

MATH 245 Homework 6

Ruby Krasnow and Tommy Thach

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Done: 1-4

Not started: 5-9

Problem 1

The function $u : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ is subharmonic if $\Delta u = u_{xx} + u_{yy} \geq 0$ and superharmonic if $\Delta u = u_{xx} + u_{yy} \leq 0$. Does the maximum and minimum principle hold for subharmonic functions or superharmonic on a connected bounded domain $D \subset \mathbb{R}^2$? If yes, state and prove it, otherwise give a counterexample.

Subharmonic

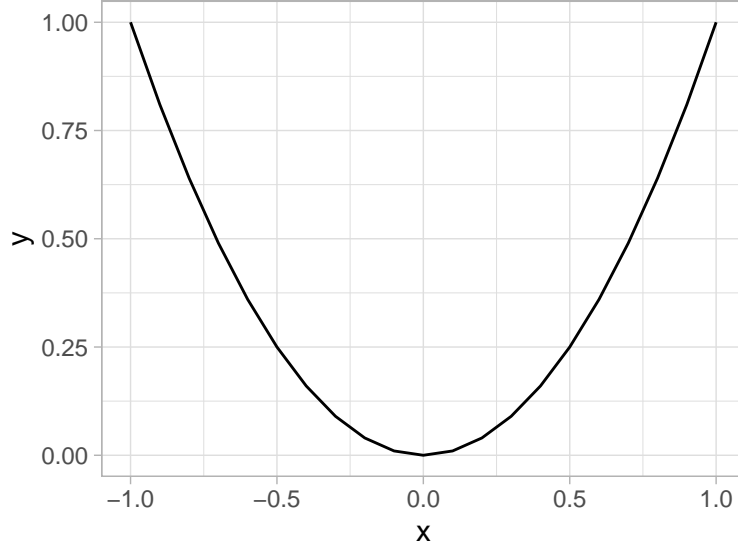
Minimum - No

Take $u(x, y) = x^2$ on the connected bounded domain $D = \{(x, y) : -1 \leq x \leq 1, -1 \leq y \leq 1\}$. Then $\Delta u = 2 > 0$, so u is subharmonic. However, the minimum of u on D occurs at $x = 0$ where $u(0, y) = 0$ and not on the boundary of D , where $u = 1$.

Maximum - Yes

Let D be a bounded domain in \mathbb{R}^2 and $u : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}$. Assume u is continuous on \bar{D} , where $\bar{D} = D \cup \partial D$, and is a continuous subharmonic function such that $\Delta u = u_{xx} + u_{yy} \geq 0$.

Then because D is bounded and u is continuous on \bar{D} , u must attain its maximum and minimum in \bar{D} . Define $M = \max_{\partial D} u = u(x_0)$ on ∂D . Our goal is to show that $\max_D u \leq \max_{\partial D} u = M$.



For every $\varepsilon > 0$, we define $v(x) = u(x) + \varepsilon|x|^2$. Then

$$\begin{aligned}
 \Delta v &= \Delta u + \varepsilon \Delta |x|^2 \\
 &= \Delta u + \varepsilon \sum_{i=1}^n \frac{\partial^2 \Delta |x|^2}{\partial x_i^2} \\
 &= \Delta u + \varepsilon(2 + 2 + \dots + 2) \\
 &= \Delta u + 2n\varepsilon
 \end{aligned}$$

And since u is subharmonic, we have that

$$\Delta v = \Delta u + 2n\varepsilon > 0 \tag{1}$$

Now assume the max of v is inside D , at some $x_1 \in D$. Then by (1), we have $\Delta v(x_1) > 0$. By generalization of the second derivative theorem, we know that at the maximum of a function $v : \mathbb{R}^n \rightarrow \mathbb{R}$, the Hessian of v , $\nabla^2 v(x_1)$, should be negative semi-definite, meaning that all eigenvalues λ_i are less than or equal to zero. But $\lambda_i \leq 0$ implies that

$$\Delta v(x_1) = \text{tr}(\nabla^2 v(x_1)) = (\lambda_1 + \lambda_2 + \dots + \lambda_n)(x_1) \leq 0$$

which is a contradiction of (1). Therefore, the maximum of v is at the boundary,

$$v(x) \leq \max_{\partial D} v \quad \forall x \in D$$

So, $\forall x \in D$,

$$\begin{aligned}
 u(x) &= v(x) - \varepsilon|x|^2 \\
 &\leq \max_{\partial D} v - \varepsilon|x|^2
 \end{aligned}$$

