EE1 and ISE1 Communications I

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Lecture four

https://www.commsp.ee.ic.ac.uk/~pld/Teaching/comms1_09/lecture_4.pdf

Lecture Aims

- Trigonometric Fourier series
- Fourier spectrum
- Exponential Fourier series

Trigonometric Fourier series

- Consider a signal set

$$\{1, \cos \omega_0 t, \cos 2\omega_0 t, ..., \cos n\omega_0 t, ..., \sin \omega_0 t, \sin 2\omega_0 t, ..., \sin n\omega_0 t, ...\}$$

- A sinusoid of frequency $n\omega_0$ is called the n^{th} harmonic of the sinusoid, where n is an integer.

- The sinusoid of frequency ω_0 is called the fundamental harmonic.

- This set is orthogonal over an interval of duration $T_0 = 2\pi/\omega_0$, which is the period of the fundamental.

Trigonometric Fourier series

The components of the set $\{1, \cos\omega_0 t, \cos 2\omega_0 t, \dots, \cos n\omega_0 t, \dots, \sin\omega_0 t, \sin 2\omega_0 t, \dots, \sin n\omega_0 t, \dots\}$ are orthogonal as

$$\int_{T_0} \cos n\omega_0 t \cos m\omega_0 t dt = \begin{cases} 0 & m \neq n \\ \frac{T_0}{2} & m = n \neq 0 \end{cases}$$

$$\int_{T_0} \sin n\omega_0 t \sin m\omega_0 t dt = \begin{cases} 0 & m \neq n \\ \frac{T_0}{2} & m = n \neq 0 \end{cases}$$

$$\int_{T_0} \sin n\omega_0 t \cos m\omega_0 t dt = 0 \quad \text{for all } m \text{ and } n$$

$\int_{T_0}$ means integral over an interval from $t = t_1$ to $t = t_1 + T_0$ for any value of t_1.

Trigonometric Fourier series

This set is also *complete* in T_0. That is, any signal in an interval $t_1 \leq t \leq t_1 + T_0$ can be written as the sum of sinusoids. Or

$$
\begin{aligned}
g(t) &= a_0 + a_1 \cos \omega_0 t + a_2 \cos 2\omega_0 t + ... + b_1 \sin \omega_0 t + b_2 \sin 2\omega_0 t + ... \\
&= a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t
\end{aligned}
$$

Series coefficients

$$
a_n = \frac{\langle g(t), \cos n\omega_0 t \rangle}{\langle \cos n\omega_0 t, \cos n\omega_0 t \rangle} \qquad b_n = \frac{\langle g(t), \sin n\omega_0 t \rangle}{\langle \sin n\omega_0 t, \sin n\omega_0 t \rangle}
$$

Trigonometric Fourier Coefficients

Therefore

$$a_n = \frac{\int_{t_1}^{t_1+T_0} g(t)\cos n\omega_0 t dt}{\int_{t_1}^{t_1+T_0} \cos^2 n\omega_0 t dt}$$

As

$$\int_{t_1}^{t_1+T_0} \cos^2 n\omega_0 t dt = T_0/2, \qquad \int_{t_1}^{t_1+T_0} \sin^2 n\omega_0 t dt = T_0/2.$$

We get

$$a_0 = \frac{1}{T_0}\int_{t_1}^{t_1+T_0} g(t) dt$$

$$a_n = \frac{2}{T_0}\int_{t_1}^{t_1+T_0} g(t)\cos n\omega_0 t dt \quad n = 1, 2, 3, ...$$

$$b_n = \frac{2}{T_0}\int_{t_1}^{t_1+T_0} g(t)\sin n\omega_0 t dt \quad n = 1, 2, 3, ...$$

Compact Fourier series

Using the identity

$$a_n \cos n\omega_0 t + b_n \sin n\omega_0 t = C_n \cos(n\omega_0 t + \theta_n)$$

where

$$C_n = \sqrt{a_n^2 + b_n^2} \qquad\qquad \theta_n = \tan^{-1}(-b_n/a_n).$$

The trigonometric Fourier series can be expressed in compact form as

$$g(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n) \quad t_1 \le t \le t_1 + T_0.$$

For consistency, we have denoted a_0 by C_0.

