

Problem Set: Introduction to Measure Theory

OSE Lab 2019

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1 Measure Spaces

Exercise 1.

- $\mathcal{G}_1 = \{A : A \subset \mathbb{R}, A \text{ open}\}$ is not closed under complements. For example, $(0, +\infty)$ is an open set in \mathbb{R} so it is included in \mathcal{G}_1 , but its complement $(-\infty, 0]$ is not open. Thus \mathcal{G}_1 is not an algebra.
- If \emptyset is not included in \mathcal{G}_2 , then it is not an algebra by definition. If \emptyset is included in $\mathcal{G}_2 = \{A : A \text{ is a finite union of intervals of the form } (a, b], (-\infty, b], \text{ and } (a, \infty)\}$, then it is an algebra as it is closed under complements (the complement would be a finite intersection of $(-\infty, a] \cup (b, \infty)$, (b, ∞) and $(-\infty, a]$, which is either empty or could be written as a finite union of the above forms) and closed under finite unions (a finite union of the sets that are finite unions of intervals of the form $(a, b], (-\infty, b],$ and (a, ∞) is still a finite union of intervals of these forms). It is not a σ -algebra. For example, the countable union

$$\bigcup_{n=2}^{\infty} \left(0, \frac{n-1}{n}\right] = (0, 1)$$

is not in \mathcal{G}_2 .

- $\mathcal{G}_3 = \{A : A \text{ is a countable union of } (a, b], (-\infty, b], \text{ and } (a, \infty)\}$ is a σ -algebra if we include \emptyset in it. The proof follows similarly as above but now “finite” applies more generally to “countable” too.

Exercise 2.

- If \mathcal{A} is a σ -algebra, then $\emptyset \in \mathcal{A}$. The complement of \emptyset , X must also be in \mathcal{A} . Thus $\{\emptyset, X\} \subset \mathcal{A}$.

- Note that if $S \subset X$, then $S^c = X \setminus S \subset X$. Also note that finite unions of subsets of X is still a subset of X . If \mathcal{A} is a σ -algebra generated from some subsets of X , then $\mathcal{A} \subset \mathcal{P}(X)$ as $\mathcal{P}(X) = \{A : A \subset X\}$ contains all the subsets of X by definition.

Exercise 3.

- Since $\emptyset \in \mathcal{S}_\alpha, \forall \alpha$, we thus have $\emptyset \in \cap_\alpha \mathcal{S}_\alpha$.
- Pick any $X \in \cap_\alpha \mathcal{S}_\alpha$ and hence $X \in \mathcal{S}_\alpha, \forall \alpha$. Since a σ -algebra is closed under complements, we have $X^c \in \mathcal{S}_\alpha, \forall \alpha$. Therefore, $X^c \in \cap_\alpha \mathcal{S}_\alpha$.
- Pick countable sets $X_1, X_2, \dots \in \cap_\alpha \mathcal{S}_\alpha$, and hence $X_i \in \mathcal{S}_\alpha, \forall \alpha, \forall i$. Since a σ -algebra is closed under countable unions, thus $\cup_i X_i \in \mathcal{S}_\alpha, \forall \alpha$. Therefore, $\cup_i X_i \in \cap_\alpha \mathcal{S}_\alpha$.

The above propositions mean that $\cap_\alpha \mathcal{S}_\alpha$ is a σ -algebra.

Exercise 4.

- Let $C = B \setminus A = B \cap A^c$. μ is a nonnegative measure and hence $\mu(C) \geq 0$. Note that $B = A \cup C$ and $A \cap C = \emptyset$. Therefore

$$\mu(B) = \mu(A) + \mu(C) \geq \mu(A).$$

- Let $B_n = \cup_{i=1}^n A_i$. Note that $B_1 \subset B_2 \subset B_3 \subset \dots$. By Theorem 1.25 (i), we have $\lim_{n \rightarrow \infty} \mu(B_n) = \mu(\cup_{i=1}^\infty B_i)$.
 - Note that $\cup_{i=1}^\infty B_i = \cup_{i=1}^\infty A_i$, hence $\mu(\cup_{i=1}^\infty B_i) = \mu(\cup_{i=1}^\infty A_i)$.
 - Note that $\lim_{n \rightarrow \infty} \mu(B_n) = \lim_{n \rightarrow \infty} \mu(\cup_{i=1}^n A_i) \leq \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(A_i) = \sum_{i=1}^\infty \mu(A_i)$.
 - Therefore, $\mu(\cup_{i=1}^\infty A_i) \leq \sum_{i=1}^\infty \mu(A_i)$.

Exercise 5.

