$\begin{cases} u_{tt} - c^2u_{xx} = F(t,x) & 0 < x < L, \quad t > 0 \\ u(0,x) = \phi(x) & 0 < x < L \\ u_t(0,x) = \psi(x) & 0 < x < L \\ u_x(t,0) = h(t), \ \ u_x(t,L) = g(t) & t > 0 \end{cases} \tag{2}$$

Step 1

We will use the principle of superposition and write $u(t,x) = v(t,x) + w(t,x)$, so $v = u - w$. Then for $v(t,x)$, we have

$$\begin{cases} \Box v = F(t,x) - \Box w = F(t,x) - w_{tt} + c^2w_{xx} = H(t,x) & 0 < x < L, \quad t > 0 \\ v(0,x) = u(0,x) - w(0,x) = \phi(x) - w(0,x) = \Phi(x) & 0 < x < L \\ v_t(0,x) = u_t(0,x) - w_t(0,x) = \psi(x) - w_t(0,x) = \Psi(x) & 0 < x < L \\ v_x(t,0) = u_x(t,0) - w_x(t,0) = h(t) - h(t) = 0 & \\ v_x(t,L) = u_x(t,L) - w_x(t,L) = g(t) - g(t) = 0 & t > 0 \end{cases} \tag{3}$$

$$\begin{cases} \Box v = H(t,x) & 0 < x < L, \quad t > 0 \\ v(0,x) = \Phi(x) & 0 < x < L \\ v_t(0,x) = \Psi(x) & 0 < x < L \\ v_x(t,0) = 0 & \\ v_x(t,L) = 0 & t > 0 \end{cases} \tag{4}$$

and for w, we have

$$\begin{cases} w_{xx} = 0 \\ w_x(t,0) = h(t) \\ w_x(t,L) = g(t) \end{cases} \tag{5}$$

For $w(t,x)$, we will use a function of the form

$$w(t,x) = \left(x - \frac{x^2}{2L}\right) h(t) + \frac{x^2}{2L} g(t)$$

$$w_x = h(t) - \frac{x}{L} h(t) + \frac{x}{L} g(t)$$

$$w_{xx} = \frac{1}{L}\Big(g(t) - h(t)\Big)$$

So that we will have $w_x(t,0) = h(t), w_x(t,L) = g(t)$.

Step 2: Solve (4)

Assume (4) has the separated solution $v(t,x) = X(x)T(t)$ and consider the following eigenvalue problem

$$\begin{cases} x'' + \lambda X = 0 \\ X'(0) = X'(L) = 0 \end{cases}$$

W know that the solution to this eigenvalue problem is

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad X_n(x) = \cos\left(\frac{n\pi x}{L}\right), \quad n = 0, 1, 2, 3...$$

Step 3

Now, let's write $v(t,x)$ as

$$v(t,x) = a_0(t) + \sum_{n=1}^{\infty} a_n(t) \cos\left(\frac{n\pi x}{L}\right)$$

and substitute this into the PDE of (4).

$$v_{tt}(t,x) = a_0''(t) + \sum_{n=1}^{\infty} a_n''(t) \cos\left(\frac{n\pi x}{L}\right)$$

$$c^2 v_{xx}(t,x) = -\sum_{n=1}^{\infty} \left(\frac{n\pi c}{L}\right)^2 a_n(t) \cos\left(\frac{n\pi x}{L}\right)$$

$$H(t,x) = a_0''(t) + \sum_{n=1}^{\infty} \left[a_n''(t) + \left(\frac{n\pi c}{L}\right)^2 a_n(t)\right] \cos\left(\frac{n\pi x}{L}\right) \tag{6}$$

Step 4

For a fixed t, we write $H(t,x)$ as the Fourier cosine series

$$H(t,x) = q_0(t) + \sum_{n=1}^{\infty} q_n(t) \cos\left(\frac{n\pi x}{l}\right) \tag{7}$$

Such that

$$q_n(t) = \frac{2}{L} \int_0^L H(t,x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$q_0(t) = \frac{1}{L} \int_0^L H(t,x) dx$$

Step 5

Variation of Parameters

Consider an ODE

$$(*) \quad y'' + a(x)y' + b(x)y = c(x)$$

and assume $\{y_1(x), y_2(x)\}$ are fundamental solutions of the associated homogeneous equation

$$(**) \quad y'' + a(x)y' + b(x)y = 0$$

Then the general solution of $(**)$ is $y_h(x) = c_1 y_1(x) + c_2 y_2(x)$ for $c_1, c_2 \in \mathbb{R}$ and the solution of $(*)$ is the sum of the general and particular solutions, $y = y_h + y_p$, where

$$y_p = -y_1(x) \int \frac{y_2(s)c(s)}{W(y_1, y_2)} ds + y_2(x) \int \frac{y_1(s)c(s)}{W(y_1, y_2)} ds$$

where

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Combine (6) and (7):

$$q_0(t) = a_0''(t)$$
$$q_n(t) = a_n''(t) + \left(\frac{n\pi c}{L}\right)^2 a_n(t)$$

