Lecture #1 – A Review of Sinusoids

Some Trigonometry

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

adding: $\sin A \cos B = \frac{1}{2}\sin(A+B) + \frac{1}{2}\sin(A-B)$

subtracting: $\cos A \sin B = \frac{1}{2}\sin(A+B) - \frac{1}{2}\sin(A-B)$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

adding: $\cos A \cos B = \frac{1}{2}\cos(A+B) + \frac{1}{2}\cos(A-B)$

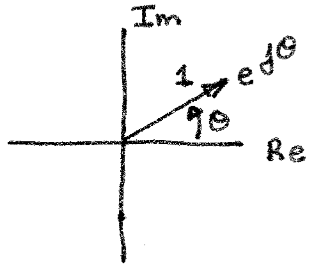
subtracting: $\sin A \sin B = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$

$$\cos^2 A = \frac{1+\cos 2A}{2} \qquad \sin^2 A = \frac{1-\cos 2A}{2}$$

Complex Exponentials

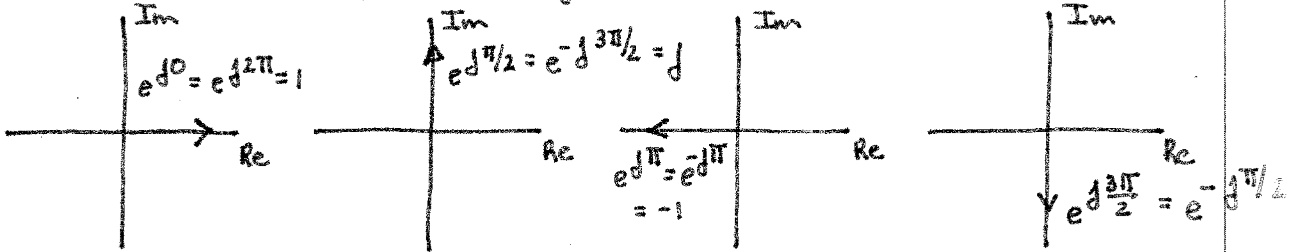
Euler Equation: $e^{j\theta} = \cos\theta + j\sin\theta$

$$e^{-j\theta} = \cos\theta - j\sin\theta$$



adding: $\cos\theta = \frac{1}{2}\left[e^{j\theta} + e^{-j\theta}\right]$

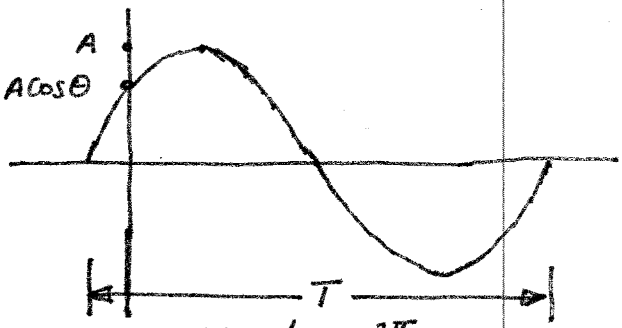
subtracting: $\sin\theta = \frac{1}{2j}\left[e^{j\theta} - e^{-j\theta}\right]$



Sinusoidal Voltages

A sinusoidal voltage is written as

$$v(t) = A\cos(\omega_0 t - \Theta)$$



$$T = \frac{1}{f_0} = \frac{2\pi}{\omega_0}$$

$$v(t) = \frac{A}{2}\left[e^{j(\omega_0 t - \Theta)} + e^{-j(\omega_0 t - \Theta)}\right]$$

$$= \frac{A}{2} e^{-j\Theta} e^{j\omega_0 t} + \frac{A}{2} e^{j\Theta} e^{-j\omega_0 t}$$

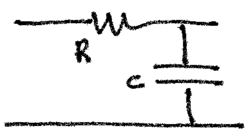
We like the exponential representation of a sinusoid.

