EXERCISES

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Abstract. These are problems for MATH 482-2, taught in Spring 2018 at Northwestern university. Some of these are simple sanity checks, while others can be quite involved. If you find mistakes or have comments please email me at krishna@math.northwestern.edu.

1. Week 1: Lectures 1-3

(1) I claimed in class that if G is a real connected Lie group, then every continuous homomorphism $\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to G$ has finite image. Prove this.

(2) What are the possible isomorphism classes of finite groups which can occur as subgroups of $\mathrm{GL}_2(\mathbb{C})$? (Hint: $\mathrm{PGL}_2(\mathbb{C})$ has a maximal compact subgroup $\pm 1 \backslash \mathrm{SU}(2)$.) Which of these are solvable?

(3) (Compactness of some arithmetic quotients $\Gamma\backslash\mathbb{H}$.) Recall that we constructed some arithmetic groups $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$, which I claimed were cocompact. Let's work through why that is. Some of this problem may require some additional information about adelic points of semisimple groups and strong approximation, so beware.

Throughout, B will be a quaternion algebra over $\mathbb{Q}$ such that $\mathrm{Ram}(\mathrm{B}) \neq \emptyset$, and so that $\infty \notin \mathrm{Ram}(\mathrm{B})$. We need to show that $\Gamma_{\mathrm{B}} := (\mathscr{O}_{\mathrm{B}})^{\mathrm{Nrd}=1}$ is discrete and cocompact in $\mathrm{SL}_2(\mathbb{R})$. Let $\mathrm{G} = \mathrm{B}^{\mathrm{Nrd}=1}$ denote the appropriate algebraic group over $\mathbb{Q}$ of reduced norm 1 elements; note that $\mathrm{G}(\mathbb{Z}) = \Gamma_{\mathrm{B}}$.

(a) Show that
$$\mathrm{G}(\mathbb{A}) = \prod_{v \in \mathrm{Ram}(\mathrm{B})} (\mathrm{B}_v)^1 \times \prod_{v \notin \mathrm{Ram}(\mathrm{B})}{}' \, \mathrm{SL}_2(\mathbb{Q}_v)$$
where $\prod'$ means restricted direct product.

(b) Show that if $v \in \mathrm{Ram}(\mathrm{B})$, then B_v^1 is a compact group. (Hint: $\mathrm{B}_v^1 \to \mathrm{PB}_v^\times$, where $\mathrm{PB}_v^\times = \mathbb{Q}_v^\times \backslash \mathrm{B}_v^\times$, has kernel $\{\pm 1\}$, so it suffices to show that $\mathrm{PB}_v^\times$ is compact. But $\mathrm{PB}_v^\times$ acts faithfully by conjugation on $B_v^{\mathrm{Trd}=0}$, so there is an embedding $\mathrm{PB}_v^\times \to \mathrm{GL}_3(\mathbb{Q}_v)$. Show that when $v \in \mathrm{Ram}(\mathrm{B})$, this lands in a conjugate of $\mathrm{GL}_3(\mathbb{Z}_v)$. Finally, show that $\mathrm{GL}_3(\mathbb{Z}_v)$ is compact. Bonus points if you show $\mathrm{GL}_n(\mathbb{Z}_v)$ is the unique up to conjugation maximal compact subgroup of $\mathrm{GL}_n(\mathbb{Q}_v)$).

(c) Show that
$$\mathrm{G}(\mathbb{Q}) \cap \prod_{v<\infty} \mathrm{G}(\mathbb{Z}_v) = \mathrm{G}(\mathbb{Z}).$$

(d) Use what you have shown, plus strong approximation, to conclude that $\mathrm{G}(\mathbb{Z})$ is discrete in $\mathrm{SL}_2(\mathbb{R})$ and that $\mathrm{G}(\mathbb{Z})\backslash\mathbb{H}$ is compact.

