MAE 207 Final: Experimental Uncertainty Analysis

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Abstract

The aim of this project is to perform meaningful uncertainty analysis from experimental data, accounting for systematic and random errors. The data was collected from a simple set of experiments that involved breaking chalk to determine the modulus of rupture. Two types of chalk, and four different loading configurations were used.

I. Introduction

When an engineer performs an experiment, the collected data does not always provide an exact answer. Instead, experimental results allow us to approximate solutions with some degree of uncertainty. Understanding, and quantifying, this uncertainty is often as important as finding the expected outcome. Several methods may be used to quantify uncertainty, we will look at two such methods: The Taylor Series Approach and The Monte Carlo Approach. The Taylor Series Approach involves linearizing a data reduction equation to estimate the uncertainty of a calculated result using uncertainty propagation. The Monte Carlo Approach involves "computationally simulating the stochastic nature of errors and their influences on experimental and calculated results." and is conducted by "generating many random values for variables of interest and calculating a result for each set of variables." [2]. There are two types of Monte Carlo Methods: Parametric and Non-Parametric Bootstrapping. The first involves assuming a distribution for the values and calculating replicates based on this known probability density function. The latter is data-driven, and does not assume any particular distribution, instead data point are re-sampled from the original data set. The Non-Parametric method has the advantage of not assuming any particular distribution for the data, but this means that enough data must be supplied to adequately represent the population.

The Modulus of Rupture for chalk when subjected to three point bending can be calculated by analyzing the maximum stress to which the chalk is subjected. For any beam: $\sigma_{max} = \frac{M\,c}{I}$. Where M, is the maximum moment, c is half the width, and I is the moment of area for the cross section. Further analysis for a circular cross section leads

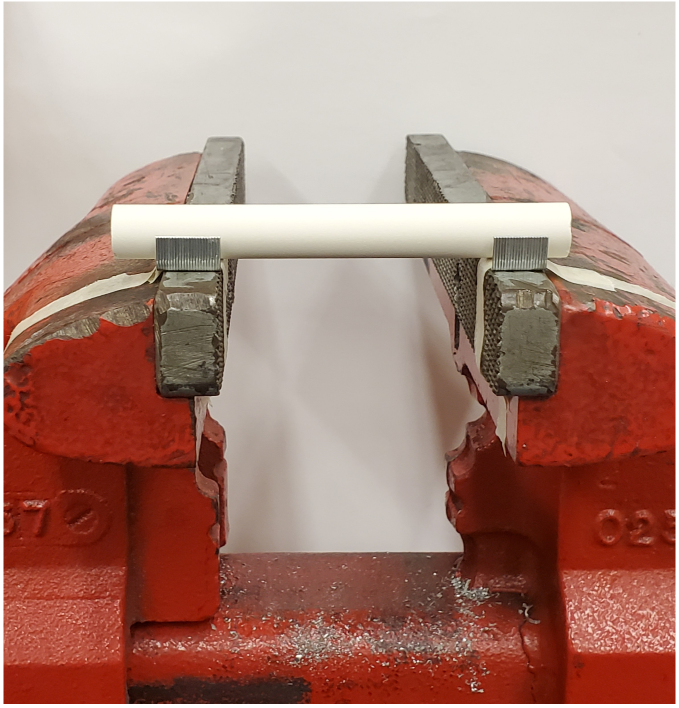


Fig. 1: This shows the experimental setup for the three point bending test. The gap was able to be varied for the different trials.

to a formula for the modulus of rupture:

$$\sigma_{max} = \frac{L\,F}{2\,\pi\,r^3} \tag{1}$$

Where L is the total length of the gap, F is the applied force, and r is the radius of the circular cross section. [1]

II. Experimental Method

The experimental method performed was designed to test the strength of rupture of chalk for several different conditions. The tools used to perform this analysis are listed below. The caliper specification sheet listed the following errors: Accuracy: $0.01mm$, Repeatability: $0.01mm$, Resolution: $0.01mm$, Digitization: $0.01mm$ and reading error was also found to be $0.0025mm$. The hanging scale listed one systematic error, which we will assume accounts for all systematic error and that was $10g$. The hanging scale also has resolution error $5g$ and reading error calculated to be $1.25g$

A. Measuring Devices

1) Calipers: Mitutoyo, Model CD-S8"CT.
2) Hanging Scale: WeiHeng, Model WH-A08.