First we find the fundamental solution of

$$a_0''(t) = 0$$
$$a_0^H = A_0 t + B_0$$

for $A_0, B_0 \in \mathbb{R}$. Call $y_1 = 1, y_2 = t$.

$$W(1, t) = \begin{vmatrix} 1 & t \\ 0 & 1 \end{vmatrix} = 1$$

$$a_0^P(t) = -\int_0^t s\, q_n(s)\, ds + t \int_0^t q_n(s)\, ds = \int_0^t (t - s) q_n(s)\, ds$$

$$a_0(t) = A_0 t + B_0 + \int_0^t (t - s) q_n(s)\, ds$$

Now we find the fundamental solutions of

$$a_n''(t) + \left(\frac{n\pi c}{L}\right)^2 a_n(t) = 0$$

This has the characteristic equation $r^2 + \left(\frac{n\pi c}{L}\right)^2 = 0$, giving us $\{y_1, y_2\} = \left\{\cos\left(\frac{n\pi ct}{L}\right), \sin\left(\frac{n\pi ct}{L}\right)\right\}$. Thus, the general solution to the homogeneous problem is

$$a_n^H(t) = A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)$$

for $A_n, B_n \in \mathbb{R}, n \in \mathbb{N}$.

$$W(y_1,y_2)(s) = \begin{vmatrix} \cos\left(\frac{n\pi cs}{L}\right) & \sin\left(\frac{n\pi cs}{L}\right) \\ -\frac{n\pi c}{L}\sin\left(\frac{n\pi cs}{L}\right) & \frac{n\pi c}{L}\cos\left(\frac{n\pi cs}{L}\right) \end{vmatrix} = \frac{n\pi c}{L}$$

$$\begin{aligned} a_n^P(t) &= -\cos\left(\frac{n\pi ct}{L}\right)\int_0^t \frac{\sin\left(\frac{n\pi cs}{L}\right)q_n(s)}{W(y_1,y_2)}ds + \sin\left(\frac{n\pi ct}{L}\right)\int_0^t \frac{\cos\left(\frac{n\pi cs}{L}\right)q_n(s)}{W(y_1,y_2)}ds \\ &= \int_0^t \frac{\sin\left(\frac{n\pi c}{L}(t-s)\right)q_n(s)}{W(y_1,y_2)}ds \\ &= \int_0^t \frac{\sin\left(\frac{n\pi c}{L}(t-s)\right)q_n(s)}{\frac{n\pi c}{L}}ds \end{aligned}$$

Therefore, $a_n(t) = a_n^H(t) + a_n^P(t)$, but the A_n, B_n in $a_n^H(t)$ and the A_0, B_0 in $a_0^H(t)$ are still unknown.

Step 6

To find A_0, B_0, A_n, B_n, we use the initial conditions from (4).

$$\Phi(x) = v(0,x) = a_0(0) + \sum_{n=1}^{\infty} a_n(0)\cos\left(\frac{n\pi x}{L}\right) = B_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right)$$

$$\Psi(x) = v_t(0,x) = A_0 + \sum_{n=1}^{\infty} \frac{n\pi c}{L}B_n\cos\left(\frac{n\pi x}{L}\right)$$

$$\begin{aligned} A_0 &= \frac{1}{L}\int_0^L \Psi(x)dx \\ B_0 &= \frac{1}{L}\int_0^L \Phi(x)dx \\ A_n &= \frac{2}{L}\int_0^L \Phi(x)\cos\left(\frac{n\pi x}{L}\right)dx \\ \frac{n\pi c}{L}B_n &= \frac{2}{L}\int_0^L \Psi(x)\cos\left(\frac{n\pi x}{L}\right)dx \end{aligned}$$

Step 7: Solution

The solution of (2) is then

$$u(t,x) = a_0(t) + \sum_{n=1}^{\infty} a_n(t)\cos\left(\frac{n\pi x}{L}\right)$$

where

$$a_0(t) = \frac{1}{L}\int_0^L [t\,\Psi(x) + \Phi(x)]\,dx + \int_0^t (t-s)q_n(s)\,ds$$

$$a_n(t) = A_n\cos\left(\frac{n\pi ct}{L}\right) + B_n\sin\left(\frac{n\pi ct}{L}\right) + \int_0^t \frac{\sin\left(\frac{n\pi c}{L}(t-s)\right)q_n(s)}{\frac{n\pi c}{L}}ds$$

$$A_n = \frac{2}{L}\int_0^L \Phi(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad B_n = \frac{2}{n\pi c}\int_0^L \Psi(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$q_n(t) = \frac{2}{L}\int_0^L H(t,x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$\begin{aligned}
H(t,x) &= F(t,x) - w_{tt} + c^2 w_{xx} \\
&= F(t,x) - \left[\left(x - \frac{x^2}{2L}\right)h''(t) + \frac{x^2}{2L}g''(t)\right] + \frac{c^2}{L}\Big(g(t) - h(t)\Big) \\
\Phi(x) &= u(0,x) - w(0,x) \\
&= \phi(x) - w(0,x) \\
&= \phi(x) - \left(x - \frac{x^2}{2L}\right)h(0) - \frac{x^2}{2L}g(0) \\
\Psi(x) &= u_t(0,x) - w_t(0,x) \\
&= \psi(x) - w_t(0,x) \\
&= \psi(x) - \left(x - \frac{x^2}{2L}\right)h'(0) - \frac{x^2}{2L}g'(0)
\end{aligned}$$

