BIOL 274 Homework 1

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Question 1

> Assume that once a drug is administered to a patient, it leaves the body following exponential decay. After administration of a dose, the concentration of the drug in the body decreases by 50% in 30 hours. Approximately how long does it take for the drug to decrease to 1% of its initial value (round to the nearest hour)?

We can represent the concentration of drug in the body at time t as $x(t) = x_0 e^{kt}$. Since the drug leaves the body according to an exponential decay model, that means k should be negative.

We are given $\frac{x}{x_0} = 0.5$ at $t = 30$ hours. Plugging into our equation, we get

$$0.5 = e^{30k}$$

$$\ln(0.5) = \ln(e^{30k})$$

$$\ln(0.5) = 30k$$

$$k = \frac{\ln(0.5)}{30}$$

```
k = log(0.5)/30
k
```

```
## [1] -0.02310491
```

Now, we want to know how long it will take for x to be 1% of x_0, or at what time t we will have $\frac{x(t)}{x(0)} = 0.01$.

$$0.01 = e^{kt}$$

$$t = \frac{\ln(0.01)}{k}$$

```
t = log(0.01)/k
round(t, 0)
```

```
## [1] 199
```

It will take approximately 199 hours for the drug to decrease to 1% of its initial value.

Question 2

Consider the population model: $\frac{dP}{dt} = 0.3\left(1 - \frac{P}{200}\right)\left(\frac{P}{50} - 1\right)P$.

a. For what values of P is the population increasing?
b. For what values of P is the population decreasing?

If the population is increasing, that means the rate of change $\frac{dP}{dt}$ is positive, while a decreasing population means that $\frac{dP}{dt}$ is negative. First we need to find the equilibrium points, or the values of P for which $\frac{dP}{dt} = 0.3\left(1 - \frac{P}{200}\right)\left(\frac{P}{50} - 1\right)P = 0$.

We can see that this will clearly occur if $P = 0$; i.e., the population has no organisms in it. The other equilibrium points occur when

$$1 - \frac{P}{200} = 0$$
$$1 = \frac{P}{200}$$
$$P = 200$$

and similarly when $1 = \frac{P}{50}$ or $P = 50$. So we need to determine the sign of $\frac{dP}{dt}$ on the intervals $(0, 50)$, $(50, 200)$, and $(200, \infty)$. Let's evaluate some convenient values within these intervals: 25, 100, and 400.

```
eqn <- function(P){
  0.3*(1-(P/200))*((P/50-1))*P
}
eqn(25)
```

```
## [1] -3.28125
```

```
eqn(100)
```

```
## [1] 15
```

