MATH 245 Homework 2

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Question 1

Determine the region in which the given equation is hyperbolic, parabolic, elliptic, or singular.

a)

$$u_{xx} + y^2 u_{yy} + u_x - u + x^2 = 0$$

$a = 1, b = 0, c = -y^2$, so we have $b^2 - ac = 0 - (-y^2) = y^2$. This will be positive everywhere except for $y = 0$, so the equation is hyperbolic where $y \neq 0$ and parabolic for $y = 0$.

b)

$$u_{xx} - y u_{yy} + x u_x + y u_y + u = 0$$

$a = 1, b = 0, c = -y$, so we have $b^2 - ac = 0 - (-y) = y$. Thus, the equation will be hyperbolic where $y > 0$, parabolic where $y = 0$, and elliptic where $y < 0$.

Question 2

Using a factorization similar to the wave equation, solve the following IVP:

$$\begin{cases} u_{xx} + 2u_{xy} - 3u_{yy} = 0 & x \subset \mathbb{R},\ y > 0 \\ u(0,x) = \sin x & x \subset \mathbb{R} \\ u_y(0,x) = x & x \subset \mathbb{R} \end{cases} \tag{1}$$

First, we can factor the equation as follows:

$$\left(\frac{\partial}{\partial x} + 3\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) u = 0$$

or

$$\left(\partial_x + 3\partial_y\right)\left(\partial_x - \partial_y\right) u = 0$$

Then set $\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) u = v$, giving us

$\left(\frac{\partial}{\partial x} + 3\frac{\partial}{\partial y}\right) v = v_x + 3v_t = 0$

which we know has the solution $v(x,y) = f(3x - y)$, so

$$u_x - u_y = f(3x - y)$$

On (characteristic) lines with the slope $y=-x+c$, or $y+x=$ constant, we must have $u_x-u_y=f(3x-y)=0$. Set $\eta=x+y$ and $\xi=x$. Then by the chain rule,

$$u_x=u_\eta+u_\xi,\ u_y=u_\eta$$

And let's rewrite y as $y=\eta-x=\eta-\xi$.

So

$$u_x-u_y=f(3x-y)\quad\longrightarrow\quad u_\xi=f(3\xi-\eta+\xi)$$

$$u_\xi=f(4\xi-\eta)$$

Now integrate with respect to ξ:

$$u(\eta,\xi)=F(4\xi-\eta)+G(\eta)$$

where F is the antiderivative of f with respect to ξ.

Now convert back to our original variables:

$$u(x,y)=F(3x-y)+G(x+y)$$

Using the fact that $u(0,x)=\sin x$,

$$u(0,x)=\sin x=F(3x)+G(x) \tag{2}$$

now replace x with a new neutral variable, α and differentiate:

$$\sin\alpha=F(3\alpha)+G(\alpha)$$

$$\cos\alpha=3F'(3\alpha)+G'(\alpha) \tag{3}$$

But we can also differentiate $u(x,y)=F(3x-y)+G(x+y)$ with respect to y to get

$$u_y(x,y)=-F'(3x-y)+G'(x+y)$$

but from our initial conditions, we know

$$u_y(0,x)=-F'(3x-0)+G'(x+0)=x$$

Let's replace x by our neutral variable α and solve for F':

$$F'(\alpha)=G'(3\alpha)-\alpha$$

Now plug this into 3:

$$\cos\alpha=3G'(\alpha)-3\alpha+G'(\alpha)$$

$$G(\alpha)=\frac{1}{4}\int\cos\alpha+3\alpha=\frac{\sin\alpha}{4}+\frac{3\alpha^2}{8}$$

So that means 2 becomes:

$$\sin\alpha=F(3\alpha)+\frac{\sin\alpha}{4}+\frac{3\alpha^2}{8}$$

$$F(\alpha)=\frac{3\sin\left(\frac{\alpha}{3}\right)}{4}-\frac{\alpha^2}{24}$$

Which means $u(x,y)=F(3x-y)+G(x+y)$ becomes

$$u(x,y) = \frac{3}{4}\sin\left(x - \frac{y}{3}\right) - \frac{(3x-y)^2}{24} + \frac{\sin(x+y)}{4} + \frac{3(x+y)^2}{8} =$$

Solution

$$u(x,y) = \frac{3}{4}\sin\left(x - \frac{y}{3}\right) + \frac{\sin(x+y)}{4} + xy + \frac{y^2}{3}$$

