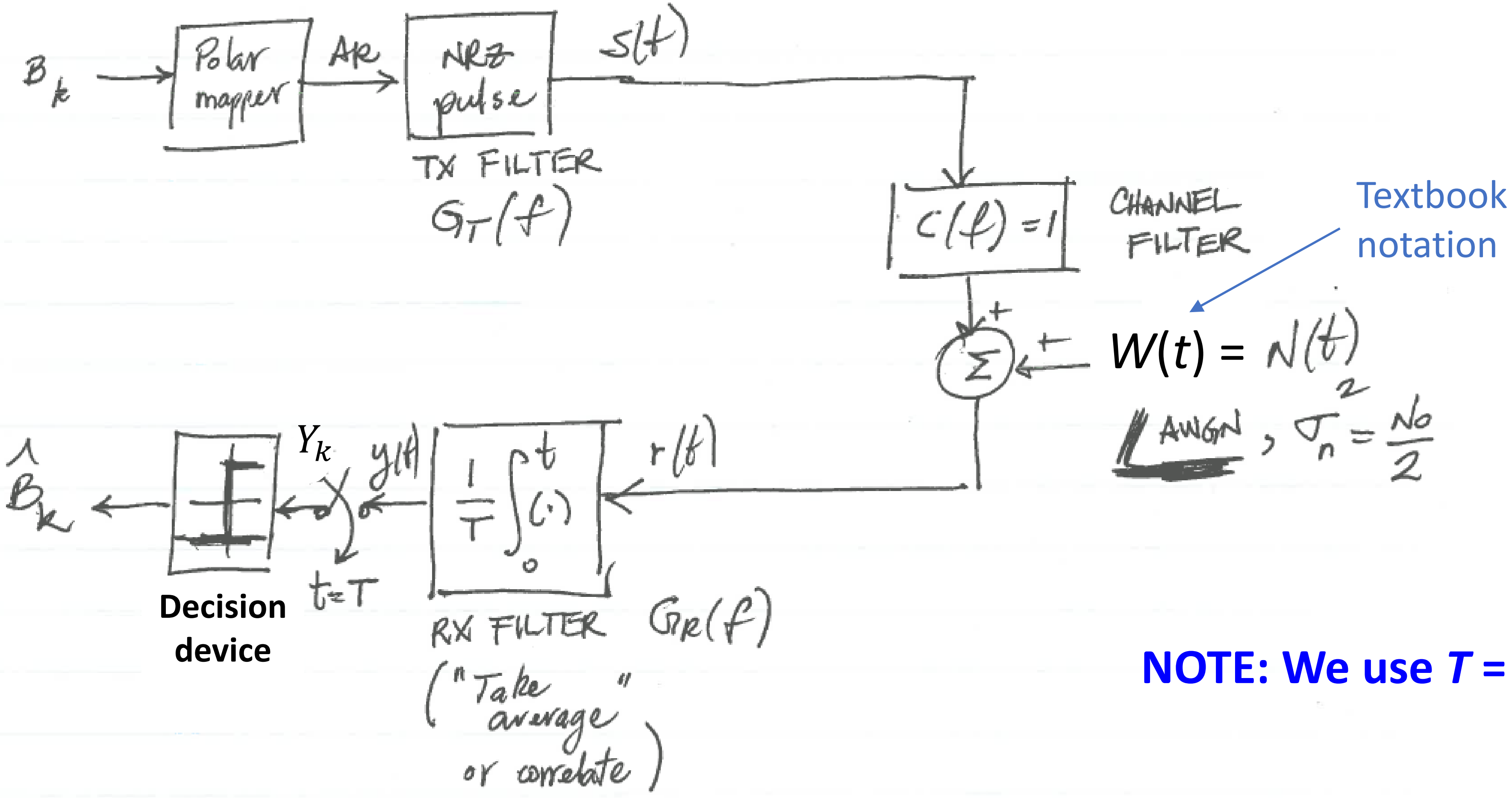
Introduction to binary communication (Polar NRZ)

- Polar NRZ signal
- Noise AWGN



NOTE: We use $T = T_b$ interchangeably

Additive White Gaussian Noise (AWGN)

Due to Brownian motion in resistors, $N(t)$. Random unpredictable

Properties:

(1) Addition: $E\{N(t)\} = 0 = \mu_N$

(2) Wideband: $\boxed{S_N(f) = kT}$

$k = 1.380649 \times 10^{-23}$ (Boltzmann's constant)
$T = 290$ K (Room temperature in Kelvin)

Show ⇒ Lecture note "2c_AWGN.pdf"

(3) Gaussian: $\boxed{f_{N(t)}(n) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{n^2}{N_0}}}$

