FM modulation, power spectral density, and noise performance of AM/FM systems

EE 160

Principles of Communication Systems

San Jose State University

Summary of FM modulation

- Modulated signal

$$u(t) = A_c \cos(2\pi f_c t + \phi(t))$$

$u_{FM}(t)$

- Frequency deviation

$$\frac{d\phi(t)}{dt} = 2\pi k_f m(t), \quad \Delta f_{\max} = k_f \cdot \max\{|m(t)|\}$$

- Modulation index and bandwidth

$$\beta_f = \frac{\Delta f_{\max}}{W}, \quad B_c = 2(\beta_f + 1)W$$

Narrowband FM

- Small deviation constant $k_f \ll 1$ so that

$$\sin(\phi(t)) \approx \phi(t), \quad \cos(\phi(t)) \approx 1$$

- Quadrature representation:

$$\begin{aligned} u(t) &= A_c \cos\left[\phi(t)\right]\cos(2\pi f_c t) - A_c \sin\left[\phi(t)\right]\sin(2\pi f_c t) \qquad u_{FM}(t) \\ &\approx A_c \cos(2\pi f_c t) - A_c \phi(t) \sin(2\pi f_c t) \end{aligned}$$

  – This is a conventional AM signal with message $\phi(t)$

- *Performance similar to AM modulation*
- *Not used in practice*

Wideband FM

- Assuming a modulating signal $m(t) = \cos(2\pi f_m t)$

$$u(t) = A_c \cos\left[2\pi f_c t + \beta_f \sin(2\pi f_m t)\right] = \mathrm{Re}\left\{A_c e^{j2\pi f_c t} e^{j\beta_f \sin(2\pi f_m t)}\right\}$$ $u_{FM}(t)$

- Fourier series coefficients of complex envelope:

$$c_n = f_m \int_0^{1/f_m} e^{j\beta_f \sin(2\pi f_m t)} \cdot e^{-j2\pi n f_m t} dt = \frac{1}{2\pi}\int_0^{2\pi} e^{j\beta_f (\sin(u) - nu)} du = J_n(\beta_f)$$

- Modulated signal can be written as

$$u(t) = \mathrm{Re}\left\{A_c \sum_{n=-\infty}^{\infty} J_n(\beta_f)\, e^{j2\pi(f_c + nf_m)t}\right\}$$ $u_{FM}(t)$

