# 12주차 실습

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# Continuous Random Variable

# Discrete r.v vs Continuous r.v

$$p_X(x) = \mathbf{P}(X = x)$$

$$f_X(x) \cdot \delta \approx \mathbf{P}(x \le X \le x + \delta)$$

$$p_{X|A}(x) = \mathbf{P}(X = x \mid A)$$

$$f_{X|A}(x) \cdot \delta \approx \mathbf{P}(x \le X \le x + \delta \mid A)$$

$$\mathbf{P}(X \in B) = \sum_{x \in B} p_X(x)$$

$$\mathbf{P}(X \in B) = \int_B f_X(x)\, dx$$

$$\mathbf{P}(X \in B \mid A) = \sum_{x \in B} p_{X|A}(x)$$

$$\mathbf{P}(X \in B \mid A) = \int_B f_{X|A}(x)\, dx$$

$$\sum_x p_{X|A}(x) = 1$$

$$\int f_{X|A}(x)\, dx = 1$$

# Expectation

**Conditional expectation of $X$, given an event**

$$\mathbf{E}[X] = \sum_x x p_X(x) \qquad \mathbf{E}[X] = \int x f_X(x)\, dx$$

$$\mathbf{E}[X \mid A] = \sum_x x p_{X|A}(x) \qquad \mathbf{E}[X \mid A] = \int x f_{X|A}(x)\, dx$$

**Expected value rule:**

$$\mathbf{E}[g(X)] = \sum_x g(x) p_X(x) \qquad \mathbf{E}[g(X)] = \int g(x) f_X(x)\, dx$$

$$\mathbf{E}[g(X) \mid A] = \sum_x g(x) p_{X|A}(x) \qquad \mathbf{E}[g(X) \mid A] = \int g(x) f_{X|A}(x)\, dx$$

# Variance

## Variance and its properties

- **Definition of variance:** $\text{var}(X) = \mathbf{E}[(X-\mu)^2]$

- Calculation using the expected value rule, $\mathbf{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$

$$\text{var}(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)\, dx$$

**Standard deviation:** $\sigma_X = \sqrt{\text{var}(X)}$

$$\text{var}(aX+b) = a^2 \text{var}(X)$$

A useful formula: $\text{var}(X) = \mathbf{E}[X^2] - \left(\mathbf{E}[X]\right)^2$

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# Distribution

# Uniform distribution

| | Uniform distribution | |
|---|---|---|
| r.v type | discrete | continuous |
| parameter | $a \in \{ \dots, -2, -1, 0, 1, 2, \dots \}$<br>$b \in \{\dots, -2, -1, 0, 1, 2, \dots\}$<br>$b \geq a$<br>$n = b - a + 1$ | $-\infty < a < b < \infty$<br>$x \in [a, b]$ |
| PMF or PDF | $\frac{1}{n}$ | $\begin{cases} \frac{1}{b-a} & for\ x \in [a.b] \\ 0 & otherwise \end{cases}$ |
| mean | $\frac{a+b}{2}$ | $\frac{a+b}{2}$ |
| variance | $\frac{(b-a+1)^2 - 1}{12}$ | $\frac{(b-a+1)^2}{12}$ |

# Geometric distribution vs Exponential distribution

| | Geometric distribution | Exponential distribution |
|---|---|---|
| r.v type | discrete | continuous |
| parameter | $0 < p < 1$<br>$k \in \{1, 2, 3, \dots\}$ | $\lambda > 0$<br>$x \in [0, \infty)$ |
| PMF or PDF | $(1-p)^{k-1}p$ | $\lambda e^{-\lambda x}$ |
| mean | $\frac{1}{p}$ | $\frac{1}{\lambda}$ |
| variance | $\frac{1-p}{p^2}$ | $\frac{1}{\lambda^2}$ |

# Normal (or Gaussian) distribution

| | Normal (or Gaussian) distribution |
|---|---|
| Notation | $\mathcal{N}(\mu, \sigma^2)$ |
| parameter | $\mu \in \mathbf{R}$<br>$\sigma^2 > 0$ |
| PDF | $\frac{1}{\sqrt{2\sigma^2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ |
| mean | |
| variance | $\frac{1-p}{p^2}$ |

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# 문제

# 문제 1

Let $X$ be a continuous random variable with a PDF of the form

$$f_X(x) = \begin{cases} c(1-x), & \text{if } x \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

Find the following values.