$\boxed{\mu_N = 0, \sigma_N^2 = \frac{N_0}{2}}$

PDF: Probability density function

(4) Noise: Random, unpredictable

The Gaussian Q-function

Use Gaussian Q-function (qfunc) to compute probabilities. If X is Gaussian then

$$\boxed{P[X > x_0] = Q\left(\frac{x_0 - \mu_X}{\sigma_X}\right)}$$

Mean μ_X, variance σ_X^2,

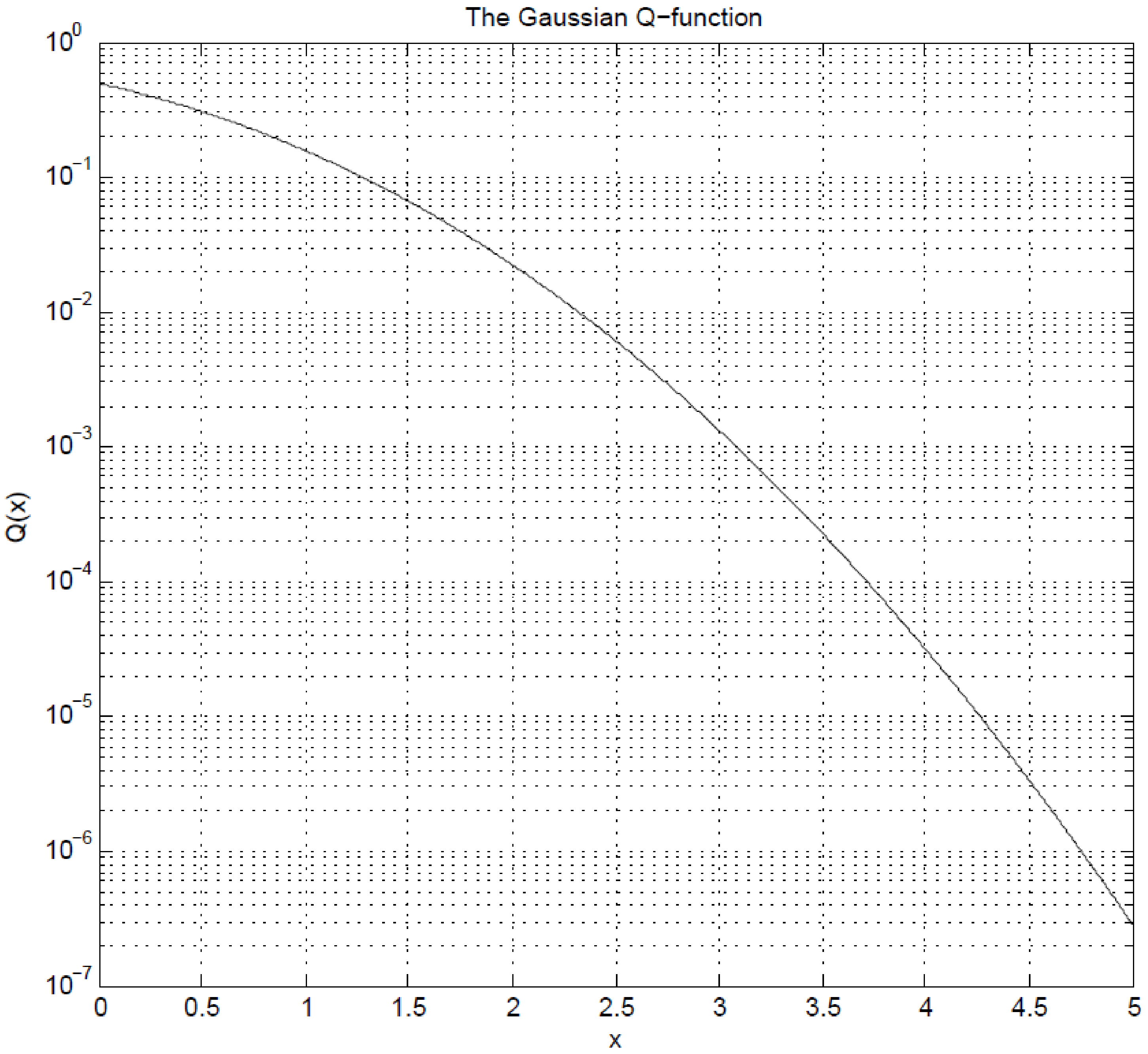
where Q(x) is the area under the tail of a zero-mean unit-variance Gaussian PDF:

$$\boxed{Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du = \frac{1}{2} \operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right)}$$

MATLAB/Octave: **qfunc**(x)