Linear Systems and Exponential Forcing Functions

$$x(t) = c\,e^{j\omega_0 t} = |c| e^{j\phi} e^{j\omega_0 t} \longrightarrow \boxed{\begin{array}{c} H(j\omega) = \\ |H(j\omega)|\, e^{j\Theta(\omega)} \end{array}} \longrightarrow y(t) = c\,H(j\omega_0)\, e^{j\omega_0 t}$$

$$= |c|\,|H(j\omega_0)|\, e^{j(\phi + \Theta(\omega_0))} e^{j\omega_0 t}$$

example #1 A lowpass RC circuit



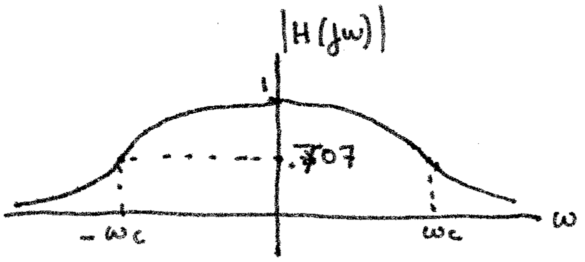
$\omega_c = 1/RC$

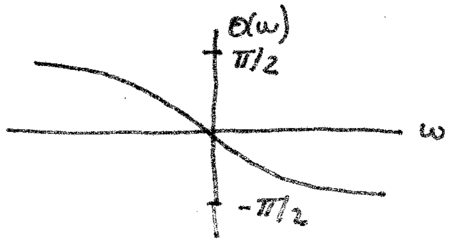
$$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$

$$H(s) = \frac{1}{s/\omega_c + 1}$$

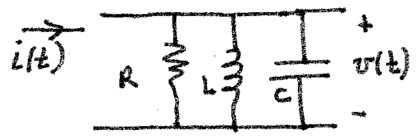
$$H(j\omega) = \frac{1}{1 + j\,\omega/\omega_c} = \frac{1}{\sqrt{1 + (\omega/\omega_c)^2}}\, e^{-j\tan^{-1}(\omega/\omega_c)}$$

$$= |H(j\omega)|\, e^{+j\Theta(\omega)}$$



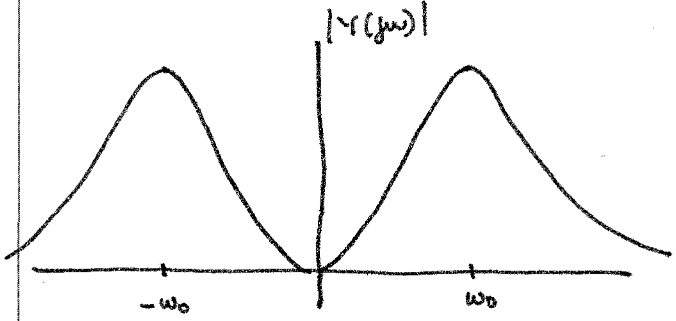


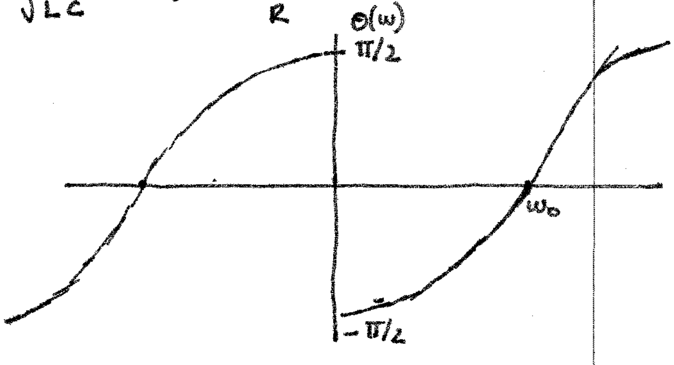
example #2 A simple bandpass circuit



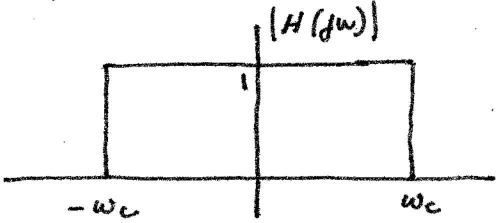
$$Y(j\omega) = \frac{1}{R} + j\,\varphi_0\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) = \frac{I(j\omega)}{V(j\omega)}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad \varphi_0 = \frac{\omega_0 L}{R}$$