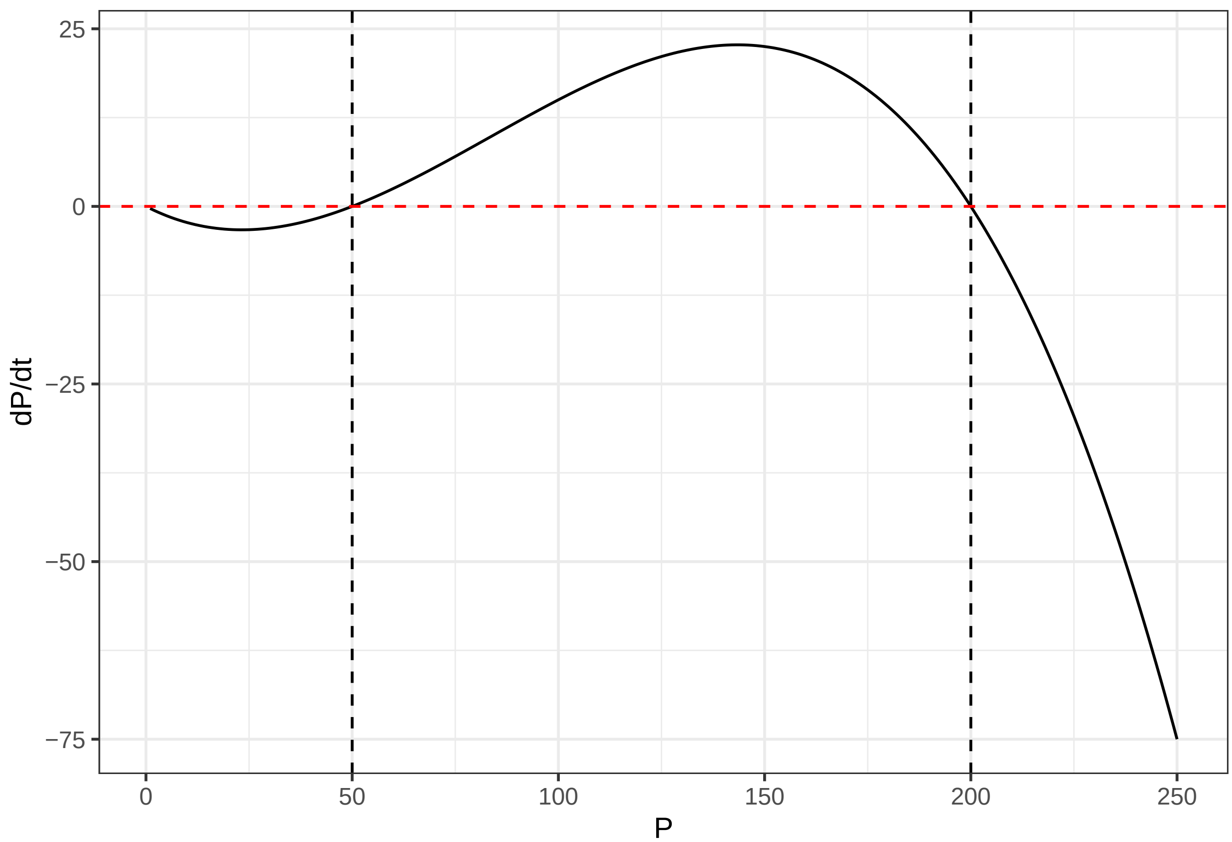
```
eqn(400)
```

```
## [1] -840
```

So the population is decreasing when $0 < P < 50$ or $P > 200$ and the population is increasing when $50 < P < 200$. Let's confirm this with a quick plot:

```
x=c(1:250)

ggplot()+
  geom_line(aes(x=x, y=0.3*(1-(x/200))*((x/50-1))*x))+
  geom_vline(xintercept = c(50, 200), linetype="dashed")+
  geom_hline(yintercept = 0, linetype="dashed", color="red")+
  theme_bw()+
  labs(x="P", y="dP/dt")
```



We can see that $\frac{dP}{dt}$ is indeed positive and negative over the expected intervals.

Question 3

Find the general solution to each of the following DEs:

a)

$$\frac{dy}{dt} = \frac{3t+1}{2y}$$
$$2y\,dy = (3t+1)dt$$
$$\int 2y\,dy = \int (3t+1)dt$$
$$y^2 = \frac{3t^2}{2} + t + C$$
$$y = \sqrt{\frac{3t^2}{2} + t + C}$$

b)

$$\frac{dy}{dx} = x\sqrt[3]{y}$$

$$y^{-\frac{1}{3}}\,dy = x\,dx$$
$$\int y^{-\frac{1}{3}}\,dy = \int x\,dx$$
$$\frac{3}{2}y^{\frac{2}{3}} = \frac{x^2}{2} + C_1$$
$$y^{\frac{2}{3}} = \frac{x^2 + C_2}{3}$$
$$y(x) = \left(\frac{x^2 + C_2}{3}\right)^{\frac{3}{2}} = \frac{(x^2 + C_2)^{\frac{3}{2}}}{3\sqrt{3}}$$

Note that $y(x) = 0$ is also a solution to the ODE, because $\frac{dy}{dx}$ will equal 0 for all x and $x\sqrt[3]{0} = 0$.

c)

$$\frac{dp}{dt} = \frac{1}{3tp^2}$$
$$3p^2\,dp = \frac{1}{t}dt$$
$$\int 3p^2\,dp = \int \frac{1}{t}dt$$
$$p^3 = \ln(t) + C$$
$$p = \sqrt[3]{\ln(t) + C}$$

Question 4

According to Newton's law of cooling, the rate of decrease of temperature of a body is proportional to the difference between its temperature and that of its environment. If the temperature in your living room is 20°C, you remove a log from the fireplace at 100°C, and it takes 10 minutes to cool to 60°C, how long will it take to decrease the temperature of the log to 25°C?

$$\frac{dT}{dt} = k(T_{body} - T_{env}) = k(T_{body} - 20)$$
$$\frac{1}{(T_{body} - 20)}dT = k\,dt$$
$$\int \frac{1}{(T_{body} - 20)}dT = \int k\,dt$$
$$\ln\left(T_{body} - 20\right) = kt + C_1$$
$$T_{body} = e^{kt + C_1} + 20 = C_2 e^{kt} + 20$$

where $C_2 = e^{C_1}$.