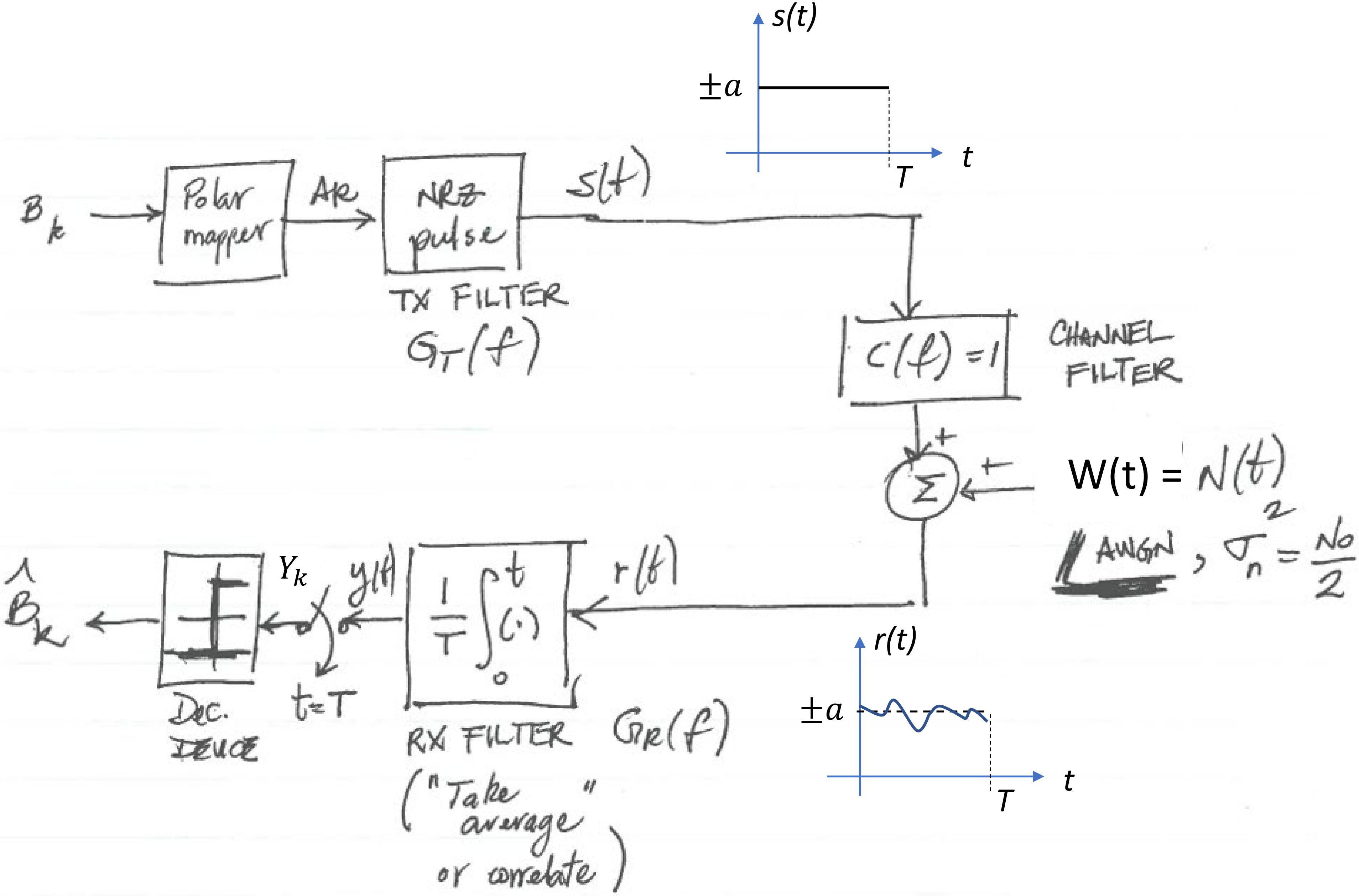
Zero-mean, unit-variance Gaussian PDF

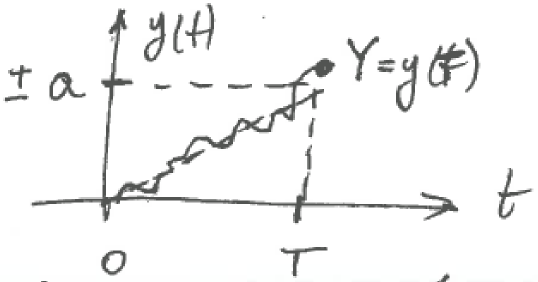
Python: **scipy.special.erfc** (x)



The Gaussian Q−function
Q(x)
x
10^0
10^-1
10^-2
10^-3
10^-4
10^-5
10^-6
10^-7
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

Back to the binary communication system:





Sampled output of averager (correlator).

$$y(T) = Y_k = \frac{1}{T}\int_0^T r(t)\,dt = \frac{1}{T}\int_0^T [s(t) + N(t)]\,dt$$

$$\boxed{Y_k = A_k + N_k}, \quad \text{where} \quad N_k = \frac{1}{T}\int_0^T N(t)\,dt.$$

