Digital Data Transmission and Quadrature Modulation

EE 161: Digital Communication Systems

Prof. Robert Morelos-Zaragoza
Department of Electrical Engineering
San Jose State University

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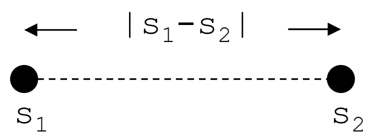
Increasing the data rate

- Decreasing the bit (signal) duration
  - Increases bandwidth
  - In power-limited cases, decreases energy, i.e., increases the probability of error
- Increasing the number of pulse amplitudes (e.g., M-ary PAM)
  - Probability of error increases, as amplitude level difference decrease for the same average energy
- Increasing the number of pulses
  - $N>1$ orthogonal pulses
  - Increases bandwidth

Probability of error analysis tools

- Pairwise error probability:



$$\text{P[decision error]} = P_{12} = Q\left(\sqrt{\frac{|s_1 - s_2|^2}{2N_0}}\right)$$

- Union bound:

$$\Pr\{\bigcup_{l=1}^{L} error_l\} \leq \sum_{l=1}^{L} \Pr\{error_l\}$$

Pairwise error probabilities

Example: 4-PAM

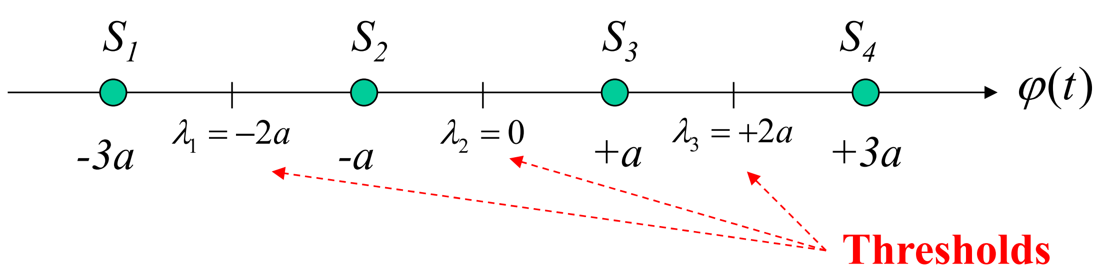
- Mapping: ℓ=2 bits/signal

Bits	*m*	S_m
00	*1*	*-3a*
01	*2*	*-a*
11	*3*	*+a*
10	*4*	*+3a*

Average energy:

$$E_s = 5a^2 \quad \Rightarrow \quad a = \sqrt{\frac{E_s}{5}}$$

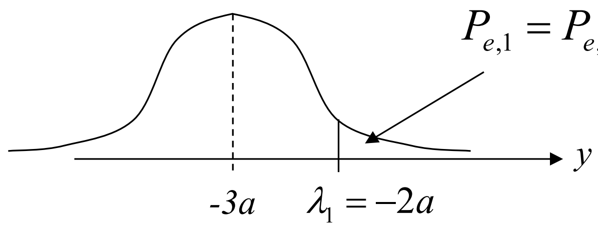
- 4-PAM signal set (constellation):



Example (cont): 4-PAM

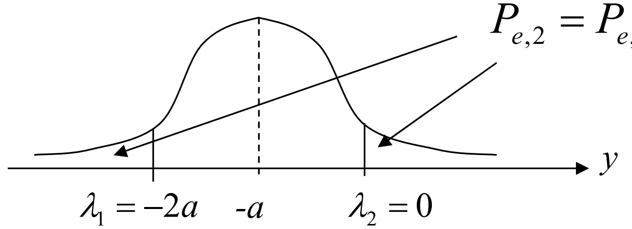
- Probability of error: Let $P_{e,i} = \Pr\{\text{error}|S_i \text{ sent}\}$
  - **Two types of signal points:**
    - Outer points (S_1 and S_4):

$$P_{e,1} = P_{e,4} = Q\left(\sqrt{\frac{2a^2}{N_0}}\right)$$



    - Inner points (S_2 and S_3):

$$P_{e,2} = P_{e,3} = 2\,Q\left(\sqrt{\frac{2a^2}{N_0}}\right)$$



Example (cont): 4-PAM

- Average probability of error

$$P_e = \frac{1}{4}\sum_{i=1}^{4} P_{e,i} = \frac{3}{2} Q\left(\sqrt{\frac{2E_s}{5N_0}}\right)$$

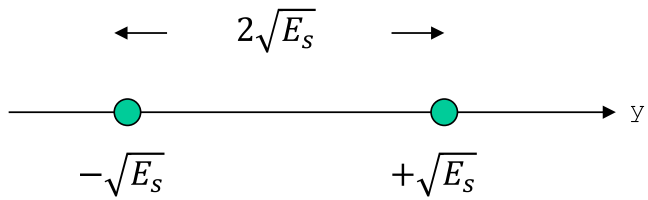
- Average probability of a bit error with Gray mapping:

$$\boxed{P_b \approx \frac{1}{2} P_e = \frac{3}{4} Q\left(\sqrt{\frac{2E_s}{5N_0}}\right)} = \frac{3}{4} Q\left(\sqrt{\frac{4E_b}{5N_0}}\right)$$

The last equality follows because $E_s = 2E_b$ (ℓ=2 bits per symbol)

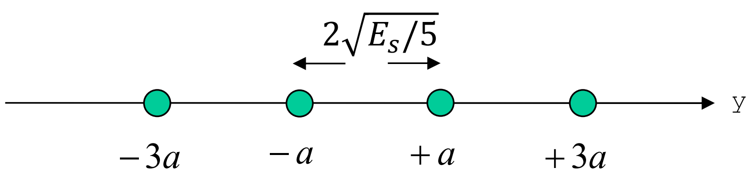
BPSK versus 4-PAM (1)

- BPSK:



$$\boxed{P_b = Q\left(\sqrt{\frac{2E_s}{N_0}}\right)}$$

- 4-PAM:



$E_s = 2E_b$: Average signal energy

$$P_e = \frac{3}{2}Q\left(\sqrt{\frac{2E_s}{5N_0}}\right)$$

Symbol error probability

$$\boxed{P_b \approx \frac{3}{4}Q\left(\sqrt{\frac{2E_s}{5N_0}}\right)}$$

Bit error probability
(Gray mapping)

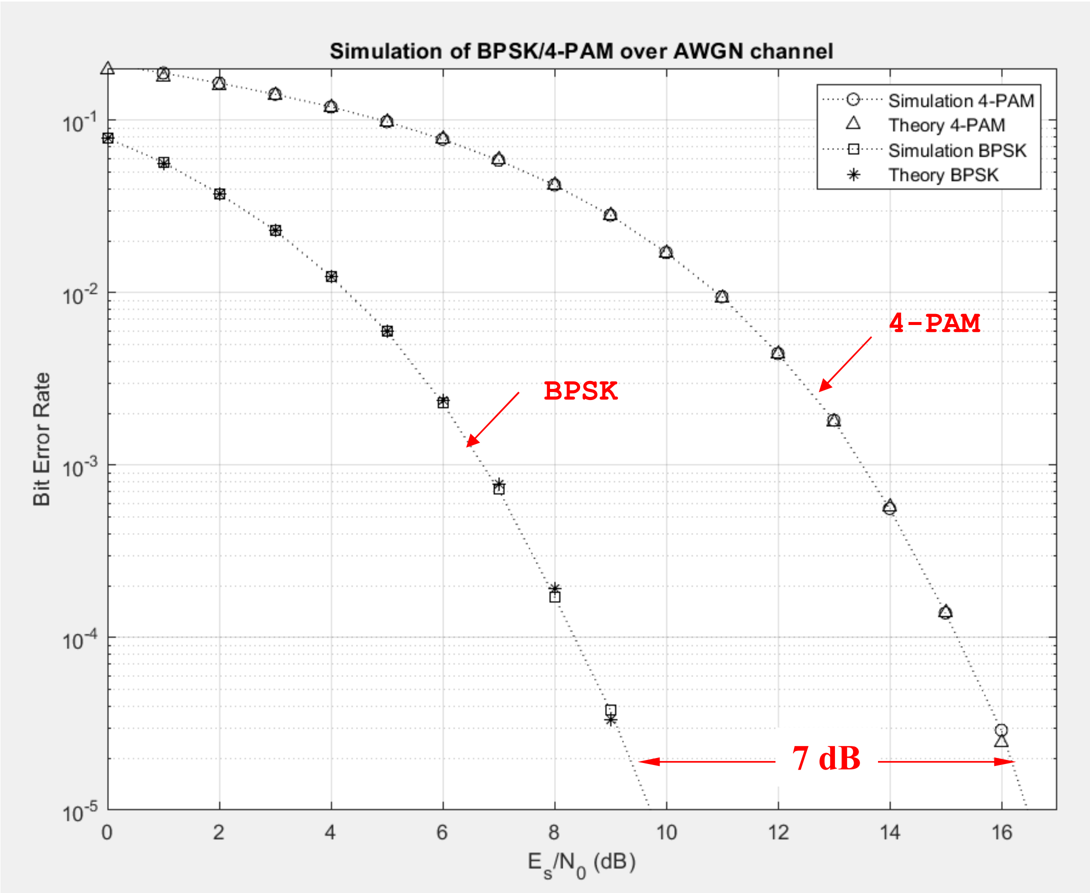
BPSK versus 4-PAM (2)

