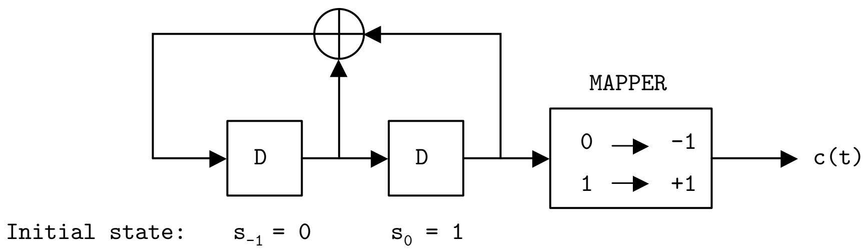
Autocorrelation function and power spectral density of a PN sequence

Consider an LFSR with connection polynomial $g(D) = 1 + D + D^2$ and a BPSK mapper as shown in the figure below.



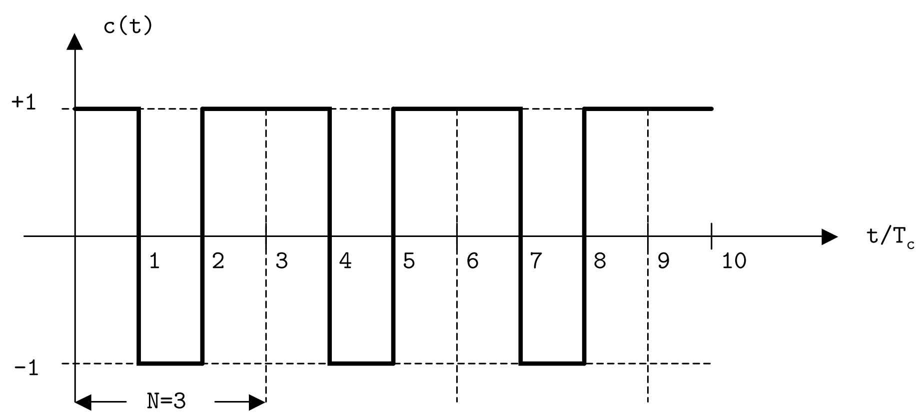
The transmitted signal is a random sequence of rectangular pulses

$$c(t) = \sum_{n=0}^{\infty} c_n \Pi\left(\frac{t - T_c/2 - nT_c}{T_c}\right).$$

The values of the coefficients c_n are obtained mapping the bit s_n as shown in the table below:

Time, n	State, $s_{n-1}s_n$	Output, c_n
0	0 1	+1
1	1 0	-1
2	1 1	+1
3	0 1	+1
4	1 0	-1
5	1 1	+1
6	0 1	+1
7	1 0	-1
8	1 1	+1
9	0 1	+1

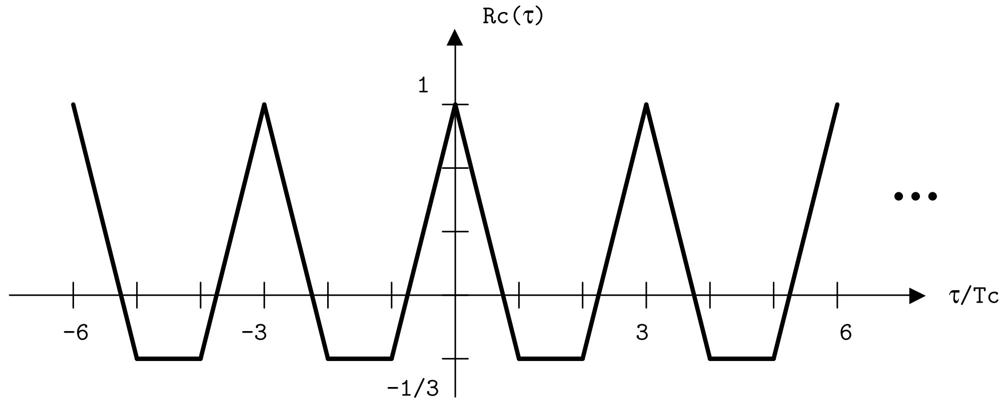
Evidently, the sequence $c(t)$ is periodic with period $T_b = 3T_c$ and is sketched in the figure below.