But since we can write

$$\begin{aligned}\max_{\partial D} v - \varepsilon |x|^2 &= \max_{\partial D} (u(x) + \varepsilon |x|^2) - \varepsilon |x|^2 \\ &= \max_{\partial D} u + \varepsilon (R - |x|^2),\end{aligned}$$

where R is the largest radius of the domain, we must have

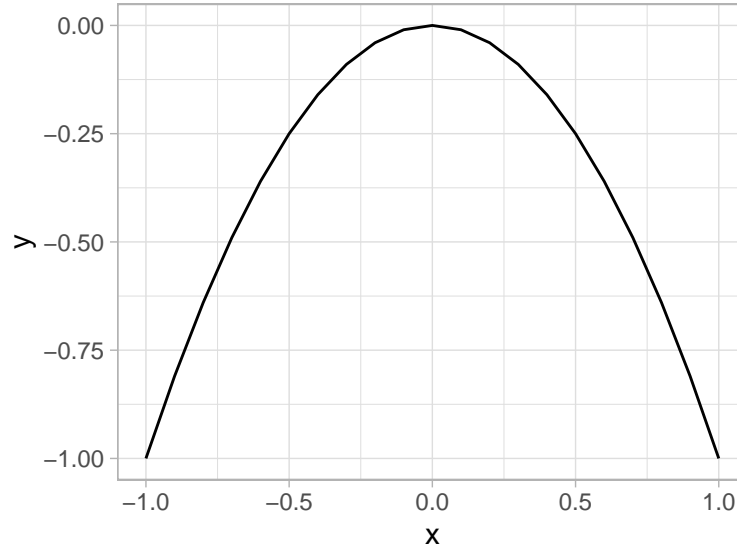
$$u(x) \leq \max_{\partial D} u + \varepsilon (R - |x|^2).$$

Since this is true for any ε , we take $\varepsilon \rightarrow 0$ to find that $u(x) \leq \max_{\partial D} u$. \square

Superharmonic

Maximum - No

Take $u(x, y) = -x^2$ on the connected bounded domain $D = \{(x, y) : -1 \leq x \leq 1, -1 \leq y \leq 1\}$. Then $\Delta u = -2 < 0$, so u is superharmonic. However, the maximum of u on D occurs at $x = 0$ where $u(0, y) = 0$ and not on the boundary of D , where $u = -1$.



Minimum - Yes

Let D be a bounded domain in \mathbb{R}^2 and $u : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}$. Assume u is continuous on \bar{D} , where $\bar{D} = D \cup \partial D$, and is a continuous superharmonic function such that $\Delta u = u_{xx} + u_{yy} \leq 0$.

Then because D is bounded and u is continuous on \bar{D} , u must attain its maximum and minimum in \bar{D} . Define $m = \min_{\partial D} u = u(x_0)$ on ∂D . Our goal is to show that $\min_D u \geq$

$\min_{\partial D} u = m$. For every $\varepsilon > 0$, we define $v(x) = u(x) - \varepsilon|x|^2$. Then

$$\begin{aligned}\Delta v &= \Delta u - \varepsilon \Delta |x|^2 \\ &= \Delta u - \varepsilon \sum_{i=1}^n \frac{\partial^2 \Delta |x|^2}{\partial x_i^2} \\ &= \Delta u - \varepsilon(2 + 2 + \dots + 2) \\ &= \Delta u - 2n\varepsilon\end{aligned}$$

And since u is superharmonic, we have that

$$\Delta v = \Delta u - 2n\varepsilon < 0 \tag{2}$$

Now assume the min of v is inside D , at some $x_1 \in D$. Then by (2), we have $\Delta v(x_1) < 0$. By generalization of the second derivative theorem, we know that at the minimum of a function $v : \mathbb{R}^n \rightarrow \mathbb{R}$, the Hessian of v , $\nabla^2 v(x_1)$, should be positive semi-definite, meaning that all eigenvalues λ_i are greater than or equal to zero. But $\lambda_i \geq 0$ implies that

$$\Delta v(x_1) = \text{tr}(\nabla^2 v(x_1)) = (\lambda_1 + \lambda_2 + \dots + \lambda_n)(x_1) \geq 0$$

which is a contradiction of (2). Therefore, the minimum of v is at the boundary,

$$v(x) \geq \min_{\partial D} v \quad \forall x \in D$$

So, $\forall x \in D$,

$$\begin{aligned}u(x) &= v(x) + \varepsilon|x|^2 \\ &\leq \min_{\partial D} v + \varepsilon|x|^2\end{aligned}$$

