

# BIOL 274 Homework 3

Ruby Krasnow

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## Question 1

For each of the following DEs, state the order of the DE and whether it is linear or nonlinear. If it is linear, state whether the coefficients are constant or not:

- (a)  $\frac{dy}{dt} = t^2 - y^2$ . Rewriting as  $\frac{dy}{dt} + y^2 = t^2$ , this is a first order nonlinear ODE.
- (b)  $17y'' = y' + t^2 - 3y$ . Rewriting as  $17y'' - y' + 3y = t^2$ , this is a second order linear ODE with constant coefficients.
- (c)  $(\sin t)y = \frac{d^3y}{dt^3} - \frac{d^2y}{dt^2} + \cos t$ . Rewriting as  $\frac{d^3y}{dt^3} - \frac{d^2y}{dt^2} - (\sin t)y = -\cos t$ , this is a third order linear ODE with non-constant coefficients.
- (d)  $yy' = 14e^t$  is a first order nonlinear ODE.

## Question 2

A certain drug is being administered intravenously to a hospital patient who has had no prior drug treatments. Fluid containing  $5 \text{ mg/cm}^3$  of the drug enters the patient's bloodstream at a rate of  $100 \text{ cm}^3/\text{hr}$ . The drug is absorbed by tissues or otherwise leaves the bloodstream at a rate proportional to the amount present, with a rate constant of  $0.4 (\text{hr})^{-1}$ .

- (a) Assuming that the drug is always uniformly distributed throughout the bloodstream, write a differential equation for the amount of the drug that is present in the bloodstream at any time and state the initial condition.

Let  $y(t)$  be the amount of the drug in the bloodstream at any time  $t$ . Then the rate of change of the drug,  $\frac{dy}{dt}$ , is equal to the rate of the drug entering the bloodstream minus the rate of the drug leaving the bloodstream:

$$\begin{aligned} \frac{dy}{dt} &= \text{rate entering} - \text{rate leaving} = \\ \left(\frac{5 \text{ mg}}{\text{cm}^3}\right) \left(\frac{100 \text{ cm}^3}{\text{hr}}\right) - (y \text{ mg}) \left(\frac{0.4}{\text{hr}}\right) &= 500 - 0.4y \text{ mg hr}^{-1} \end{aligned}$$

So

$$\frac{dy}{dt} = 500 - 0.4y$$

Since the patient has had no previous drug treatments, we can assume that there is no drug in the bloodstream at time  $t = 0$ , i.e., we have an initial condition of  $y(0) = 0$ .

(b) Using the method of integrating factor find the solution to this initial value problem.

$$\begin{aligned}\frac{dy}{dt} &= 500 - 0.4y \quad \longrightarrow \quad \frac{dy}{dt} + 0.4y = 500 \\ h(t) &= e^{\int 0.4 dt} = e^{0.4t} \\ y(t) &= \frac{1}{h(t)} \int h(t)g(t)dt = \\ e^{-0.4t} \int 500e^{0.4t} &= 500e^{-0.4t} \left( \frac{1}{0.4}e^{0.4t} + C_1 \right) = 1250 + C_2e^{-0.4t}\end{aligned}$$

where  $C_2 = 500C_1$ .

$$y(t) = 1250 + C_2e^{-0.4t}$$

Plugging in our initial condition,  $0 = 1250 + C_2e^0 \quad \longrightarrow \quad C_2 = -1250$

$$y(t) = 1250 - 1250e^{-0.4t}$$

Check by differentiating:

$$y'(t) = -0.4(-1250e^{-0.4t}) = 500e^{-0.4t}$$

Now we rewrite our initial DE:

$$\begin{aligned}500 - 0.4y &= \\ 500 - 0.4(1250 - 1250e^{-0.4t}) &= \\ 500 - 500 + 500e^{-0.4t} &= \\ 500e^{-0.4t} &= y'(t)\end{aligned}$$

So we have confirmed that our solution satisfies the original differential equation.

(c) How much of the drug is present in the bloodstream after a long time (i.e., as  $t \rightarrow \infty$ )?