- $\lambda(\emptyset) = \mu(\emptyset \cap B) = \mu(\emptyset) = 0$.
- For any $\{A_i\}_{i=1}^\infty \subset \mathcal{S}$ s.t. $A_i \cap A_j = \emptyset \forall i \neq j$, we have

$$\begin{aligned} \lambda(\cup_{i=1}^\infty A_i) &= \mu((\cup_{i=1}^\infty A_i) \cap B) = \mu(\cup_{i=1}^\infty (A_i \cap B)) \\ &= \sum_{i=1}^\infty \mu(A_i \cap B) = \sum_{i=1}^\infty \lambda(A_i). \end{aligned}$$

Exercise 6. Let $B_n = A_1 \cap A_n^c$. Since

$$\cup_{i=1}^{\infty} B_i = \cup_{i=1}^{\infty} (A_1 \cap A_i^c) = A_1 \cap (\cup_{i=1}^{\infty} A_i^c) = A_1 \setminus (\cap_{i=1}^{\infty} A_i),$$

we have $\mu(\cup_{i=1}^{\infty} B_i) = \mu(A_1) - \mu(\cap_{i=1}^{\infty} A_i)$. In addition,

$$\lim_{n \rightarrow \infty} \mu(B_n) = \lim_{n \rightarrow \infty} \mu(A_1 \cap A_n^c) = \lim_{n \rightarrow \infty} (\mu(A_1) - \mu(A_n)) = \mu(A_1) - \lim_{n \rightarrow \infty} \mu(A_n).$$

Note that $B_1 \subset B_2 \subset B_3 \subset \dots$. By Theorem 1.25 (i) we have

$$\lim_{n \rightarrow \infty} \mu(B_n) = \mu(\cup_{i=1}^{\infty} B_i).$$

Therefore,

$$\mu(A_1) - \lim_{n \rightarrow \infty} \mu(A_n) = \mu(A_1) - \mu(\cap_{i=1}^{\infty} A_i) \implies \lim_{n \rightarrow \infty} \mu(A_n) = \mu(\cap_{i=1}^{\infty} A_i).$$

2 Construction of Lebesgue Measure

Exercise 7. Since μ^* is an outer measure, it is countably subadditive. Note that $(B \cap E) \cup (B \cap E^c) = B$, thus

$$\mu^*(B) \leq \mu^*(B \cap E) + \mu^*(B \cap E^c).$$

If in addition $\mu^*(B) \geq \mu^*(B \cap E) + \mu^*(B \cap E^c)$, it must be that $\mu^*(B) = \mu^*(B \cap E) + \mu^*(B \cap E^c)$.

Exercise 8. Denote \mathcal{O} the collection of open sets of \mathbb{R} . By definition, the Borel σ -algebra of \mathbb{R} is the σ -algebra generated by \mathcal{O} , i.e., $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{O})$. Let ν be a premeasure on \mathbb{R} and μ^* the outer measure generated by ν , and \mathcal{M} the σ -algebra from the Caratheodory construction. By Theorem 2.12, we have $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{O}) \subset \mathcal{M}$.

3 Measurable Functions

Exercise 9. Consider a countable set $\{x_n\}_{n=1}^{\infty}$. Note that $\{x_n\} \subset (x_n - \frac{\varepsilon}{2^{n+1}}, x_n + \frac{\varepsilon}{2^{n+1}}], \forall \varepsilon > 0$. Note that

$$\sum_{n=1}^{\infty} \left[\left(x_n + \frac{\varepsilon}{2^{n+1}} \right) - \left(x_n - \frac{\varepsilon}{2^{n+1}} \right) \right] = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

By definition of Lebesgue measure,

$$\begin{aligned}\lambda(\{x_n\}_{n=1}^\infty) &= \inf \left\{ \sum_{n=1}^\infty (b_n - a_n) : \{x_n\}_{n=1}^\infty \subset \bigcup_{i=1}^\infty (a_i, b_i] \right\} \\ &\leq \inf \left\{ \sum_{n=1}^\infty \left[\left(x_n + \frac{\varepsilon}{2^{n+1}}\right) - \left(x_n - \frac{\varepsilon}{2^{n+1}}\right) \right] : \{x_n\}_{n=1}^\infty \subset \bigcup_{i=1}^\infty \left(x_n - \frac{\varepsilon}{2^{n+1}}, x_n + \frac{\varepsilon}{2^{n+1}}\right], \varepsilon > 0 \right\} \\ &= \inf \{ \varepsilon : \varepsilon > 0 \} = 0.\end{aligned}$$

Therefore, $\lambda(\{x_n\}_{n=1}^\infty) = 0$.

Exercise 10. Since \mathcal{M} is a σ -algebra, it is closed under complements and countable unions.

1. If $\forall a, \{x \in X : f(x) \geq a\} \in \mathcal{M}$, then its complement $\{x \in X : f(x) < a\} \in \mathcal{M}$ too.
So (*) can be replaced by $\{x \in X : f(x) \geq a\}$.
2. If $\forall a, \{x \in X : f(x) > a\} \in \mathcal{M}$, then a countable union

$$\bigcup_{n=1}^\infty \{x \in X : f(x) > a + \frac{1}{n}\} = \{x \in X : f(x) \geq a\}$$

is also in \mathcal{M} . By (1), (*) can be replaced by $\{x \in X : f(x) > a\}$.

