2.4 Curvature

2.4.1 Definitions and Examples

The notion of curvature measures how sharply a curve bends. We would expect the curvature to be 0 for a straight line, to be very small for curves which bend very little and to be large for curves which bend sharply. If we move along a curve, we see that the direction of the tangent vector will not change as long as the curve is flat. Its direction will change if the curve bends. The more the curve bends, the more the direction of the tangent vector will change. So, it makes sense to study how the tangent vector changes as we move along a curve. But because we are only interested in the direction of the tangent vector, not its magnitude, we will consider the unit tangent vector. Curvature is defined as follows:

Definition 150 (Curvature) *Let C be a smooth curve with position vector $\overrightarrow{r}(s)$ where s is the arc length parameter. The* ***curvature*** *κ of C is defined to be:*

$$\kappa = \left\| \frac{d\overrightarrow{T}}{ds} \right\| \tag{2.11}$$

where $\overrightarrow{T}$ is the unit tangent vector. Note the letter used to denote the curvature is the greek letter ***kappa*** *denoted κ.*

Remark 151 *The above formula implies that $\overrightarrow{T}$ be expressed in terms of s, arc length. And therefore, we must have the curve parametrized in terms of arc length.*

Example 152 *Find the curvature of a circle of radius a.*
We saw earlier that the parametrization of a circle of radius a with respect to arc length length was $\overrightarrow{r}(s) = \left\langle a\cos\frac{s}{a}, a\sin\frac{s}{a}\right\rangle$. First, we need to compute $\overrightarrow{T}(s)$. By definition,

$$\overrightarrow{T}(s) = \frac{\overrightarrow{r}'(s)}{\|\overrightarrow{r}'(s)\|}$$

So, we must first compute $\overrightarrow{r}'(s)$.

$$\begin{aligned}\overrightarrow{r}'(s) &= \left\langle a\left(-\frac{1}{a}\sin\frac{s}{a}\right), a\left(\frac{1}{a}\cos\frac{s}{a}\right)\right\rangle \\ &= \left\langle -\sin\frac{s}{a}, \cos\frac{s}{a}\right\rangle\end{aligned}$$

We can see that $\|\overrightarrow{r}'(s)\| = 1$ (but we already knew that from theorem 148. Thus

$$\overrightarrow{T}(s) = \left\langle -\sin\frac{s}{a}, \cos\frac{s}{a}\right\rangle$$

It follows that

$$\frac{d\overrightarrow{T}}{ds} = \left\langle -\frac{1}{a}\cos\frac{s}{a}, -\frac{1}{a}\sin\frac{s}{a}\right\rangle$$

and therefore

$$\begin{aligned}\kappa &= \left\Vert \frac{d\overrightarrow{T}}{ds} \right\Vert \\ &= \frac{1}{a}\end{aligned}$$

In other words, the curvature of a circle is the inverse of its radius. This agrees with our intuition of curvature. Curvature is supposed to measure how sharply a curve bends. The larger the radius of a circle, the less it will bend, that is the less its curvature should be. This is indeed the case. The larger the radius, the smaller its inverse.

Example 153 *Find the curvature of the circular helix.*
Earlier, we found that the parametrization of the circular helix with respect to arc length was $\overrightarrow{r}(s) = \left\langle \cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}} \right\rangle$*. As before, we need to compute* $\overrightarrow{T}(s)$ *which can be obtained from* $\overrightarrow{r}'(s)$*.*

$$\overrightarrow{r}'(s) = \left\langle \frac{-1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

Thus

$$\begin{aligned}\overrightarrow{T}(s) &= \left\langle \frac{-1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \\ &= \frac{1}{\sqrt{2}}\left\langle -\sin\frac{s}{\sqrt{2}}, \cos\frac{s}{\sqrt{2}}, 1 \right\rangle\end{aligned}$$

Therefore

$$\frac{d\overrightarrow{T}}{ds} = \frac{1}{\sqrt{2}}\left\langle \frac{-1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}}, 0 \right\rangle$$

It follows that

$$\begin{aligned}\kappa &= \left\Vert \frac{d\overrightarrow{T}}{ds} \right\Vert \\ &= \frac{1}{2}\end{aligned}$$

The definition for the curvature works well when the curve is parametrized with respect to arc length, or when this can be done easily. However, since s can be expressed as a function as t, there should also be formulas for the curvature in term of t. The next theorems give us various formulas for the curvature.