$\lim_{t \rightarrow \infty} e^{-0.4t} = 0$ , which means that  $\lim_{t \rightarrow \infty} y(t) = 1250 + 0 = 1250$ . Thus, 1250 mg of the drug is present in the bloodstream after a long time.

### Question 3

$$\begin{aligned}\frac{dy}{dx} &= \frac{-y}{x} + 2, \quad y(1) = 3 \\ \frac{dy}{dx} + \frac{1}{x}y &= 2 \\ h(x) &= e^{\int \frac{1}{x} dx} = e^{\ln x} = x \\ y(x) &= \frac{1}{h(x)} \int h(x)g(x)dx = \\ y(x) &= \frac{1}{x} \int 2x dx = \frac{1}{x}(x^2 + C) \quad \longrightarrow \quad y(x) = x + \frac{C}{x}\end{aligned}$$

Plugging in our initial conditions, we get  $y(1) = 3 = 1 + C$ , which means  $C = 2$  and our final solution to the IVP is:

$$y(x) = x + \frac{2}{x}$$

Checking our solution by differentiating:

$$y'(x) = 1 - \frac{2}{x^2}$$

If we rewrite our original DE, we can see that it is equivalent:

$$\frac{dy}{dx} = \frac{-y}{x} + 2 = \frac{-x - \frac{2}{x}}{x} + 2 = -1 - \frac{2}{x^2} + 2 = 1 - \frac{2}{x^2}$$

## Question 4

$$\frac{dp}{dt} = \frac{1}{t+1}p + 4t^2 + 4t, \quad p(1) = 10$$

$$\frac{dp}{dt} - \frac{1}{t+1}p = 4t^2 + 4t$$

$$h(t) = e^{\int \frac{-1}{t+1} dx} = e^{-\ln(t+1)} = e^{\ln(\frac{1}{t+1})} = \frac{1}{t+1}$$

$$p(t) = \frac{1}{h(t)} \int h(t)g(t)dt =$$

$$(t+1) \int \frac{4t^2 + 4t}{t+1} dt =$$

$$(t+1) \int \frac{4t(t+1)}{t+1} dt = 4(t+1) \int t dt =$$

$$4(t+1) \left( \frac{t^2}{2} + C_1 \right) = 2t^2(t+1) + C_2(t+1)$$

Where  $C_2 = 4C_1$ . Therefore, our general solution is:

$$p(t) = (t+1)(2t^2 + C_2)$$

Now plugging in our initial conditions  $p(1) = 10$ , we get:

$$10 = (2)(2 + C_2) \quad \longrightarrow \quad 5 = 2 + C_2 \quad \longrightarrow \quad C_2 = 3$$

So our final solution to the IVP is:

$$p(t) = (t+1)(2t^2 + 3)$$

Check by differentiating:

$$p'(t) = (t+1)(4t) + (2t^2 + 3) = 4t^2 + 4t + 2t^2 + 3$$

And since

$$2t^2 + 3 = \frac{(t+1)(2t^2 + 3)}{t+1} = \frac{p(t)}{t+1}$$

we get  $p'(t) = \frac{1}{t+1}p + 4t^2 + 4t$ , which exactly matches our original ODE.

## Question 5

A 400-gallon tank initially contains 200 gallons of water containing 2 parts per billion by weight of dioxin, an extremely potent carcinogen. Suppose water containing 5 parts per billion of dioxin flows into the top of the tank at a rate of 4 gallons per minute. The water in the tank is kept well mixed, and 2 gallons per minute are removed from the bottom of the tank. How much dioxin is in the tank when the tank is full?