$$= A_c \sum_{n=0}^{\infty} J_n(\beta_f) \cos\left[2\pi(f_c + nf_m)t\right]$$

See Table 4.1 of textbook for values of $J_n(\beta)$

Bessel functions of first kind

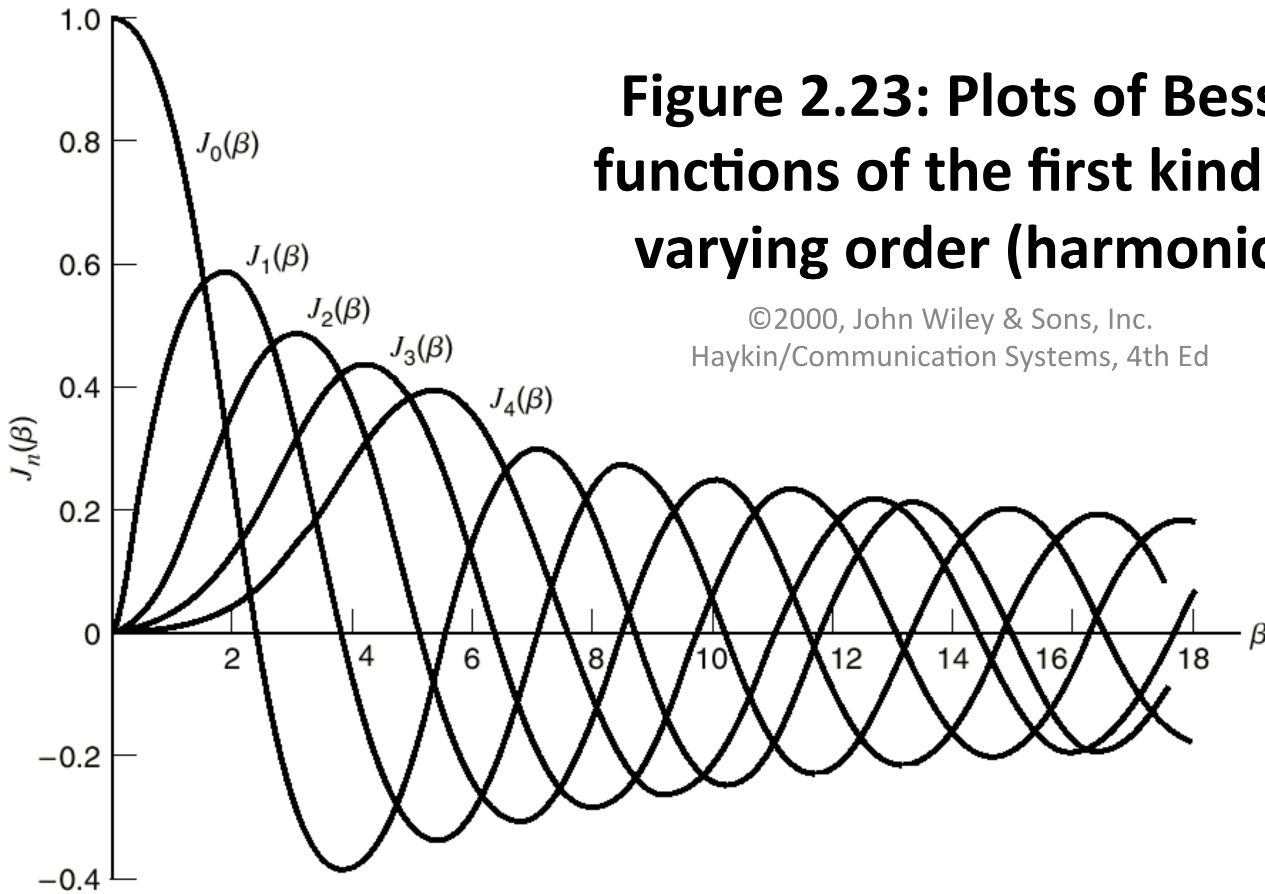


Figure 2.23: Plots of Bessel functions of the first kind for varying order (harmonic).

