

# BIOL 274 Homework 4

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## Question 1

Solve each of the following IVPs:

(a)

$$\begin{cases} y'' + 6y' + 8y = 0 \\ y(0) = 3 \\ y'(0) = -16 \end{cases} \quad (1)$$

The characteristic equation is

$$\lambda^2 + 6\lambda + 8 = 0 \quad \longrightarrow \quad (\lambda + 4)(\lambda + 2) = 0$$

Which means we have two distinct real roots,  $\lambda_1 = -4$  and  $\lambda_2 = -2$ .

Thus, our general solution is

$$y = C_1 e^{-4x} + C_2 e^{-2x}$$

We differentiate to find that

$$y' = -4C_1 e^{-4x} - 2C_2 e^{-2x}$$

Now we plug in our initial conditions:

$$\begin{aligned} 3 &= C_1 + C_2 \\ -16 &= -4C_1 - 2C_2 \end{aligned}$$

Multiply both sides of the first equation by 2 and add to the second equation:

$-10 = -2C_1 \quad \longrightarrow \quad C_1 = 5$ , so  $C_2 = 3 - 5 = -2$ . Which means the solution to the IVP is

Solution

$$y(x) = 5e^{-4x} - 2e^{-2x}$$

Check solution

$$y'(x) = -20e^{-4x} + 4e^{-2x}$$

$$y''(x) = 80e^{-4x} - 8e^{-2x}$$

$$y'' + 6y' + 8y = (80 - 120 + 40)e^{-4x} + (-8 + 24 - 16)e^{-2x} = 0$$

(b)

$$\begin{cases} y'' + 6y' + 9y = 0 \\ y(0) = \frac{1}{3} \\ y'(0) = \frac{1}{2} \end{cases} \quad (2)$$

The characteristic equation is

$$\lambda^2 + 6\lambda + 9 = 0 \quad \longrightarrow \quad (\lambda + 3)(\lambda + 3) = 0$$

So we have a repeated real root,  $\lambda_1 = \lambda_2 = -3$ . Thus, our general solution is

$$y = C_1 e^{-3x} + C_2 x e^{-3x}$$

We differentiate to find that

$$y' = -3C_1 e^{-3x} + C_2 e^{-3x} - 3C_2 x e^{-3x}$$

Now we plug in our initial conditions:

$$\frac{1}{3} = C_1$$

$$\frac{1}{2} = -3C_1 + C_2 \quad \longrightarrow \quad C_2 = \frac{3}{2}$$

Which means the solution to the IVP is

Solution

$$y(x) = \frac{1}{3} e^{-3x} + \frac{3}{2} x e^{-3x}$$

Check solution

$$y'(x) = -e^{-3x} - \frac{9}{2} x e^{-3x} + \frac{3}{2} e^{-3x} = \frac{1}{2} e^{-3x} - \frac{9}{2} x e^{-3x}$$

$$y''(x) = -\frac{3}{2} e^{-3x} + \frac{27}{2} x e^{-3x} - \frac{9}{2} e^{-3x} = \frac{27}{2} x e^{-3x} - 6e^{-3x}$$

Plugging these expressions into our original ODE, we see that

$$y'' + 6y' + 9y = \left( \frac{27}{2} - \frac{54}{2} + \frac{27}{2} \right) x e^{-3x} + (-6 + 3 + 3) e^{-3x} = 0$$

## Question 2

Consider the equation

$$\frac{d^2 y}{dt^2} + \frac{k}{m} y = 0$$

for the motion of a single (undamped) harmonic oscillator.

(a) Consider the function  $y(t) = \cos(\beta t)$ . Under what conditions on  $\beta$  is  $y(t)$  a solution?

$$\lambda^2 + \frac{k}{m} = 0$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{0 \pm \sqrt{0 - 4\frac{k}{m}}}{2}$$

$$0 \pm \sqrt{\frac{k}{m}} i$$

$y(t) = \cos(\beta t)$  is only a solution if  $\beta = \sqrt{\frac{k}{m}}$ .

$$y(t) = \cos\left(\sqrt{\frac{k}{m}}t\right) \quad y'(t) = -\sqrt{\frac{k}{m}} \sin\left(\sqrt{\frac{k}{m}}t\right) \quad y''(t) = -\frac{k}{m} \cos\left(\sqrt{\frac{k}{m}}t\right)$$

$$\text{Then } y'' + \frac{k}{m}y = -\frac{k}{m} \cos\left(\sqrt{\frac{k}{m}}t\right) + \frac{k}{m} \cos\left(\sqrt{\frac{k}{m}}t\right) = 0$$

### Question 3

Find the general solution for each of the following.