1. $c =$

2. $\mathbf{P}(X = 1/2) =$

3. $\mathbf{P}\big(X \in \{1/k : k \text{ integer}, k \geq 2\}\big) =$

4. $\mathbf{P}(X \leq 1/2) =$

# 문제 2

Suppose that $X$ has a PDF of the form

$$f_X(x) = \begin{cases} 1/x^2, & \text{if } x \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

For any $x > 2$, the conditional PDF of $X$, given the event $X > 2$ is

# 문제 3

The random variables $X$ and $Y$ have a joint PDF of the form $f_{X,Y}(x,y) = c \cdot \exp\left\{-\frac{1}{2}(4x^2 - 8x + y^2 - 6y + 13)\right\}$.

$\mathbf{E}[X] =$

$\mathrm{var}(X) =$

$\mathbf{E}[Y] =$

$\mathrm{var}(Y) =$

# 문제 4

The bias of a coin (i.e., the probability of Heads) can take three possible values, $1/4$, $1/2$, or $3/4$, and is modeled as a discrete random variable $Q$ with PMF

$$p_Q(q) = \begin{cases} 1/6, & \text{if } q = 1/4, \\ 2/6, & \text{if } q = 2/4, \\ 3/6, & \text{if } q = 3/4, \\ 0, & \text{otherwise.} \end{cases}$$

Let $K$ be the total number of Heads in two independent tosses of the coin. Find $p_{Q|K}(3/4 \mid 2)$.

# 문제 5

The random variables $X$ and $Y$ are jointly continuous, with a joint PDF of the form

$$f_{X,Y}(x,y) = \begin{cases} cxy, & \text{if } 0 \le x \le y \le 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $c$ is a normalizing constant.

For $x \in [0, 0.5]$, the conditional PDF $f_{X|Y}(x \mid 0.5)$ is of the form $ax^b$. Find $a$ and $b$. Your answers should be numbers.

$a =$

$b =$

# 문제 6

Let $Z$ be normal with zero mean and variance equal to 4. For this case, the Chebyshev inequality yields:

$\mathbf{P}(|Z| \geq 4) \leq$ ☐

# 문제 7

The random variables $X_i$ are i.i.d. with mean $2$ and standard deviation equal to $3$. Assume that the $X_i$ are nonnegative. Let $S_n = X_1 + \cdots + X_n$. Use the CLT to find good approximations to the following quantities. You may want to refer to the normal table . In parts (a) and (b), give answers with 4 decimal digits.

a) $\mathbf{P}(S_{100} \le 245) \approx$

b) We let $N$ (a random variable) be the first value of $n$ for which $S_n$ exceeds $119$.

$\mathbf{P}(N > 49) \approx$

c) What is the largest possible value of $n$ for which we have $\mathbf{P}(S_n \le 128) \approx 0.5$?