- Loss in signal-to-noise ratio (SNR), or E_s/N_0 , of 4-PAM with respect to binary modulation (BPSK):

$$L(\text{dB}) = 10\log_{10}\left(\frac{2E_s/N_0}{2E_s/5N_0}\right) = 10\log_{10}(5) \approx 7\ \text{dB}$$

(ignoring the ¾ factor and the approximation used in the probability of bit error of 4-PAM)

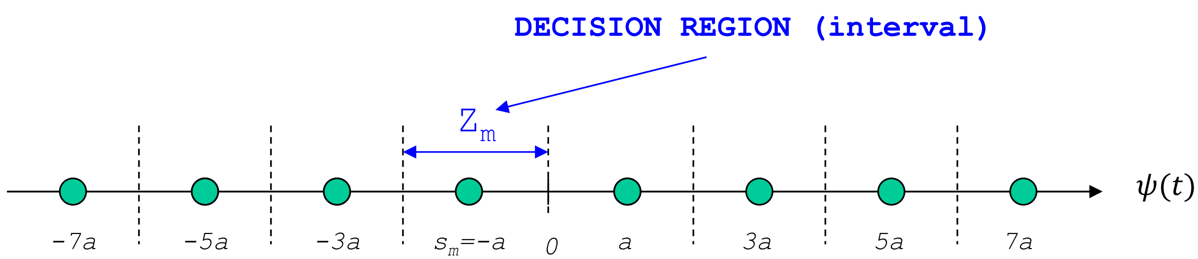
Error Performance of BPSK and 4-PAM



M-ary pulse amplitude modulation (M-PAM)

- $M=2^\ell$ signals transmitted as M amplitudes:

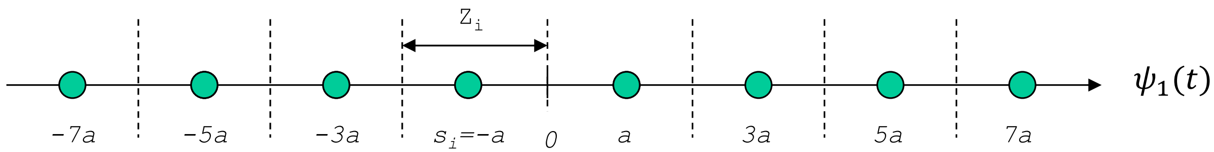
$$s_m(t) = s_m\psi(t), \quad m = 1,2,\cdots,M$$



Signal constellation for M-PAM

M-ary pulse amplitude modulation (M-PAM)

Signal constellation:

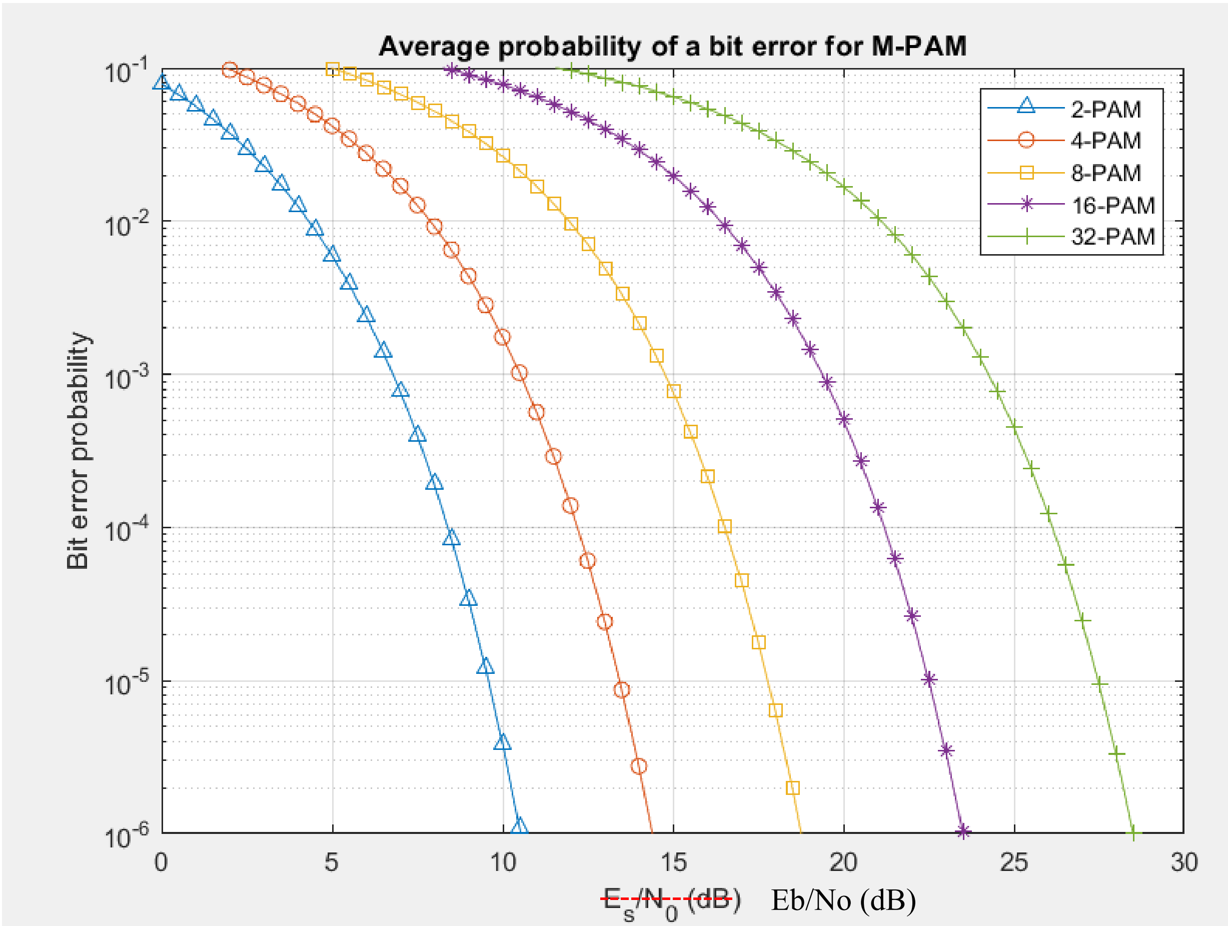


$$a = \sqrt{\frac{3E_s}{M^2-1}}$$

Average probability of a bit error:

$$\boxed{P_b \approx Q\left(\sqrt{\frac{6E_s}{(M^2-1)N_0}}\right)} = Q\left(\sqrt{\frac{6\ell E_b}{(M^2-1)N_0}}\right)$$