(a)

$$y'' + 3y' + 2y = 4t^2 - 2t \tag{3}$$

The characteristic equation is

$$\lambda^2 + 3\lambda + 2 = 0 \quad \longrightarrow \quad (\lambda + 2)(\lambda + 1) = 0$$

Which means we have two distinct real roots,  $\lambda_1 = -2$  and  $\lambda_2 = -1$ .

Thus, our complementary function (solution) is

$$y_c = C_1 e^{-2t} + C_2 e^{-t}$$

Let's assume our solution takes the form  $y = At^2 + Bt + C$ . Then  $y' = 2At + B$  and  $y'' = 2A$ , so the left-hand side of 3 becomes

$$2A + 6At + 3B + 2At^2 + 2Bt + 2C = 4t^2 - 2t$$

We can split this up by coefficients of  $t$  into

$$2At^2 = 4t^2$$

$$6At + 2Bt = -2t$$

$$2A + 3B + 2C = 0$$

Which means that  $A = 2$ ,  $B = -7$ ,  $C = \frac{17}{2}$  and so

$$y_p = 2t^2 - 7t + \frac{17}{2}$$

Combine  $y_p$  with  $y_c$  to find our general solution:

Solution

$$y(t) = C_1 e^{-2t} + C_2 e^{-t} + 2t^2 - 7t + \frac{17}{2}$$

Check solution

$$y'(t) = -2C_1 e^{-2t} - C_2 e^{-t} + 4t - 7$$

$$y''(t) = 4C_1 e^{-2t} + C_2 e^{-t} + 4$$

$$y'' + 3y' + 2y = (4C_1 - 6C_1 + 2C_1)e^{-2t} + (C_2 - 3C_1 + 2C_1)e^{-t} + 4 + 12t - 21 + 4t^2 - 14t + 17 = 4t^2 - 2t$$

(b)

$$y'' - y' - 2y = 15e^{2t} \quad (4)$$

The characteristic equation is

$$\lambda^2 - \lambda - 2 = 0 \quad \longrightarrow \quad (\lambda - 2)(\lambda + 1) = 0$$

Which means we have two distinct real roots,  $\lambda_1 = 2$  and  $\lambda_1 = -1$ .

Thus, our complementary function (solution) is

$$y_c = C_1 e^{2t} + C_2 e^{-t}$$

Let's first assume our solution takes the form  $y = Ae^{2t}$ . Then  $y' = 2Ae^{2t}$  and  $y'' = 4Ae^{2t}$ , so the left-hand side of 4 becomes

$$4Ae^{2t} - 2Ae^{2t} - 2Ae^{2t} = 0 \neq 15e^{2t}$$

Oh no! What happened? Well,  $e^{2t}$  satisfies the associated homogenous DE and so is part of the complementary solution. Instead, we will try  $y = Ate^{2t}$ . Then  $y' = 2Ate^{2t} + Ae^{2t}$  and  $y'' = 4Ate^{2t} + 4Ae^{2t}$ , so the left-hand side of 4 becomes

$$y'' - y' - 2y = (4A - 2A - 2A)te^{2t} + (4A - A)e^{2t}$$

Which gives us  $3Ae^{2t} = 15e^{2t}$ , meaning  $A = 5$  and we have

$$y_p = 5te^{2t}$$

Combine  $y_p$  with  $y_c$  to find our general solution:

Solution

$$y(t) = C_1 e^{2t} + C_2 e^{-t} + 5te^{2t}$$

or

$$y(t) = (C_1 + 5t)e^{2t} + C_2 e^{-t}$$

Check solution

$$y'(t) = (2C_1 + 10t + 5)e^{2t} - C_2e^{-t}$$

$$y''(t) = (4C_1 + 20t + 20)e^{2t} + C_2e^{-t}$$

$$y'' - y' - 2y = (4C_1 + 20t + 20 - 2C_1 - 10t - 5 - 2C_1 - 10t)e^{2t} + (C_2 + C_2 - 2C_2)e^{-t} = 15e^{2t}$$

(c)

$$y'' - 2y' - 3y = \cos(2t) \quad (5)$$

The characteristic equation is

$$\lambda^2 - 2\lambda - 3 = 0 \quad \longrightarrow \quad (\lambda - 3)(\lambda + 1) = 0$$

Which means we have two distinct real roots,  $\lambda_1 = 3$  and  $\lambda_2 = -1$ .