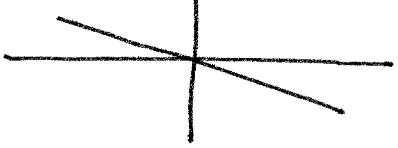




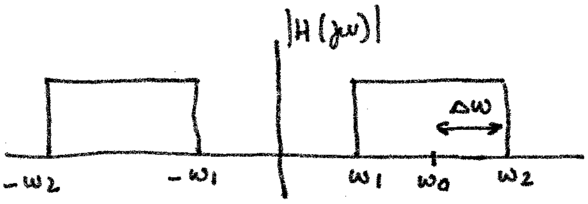
example #3 An ideal lowpass filter



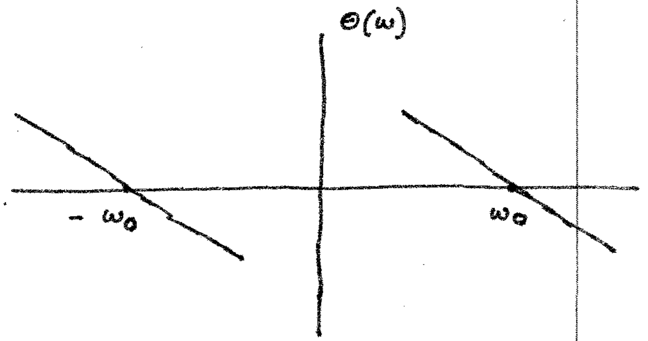
$\theta(\omega) = -\omega t_0$



example #4 An ideal bandpass filter



$$\omega_2 = \omega_0 + \Delta\omega$$
$$\omega_1 = \omega_0 - \Delta\omega$$



Note $|H(j\omega)|$ is even (symmetric) around $\omega = 0$

$\theta(\omega)$ is odd (antisymmetric) around $\omega = 0$

The Average Power of a Sinusoid

One Sinusoid

$$v(t) = A \cos \omega_0 t \qquad \omega_0 = \frac{2\pi}{T}$$

$$p(t) = v^2(t) = A^2 \cos^2 \omega_0 t = \frac{A^2}{2}\left[1 + \cos 2\omega_0 t\right]$$

$$P_{AV} = \frac{1}{T}\int_{-T/2}^{T/2} p(t)\,dt = \frac{1}{T}\frac{A^2}{2}\int_{-T/2}^{T/2}\left[1 + \cos 2\omega_0 t\right]dt = \frac{A^2}{2}$$

Two Sinusoids

$$v(t) = A \cos \omega_0 t + B \cos n \omega_0 t$$

$$p(t) = v^2(t) = A^2 \cos^2 \omega_0 t + B^2 \cos^2 n\omega_0 t + 2AB \cos \omega_0 t \cos n\omega_0 t$$

$$= \frac{A^2}{2}\left[1 + \cos 2\omega_0 t\right] + \cancel{\frac{B^2}{2}[1+\cos} + \frac{B^2}{2}\left[1 + \cos 2n\omega_0 t\right] + AB \cos(n+1)\omega_0 t + AB \cos(n-1)\omega_0 t$$

$$P_{AV} = \frac{A^2}{2} + \frac{B^2}{2}$$

But if we have $\quad v(t) = A \cos \omega_0 t + B \cos \beta \omega_0 t$

where $\beta \neq n$

then $\quad P_{AV} = \frac{1}{T}\int_{-T/2}^{T/2} p^2(t)\,dt \neq \frac{A^2}{2} + \frac{B^2}{2}$

This is the issue of orthogonality.

ω_0 is a fundamental

$n\omega_0$ is the n^{th} harmonic

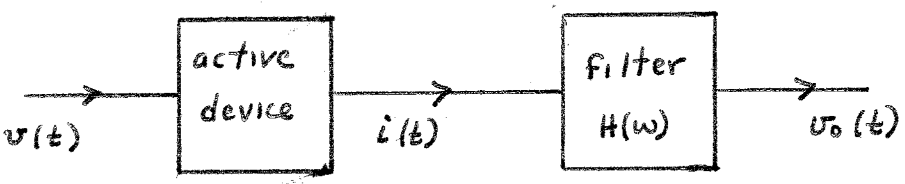
harmonics and fundamentals are orthogonal over the period of the fundamental.

Sinusoids and Nonlinearities

An active device (like a transistor) has a transfer characteristic

$$i(t) = a_0 + a_1 v(t) + a_2 v^2(t) + a_3 v^3(t) + \cdots$$

where $a_0 > a_1 > a_2 > a_3 > \cdots$. Typically, the active device is followed by a filter in order to be able to extract the desired set of frequencies.



example #1 **Amplitude Modulation**

Let $v(t) = A \cos \omega_c t + B \cos \omega_1 t$ where $\omega_1 \ll \omega_c$.

carrier frequency (1 MHz)

tone (1 KHz)

Ignore a_3, a_4, a_5, etc ...