Performance of *M*-PAM

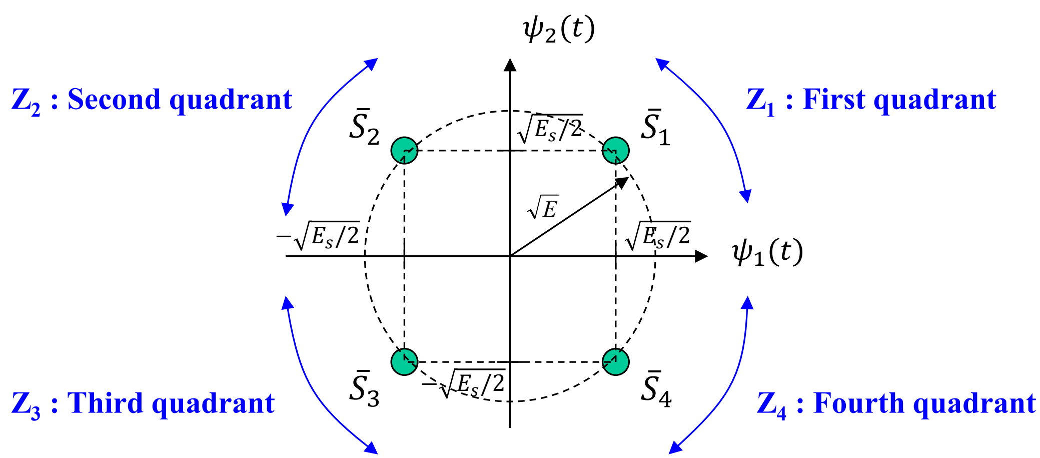


Example: QPSK modulation

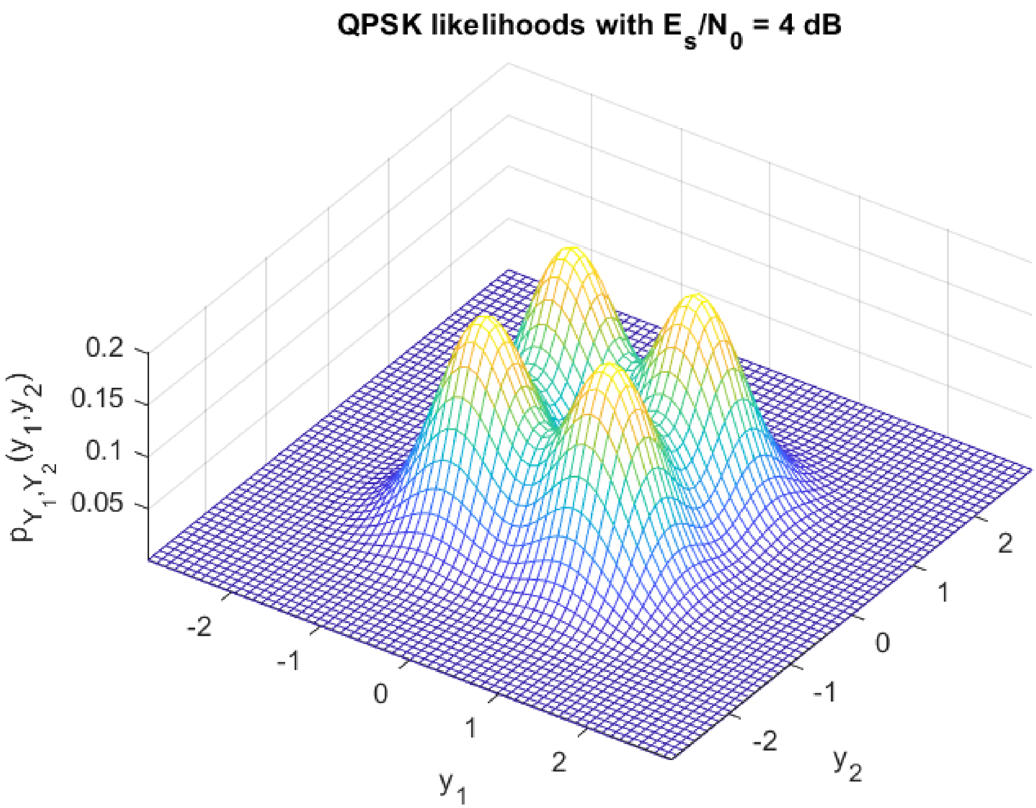
- Mapping:
  (Gray)

Bits	*m*	$S_m=(S_{m1}, S_{m2})$
00	*1*	$\left(\sqrt{E_s/2}, \sqrt{E_s/2}\right)$
01	*2*	$\left(\sqrt{E_s/2}, -\sqrt{E_s/2}\right)$
11	*3*	$\left(-\sqrt{E_s/2}, -\sqrt{E_s/2}\right)$
10	*4*	$\left(-\sqrt{E_s/2}, \sqrt{E_s/2}\right)$

- Signal set (constellation):

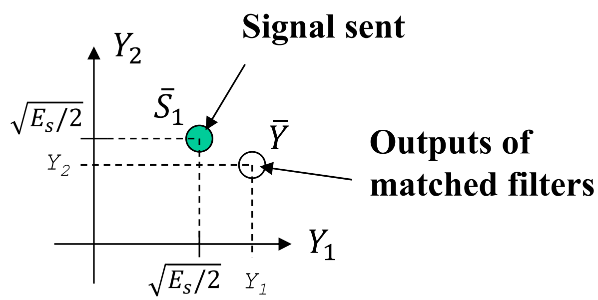


QPSK likelihoods



Example (cont.): QPSK modulation

- <u>Decision rule</u>:
  - Decide that S_i sent if (Y_1, Y_2) is in the i-th quadrant
  - Let $\boldsymbol{P_{c,i}}$=Pr{"no error"|S_i sent}
    - Assume that $i=1$:



$$P_{c,1} = \Pr\{Y_1 > 0, Y_2 > 0\} = \left[1 - Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right]^2$$

Example (cont.): QPSK modulation

- Probability of error:
  - By symmetry: $P_{c,1} = P_{c,2} = P_{c,3} = P_{c,4}$

$$P_e = 1 - P_c = 1 - \left[1 - Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right]^2$$

$$= 2Q\left(\sqrt{\frac{E_s}{N_0}}\right) - \left[Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right]^2 \approx 2Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

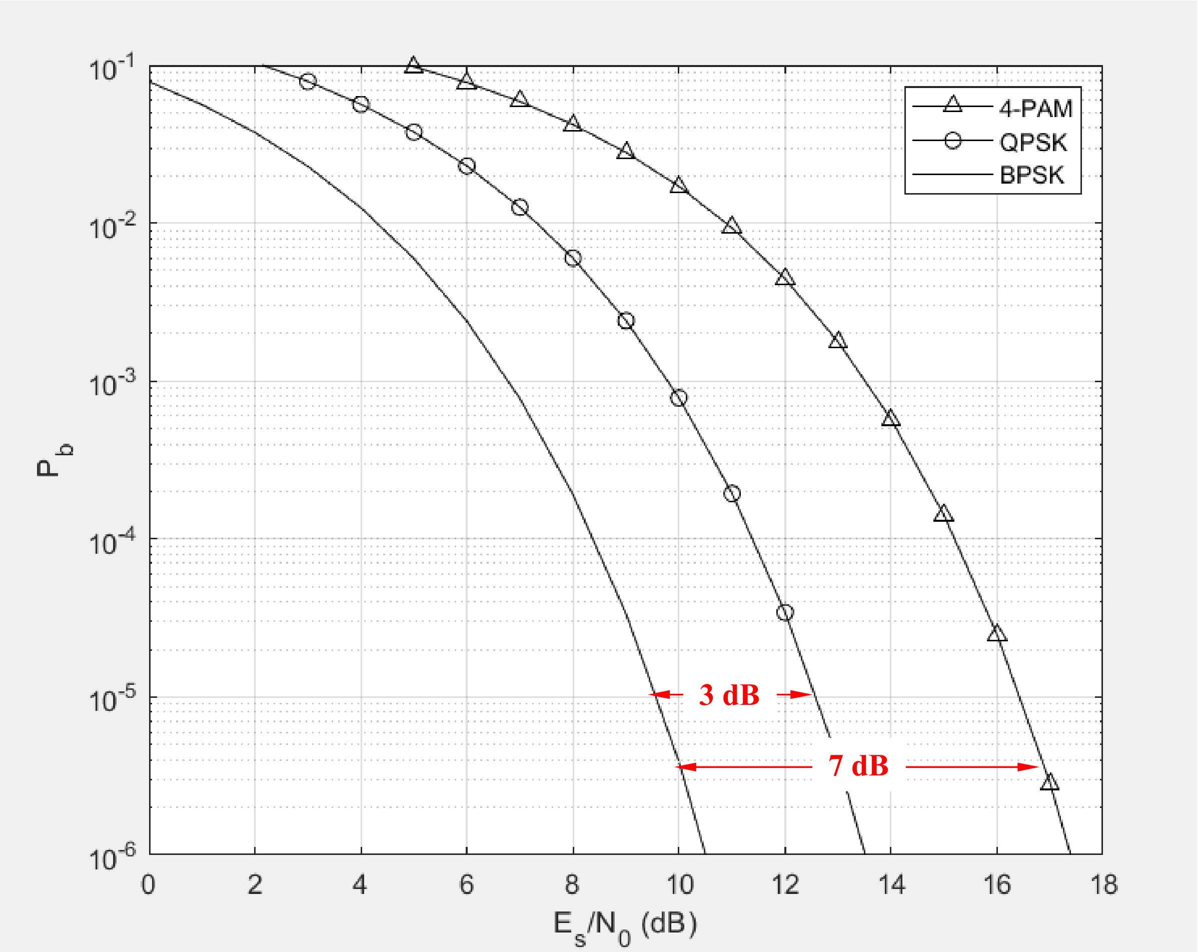
- Probability of a bit error:

$$\boxed{P_b \approx \frac{1}{2} P_e = Q\left(\sqrt{\frac{E_s}{N_0}}\right),} \quad E_s = 2E_b$$

QPSK requires twice the average signal energy E_s, compared to BPSK

(THE SAME P_b vs E_b (bit) AS WITH BINARY MODULATION!!)

BPSK / QPSK / 4-PAM



QPSK versus 4-PAM

- QPSK requires 4 dB less average bit energy to achieve the same bit error probability (see previous slide)

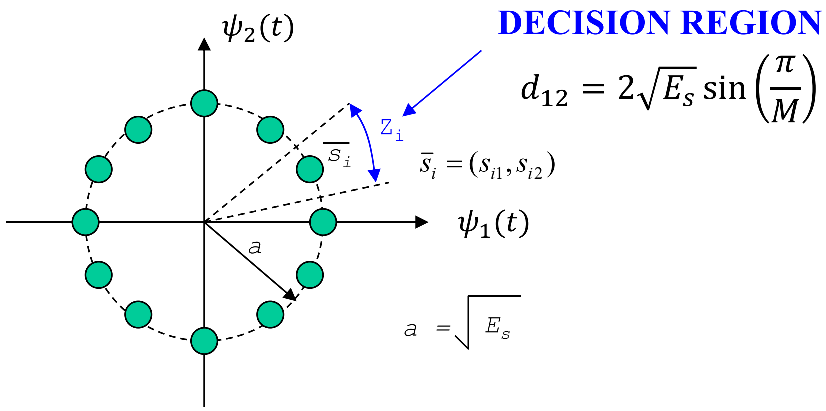
- 4-PAM transmits two levels of energy
  - $E_1 = a^2 = E_s/5$ (inner points)
  - $E_2 = 9a^2 = 9E_s/5$ (outer points)
  - Peak-to-average power ratio (PAPR):

$$\text{PAPR} = 10\log_{10}\left(\frac{E_2}{E_s}\right) = 10\log_{10}\left(\frac{9}{5}\right) = 2.55 \text{ dB.}$$

This is a <u>problem in power amplifier design</u>

M-ary Phase-Shift Keying (M-PSK)

Signal constellation:

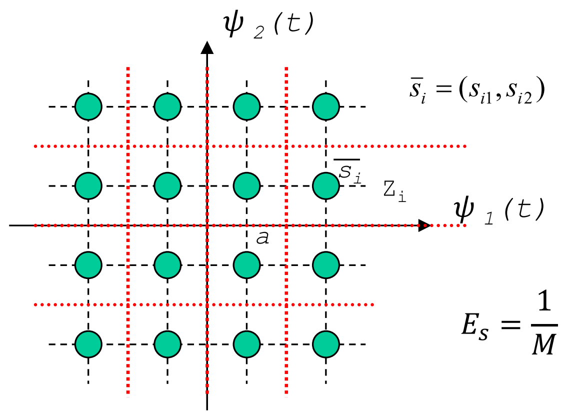


Average probability of a bit error:

$$\boxed{P_b \approx Q\left(\sqrt{\frac{2E_s}{N_0}\sin^2\left(\frac{\pi}{M}\right)}\right)} = Q\left(\sqrt{\frac{2E_b}{N_0}\ell\sin^2\left(\frac{\pi}{M}\right)}\right)$$

M-ary Quadrature Amplitude Modulation (M-QAM)

Square signal constellation (cartesian product of two $\sqrt{M}$-PAM constellations)

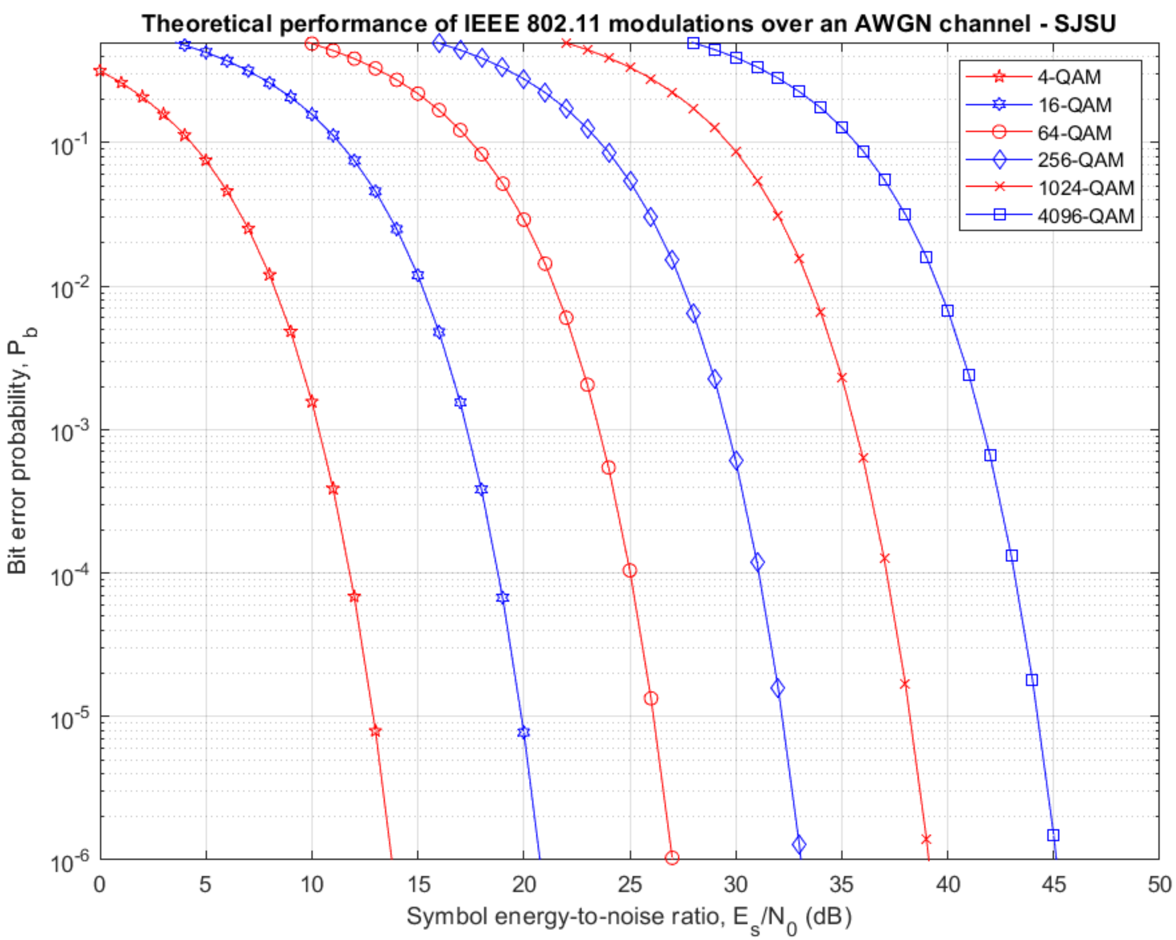


$$E_s = \frac{1}{M}\sum_{i=1}^{M} |\bar{s}_i|^2$$

Average probability of a bit error:

$$\boxed{P_b \approx 2Q\left(\sqrt{\frac{3E_s}{(M-1)N_0}}\right)} = 2Q\left(\sqrt{\frac{3\ell E_b}{(M-1)N_0}}\right)$$