Problem 3: Damped Heat Equation

Using the method of separation of variables, solve the damped heat equation:

$$\begin{cases}
u_t + au = ku_{xx} & -\pi < x < \pi, \quad t > 0 \\
u(0,x) = \phi(x) & -\pi < x < \pi \\
u(t,\pi) = u(t,-\pi) & t > 0 \\
u_x(t,\pi) = u_x(t,-\pi) & t > 0
\end{cases} \tag{8}$$

for constants a and $k > 0$

Problem 4: Beam Equation

Using the method of separation of variables, solve the beam equation:

$$\begin{cases}
u_{tt} = c^2 u_{xxxx} & 0 < x < L, \quad t > 0 \\
u(0,x) = \phi(x) & 0 < x < L \\
u_t(0,x) = \psi(x) & 0 < x < L \\
u(t,0) = u(t,L) = 0 & t > 0 \\
u_{xx}(t,0) = u_{xx}(t,L) = 0
\end{cases} \tag{9}$$

Suppose we have a separated solution of the form $u(t,x) = X(x)T(t)$.

If we plug this into the homogeneous Dirichlet BCs for u, we find $u(t,0) = T(t)X(0) = 0, u(t,L) = T(t)X(L) = 0$. In order for both of these to be true, we must have either $T(t) = 0$ or $X(0) = X(L) = 0$. But if $T(t) = 0$, then $u(t,x) = 0$ for all x, which contradicts our ICs. Thus, we must have $X(0) = X(L) = 0$. Similarly, from our other BCs we find that

$$X''(0) = X''(L) = 0$$

.

Plugging our expression $u(t,x) = X(x)T(t)$ into the PDE in (9), it becomes

$$T''(t)X(x) = c^2 T(t)X^{(4)}(x)$$

$$\frac{T''(t)}{c^2T(t)} = \frac{X^{(4)}(x)}{X(x)} = \lambda$$
$$X^{(4)} - \lambda X = 0$$
$$r^4 - \lambda = 0$$

Case 1: Zero eigenvalues, $\lambda = 0$

By the PDE, $X^{(4)}(x) = 0$ implies that $u_{xxxx} = 0$, and thus that $X(x)$ is of the form $Ax^3 + Bx^2 + Cx + D$. Now plugging in our initial conditions:

$$X(0) = 0 \quad \Longrightarrow \quad D = 0$$

$$X''(x) = 6Ax + 2B \quad \Longrightarrow \quad X''(0) = 2B = 0 \quad \Longrightarrow \quad B = 0$$

$$X''(l) = 6Al = 0 \quad \Longrightarrow \quad A = 0$$

$$X(l) = Cl = 0 \quad \Longrightarrow \quad C = 0$$

Therefore, we would have $X(x) = 0$, so there are no eigenfunctions $X(x)$ that satisfy $X^{(4)} + \lambda X = 0$ when $\lambda = 0$ and hence no zero eigenvalues.

Case 2: Negative eigenvalues, $\lambda = -\beta^4 < 0$

We will use Green's second identity to show that we cannot have any negative eigenvalues. Start with $X^{(4)} = \lambda X$, multiply by $\bar{X}$, and integrate both sides from 0 to l:

$$\begin{aligned} X^{(4)} &= \lambda X \\ \int_0^l X^{(4)}\bar{X}\,dx &= \lambda \int_0^l X\bar{X}\,dx \\ &= \lambda \int_0^l |X|^2\,dx \end{aligned}$$

Now Green's second identity states that for any $u, v \in C^2[a, b]$, we have

$$\int_a^b vu''dx = \int_a^b uv''dx + [vu' - uv']_{x=a}^{x=b}$$

Letting $u = X^{(4)}$ and $v = \bar{X}$,

$$\int_0^l X^{(4)}\bar{X}\;dx = \int_0^l |X^{''}|^2\;dx\; + \left[\bar{X}X^{(5)} - X^{(4)}\bar{X}'\right]_{x=0}^{x=l}$$

But since $X(0) = X(L) = 0$, the boundary terms disappear and we have

$$\int_0^l |X^{''}|^2\;dx = \lambda \int_0^l |X|^2\,dx$$

$$\lambda = \frac{\int_0^l |X''|^2 \, dx}{\int_0^l |X|^2 \, dx} \quad \implies \quad \lambda \geq 0$$

Therefore, we have only positive eigenvalues.