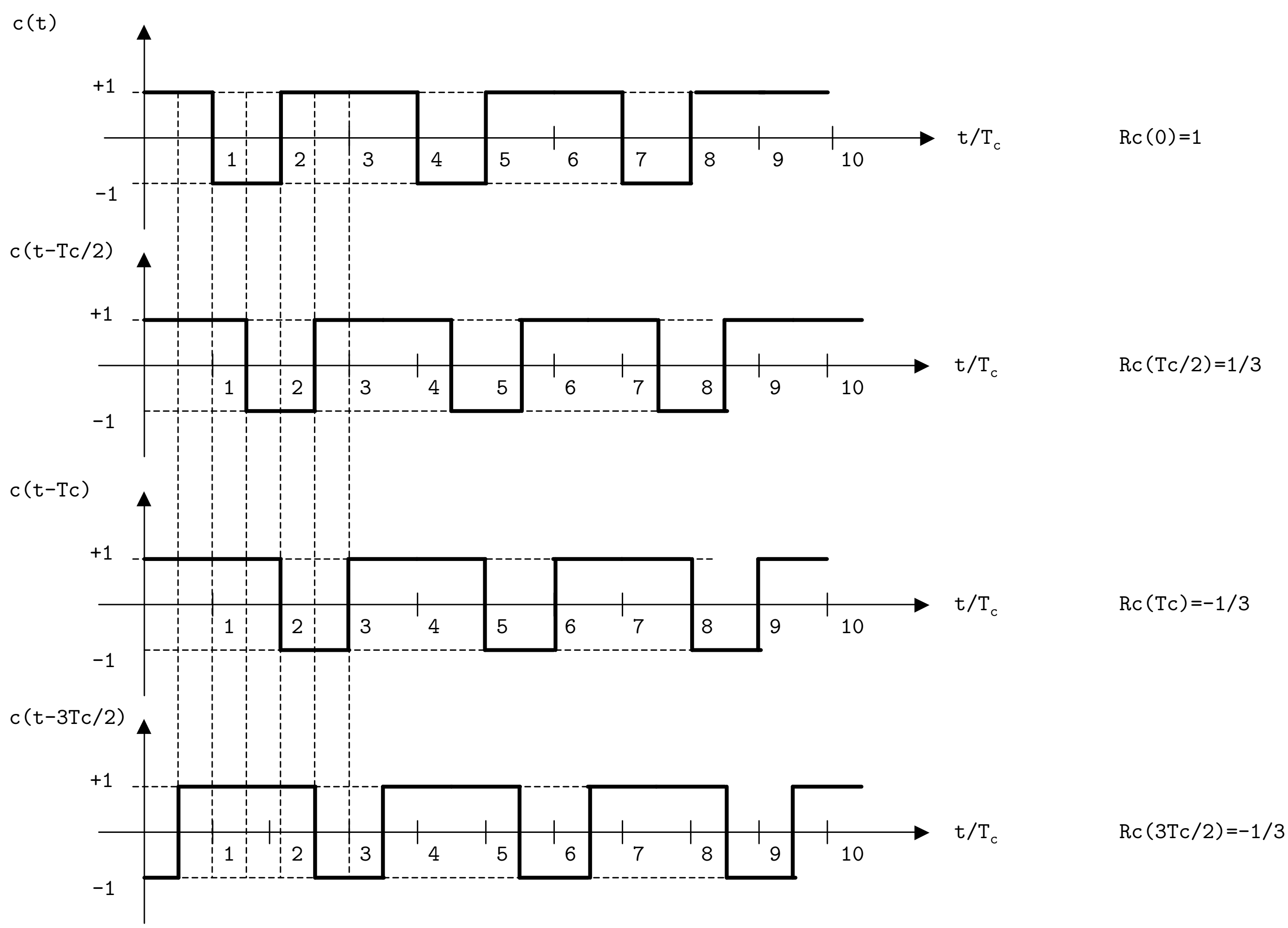
The autocorrelation function of $c(t)$, defined as

$$R_c(\tau) = \frac{1}{T_b} \int_{t_0}^{t_0+T_b} c(t)c(t-\tau)dt,$$

is also periodic with period $T_b = NT_c$, $N = 3$, and shown below



The peak values of the function $R_c(\tau)$ can be computed with the aid of the sketch below (with $t_0 = 0$).