FM spectra

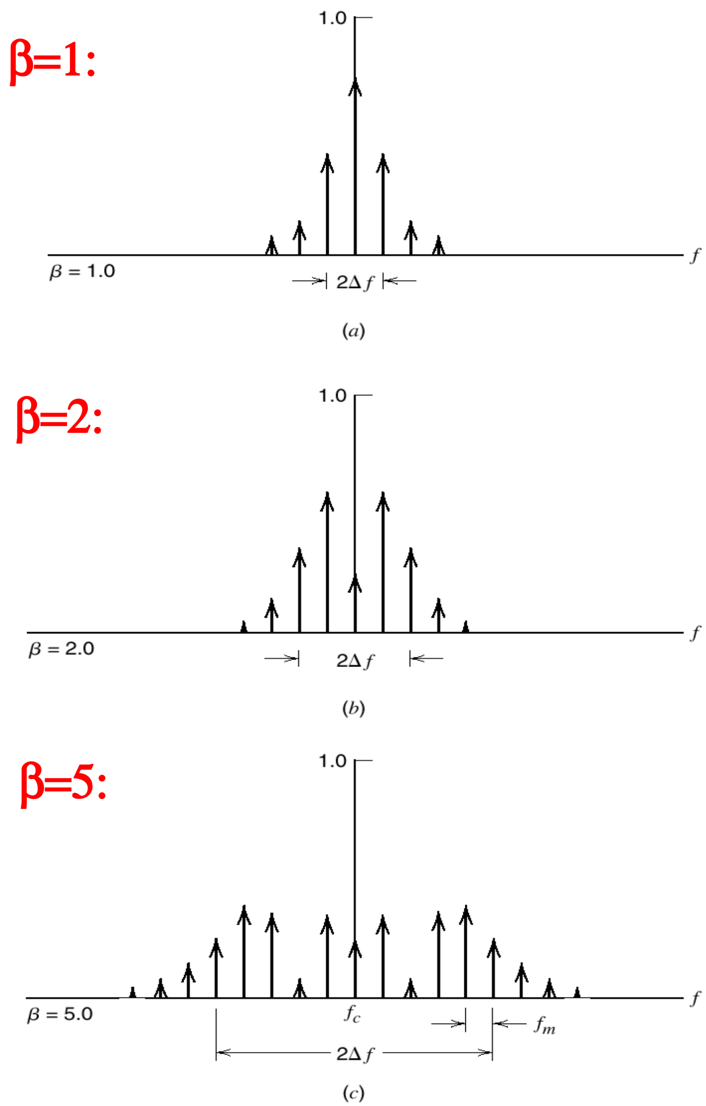


Figure 2.24
Discrete amplitude spectra of an FM signal, normalized with respect to the carrier amplitude, for the case of sinusoidal modulation of fixed frequency and varying amplitude. Only the spectra for positive frequencies are shown.

Spectral density and correlation

- Energy-type signals

$$E_x = \int_{-\infty}^{\infty} |\, x(t) \,|^2 \; dt = \int_{-\infty}^{\infty} |\, X(f) \,|^2 \; df$$

the function $|X(f)|^2$ is *energy spectral density*

$$G_x(f) = |\, X(f) \,|^2 \quad \left(\text{Joules / Hz} \right)$$

- LTI systems: energy spectral density of output signal

$$E_y = \int_{-\infty}^{\infty} |\, X(f) \,|^2 |\, H(f) \,|^2 \; df \quad \Rightarrow \quad G_y(f) = G_x(f) \,|\, H(f) \,|^2$$

Spectral density and correlation (2)

- Power-type signals: $P_x = \lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{t_0+T} |x(t)|^2 \, dt$
- Define the *autocorrelation function*

$$R_x(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{t_0+T} x(t)x(t-\tau)d\tau$$

Then $P_x = R_x(0)$ and $P_x = \int_{-\infty}^{\infty} S_x(f)\,df$ where

$$S_x(f) = F\{R_x(\tau)\} \quad \left(\text{Watts} / \text{Hz}\right)$$

(Fourier transform of the autocorrelation) is the *power spectral density*.

Spectral density and correlation (3)

- LTI systems: energy spectral density of output signal

$$S_y(f) = S_x(f)\,|\,H(f)\,|^2$$

- Periodic signals:

$$R_x(\tau) = \sum_{n=-\infty}^{\infty} |\,x_n\,|^2\, e^{j2\pi \frac{\tau}{T_0} n}, \quad S_x(f) = \sum_{n=-\infty}^{\infty} |\,x_n\,|^2\, \delta\left(f - \frac{n}{T_0}\right)$$

- LTI systems with periodic inputs

$$S_y(f) = \sum_{n=-\infty}^{\infty} |\,x_n\,|^2 \left|H\left(\frac{n}{T_0}\right)\right|^2 \delta\left(f - \frac{n}{T_0}\right), \quad P_y = \sum_{n=-\infty}^{\infty} |\,x_n\,|^2 \left|H\left(\frac{n}{T_0}\right)\right|^2$$

Noise signals

- In the case of noise, denoted $X(t)$, the values are *unpredictable* and only a probability can be associated to them: $\Pr\left\{x_1 < X(t) \le x_2\right\}$
- The most common type of noise found in a communication system is ***Gaussian*** with PDF:

$$p_{X(t)}(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2}$$

where

$$\mu_X = E\left\{X(t)\right\}: \quad \text{Mean value}$$

$$\sigma_X^2 = E\left\{\left(X(t) - \mu_X\right)^2\right\}: \quad \textit{Variance}$$

Additive white noise

- If noise is ***additive***, then $\mu_X = 0$
- *Autocorrelation* of a noise signal:

$$R_x(\tau) = E\left\{X(t)X(t+\tau)\right\}$$

- As before, the power spectral density is

$$S_x(f) = F\{R_x(\tau)\} \quad \left(\text{Watts / Hz}\right)$$

- If noise is ***white***, then

MATLAB script PSD thermal noise

$$S_x(f) = \frac{N_0}{2} \quad \left(\text{Watts / Hz}\right), \quad N_0 = kT,$$

where k is Boltzmann's constant.

