BIOL 274 Homework 8

Ruby Krasnow

2024-04-05

Visualizing Solutions of Linear Systems and Runge-Kutta

Question 1

Consider the epidemic model presented in class.

a.

For the parameters given, we found one single equilibrium point. Determine whether this equilibrium point is stable or unstable and sketch the phase plane near the equilibrium point.

Let X be the number of susceptible individuals at time t and Y be the number of infected individuals (who can transmit the disease). The epidemic model presented in class takes the following form:

$$\frac{dX}{dt} = -\mu XY - \rho$$

$$\frac{dY}{dt} = \mu XY - \nu Y$$

where $\mu = 0.025,\ \rho = 5$, and $\nu = 0.5$. We determined that given these parameters, the model has a single equilibrium point at $\left(\frac{\nu}{\mu}, \frac{\rho}{\nu}\right) = (20, 10)$.

We will determine the local stability of this equilibrium point by first computing the Jacobian matrix at $(20, 10)$.

Given a system of the type $\dot{x} = f(x, y), \quad \dot{y} = g(x, y)$, the Jacobian is defined as the following matrix:

$$J(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

For our system, the Jacobian is

$$J(X, Y) = \begin{bmatrix} -\mu Y & -\mu X \\ \mu Y & -\mu X - \nu \end{bmatrix}$$

We then evaluate the Jacobian at our equilibrium point:

$$J(20, 10) = \begin{bmatrix} -0.025(10) & -0.025(20) \\ 0.025(10) & 0.025(20) - 0.5 \end{bmatrix} = \begin{bmatrix} -0.25 & -0.5 \\ 0.25 & 0 \end{bmatrix}$$

Our next step is to find the eigenvalues of this matrix.

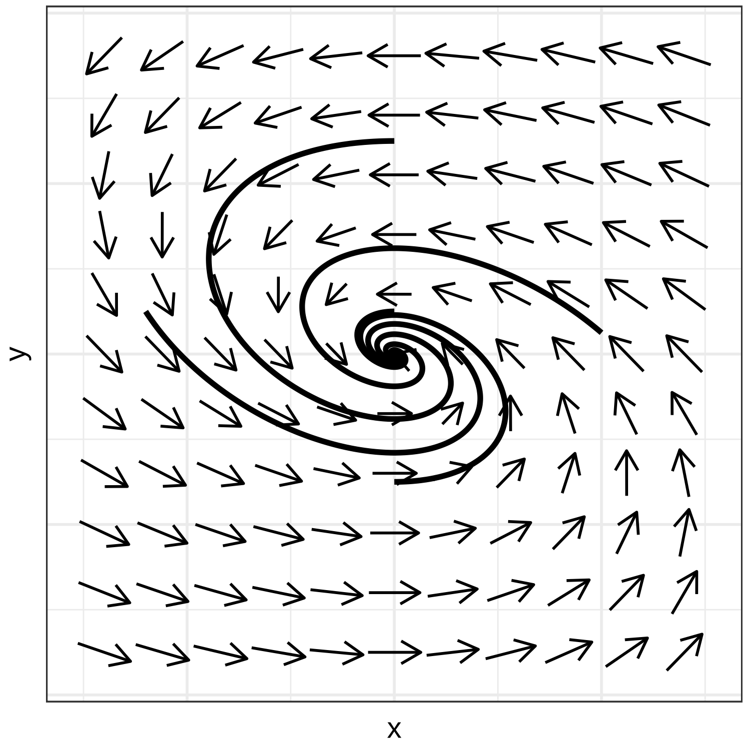
$$\det \begin{bmatrix} -0.25 - \lambda & -0.5 \\ 0.25 & 0 - \lambda \end{bmatrix} = 0$$

$$(-0.25-\lambda)(-\lambda)+0.125=0$$

$$\lambda^2+0.25\lambda+0.125=0$$

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a} \quad\longrightarrow\quad \frac{-0.25\pm\sqrt{0.0625-4(0.125)}}{2}=\frac{-0.25\pm\sqrt{-4.375}}{2}\approx -0.125\pm 0.331\,i$$

We found complex eigenvalues with a negative real part, which means that we have a (stable) spiral sink. Near our equilibrium point, the phase portrait looks like:



b.

Modify the model given in class, assuming that the Susceptible population exhibits exponential growth with an intrinsic growth rate of 0.10 in the absence of any infected individuals.

c.

Find all equilibria of your new Epidemic model from part (b) and determine their stability by sketching the phase plane near each equilibrium.

Question 2

Consider the IVP from class:

$$\frac{dy}{dt}=-2ty^2, \quad y(0)=1$$

over the interval $0\le t\le 2$.

a.

Calculate the Runge-Kutta approximation to the solution with $n = 4$ steps.

b.

Calculate the total error e_4 associated with this approximation.

c.

How many steps are necessary to approximate the solution with an error of less than 0.0001? Make sure to justify your answer.