$n =$

| | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | .5000 | .5040 | .5080 | .5120 | .5160 | .5199 | .5239 | .5279 | .5319 | .5359 |
| 0.1 | .5398 | .5438 | .5478 | .5517 | .5557 | .5596 | .5636 | .5675 | .5714 | .5753 |
| 0.2 | .5793 | .5832 | .5871 | .5910 | .5948 | .5987 | .6026 | .6064 | .6103 | .6141 |
| 0.3 | .6179 | .6217 | .6255 | .6293 | .6331 | .6368 | .6406 | .6443 | .6480 | .6517 |
| 0.4 | .6554 | .6591 | .6628 | .6664 | .6700 | .6736 | .6772 | .6808 | .6844 | .6879 |
| 0.5 | .6915 | .6950 | .6985 | .7019 | .7054 | .7088 | .7123 | .7157 | .7190 | .7224 |
| 0.6 | .7257 | .7291 | .7324 | .7357 | .7389 | .7422 | .7454 | .7486 | .7517 | .7549 |
| 0.7 | .7580 | .7611 | .7642 | .7673 | .7704 | .7734 | .7764 | .7794 | .7823 | .7852 |
| 0.8 | .7881 | .7910 | .7939 | .7967 | .7995 | .8023 | .8051 | .8078 | .8106 | .8133 |
| 0.9 | .8159 | .8186 | .8212 | .8238 | .8264 | .8289 | .8315 | .8340 | .8365 | .8389 |
| 1.0 | .8413 | .8438 | .8461 | .8485 | .8508 | .8531 | .8554 | .8577 | .8599 | .8621 |
| 1.1 | .8643 | .8665 | .8686 | .8708 | .8729 | .8749 | .8770 | .8790 | .8810 | .8830 |
| 1.2 | .8849 | .8869 | .8888 | .8907 | .8925 | .8944 | .8962 | .8980 | .8997 | .9015 |
| 1.3 | .9032 | .9049 | .9066 | .9082 | .9099 | .9115 | .9131 | .9147 | .9162 | .9177 |
| 1.4 | .9192 | .9207 | .9222 | .9236 | .9251 | .9265 | .9279 | .9292 | .9306 | .9319 |
| 1.5 | .9332 | .9345 | .9357 | .9370 | .9382 | .9394 | .9406 | .9418 | .9429 | .9441 |
| 1.6 | .9452 | .9463 | .9474 | .9484 | .9495 | .9505 | .9515 | .9525 | .9535 | .9545 |
| 1.7 | .9554 | .9564 | .9573 | .9582 | .9591 | .9599 | .9608 | .9616 | .9625 | .9633 |
| 1.8 | .9641 | .9649 | .9656 | .9664 | .9671 | .9678 | .9686 | .9693 | .9699 | .9706 |
| 1.9 | .9713 | .9719 | .9726 | .9732 | .9738 | .9744 | .9750 | .9756 | .9761 | .9767 |
| 2.0 | .9772 | .9778 | .9783 | .9788 | .9793 | .9798 | .9803 | .9808 | .9812 | .9817 |
| 2.1 | .9821 | .9826 | .9830 | .9834 | .9838 | .9842 | .9846 | .9850 | .9854 | .9857 |
| 2.2 | .9861 | .9864 | .9868 | .9871 | .9875 | .9878 | .9881 | .9884 | .9887 | .9890 |
| 2.3 | .9893 | .9896 | .9898 | .9901 | .9904 | .9906 | .9909 | .9911 | .9913 | .9916 |
| 2.4 | .9918 | .9920 | .9922 | .9925 | .9927 | .9929 | .9931 | .9932 | .9934 | .9936 |
| 2.5 | .9938 | .9940 | .9941 | .9943 | .9945 | .9946 | .9948 | .9949 | .9951 | .9952 |
| 2.6 | .9953 | .9955 | .9956 | .9957 | .9959 | .9960 | .9961 | .9962 | .9963 | .9964 |
| 2.7 | .9965 | .9966 | .9967 | .9968 | .9969 | .9970 | .9971 | .9972 | .9973 | .9974 |
| 2.8 | .9974 | .9975 | .9976 | .9977 | .9977 | .9978 | .9979 | .9979 | .9980 | .9981 |
| 2.9 | .9981 | .9982 | .9982 | .9983 | .9984 | .9984 | .9985 | .9985 | .9986 | .9986 |
| 3.0 | .9987 | .9987 | .9987 | .9988 | .9988 | .9989 | .9989 | .9989 | .9990 | .9990 |
| 3.1 | .9990 | .9991 | .9991 | .9991 | .9992 | .9992 | .9992 | .9992 | .9993 | .9993 |
| 3.2 | .9993 | .9993 | .9994 | .9994 | .9994 | .9994 | .9994 | .9995 | .9995 | .9995 |
| 3.3 | .9995 | .9995 | .9995 | .9996 | .9996 | .9996 | .9996 | .9996 | .9996 | .9997 |
| 3.4 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9998 |