Periodicity of the Trigonometric series

We have seen that an arbitrary signal $g(t)$ may be expressed as a trigonometric Fourier series over any interval of T_0 seconds.
What happens to the Trigonometric Fourier series outside this interval?
Answer: The Fourier series is periodic of period T_0 (the period of the fundamental harmonic).
Proof:

$$\phi(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n) \quad \text{for all } t$$

and

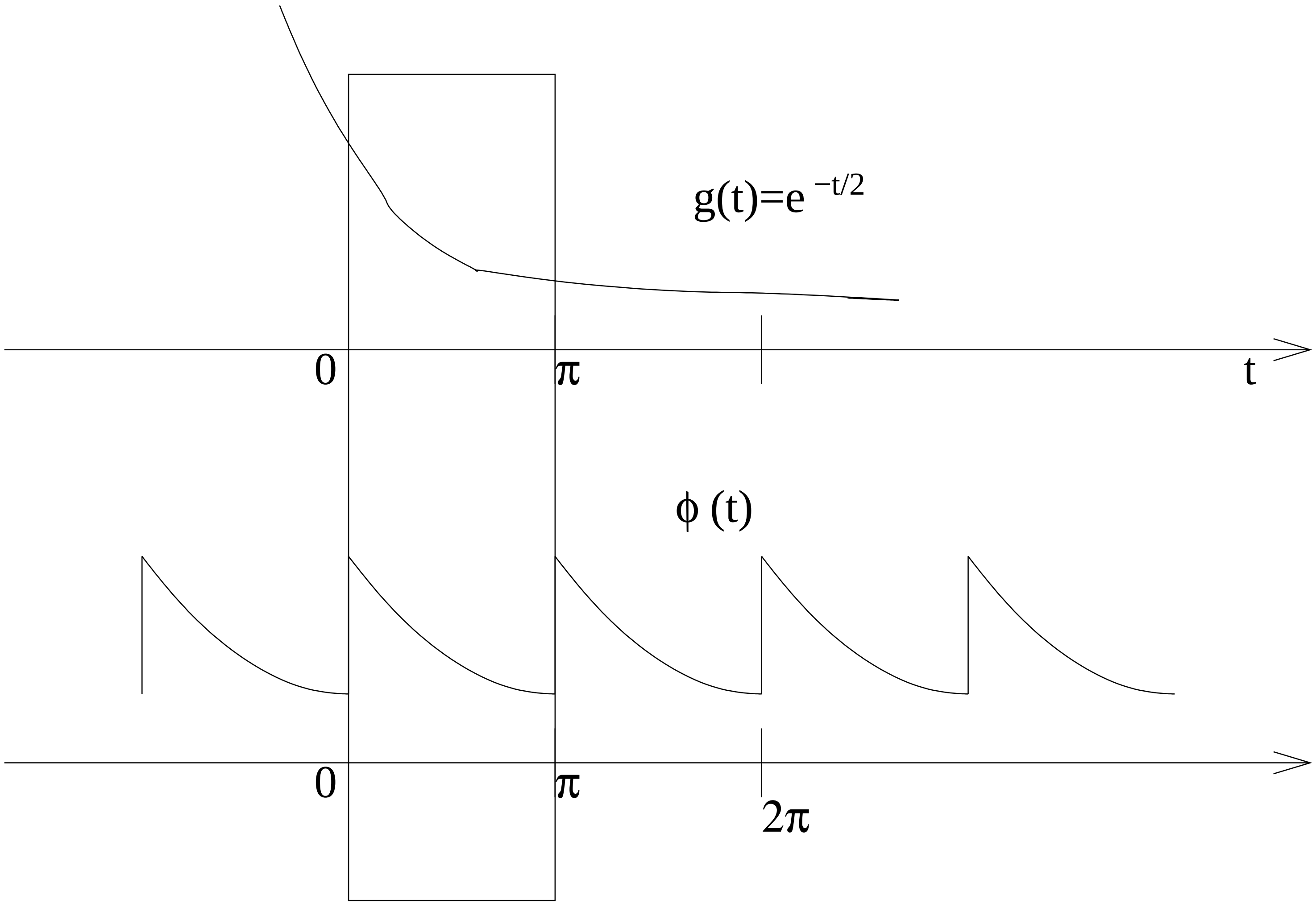
$$\begin{aligned}
\phi(t+T_0) &= C_0 + \sum_{n=1}^{\infty} C_n \cos[n\omega_0(t+T_0) + \theta_n] \\
&= C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + 2n\pi + \theta_n) \\
&= C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n) \\
&= \phi(t) \quad \text{for all } t
\end{aligned}$$

Properties of trigonometric series

- The trigonometric Fourier series is a periodic function of period $T_0 = 2\pi/\omega_0$.