Section 5.3 of textbook

Additive white noise (2)

- In our laboratory, white noise is noticed as a floor in the spectrum analyzer display.
  - *Assignment: Measure the noise level in the SA*
- The autocorrelation function of ***white noise*** is

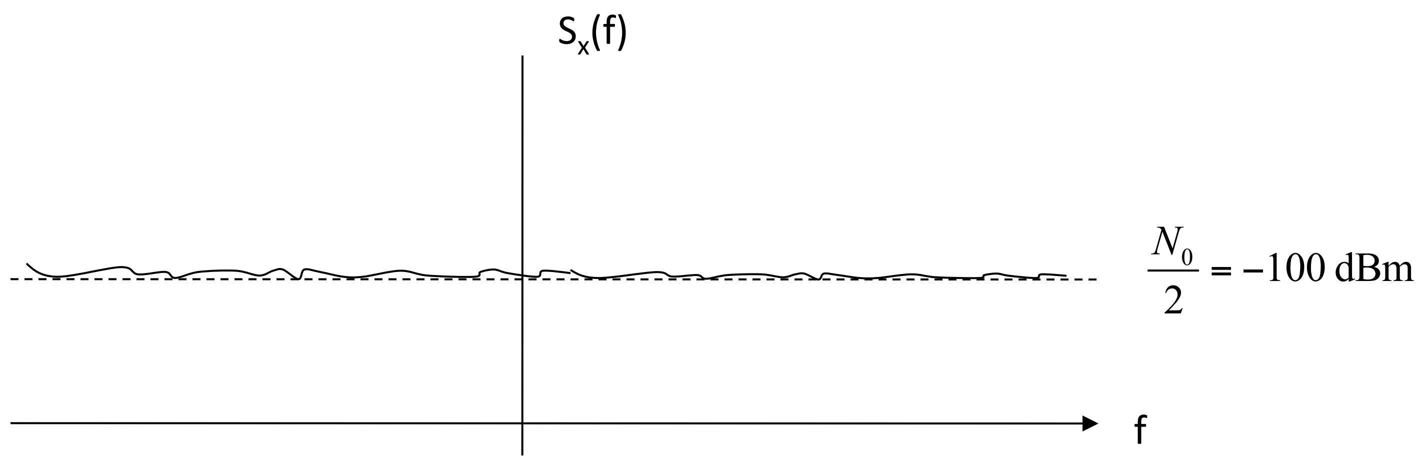
$$R_x(\tau) = \frac{N_0}{2}\delta(\tau).$$

This means that any two white noise values are ***uncorrelated***:

$$R_x(t_2 - t_1) = E\left\{X(t_1)X(t_2)\right\} = 0, \quad t_1 \neq t_2.$$

Noise in spectrum analyzer

- Example from lab: Span=2 MHz, BW=3 kHz



- The measurement of noise floor depends on resolution bandwidth. (Why?)

AWGN: Additive White Gaussian Noise

- Mean and variance:

$$E\left\{X(t)\right\} = 0, \quad \text{var}\left\{X(t)\right\} = \sigma_X^2 = \frac{N_0}{2}.$$

- PDF: Gaussian

$$p_{X(t)}(x) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{x^2}{N_0}}$$

- Probability computation:

$$\Pr\left\{X(t) > x_0\right\} = Q\left(\frac{x_0}{\sigma_X}\right) = Q\left(\frac{2x_0}{N_0}\right), \quad Q\left(x\right) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$$

in terms of the ***Gaussian Q-function***. (Table 5.1)

