MATH 245 Homework 2

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Question 1

Determine the region in which the given equation is hyperbolic, parabolic, elliptic, or singular.

a)

$$u_{xx} + y^2 u_{yy} + u_x - u + x^2 = 0$$

$a = 1, b = 0, c = -y^2$, so we have $b^2 - ac = 0 - (-y^2) = y^2$. This will be positive everywhere except for $y = 0$, so the equation is hyperbolic where $y \neq 0$ and parabolic for $y = 0$.

b)

$$u_{xx} - y u_{yy} + x u_x + y u_y + u = 0$$

$a = 1, b = 0, c = -y$, so we have $b^2 - ac = 0 - (-y) = y$. Thus, the equation will be hyperbolic where $y > 0$, parabolic where $y = 0$, and elliptic where $y < 0$.

Question 2

Using a factorization similar to the wave equation, solve the following IVP:

$$\begin{cases} u_{xx} + 2u_{xy} - 3u_{yy} = 0 & x \subset \mathbb{R},\ y > 0 \\ u(0, x) = \sin x & x \subset \mathbb{R} \\ u_y(0, x) = x & x \subset \mathbb{R} \end{cases}$$

First, we can factor the equation as follows:

$$\left(\frac{\partial}{\partial x} + 3\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) u = 0$$

or

$$\left(\partial_x + 3\partial_y\right)\left(\partial_x - \partial_y\right) u = 0$$

Then set $\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) u = v$, giving us

$\left(\frac{\partial}{\partial x} + 3\frac{\partial}{\partial y}\right) v = v_x + 3v_t = 0$

which we know has the solution $v(x, y) = f(3x - y)$, so

$$u_x - u_y = f(3x - y)$$

Now we can reorder our equation as

$$\left(\partial_x - \partial_y\right)\left(\partial_x + 3\partial_y\right) u = 0$$

and set $w = \left(\partial_x + 3\partial_y\right) u$

Then

$$w_x - w_y = 0$$

Which we know has the solution $w(x, y) = g(x + y)$. So $u_x + 3u_y = g(x + y)$, which gives us a system of two equations:

$$\Big\{ u_x - u_y = f(3x - y) u_x + 3u_y = g(x + y)$$

Subtract the first equation from the second:

$$4u_y = -f(3x - y) + g(x + y)$$

Now we can integrate with respect to y to find that:

$$u(x, y) = F(3x - y) + G(x + y)$$

where F is the antiderivative of $-f$ with respect to y and G is the antiderivative of g with respect to y.

Using the fact that $u(0, x) = \sin x$,

$$u(0, x) = \sin x = F(3x) + G(x) \tag{1}$$

now replace x with a new neutral variable, α and differentiate:

$$\begin{aligned} \sin\alpha &= F(3\alpha) + G(\alpha) \\ \cos\alpha &= 3F'(3\alpha) + G'(\alpha) \end{aligned} \tag{2}$$

But we can also differentiate $u(x, y) = F(3x - y) + G(x + y)$ with respect to y to get

$$u_y(x, y) = -F'(3x - y) + G'(x + y)$$

but from our initial conditions, we know

$$u_y(0, x) = -F'(3x - 0) + G'(x + 0) = x$$

Let's replace x by our neutral variable α and solve for F':

$$F'(\alpha) = G'(3\alpha) - \alpha$$

Now plug this into (2):

$$\cos\alpha = 3G'(\alpha) - 3\alpha + G'(\alpha)$$

$$G(\alpha) = \frac{1}{4}\int \cos\alpha + 3\alpha = \frac{\sin\alpha}{4} + \frac{3\alpha^2}{8}$$

So that means (1) becomes:

$$\sin\alpha = F(3\alpha) + \frac{\sin\alpha}{4} + \frac{3\alpha^2}{8}$$

$$F(\alpha) = \frac{3\sin\left(\frac{\alpha}{3}\right)}{4} - \frac{\alpha^2}{24}$$

Which means $u(x,y) = F(3x-y) + G(x+y)$ becomes

$$u(x,y) = \frac{3}{4}\sin\left(x - \frac{y}{3}\right) - \frac{(3x-y)^2}{24} + \frac{\sin(x+y)}{4} + \frac{3(x+y)^2}{8} =$$

Solution

$$u(x,y) = \frac{3}{4}\sin\left(x - \frac{y}{3}\right) + \frac{\sin(x+y)}{4} + xy + \frac{y^2}{3}$$

Check solution

$$u_y = \frac{-1}{4}\cos\left(x - \frac{y}{3}\right) + \frac{\cos(x+y)}{4} + x + \frac{2y}{3}$$
$$u_{yy} = \frac{-1}{12}\sin\left(x - \frac{y}{3}\right) - \frac{\sin(x+y)}{4} + \frac{2}{3}$$
$$u_x = \frac{3}{4}\cos\left(x - \frac{y}{3}\right) + \frac{\cos(x+y)}{4} + y$$
$$u_{xx} = \frac{-3}{4}\sin\left(x - \frac{y}{3}\right) - \frac{\sin(x+y)}{4}$$
$$u_{xy} = \frac{1}{4}\sin\left(x - \frac{y}{3}\right) - \frac{\sin(x+y)}{4} + 1$$

Check that $u_{xx} + 2u_{xy} - 3u_{yy} = 0$

$$\left(\frac{-3}{4} + \frac{2}{4} + \frac{1}{4}\right)\sin\left(x - \frac{y}{3}\right) + \left(\frac{-1}{4} + \frac{-2}{4} + \frac{3}{4}\right)\sin(x+y) + (0 + 2 - 3) = 0$$

Question 3

Solve the Neumann boundary value problem for the wave equation on half line:

$$\begin{cases} u_{tt} = c^2 u_{xx} + f(t,x) & 0 < x < \infty \\ u(0,x) = \phi(x) & 0 < x < \infty \\ u_t(0,x) = \psi(x) & 0 < x < \infty \\ u_x(t,0) = h(t) & t > 0 \end{cases}$$

The method we will use is to first convert to homogenous Neumann boundary conditions, and then extend the equation to the entire line using the reflection method.

First, define $w(t,x) = u(t,x) - H(t)$, where $H_x(t) = h(t)$.

