MATH 245 Homework 6

Ruby Krasnow and Tommy Thach

2024-04-12

Done: 1

Not started: 2-9

Problem 1

The function $u : D \subset \mathbb{R}^2 \to \mathbb{R}$ is subharmonic if $\Delta u = u_{xx} + u_{yy} \geq 0$ and superharmonic if $\Delta u = u_{xx} + u_{yy} \leq 0$. Does the maximum and minimum principle hold for subharmonic functions or superharmonic on a connected bounded domain $D \subset \mathbb{R}^2$? If yes, state and prove it, otherwise give a counterexample.

Subharmonic

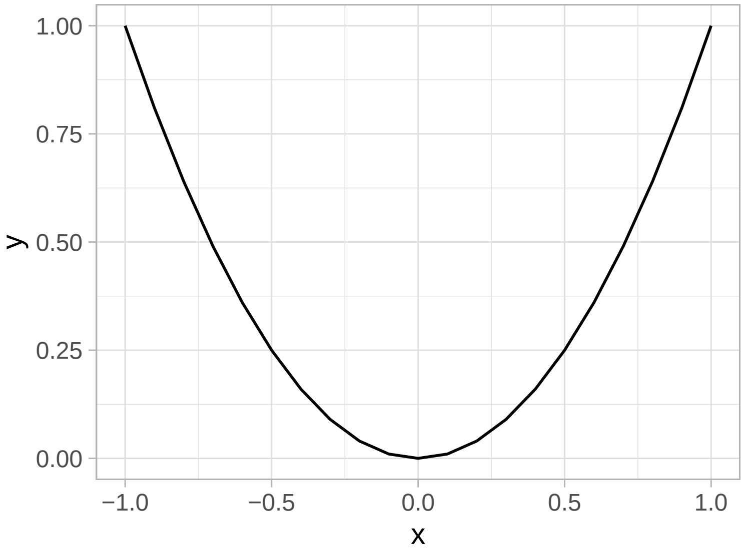
Minimum - No

Take $u(x, y) = x^2$ on the connected bounded domain $D = \{(x, y) : -1 \leq x \leq 1, -1 \leq y \leq 1\}$. Then $\Delta u = 2 > 0$, so u is subharmonic. However, the minimum of u on D occurs at $x = 0$ where $u(0, y) = 0$ and not on the boundary of D, where $u = 1$.

Maximum - Yes

Let D be a bounded domain in $\mathbb{R}^2$ and $u : D \subset \mathbb{R}^2 \to \mathbb{R}$. Assume u is continuous on $\bar{D}$, where $\bar{D} = D \cup \partial D$, and is a continuous subharmonic function such that $\Delta u = u_{xx} + u_{yy} \geq 0$.

Then because D is bounded and u is continuous on $\bar{D}$, u must attain its maximum and minimum in $\bar{D}$. Define $M = \max_{\partial D} u = u(x_0)$ on ∂D. Our goal is to show that $\max_D u \leq \max_{\partial D} u = M$.



For every $\varepsilon > 0$, we define $v(x) = u(x) + \varepsilon |x|^2$. Then

$$\begin{aligned}
\Delta v &= \Delta u + \varepsilon \Delta |x|^2 \\
&= \Delta u + \varepsilon \sum_{i=1}^{n} \frac{\partial^2 \Delta |x|^2}{\partial x_i^2} \\
&= \Delta u + \varepsilon (2 + 2 + \dots + 2) \\
&= \Delta u + 2n\varepsilon
\end{aligned}$$

And since u is subharmonic, we have that

$$\Delta v = \Delta u + 2n\varepsilon > 0 \tag{1}$$

Now assume the max of v is inside D, at some $x_1 \in D$. Then by (1), we have $\Delta v(x_1) > 0$. By generalization of the second derivative theorem, we know that at the maximum of a function $v : \mathbb{R}^n \to \mathbb{R}$, the Hessian of v, $\nabla^2 v(x_1)$, should be negative semi-definite, meaning that all eigenvalues λ_i are less than or equal to zero. But $\lambda_i \leq 0$ implies that

$$\Delta v(x_1) = \operatorname{tr}\left(\nabla^2 v(x_1)\right) = (\lambda_1 + \lambda_2 + \dots + \lambda_n)(x_1) \leq 0$$

which is a contradiction of (1). Therefore, the maximum of v is at the boundary,

$$v(x) \leq \max_{\partial D} v \quad \forall\, x \in D$$

So, $\forall\, x \in D$,

$$\begin{aligned}
u(x) &= v(x) - \varepsilon |x|^2 \\
&\leq \max_{\partial D} v - \varepsilon |x|^2
\end{aligned}$$

But since we can write

$$\begin{aligned}\max_{\partial D} v - \varepsilon|x|^2 &= \max_{\partial D} \left(u(x) + \varepsilon|x|^2\right) - \varepsilon|x|^2 \\ &= \max_{\partial D} u + \varepsilon\left(R - |x|^2\right),\end{aligned}$$

where R is the largest radius of the domain, we must have

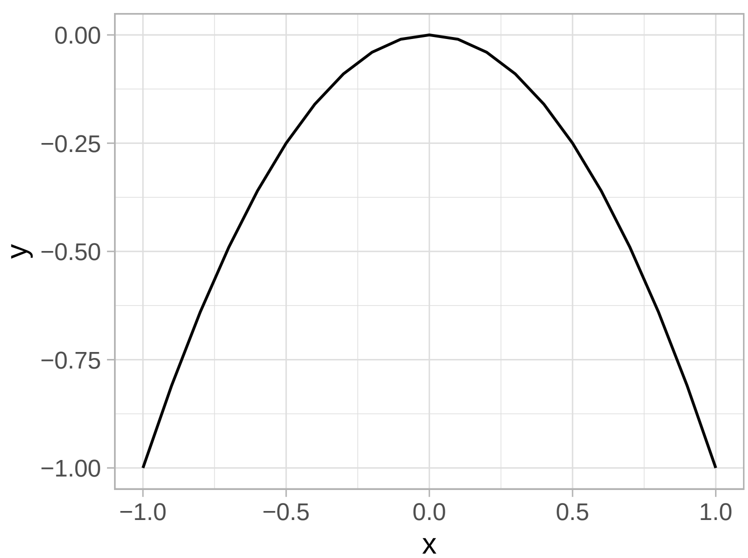
$$u(x) \leq \max_{\partial D} u + \varepsilon\left(R - |x|^2\right).$$