Since $T_{body} = C_2 + 20$ at time $t = 0$, we can write the equation as

$$T_{body}(t) = (T_0 - 20)e^{kt} + 20$$

for the temperature of the body at time t. Now using the information we are given,

$$60 = (100 - 20)e^{10k} + 20$$

$$
\begin{aligned}
40 &= 80e^{10k} \\
\frac{1}{2} &= e^{10k} \\
\frac{\ln(1/2)}{10} &= k
\end{aligned}
$$

```
k2 = log(1/2)/(10)
k2
```

```
## [1] -0.06931472
```

Now we want to solve for t in the following equation:

$$25 = (60 - 20)e^{kt} + 20$$

$$
\begin{aligned}
5 &= 40e^{kt} \\
\frac{\ln(1/8)}{k} &= t
\end{aligned}
$$

```
t2 = (log(1/8))/(k2)
t2
```

```
## [1] 30
```

Quick sanity check: If it took 10 minutes to cool from 100°C to 60°C and 30 minutes to cool from 60°C to 25°C, it should take 40 minutes to cool from 100°C to 25°C:

```
# 25=(100-20)e^kt+20
# 5=80e^kt
# ln(1/16)=kt

t3=(log(1/16))/(k2)
t3
```

```
## [1] 40
```

Question 5

Solve each of the following IVPs:

a)

$$\frac{dx}{dt} = -xt, \; x(0) = 1/\sqrt{\pi}$$

$$
\begin{aligned}
\frac{1}{x} dx &= -t \, dt \\
\int \frac{1}{x} dx &= \int -t \, dt \\
\ln(x) &= -\frac{t^2}{2} + C_1
\end{aligned}
$$

$$x(t) = C_2 e^{-t^2/2}$$

is our general solution to the differential equation, where $C_2 = e^{C_1}$. Now we plug in the initial conditions:

$$1/\sqrt{\pi} = C_2 e^{-(0^2/2)} = C_2 e^0 = C_2$$

$$x(t) = \frac{e^{-t^2/2}}{\sqrt{\pi}}$$

is the solution to the IVP.

b)

$$\frac{dy}{dx} = xy^2 + 2y^2,\ y(0) = 1$$

$$\frac{dy}{dx} = y^2(x+2)$$
$$y^{-2}dy = (x+2)dx$$

$$-\frac{1}{y} = \frac{x^2}{2} + 2x + C$$

We can plug in our initial conditions now:

$$-\frac{1}{1} = 0 + 0 + C$$

so $C = -1$.

$$-\frac{1}{y} = \frac{x^2}{2} + 2x - 1$$
$$\frac{1}{y} = 1 - \frac{x^2}{2} - 2x$$
$$\frac{1}{y} = 1 - \frac{x^2}{2} - 2x$$

or

$$\frac{2}{y} = 2 - x^2 - 4x$$

$$y(x) = \frac{2}{2 - x^2 - 4x}$$

is the solution to the IVP.

c)

$$\frac{du}{dt} = \frac{\cos(t)}{9u^2},\ u(0) = 2$$

$$u^2 du = \frac{\cos(t)}{9} dt$$

$$\int u^2 du = \frac{1}{9} \int \cos(t) dt$$

$$\frac{u^3}{3} = \frac{1}{9} \sin(t) + C_1$$

$$u^3 = \frac{\sin(t)}{3} + C_2$$

Where $C_2 = 3C_1$. Now we plug in the initial conditions:

$$2^3 = \frac{\sin(0)}{3} + C_2 = 0 + C_2$$

So $C_2 = 8$ and we have

$$u(t) = \sqrt[3]{\frac{\sin(t)}{3} + 8}$$

as the solution to the IVP.