Then our IBVP becomes:

$$\begin{cases} \Box w = \Box u - H''(t) = f(t,x) - H''(t) \equiv F(t,x) & x \geq 0, t > 0 \\ w(0,x) = u(0,x) - H(0) = \phi(x) - H(0) \equiv \Phi(x) & x \geq 0 \\ w_t(0,x) = u_t(0,x) - H_t(0) = \psi(x) - H_t(0) \equiv \Psi(x) & x \geq 0 \\ w_x(0,x) = u_x(t,0) - H_x(0) = u_x(t,0) - h(t) \equiv 0 & t \geq 0 \end{cases}$$

Now, since we have homogeneous Neumann boundary conditions, we will use even extensions of the initial data to the whole line, namely:

$$\Phi_{\text{even}} = \begin{cases} \Phi(x) & \text{for } x \geq 0 \\ \Phi(-x) & \text{for } x \leq 0 \end{cases}, \quad \Psi_{\text{even}} = \begin{cases} \Psi(x) & \text{for } x \geq 0 \\ \Psi(-x) & \text{for } x \leq 0 \end{cases}, \quad F_{\text{even}} = \begin{cases} F(t,x) & \text{for } x \geq 0 \\ F(t,-x) & \text{for } x \leq 0 \end{cases} \tag{3}$$

Then we assume $U(t,x)$ is an extension of $w(t,x)$ to the whole line. Since the initial condition and non-homogeneous parts are both even, the solution U is even, so we automatically have $U_x(t,0) = 0$.

Consider

$$\begin{cases} \Box U(t,x) = F_{\text{even}}(t,x) & \text{for } x \in \mathbb{R}, t > 0 \\ U(0,x) = \Phi_{\text{even}}(x) & \text{for } x \in \mathbb{R} \\ U_t(0,x) = \Psi_{\text{even}}(x) & \text{for } x \in \mathbb{R} \end{cases} \tag{4}$$

The solution at point (t_0, x_0) is

$$U(t_0,x_0) = \frac{\Phi_{\text{even}}(x_0+ct_0) + \Phi_{\text{even}}(x_0-ct_0)}{2} + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} \Psi_{\text{even}}(x)dx + \frac{1}{2c}\int_0^{t_0}\int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} F_{\text{even}}(t,x)dx\,dt$$

Assume $t_0 > 0, x_0 > 0 \Rightarrow x_0 + ct_0 > 0$. Now assume $x_0 > ct_0$, so the domain of dependence is entirely to the right of the line $x = ct$. This means $w(t_0,x_0) = U(t_0,x_0)$ and our solution is

$$w(t_0,x_0) = \frac{\Phi(x_0+ct_0) + \Phi(x_0-ct_0)}{2} + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} \Psi(x)dx + \frac{1}{2c}\int_0^{t_0}\int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} F(t,x)dx\,dt$$

$$\frac{\phi(x_0+ct_0) - H(0) + \phi(x_0-ct_0) - H(0)}{2} + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} \psi(x) - H'(0)dx + \frac{1}{2c}\int_0^{t_0}\int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} f(t,x) - H''(t)dx\,dt$$

$$w(t_0,x_0) = \frac{\phi(x_0+ct_0) + \phi(x_0-ct_0)}{2} - H(0) + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} \psi(x)dx - \frac{H'(0)}{2c}\left[(x_0+ct_0) - (x_0-ct_0)\right] +$$
$$\frac{1}{2c}\int_0^{t_0}\int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} f(t,x)dx\,dt - \frac{1}{2c}H''(t)\int_0^{t_0}\int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} dx\,dt$$

We can rewrite this as

$$w(t_0,x_0) = \frac{\phi(x_0+ct_0) + \phi(x_0-ct_0)}{2} + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} \psi(x)dx + \frac{1}{2c}\int_0^{t_0}\int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} f(t,x)dx\,dt$$
$$- H(0) - H'(0)t_0 - \frac{1}{2c}\int_0^{t_0} H''(t)2c(t_0-t)dt.$$

Call the section in blue “A”.

This means we have

$$w(t_0,x_0) = A - H(0) - H'(0)t_0 - \int_0^{t_0} H''(t)(t_0-t)\,dt \tag{5}$$

Now we use integration by parts on the final integral in (5), setting

$$u = (t_0 - t), \quad du = dt, \quad dv = H''(t)dt, \quad v = H'(t)$$

,

so that

$$\int_0^{t_0} H''(t)(t_0 - t)\,dt = H'(t)(t_0 - t)|_0^{t_0} + \int_0^{t_0} H'(t)dt = -H'(0)t_0 + H(t_0) - H(0)$$

So (5) becomes

$$w(t_0, x_0) = A - H(0) - H'(0)t_0 + H'(0)t_0 - H(t_0) + H(0)$$
$$w(t_0, x_0) = A - H(t_0)$$

Since $u(t, x) = u(t, x) + H(t)$, we get $u(t_0, x_0) = A$, or

$$u(t_0, x_0) = \frac{\phi(x_0 + ct_0) + \phi(x_0 - ct_0)}{2} + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} \psi(x)dx + \frac{1}{2c}\int_0^{t_0}\int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} f(t, x)dx\,dt \qquad (6)$$

when $x_0 > ct_0$.

Now we must consider what happens when $x_0 < ct_0$, such that $x_0 - ct_0 < 0$. Then we have

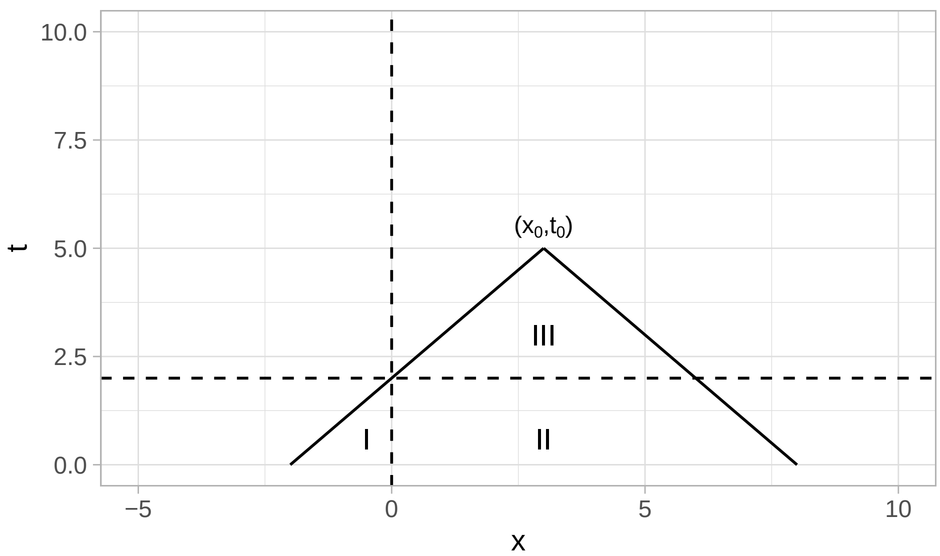
$$\begin{aligned}&w(t_0, x_0) = \frac{\phi(x_0 + ct_0) - H(0) + \phi(ct_0 - x_0) - H(0)}{2} + \\ &\frac{1}{2c}\int_{x_0-ct_0}^{0} \psi(-x) - H'(0)\,dx \; + \frac{1}{2c}\int_0^{x_0+ct_0} \psi(x) - H'(0)\,dx + \\ &\frac{1}{2c}\iint\limits_{\Delta} F_{\text{even}}(t, x)\,dx\,dt\end{aligned}$$