The Gaussian Q-function

- Properties: (a) $Q(-x)=1-Q(x)$; (b) $Q(0)=0.5$; (c) $Q(\infty)=0$

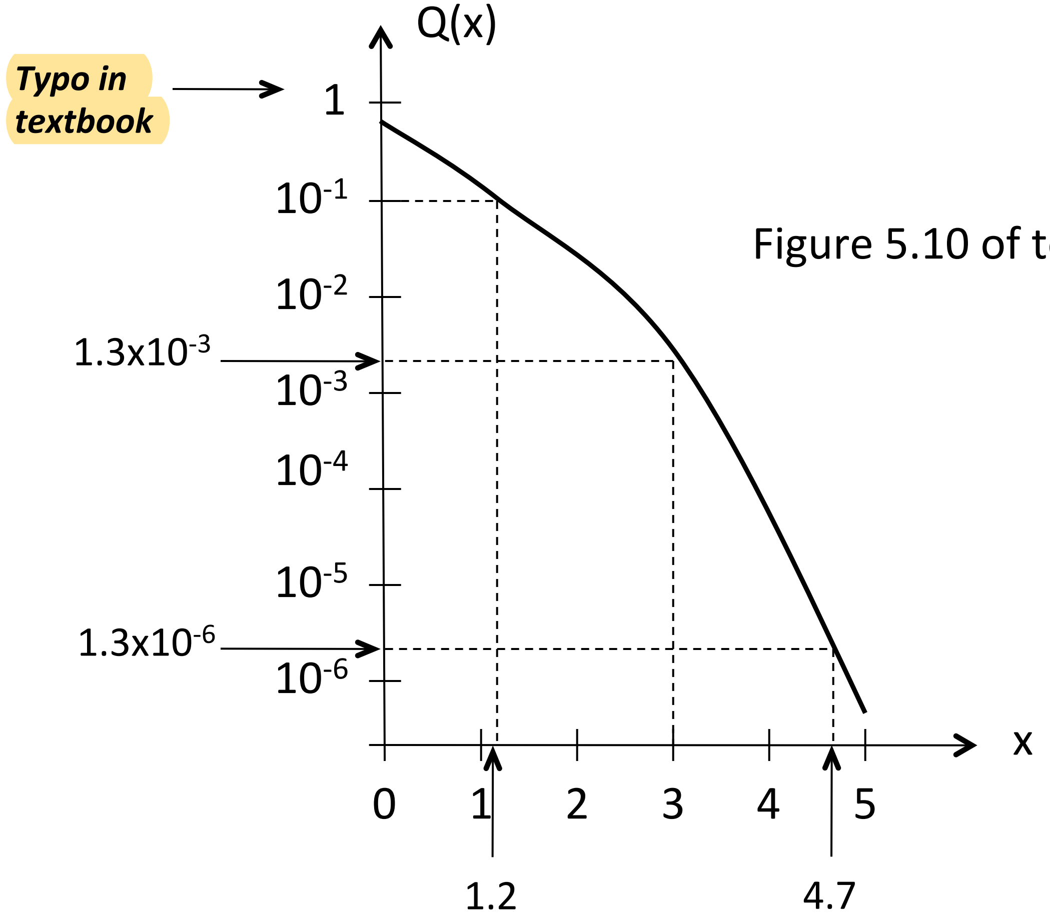
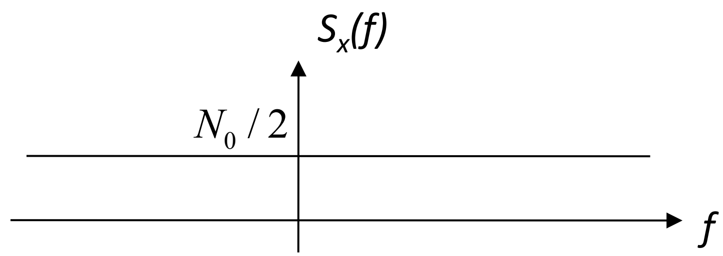


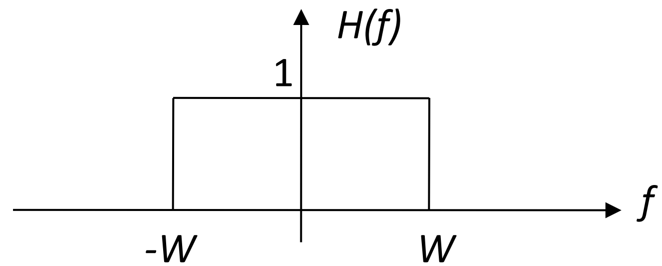
Figure 5.10 of textbook

Baseband Signal-to-Noise ratio (SNR)