Theorem 154 *Let C be a smooth curve with position vector $\overrightarrow{r}(t)$ where t is any parameter. Then the following formulas can be used to compute κ.*

$$\kappa = \frac{\left\Vert \overrightarrow{T}'(t) \right\Vert}{\left\Vert \overrightarrow{r}'(t) \right\Vert} \tag{2.12}$$

$$\kappa = \frac{\left\| \overrightarrow{r}'(t) \times \overrightarrow{r}''(t) \right\|}{\left\| \overrightarrow{r}'(t) \right\|^3} \tag{2.13}$$

Proof. *We prove each formula separately.*

1. *Proof of* $\kappa = \dfrac{\left\| \overrightarrow{T}'(t) \right\|}{\left\| \overrightarrow{r}'(t) \right\|}$

   Using the chain rule, we have $\dfrac{d\overrightarrow{T}}{dt} = \dfrac{d\overrightarrow{T}}{ds}\dfrac{ds}{dt} = \dfrac{d\overrightarrow{T}}{ds}\left\| \overrightarrow{r}'(t) \right\|$ *from formula 2.10. Thus,*

$$\begin{aligned} \frac{d\overrightarrow{T}}{ds} &= \frac{\frac{d\overrightarrow{T}}{dt}}{\left\| \overrightarrow{r}'(t) \right\|} \\ &= \frac{\overrightarrow{T}'(t)}{\left\| \overrightarrow{r}'(t) \right\|} \end{aligned}$$

   Hence

$$\begin{aligned} \kappa &= \left\| \frac{d\overrightarrow{T}}{ds} \right\| \\ &= \frac{\left\| \overrightarrow{T}'(t) \right\|}{\left\| \overrightarrow{r}'(t) \right\|} \end{aligned}$$

2. *Proof of* $\kappa = \dfrac{\left\| \overrightarrow{r}'(t) \times \overrightarrow{r}''(t) \right\|}{\left\| \overrightarrow{r}'(t) \right\|^3}$

   We express $\overrightarrow{r}'(t)$ *and* $\overrightarrow{r}''(t)$ *in terms of* T *and* T'*.then compute their cross product.*

   - *Computation of* $\overrightarrow{r}'$*. Since* $\overrightarrow{T} = \dfrac{\overrightarrow{r}'}{\left\| \overrightarrow{r}' \right\|}$ *and* $\dfrac{ds}{dt} = \left\| \overrightarrow{r}' \right\|$, *we get that*

$$\overrightarrow{r}' = \frac{ds}{dt}\overrightarrow{T}$$

   - *Computation of* $\overrightarrow{r}''$ *Taking the derivative with respect to* t *of the previous formula gives us*

$$\overrightarrow{r}'' = \frac{d^2 s}{dt^2}\overrightarrow{T} + \frac{ds}{dt}\overrightarrow{T}'$$

   - *Computation of* $\overrightarrow{r}'(t) \times \overrightarrow{r}''(t)$*. From the two previous formulas and using the properties of cross products, we see that*

$$\overrightarrow{r}' \times \overrightarrow{r}'' = \frac{ds}{dt}\frac{d^2 s}{dt^2}\left(\overrightarrow{T} \times \overrightarrow{T} \right) + \left(\frac{ds}{dt} \right)^2 \left(\overrightarrow{T} \times \overrightarrow{T}' \right)$$