$$\begin{aligned}&w(t_0, x_0) = \frac{\phi(x_0 + ct_0) + \phi(ct_0 - x_0)}{2} - H(0) + \\ &\frac{1}{2c}\int_{x_0-ct_0}^{0} \psi(-x) - H'(0)\,dx \; + \frac{1}{2c}\int_0^{x_0+ct_0} \psi(x) - H'(0)\,dx + \\ &\frac{1}{2c}\iint\limits_{\Delta} F_{\text{even}}(t, x)\,dx\,dt\end{aligned}$$

Violet section:

$$\begin{aligned}&\frac{1}{2c}\int_{x_0-ct_0}^{0} \psi(-x)\,dx \; + \frac{1}{2c}\int_0^{x_0+ct_0} \psi(x)\,dx \; - \frac{1}{2c}\left[H'(0)(0 - (x - ct_0)) + H'(0)(x + ct_0 - 0)\right] = \\ &\frac{1}{2c}\left[\int_0^{ct_0-x_0} \psi(x)\,dx + \int_0^{x_0+ct_0} \psi(x)\,dx\right] - H'(0)t_0\end{aligned}$$

Now we investigate $\frac{1}{2c}\iint\limits_{\Delta} F_{\text{even}}(t, x)\,dx\,dt$.



$$\mathrm{I} = \frac{1}{2c}\int_0^{t_0-\frac{x_0}{c}}\int_{x_0-c(t_0-t)}^{0} F(t,-x)\,dx\,dt, \quad \mathrm{II} = \frac{1}{2c}\int_0^{t_0-\frac{x_0}{c}}\int_0^{x_0+c(t_0-t)} F(t,x)\,dx\,dt,$$

$$\mathrm{III} = \frac{1}{2c}\int_{t_0-\frac{x_0}{c}}^{t_0}\int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} F(t,x)\,dx\,dt$$

Now we will evaluate each integral separately:

$$\begin{aligned}
\mathrm{I} &= \frac{1}{2c}\int_0^{t_0-\frac{x_0}{c}}\int_{x_0-c(t_0-t)}^{0} f(t,-x) - H''(t)\,dx\,dt \\
&= \frac{1}{2c}\int_0^{t_0-\frac{x_0}{c}}\int_{x_0-c(t_0-t)}^{0} f(t,-x)\,dx\,dt - \frac{1}{2c}\int_0^{t_0-\frac{x_0}{c}} H''(t)\int_{x_0-c(t_0-t)}^{0} dx\,dt \\
&= -\frac{1}{2c}\int_0^{t_0-\frac{x_0}{c}}\int_{-x_0+c(t_0-t)}^{0} f(t,x)\,dx\,dt + \frac{1}{2c}\int_0^{t_0-\frac{x_0}{c}} H''(t)(x_0-c(t_0-t))\,dt \\
&= \frac{1}{2c}\int_0^{t_0-\frac{x_0}{c}}\int_0^{c(t_0-t)-x_0} f(t,x)\,dx\,dt + \frac{1}{2c}\int_0^{t_0-\frac{x_0}{c}} H''(t)(x_0-c(t_0-t))\,dt
\end{aligned}$$

$$\begin{aligned}
\mathrm{II} &= \frac{1}{2c}\int_0^{t_0-\frac{x_0}{c}}\int_0^{x_0+c(t_0-t)} f(t,x)\,dx\,dt - \frac{1}{2c}\int_0^{t_0-\frac{x_0}{c}} H''(t)\int_0^{x_0+c(t_0-t)} dx\,dt \\
&= \frac{1}{2c}\int_0^{t_0-\frac{x_0}{c}}\int_0^{x_0+c(t_0-t)} f(t,x)\,dx\,dt - \frac{1}{2c}\int_0^{t_0-\frac{x_0}{c}} H''(t)(x_0+c(t_0-t))\,dt
\end{aligned}$$

$$\begin{aligned}
\mathrm{III} &= \frac{1}{2c}\int_{t_0-\frac{x_0}{c}}^{t_0}\int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} f(t,x)\,dx\,dt - \frac{1}{2c}\int_{t_0-\frac{x_0}{c}}^{t_0} H''(t)\int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} dx\,dt \\
&= \frac{1}{2c}\int_{t_0-\frac{x_0}{c}}^{t_0}\int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} f(t,x)\,dx\,dt - \frac{1}{2c}\int_{t_0-\frac{x_0}{c}}^{t_0} H''(t)\;2c(t_0-t)\,dt \\
&= \frac{1}{2c}\int_{t_0-\frac{x_0}{c}}^{t_0}\int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} f(t,x)\,dx\,dt - \int_{t_0-\frac{x_0}{c}}^{t_0} H''(t)\;(t_0-t)\,dt
\end{aligned}$$

Now we use integration by parts on the final integral in the above expression, setting

$$u = (t_0 - t), \quad du = dt, \quad dv = H''(t)dt, \quad v = H'(t)$$

so that

$$\begin{aligned}\int_{t_0-\frac{x_0}{c}}^{t_0} H''(t)(t_0-t)\,dt &= H'(t)(t_0-t)\Big|_{t_0-\frac{x_0}{c}}^{t_0} + \int_{t_0-\frac{x_0}{c}}^{t_0} H'(t)dt \\ &= -\frac{x_0}{c}H'\left(t_0-\frac{x_0}{c}\right) + H(t_0) - H\left(t_0-\frac{x_0}{c}\right)\end{aligned}$$

So III is

$$\frac{1}{2c}\int_{t_0-\frac{x_0}{c}}^{t_0}\int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} f(t,x)\,dx\,dt + \frac{x_0}{c}H'\left(t_0-\frac{x_0}{c}\right) - H(t_0) + H\left(t_0-\frac{x_0}{c}\right)$$