QAM modulations used in IEEE 802 standards



General *M*-ary Digital Modulation

- M information signals $\{s_m(t), m = 1, \cdots, M\}$ transmitted, $M = 2^{\ell}$
- Each signal carries $\ell = \log_2(M)$ bits
- N *orthonormal* pulses are used, $\{\psi_n(t), n = 1, \cdots, N\}$ over the time interval $[0, T]$ such that

$$\int_0^T \psi_n(t)\psi_m(t)\ dt = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$$

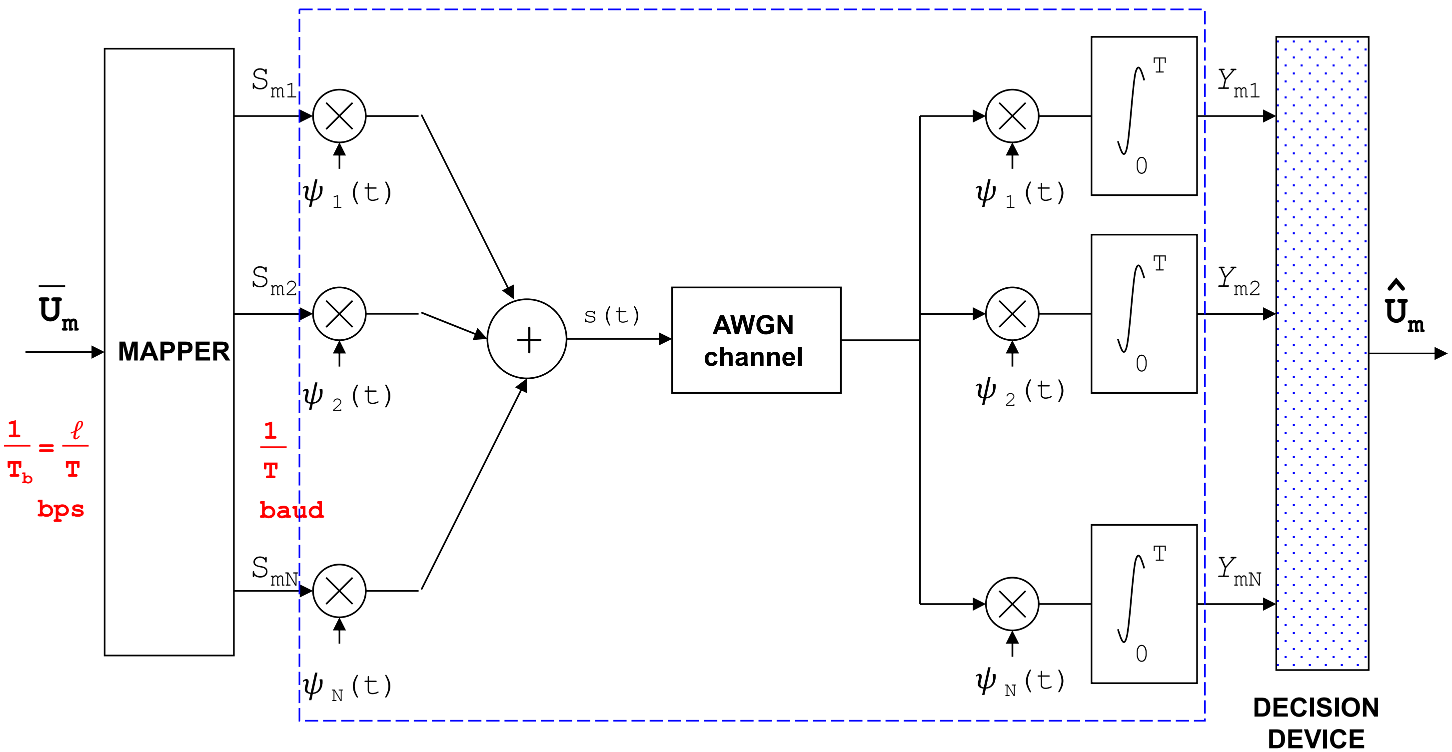
and

$$s_m(t) = \sum_{n=1}^{N} s_{mn}\, \psi_n(t), \quad 0 < t \leq T. \quad \leftrightarrow \quad \bar{s}_m = (s_{m1}, s_{m2}, \ldots, s_{mN})$$

- Energy <u>per bit</u> and bit rate:

$$E_b = \frac{E_s}{\ell}, \quad R_b = \ell R, \quad R = \frac{1}{T}$$

General Digital Modulation System

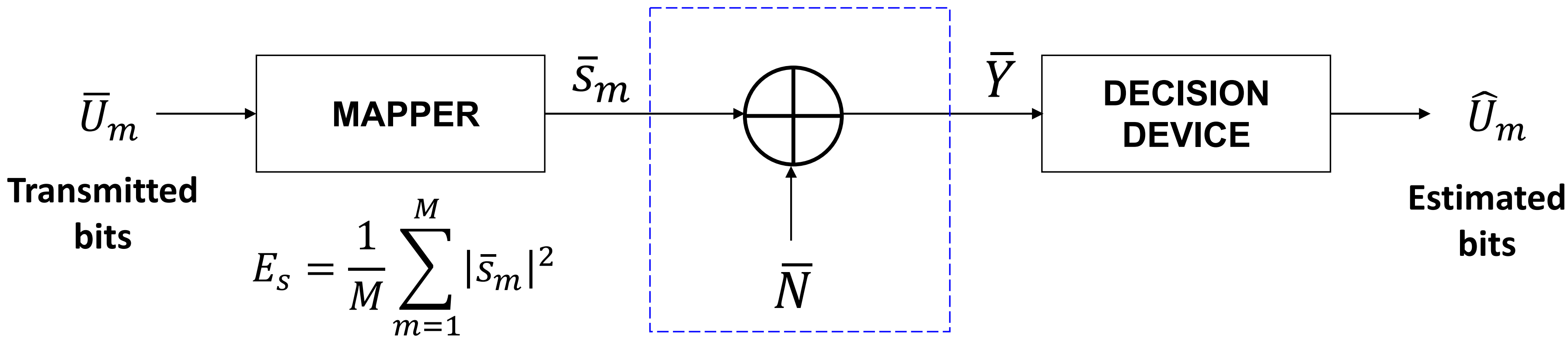


- *N* virtual parallel channels are created
- Correlator (or matched-filter) outputs Y_{mn} are <u>Gaussian</u> $N(S_{mn}\,;\,N_0/2)$

Examples of digital modulation formats

Message size M	Dimension N	Modulation Scheme	Other names
M	1	M-PAM	
M	2	M-PSK M-QAM	**Quadrature modulation**
M	M	M-FSK M-PPM	Orthogonal modulation