A centered three point bending was used for every trial as shown in Figure 1. However, we have four different sets of experimental parameters comprising four "conditions" described below.

B. Testing Conditions

1) "Small Chalk" (Mean Diameter $9.64mm$), "Short Gap" (Length $46.91mm$)
2) "Small Chalk" (Mean Diameter $9.64mm$), "Large Gap" (Length $68.1mm$)
3) "Large Chalk" (Mean Diameter $14.8mm$), "Large Gap" (Length $45.94mm$)
4) "Large Chalk" (Mean Diameter $14.8mm$), "Short Gap" (Length $35.61mm$)

For each of the conditions, the experimental method is described below.

C. Method

1) Prepare loading setup by adjusting gap to desired length, measure gap using a pair of calipers. This will be the measured gap

MAE 207 Data								
	Condition #1				Condition #2			
	Trial	**Diameter(mm)**	**Length(mm)**	**Break Weight(kg)**	**Trial**	**Diameter(mm)**	**Length(mm)**	**Break Weight(kg)**
	1	9.4	46.91	2.375	1	9.73	68.1	1.455
	2	9.71	46.91	2.005	2	9.7	68.1	1.39
	3	9.63	46.91	2.47	3	9.58	68.1	1.04
	4	9.64	46.91	2.325	4	9.67	68.1	1.175
	5	9.62	46.91	2.48	5	9.55	68.1	1.175
	6	9.72	46.91	2.145	6	9.7	68.1	1.385
	7	9.63	46.91	2.04	7	9.6	68.1	1.075
	8	9.54	46.91	2.39	8	9.58	68.1	1.21
	9	9.57	46.91	2.115	9	9.65	68.1	1.46
	10	9.97	46.91	2.27	10	9.57	68.1	1.375
	11	9.7	46.91	2.515	11	9.61	68.1	0.84
	Mean	9.648181818	46.91	2.28454545		9.630909091	68.1	1.234545455
	SD	0.019616364	5.55358E-29	0.03330227		0.003809091	2.22143E-28	0.039262273
	Condition #3				Condition #4			
	Trial	**Diameter(mm)**	**Length(mm)**	**Break Weight(kg)**	**Trial**	**Diameter(mm)**	**Length(mm)**	**Break Weight(kg)**
	1	14.7	45.94	4.95	1	14.68	35.62	7.135
	2	14.68	45.94	4.92	2	14.67	35.62	2.385
	3	14.71	45.94	4.7	3	14.98	35.62	5.825
	4	14.78	45.94	6.515	4	14.66	35.62	6.22
	5	14.72	45.94	5.86	5	15.03	35.62	6.025
	6	14.87	45.94	2.835	6	14.7	35.62	5.28
	7	14.78	45.94	3.855	7	14.81	35.62	7.345
	8	14.7	45.94	6.005	8	15.09	35.62	5.17
	9	14.75	45.94	4.53	9	14.67	35.62	5.32
	10	14.71	45.94	5.64	10	15.03	35.62	6.255
	11	14.69	45.94	5.605	11	14.97	35.62	5.54
	Mean	14.73545455	45.94	5.03772727		14.84454545	35.62	5.681818182
	SD	0.003147273	0	1.11228682		0.030727273	0	1.707781364

Fig. 2: Here we have the data collected from the four sets of experimental trials.

for each trial of the specific loading condition (assuming no random fluctuation of this length measured.)