(4) (The local trace formula for a finite group.) Let H be a finite group, and set $G = H \times H$. Set $V = L^2(H)$. Consider the biregular representation
$$B : G \to \mathrm{GL}(V)$$
given by
$$B(g) = B(h_1, h_2) : \phi \mapsto (x \mapsto \phi(h_1^{-1} x h_2).$$

(a) Show that this representation decomposes as a $G = H \times H$ representation as
$$V = \bigoplus_{\pi \in \mathrm{Irr}(H)} \pi^\vee \boxtimes \pi.$$

(Hint: Every irreducible representation Π of $G = H \times H$ is of the form $\sigma \otimes \pi$ for σ, π irreducibles of H. Show that since Π appears in V, we must have $\sigma \cong \pi^\vee$ using Schur's lemma. On the other hand, every sub-representation of V is of this form because...)

(b) Let $f = f_1 \otimes f_2 \in \mathscr{S}(G) = \mathscr{S}(H) \otimes \mathscr{S}(H)$ be a pure tensor. Consider the operator

$$B(f)\phi(x) = \sum_{h_1, h_2 \in H} \phi(h_1^{-1} x h_2) f(x_1, x_2).$$

Compute $\mathrm{Tr}(B(f))$ in terms of the decomposition of (a). This is the spectral side of the "local trace formula."

(c) There is an obvious basis for V. Compute the trace of $B(f)$ on V using this basis. This is the geometric side of the "local trace formula."

(d) Suppose $f_i = \mathbb{1}_{\{h_i\}}$ are just the characteristic functions of points. What character relation does the local trace formula give?

(5) (Easy) Show that if S is a symmetric space, and G is the connected component of the identity in $\mathrm{Isom}(S)$, then G acts transitively on S.

(6) Show that every invariant linear differential operator D on $\mathbb{R}^n$ has constant coefficients.

(7) (Easy) Using the model of $\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$ with the metric $ds^2 = y^{-2}(dx^2 + dy^2)$, show that $\Delta = -\mathrm{divgrad}$ is given in coordinates by

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

(8) Show that on a Riemannian manifold X, a diffeomorphism $F : X \to X$ commutes with Δ if and only if F is an isometry. (Hint: follow your nose, and compare Δ in local coordinates around x and $y = F(x)$.)

2. Week 2: Lectures 4-6

So far, we have given no example of a high rank symmetric space (we have also not formally defined rank!). Here is an important one. The Siegel upper half space $\mathbb{H}_g$ is defined by

$$\mathbb{H}_g = \{Z = X + iY \in \mathrm{Mat}_{g\times g}(\mathbb{C}) : {}^tZ = Z, Y \text{ is positive definite}\}.$$

This has the structure of a Riemannian manifold if after identifying all tangent spaces with $\{Z : {}^tZ = Z\}$, the metric is given by

$$ds^2 = \mathrm{Tr}(Y^{-1}dZY^{-1}\overline{dZ}).$$

(1) (a) Show that this formula really defines a metric on $\mathbb{H}_g$.
  (b) Show that, if we define $\mathrm{G} = \mathrm{Sp}_{2g}(\mathbb{R})$ where

$$\mathrm{Sp}_{2g}(\mathbb{R}) = \left\{ g \in \mathrm{GL}_{2g}(\mathbb{R}) : {}^tg \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix} \right\}$$

  then G acts on $\mathbb{H}_g$ via

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} .Z = (AZ + B)(CZ + D)^{-1}$$

  and that this action is transitive. Compute too that G acts by isometries, and identify the stabilizer K of the point $x_0 = i1_g$.
  (c) Show that the geodesic inversion around x_0 comes from an element of G, hence from an element of K.
  (d) Show that $\mathrm{G} = \mathrm{Isom}(\mathbb{H}_g)^\circ$ up to center.

