Can you save the world?

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This writeup derives an approximate answer to the 5/12/23 riddler: https://fivethirtyeight.com/features/can-you-save-the-world/.

We assume here that the best question to ask everyone is of the following form:

"Is your number larger than p?"

Then the problem becomes what should we make p to achieve the largest probability of choosing the correct person, and what that probability is.

Let m be the number of people in the world (later, we will set $m = 8$ billion).

If $n \geq 1$ people say yes, then we have a $\frac{1}{n}$ probability of picking the correct person (by choosing a random person who said yes). If 0 people say yes, then we have to choose a random person who said no, so we have a $\frac{1}{n}$ probability of picking the correct person. Since each person's number is uniformly distributed between 0 and 1, the probability a person says yes is $1 - p$. Each person is then a Bernouli trial with probability $1 - p$, and thus the probability that n people say yes is $\binom{m}{n}(1-p)^n p^{m-n}$ (this is the Probability Mass Function of the binomial distribution). If we let X be the random variable that is 1 upon a successful pick of a person and 0 otherwise, then we can multiply the probability of each outcome (the number of people that say yes) by the probability X is 1 in that case to get the expected value of X:

$$E(X) = \frac{1}{m}p^m + \sum_{n=1}^{m} \frac{1}{n}\binom{m}{n}(1-p)^n p^{m-n}$$

We want to maximize $E(X)$ on the interval $p \in [0, 1]$. One strategy is to find the derivative and set it equal to 0. First, we write out the case with all yeses for $E(X)$ to make taking the derivative easier:

$$E(X) = \frac{1}{m}p^m + \frac{1}{m}(1-p)^m + \sum_{n=1}^{m-1} \frac{1}{n}\binom{m}{n}(1-p)^n p^{m-n}$$

We now find the derivative and set it equal to 0:

$$E'(X) = p^{m-1} + (1-p)^{m-1} + \sum_{n=1}^{m-1} -\binom{m}{n}(1-p)^{n-1}p^{m-n} + \frac{m-n}{n}\binom{m}{n}(1-p)^{n}p^{m-n-1} = 0$$

Although there are some tricks you can do to simplify, we still end up with a polynomial of degree $m-1$, which is not solvable analytically for $m > 5$. Thus, we must turn to numerical methods to maximize $E(X)$.

Numerical methods at first seem challenging because even evaluating the entire function for a single p is nearly impossible for $m = 8$ billion: there are 8 billion terms, each of which require on the order of 8 billion multiplications. However, we can be clever and realize that for $n = 8$ billion, we will be selecting a large p, and it is extremely unlikely that there will be that many yeses, and so most of the terms will be very close to 0. Thus, we can consider the sum only up to k, for $k = 100$ (we can experimentally continue increasing k until the answer remains stable). Another good observation is that the polynomial has just one maximum in the range $[0, 1]$; for now, this is just an observation, and proving it is left as an exercise to the reader. With this observation there is only one 0 in the range $[0, 1]$, so it is easy to find with binary search on the derivative. We also can divide by p^{m-k-1}. We still must write a program that can handle the arbitrary precision that we need (and again we can increase the precision until the answer remains stable), which (after some sweat blood and tears) is here:

`https://github.com/JoshEngels/Riddles/blob/main/save_final.py`

I am not 100% sure this is correct, but I did test it on small numbers and the approximate and exact functions returned the same result. For $n = 8$ billion, the program returns that the best p is $0.99999999981214237285511806003514245 23$ (followed by some more digits) and the corresponding value for $E(X)$ is 0.51735143694988. For what it's worth, just choosing the "naive" value of $1 - 1/8000000000$ (which is slightly higher) has $E(X) = 0.485$. So for all this work we get a 3% better chance of humanity surviving. Not bad!