(b) Extend (a) to prove that column j of AB is a linear combination of the columns of A with the coefficients in the linear combination being the entries of column j of B.

(c) For any row vector $w \in \mathsf{F}^m$, prove that wA is a linear combination of the rows of A with the coefficients in the linear combination being the coordinates of w. *Hint:* Use properties of the transpose operation applied to (a).

(d) Prove the analogous result to (b) about rows: Row i of AB is a linear combination of the rows of B with the coefficients in the linear combination being the entries of row i of A.

15.† Let A and B be matrices for which the product matrix AB is defined, and let u_j and v_j denote the jth columns of AB and B, respectively. If $v_p = c_1v_{j_1} + c_2v_{j_2} + \cdots + c_kv_{j_k}$ for some scalars $c_1, c_2, \ldots c_k$, prove that $u_p = c_1u_{j_1} + c_2u_{j_2} + \cdots + c_ku_{j_k}$. Visit `goo.gl/sRpves` for a solution.

16. Let V be a finite-dimensional vector space, and let $\mathsf{T}\colon \mathsf{V} \to \mathsf{V}$ be linear.

(a) If $\operatorname{rank}(\mathsf{T}) = \operatorname{rank}(\mathsf{T}^2)$, prove that $\mathsf{R}(\mathsf{T}) \cap \mathsf{N}(\mathsf{T}) = \{\mathit{0}\}$. Deduce that $\mathsf{V} = \mathsf{R}(\mathsf{T}) \oplus \mathsf{N}(\mathsf{T})$ (see the exercises of Section 1.3).

(b) Prove that $\mathsf{V} = \mathsf{R}(\mathsf{T}^k) \oplus \mathsf{N}(\mathsf{T}^k)$ for some positive integer k.

17. For the definition of *projection* and related facts, see pages 76–77. Let V be a vector space and $\mathsf{T}\colon \mathsf{V} \to \mathsf{V}$ be a linear transformation. Prove that $\mathsf{T} = \mathsf{T}^2$ if and only if T is a projection on $\mathsf{W}_1 = \{y : \mathsf{T}(y) = y\}$ along $\mathsf{N}(\mathsf{T})$.

18. Let β be an ordered basis for a finite-dimensional vector space V, and let $\mathsf{T}\colon \mathsf{V} \to \mathsf{V}$ be linear. Prove that, for any nonnegative integer k, $[\mathsf{T}^k]_\beta = ([\mathsf{T}]_\beta)^k$.

19. Using only the definition of matrix multiplication, prove that, multiplication of matrices is associative.

20. For an incidence matrix A with related matrix B defined by $B_{ij} = 1$ if i is related to j and j is related to i, and $B_{ij} = 0$ otherwise, prove that i belongs to a clique if and only if $(B^3)_{ii} > 0$.

21. Use Exercise 20 to determine the cliques in the relations corresponding to the following incidence matrices.

(a) $\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$ **(b)** $\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$

22. Let A be an incidence matrix that is associated with a dominance relation. Prove that the matrix $A + A^2$ has a row [column] in which each entry is positive except for the diagonal entry.

23. Prove that the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

corresponds to a dominance relation. Use Exercise 22 to determine which persons dominate [are dominated by] each of the others within two stages.

24. Let A be an $n \times n$ incidence matrix that corresponds to a dominance relation. Determine the number of nonzero entries of A.

2.4 INVERTIBILITY AND ISOMORPHISMS

The concept of invertibility is introduced quite early in the study of functions. Fortunately, many of the intrinsic properties of functions are shared by their inverses. For example, in calculus we learn that the properties of being continuous or differentiable are generally retained by the inverse functions. We see in this section (Theorem 2.17) that the inverse of a linear transformation is also linear. This result greatly aids us in the study of *inverses* of matrices. As one might expect from Section 2.3, the inverse of the left-multiplication transformation L_A (when it exists) can be used to determine properties of the inverse of the matrix A.

In the remainder of this section, we apply many of the results about invertibility to the concept of *isomorphism*. We will see that finite-dimensional vector spaces (over F) of equal dimension may be identified. These ideas will be made precise shortly.

The facts about inverse functions presented in Appendix B are, of course, true for linear transformations. Nevertheless, we repeat some of the definitions for use in this section.

Definition. *Let V and W be vector spaces, and let $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ be linear. A function $\mathsf{U}\colon \mathsf{W} \to \mathsf{V}$ is said to be an* **inverse** *of T if $\mathsf{TU} = \mathsf{I_W}$ and $\mathsf{UT} = \mathsf{I_V}$. If T has an inverse, then T is said to be* **invertible***. As noted in Appendix B, if T is invertible, then the inverse of T is unique and is denoted by T^{-1}.*

The following facts hold for invertible functions T and U.

1. $(\mathsf{TU})^{-1} = \mathsf{U}^{-1}\mathsf{T}^{-1}$.
2. $(\mathsf{T}^{-1})^{-1} = \mathsf{T}$; in particular, T^{-1} is invertible.

We often use the fact that a function is invertible if and only if it is both one-to-one and onto. We can therefore restate Theorem 2.5 as follows.

3. Let $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ be a linear transformation, where V and W are finite-dimensional spaces of equal dimension. Then T is invertible if and only if $\operatorname{rank}(\mathsf{T}) = \dim(\mathsf{V})$.