And now we can find the sum of the three regions:

$$\begin{aligned}\mathrm{I}+\mathrm{II}+\mathrm{III} = &\frac{1}{2c}\int_0^{t_0-\frac{x_0}{c}}\int_0^{c(t_0-t)-x_0} f(t,x)\,dx\,dt + \frac{1}{2c}\int_0^{t_0-\frac{x_0}{c}} H''(t)(x_0-c(t_0-t))\,dt + \\ &\frac{1}{2c}\int_0^{t_0-\frac{x_0}{c}}\int_0^{x_0+c(t_0-t)} f(t,x)\,dx\,dt - \frac{1}{2c}\int_0^{t_0-\frac{x_0}{c}} H''(t)(x_0+c(t_0-t))\,dt + \\ &\frac{1}{2c}\int_{t_0-\frac{x_0}{c}}^{t_0}\int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} f(t,x)\,dx\,dt + \\ &\frac{x_0}{c}H'\left(t_0-\frac{x_0}{c}\right) - H(t_0) + H\left(t_0-\frac{x_0}{c}\right)\end{aligned}$$

$$\begin{aligned}= &\frac{1}{2c}\int_0^{t_0-\frac{x_0}{c}}\int_0^{c(t_0-t)-x_0} f(t,x)\,dx\,dt + \frac{1}{2c}\int_0^{t_0-\frac{x_0}{c}}\int_0^{x_0+c(t_0-t)} f(t,x)\,dx\,dt + \frac{1}{2c}\int_{t_0-\frac{x_0}{c}}^{t_0}\int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} f(t,x)\,dx\,dt + \\ &\frac{1}{2c}\int_0^{t_0-\frac{x_0}{c}} H''(t)(x_0-c(t_0-t))\,dt - \frac{1}{2c}\int_0^{t_0-\frac{x_0}{c}} H''(t)(x_0+c(t_0-t))\,dt + \\ &\frac{x_0}{c}H'\left(t_0-\frac{x_0}{c}\right) - H(t_0) + H\left(t_0-\frac{x_0}{c}\right)\end{aligned}$$

$$\begin{aligned}\frac{1}{2c}\int_0^{t_0-\frac{x_0}{c}} H''(t)(x_0-c(t_0-t))\,dt - \frac{1}{2c}\int_0^{t_0-\frac{x_0}{c}} H''(t)(x_0+c(t_0-t))\,dt &= \\ \frac{1}{2c}\int_0^{t_0-\frac{x_0}{c}} H''(t)(-c(t_0-t))\,dt - \frac{1}{2c}\int_0^{t_0-\frac{x_0}{c}} H''(t)c(t_0-t)\,dt &= \\ -\frac{1}{2c}\int_0^{t_0-\frac{x_0}{c}} H''(t)c(t_0-t)\,dt - \frac{1}{2c}\int_0^{t_0-\frac{x_0}{c}} H''(t)c(t_0-t)\,dt &= \\ -\int_0^{t_0-\frac{x_0}{c}} H''(t)(t_0-t)\,dt\end{aligned}$$

Now we use integration by parts, so that

$$\int_0^{t_0-\frac{x_0}{c}} H''(t)(t_0-t)\,dt = H'(t)(t_0-t)\Big|_0^{t_0-\frac{x_0}{c}} + \int_0^{t_0-\frac{x_0}{c}} H'(t)dt = \frac{x_0}{c}H'\left(t_0-\frac{x_0}{c}\right) + H\left(t_0-\frac{x_0}{c}\right) - H(0)$$

$\mathrm{I} + \mathrm{II} + \mathrm{III} =$

$$\frac{1}{2c}\int_0^{t_0-\frac{x_0}{c}}\int_0^{c(t_0-t)-x_0} f(t,x)\,dx\,dt + \frac{1}{2c}\int_0^{t_0-\frac{x_0}{c}}\int_0^{x_0+c(t_0-t)} f(t,x)\,dx\,dt + \frac{1}{2c}\int_{t_0-\frac{x_0}{c}}^{t_0}\int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} f(t,x)\,dx\,dt$$
$$-\frac{x_0}{c}H'\left(t_0-\frac{x_0}{c}\right) - H\left(t_0-\frac{x_0}{c}\right) + H(0)+$$
$$\frac{x_0}{c}H'\left(t_0-\frac{x_0}{c}\right) - H(t_0) + H\left(t_0-\frac{x_0}{c}\right)$$

$\mathrm{I} + \mathrm{II} + \mathrm{III} =$

$$\frac{1}{2c}\int_0^{t_0-\frac{x_0}{c}}\int_0^{c(t_0-t)-x_0} f(t,x)\,dx\,dt + \frac{1}{2c}\int_0^{t_0-\frac{x_0}{c}}\int_0^{x_0+c(t_0-t)} f(t,x)\,dx\,dt + \frac{1}{2c}\int_{t_0-\frac{x_0}{c}}^{t_0}\int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} f(t,x)\,dx\,dt+$$
$$H(0) - H(t_0)$$

$$w(t_0,x_0) = \frac{\phi(x_0+ct_0)+\phi(ct_0-x_0)}{2} - H(0)+$$
$$\frac{1}{2c}\left[\int_0^{ct_0-x_0}\psi(x)\,dx + \int_0^{x_0+ct_0}\psi(x)\,dx\right] - H'(0)t_0+$$
$$\frac{1}{2c}\int_0^{t_0-\frac{x_0}{c}}\int_0^{c(t_0-t)-x_0} f(t,x)\,dx\,dt + \frac{1}{2c}\int_0^{t_0-\frac{x_0}{c}}\int_0^{x_0+c(t_0-t)} f(t,x)\,dx\,dt + \frac{1}{2c}\int_{t_0-\frac{x_0}{c}}^{t_0}\int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} f(t,x)\,dx\,dt+$$
$$H(0) - H(t_0)$$