2) Measure the diameter of the chalk with a pair of calipers and record on datasheet.
3) Place chalk and hanging scale on the mount with the scale centered in the gap. Tare the scale.
4) Pull the hanging scale from below until the chalk breaks, use the auto-save feature of the hanging scale to record the maximum force applied. Record this force on the datasheet.
5) Repeat the 2-4 for every trial for specified loading condition.
6) Once every trial is complete for a given loading condition, begin again at step 1 for a new loading condition.

III. Experimental Results

1) Discussion of Experimental Methods: Each trial was conducted successfully. Exactly one piece of chalk was rejected from each 12-piece box because of obvious defects, this left eleven total trials for each condition. This is a good number, but could be increased to reduce uncertainty. The scale used was very effective because it allowed the trials to be completed quickly. However, pulling on the end of the scale seemed somewhat unscientific, forces in multiple directions were introduced by doing this, and the weight was not added as slowly as we would have liked.

2) Discussion of Collected Data: The data collected can be seen in Fig. 1. We can see the mean diameter for the smaller chalk is $9.64mm$, and the variance of that measurement (across all trials) is $0.01mm$. The larger chalk had a mean diameter of $14.79mm$ and a variance of $0.018mm$. We see that both of the chalks had a similar variance of diameters, on the order of $0.01mm$. For condition 1 the mean break weight was $2.28kg$, for condition 2 the mean break weight was $1.23kg$, for condition 3 the mean breaking weight was $5.04kg$ and for condition 4 the mean break weight was $5.68kg$.

When we compare the four trials, other trends emerge. Firstly, phenomena that we would expect to see did occur: increasing the gap distance results in a lower breaking force, this is expected because with a larger gap distance, the chalk is subjected to a larger moment relative to the force applied. Increasing the diameter of the chalk resulted in a higher breaking force, this is expected because a larger diameter means a higher moment of area, and so more stress is required to cause rupture.

The variance of breaking force was much lower for the smaller chalk, showing that the measurements are more similar for this smaller chalk. Additionally, we do not immediately see any significant trends in variance associated with the gap distance: meaning that changing the gap distance did not seem to affect the variance for that condition. We can draw this conclusion because the variance increased for the smaller chalk when we increased the gap distance, but the variance decreased for the larger chalk when we increased the gap distance. We may need more trials to successfully conclude this, however.

There were outliers detected in the data set using the Modified Thompson Tau Technique at the 95% confidence level. We did not choose to remove the outlier from the data set because we did not want to reduce the number of data points for uncertainty calculation. For condition 1, the radius value of $0.00499m$ was detected to be an outlier. For condition 2 the break weight of $0.84kg$ was detected to be an outlier. For condition three both the radius value of $0.00744m$ and the weight of $2.84kg$ were detected to be outliers. For condition 4, the weight value of $2.39kg$ was detected to be an outlier.

3) Analysis of Chalk Strength: The rupture strength of the chalk can be calculated using Eqn. 1. The average rupture strength for condition 1 is

$1.52 * 10^5 N/m^2$. The average rupture strength for condition 2 is $1.20*10^4 N/m^2$. The average rupture strength for condition 3 is $9.22 * 10^4 N/m^2$. The average rupture strength for condition 4 is $7.87 * 10^4 N/m^2$. There may be some explanation for the variation in rupture strength across the various conditions. Firstly, the chalk used in conditions 1 and 2 are different than conditions 3 and 4. In 1 and 2 we used "normal" crayola chalk, however, in conditions 3 and 4 we used alpha-color "low-dust chalk". We see that conditions 1 and 2 are more similar to each other and conditions 3 and 4 are similar to each other. However, there is not a good explanation for why condition 1 would result in a dramatically higher rupture strength than the other trials.

IV. Uncertainty Analysis

To analyze the uncertainty of the data we used two different methods, a Taylor Series Approach and a Monte Carlo Approach. What we found was that the Taylor Series approach did not appropriately capture the uncertainty of the calculated result. and Monte Carlo Analysis seems to be much more appropriate for this specific data reduction equation (Eqn. 1).