But since we can write

$$\begin{aligned}\min_{\partial D} v + \varepsilon|x|^2 &= \min_{\partial D} (u(x) - \varepsilon|x|^2) + \varepsilon|x|^2 \\ &= \min_{\partial D} u + \varepsilon(|x|^2 - R),\end{aligned}$$

where R is the largest radius of the domain, we must have

$$u(x) \leq \min_{\partial D} u + \varepsilon(|x|^2 - R).$$

Since this is true for any ε , we take $\varepsilon \rightarrow 0$ to find that $u(x) \geq \min_{\partial D} u$. \square

Problem 2

Suppose u is harmonic on the disk $D = \{(x, y) : x^2 + y^2 < 4\}$ and $u = \sin(\theta) + 1$ on ∂D . Without finding the solution u , find the maximum value of u in $D \cup \partial D$ and value of u at the origin.

By the maximum principle, u obtains its maximum in $D \cup \partial D$ on the boundary ∂D and not inside. The maximum of $u = \sin \theta + 1$ on ∂D is at $\theta = \frac{\pi}{2}$, where $\sin \theta$ is at its maximum. Therefore, the maximum value of u in $D \cup \partial D$ is $u = \sin(\frac{\pi}{2}) + 1 = 1 + 1 = 2$.

The mean value property says that for a harmonic function u in a disk D , continuous in its closure \bar{D} , the value of u at the center of D equals the average of u on its circumference. The average value of a continuous function $f(x)$ over the interval $[a, b]$ is given by

$$f_{avg}(x) = \frac{1}{b-a} \int_a^b f(x) dx$$

Here,

$$\begin{aligned} u_{avg}(\theta) &= \frac{1}{2\pi} \int_0^{2\pi} (\sin \theta + 1) d\theta \\ &= \frac{1}{2\pi} \left[(\cos \theta + \theta) \right]_0^{2\pi} \\ &= \frac{1}{2\pi} \left[(\cos 2\pi + 2\pi) - \cos 0 \right] \\ &= \frac{1}{2\pi} (1 + 2\pi - 1) = 1 \end{aligned}$$

Therefore, the value of u at the origin is 1.

Problem 3

Use the maximum principle to show that the solution of the Dirichlet problem for $\Delta u = 0$ depends continuously on the boundary data.

Define u_1 and v to be two solutions to the Dirichlet problems

$$\begin{cases} \Delta u_1 = 0 & \text{in } D \\ u_1 = h(x) & \text{on } \partial D \end{cases}$$

$$\begin{cases} \Delta u_2 = 0 & \text{in } D \\ u_2 = h(x) + \varepsilon(x) & \text{on } \partial D \end{cases}$$

where $\varepsilon(x)$ is small $\forall x$ on ∂D . Now let $w = u_1 - u_2$. Then w solves the following Dirichlet problem:

$$\begin{cases} \Delta w = 0 & \text{in } D \\ w = \varepsilon(x) & \text{on } \partial D \end{cases}$$

By the maximum and minimum principles,

$$\min_{\partial D} \varepsilon(x) \leq w \leq \max_{\partial D} \varepsilon(x)$$

If $\varepsilon \rightarrow 0$ for all x on ∂D , then $w = u_1 - u_2$ approaches zero for all $x \in D$.