Other values can be found by the even symmetry of $R_c(\tau)$, e.g., $R_c(-T_c/2) = R_c(T_c/2) = 1/3$.

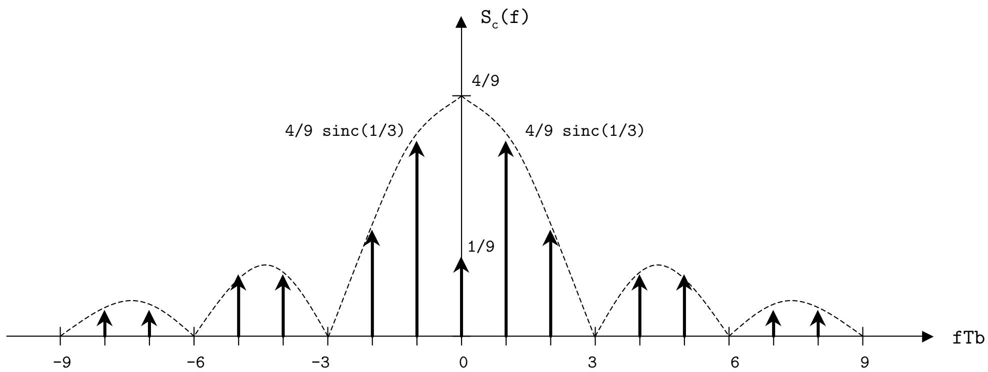
To compute the power spectral density of $c(t)$, write the autocorrelation function as a sequence of triangular and rectangular pulses as follows

$$\begin{aligned}
R_c(\tau) &= \sum_{n=-\infty}^{\infty} \frac{4}{3}\,\Lambda\left(\frac{t-3nT_c}{T_c}\right) - \frac{1}{3}\,\Pi\left(\frac{t-3nT_c}{3T_c}\right) \\
&= \left[\frac{4}{3}\,\Lambda\left(\frac{t}{T_c}\right) - \frac{1}{3}\,\Pi\left(\frac{t}{3T_c}\right)\right] \star \sum_{n=-\infty}^{\infty} \delta(t-3nT_c).
\end{aligned}$$

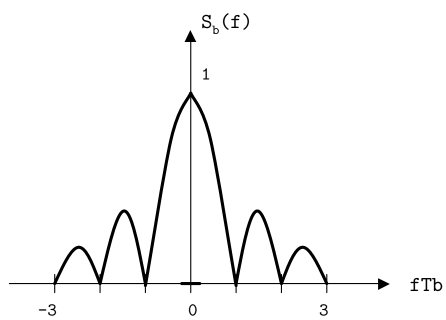
Then,

$$\begin{aligned}
S_c(f) = \mathcal{F}\{R_c(\tau)\} &= \left[\frac{4}{3}\,T_c\,\mathrm{sinc}^2\left(fT_c\right) - \frac{1}{3}\,3T_c\,\mathrm{sinc}\left(3T_c f\right)\right]\frac{1}{3T_c}\sum_{n=-\infty}^{\infty}\delta\left(f-\frac{n}{3T_c}\right) \\
&= \sum_{n=-\infty}^{\infty}\left[\frac{4}{9}\,\mathrm{sinc}^2\left(\frac{n}{3}\right) - \frac{1}{3}\,\mathrm{sinc}\left(n\right)\right]\delta\left(f-\frac{n}{3T_c}\right) \\
&= \frac{1}{9}\,\delta(f) + \frac{4}{9}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\mathrm{sinc}^2\left(\frac{n}{3}\right)\delta\left(f-\frac{n}{3T_c}\right).
\end{aligned}$$

The power spectral density (PSD) function is shown in the figure below.



Now compare this with the PSD function of BPSK modulation using rectangular pulses and signaling rate $1/T_b$:



We have that the ratio of peak PSD values is $\frac{S_{b,\max}}{S_{c,\max}} = \left(\frac{4}{9}\,\mathrm{sinc}(\frac{1}{3})\right)^{-1} = 2.72$. In general,

$$\frac{S_{b,\max}}{S_{c,\max}} = \frac{N^2}{(N+1)\,\mathrm{sinc}\left(\frac{1}{N}\right)} \to N, \qquad \text{as} \quad N\to\infty.$$