BIOL 274 Homework 6

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Question 1

Write each of the following systems in matrix/vector notation:

(a)

$$x' = 4x - 6y$$
$$y' = 0.7x + y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ 0.7 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{or}$$

$$\dot{X} = \begin{bmatrix} 4 & -6 \\ 0.7 & 1 \end{bmatrix} X$$

(b)

$$\frac{dx_1}{dt} - 16x_2 = -2x_1$$
$$\frac{dx_2}{dt} + 4x_1 = 0$$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -2 & 16 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \text{or}$$

$$\dot{X} = \begin{bmatrix} -2 & 16 \\ -4 & 0 \end{bmatrix} X$$

(c)

$$\dot{v} = v - u + \sin t$$
$$\dot{u} = 2v + u - \cos t$$

$$\begin{bmatrix} \dot{v} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix} + \begin{bmatrix} \sin t \\ -\cos t \end{bmatrix}, \quad \text{or}$$

$$\dot{V} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} V + \begin{bmatrix} \sin t \\ -\cos t \end{bmatrix}$$

Question 2

Find the general solution for each of the following systems:
(a)

$$\dot{x} = 9x - 8y$$
$$\dot{y} = 24x - 19y$$

$$A = \begin{bmatrix} 9 & -8 \\ 24 & -19 \end{bmatrix}$$

$$\det \begin{bmatrix} 9-\lambda & -8 \\ 24 & -19\lambda \end{bmatrix} = 0$$

$$(9-\lambda)(-19-\lambda) + 192 = 0$$
$$-171 - 9\lambda + 19\lambda + \lambda^2 + 192 = \lambda^2 + 10\lambda + 21 = (\lambda+7)(\lambda+3)$$

So we have found two distinct real eigenvalues, $\lambda_1 = -7$, $\lambda_2 = -3$. Now we need to solve $AX = \lambda X$, or $(A - \lambda I)X = 0$ with these two values of λ. Let's start with the first eigenvalue, -7.

$$A + 7I = \begin{bmatrix} 9+7 & -8 \\ 24 & -19+7 \end{bmatrix} = \begin{bmatrix} 16 & -8 \\ 24 & -12 \end{bmatrix}$$

Now we use Gauss-Jordan elimination:

$$\begin{bmatrix} 16 & -8 \\ 24 & -12 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}$$

$$-2x_1 + x_2 = 0 \quad \rightarrow \quad x_2 = 2x_1 \quad \rightarrow \quad v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

And repeat for the second eigenvalue, -3:

$$A + 3I = \begin{bmatrix} 9+3 & -8 \\ 24 & -19+3 \end{bmatrix} = \begin{bmatrix} 12 & -8 \\ 24 & -16 \end{bmatrix}$$

$$\begin{bmatrix} 12 & -8 \\ 24 & -16 \end{bmatrix} \longrightarrow \begin{bmatrix} -3 & 2 \\ 0 & 0 \end{bmatrix}$$

$$-3x_1 + 2x_2 = 0 \quad \rightarrow \quad 2x_2 = 3x_1 \quad \rightarrow \quad v_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

So we have found $\lambda_1 = -7$ with eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\lambda_2 = -3$ with eigenvector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. We can plug these values into our formula for the general solution of a system of linear first-order ODEs with constant coefficients:

$$y = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2$$

Solution

$$y = C_1 e^{-7t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{-3t} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Let's also do a quick check using R:

```r
A <- matrix(c(9, -8, 24, -19), 2, 2, byrow=TRUE)

lambda1 <- eigen(A)$values[1] # eigenvalue 1
v1<- eigen(A)$vectors[,1] / eigen(A)$vectors[1,1] #eigenvector 1
# divided by the first element to give integer values

lambda2 <- eigen(A)$values[2] # eigenvalue 2
v2 <- eigen(A)$vectors[,2] / eigen(A)$vectors[1,2]*2 #eigenvector 2
# divided by the first element and multiplied by 2 to give integer values
```

```r
paste("lambda 1 = ", lambda1, " with eigenvector [", v1[1], ",", v1[2], "]", sep="")
```

```
## [1] "lambda 1 = -7 with eigenvector [1,2]"
```

```r
paste("lambda 2 = ", lambda2, " with eigenvector [", v2[1], ",", v2[2], "]", sep="")
```

```
## [1] "lambda 2 = -3 with eigenvector [2,3]"
```

(b)

$$y = C_1 e^{-t} \begin{bmatrix} \sin 2t \\ \cos 2t \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} -\cos 2t \\ \sin 2t \end{bmatrix}$$