Equivalent discrete vector channel



$$E_s = \frac{1}{M}\sum_{m=1}^{M} |\bar{s}_m|^2$$

$$\bar{Y} = \bar{s}_m + \bar{N}$$

Correlator/MF outputs

Mapper outputs

AWGN samples

Decision regions

The likelihood functions are (AWGN):

$$\boxed{L(m) = p_{\bar{Y}|\bar{s}_m}(\bar{y}|\bar{s}_m) = \frac{1}{\left(\sqrt{\pi N_0}\right)^N} \exp\left[-\frac{1}{N_0}\sum_{n=1}^{N}(y_n - s_{mn})^2\right]}$$

Maximizing $L(m)$ is equivalent to minimizing the distance between $\bar{y}$ and $\bar{s}_m$. This gives **Voronoi regions** as decision regions:

$$Z_m = \{\bar{y} \mid : \ |\bar{y} - \bar{s}_m| \text{ is smallest}, 1 \leq m \leq M\}$$

Quadrature Modulation

- N=2 pulses chosen as:

$$\psi_1(t) = \sqrt{\frac{2}{T}}\, \psi(t) \cos(\, 2\pi f_c t)$$

$$\psi_2(t) = \sqrt{\frac{2}{T}}\, \psi(t) \sin(\, 2\pi f_c t)$$

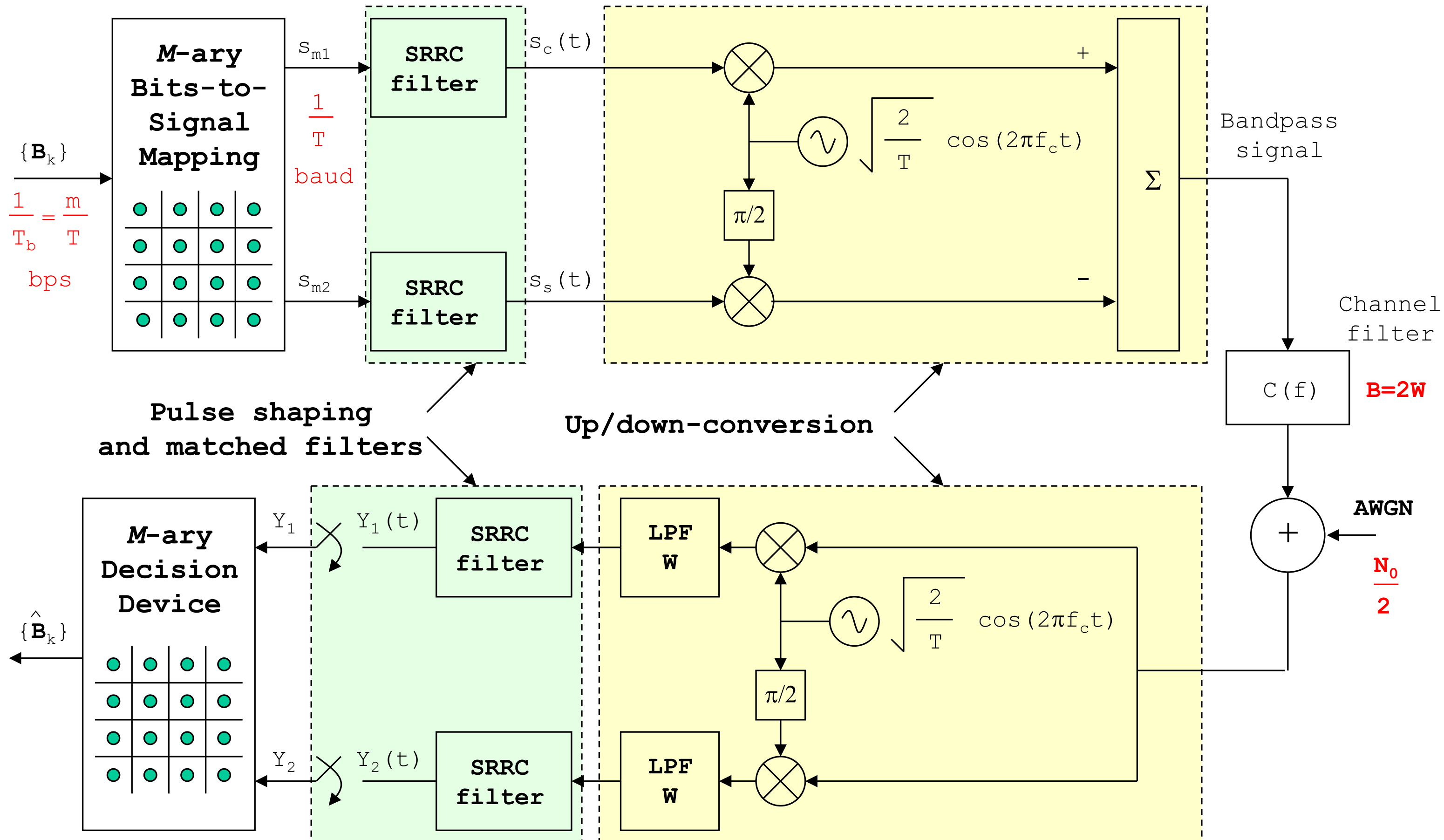
Bandpass pulses

$$\boxed{s_m(t) = s_{m1}\psi_1(t) + s_{m2}\psi_2(t), \quad 0 < t \leq T.}$$

- ISI (**i**nter**s**ymbol **i**nterference) removed by a *raised-cosine spectrum* $X(f) = P^2(f)$ with rolloff factor α. This is, square-root raised cosine (SRRC) filters (more on this later in the course)

$$G_T(f) = G_R(f) = P(f) = \sqrt{X(f)}$$

M-ary Digital Communication System



General procedure to analyze a signal constellation

1. Find the average energy as a function of amplitude:

$$E_s = f(a) \quad \rightarrow \quad a = f^{-1}(E_s)$$

2. Decision regions are Voronoi regions:

$$Z_m = \{\bar{y} \mid : \ |\bar{y} - \bar{s}_m| \text{ is smallest, } 1 \le m \le M\}$$

3. Determine the smallest pairwise distance between signal points

$$d_{12} = \min_{m \neq n}\{|\bar{s}_m - \bar{s}_n|\}$$

General procedure to analyze a signal constellation (cont.)

4. Determine the average number of nearest-neighbor signal points A_d at minimum distance d_{12}

5. Nearest neighbor approximation on the bit error probability

Natural labeling:

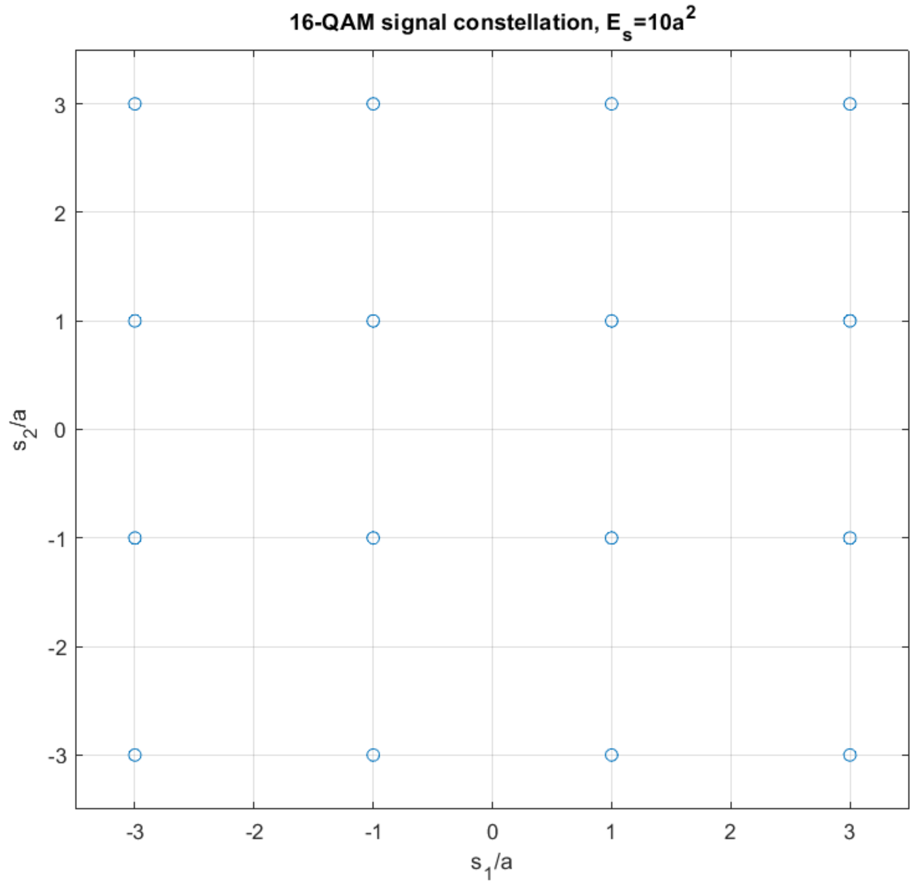
$$\boxed{P_b \approx \frac{1}{2} A_d \, Q\left(\sqrt{\frac{{d_{12}}^2}{2N_0}}\right)}$$