Check solution

$$u_y = \frac{-1}{4}\cos\left(x - \frac{y}{3}\right) + \frac{\cos(x+y)}{4} + x + \frac{2y}{3}$$

$$u_{yy} = \frac{-1}{12}\sin\left(x - \frac{y}{3}\right) - \frac{\sin(x+y)}{4} + \frac{2}{3}$$

$$u_x = \frac{3}{4}\cos\left(x - \frac{y}{3}\right) + \frac{\cos(x+y)}{4} + y$$

$$u_{xx} = \frac{-3}{4}\sin\left(x - \frac{y}{3}\right) - \frac{\sin(x+y)}{4}$$

$$u_{xy} = \frac{1}{4}\sin\left(x - \frac{y}{3}\right) - \frac{\sin(x+y)}{4} + 1$$

Check that $u_{xx} + 2u_{xy} - 3u_{yy} = 0$

$$\left(\frac{-3}{4} + \frac{2}{4} + \frac{1}{4}\right)\sin\left(x - \frac{y}{3}\right) + \left(\frac{-1}{4} + \frac{-2}{4} + \frac{3}{4}\right)\sin(x+y) + (0+2-3) = 0$$

Question 3

Solve the Neumann boundary value problem for the wave equation on half line:

$$\begin{cases} u_{tt} = c^2 u_{xx} + f(t,x) & 0 < x < \infty \\ u(0,x) = \phi x & 0 < x < \infty \\ u_t(0,x) = \psi x & 0 < x < \infty \\ u_x(t,0) = h(t) & t > 0 \end{cases} \tag{4}$$

Question 4

Consider the 3D wave equation for $u(t,x,y,z)$:

$$u_{tt} = c^2 \Delta u \qquad (x,y,z) \in \mathbb{R}^3, \quad t > 0$$

Change the coordinates to spherical coordinates. Assume the solution is spherically symmetric, so that $u(t,x,y,z) = u(t,r)$ and does not depend on θ and ϕ. Find the solution for $u(0,r) = 0$ and

$$u_t(0,r) = \begin{cases} 1 & |r| \le 1 \\ 0 & |r| > 1 \end{cases} \tag{5}$$

Hint: use the substitution $u(t,r) = \frac{1}{r}w(t,r)$.

First, we need to derive the formula for the Laplacian in spherical coordinates.

We know the equation for the Laplacian in polar coordinates is:

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Now let's convert to spherical coordinates:

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{s^2 + z^2}$$

$$\begin{aligned} x &= s\cos\phi \\ y &= s\sin\phi \\ z &= r\cos\theta \\ s &= r\sin\theta \end{aligned}$$

By the two-dimensional Laplacian, we have

$$u_{zz} + u_{ss} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$

and

$$u_{xx} + u_{yy} = u_{ss} + \frac{1}{s}u_s + \frac{1}{s^2}u_{\phi\phi}$$

We add these two equations and cancel u_s to get

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \frac{1}{s}u_s + \frac{1}{s^2}u_{\phi\phi}$$

Now since u doesn't depend on θ or ϕ, we have

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{s}u_s = u_{rr} + \frac{1}{r}u_r + \frac{1}{r\sin\theta}u_s$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial s} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial s} + \frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial s} = u_r\frac{1}{\sin\theta} + 0 + 0 = u_r\frac{s}{r}$$

So with our change of variables, we have

$$u_{tt} = c^2\left(u_{rr} + \frac{2}{r}u_r\right)$$

Now set $w = ru$, or $u = \frac{w}{r}$. Then

$$w_t = ru_t, \quad w_{tt} = ru_{tt}, \quad u_{tt} = \frac{w_{tt}}{r}$$

$$w_t = ru_t, \quad w_{tt} = ru_{tt}, \quad u_{tt} = \frac{w_{tt}}{r}$$

$$u_r = \frac{w_r}{r} - \frac{w}{r^2}$$

$$u_{rr} = \frac{w_{rr}}{r} - \frac{2w_r}{r^2} + \frac{2w}{r^3}$$

So

$$u_{tt} = c^2 \left(u_{rr} + \frac{2}{r} u_r \right)$$

becomes

$$\frac{w_{tt}}{r} = c^2 \left(\frac{w_{rr}}{r} - \frac{2w_r}{r^2} + \frac{2w}{r^3} + \frac{2}{r} \left(\frac{w_r}{r} - \frac{w}{r^2} \right) \right)$$

which simplifies to

$$w_{tt} = c^2 w_{rr}$$

but this is just the wave equation, which we know has the solution

$$w(t,r) = \frac{\varphi(r+ct) + \varphi(r-ct)}{2} + \frac{1}{2c} \int_{r-ct}^{r+ct} \psi(s) ds$$