- If the function $g(t)$ is periodic with period T_0, then a Fourier series representing $g(t)$ over an interval T_0 will also represent $g(t)$ for all t.

Example



Example

$\omega_0 = 2\pi/T_0 = 2$ rad/s.

$$g(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(2nt + \theta_n)$$

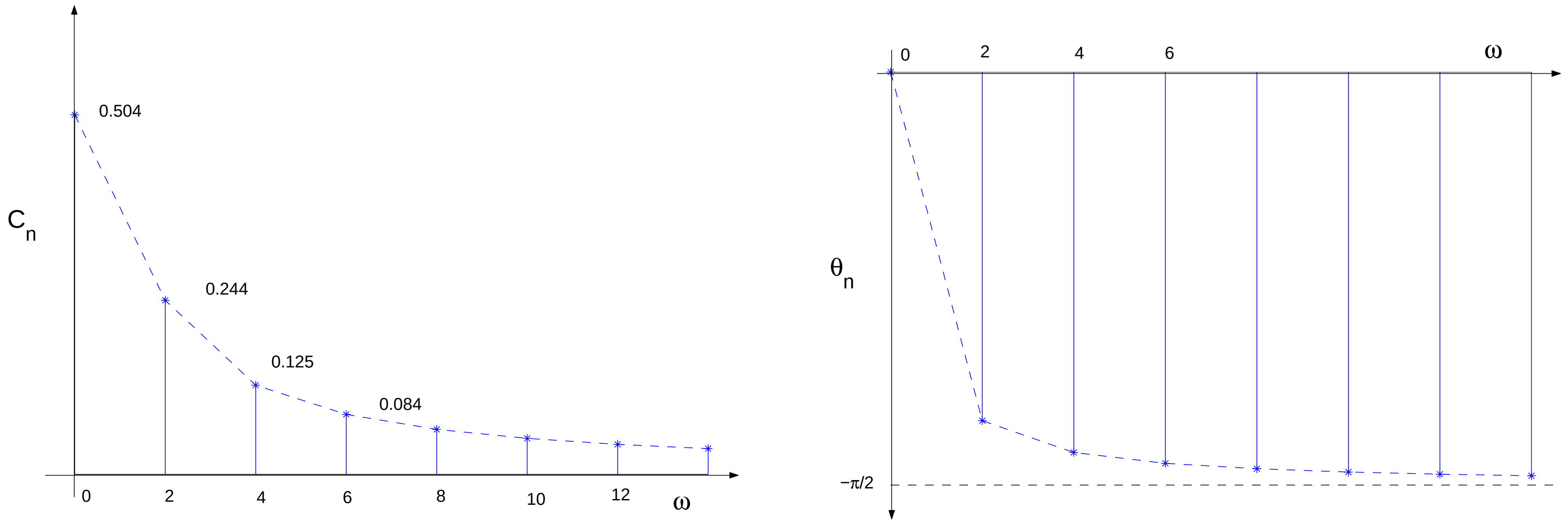
n	0	1	2	3	4
C_n	0.504	0.244	0.125	0.084	0.063
θ_n	0	-75.96	-82.87	-85.84	-86.42

We can plot

- the amplitude C_n versus ω this gives us the **amplitude spectrum**
- the phase θ_n versus ω (**phase spectrum**).

This two plots together are the **frequency spectra** of $g(t)$.

Amplitude and phase spectra



Exponential Fourier Series

Consider a set of exponentials

$$e^{jn\omega_0 t} \qquad n = 0, \pm 1, \pm 2, ...$$

The components of this set are orthogonal.

A signal $g(t)$ can be expressed as an exponential series over an interval T_0:

$$g(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \qquad D_n = \frac{1}{T_0}\int_{T_0} g(t) e^{-jn\omega_0 t}\mathrm{d}t$$