Case 3: Positive eigenvalues, $\lambda = \beta^4 > 0$

This case gives us the characteristic equation: $r^4 - \beta^4 = 0$, or $r^2 = \pm\beta^2, r = \pm\beta, \pm\beta i$

$$X(x) = Ae^{\beta x} + Be^{-\beta x} + C\sin(\beta x) + D\cos(\beta x)$$

$$X(0) = A + B + D = 0 \tag{10}$$

$$X''(x) = \beta^2 Ae^{\beta x} + \beta^2 Be^{-\beta x} - \beta^2 C\sin(\beta x) - \beta^2 D\cos(\beta x)$$

Since we are in a case where we defined β to be non-zero, we can divide by it to find that

$$X''(0) = \beta^2 A + \beta^2 B - \beta^2 D = 0 \quad \implies \quad A + B = D$$

Combining this with (10), we find that $D = 0$.

$$X(L) = Ae^{\beta L} + Be^{-\beta L} + C\sin(\beta L) = 0$$

$$X''(L) = \beta^2 Ae^{\beta L} + \beta^2 Be^{-\beta L} - \beta^2 C\sin(\beta L) = 0$$

$$C\sin(\beta L) = -C\sin(\beta L), \quad 2C\sin(\beta L) = 0$$

Either $C = 0$ or $\sin(\beta L) = 0$. Say $C = 0$, then $X(0) = A + B = 0$ implies $A = -B$. Then the boundary condition gives

$$X(L) = Be^{\beta L} - Be^{-\beta L} = 0, \quad B(e^{\beta L} - e^{-\beta L}) = 0$$

Since $e^{\beta L}$ and $e^{-\beta L}$ are only equal when $L = 0$ and we defined $L > 0$, we must have $B = A = 0$, which would imply that $X(x) = 0$, the trivial solution. Therefore, we must have $\sin(\beta L) = 0$, such that $\beta = \frac{n\pi}{L}$.

Returning to our boundary conditions,

$$X(0) = Ae^0 + Be^0 + C\sin 0 = 0$$

$$A + B = 0, \quad A = -B$$

$$X(L) = Ae^{\left(\frac{n\pi}{L}\right)L} - Ae^{-\left(\frac{n\pi}{L}\right)L} + C\sin\left(\left(\frac{n\pi}{L}\right)L\right) = 0$$

$$X(L) = Ae^{n\pi} - Ae^{-n\pi} + C\sin(n\pi) = 0$$

$$Ae^{n\pi} = Ae^{-n\pi}$$

But for $n = 1, 2, 3...$, this is only true if $A = 0$, which means B is also zero.

Thus, we have eigenvalues of $\lambda_n = \left(\frac{n\pi}{L}\right)^4$ with eigenfunctions

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3...$$

Now we return to the time equation,

$$T''(t) - \lambda c^2 T(t) = 0$$

$$T'' - \left(\frac{n\pi}{L}\right)^4 c^2 T = 0$$

$$r^2 - \left(\frac{n\pi}{L}\right)^4 c^2 = 0$$

$$T_n(t) = Ae^{\left(\frac{n\pi}{L}\right)^2 ct} + Be^{-\left(\frac{n\pi}{L}\right)^2 ct}$$

$$u_n(t,x) = \left[Ae^{\left(\frac{n\pi}{L}\right)^2 ct} + Be^{-\left(\frac{n\pi}{L}\right)^2 ct}\right] \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3...$$

Check if this actually satisfies our PDE:

$$u_{tt} = c^2 u_{xxxx}$$

$$u\left(\frac{n\pi}{L}\right)^4 c^2 = c^2\left(\frac{n\pi}{L}\right)^4 u$$

Now we can use the principle of superposition to find our general solution:

$$u(t,x) = \sum_{n=1}^{\infty} \left[Ae^{\left(\frac{n\pi}{L}\right)^2 ct} + Be^{-\left(\frac{n\pi}{L}\right)^2 ct}\right] \sin\left(\frac{n\pi x}{L}\right)$$

$$u(0,x) = \phi(x) = \sum_{n=1}^{\infty} [A + B] \sin\left(\frac{n\pi x}{L}\right)$$

$$A + B = \frac{2}{L}\int_0^L \phi(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$u_t(0,x) = \psi(x) = \sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 c\,[A - B] \sin\left(\frac{n\pi x}{L}\right)$$

$$A - B = \left(\frac{n\pi}{L}\right)^2 \frac{2c}{L}\int_0^L \psi(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$-2B + \frac{2}{L}\int_0^L \phi(x) \sin\left(\frac{n\pi x}{L}\right) dx = \left(\frac{n\pi}{L}\right)^2 \frac{2c}{L}\int_0^L \psi(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$\begin{aligned} B &= \frac{1}{L}\int_0^L \phi(x) \sin\left(\frac{n\pi x}{L}\right) dx - \left(\frac{n\pi}{L}\right)^2 \frac{c}{L}\int_0^L \psi(x) \sin\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{1}{L}\int_0^L \left[\phi(x) - c\left(\frac{n\pi}{L}\right)^2 \psi(x)\right] \sin\left(\frac{n\pi x}{L}\right) dx \end{aligned}$$

$$A = -B + \frac{2}{L}\int_0^L \phi(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Therefore, our solution to (9) is:

Solution

$$u(t,x) = \sum_{n=1}^{\infty}\left[Ae^{\left(\frac{n\pi}{L}\right)^2 ct} + Be^{-\left(\frac{n\pi}{L}\right)^2 ct}\right]\sin\left(\frac{n\pi x}{L}\right), \quad \text{where}$$
$$B = \frac{1}{L}\int_0^L \left[\phi(x) - c\left(\frac{n\pi}{L}\right)^2\psi(x)\right]\sin\left(\frac{n\pi x}{L}\right)dx$$
$$A = \frac{1}{L}\int_0^L \left[\phi(x) + c\left(\frac{n\pi}{L}\right)^2\psi(x)\right]\sin\left(\frac{n\pi x}{L}\right)dx$$

Problem 5: Radioactive Decay Problem

Using the method of separation of variables, solve the radioactive decay problem, for constants $A, a > 0$.

$$\begin{cases} u_t - u_{xx} = Ae^{-ax} \\ u(0,x) = \sin x \\ u(t,0) = u(t,\pi) = 0 \end{cases} \tag{11}$$

We want to find a separated solution of the form $u(t,x) = X(x)T(t)$. Recall that for the analogous homogeneous PDE with homogeneous Dirichlet boundary conditions, we consider the following eigenvalue problem $X'' + \lambda X = 0, \quad X(0) = X(l) = 0$, which we have shown to have the eigenvalues and eigenvectors

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \sin\left(\frac{n\pi x}{l}\right), \quad n = 1, 2, 3...$$

Giving us the Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \tag{12}$$

where

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Now, we assume that our solution to (11) will take a similar form as (12), where $l = \pi$ and $f(t,x) = Ae^{-ax}$:

$$u(t,x) = b_0(t) + \sum_{n=1}^{\infty} b_n(t)\sin(nx)$$

We can differentiate this and plug it into (11) as follows:

$$u_t(t,x) = b_0'(t) + \sum_{n=1}^{\infty} b_n'(t)\sin(nx)$$

$$u_{xx}(t,x) = -\sum_{n=1}^{\infty} b_n(t)\, n^2 \sin(nx)$$

$$b_0'(t) + \sum_{n=1}^{\infty} b_n'(t)\sin(nx) + \sum_{n=1}^{\infty} b_n(t)\, n^2 \sin(nx) = Ae^{-ax} \tag{13}$$

For each fixed t, we write Ae^{-ax} as a Fourier sine series:

$$Ae^{-ax} = q_0(t) + \sum_{n=1}^{\infty} q_n(t)\sin(nx) \tag{14}$$

where

$$\begin{aligned} q_0(t) &= \frac{1}{l}\int_0^l f(t,x)dx \\ &= \frac{A}{\pi}\int_0^{\pi} e^{-ax}dx \\ &= \frac{-A}{a\pi}\left[e^{-ax}\right]_0^{pi} \\ &= \frac{-A}{a\pi}\left(e^{-a\pi} - 1\right) \end{aligned}$$

$$\begin{aligned} q_n(t) &= \frac{2}{l}\int_0^l f(t,x)\sin\left(\frac{n\pi x}{l}\right)dx \\ &= \frac{2A}{\pi}\int_0^{\pi} e^{-ax}\sin(nx)dx \end{aligned}$$

Now we do integration by parts twice on $\int_0^{\pi} e^{-ax}\sin(nx)dx$

First with $u = \sin(nx),\ \ du = n\cos(nx)\,dx,\ \ dv = e^{-ax}\,dx,\ \ v = \frac{-1}{a}e^{-ax}$,

and the second time with $u = n\cos(nx),\ \ du = -n^2\sin(nx)\,dx,\ \ dv = \frac{1}{a}e^{-ax}\,dx,\ \ v = \frac{-1}{a^2}e^{-ax}$

$$\begin{aligned} \int_0^{\pi} e^{-ax}\sin(nx)dx &= \left[\frac{-\sin(nx)}{a}e^{-ax}\right]_0^{\pi} + \frac{n}{a}\int_0^{\pi} e^{-ax}\cos\left(\frac{n\pi x}{l}\right)dx \\ &= 0 - 0 + \frac{n}{a}\int_0^{\pi} e^{-ax}\cos(nx)dx \\ &= \left[\frac{-n\cos(nx)}{a^2}e^{-ax}\right]_0^{\pi} - \frac{n^2}{a^2}\int_0^{\pi} e^{-ax}\sin(nx)dx \end{aligned}$$