Since $\varphi = 0$,

$$w(t,r) = \frac{1}{2c} \int_{r-ct}^{r+ct} \psi(s) ds$$

Now we have 4 cases:

Case 1: $r - ct \geq -1, r + ct \leq 1$

$$w(t,r) = \frac{1}{2c} \int_{r-ct}^{r+ct} s \; ds$$

Case 2: $r - ct < -1, r + ct > 1$

$$w(t,r) = \frac{1}{2c} \int_{-1}^{1} s \; ds$$

Case 3: $r - ct < -1, r + ct \leq 1$

$$w(t,r) = \frac{1}{2c} \int_{-1}^{r+ct} s \; ds$$

Case 4: $r - ct \geq -1, r + ct > 1$

$$w(t,r) = \frac{1}{2c} \int_{r-ct}^{1} s \; ds$$

Since $u = \frac{w}{r}$, this means we have

$$u(t,r) = \begin{cases} \frac{1}{2c} \int_{r-ct}^{r+ct} s \; ds & r - ct \geq -1, r + ct \leq 1 \\ \\ \frac{1}{2c} \int_{-1}^{1} s \; ds & r - ct < -1, r + ct > 1 \\ \\ \frac{1}{2c} \int_{-1}^{r+ct} s \; ds & r - ct < -1, r + ct \leq 1 \\ \\ \frac{1}{2c} \int_{r-ct}^{1} s \; ds & r - ct \geq -1, r + ct > 1 \end{cases} \tag{6}$$

Question 5

Consider the following Dirichlet boundary value problem:

$$\begin{cases} u_{tt} + x(t,x)u_t = u_{xx} & 0 < x < 1 \\ u(0,x) = \phi(x) & 0 < x < 1 \\ u_t(0,x) = \psi(x) & 0 < x < 1 \\ u(t,0) = u(t,1) = 0 & t \geq 0 \end{cases} \tag{7}$$

Assume that $|a(t,x)| \leq m$ for some constant m and all $0 < x < 1$ and $t \geq 0$. Let

$$E(t) = \frac{1}{2}\int_0^1 \left(u_t(t,x)^2 + u_x(t,x)^2\right) dx$$

(1) Show that $\frac{dE(t)}{dt} \leq 2mE(t)$ for $t \geq 0$.
(2) Use part (a) and show that $\frac{d}{dt}\left(e^{-2mE(t)}\right) \leq 0$ for all $t \geq 0$. Hence, by integration from $[0,t]$, we get that

$$E(t) \leq e^{2mt}E(0) \quad \text{for all } t \geq 0$$

(3) If $\phi(x) = \psi(x) = 0$ for all $0 < x < 1$, what does this say about $E(t)$ for $t \geq 0$ and hence about $u(t,x)$ for $t \geq 0$?

(4) Use the previous part to prove uniqueness of the following problem:

$$\begin{cases} u_{tt} + a(t,x)u_t = u_{xx} & 0 < x < 1, t > 0 \\ u(0,x) = \phi(x) & 0 < x < 1 \\ u_t(0,x) = \psi(x) & 0 < x < 1 \\ u(t,0) = f(t) & t \geq 0 \\ u(t,1) = g(t) & t \geq 0 \end{cases} \tag{8}$$

Problem 6

Does the D'Alembert method work for the wave equation $u_{tt} = c(x)^2u_{xx}$? What about $u_{tt} = c(t)^2u_{xx}$? Why?

Problem 7 (The Poisson-Darboux Equation)

Solve the initial value problem

$$\begin{cases} u_{tt} - u_{xx} - \frac{2}{x}u_x = 0 & -\infty < x < \infty, t > 0 \\ u(0,x) = 0 & -\infty < x < \infty \\ u_t(0,x) = g(x) & -\infty < x < \infty \end{cases} \tag{9}$$

where $g(x) = -g(x)$ is an even function. Hint: set $w = xu$.

Problem 8

Solve the following characteristic initial value problem:

$$\begin{cases} y^3 u_{xx} - y u_{yy} + u_y = 0 & 0 < x < 4, \quad |y| \leq 2\sqrt{2} \\ u(x,y) = f(x) & x + \frac{y^2}{2} = 4 \text{for} 2 \leq x \leq 4 \\ u(x,y) = g(x) & x - \frac{y^2}{2} = 0 \text{for} 0 \leq x \leq 2 \end{cases} \tag{10}$$

where $f(2) = g(2)$. Hint: Use the transformation $\eta = x - \frac{y^2}{2}$ and $\xi = x + \frac{y^2}{2}$ and express the PDE in the coordinates (ξ, η).