The uncertainties present in the calipers are summarized:

1) Accuracy β_1: $0.01mm$ (Systematic)
2) Digitization β_2: $0.01mm$ (Systematic)
3) Repeatability ϵ_1: $0.01mm$ (Random)
4) Reading ϵ_2: $0.025mm$ (Random)

The uncertainties present in the hanging scale are summarized:

1) Accuracy β_3: $10g$ (Systematic)
2) Resolution ϵ_3: $5g$ (Random)
3) Reading ϵ_4: $1.25g$ (Random)

All other errors are either not present or not known.

The data reduction equation was inputted into the calculations as Eq.2 where g, the acceleration of gravity, was included because the breaking weight was found in units of kg, not N:

$$r = \frac{L\,F\,g}{2\,\pi\,rad^3} \tag{2}$$

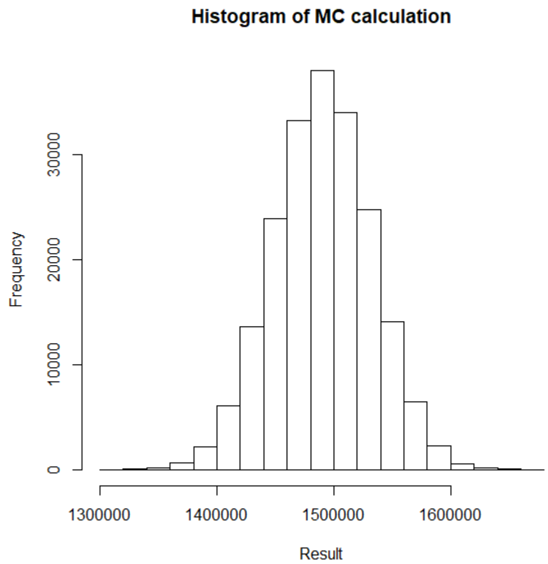
A. Taylor Series Analysis

1) Discussion: The Taylor series analysis relies on the linearization of the data reduction equation, and calculating the total uncertainty using the following equations:

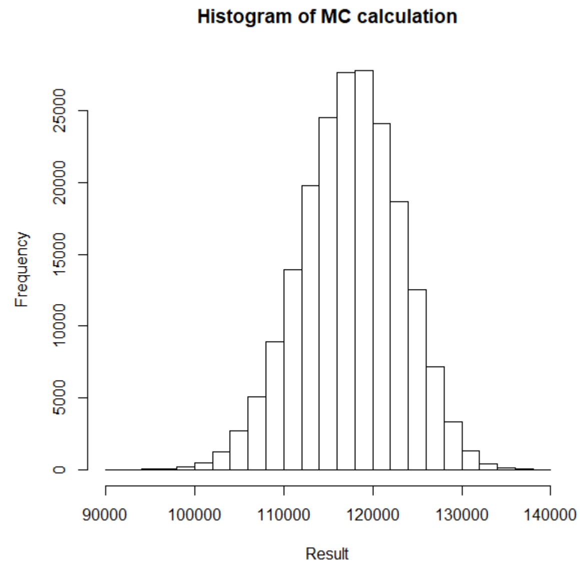
$$u_{\bar{r}} = \sqrt{b_{\bar{r}}^2 + s_{\bar{r}}^2}$$

$$U_{\bar{r}} = k\,u_{\bar{r}}$$

Where $b_{\bar{r}}^2$ is the calculated systematic uncertainty of the average value of r, $s_{\bar{r}}^2$ is the random uncertainty of the average value of r, k is the coverage factor (here assumed to be equal to 2).