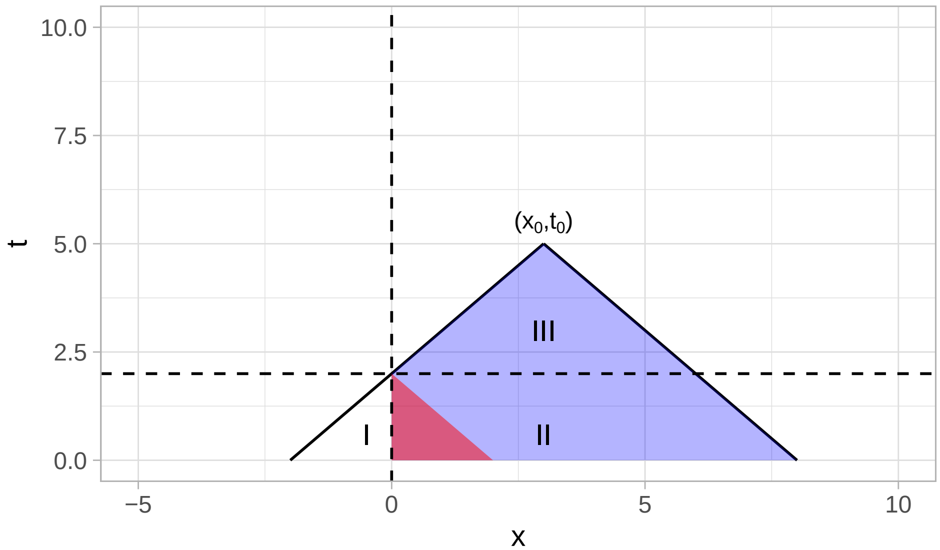
Solution

So for $ct_0 > x_0$, $u(t_0,x_0) = w(t_0,x_0) + H(t_0)$, which means

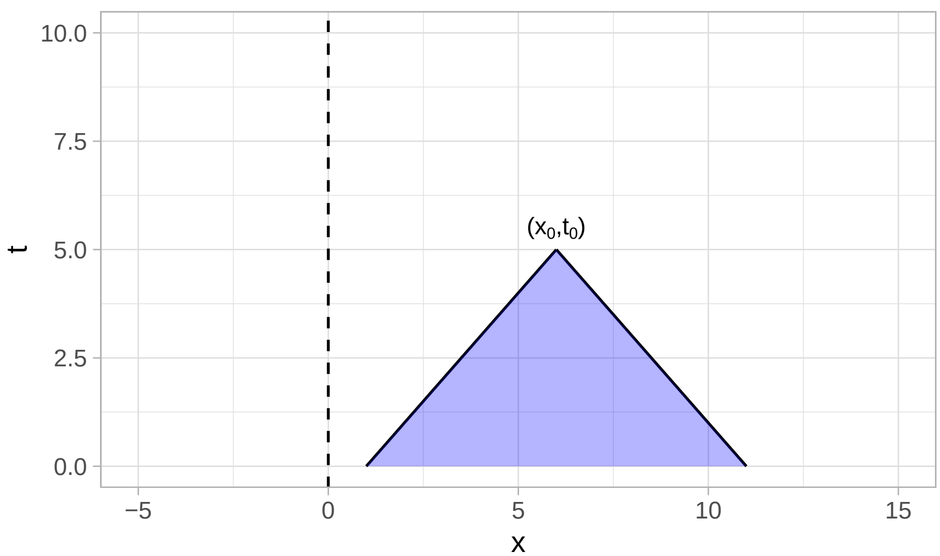
$$u(t_0,x_0) = \frac{\phi(x_0+ct_0)+\phi(ct_0-x_0)}{2}+$$
$$\frac{1}{2c}\left[\int_0^{ct_0-x_0}\psi(x)\,dx + \int_0^{x_0+ct_0}\psi(x)\,dx\right] - H'(0)t_0+$$
$$\frac{1}{2c}\int_0^{t_0-\frac{x_0}{c}}\int_0^{c(t_0-t)-x_0} f(t,x)\,dx\,dt + \frac{1}{2c}\int_0^{t_0-\frac{x_0}{c}}\int_0^{x_0+c(t_0-t)} f(t,x)\,dx\,dt+$$
$$\frac{1}{2c}\int_{t_0-\frac{x_0}{c}}^{t_0}\int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} f(t,x)\,dx\,dt$$

While for $ct_0 > x_0$, from (6), we have

$$u(t_0,x_0) = \frac{\phi(x_0+ct_0)+\phi(x_0-ct_0)}{2} + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0}\psi(x)dx + \frac{1}{2c}\int_0^{t_0}\int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} f(t,x)dx\,dt$$



When $x_0 < ct_0$, we're integrating f over the domain of dependence (the shaded blue region, including the red triangle) but then adding f integrated over the red triangle. When $x_0 < ct_0$, we just integrate f over the blue triangle.



Question 4

Consider the 3D wave equation for $u(t,x,y,z)$:

$$u_{tt} = c^2 \Delta u \qquad (x,y,z) \in \mathbb{R}^3, \quad t > 0$$

Change the coordinates to spherical coordinates. Assume the solution is spherically symmetric, so that $u(t,x,y,z) = u(t,r)$ and does not depend on θ and ϕ. Find the solution for $u(0,r) = 0$ and

$$u_t(0,r) = \begin{cases} 1 & |r| \leq 1 \\ 0 & |r| > 1 \end{cases}$$

Hint: use the substitution $u(t,r) = \frac{1}{r}w(t,r)$.

First, we need to derive the formula for the Laplacian in spherical coordinates.

We know the equation for the Laplacian in polar coordinates is:

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Now let's convert to spherical coordinates:

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{s^2+z^2}$$

$$x = s\cos\phi, \quad y = s\sin\phi, \quad z = r\cos\theta$$
$$s = r\sin\theta$$

By the two-dimensional Laplacian, we have

$$u_{zz} + u_{ss} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta},$$

$$u_{xx} + u_{yy} = u_{ss} + \frac{1}{s}u_s + \frac{1}{s^2}u_{\phi\phi}$$

We add these two equations and cancel u_s to get

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \frac{1}{s}u_s + \frac{1}{s^2}u_{\phi\phi}$$

Now since u doesn't depend on θ or ϕ, we have

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{s}u_s = u_{rr} + \frac{1}{r}u_r + \frac{1}{r\sin\theta}u_s$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial s} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial s} + \frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial s} = u_r\frac{1}{\sin\theta} + 0 + 0 = u_r\frac{s}{r}$$

So with our change of variables, we have

$$u_{tt} = c^2\left(u_{rr} + \frac{2}{r}u_r\right)$$

Now set $w = ru$, or $u = \frac{w}{r}$. Then

$$w_t = ru_t, \quad w_{tt} = ru_{tt}, \quad u_{tt} = \frac{w_{tt}}{r}$$

$$w_t = ru_t, \quad w_{tt} = ru_{tt}, \quad u_{tt} = \frac{w_{tt}}{r}$$
$$u_r = \frac{w_r}{r} - \frac{w}{r^2}$$

$$u_{rr} = \frac{w_{rr}}{r} - \frac{2w_r}{r^2} + \frac{2w}{r^3}$$

So $u_{tt} = c^2 \left(u_{rr} + \frac{2}{r}u_r\right)$ becomes

$$\frac{w_{tt}}{r} = c^2 \left(\frac{w_{rr}}{r} - \frac{2w_r}{r^2} + \frac{2w}{r^3} + \frac{2}{r}\left(\frac{w_r}{r} - \frac{w}{r^2}\right)\right),$$

which simplifies to

$$w_{tt} = c^2 w_{rr}.$$

But this is just the wave equation, and we can use d'Alembert's formula to find the solution:

$$w(t,r) = \frac{\varphi(r+ct) + \varphi(r-ct)}{2} + \frac{1}{2c}\int_{r-ct}^{r+ct} \psi(s)ds$$