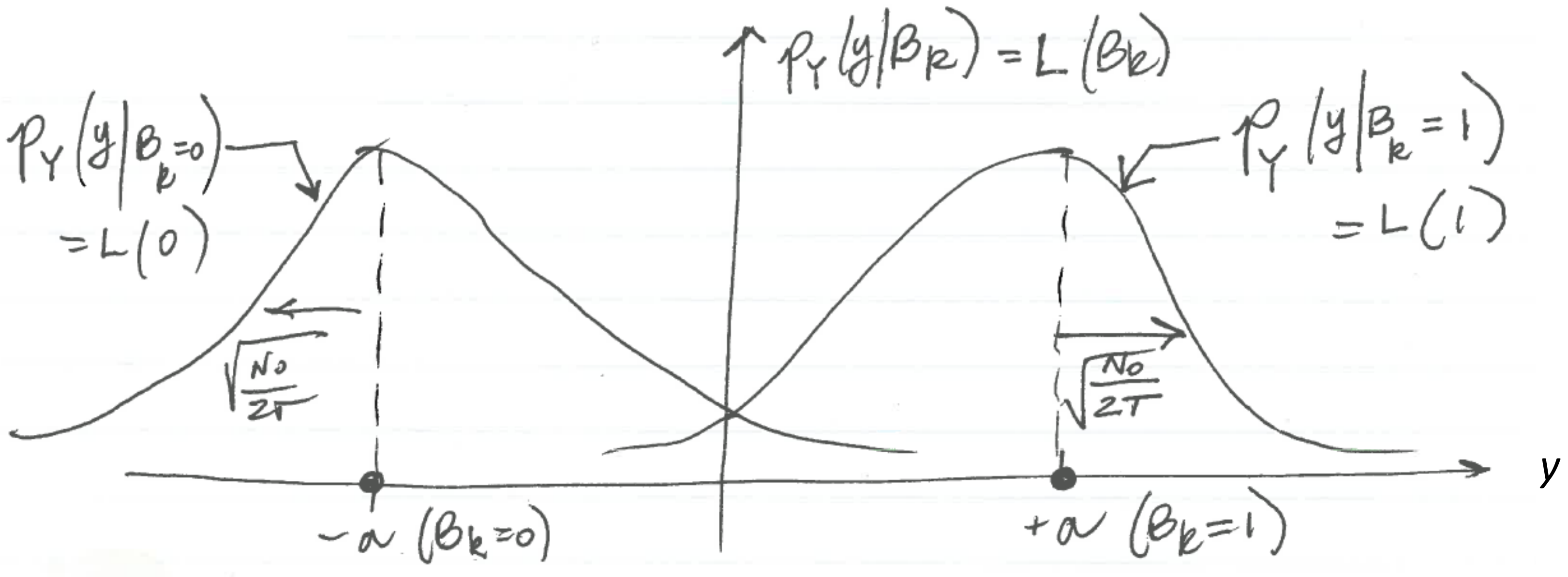
Mapper output

AWGN sample

$$E\{N_k\} = \frac{1}{T}\int_0^T E\{N(t)\}\,dt = \underline{\underline{0}} \quad \checkmark$$

$$\mathrm{Var}\{N_k\} = E\{N_k^2\} = \frac{1}{T^2}\int_0^T E\{N_k^2\}\,dt = \frac{N_0}{2T}$$

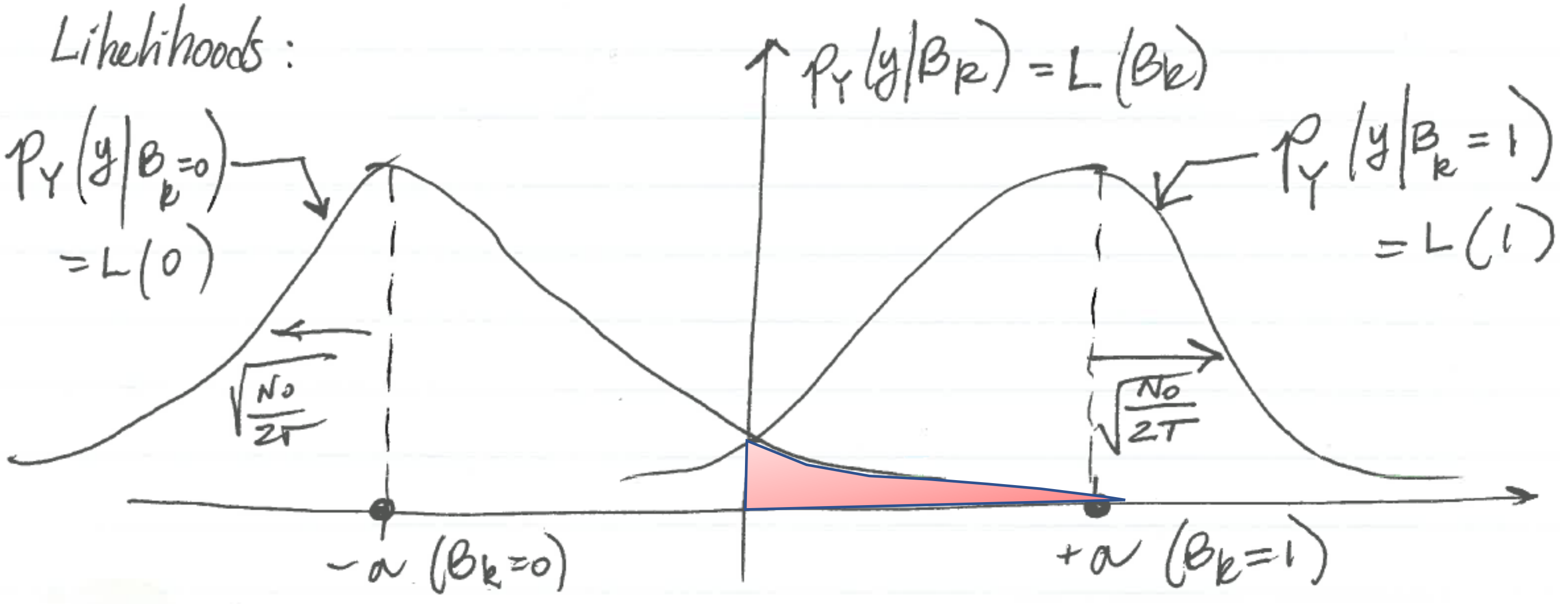
The two conditional PDF's of Y, the "averager" output, are known as the **likelihoods**:



$$\boxed{Y_k = A_k + N_k = \pm a + N_k}$$

Use for <u>computer simulations</u>

Likelihoods :



Average P_b :

$$P_b = P[\hat{B}_k \neq B_k] = P[\hat{B}_k=1 / B_k=0] \cdot P[B_k=0]$$

$$+ P[\hat{B}_k=0 / B_k=1] \cdot P[B_k=1]$$

Decision rule :
Maximum likelihood

$$\hat{B}_k = \begin{cases} 0, & L(0) > L(1), \text{ or } \boxed{Y \leq 0}, \\ 1, & \text{otherwise. }, \text{ or } Y > 0 \end{cases}$$

Assuming that $P[B_k=0]=P[B_k=1]=\frac{1}{2}$:

$$P_b = \frac{1}{2}\left(P[\hat{B}_k=1 / B_k=0] + P[\hat{B}_k=0 / B_k=1]\right)$$

$$= \frac{1}{2}\left(P[Y>0 / B_k=0] + P[Y\leq 0 / B_k=0]\right).$$

By symmetry:

$$P_b = P[Y>0 / B_k=0]$$ (Red shaded area in the previous slide)

$$= Q\left(\frac{0-(-a)}{\sqrt{N_0/2T}}\right) = Q\left(\sqrt{\frac{2a^2T}{N_0}}\right)$$

Signal energy:

$$\boxed{E_s = a^2 T}$$

$$Q\left(\sqrt{\frac{0-\mu_Y}{\sigma_Y}}\right), \mu_Y = -a, \sigma_Y = \sqrt{N_0/_{2T}}$$

Remove sqrt

Signal energy: $$E_s = \int_{-\infty}^{\infty} s^2(t)\, dt$$

For the NRZ pulse: $\boxed{E_s = a^2 T}$

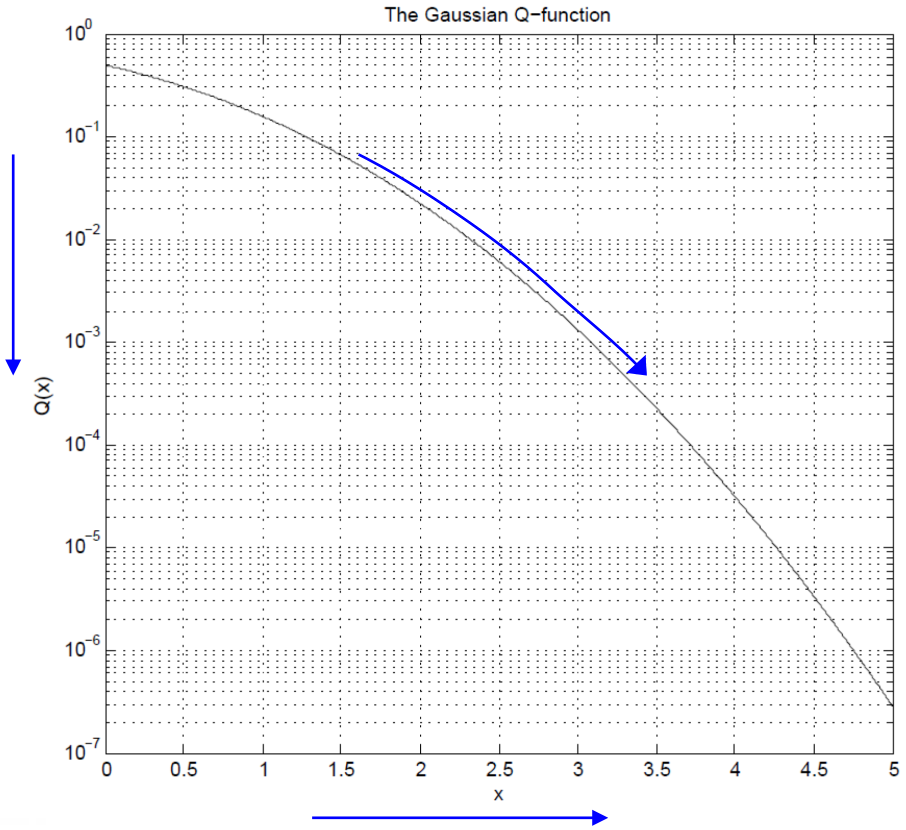
$$\therefore \boxed{P_b = Q\left(\sqrt{\frac{2E_s}{N_0}}\right)}$$