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## [1] "1 part per billion = 1 µg/L = 1 µg per 0.2641720524 gal = 3.78541178340029 µg/gal"
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## [1] "5 ppb = 18.9270589170015 µg/gal"
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The volume of water at any time  $t$  is

$$200 \text{ gal} + \left( \frac{4 \text{ gal}}{\text{min}} - \frac{2 \text{ gal}}{\text{min}} \right) (t \text{ min}) = 200 + 2t$$

Let  $y(t)$  be the amount of dioxin in the tank at any time  $t$ .

$$\frac{dy}{dt} = \text{rate entering} - \text{rate leaving} =$$

$$\left( \frac{18.927 \text{ µg}}{\text{gal}} \right) \left( \frac{4 \text{ gal}}{\text{min}} \right) - \left( \frac{y(t) \text{ µg}}{200 + 2t \text{ gal}} \right) \left( \frac{2 \text{ gal}}{\text{min}} \right)$$

$$\frac{dy}{dt} = 4(18.927) - \frac{y}{100 + t} = 75.708 - \frac{y}{100 + t}$$

$$\frac{dy}{dt} + \frac{1}{100 + t}y = 75.708$$

$$h(t) = e^{\int \frac{1}{t+100} dx} = e^{\ln(t+100)} = t + 100$$

$$y(t) = \frac{1}{h(t)} \int h(t)g(t)dt =$$

$$\frac{1}{t + 100} \int 75.708(t + 100)dt = \frac{75.708}{t + 100} \left( \frac{t^2}{2} + 100t + C_1 \right)$$

$$y(t) = \frac{37.854t^2 + 7570.8t + C_2}{t + 100}$$

$$y(0) = \frac{C_2}{100}$$

The starting amount of dioxin, in µg ( $y_0$ ), is:

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200*2*c #200 gallons * 2 ppb * 3.785 (µg/gal)/(1 ppb)
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## [1] 1514.165
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Which means that  $C_2$  is 151416.5 and the solution to the IVP is

$$y(t) = \frac{37.9t^2 + 7570.8t + 151416.5}{t + 100}$$

Check our solution by differentiating:

$$\begin{aligned} y'(t) &= \frac{(t + 100)[(37.9)(2t) + 7570.8] - (37.9t^2 + 7570.8t + 151416.5)}{(t + 100)^2} = \\ &= \frac{75.708t^2 + 7570.8t + 7570.8t + 757082 - (37.9t^2 + 7570.8t + 151416.5)}{(t + 100)^2} = \\ &= \frac{75.708t^2 + 15141.6t + 757082 - (37.9t^2 + 7570.8t + 151416.5)}{(t + 100)^2} \end{aligned}$$

Our original ODE was

$$\begin{aligned} \frac{dy}{dt} &= 75.708 - \frac{y}{100 + t} = \\ 75.708 - \frac{37.9t^2 + 7570.8t + 151416.5}{(t + 100)^2} &= \\ 75.708 \frac{(t + 100)^2}{(t + 100)^2} - \frac{37.9t^2 + 7570.8t + 151416.5}{(t + 100)^2} &= \\ \frac{75.708(t + 100)^2 - (37.9t^2 + 7570.8t + 151416.5)}{(t + 100)^2} &= \\ \frac{75.708t^2 + 15141.6t + 757082 - (37.9t^2 + 7570.8t + 151416.5)}{(t + 100)^2} \end{aligned}$$

Which is exactly what we got by differentiating our solution.

However, the question does not ask for the solution to the IVP, it asks how much dioxin is in the tank when the tank is full. The tank will be full at the time  $t$  such that  $200 + 2t = 400$ , which means the tank is full at  $t = 100$  minutes. Now we can plug  $t = 100$  into our solution:

$$\begin{aligned} y(t) &= \frac{37.9t^2 + 7570.8t + 151416.5}{t + 100} \\ y(100) &= \frac{379000 + 757080 + 151416.5}{200} = 6437.5 \text{ } \mu\text{g} = 6.44 \text{ mg} \end{aligned}$$

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# Using unrounded values gives us the same answer, 6.44 mg
((c*10*100*100)+(c*2*1000*100)+(c*4*10000))/200/1000
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## [1] 6.4352
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# Divide by 400 gal and convert units to ppb
((c*10*100*100)+(c*2*1000*100)+(c*4*10000))/200/(400*c)
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## [1] 4.25
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So, there are approximately 6.44 mg of dioxin in the tank when it is full (containing 400 gallons), corresponding to around 4.25 parts per billion. Given that the initial concentration was 2 parts per billion and the water entering the tank is 5 parts per billion, this seems like a reasonable value.