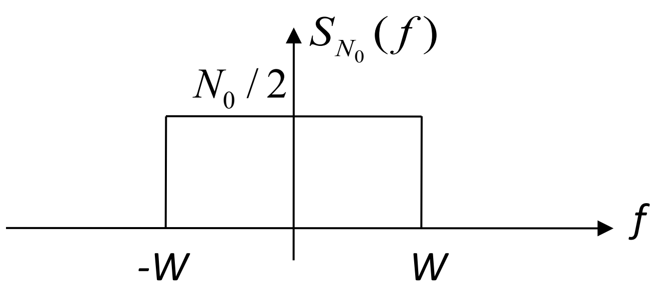
Baseband (lowpass) system



White noise input



Receive filter



Filter output

Output power:

$$P_{N_0} = \int_{-\infty}^{\infty} S_{N_0}(f)\,df = \int_{-W}^{W} \frac{N_0}{2}\,df = N_0 W$$

NOTE: Noise power is proportional to bandwidth

Noise floor in SA slides

Baseband SNR: $\boxed{\left(\frac{S}{N}\right)_b = \frac{P_R}{N_0 W}}$ P_R : Received signal power

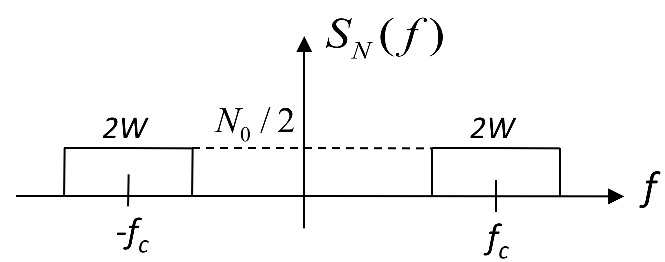
SNR in DSB-SC AM systems

- Transmitted signal: $u(t) = A_c\, m(t)\cos\left(2\pi f_c t\right)$
- Received signal:

$$r(t) = u(t) + N(t) = A_c\, m(t)\cos\left(2\pi f_c t\right) + \left[N_c(t)\cos\left(2\pi f_c t\right) - N_s(t)\sin\left(2\pi f_c t\right)\right]$$

Bandpass noise ***In-phase noise*** ***Quadrature noise***

- After mixing and lowpass filtering: $y_l(t) = \frac{1}{2}\left[A_c\, m(t) + N_c(t)\right]$
- Modulating signal power: $P_O = \frac{A_c^2}{4} P_m, \quad \text{where } P_m = \lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{t_0+T} |\, m(t)\,|^2\, dt$
- Noise power: $P_{N_0} = \frac{1}{4} P_N = \frac{1}{4}\left(2WN_0\right)$



- Output SNR:

$$\left(\frac{S}{N}\right)_{O,DSB} = \frac{P_O}{P_{N_0}} = \frac{\frac{A_c^2}{4}P_m}{\frac{1}{4}(2N_0W)} = \frac{\frac{A_c^2}{2}P_m}{N_0W} = \boxed{\frac{P_R}{N_0W} = \left(\frac{S}{N}\right)_b}$$

SAME AS BASEBAND SYSTEM !!!

SNR in SSB-SC AM systems

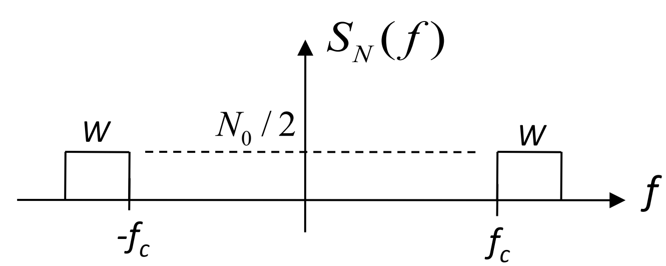
- Transmitted signal: $u(t) = A_c\left[m(t)\cos\left(2\pi f_c t\right) \pm \underbrace{\hat{m}(t)}_{\textit{Hilbert transform}}\sin\left(2\pi f_c t\right)\right]$

- After mixing and lowpass filtering:

$$y_l(t) = \frac{1}{2}\left[A_c\, m(t) + N_c(t)\right]$$

Same as in DSB-SC AM systems!