(a)



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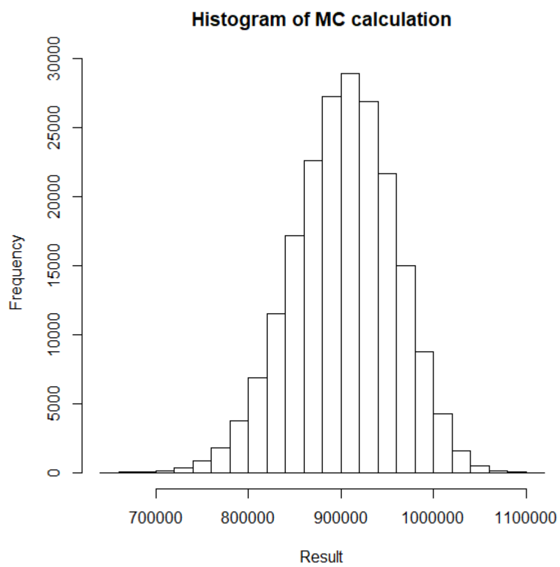

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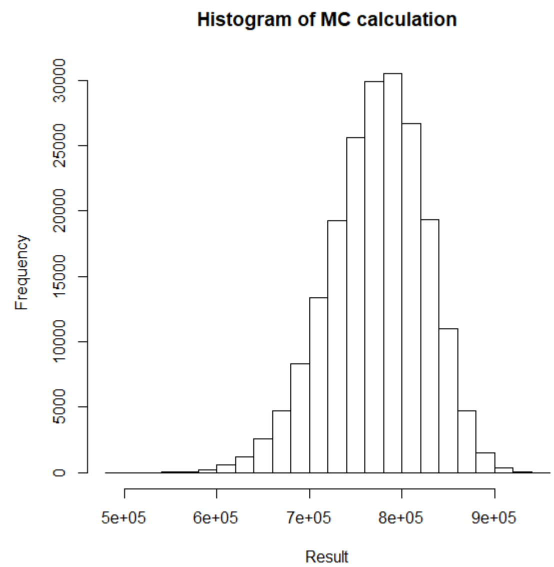

(d)

Fig. 3: Pictured above are the four histograms produced for each Monte Carlo Analysis. Figure 3a corresponds to the analysis for condition 1, 3b corresponds to condition 2, etc.

First, we chose to calculate $s_{\bar{r}}^2$ via direct calculation from the sample, in this way we avoid calculating the random error uncertainty and covariance of the input parameters. This was done in R with the following function:

$$s_{\bar{r}}^2 = \mathit{sd(r)/sqrt(length(r))}$$

In order to calculate $b_{\bar{r}}^2$, we used the following equation for $L = 3$:

$$b_{\bar{r}}^2 = \theta_1^2 b_{11} + \theta_2^2 b_{22} + \theta_3^2 b_{33} + 2\theta_1\theta_2 b_{12} +$$

$$+2\theta_1\theta_3 b_{13} + 2\theta_2\theta_3 b_{23}$$

Where the b_{mm} variables are the sum of variances of elemental systematic errors of $\bar{x}_m$. b_{mn} variables are the sum of variances of the elemental systematic error that are common to $\bar{x}_m$ and $\bar{x}_n$. Finally, the θ variables are the sensitivity coefficients for the respective $\bar{x}$ values. The summary of results are below:

$$\theta_1 = \frac{\partial r}{\partial L}_{F=\bar{F}\cdots} = \frac{\bar{F}}{2\,\pi\,\bar{r}^3}$$

$$\theta_2 = \frac{\partial r}{\partial F}_{L=\bar{L}\cdots} = \frac{\bar{L}}{2\,\pi\,\bar{r}^3}$$

$$\theta_3 = \frac{\partial r}{\partial rad}_{rad=r\bar{a}d\cdots} = \frac{3\,\bar{F}\,\bar{L}}{2\,\pi\,\bar{r}^4}$$

$$b_{11} = \frac{\beta_1^2 + \beta_2^2}{2^2}$$

$$b_{22} = \frac{\beta_3^2}{2^2}$$

$$b_{33} = b_{13} = \frac{\beta_1^2 + \beta_2^2}{2^2}$$

$$b_{12} = b_{23} = 0$$

The b values were chosen because β_1 and β_2 are associated with the the length measurements and β_3 and β_4 are associated with force measurements. b_{13} is common while b_{12} and b_{23} share no commonality, thus are zero.