Evidently,

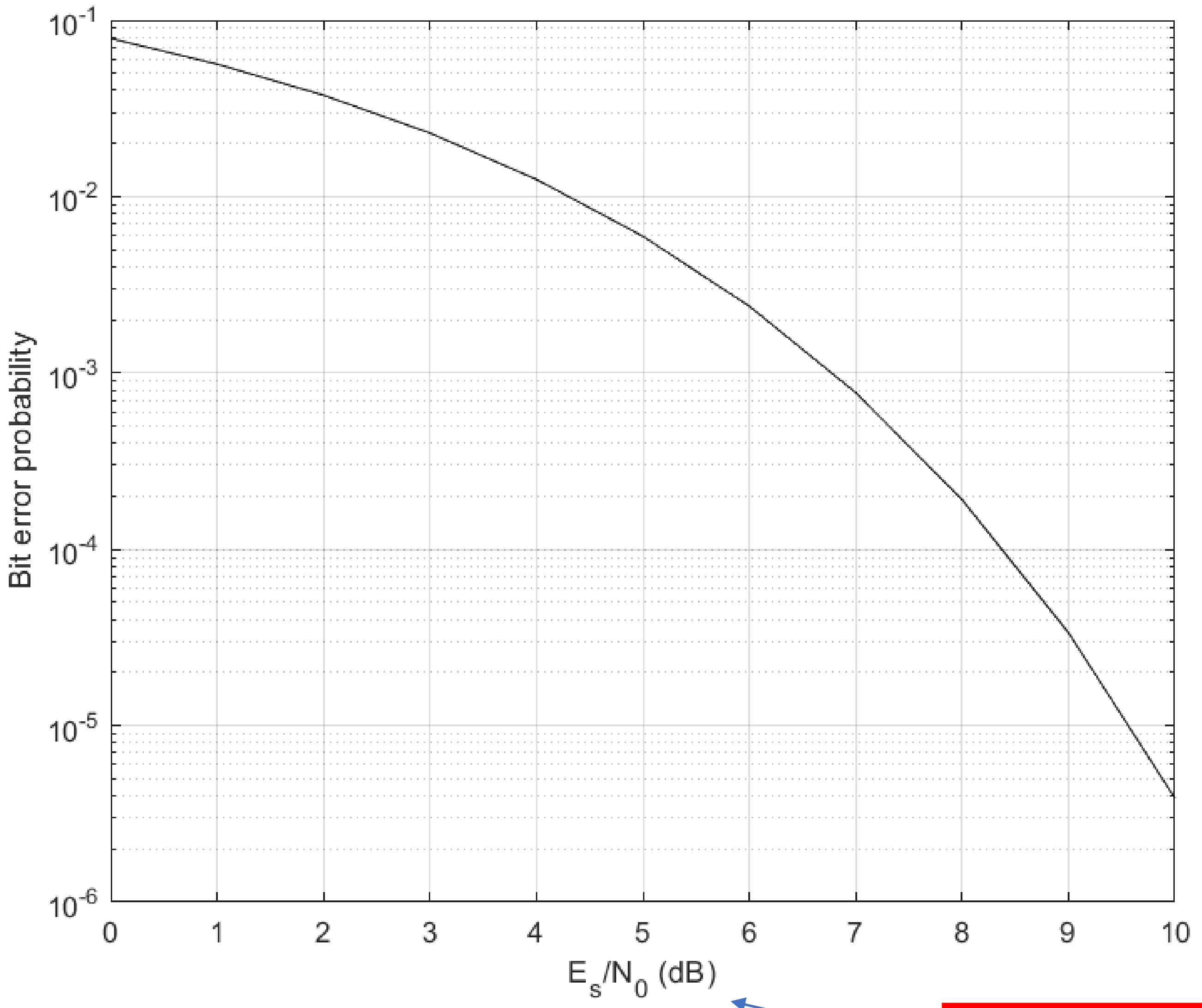
$P_b \to 0$

as $\underline{T \to \infty}$

This makes sense, since the longer the observation interval, the closer the output is the $\pm a$