Then moving the integrals to the same side,

$$\begin{aligned} \left(1 + \frac{n^2}{a^2}\right)\int_0^{\pi} e^{-ax}\sin(nx)dx &= \left[\frac{-n\cos(nx)}{a^2}e^{-ax}\right]_0^{\pi} \\ &= \frac{-n\cos(n\pi)}{a^2}e^{-a\pi} + \frac{n}{a^2} \\ (n^2 + a^2)\int_0^{\pi} e^{-ax}\sin(nx)dx &= n(-1)^{n+1}e^{-a\pi} + n \end{aligned}$$

Thus,

$$\int_0^{\pi} e^{-ax}\sin(nx)dx = \frac{n(-1)^{n+1}e^{-a\pi} + n}{n^2 + a^2}$$

Which means that

$$q_n = \frac{2A}{\pi} \frac{n(-1)^{n+1}e^{-a\pi} + n}{n^2 + a^2}$$

Now, by (13) and (14), we get the following equations:

$$\begin{cases} b_0'(t) = q_0(t) \\ b_n'(t) + b_n(t)n^2 = q_n(t) \end{cases}$$

From $b_0'(t) = q_0(t)$ we get

$$b_0(t) = \int_0^t q_0(s)\ ds$$

Since $q_0 = \frac{-A}{a\pi}\left(e^{-a\pi} - 1\right)$, this means

$$b_0(t) = \frac{-At}{a\pi}\left(e^{-a\pi} - 1\right) + b_0(0)$$

On the other hand, we have $b_n'(t) + b_n(t)n^2 = q_n(t)$, which we solve as follows:

$$\mu(t) = \exp\left(\int_0^t n^2\ ds\right) = \exp\left(n^2 t\right)$$

$$b_n(t) = \frac{1}{\mu(t)}\left[\int_0^t \mu(s) q_n(s)\ ds + b_n(0)\right]$$

$$b_n(t) = b_n(0)\,\mu(t)^{-1} + \int_0^t \frac{\mu(s)}{\mu(t)} q_n(s)\ ds$$

$$b_n(t) = e^{-n^2} b_n(0) + \int_0^t \frac{\exp\left(n^2 s\right)}{\exp\left(n^2 t\right)} q_n(s)\ ds$$

$$b_n(t) = e^{-n^2} b_n(0) + \int_0^t e^{n^2(s-t)} q_n(s)\ ds$$

$$u(0, x) = \sin x = b_0(0) + \sum_{n=1}^{\infty} b_n(0) \sin\left(nx\right)$$

so using our equations to find the coefficients of a Fourier sine series,

$$b_0(0) = \frac{2}{\pi}\int_0^\pi \sin\left(nx\right) dx$$

$$= \frac{-2}{n\pi}\cos\left(nx\right)\Big|_0^\pi\ dx = \frac{2}{n\pi}\left(1 + (-1)^{n+1}\right)$$

$$b_n(0) = \frac{1}{\pi}\int_0^\pi \sin\left(x\right)\sin\left(nx\right) dx$$

We evaluate this integral using the following trigonometric identity:

$$\sin\alpha \sin\beta = \frac{\cos\left(\alpha - \beta\right) - \cos\left(\alpha + \beta\right)}{2} \quad\Longrightarrow\quad \sin\left(x\right)\sin\left(nx\right) = \frac{\cos\left(-x\right) - \cos\left(3x\right)}{2}$$

$$b_n(0) = \frac{1}{2\pi}\int_0^\pi \cos(x) - \cos(3x)dx = 0$$

Because $\sin 0, \sin\pi$, and $\sin 3\pi$ all equal zero.

Therefore, our solution to (11) is:

Solution

$$\begin{aligned}
u(t,x) &= b_0(t) + \sum_{n=1}^{\infty} b_n(t)\sin(nx), \quad \text{where} \\
b_n(t) &= \int_0^t e^{n^2(s-t)} q_n \, ds, \\
q_n &= \frac{2A}{\pi}\frac{n(-1)^{n+1}e^{-a\pi} + n}{n^2 + a^2}, \\
b_0(t) &= \frac{-At}{a\pi}\left(e^{-a\pi} - 1\right) + b_0(0), \\
b_0(0) &= \frac{2}{n\pi}(1 + (-1)^{n+1})
\end{aligned}$$

Problem 6: Telegraph Equation

Using the method of separation of variables, solve the telegraph equation:

$$\begin{cases} u_{tt} + au_t + bu = c^2 u_{xx} & 0 < x < l, \quad t > 0 \\ u(0,x) = \phi(x) & 0 \le x \le l \\ u_t(0,x) = \psi(x) & 0 \le x \le l \\ u(t,0) = u(t,l) = 0 & t > 0 \end{cases} \tag{15}$$

for constants $a, b > 0$. Only find the solution when the characteristic equation of the time problem has real roots. Define the following energy:

$$E(t) = \frac{1}{2}\int_0^l \left(u_t^2 + c^2 u_x^2 + bu^2\right) dx$$

Show that $E(t) \le E(0)$ for all $t > 0$. Then prove that the telegraph equation has a unique solution.