Since $\varphi = 0$,

$$w(t,r) = \frac{1}{2c}\int_{r-ct}^{r+ct} \psi(s)ds$$

Now we have 4 cases:

Case 1: $r - ct \geq -1, r + ct \leq 1$

$$w(t,r) = \frac{1}{2c}\int_{r-ct}^{r+ct} s\ ds$$

$$= \frac{1}{4c}((r+ct)^2 - (r-ct)^2)$$

$$= \frac{1}{4c}(r^2 + 2crt + c^2t^2 - r^2 + 2crt - c^2t^2) = \frac{4crt}{4c} = rt$$

Case 2:$r - ct < -1, r + ct > 1$

$$w(t,r) = \frac{1}{2c}\int_{-1}^{1} s\ ds =$$

$$w(t,r) = \frac{1}{4c}(1-1) = 0$$

Case 3:$r - ct < -1, r + ct \leq 1$

$$w(t,r) = \frac{1}{2c}\int_{-1}^{r+ct} s\ ds$$

$$= \frac{1}{4c}((r+ct)^2 - 1)$$

Case 4:$r - ct \geq -1, r + ct > 1$

$$w(t,r) = \frac{1}{2c}\int_{r-ct}^{1} s\ ds$$

$$= \frac{1}{4c}(1 - (r-ct)^2)$$

Since $u = \frac{w}{r}$, this means we have

Solution

$$u(t,r) = \begin{cases} t & \text{if } r - ct \geq -1, r + ct \leq 1 \\ 0 & \text{if } r - ct < -1, r + ct > 1 \\ \frac{1}{4rc}((r+ct)^2 - 1) & \text{if } r - ct < -1, r + ct \leq 1 \\ \frac{1}{4rc}(1 - (r-ct)^2) & \text{if } r - ct \geq -1, r + ct > 1 \end{cases}$$

Question 5

Consider the following Dirichlet boundary value problem:

$$\begin{cases} u_{tt} + x(t,x)u_t = u_{xx} & 0 < x < 1 \\ u(0,x) = \phi(x) & 0 < x < 1 \\ u_t(0,x) = \psi(x) & 0 < x < 1 \\ u(t,0) = u(t,1) = 0 & t \geq 0 \end{cases}$$

Assume that $|a(t,x)| \leq m$ for some constant m and all $0 < x < 1$ and $t \geq 0$. Let

$$E(t) = \frac{1}{2}\int_0^1 \left(u_t(t,x)^2 + u_x(t,x)^2\right)\,dx$$

(1) Show that

$$\frac{dE(t)}{dt} \leq 2mE(t) \tag{7}$$

for $t \geq 0$.

First differentiate $E(t)$:

$$\frac{dE}{dt} = \frac{d}{dt}\left[\frac{1}{2}\int_0^1 \left(u_t^2 + u_x^2\right)\,dx\right]$$

$$= \frac{1}{2}\int_0^1 \frac{\partial}{\partial t}\left(u_t^2 + u_x^2\right)\,dx \tag{8}$$

$$= \frac{1}{2}\int_0^1 2u_tu_{tt} + 2u_xu_{xt}\,dx \tag{9}$$

$$= \int_0^1 u_tu_{tt}\,dx + \int_0^1 2u_xu_{xt}\,dx =: I + J. \tag{10}$$

The equality (8) follows from differentiation under the integral sign, while (9) follows from the chain rule for partial derivatives.

Consider the integral J in (10). By integrating by parts, we can move one of the partials $\frac{\partial}{\partial x}$ to the other factor, at the cost of introducing a minus sign and a boundary term. Hence J becomes

$$J = \int_0^1 u_xu_{xt}\,dx = -\int_0^1 u_xxu_t\,dx + [u_xu_t]_{x=0}^{x=1}. \tag{11}$$

The boundary term vanishes, since $u(t,0) \equiv u(t,1) \equiv 0$ for $t > 0$ implies that u_t is identically zero at $x = 0, 1$. So, substituting (11) for J, equation (10) becomes

$$I + J = \int_0^1 u_tu_{tt}\,dx - \int_0^1 u_{xx}u_t\,dx$$

$$= \int_0^1 u_t(u_{tt} - u_{xx})\,dx$$

$$= \int_0^1 u_t(-au_t)\,dx \tag{12}$$

$$= \int_0^1 (-a)u_t^2 \, dx. \tag{13}$$

Here, equality (12) just uses the PDE. Since $u_t^2 \geq 0$ and $-a \leq |-a| \leq m$, we see that $(-a)u_t^2 \leq mu_t^2$. Hence, the expression in (13) satisfies the inequality

$$I + J = \int_0^1 (-a)u_t^2 \, dx \leq \int_0^1 mu_t^2 \, dx \leq m \int_0^1 u_t^2 + u_x^2 \, dx = 2mE. \tag{14}$$

Where we also used the fact that $u_x^2 \geq 0$ means that $m \int_0^1 u_t^2 \, dx \leq m \int_0^1 (u_t^2 + u_x^2) \, dx$.

The desired inequality (7) follows from (14) and the fact that $\frac{dE}{dt} = I + J$.

(2) Use part (1) and show that $\frac{d}{dt}\left(e^{-2mE(t)}\right) \leq 0$ for all $t \geq 0$. Hence, by integration from $[0, t]$, we get that

$$E(t) \leq e^{2mt}E(0) \text{ for all } t \geq 0. \tag{15}$$

By the product rule,

$$\begin{aligned}
\frac{d}{dt}\left(e^{-2mt}E\right) &= -2me^{-2mt}E + e^{-2mt}\frac{dE}{dt} \\
&= e^{-2mt}\left(\frac{dE}{dt} - 2mE\right) \\
&\leq e^{-2mt} \cdot 0 = 0.
\end{aligned}$$

(3) If $\phi(x) = \psi(x) = 0$ for all $0 < x < 1$, what does this say about $E(t)$ for $t \geq 0$ and hence about $u(t, x)$ for $t \geq 0$?