Bit error probability, P_b, as a function of signal energy-to-noise ratio, E_s/N_0 (dB):



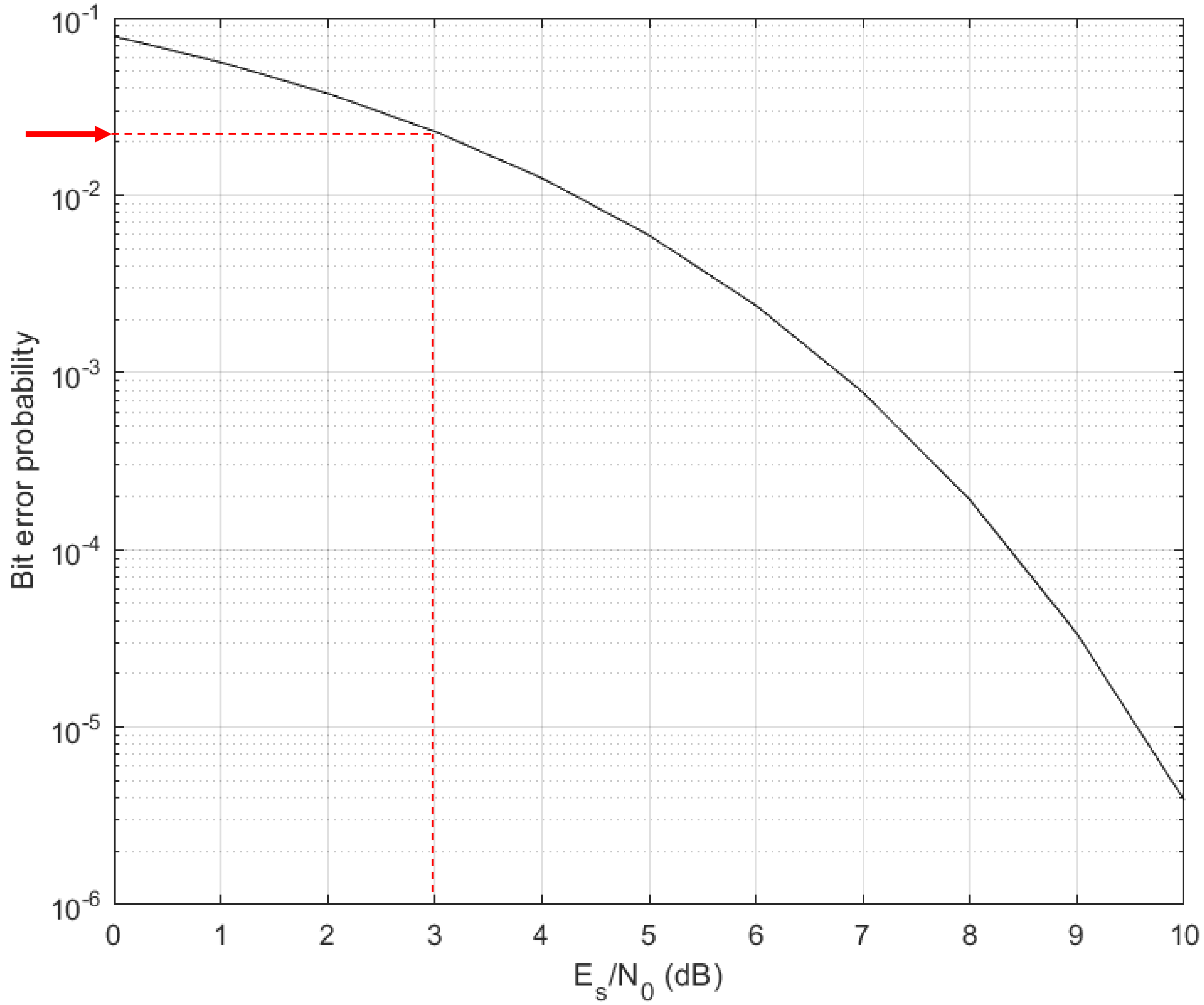
$$\boxed{P_b = Q\left(\sqrt{\frac{2E_s}{N_0}}\right)}$$

MATLAB script:

```
clear
EsNo_dB = 0:1:10;
perr = qfunc(sqrt(2*10.^(EsNo_dB/10)));
semilogy(EsNo_dB,perr,'-k')
xlabel('E_s/N_0 (dB)');
ylabel('Bit error probability');
grid on
```

$$\boxed{E_s/N_0\,(\mathrm{dB}) = 10\log_{10}(E_s/N_0)}$$

Example 1: What is value of P_b at E_s/N_0 = 3 dB?