(2) The space $\mathbb{H}_g$ plays a fundamental role in the theory of Shimura varieties, since the $\mathbb{C}$ points of the (coarse!) moduli space of principally polarized Abelian varieties A_g can be identified with the quotient (a locally symmetric space!)

$$A_g(\mathbb{C}) = \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathbb{H}_g.$$

Show this. (Hint: Of course, this will require a bit of knowledge about Abelian varieties. If this makes you unhappy, then just think about the case $g = 1$, which identifies $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ with the coarse moduli of elliptic curves, i.e. genus 1 curves with a marked point. This is already interesting. If you do want to think about the high dimensional case, proceed as follows: first, there is a map $\mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathbb{H}_g \to A_g(\mathbb{C})$ given by associating to $Z \in \mathbb{H}_g$ the lattice $Z.\mathbb{Z}^g + \mathbb{Z}^g \subset \mathbb{C}^g$. This gives a way to associate to a point in $\mathbb{H}_g$ a complex torus. Show this torus is algebraizable, and that every algebraizable torus can be found this way. You need to use Riemann's bilinear relations. What role does the polarization play?)

(3) (This may require some knowledge of representation theory of Lie algebras.) Let G be a Lie group, and let $\mathfrak{g}_\mathbb{C}$ be the complexified Lie algebra. Show that the center of the universal enveloping algebra $\mathfrak{Z}(\mathfrak{g}_\mathbb{C})$ can be identified with left G-invariant differential operators on G. Use this knowledge, together with Harish-Chandra's identification of this ring, to write down an element of $\mathscr{D}(\mathbb{H}_g)$ which is not a polynomial in the Laplacian.

Here is another way to think about Laplace eigenfunctions on S^2.

(4) Show that H_n, the space of harmonic homogeneous polynomials on $\mathbb{R}^3$ of degree n, is an irreducible representation of SO(3) of dimension $2n + 1$.
(5) Show that H_n exhaust all irreducible representations of SO(3).
(6) (Easy) For each n, why is there always an $f \in H_n$ satisfying $f(x_0) \neq 0$?
(7) Deduce the spectral decomposition

$$L^2(S^2) = \bigoplus_{n=0}^{\infty} H_n$$

by taking the Peter-Weyl decomposition of $L^2(\mathrm{SO}(3))$ and looking at fixed points for the right regular action of K on G. Why does this show that these are the only possible eigenfunctions of Δ on S^2?

Finally, here are some odds and ends from spectral theory.

(8) Show that if T is trace class, then

$$\mathrm{Tr}(T) := \sum_{i=1}^{\infty} \langle T\phi_i, \phi_i \rangle$$

is independent of choice of orthonormal basis ϕ_i of our Hilbert space H.

(9) Show that the composition of two Hilbert-Schmidt operators is always trace class.

3. Week 3: Lectures 7-9

(1) Recall that $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is said to be a congruence subgroup if it contains $\Gamma(N)$ for any N. We often consider congruence subgroups of the form $\Gamma_0(N)$ and $\Gamma_1(N)$.
  (a) Show that reduction mod N is surjective.
  (b) Compute the index of $\Gamma(N), \Gamma_0(N), and \Gamma_1(N)$ in $\mathrm{SL}_2(\mathbb{Z})$.
  (c) Find the number of cusps for these groups.

(2) Using the presentation of $\mathrm{SL}_2(\mathbb{Z})$, find an example of a non-congruence subgroup of finite index.

(3) Show that if Γ is a cocompact Fuchsian group and $\gamma \in \Gamma$ is parabolic, then $\gamma = \pm 1$.

(4) Follow this sketch to show that every finitely generated Fuchsian group of the first kind Γ (i.e. good Fuchsian group) has a polygonal fundamental domain.

(5) Write the TF (for compact quotient) without mention of k, only using the Selberg transform h and inversion formulas.

(6) The Fourier expansion of $E(z, s)$ for $\Gamma = \Gamma(1)$

(7) Show that for a good Fuchsian group Γ with distinct cusps c_1, c_2, we have

$$\langle \mathrm{Eis}_{c_1}(\psi_1), \mathrm{Eis}_{c_2}(\psi_2) \rangle = 0$$