Suppose we have a separated solution of the form $u(t,x) = X(x)T(t)$.

If we plug this into the homogeneous Dirichlet BCs, we find $u(t,0) = T(t)X(0) = 0, u(t,l) = T(t)X(l) = 0$. In order for both of these to be true, we must have either $T(t) = 0$ or $X(0) = X(l) = 0$.

But if $T(t) = 0$, then $u(t,x) = 0$ for all x, which contradicts our ICs. Thus, we must have $X(0) = X(l) = 0$.

Plugging our expression $u(t,x) = X(x)T(t)$ into the PDE in (15), it becomes

$$T''(t)X(x) + aT'(t)X(x) + bT(t)X(x) = c^2 T(t)X''(x)$$

$$-\frac{T''(t) + aT'(t) + bT(t)}{c^2 T(t)} = -\frac{X''(x)}{X(x)} = \lambda$$

Based on our assumptions thus far and the IBVP (15), we have three problems:

Spatial problem

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(l) = 0 \end{cases}$$

Time problem

$$T'' + aT' + (b + \lambda c^2)T = 0 \tag{16}$$

IVP

$$u(0, x) = T(0)X(x) = \phi(x)$$

The spatial problem is an eigenvalue problem with homogenous Dirichlet BCs, which we already know has the eigenvalues and eigenvectors

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \sin\left(\frac{n\pi x}{l}\right), \quad n = 1, 2, 3...$$

Now moving to the time problem, the equation (16) becomes the characteristic equation $r^2+ar+(b+\lambda c^2) = 0$. From the Pythagorean Theorem, the roots of this equation are

$$\frac{-a \pm \sqrt{a^2 - 4(b + \lambda c^2)}}{2}$$

In order for the characteristic equation of the time problem to have only real roots, we must have $a^2 - 4(b + \lambda c^2) \geq 0$, or

$$\lambda \leq \frac{a^2 - 4b}{4c^2}$$

For each λ_n associated with the spatial problem, we get a solution to the time problem:

$$T_n(t) = C_n \exp\left[\frac{t}{2}(-a - a^2 - 4b - 4\lambda_n c^2)\right] + D_n \exp\left[\frac{t}{2}(-a + a^2 - 4b - 4\lambda_n c^2)\right]$$

where $C_n, D_n \in \mathbb{R}$

Therefore, the following are solutions of the PDE in (15): $u_n(t, x) = X_n(x)\, T_n(t), \quad n = 1, 2, 3...$

By the principle of superposition,

$$\begin{aligned} u(t, x) &= \sum_{n=1}^{\infty} c'_n\, u_n(t, x) \\ &= \sum_{n=1}^{\infty} c'_n\, T_n(t)\, X_n(x) \\ &= \sum_{n=1}^{\infty} A_n\, \exp\left[\frac{t}{2}(-a - a^2 - 4b - 4\lambda_n c^2)\right]\, \sin\left(\frac{n\pi x}{l}\right) + \\ &\sum_{n=1}^{\infty} B_n\, \exp\left[\frac{t}{2}(-a + a^2 - 4b - 4\lambda_n c^2)\right]\, \sin\left(\frac{n\pi x}{l}\right) \end{aligned}$$

where $A_n = c'_n C_n$ and $B_n = c'_n D_n$, with $A_n, B_n \in \mathbb{R}$.

We can get rid of these arbitrary constants by using our initial conditions, and . First, let us simplify the notation by defining $\gamma = \frac{t}{2}(-a - a^2 - 4b - 4\lambda_n c^2)$ and $\zeta = \frac{t}{2}(-a - a^2 - 4b - 4\lambda_n c^2)$.

Using our first initial condition, $u(0,x) = \phi(x)$:

$$u(t,x) = \sum_{n=1}^{\infty} A_n\, e^{\gamma t} \sin\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} B_n\, e^{\zeta t} \sin\left(\frac{n\pi x}{l}\right)$$

$$u(0,x) = \phi(x) = \sum_{n=1}^{\infty} (A_n + B_n) \sin\left(\frac{n\pi x}{l}\right)$$

We can use the equation for the coefficients inside a Fourier sine series to find $A_n + B_n$

$$A_n + B_n = \frac{2}{l}\int_0^l \phi(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

Now repeating for our second initial condition, $u_t(0,x) = \psi(x)$:

$$u_t(t,x) = \sum_{n=1}^{\infty} \gamma\, A_n\, e^{\gamma t}\ \sin\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} \zeta\, B_n\, e^{\zeta t}\ \sin\left(\frac{n\pi x}{l}\right)$$

$$u_t(0,x) = \psi(x) = \sum_{n=1}^{\infty} (\gamma A_n + \zeta B_n) \sin\left(\frac{n\pi x}{l}\right)$$

$$(\gamma A_n + \zeta B_n) = \frac{2}{l}\int_0^l \psi(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