- Noise power: $P_{N_0} = \frac{1}{4}P_N = \frac{1}{4}\left(WN_0\right)$



- Output SNR:

$$\boxed{\left(\frac{S}{N}\right)_{O,SSB} = \left(\frac{S}{N}\right)_{O,DSB}} = \frac{P_O}{P_{N_0}} = \frac{\frac{A_c^2}{2}P_m}{\frac{1}{4}(N_0W)} = \frac{\frac{A_c^2}{2}P_m}{N_0W} = \frac{P_R}{N_0W} = \left(\frac{S}{N}\right)_b$$

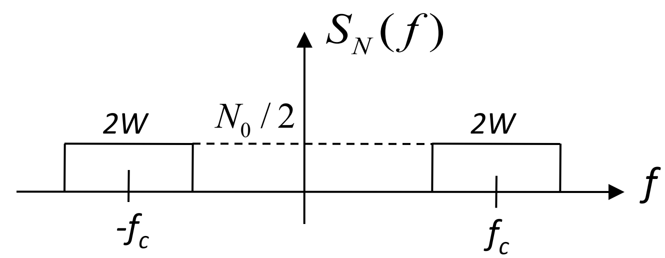
SNR in conventional AM systems

- Transmitted signal: $u(t) = A_c\left[1 + am_n(t)\right]\cos\left(2\pi f_c t\right)$
- Received signal:

$$r(t) = u(t) + N(t) = A_c\left[1 + am_n(t)\right]\cos\left(2\pi f_c t\right) + \left[N_c(t)\cos\left(2\pi f_c t\right) - N_s(t)\sin\left(2\pi f_c t\right)\right]$$

Bandpass noise ($N(t)$), ***In-phase noise*** ($N_c(t)$), ***Quadrature noise*** ($N_s(t)$)

- After mixing, lowpass filtering, DC removal: $y_l(t) = \frac{1}{2}\left[A_c\, am_n(t) + N_c(t)\right]$
- Received power: $P_R = \frac{A_c^2}{2}\left[1 + a^2 P_m\right]$
- Noise power: $P_{N_0} = \frac{1}{4}P_N = \frac{1}{4}\left(2WN_0\right)$



- Output SNR:

$$\left(\frac{S}{N}\right)_{O,AM} = \frac{P_O}{P_{N_0}} = \frac{\frac{A_c^2}{4}a^2 P_{M_n}}{\frac{1}{4}(2N_0W)} = \frac{A_c^2 a^2 P_{M_n}}{2N_0W} = \frac{a^2 P_{M_n}}{1 + a^2 P_{M_n}}\left(\frac{S}{N}\right)_b = \boxed{\eta\left(\frac{S}{N}\right)_b}, \quad \eta : \text{efficiency}$$

SNR in conventional AM systems(2)

- Conventional AM has much lower SNR in exchange for a simple receiver
- ***Envelope detector*** (read section 6.1.4 of textbook)