Since $u(0, x) \equiv \varphi(x) \equiv 0$ for $0 < x < 1$, we see that u_x is identically zero at time $t = 0$. Similarly, $u_t(0, x) \equiv \psi(x) \equiv 0$ for $0 < x < 1$. Thus at $t = 0$,

$$E(0) = \frac{1}{2}\int_0^1 \left(u_t(t, 0)^2 + u_x(t, 0)^2\right) dx = \frac{1}{2}\int_0^1 0^2 + 0^2 \, dx = 0. \tag{16}$$

But (16) together with the inequality (15) implies that

$$0 \leq E(t) \leq e^{2mt}E(0) \leq 0$$

for all $t \geq 0$. Hence the energy E is identically zero. Since the integrand $u_t^2 + u_x^2$ is nonnegative, this is only possible if $u_t \equiv u_x \equiv 0$ for $t > 0, 0 < x < 1$, meaning that u varies with neither time nor position. But this implies that u must be identically zero everywhere.

(4) Use the previous part to prove uniqueness of the following problem:

$$\begin{cases}
u_{tt} + a(t, x)u_t = u_{xx} & 0 < x < 1, t > 0 \\
u(0, x) = \phi(x) & 0 < x < 1 \\
u_t(0, x) = \psi(x) & 0 < x < 1 \\
u(t, 0) = f(t) & t \geq 0 \\
u(t, 1) = g(t) & t \geq 0
\end{cases}$$

Let u and v be two solutions, and define $w := u - v$. Observe that w satisfies the original boundary value problem along with the conditions specified in part (3), implying that $w \equiv 0$. Hence $u - v \equiv 0$ and so any solution u must be unique. □

Problem 6

Does the D'Alembert method work for the wave equation $u_{tt} = c(x)^2 u_{xx}$? What about $u_{tt} = c(t)^2 u_{xx}$? Why?

Let's assume that we can factor this wave equation with $c = c(x)$ as we could with constant c:

$$(\partial_t + c(x)\partial_x)(\partial_t - c(x)\partial_x)u = 0$$

Now, let's distribute:

$$(\partial_t + c(x)\partial_x)(u_t - c(x)u_x) = 0$$
$$u_{tt} - c(x)u_{xt} + c(x)u_{xt} - c(x)(c'(x)u_x + c(x)u_{xx}) = 0$$
$$u_{tt} - c(x)^2 u_{xx} - c(x)c'(x)u_x = 0$$

But this is only equivalent to our original equation if $c'(x) = 0$ for all x, i.e., if c is constant.

We can repeat the process for $c = c(t)$:

$$(\partial_t + c(t)\partial_x)(\partial_t - c(t)\partial_x)u = 0$$

$$(\partial_t + c(t)\partial_x)(u_t - c(t)u_x) = 0$$
$$u_{tt} - (c'(t)u_x + c(t)u_{xt}) + c(t)u_{xt} - c(t)^2 u_{xx} = 0$$
$$u_{tt} - c(t)^2 u_{xx} - c'(t)u_x = 0$$

Which, as before, is only equivalent to our original equation if $c'(t) = 0$ for all t, i.e., if c is constant.

Problem 7 (The Poisson-Darboux Equation)

Solve the initial value problem

$$\begin{cases} u_{tt} - u_{xx} - \frac{2}{x}u_x = 0 & -\infty < x < \infty, t > 0 \\ u(0,x) = 0 & -\infty < x < \infty \\ u_t(0,x) = g(x) & -\infty < x < \infty \end{cases}$$

where $g(x) = g(-x)$ is an even function. Hint: set $w = xu$.

Using the results from when we set $w = ru$ in Problem 4,

$$u_t = \frac{w_t}{x}, u_{tt} = \frac{w_{tt}}{x}$$

$$u_x = \frac{w_x}{x} - \frac{w}{x^2}$$

$$u_{xx} = \frac{w_{xx}}{x} - \frac{2w_x}{x^2} + \frac{2w}{x^3}$$

So $u_{tt} - u_{xx} - \frac{2}{x}u_x = 0$ becomes

$$\frac{w_{tt}}{x} - \frac{w_{xx}}{x} + \frac{2w_x}{x^2} - \frac{2w}{x^3} - \frac{2}{x}\left(\frac{w_x}{x} - \frac{w}{x^2}\right) = 0$$

Which simplifies to $w_{tt} - w_{xx} = 0$, and so w solves the wave equation when $-\infty < x < \infty, t > 0$.

Now observe that $w(0,x) = xu(0,x) = 0$ and $w_t(0,x) = xu_t(0,x) = xg(x)$. Hence, w solves the initial value problem:

$$
\begin{cases}
w_{tt} - w_{xx} & -\infty < x < \infty, t > 0 \\
w(0,x) = 0 & -\infty < x < \infty, t > 0 \\
w_t(0,x) = xg(x) & -\infty < x < \infty, t > 0
\end{cases} \tag{17}
$$

whre $g(x)$ is an even function.

By d'Alembert's formula, the solution to (17) is given by:

$$
w(t,x) = \frac{1}{2}\int_{x-t}^{x+t} sg(s)ds
$$

Hence since $w_t = xu_t$,

$$
w(t,x) = \frac{1}{2x}\int_{x-t}^{x+t} sg(s)ds, \tag{18}
$$

assuming $x \neq 0$.

To handle the case when $x = 0$, assume that u is continuous in x, so that $u(t,0) = \lim_{x\to 0} u(t,x)$. Define

$$
I(y) = \int_0^y sg(s)\, ds.
$$

This allows us to rewrite (18) as

$$
u(t,x) = \frac{I(x+t) - I(x-t)}{2x} = \frac{I(x+t) - I(t-x)}{2x},
$$

Noting that the evenness of $g(s)$ means that $sg(s)$ is odd, and so $I(y)$ is even (one can check by substitution that $I(y) = I(-y)$). Hence, letting $x \to 0$, we see that

$$
u(t,0) = \lim_{x\to 0} \frac{I(t+x) - I(t-x)}{2x}. \tag{19}
$$

But this is just the form of a derivative, namely the symmetric derivative of I at t. By the Fundamental Theorem of Calculus, assuming g is continuous, we know the limit (19) exists and is necessarily equal to $I'(t) = tg(t)$.

Thus $u(t,x)$ is given for any $t \geq 0, x \in \mathbb{R}$ by

$$
u(t,x) = \begin{cases}
\frac{1}{2x}\int_{x-t}^{x+t} sg(s)ds, & x \neq 0, \\
tg(t), & x = 0
\end{cases} \tag{20}
$$

Problem 8

Solve the following characteristic initial value problem:

$$
\begin{cases}
y^3 u_{xx} - yu_{yy} + u_y = 0 & 0 < x < 4, \quad |y| \leq 2\sqrt{2} \\
u(x,y) = f(x) & x + \frac{y^2}{2} = 4 \text{ for } 2 \leq x \leq 4 \\
u(x,y) = g(x) & x - \frac{y^2}{2} = 0 \text{ for } 0 \leq x \leq 2
\end{cases}
$$

where $f(2) = g(2)$. Hint: Use the transformation $\eta = x - \frac{y^2}{2}$ and $\xi = x + \frac{y^2}{2}$ and express the PDE in the coordinates (ξ, η).

