# MATH 245 Homework 5

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## Question 5: Radioactive Decay Problem

$$f(x) = \begin{cases} u_t - u_{xx} = Ae^{-ax} \\ u(0,x) = \sin x \\ u(t,0) = u(t,\pi) = 0 \end{cases} \tag{1}$$

We want to find a separated solution of the form $u(t,x) = X(x)T(t)$. Recall that for the analogous homogeneous PDE with homogeneous Dirichlet boundary conditions, we consider the following eigenvalue problem $X'' + \lambda X = 0, \quad X(0) = X(l) = 0$, which we have shown to have the eigenvalues and eigenvectors

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \sin\left(\frac{n\pi x}{l}\right), \quad n = 1, 2, 3...$$

Giving us the Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \tag{2}$$

where

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Now, we assume that our solution to (1) will take a similar form as (2), where $l = \pi$ and $f(t,x) = Ae^{-ax}$:

$$u(t,x) = b_0(t) + \sum_{n=1}^{\infty} b_n(t)\sin(nx)$$

We can differentiate this and plug it into (1) as follows:

$$u_t(t,x) = b_0'(t) + \sum_{n=1}^{\infty} b_n'(t)\sin(nx)$$

$$u_{xx}(t,x) = -\sum_{n=1}^{\infty} b_n(t)\, n^2 \sin(nx)$$

$$b_0'(t) + \sum_{n=1}^{\infty} b_n'(t)\sin(nx) + \sum_{n=1}^{\infty} b_n(t)\, n^2\sin(nx) = Ae^{-ax} \tag{3}$$

For each fixed $t$, we write $Ae^{-ax}$ as a Fourier sine series:

$$Ae^{-ax} = q_0(t) + \sum_{n=1}^{\infty} q_n(t)\sin(nx) \tag{4}$$

where

$$\begin{aligned}
q_0(t) &= \frac{1}{l}\int_0^l f(t,x)dx \\
&= \frac{A}{\pi}\int_0^{\pi} e^{-ax}dx \\
&= \frac{-A}{a\pi}\left[e^{-ax}\right]_0^{pi} \\
&= \frac{-A}{a\pi}\left(e^{-a\pi}-1\right)
\end{aligned}$$

$$\begin{aligned}
q_n(t) &= \frac{2}{l}\int_0^l f(t,x)\sin\left(\frac{n\pi x}{l}\right)dx \\
&= \frac{2A}{\pi}\int_0^{\pi} e^{-ax}\sin(nx)dx
\end{aligned}$$

Now we do integration by parts twice on $\int_0^{\pi} e^{-ax}\sin(nx)dx$

First with $u=\sin(nx),\quad du = n\cos(nx)\ dx,\quad dv = e^{-ax}\ dx,\quad v = \frac{-1}{a}e^{-ax}$,

and the second time with $u = n\cos(nx),\quad du = -n^2\sin(nx)\ dx,\quad dv = \frac{1}{a}e^{-ax}\ dx,\quad v = \frac{-1}{a^2}e^{-ax}$

$$\begin{aligned}
\int_0^{\pi} e^{-ax}\sin(nx)dx &= \left[\frac{-\sin(nx)}{a}e^{-ax}\right]_0^{\pi} + \frac{n}{a}\int_0^{\pi} e^{-ax}\cos\left(\frac{n\pi x}{l}\right)dx \\
&= 0-0+\frac{n}{a}\int_0^{\pi} e^{-ax}\cos(nx)dx \\
&= \left[\frac{-n\cos(nx)}{a^2}e^{-ax}\right]_0^{\pi} - \frac{n^2}{a^2}\int_0^{\pi} e^{-ax}\sin(nx)dx
\end{aligned}$$

Then moving the integrals to the same side,

$$\begin{aligned}
\left(1+\frac{n^2}{a^2}\right)\int_0^{\pi} e^{-ax}\sin(nx)dx &= \left[\frac{-n\cos(nx)}{a^2}e^{-ax}\right]_0^{\pi} \\
&= \frac{-n\cos(n\pi)}{a^2}e^{-a\pi} + \frac{n}{a^2} \\
(n^2+a^2)\int_0^{\pi} e^{-ax}\sin(nx)dx &= n(-1)^{n+1}e^{-a\pi}+n
\end{aligned}$$

Thus,

$$\int_0^{\pi} e^{-ax}\sin(nx)dx = \frac{n(-1)^{n+1}e^{-a\pi}+n}{n^2+a^2}$$

Which means that

$$q_n = \frac{2A}{\pi}\frac{n(-1)^{n+1}e^{-a\pi}+n}{n^2+a^2}$$

Now, by (3) and (4), we get the following equations:

$$\begin{cases} b_0'(t) = q_0(t) \\ b_n'(t) + b_n(t)n^2 = q_n(t) \end{cases}$$

From $b_0'(t) = q_0(t)$ we get $b_0(t) = \int_0^t q_0(s)\, ds + b_0(0)$, where $b_0(0)$ is our integration constant.

On the other hand, we have $b_n'(t) + b_n(t)n^2 = q_n(t)$, which we solve as follows:

$$\mu(t) = \exp\left(\int_0^t n^2\, ds\right) = \exp\left(n^2 t\right)$$

$$b_n(t) = \frac{1}{\mu(t)}\left[\int_0^t \mu(s) q_n(s)\, ds + b_n(0)\right]$$

$$b_n(t) = b_n(0)\, \mu(t)^{-1} + \int_0^t \frac{\mu(s)}{\mu(t)} q_n(s)\, ds$$

$$b_n(t) = e^{-n^2} b_n(0) + \int_0^t \frac{\exp\left(n^2 s\right)}{\exp\left(n^2 t\right)} q_n(s)\, ds$$

$$b_n(t) = e^{-n^2} b_n(0) + \int_0^t e^{n^2(s-t)} q_n(s)\, ds$$

Therefore, our solution to (1) is:

$$u(t,x) = b_0(t) + \sum_{n=1}^{\infty} b_n(t) \sin(nx), \quad \text{where} b_n(t) = e^{-n^2} b_n(0) + \int_0^t e^{n^2(s-t)} q_n\, ds, \quad q_n = \frac{2A}{\pi} \frac{n(-1)^{n+1} e^{-a\pi} + n}{n^2 + a^2}, q_0 = \frac{-A}{a\pi}\left(e^{-a\pi}\right.$$

$$u(0,x) = b_0(0) = \sin(x)$$