3. If $\forall a, \{x \in X : f(x) \leq a\} \in \mathcal{M}$, then its complement $\{x \in X : f(x) > a\} \in \mathcal{M}$ too.
By (2), (*) can be replaced by $\{x \in X : f(x) \leq a\}$.

Exercise 11.

- Let $F(f, g) = f + g$ which is continuous. By (4), $f + g$ is measurable.
- Let $F(f, g) = f \cdot g$ which is continuous. By (4), $f \cdot g$ is measurable.
- Let $f_n = f$ for n odd and $f_n = g$ for n even. Thus $\sup_{n \in \mathbb{N}} f_n(x) = \max(f, g)$. By (2), $\max(f, g)$ is measurable.
- Let $f_n = f$ for n odd and $f_n = g$ for n even. Thus $\inf_{n \in \mathbb{N}} f_n(x) = \min(f, g)$. By (2), $\min(f, g)$ is measurable.
- First, $g(x) = -1$ is measurable. Thus $-f = f \cdot g$ is measurable. Next note that $|f| = \max(f, -f)$, so $|f|$ is also measurable.

Exercise 12. We construct a partition as in the proof. If f is bounded, $\exists M$ s.t. $f(x) < M, \forall x$. For $n > M$, $\forall x \in X$, there exists some i s.t. $x \in E_i^n$. Thus $s_n(x) = \frac{i-1}{2^n}$ for

this i and $|f(x) - s_n(x)| < \frac{1}{2^n}$. $\forall \varepsilon > 0$, there exists $N \in \mathbb{N}$ s.t. $\frac{1}{2^N} < \varepsilon$. Therefore, $\forall n > \max(N, M)$, we have

$$|f(x) - s_n(x)| < \frac{1}{2^n} < \varepsilon,$$

hence the convergence in is uniform.

4 Lebesgue Integration

Exercise 13. Since $f^+ = \max\{f(x), 0\} \in [0, M)$, and $\mu(E) < \infty$, then

$$0 \leq \int_E f^+ d\mu \leq M\mu(E) < \infty.$$

Similarly, $f^- = \max\{-f(x), 0\} \in [0, M)$, then

$$0 \leq \int_E f^- d\mu \leq M\mu(E) < \infty.$$

Both $\int_E f^+ d\mu$ and $\int_E f^- d\mu$ are finite, so $f \in \mathcal{L}^1(\mu, E)$.

Exercise 14. Suppose there exists $X \subset E$ with $\mu(X) > 0$, and $f(x) = \infty, \forall x \in X$. Then

$$\int_E |f| d\mu \geq \int_X |f| d\mu \geq \int_X f d\mu = \infty,$$

contradictory to f being integrable.

Exercise 15. $f \leq g$ implies that $\{s : 0 \leq s \leq f, s \text{ simple, measurable}\} \subset \{s : 0 \leq s \leq g, s \text{ simple, measurable}\}$, and thus $\sup\{\int_E s d\mu : 0 \leq s \leq f, s \text{ simple, measurable}\} \leq \sup\{\int_E s d\mu : 0 \leq s \leq g, s \text{ simple, measurable}\}$. By definition this is $\int_E f d\mu \leq \int_E g d\mu$.

Exercise 16. Consider any $s(x) = \sum_{i=1}^N c_i \chi_{E_i}$ simple, measurable. $A \subset E$ implies that $A \cap E_i \subset E \cap E_i, \forall i$. Hence $\mu(A \cap E_i) \leq \mu(E \cap E_i)$. Thus

$$\int_A s d\mu = \sum_{i=1}^N c_i \mu(A \cap E_i) \leq \sum_{i=1}^N c_i \mu(E \cap E_i) = \int_E s d\mu.$$

Therefore,

$$\begin{aligned}\int_A f^+ d\mu &= \sup \left\{ \int_A s d\mu : 0 \leq s \leq f^+, s \text{ simple, measurable} \right\} \\ &\leq \sup \left\{ \int_E s d\mu : 0 \leq s \leq f^+, s \text{ simple, measurable} \right\} \\ &= \int_E f^+ d\mu < \infty,\end{aligned}$$

and similarly $\int_A f^- d\mu < \infty$. So $f \in \mathcal{L}^1(\mu, A)$.

Exercise 17. Let $X_1 = A \cap B$ and $X_2 = A - B$. Note that $X_1 \cap X_2 = \emptyset$ and $A = X_1 \cup X_2$. Since $B \subset A$, we have $A \cap B = B$. Thus

$$\int_A f d\mu = \int_B f d\mu + \int_{A-B} f d\mu.$$

Since $\mu(A - B) = 0$, by Proposition 4.6 we have $\int_{A-B} f d\mu = 0$. Therefore $\int_A f d\mu = \int_B f d\mu$.