Set $\eta = x - \frac{y^2}{2}$ and $\xi = x + \frac{y^2}{2}$. Then

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi}\right) + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \eta}\right)$$

$$= \frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \eta \partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \eta \partial \xi}\frac{\partial \eta}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x}$$

Which we can simplify to

$$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

Now we do the same for y:

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = y\frac{\partial u}{\partial \xi} - y\frac{\partial u}{\partial \eta}$$

But since $\eta = x - \frac{y^2}{2}$ and $\xi = x + \frac{y^2}{2}$, we can rewrite y as $\sqrt{\xi - \eta}$.

$$\frac{\partial u}{\partial y} = \sqrt{\xi - \eta}\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right)$$

Now using the product rule,

$$\frac{\partial^2 u}{\partial y^2} = \sqrt{\xi - \eta}\,\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right) + \left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right)$$

Where the second term has been simplified because $\frac{\partial}{\partial y}$ of $\sqrt{\xi - \eta}$ is simply $\frac{\partial}{\partial y}y = 1$.

Now we use the chain rule again for the first term:

Now using the product rule,

$$\frac{\partial^2 u}{\partial y^2} = \sqrt{\xi - \eta}\left(\frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \eta \partial \xi}\frac{\partial \eta}{\partial y} - \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial y} - \frac{\partial^2 u}{\partial \eta \partial \xi}\frac{\partial \xi}{\partial y}\right) + \left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right)$$

Which simplifies to

$$u_{yy} = (\xi - \eta)\left(u_{\xi\xi} + u_{\eta\eta} - 2u_{\xi\eta}\right) + u_\xi - u_\eta$$

This means $y^3 u_{xx} - y u_{yy} + u_y = 0$ becomes

$$(\xi - \eta)^{\frac{3}{2}}\left(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}\right) - (\xi - \eta)^{\frac{1}{2}}\left[(\xi - \eta)\left(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}\right) + u_\xi - u_\eta\right] + (\xi - \eta)^{\frac{1}{2}}\left(u_\xi - u_\eta\right) = 0$$

$$(\xi - \eta)^{\frac{3}{2}}\left(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}\right) - (\xi - \eta)^{\frac{3}{2}}\left(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}\right) - (\xi - \eta)^{\frac{1}{2}}\left(u_\xi - u_\eta\right) + (\xi - \eta)^{\frac{1}{2}}\left(u_\xi - u_\eta\right) = 0$$

$$(\xi - \eta)^{\frac{3}{2}} (4\, u_{\xi\eta}) = 0$$

But by integrating with respect to both variables, we recognize $u_{\xi\eta} = 0$ as having the solution

$$u(\xi, \eta) = h_1(\xi) + h_2(\eta),$$

or in our original coordinates,

$$u(x, y) = h_1\left(x + \frac{y^2}{2}\right) + h_2\left(x - \frac{y^2}{2}\right)$$

Now we use our given conditions. When $2 \le x \le 4$, we have $g(x) = h_1(2x) + h_2(0)$, which means $h_1(z_1) = g\left(\frac{z_1}{2}\right) - h_2(0)$.

Similarly, when $0 \le x \le 2$, we have $f(x) = h_1(4) + h_2(2x - 4)$, which means $h_2(z_2) = f\left(\frac{z_2}{2} + 2\right) - h_1(4)$.

Thus,

$$h_1(z_1) + h_2(z_2) = g\left(\frac{z_1}{2}\right) - h_2(0) + f\left(\frac{z_2}{2} + 2\right) - h_1(4)$$

But $h_2(0) + h_1(4)$ is just $g(x)$ when $x = 2$ (or equivalently, $f(x)$ when $x = 2$),

so after replacing z_1 by $x + \frac{y^2}{2}$ and z_2 by $x - \frac{y^2}{2}$, we get

Solution

$$u(x, y) = g\left(\frac{x}{2} + \frac{y^2}{4}\right) + f\left(\frac{x}{2} + \frac{y^2}{4} + 2\right) - f(2)$$

Check solution

$$u_y(x, y) = \frac{y}{2} g'\left(\frac{x}{2} + \frac{y^2}{4}\right) + \frac{y}{2} f'\left(\frac{x}{2} + \frac{y^2}{4} + 2\right)$$

$$u_{yy}(x, y) = \frac{y^2}{4} g''\left(\frac{x}{2} + \frac{y^2}{4}\right) + \frac{1}{2} g'\left(\frac{x}{2} + \frac{y^2}{4}\right) + \frac{y^2}{4} f''\left(\frac{x}{2} + \frac{y^2}{4} + 2\right) + \frac{1}{2} f'\left(\frac{x}{2} + \frac{y^2}{4} + 2\right)$$

$$u_x(x, y) = \frac{1}{2} g'\left(\frac{x}{2} + \frac{y^2}{4}\right) + \frac{1}{2} f'\left(\frac{x}{2} + \frac{y^2}{4} + 2\right) - f'(2)$$

$$u_{xx}(x, y) = \frac{1}{4} g''\left(\frac{x}{2} + \frac{y^2}{4}\right) + \frac{1}{4} f''\left(\frac{x}{2} + \frac{y^2}{4} + 2\right) - f''(2)$$

This means $y^3 u_{xx} - y u_{yy} + u_y = 0$ becomes

$$\begin{aligned}
&\left[\frac{y^3}{4} g''\left(\frac{x}{2} + \frac{y^2}{4}\right) + \frac{y^3}{4} f''\left(\frac{x}{2} + \frac{y^2}{4} + 2\right)\right] - \\
&\left[\frac{y^3}{4} g''\left(\frac{x}{2} + \frac{y^2}{4}\right) + \frac{y}{2} g'\left(\frac{x}{2} + \frac{y^2}{4}\right) + \frac{y^3}{4} f''\left(\frac{x}{2} + \frac{y^2}{4} + 2\right) + \frac{y}{2} f'\left(\frac{x}{2} + \frac{y^2}{4} + 2\right)\right] + \\
&\left[\frac{y}{2} g'\left(\frac{x}{2} + \frac{y^2}{4}\right) + \frac{y}{2} f'\left(\frac{x}{2} + \frac{y^2}{4} + 2\right)\right] = 0
\end{aligned}$$