Since the cross product of a vector by itself is always the zero vector, we see that

$$\begin{aligned} \|\overrightarrow{r}' \times \overrightarrow{r}''\| &= \left(\frac{ds}{dt}\right)^2 \left\|\overrightarrow{T} \times \overrightarrow{T}'\right\| \\ &= \left(\frac{ds}{dt}\right)^2 \left\|\overrightarrow{T}\right\| \left\|\overrightarrow{T}'\right\| \sin\theta \end{aligned}$$

where θ is the angle between $\overrightarrow{T}$ and $\overrightarrow{T}'$. Since $\left\|\overrightarrow{T}\right\| = 1$, by proposition 125, we know that $\overrightarrow{T} \perp \overrightarrow{T}'$ thus

$$\begin{aligned} \|\overrightarrow{r}' \times \overrightarrow{r}''\| &= \left(\frac{ds}{dt}\right)^2 \left\|\overrightarrow{T}'\right\| \\ &= \|\overrightarrow{r}'(t)\|^2 \left\|\overrightarrow{T}'\right\| \text{ from formula } 2.10. \end{aligned}$$

Therefore

$$\left\|\overrightarrow{T}'\right\| = \frac{\|\overrightarrow{r}' \times \overrightarrow{r}''\|}{\|\overrightarrow{r}'(t)\|^2}$$

So

$$\begin{aligned} \kappa &= \frac{\left\|\overrightarrow{T}'(t)\right\|}{\|\overrightarrow{r}'(t)\|} \\ \kappa &= \frac{\|\overrightarrow{r}' \times \overrightarrow{r}''\|}{\|\overrightarrow{r}'(t)\|^3} \end{aligned}$$

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Remark 155 *Formula 2.12 is consistent with the definition of curvature. It says that if t is any parameter used for a curve C, then the curvature of C is $\kappa = \dfrac{\left\|\overrightarrow{T}'(t)\right\|}{\|\overrightarrow{r}'(t)\|}$. In the case the parameter is s, then the formula and using the fact that $\|\overrightarrow{r}'(s)\| = 1$, the formula gives us the definition of curvature.*

Theorem 156 *If C is a curve with equation $y = f(x)$ where f is twice differentiable then*

$$\kappa = \frac{|f''(x)|}{\left(1 + (f'(x))^2\right)^{\frac{3}{2}}} \tag{2.14}$$

Proof. *The proof follows easily from formula 2.13. First, let us remark that it is easy to parametrize the curve given by $y = f(x)$ as a 3-D parametric curve. We can simply use*

$$\begin{cases} x = x \\ y = f(x) \\ z = 0 \end{cases}$$

Using x as the name of the parameter. Thus, the position vector of our curve is $\overrightarrow{r}(x) = \langle x, f(x), 0\rangle$. It follows that

$$\overrightarrow{r}'(x) = \langle 1, f'(x), 0\rangle$$

and

$$\overrightarrow{r}''(x) = \langle 0, f''(x), 0\rangle$$

Thus

$$\overrightarrow{r}' \times \overrightarrow{r}'' = \langle 0, 0, f''(x)\rangle$$

Hence

$$\|\overrightarrow{r}' \times \overrightarrow{r}''\| = |f''(x)|$$

and

$$\|\overrightarrow{r}'\| = \sqrt{1 + (f'(x))^2}$$

Therefore

$$\begin{aligned}
\kappa &= \frac{\|\overrightarrow{r}' \times \overrightarrow{r}''\|}{\|\overrightarrow{r}'(t)\|^3} \\
&= \frac{|f''(x)|}{\left(\sqrt{1 + (f'(x))^2}\right)^3} \\
&= \frac{|f''(x)|}{\left(1 + (f'(x))^2\right)^{\frac{3}{2}}}
\end{aligned}$$

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We illustrate these formulas with some examples. Of course, the formula we use depends on the information given.

Example 157 *Find the curvature of the curve given by $\overrightarrow{r}(t) = \left\langle 2t, t^2, -\dfrac{1}{3}t^3 \right\rangle$ as a function of t*

Since we have the position vector describing the curve but it is not given with respect to arc length, we can find κ by using either formula 2.12 or formula 2.13. Since this is an example, we show both methods. Since both methods require $\overrightarrow{r}'$ and $\|\overrightarrow{r}'\|$, we compute both here.

$$\overrightarrow{r}'(t) = \left\langle 2, 2t, -t^2 \right\rangle$$

Therefore

$$\begin{aligned}
\|\overrightarrow{r}'(t)\| &= \sqrt{4 + 2t^2 + t^4} \\
&= \sqrt{(2 + t^2)^2} \\
&= 2 + t^2 \text{ since } 2 + t^2 > 0
\end{aligned}$$