Which can be rearranged as

$$A_n = \frac{1}{\gamma}\left[-\zeta B_n + \frac{2}{l}\int_0^l \psi(x) \sin\left(\frac{n\pi x}{l}\right) dx\right]$$

Now we can solve for B_n:

$$\frac{-\zeta B_n}{\gamma} + \frac{2}{\gamma l}\int_0^l \psi(x) \sin\left(\frac{n\pi x}{l}\right) dx = -B_n + \frac{2}{l}\int_0^l \phi(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$B_n - \frac{\zeta B_n}{\gamma} = \left(1 - \frac{\zeta}{\gamma}\right) B_n = \frac{2}{l}\int_0^l \phi(x) \sin\left(\frac{n\pi x}{l}\right) dx - \frac{2}{\gamma l}\int_0^l \psi(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

Therefore, our solution is:

Solution

$$u(t,x) = \sum_{n=1}^{\infty} A_n\, e^{\gamma t} \sin\Big(\frac{n\pi x}{l}\Big) + \sum_{n=1}^{\infty} B_n\, e^{\zeta t} \sin\Big(\frac{n\pi x}{l}\Big), \tag{17}$$

$$\text{where} \quad A_n = \frac{2}{l}\int_0^l \phi(x) \sin\Big(\frac{n\pi x}{l}\Big) dx - B_n$$

$$B_n = \frac{2\gamma}{(\gamma-\zeta)l}\int_0^l \phi(x) \sin\Big(\frac{n\pi x}{l}\Big) dx - \frac{2(\gamma-\zeta)}{l}\int_0^l \psi(x) \sin\Big(\frac{n\pi x}{l}\Big) dx,$$

$$\gamma = \frac{t}{2}(-a - a^2 - 4b - 4\lambda_n c^2)$$

$$\zeta = \frac{t}{2}(-a - a^2 - 4b - 4\lambda_n c^2)$$

$$\lambda_n = \Big(\frac{n\pi}{l}\Big)^2, \quad n = 1, 2, 3...$$

Now we use an energy argument to show that the telegraph equation has a unique solution. We start with the given energy equation and differentiate:

$$\begin{aligned}
\frac{d}{dt}E(t) &= \frac{1}{2}\int_0^l \frac{d}{dt}(u_t^2 + c^2 u_x^2 + bu^2)\; dx \\
&= \frac{1}{2}\int_0^l 2u_t u_{tt} + 2c^2 u_x u_{xt} + 2buu_t)\; dx \\
&= \int_0^l u_t u_{tt} + \int_0^l c^2 u_x u_{xt} + \int_0^l buu_t \\
&= \int_0^l c^2 u_x u_{xt}\; dx = c^2 u_x u_t\Big|_0^l - \int_0^l c^2 u_t u_{xx}\; dx
\end{aligned}$$

The term $c^2 u_x u_t\Big|_0^l$ disappears because of our boundary conditions, so we now have

$$\frac{d}{dt}E(t) = \int_0^l u_t u_{tt} + buu_t - c^2 u_t u_{xx}\; dx = \int_0^l u_t(u_{tt} + bu - c^2 u_{xx})\; dx$$

But our PDE, $u_{tt} + au_t + bu = c^2 u_{xx}$, can be re-written as $u_{tt} + bu - c^2 u_{xx} = -au_t$, which means

$$\frac{d}{dt}E(t) = -\int_0^l a(u_t^2)\; dx$$

Since a and u_t^2 are necessarily non-negative, $\int_0^l a(u_t^2)\; dx \geq 0$ which means that

$$\frac{d}{dt}E(t) = -\int_0^l a(u_t^2)\; dx \leq 0$$

.

Since we just showed that the derivative of $E(t)$ is less than or equal to zero for all t, we have that $E(t) \leq E(0)$ for all $t > 0$.

Let u and v be two solutions to (15) and define $w = u - v$. Then w satisfies the problem

$$\begin{cases} w_{tt} + aw_t + bw = c^2 w_{xx} & 0 < x < l, \quad t > 0 \\ w(0,x) = 0 & 0 \le x \le l \\ w_t(0,x) = 0 & 0 \le x \le l \\ w(t,0) = w(t,l) = 0 & t > 0 \end{cases} \tag{18}$$

$$E(0) = \frac{1}{2}\int_0^l (w_t(0,x))^2 + c^2(w_x(0,x))^2 + b(w(0,x)^2)\ dx = \frac{1}{2}\int_0^l (0+0+0)\ dx = 0$$

So $E(t) \le E(0)$ for all $t > 0$ but $E(0) = 0$ for our solution $w(t,x)$, which means that

$$E(t) = \frac{1}{2}\int_0^l (w_t^2 + c^2 w_x^2 + bw^2)\ dx \le 0$$

Since all of the terms under the integrand are non-negative, this is only possible if $w_t = w_x = w = 0$ for all $t > 0$. Thus, $w = u - v = 0 \quad \Rightarrow \quad u = v$, meaning any solution to (15) is in fact the unique solution.