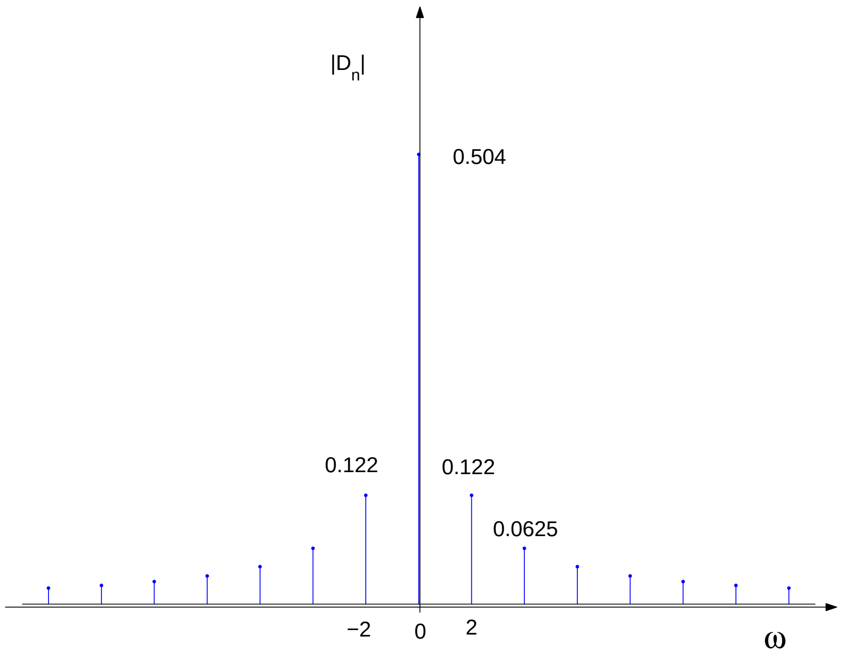
Trigonometric and exponential Fourier series

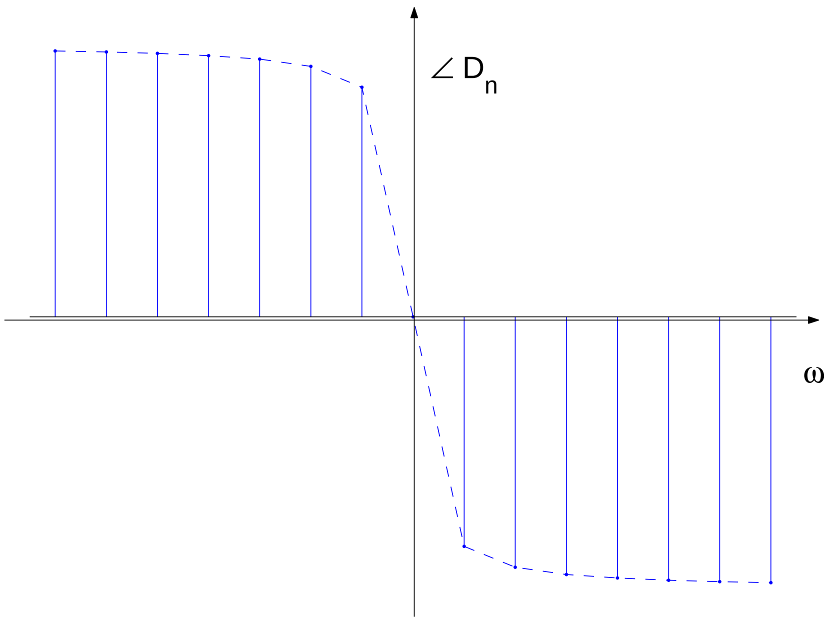
Trigonometric and exponential Fourier series are related. In fact, a sinusoid in the trigonometric series can be expressed as a sum of two exponentials using Euler's formula.

$$\begin{aligned} C_n \cos(n\omega_0 t + \theta_n) &= \tfrac{C_n}{2}\left[e^{j(n\omega_0 t + \theta_n)} + e^{-j(n\omega_0 t + \theta_n)}\right] \\ &= \left(\tfrac{C_n}{2} e^{j\theta_n}\right) e^{jn\omega_0 t} + \left(\tfrac{C_n}{2} e^{-j\theta_n}\right) e^{-jn\omega_0 t} \\ &= D_n e^{jn\omega_0 t} + D_{-n} e^{-jn\omega_0 t} \end{aligned}$$

$$D_n = \frac{1}{2} C_n e^{j\theta_n} \qquad D_{-n} = \frac{1}{2} C_n e^{-j\theta_n}$$

Amplitude and phase spectra. Exponential case





Parseval's Theorem

Trigonometric Fourier series representation $g(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$.
The power is given by

$$P_g = C_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} C_n^2.$$

Exponential Fourier series representation $g(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$.
Power for the exponential representation

$$P_g = \sum_{n=-\infty}^{\infty} |D_n|^2$$

Conclusions

- Trigonometric Fourier series
- Exponential Fourier series
- Amplitude and phase spectra
- Parseval's theorem