Solution of Homework # 3

1. (a) The signal $x(t)$ is periodic with period $T_0 = 4$. The derivative of $x(t)$, denoted $y(t) = \frac{d}{dt}\,x(t)$ is a train of rectangular pulses of amplitude $\frac{3}{2}$ delayed by 1 sec with a duty cycle $d = \frac{1}{2}$. Its Fourier series coefficients (FSC) are

$$y_n = \frac{3}{4}\,\text{sinc}\left(\frac{n}{2}\right)\; e^{-j\frac{\pi}{2}n}, \quad n \neq 0, \tag{1}$$

where the time delay property has been used.

As discussed in class, the FSC of $x(t)$ are related to those of $y(t)$ by the differentiation property:

$$y_n = j2\pi\frac{n}{T_0}\cdot x_n = j\frac{\pi}{2}\,n\cdot x_n, \quad n \neq 0. \tag{2}$$

From (1) and (2), the FSC of $x(t)$ are found to be, for $n \neq 0$,

$$x_n = \frac{\frac{3}{4}\,\text{sinc}\left(\frac{n}{2}\right)\; e^{-j\frac{\pi}{2}n}}{j\frac{\pi}{2}\,n} = \frac{3}{2\pi n}\;\text{sinc}\left(\frac{n}{2}\right)\; e^{-j\frac{\pi}{2}(n+1)}.$$

For $n = 0$, x_0 is simply the average of $x(t)$ which is given by

$$x_0 = \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)\,dt = \frac{1}{4}\int_0^2 \left(\frac{3t}{2}\right)\,dt = \frac{3}{4}.$$

(b) From part (a), the FSC amplitudes are given by

$$|x_n| = \begin{cases} \dfrac{3}{4}, & n = 0, \\[2ex] \left|\dfrac{3}{2\pi n}\,\text{sinc}\left(\dfrac{n}{2}\right)\right|, & n \neq 0. \end{cases}$$

For $n = \pm 1$,

$$|x_n| = \frac{3}{2\pi}\,\text{sinc}\left(\frac{1}{2}\right) = \frac{3}{\pi^2},$$

while for $n = \pm 2$, $x_n = 0$. The sketch is shown in the next page.

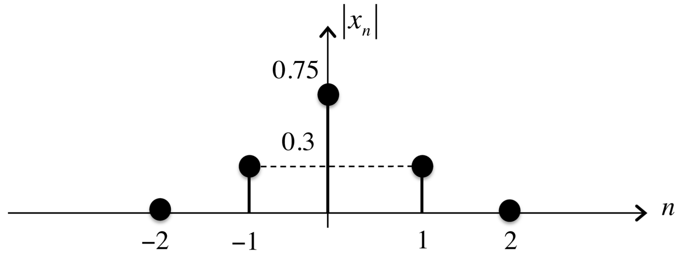
(c) Notice that in order to compute the THD, the signal needs to have *zero average*. This is done by subtracting x_0 before computing the power:

$$P_x = \frac{1}{4}\int_0^4 \left[x(t) - x_0\right]^2 dt = \frac{1}{4}\left[\int_0^2 \left(\frac{3}{2}\,t - \frac{3}{4}\right)^2 dt + \int_2^4 \left(-\frac{3}{4}\right)^2 dt\right] = \frac{33}{32}.$$

Finally,

$$\text{THD} = \frac{\frac{1}{2}\,P_x - |x_1|^2}{|x_1|^2} = \frac{\frac{1}{2}\,\left(\frac{33}{32}\right) - \left(\frac{3}{\pi^2}\right)^2}{\left(\frac{3}{\pi^2}\right)^2} = 4.5726$$

This value of THD indicates that waveform $x(t)$ is very different from a sinusoidal.



The discrete amplitude spectrum of $x(t)$ in problem 1

2. We have that

$$X(f) = \Pi\left(\frac{f}{1400}\right) + \Lambda\left(\frac{f}{700}\right),.$$

where $\Pi(\cdot)$ and $\Lambda(\cdot)$ denote respectively unit rectangular and triangular pulses. Using the properties of additivity and modulation of the Fourier transform,

$$Y(f) = -X(f) + \frac{1}{2}\left[X(f+1000) + X(f-1000)\right] + \left[X(f+1500) + X(f-1500)\right].$$

A computer plot of $Y(f)$ is shown in the last page.