Gray bit labeling:

$$\boxed{P_b \approx \frac{1}{\ell} A_d \, Q\left(\sqrt{\frac{{d_{12}}^2}{2N_0}}\right)}$$

Example: 16-QAM constellation

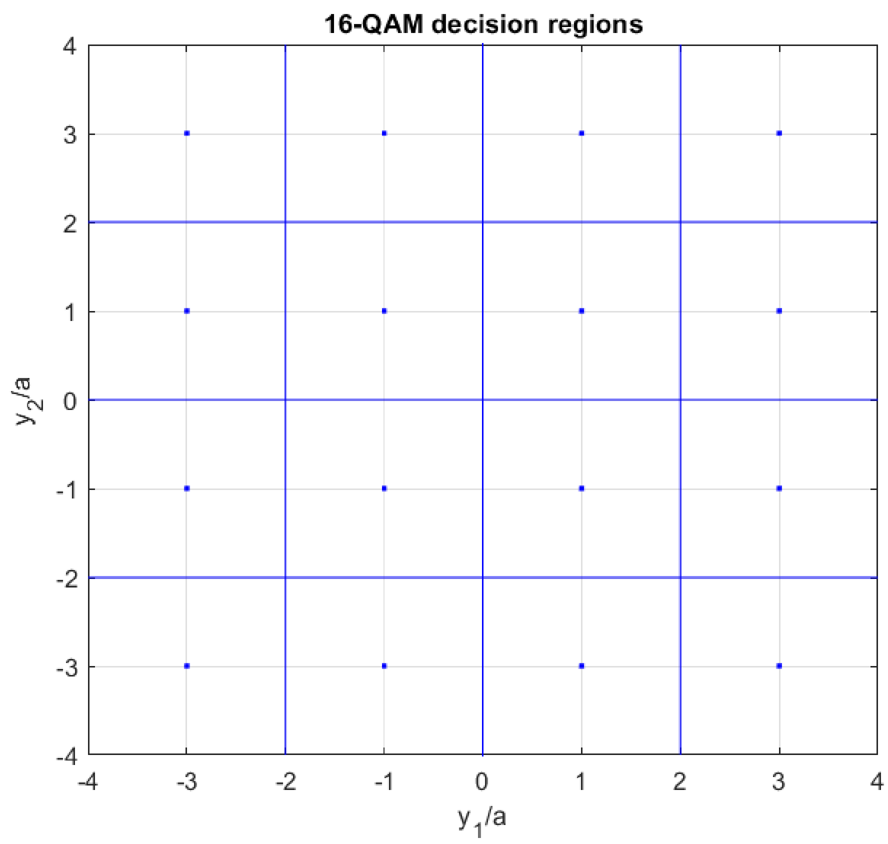
1. Average signal energy



$$\boxed{a = \sqrt{E_s/10}}$$

Example: 16-QAM constellation (cont.)

2. Decision regions



Example: 16-QAM constellation (cont.)

3. Smallest pairwise distance:

$$\boxed{d_{12} = 2a = 2\sqrt{E_s/10}}$$

4. Average number of nearest-neighbor signal points:

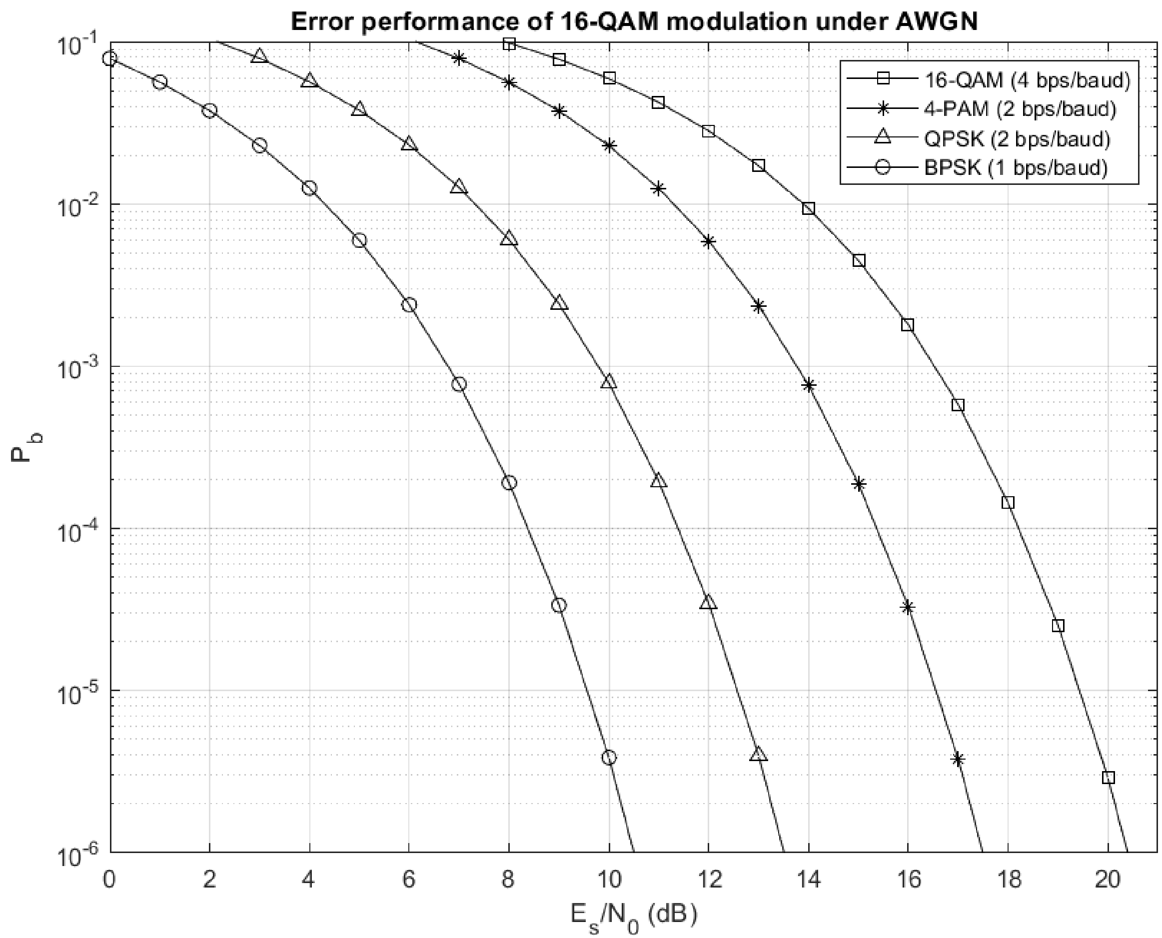
$$\boxed{A_d = \frac{1}{16}[4(2) + 8(3) + 4(4)] = 3}$$

4 corner points (2 neighbors) | 8 wall points (3 neighbors) | 4 inner points (4 neighbors)

5. Nearest neighbor approximation (Gray bit labeling):

$$\boxed{P_b \approx \frac{3}{4}\, Q\left(\sqrt{\frac{E_s}{5N_0}}\right)}$$

Example: 16-QAM constellation (cont.)