Then

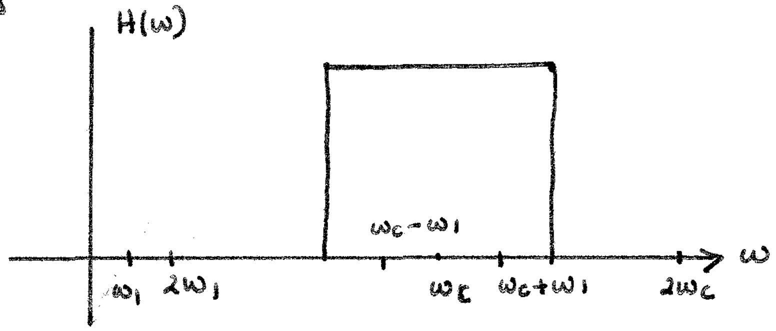
$$i(t) = a_0 + a_1 (A \cos \omega_c t + B \cos \omega_1 t)$$

$$+ a_2 (A^2 \cos^2 \omega_c t + 2AB \cos \omega_c t \cos \omega_1 t + B^2 \cos^2 \omega_1 t)$$

$$i(t) = \left[a_0 + \frac{a_2 A^2}{2} + \frac{a_2 B^2}{2}\right] + a_1 B \cos\omega_1 t + \frac{a_2 B^2}{2}\cos 2\omega_1 t$$

$$\underline{+ a_1 A \cos\omega_c t + 2a_2 AB\cos\omega_1 t \cos\omega_c t} + \frac{a_2 B^2}{2}\cos 2\omega_c t$$

A bandpass filter centered at ω_c, ~~will~~ and extending a bit beyond $\omega_c \pm \omega_1$ will include the underlined portion and exclude everything else.



The filter output can then be written as

$$v_o(t) = a_1 A\left[1 + 2B\frac{a_2}{a_1}\cos\omega_1 t\right]\cos\omega_c t$$

This is an <u>amplitude</u> - <u>modulated</u> signal.

<u>example #2</u> <u>Heterodyning</u> or <u>Mixing</u>

This is similar to amplitude modulation except that

$$v(t) = A\cos\omega_1 t + B\cos\omega_2 t$$

where ω_1 and ω_2 are relatively close. For example, $\omega_1 = 2$ MHz and $\omega_2 = 5$ MHz.

In this instance, we want to use the term

$$2ABa_2 \cos\omega_1 t \cos\omega_2 t = a_2 AB\left[\cos(\omega_1+\omega_2)t + \cos(\omega_1-\omega_2)t\right]$$

to generate the sum and the difference frequencies.

The filter is used to extract the desired one of these frequencies from the complitcated set of non-linear device outputs.

example #3 Intermodulation Distortion

In this example, we will focus on the 3rd order term in the non-linearity $a_3 v^3(t)$
and we will have the input be composed of two frequencies that are very close to each other

$$v(t) = A\cos\omega_1 t + B\cos\omega_2 t \qquad \text{where } \omega_2 = \omega_1 + \Delta$$

$$\Rightarrow \quad v^3(t) = A^3\cos^3\omega_1 t + 3A^2B\cos^2\omega_1 t\cos\omega_2 t + 3AB^2\cos\omega_1 t\cos^2\omega_2 t + B^3\cos^3\omega_2 t$$

Now

$$A^3\cos^3\omega_1 t = \frac{A^3}{2}\cos\omega_1 t\,(1+\cos 2\omega_1 t)$$

$$= \frac{3A^3}{4}\cos\omega_1 t + \frac{A^3}{4}\cos 3\omega_1 t$$

$$3A^2B\cos^2\omega_1 t\cos\omega_2 t = \frac{3A^2B}{2}\left(1+\cos 2\omega_1 t\right)\cos\omega_2 t$$

$$= \frac{3A^2B}{2}\cos\omega_2 t + \frac{3A^2B}{4}\cos(2\omega_1+\omega_2)t + \frac{3A^2B}{4}\cos(2\omega_1-\omega_2)t$$