MATH 245 Homework 3

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Question 1 (Parity of Solution)

Prove the following.

(a) If initial conditions of the wave equation on the whole line are even(odd), the solution is even(odd).

Consider the IVP

$$\begin{cases} u_{tt} = c^2 u_{xx} & -\infty < x < \infty \\ u(0,x) = \varphi(x) & -\infty < x < \infty \\ u_t(0,x) = \psi(x) & -\infty < x < \infty \end{cases}$$

where $\varphi(x)$ and $\psi(x)$ are even functions, i.e., $\varphi(-x) = \varphi(x)$ and $\psi(-x) = \psi(x)$. Then using D'Alembert's formula, we know the solution u is

$$u(x,t) = \frac{\varphi(x+ct) + \varphi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds$$

Now plug in $-x$:

$$\begin{aligned} u(-x,t) &= \frac{\varphi(-x+ct) + \varphi(-x-ct)}{2} + \frac{1}{2c}\int_{-x-ct}^{-x+ct} \psi(s)ds \\ &= \frac{\varphi(x-ct) + \varphi(x+ct)}{2} + \frac{1}{2c}\int_{-x-ct}^{-x+ct} \psi(-s)ds \end{aligned}$$

After setting $y = -s, dy = -ds$ in the integral of ψ:

$$u(-x,t) = \frac{\varphi(x-ct) + \varphi(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy = u(x,t).$$

Therefore, even initial conditions imply that the solution of the wave equation is even.

Similarly, assume $\varphi(x)$ and $\psi(x)$ are odd functions, i.e., $\varphi(-x) = -\varphi(x)$ and $\psi(-x) = -\psi(x)$. Then using D'Alembert's formula, we know $u(-x,t)$ is

$$\begin{aligned} u(-x,t) &= \frac{\varphi(-x+ct) + \varphi(-x-ct)}{2} + \frac{1}{2c}\int_{-x-ct}^{-x+ct} \psi(s)ds \\ &= \frac{-\varphi(x-ct) - \varphi(x+ct)}{2} - \frac{1}{2c}\int_{-x-ct}^{-x+ct} \psi(-s)ds \end{aligned}$$

After setting $y = -s, dy = -ds$ in the integral of ψ:

$$u(-x,t) = \frac{-\varphi(x-ct) - \varphi(x+ct)}{2} - \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(-y)dy = -u(x,t).$$

Therefore, odd initial conditions imply that the solution of the wave equation is odd.

(b) If the initial condition of the heat equation on the whole line is even(odd), the solution is even(odd).

Question 2 (Speed of Heat vs Wave)

Consider the traveling wave $u(x,t) = f(x - at)$ where f is a given function of one variable.

(a) If it is a solution of the wave equation, show that the speed must be $a = \pm c$ (unless f is a linear function).

(b) If it is a solution of the diffusion equation, find f and show that the speed a is arbitrary.

Question 4 (Maximum Principle)

Consider two solutions $u(t,x)$ and $v(t,x)$ of the diffusion equation in $\Omega_T = \{0 \leq x \leq l, 0 \leq t \leq \infty\}$

(a) Let $M(T) =$ the maximum of $u(t,x)$ in the closed rectangle $\Omega_T = \{0 \leq x \leq l, 0 \leq t \leq \infty\}$. Does $M(T)$ increase or decrease as a function of T? Explain.

(b) Let $m(T) =$ the minimum of $u(t,x)$ in the closed rectangle $\Omega_T = \{0 \leq x \leq l, 0 \leq t \leq \infty\}$. Does $m(T)$ increase or decrease as a function of T? Explain.

(c) **Comparison Principle** If $u \leq v$ for $t = 0$, for \$x=0, \$ for $x = l$, then $u \leq v$ for $0 \leq t \leq \infty$ and $0 \leq x \leq l$.

Question 5 (Diffusion Equation with Dissipation).

Solve the following IVP for constant dissipation $b > 0$.

$$\begin{cases} u_t - ku_{xx} + bu = 0 & -\infty \leq x \leq \infty,\ t > 0 \\ u(0,x) = \psi(x), & -\infty \leq x \leq \infty \end{cases} \tag{1}$$

We will use the change of variables $u(t,x) = e^{-bt}v(t,x)$

Question 6

Prove that there is no maximum principle for the wave equation.