Spectral Efficiency (RC spectrum)

- Band-limited channel:

$$W = \frac{B}{2} = \frac{R}{2}(1+\alpha) = \frac{R_b}{2\ell}(1+\alpha)$$

- Spectral Efficiency:

$$\boxed{\rho = \frac{R_b}{B} = \frac{\ell}{(1+\alpha)} \qquad [\boldsymbol{bps}/Hz]}$$

Note: Most references (including the textbook) use $\alpha = 0$.

Example

- Bandpass channel bandwidth: B=10 MHz
- Raised-cosine spectrum with rolloff factor: $\alpha = 0.22$
- **QPSK modulation** gives a bit rate

$$R_b = \frac{\ell B}{(1+\alpha)} = \frac{20 \times 10^6}{1.22} = 16.4 \times 10^6 \text{ bps}$$

- Compare with **binary modulation**:

$$R_b = \frac{B}{(1+\alpha)} = \frac{10 \times 10^6}{1.22} = 8.2 \times 10^6 \text{ bps}$$

- With **8-PSK modulation**:

$$R_b = \frac{\ell B}{(1+\alpha)} = \frac{30 \times 10^6}{1.22} = 24.6 \times 10^6 \text{ bps}$$

Example (cont.)

- Required average E_b/N_0 to obtain $P_b=10^{-5}$
  - *QPSK and binary modulation (BPSK):*

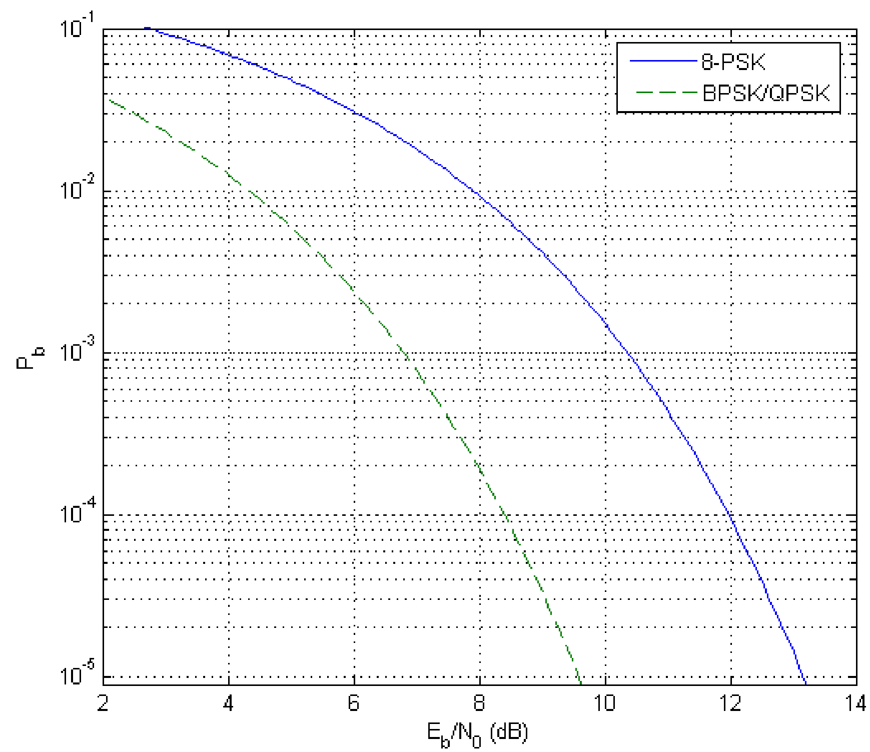
$$P_b = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = 10^{-5} \Rightarrow \frac{E_b}{N_0} = \frac{1}{2}(4.265)^2 \Rightarrow \frac{E_b}{N_0}(dB) = 9.6$$

  - *8-PSK:*

$$P_b = Q\left(\sqrt{\frac{2E_b}{N_0}(3)\sin^2(22.5°)}\right) = 10^{-5} \Rightarrow \frac{E_b}{N_0} = \frac{1}{0.879}(4.265)^2$$

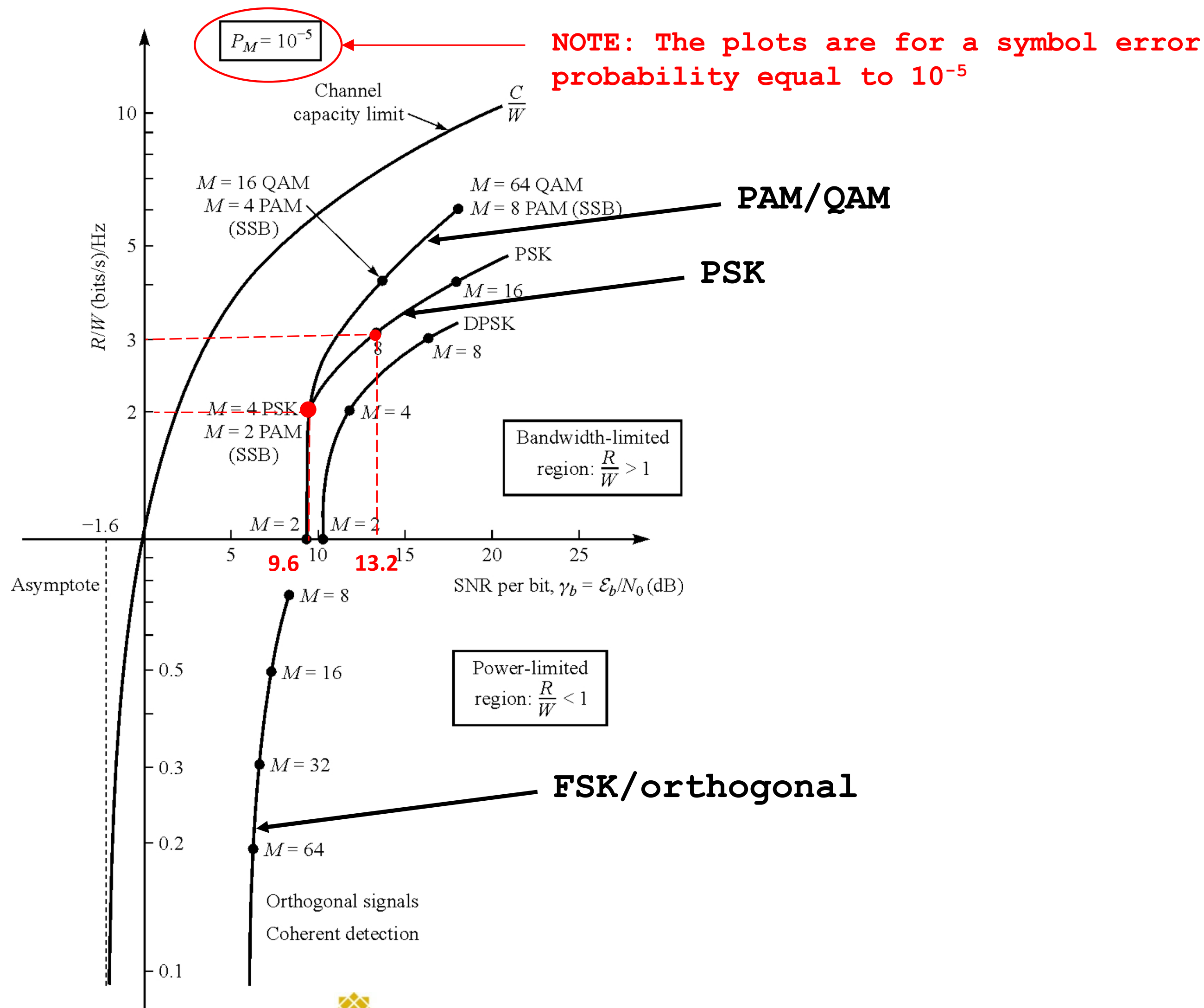
$$\Rightarrow \frac{E_b}{N_0}(dB) = 13.2$$

Example (cont.): Error Performance



Spectral Efficiency vs. E_b/N_0 (dB) (with $\alpha = 0$)

(Fig. 9.29 of textbook)



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