Thus, our complementary function (solution) is

$$y_c = C_1e^{3t} + C_2e^{-t}$$

Let's assume our solution takes the form  $y = A \sin(2t) + B \cos(2t)$ . Then

$$y' = 2A \cos(2t) - 2B \sin(2t)$$

$$y'' = -4A \sin(2t) - 4B \cos(2t)$$

So the left-hand side of 5 becomes

$$y'' - 2y' - 3y = (-4A + 4B - 3A) \sin(2t) + (-4B - 4A - 3B) \cos(2t)$$

Compare the coefficients of the sin and cos terms to the original DE:

$$-7A + 4B = 0$$

$$-4A - 7B = 1$$

$$A = \frac{4B}{7}$$

$$-\frac{16B}{7} - \frac{49B}{7} = 1$$

Which gives us that  $B = \frac{-7}{65}$ ,  $A = \frac{-4}{65}$ . Thus, our particular solution is

$$y_p = -\frac{4}{65} \sin(2t) - \frac{7}{65} \cos(2t)$$

Combine  $y_p$  with  $y_c$  to find our general solution:

Solution

$$y(t) = C_1e^{3t} + C_2e^{-t} - \frac{4}{65} \sin(2t) - \frac{7}{65} \cos(2t)$$

Check solution

$$y'(t) = 3C_1 e^{3t} - C_2 e^{-t} + \frac{14}{65} \sin(2t) - \frac{8}{65} \cos(2t)$$

$$y''(t) = 9C_1 e^{3t} + C_2 e^{-t} + \frac{16}{65} \sin(2t) + \frac{28}{65} \cos(2t)$$

$$y'' - 2y' - 3y = (9-6-3)C_1 e^{3t} + (1+2-3)C_2 e^{-t} + \left(\frac{16}{65} - \frac{28}{65} + \frac{12}{65}\right) \sin(2t) + \left(\frac{28}{65} + \frac{16}{65} + \frac{21}{65}\right) \cos(2t) = \cos(2t)$$

(d)

$$y'' + 4y = e^t + \sin(2t) \quad (6)$$

The characteristic equation is

$$\lambda^2 + 4 = 0 \quad \longrightarrow \quad \lambda = 0 \pm 2i$$

Thus, our complementary function (solution) is

$$y_c = C_1 \sin(2t) + C_2 \cos(2t)$$

Let's break up our right hand side to find  $y_{p_1}$  and  $y_{p_2}$ .

For  $y_{p_1}$ , we assume our solution takes the form  $y = Ae^t$ . Then  $Ae^t + 4Ae^t = e^t$ , so  $A = \frac{1}{5}$  and we have

$$y_{p_1} = \frac{e^t}{5}$$

For  $y_{p_2}$ , we cannot assume our solution takes the form  $y = A \sin(2t) + B \cos(2t)$  because we had complex roots  $\pm 2i$  for our characteristic equation. Instead, we try

$$y = At \sin(2t) + Bt \cos(2t)$$

$$y' = 2At \cos(2t) + A \sin(2t) + B \cos(2t) - 2Bt \sin(2t) = (2At + B) \cos(2t) + (-2Bt + A) \sin(2t)$$

$$y'' = (-4At - 2B) \sin(2t) + 2A \cos(2t) + (-4Bt + 2A) \cos(2t) - 2B \sin(2t) =$$

$$(-4At - 4B) \sin(2t) + (-4Bt + 4A) \cos(2t)$$

$$(-4At - 4B + 4At) \sin(2t) + (-4Bt + 4A + 4Bt) \cos(2t) = \sin(2t)$$

$$-4B \sin(2t) + 4A \cos(2t) = \sin(2t)$$

$$A = 0, B = -\frac{1}{4}$$

$$y_{p_2} = -t \frac{\cos(2t)}{4}$$

$$y_p = \frac{e^t}{5} - \frac{t \cos(2t)}{4}$$

Now adding this to our complementary function, we get

Solution

$$y(t) = C_1 \sin(2t) + C_2 \cos(2t) + \frac{e^t}{5} - \frac{t \cos(2t)}{4}$$

Check solution

$$y'(t) = 2C_1 \cos(2t) - 2C_2 \sin(2t) + \frac{e^t}{5} - \frac{\cos(2t)}{4} + \frac{t \sin(2t)}{2}$$

$$y''(t) = -4C_1 \sin(2t) - 4C_2 \cos(2t) + \frac{e^t}{5} + \frac{\sin(2t)}{2} + \frac{\sin(2t)}{2} + t \cos(2t)$$

$$y'' + 4y = (-4C_1 + 1 + 4C_1) \sin(2t) + (-4C_2 + t + 4C_2 - t) \cos(2t) + \frac{e^t}{5} + \frac{4e^t}{5} = e^t + \sin(2t)$$