Since this is true for any ε, we take $\varepsilon \to 0$ to find that $u(x) \leq \max_{\partial D} u$. $\square$

Superharmonic

Maximum - No

Take $u(x,y) = -x^2$ on the connected bounded domain $D = \{(x,y) : -1 \leq x \leq 1, -1 \leq y \leq 1\}$. Then $\Delta u = -2 < 0$, so u is superharmonic. However, the maximum of u on D occurs at $x = 0$ where $u(0,y) = 0$ and not on the boundary of D, where $u = -1$.



Minimum - Yes

Let D be a bounded domain in $\mathbb{R}^2$ and $u : D \subset \mathbb{R}^2 \to \mathbb{R}$. Assume u is continuous on $\bar{D}$, where $\bar{D} = D \cup \partial D$, and is a continuous superharmonic function such that $\Delta u = u_{xx} + u_{yy} \leq 0$.

Then because D is bounded and u is continuous on $\bar{D}$, u must attain its maximum and minimum in $\bar{D}$. Define $m = \min_{\partial D} u = u(x_0)$ on ∂D. Our goal is to show that $\min_D u \geq$

$\min_{\partial D} u = m$. For every $\varepsilon > 0$, we define $v(x) = u(x) - \varepsilon|x|^2$. Then

$$\begin{aligned}
\Delta v &= \Delta u - \varepsilon \Delta |x|^2 \\
&= \Delta u - \varepsilon \sum_{i=1}^{n} \frac{\partial^2 \Delta |x|^2}{\partial x_i^2} \\
&= \Delta u - \varepsilon (2 + 2 + ... + 2) \\
&= \Delta u - 2n\varepsilon
\end{aligned}$$

And since u is superharmonic, we have that

$$\Delta v = \Delta u - 2n\varepsilon < 0 \tag{2}$$

Now assume the min of v is inside D, at some $x_1 \in D$. Then by (2), we have $\Delta v(x_1) < 0$. By generalization of the second derivative theorem, we know that at the minimum of a function $v : \mathbb{R}^n \to \mathbb{R}$, the Hessian of v, $\nabla^2 v(x_1)$, should be positive semi-definite, meaning that all eigenvalues λ_i are greater than or equal to zero. But $\lambda_i \geq 0$ implies that

$$\Delta v(x_1) = \operatorname{tr}\left(\nabla^2 v(x_1)\right) = (\lambda_1 + \lambda_2 + ... + \lambda_n)(x_1) \geq 0$$

which is a contradiction of (2). Therefore, the minimum of v is at the boundary,

$$v(x) \geq \min_{\partial D} v \quad \forall\ x \in D$$

So, $\forall\ x \in D$,

$$\begin{aligned}
u(x) &= v(x) + \varepsilon |x|^2 \\
&\leq \min_{\partial D} v + \varepsilon |x|^2
\end{aligned}$$

But since we can write

$$\begin{aligned}
\min_{\partial D} v + \varepsilon |x|^2 &= \min_{\partial D} \left(u(x) - \varepsilon |x|^2\right) + \varepsilon |x|^2 \\
&= \min_{\partial D} u + \varepsilon \left(|x|^2 - R\right),
\end{aligned}$$

where R is the largest radius of the domain, we must have

$$u(x) \leq \min_{\partial D} u + \varepsilon \left(|x|^2 - R\right).$$

Since this is true for any ε, we take $\varepsilon \to 0$ to find that $u(x) \geq \min_{\partial D} u$. $\square$

Problem 2

Suppose u is harmonic on the disk $D = \{(x, y) : x^2 + y^2 < 4\}$ and $u = \sin(\theta) + 1$ on ∂D. Without finding the solution u, find the maximum value of u in $D \cup \partial D$ and value of u at the origin.

By the maximum principle, u obtains its maximum in $D \cup \partial D$ on the boundary ∂D and not inside. The maximum of $u = \sin\theta + 1$ on ∂D is at $\theta = \frac{\pi}{2}$, where $\sin\theta$ is at its maximum. Therefore, the maximum value of u in $D \cup \partial D$ is $u = \sin\left(\frac{\pi}{2}\right) + 1 = 1 + 1 = 2$.

The mean value property says that for a harmonic function u in a disk D, continuous in its closure $\bar{D}$, the value of u at the center of D equals the average of u on its circumference. The average value of a continuous function $f(x)$ over the interval $[a, b]$ is given by

$$f_{avg}(x) = \frac{1}{b-a} \int_a^b f(x)\, dx$$

Here,

$$\begin{aligned} u_{avg}(\theta) &= \frac{1}{2\pi} \int_0^{2\pi} (\sin\theta + 1)\, d\theta \\ &= \frac{1}{2\pi} \Big[(\cos\theta + \theta)\Big]_0^{2\pi} \\ &= \frac{1}{2\pi} \Big[(\cos 2\pi + 2\pi) - \cos 0\Big] \\ &= \frac{1}{2\pi}(1 + 2\pi - 1) = 1 \end{aligned}$$

Therefore, the value of u at the origin is 1.