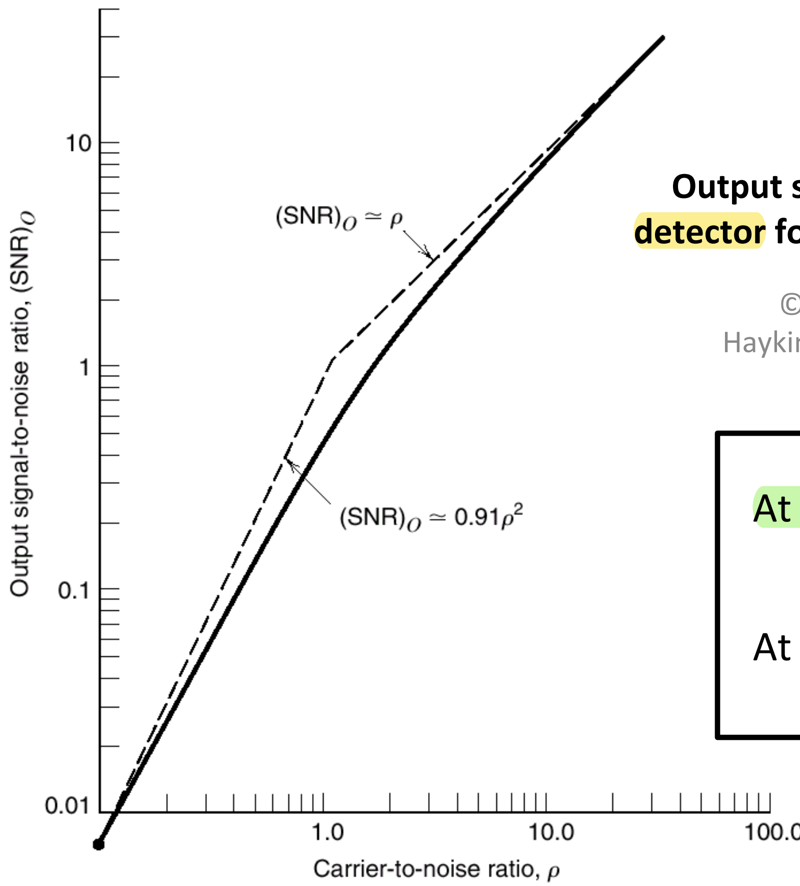


Figure 2.39
Output signal-to-noise ratio of an envelope detector for varying carrier-to-noise ratio ($\rho=\eta$).

At low noise power: $\left(\frac{S}{N}\right)_{O,AM} = \eta\left(\frac{S}{N}\right)_b$

At high noise power: Multiplicative noise with Rayleigh distribution

SNR in FM systems

- Transmitted signal:

$$u(t) = A_c \cos(2\pi f_c t + \phi(t)) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau\right]$$

- Demodulator output: $y_l(t) = k_f\, m(t) + \frac{1}{2\pi}\frac{d}{dt}\frac{V_n(t)}{A_c}\sin\left[\Phi_n(t) - \phi(t)\right],$

with $V_n(t) = \sqrt{N_c^2(t) + N_s^2(t)}, \quad \Phi_n(t) = \tan^{-1}\left[\frac{N_s(t)}{N_c(t)}\right].$

- Noise component: $Y_n(t) = \frac{1}{A_c}\left\{N_s(t)\sin\left[\Phi(t)\right] - N_c(t)\cos\left[\Phi(t)\right]\right\},$

$$S_{Y_n}(f) = \frac{N_0}{A_c^2} f \cdot \Pi\left(\frac{2f}{B_c}\right), \quad S_{N_0}(f) = \frac{N_0}{A_c^2} f^2 \text{ (parabolic)}, \quad P_{N_0} = \frac{2N_0 W^3}{3A_c^2}$$

- Output SNR:

$$\boxed{\left(\frac{S}{N}\right)_{O,FM} = \frac{3\beta_f^2 P_M}{\left(\max\left\{|m(t)|\right\}\right)^2}\cdot\left(\frac{S}{N}\right)_b}$$

Larger SNR = Larger bandwidth (β_f)

Noise figure of an amplifier

- Output power of a (linear) amplifier:

$$P_O = G_A P_i, \quad \text{where } \; G_A = |H(f)|_{\max}^2$$

- Output SNR:

$$\left(\frac{S}{N}\right)_O = \frac{1}{F} \cdot \left(\frac{S}{N}\right)_I$$

- The ***noise figure*** *F* represents the loss in SNR due to additional noise introduced by the amplifier

$$\boxed{10\log_{10}\left(\frac{S}{N}\right)_O = 10\log_{10}\left(\frac{S}{N}\right)_I - 10\log_{10} F}$$

Low-noise amplifiers (LNA) have NF below 3 dB, while some integrated circuit amplifiers have typical values of NF of about 7 dB.

Cascade of amplifiers: $F = F_1 + \dfrac{F_2 - 1}{G_{A,1}} + \dfrac{F_3 - 1}{G_{A,1} G_{A,2}} + \; \dots \; + \dfrac{F_k - 1}{G_{A,1} G_{A,2} \cdots G_{A,k-1}}$