Problem 4

Solve the following problem on semi-annulus domain:

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 & 0 < a < r < b, 0 < \theta < \pi \\ u(r, 0) = u(r, \pi) = 0 & 0 < a < r < b \\ u_r(a, \theta) = \theta & 0 < \theta < \pi \\ u_r(b, \theta) = 0 & 0 < \theta < \pi \end{cases} \quad (3)$$

We look for a separated solution of the form $u = \Theta(\theta)R(r)$. First, we rewrite $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ as

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

Separating variables, we get

$$\frac{R''}{R}r^2 + \frac{R'}{R}r = -\frac{\Theta''}{\Theta}$$

Since the left-hand side is a function of only R and the right-hand side is a function of only Θ , we can set both sides equal to a constant λ and separate this into two problems, one in Θ and one in R . Start with θ :

$$\Theta'' + \Theta\lambda = 0$$

From our boundary conditions,

$$R(r)\Theta(0) = R(r)\Theta(\pi) = 0 \implies \Theta(\pi) = \Theta(0) = 0$$

So we have an eigenvalue problem with homogenous Dirichlet BCs, which we have shown to have the eigenvalues and eigenvectors

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \sin\left(\frac{n\pi x}{l}\right), \quad n = 1, 2, 3, \dots$$

Here, that means

$$\lambda_n = n^2, \quad X_n(x) = \sin(n\theta), \quad n = 1, 2, 3, \dots$$

Now onto the R problem:

$$\begin{aligned} \frac{R''}{R}r^2 + \frac{R'}{R}r &= \lambda \\ R''r^2 + R'r - \lambda R &= 0 \end{aligned}$$

This is an Euler ODE of the form $ax^2y'' + bxy' + cy = 0$, which has the characteristic equation $am(m-1) + bm + c = 0$ and, if m_1, m_2 are distinct real roots, the general solution is $y = Cx^{m_1} + Dx^{m_2}$. (Note that I have used m instead of the standard r to avoid confusion with the variable r from (3).

Here, $a = 1, b = 1, c = -\lambda$, so the characteristic equation becomes $m(m-1) + m - \lambda = 0$, which simplifies to $m^2 - \lambda = 0$. Thus, our roots are $\pm\sqrt{\lambda} = \pm n$. Thus, $R(r) = Cr^n + Dr^{-n}$.

Our separated solutions thus take the form

$$u_n(r, \theta) = (Cr^n + Dr^{-n}) \sin(n\theta), \quad n = 1, 2, 3, \dots$$

$$u(r, \theta) = \sum_{n=1}^{\infty} (Cr^n + Dr^{-n}) \sin(n\theta)$$

Now we use our boundary conditions $u_r(a, \theta) = \theta$ and $u_r(b, \theta) = 0$:

$$u_r(r, \theta) = \sum_{n=1}^{\infty} (nCn^{n-1} - nDr^{-n-1}) \sin(n\theta)$$

$$u_r(b, \theta) = 0 = \sum_{n=1}^{\infty} (nCn^{n-1} - nDb^{-n-1}) \sin(n\theta)$$

$$Cb^n = Db^{-n} \rightarrow C = Db^{-2n}$$

$$u_r(a, \theta) = \theta = \sum_{n=1}^{\infty} (nDb^{-2n}a^{n-1} - nDa^{-n-1}) \sin(n\theta)$$

$$\frac{nD}{a} (b^{-2n}a^n - a^{-n}) = \int_0^\pi \theta \sin(n\theta) d\theta$$

$$\int_0^\pi \theta \sin(n\theta) d\theta = \theta(-1/n \cos n\theta) \Big|_0^\pi = \pi(-1/n)(-1)^n = \frac{\pi(-1)^{n+1}}{n}$$

$$\frac{nD}{a} (b^{-2n}a^n - a^{-n}) = \frac{\pi(-1)^{n+1}}{n}$$

Therefore, our solution to (3) is

Solution

$$u(r, \theta) = \sum_{n=1}^{\infty} (Cr^n + Dr^{-n}) \sin(n\theta)$$

where

$$D = \frac{a\pi(-1)^{n+1}}{n^2(b^{-2n}a^n - a^{-n})}$$
$$C = Db^{-2n}$$

Problem 5

Consider the Neumann problem with periodic BC:

$$\begin{cases} \Delta u = 0 & r^2 > a^2 \\ u_r(a, \theta) = g(\theta) & r^2 = a^2 \end{cases} \quad (4)$$

Find the necessary condition for the existence of a solution and then solve the problem. Hint: Use Green's identity

$$\iint_D (u\Delta v - v\Delta u) dA = \int_{\partial D} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) ds$$

where D is a domain and n is a unit normal vector. Note that for a 2-dimensional domain D , $dA = dx dy$ and ds is arc length.