2) Results: The Taylor series analysis did not provide a good understanding of the uncertainties present in the calculation of the mean value for rupture strength. This is because of the non-linear nature of the DRE. We saw that the uncertainty calculated from the Taylor Series Approach was actually three orders of magnitude larger than the mean value of the calculation! The results for each calculation are summarized below.

Condition 1: $\mu = 1.52 * 10^5 \pm 4.07 * 10^8 N/m^2$ (95%)

Condition 2:$\mu = 1.20 * 10^4 \pm 9.04 * 10^6 N/m^2$ (95%)

Condition 3:$\mu = 9.22 * 10^4 \pm 1.02 * 10^8 N/m^2$ (95%)

Condition 4:$\mu = 7.87 * 10^4 \pm 9.76 * 10^7 N/m^2$ (95%)

B. Monte Carlo Approach

1) Discussion: The uncertainty analysis using the Monte Carlo Approach was more successful than the Taylor Series Approach. The uncertainty for each condition was much more reasonable. We chose to use a parametric method for this analysis because we only had eleven data points. We assumed that every systematic error was uniformly distributed because we were given no indication of their distribution from the specification sheets.

2) Results: The results for each condition are summarized below. The uncertainty interval is reported as a range, but from these ranges we can estimate the magnitude of the total uncertainty. For each condition, the uncertainty is on the order of $10^4 N/m^2$.This is still very large, but much more reasonable than the $10^6 N/m^2$ values we were getting using the Taylor Series Approach. *Condition 1*: μ UI: $\{1412339, 1566990\} N/m^2$ (95%)

Condition 2: μ UI: $\{106369.2, 127911.8\} N/m^2$ (95%)

Condition 3: μ UI: $\{791940.8, 1004863.1\} N/m^2$ (95%)

Condition 4: μ UI: $\{661878.3, 864525.2\} N/m^2$ (95%)

V. Conclusion

The experiment presented provides a good learning tool for the understanding of uncertainty calculation. We saw how the Taylor Series Approach provides a much different answer than the Monte Carlo Approach. The Taylor series approach provided an uncertainty interval for the mean of rupture strength that was not realistic because it was several orders of magnitude larger than the mean values calculated. The large error was the result of the sensitivity coefficients $\theta_1, \cdots$. The sensitivity coefficients were all very large because of $\frac{1}{r^3}$ terms present in every one. Although the systematic errors present in our measurement tools were very small compared to the values we were measuring, this cubed term amplified this uncertainty greatly.

The Monte Carlo Approach, while better, still calculated that the uncertainty interval for the mean value of rupture strength to be quite large, each on the order of $10^4 N/m^2$. Again, this is likely due to the nature of the DRE.

There are a few ways we could improve the experiment and reduce the uncertainty of our calculated result. Obviously, the first thing to do would be reduce the uncertainty of the measurement tools. Again, the radius term is cubed and so our uncertainty is particularly sensitive to errors in the length measurement. However, the calipers used were some of the most accurate calipers one would be expected to find in an engineering laboratory. Decreasing this error would require a large scale machine. Furthermore, the UI sensitivity to errors in length measurement is so great that it would require the experimenter to go to great lengths (and cost) to reduce the error terms to levels that would reduce the overall uncertainty significantly.

However, the scale used to calculate the breaking force was not particularly expensive or accurate, it was simply a luggage scale. A more accurate tool for calculating this force could easily be acquired to reduce the uncertainty in that term.

The experiment itself could be improved to reduce many of the random errors. For example, using very fine weights to load the chalk until rupture would likely reduce the random errors from the experimentation method.

Finally, the uncertainty may always remain relatively high because whiteboard chalk is not particularly well standardized or regulated. There is no incentive to carefully prepare chalk to be uniform, or isotropic, etc. so there should not be an expectation that the rupture strength is uniform across all samples.

References

[1] Bansal, R. K. *A textbook of strength of materials.* Laxmi Publications, 2010.

[2] Shaw, Benjamin David. *Uncertainty Analysis of Experimental Data with R.* Chapman and Hall/CRC, 2017.