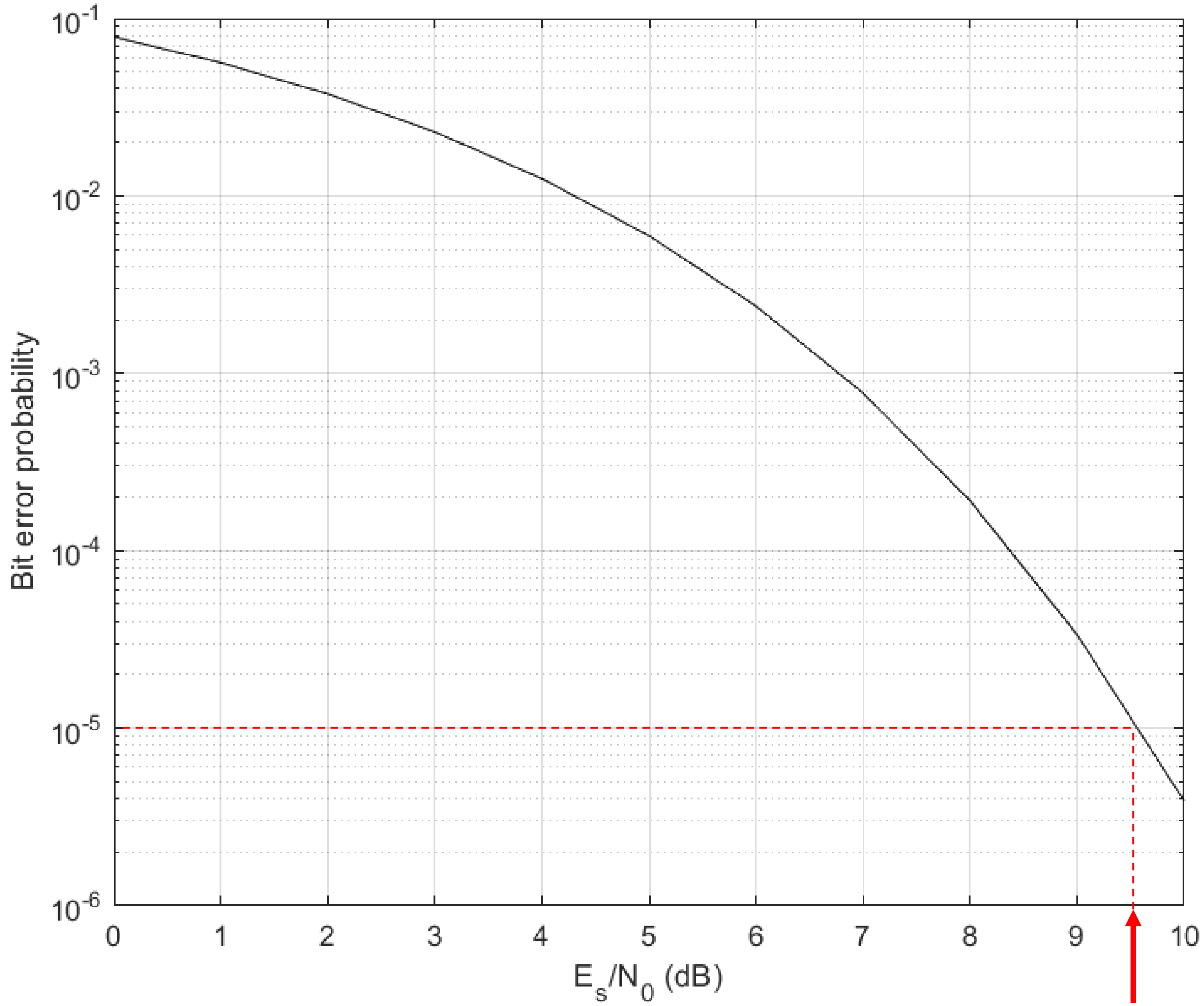


$$P_b = Q\left(\sqrt{\frac{2E_s}{N_0}}\right)$$

$$= Q\left(\sqrt{2 \cdot 10^{(3\text{ dB})/10}}\right) = Q(2)$$

$$= 0.0228 = \mathbf{2.3 \times 10^{-2}}$$

Example 2: What is value of E_s/N_0 (dB) is required to achieve $P_b=10^{-5}$?



Use: $Q^{-1}(10^{-5})$ = qfuncinv(1e-5) = 4.2649

$$\frac{E_s}{N_0} = \frac{1}{2}[Q^{-1}(P_b)]^2 = \frac{1}{2}[Q^{-1}(10^{-5})]^2$$

$$= \frac{(4.2649)^2}{2} = 9.095$$

$$\rightarrow \frac{E_s}{N_0}\ (\mathrm{dB}) = 10\log_{10}(9.095) = \mathbf{9.6\ dB}$$