3. (a) Let $x(t) = x_1(t)\cos(1000\pi t)$, where $x_1(t) = 10\,\mathrm{sinc}(2000t)$ with Fourier transform

$$X_1(f) = \frac{10}{2000}\,\Pi\left(\frac{f}{2000}\right).$$

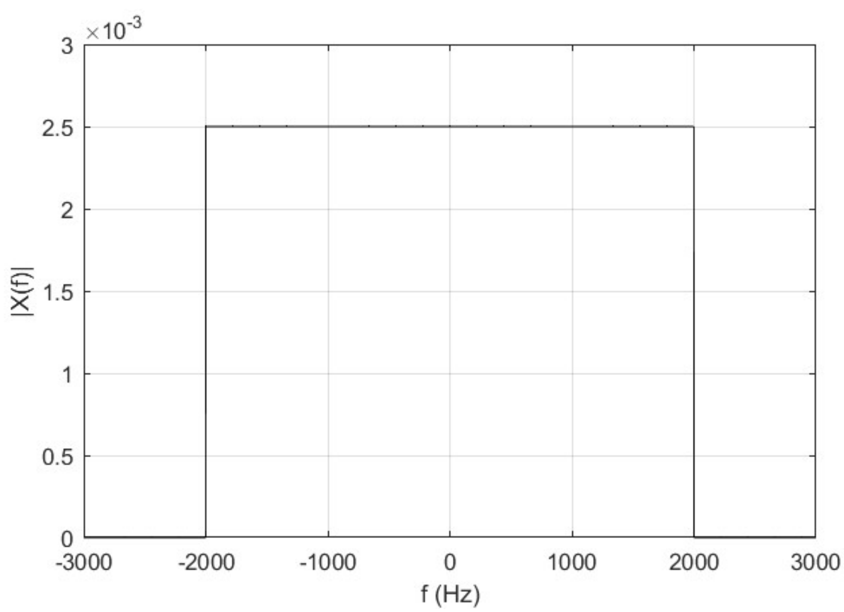
Using the modulation property.

$$X(f) = \frac{1}{2}\left[X_1(f+1000) + X_1(f-1000)\right] = \frac{1}{400}\left[\Pi\left(\frac{f+1000}{2000}\right) + \Pi\left(\frac{f-1000}{2000}\right)\right].$$

As seen in the sketch below, the two rectangular pulses are contiguous and thus we can also write:

$$X(f) = \frac{1}{400}\,\Pi\left(\frac{f}{4000}\right).$$

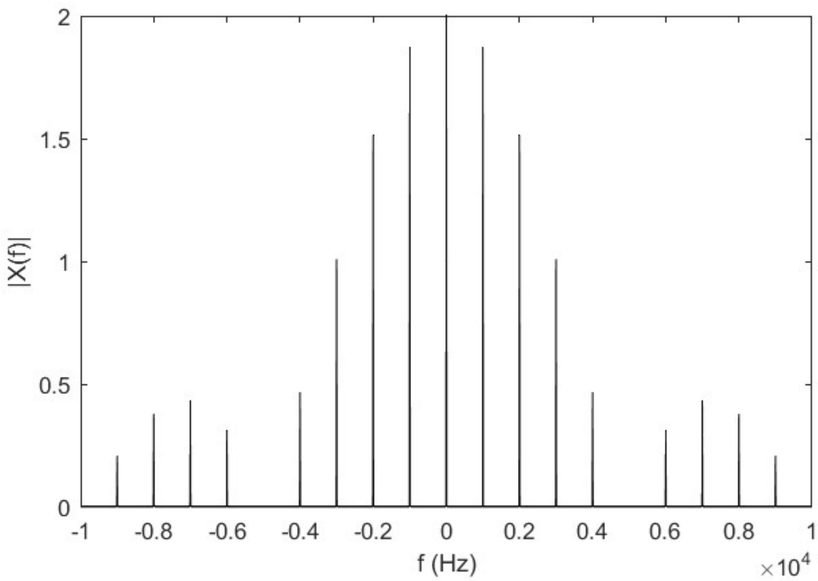
Amplitude spectrum sketch:

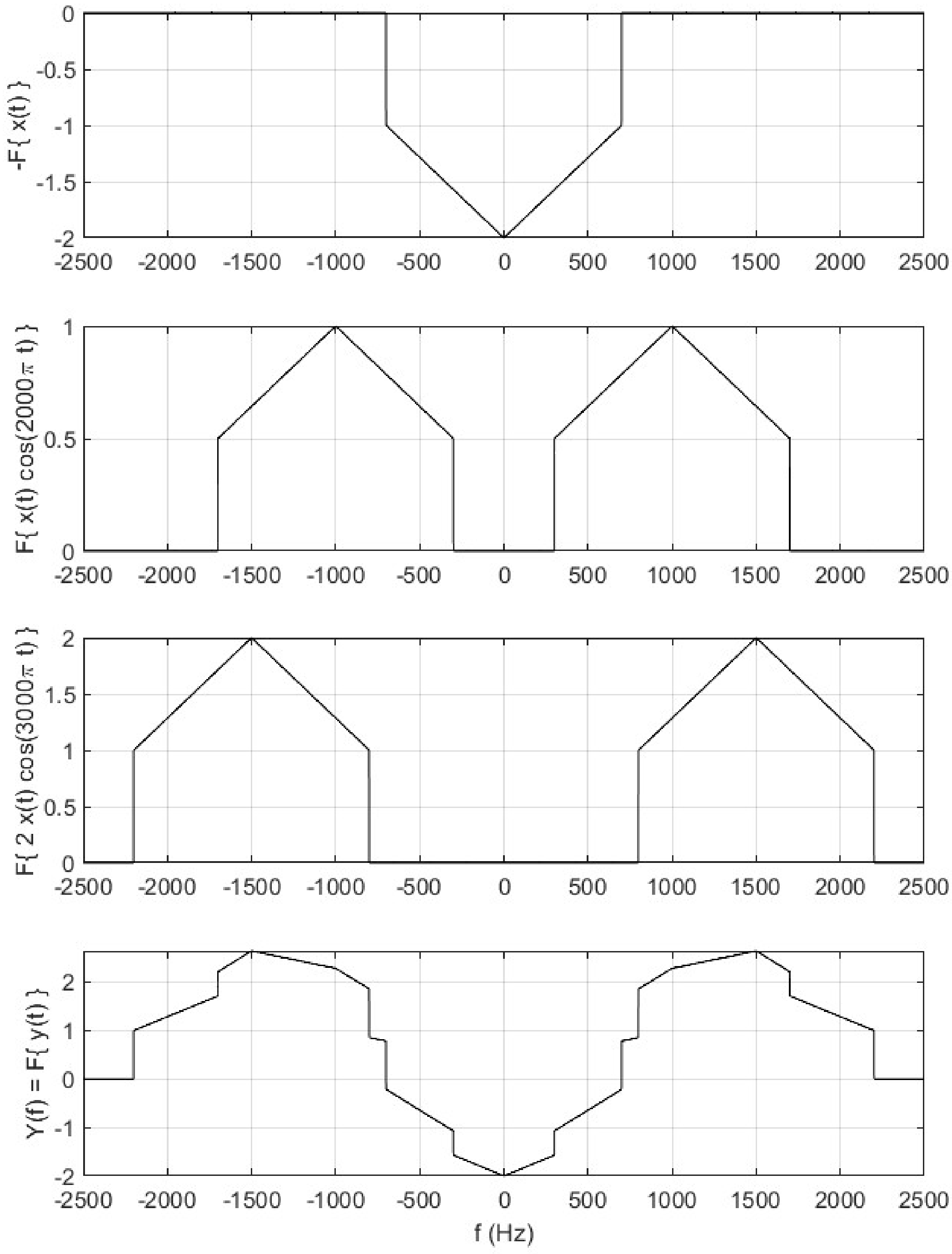


(b) The signal is periodic with fundamental frequency $f_0 = 1000$ Hz. The Fourier transform is

$$X(f) = \sum_{n=-\infty}^{\infty} x_n\, \delta(f - n f_0) = \frac{10}{5} \sum_{n=-\infty}^{\infty} \operatorname{sinc}\left(\frac{n}{5}\right) \delta(f - 1000n).$$

Amplitude spectrum sketch:





Amplitude spectrum $|Y(f)|$ of problem 2