Method 1 *Here, we use formula 2.12. Using this formula, we need to find* $\left\Vert \overrightarrow{T}'(t)\right\Vert$

$$\begin{aligned}\overrightarrow{T}(t) &= \frac{\overrightarrow{r}'(t)}{\left\Vert \overrightarrow{r}'(t)\right\Vert} \\ &= \frac{\left\langle 2, 2t, -t^2\right\rangle}{2+t^2} \\ &= \left\langle \frac{2}{2+t^2}, \frac{2t}{2+t^2}, \frac{-t^2}{2+t^2}\right\rangle\end{aligned}$$

So,

$$\overrightarrow{T}'(t) = \frac{1}{\left(2+t^2\right)^2}\left\langle -4t, 4-2t^2, -4t\right\rangle$$

therefore

$$\begin{aligned}\left\Vert \overrightarrow{T}'(t)\right\Vert &= \frac{\sqrt{16t^2+16-16t^2+4t^4+16t^2}}{\left(2+t^2\right)^2} \\ &= \frac{\sqrt{4\left(t^4+4t^2+4\right)}}{\left(2+t^2\right)^2} \\ &= \frac{2}{2+t^2}\end{aligned}$$

It follows that

$$\begin{aligned}\kappa &= \frac{\left\Vert \overrightarrow{T}'(t)\right\Vert}{\left\Vert \overrightarrow{r}'(t)\right\Vert} \\ &= \frac{2}{\left(2+t^2\right)^2}\end{aligned}$$

Method 2 *Here, we use formula 2.13.Using this formula, we need to find* $\overrightarrow{r}''(t)$.

$$\overrightarrow{r}''(t) = \langle 0, 2, -2t\rangle$$

Next, we find $\left\Vert \overrightarrow{r}'(t) \times \overrightarrow{r}''(t)\right\Vert$.

$$\begin{aligned}\overrightarrow{r}'(t) \times \overrightarrow{r}''(t) &= \left\langle 2, 2t, -t^2\right\rangle \times \langle 0, 2, -2t\rangle \\ &= \left\langle -2t^2, 4t, 4\right\rangle\end{aligned}$$

Therefore

$$\begin{aligned}\left\Vert \overrightarrow{r}'(t) \times \overrightarrow{r}''(t)\right\Vert &= \left\Vert\left\langle -2t^2, 4t, 4\right\rangle\right\Vert \\ &= \sqrt{4t^4+16t^2+16} \\ &= 2\left(t^2+2\right)\end{aligned}$$

Finally,

$$\begin{aligned}\kappa &= \frac{\left\|\overrightarrow{r}'(t)\times\overrightarrow{r}''(t)\right\|}{\left\|\overrightarrow{r}'(t)\right\|^3}\\ &= \frac{2\left(t^2+2\right)}{\left(t^2+2\right)^3}\\ &= \frac{2}{\left(2+t^2\right)^2}\end{aligned}$$

The same answer as above.

Example 158 *Find the curvature of $y=x^2$ as a function of x. Then, find the curvature of the same curve when $x=0$, $x=1$.*
Here, we use formula 2.14. Let $f(x)=x^2$. First, we need to find $f'(x)$ and $f''(x)$.

$$\begin{aligned}f'(x) &= 2x\\ f''(x) &= 2\end{aligned}$$

So

$$\begin{aligned}\kappa &= \frac{\left|f''(x)\right|}{\left(1+\left(f'(x)\right)^2\right)^{\frac{3}{2}}}\\ \kappa &= \frac{2}{\left(1+4x^2\right)^{\frac{3}{2}}}\end{aligned}$$

When $x=0$, we get

$$\kappa = 2$$

When $x=1$, we get

$$\kappa = \frac{2}{5^{\frac{3}{2}}}$$

Make sure you have read, studied and understood what was done above before attempting the problems.

2.4.2 Problems

Do odd # 1, 3 (only find $\overrightarrow{T}$ and κ), 5, 9 - 15 (only find $\overrightarrow{T}$ and κ), 19, 23, 25 at the end of 10.4 in your book.