Example 1

Let $\mathsf{T}\colon \mathsf{P}_1(R) \to \mathsf{R}^2$ be the linear transformation defined by $\mathsf{T}(a+bx) = (a, a+b)$. The reader can verify directly that $\mathsf{T}^{-1}\colon \mathsf{R}^2 \to \mathsf{P}_1(R)$ is defined by $\mathsf{T}^{-1}(c,d) = c + (d-c)x$. Observe that T^{-1} is also linear. As Theorem 2.17 demonstrates, this is true in general. ◆

Theorem 2.17. *Let* V *and* W *be vector spaces, and let* $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ *be linear and invertible. Then* $\mathsf{T}^{-1}\colon \mathsf{W} \to \mathsf{V}$ *is linear.*

Proof. Let $y_1, y_2 \in \mathsf{W}$ and $c \in F$. Since T is onto and one-to-one, there exist unique vectors x_1 and x_2 such that $\mathsf{T}(x_1) = y_1$ and $\mathsf{T}(x_2) = y_2$. Thus $x_1 = \mathsf{T}^{-1}(y_1)$ and $x_2 = \mathsf{T}^{-1}(y_2)$; so

$$\begin{aligned}\mathsf{T}^{-1}(cy_1 + y_2) &= \mathsf{T}^{-1}[c\mathsf{T}(x_1) + \mathsf{T}(x_2)] = \mathsf{T}^{-1}[\mathsf{T}(cx_1 + x_2)] \\ &= cx_1 + x_2 = c\mathsf{T}^{-1}(y_1) + \mathsf{T}^{-1}(y_2).\end{aligned}$$ ∎

Corollary. *Let* T *be an invertible linear transformation from* V *to* W. *Then* V *is finite-dimensional if and only if* W *is finite-dimensional. In this case,* $\dim(\mathsf{V}) = \dim(\mathsf{W})$.

Proof. Suppose that V is finite-dimensional. Let $\beta = \{x_1, x_2, \ldots, x_n\}$ be a basis for V. By Theorem 2.2 (p. 68), $\mathsf{T}(\beta)$ spans $\mathsf{R}(\mathsf{T}) = \mathsf{W}$; hence W is finite-dimensional by Theorem 1.9 (p. 45). Conversely, if W is finite-dimensional, then so is V by a similar argument, using T^{-1}.

Now suppose that V and W are finite-dimensional. Because T is one-to-one and onto, we have

$$\operatorname{nullity}(\mathsf{T}) = 0 \quad \text{and} \quad \operatorname{rank}(\mathsf{T}) = \dim(\mathsf{R}(\mathsf{T})) = \dim(\mathsf{W}).$$

So by the dimension theorem (p. 70), it follows that $\dim(\mathsf{V}) = \dim(\mathsf{W})$. ∎

It now follows immediately from Theorem 2.5 (p. 71) that if T is a linear transformation between vector spaces of equal (finite) dimension, then the conditions of being invertible, one-to-one, and onto are all equivalent.

We are now ready to define the inverse of a matrix. The reader should note the analogy with the inverse of a linear transformation.

Definition. *Let A be an $n \times n$ matrix. Then A is* **invertible** *if there exists an $n \times n$ matrix B such that $AB = BA = I$.*

If A is invertible, then the matrix B such that $AB = BA = I$ is unique. (If C were another such matrix, then $C = CI = C(AB) = (CA)B = IB = B$.) The matrix B is called the **inverse** of A and is denoted by A^{-1}.

Example 2

The reader should verify that the inverse of

$$\begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix} \quad \text{is} \quad \begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix}. \quad \blacklozenge$$

In Section 3.2, we will learn a technique for computing the inverse of a matrix. At this point, we develop a number of results that relate the inverses of matrices to the inverses of linear transformations.

Theorem 2.18. *Let* V *and* W *be finite-dimensional vector spaces with ordered bases* β *and* γ, *respectively. Let* $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ *be linear. Then* T *is invertible if and only if* $[\mathsf{T}]_\beta^\gamma$ *is invertible. Furthermore,* $[\mathsf{T}^{-1}]_\gamma^\beta = ([\mathsf{T}]_\beta^\gamma)^{-1}$.

Proof. Suppose that T is invertible. By the Corollary to Theorem 2.17, we have $\dim(\mathsf{V}) = \dim(\mathsf{W})$. Let $n = \dim(\mathsf{V})$. So $[\mathsf{T}]_\beta^\gamma$ is an $n \times n$ matrix. Now $\mathsf{T}^{-1}\colon \mathsf{W} \to \mathsf{V}$ satisfies $\mathsf{T}\mathsf{T}^{-1} = \mathsf{I}_\mathsf{W}$ and $\mathsf{T}^{-1}\mathsf{T} = \mathsf{I}_\mathsf{V}$. Thus

$$I_n = [\mathsf{I}_\mathsf{V}]_\beta = [\mathsf{T}^{-1}\mathsf{T}]_\beta = [\mathsf{T}^{-1}]_\gamma^\beta [\mathsf{T}]_\beta^\gamma.$$

Similarly, $[\mathsf{T}]_\beta^\gamma [\mathsf{T}^{-1}]_\gamma^\beta = I_n$. So $[\mathsf{T}]_\beta^\gamma$ is invertible and $\left([\mathsf{T}]_\beta^\gamma\right)^{-1} = [\mathsf{T}^{-1}]_\gamma^\beta$.

Now suppose that $A = [\mathsf{T}]_\beta^\gamma$ is invertible. Then there exists an $n \times n$ matrix B such that $AB = BA = I_n$. By Theorem 2.6 (p. 73), there exists $\mathsf{U} \in \mathcal{L}(\mathsf{W}, \mathsf{V})$ such that

$$\mathsf{U}(w_j) = \sum_{i=1}^{n} B_{ij} v_i \quad \text{for } j = 1, 2, \ldots, n,$$

where $\gamma = \{w_1, w_2, \ldots, w_n\}$ and $\beta = \{v_1, v_2, \ldots, v_n\}$. It follows that $[\mathsf{U}]_\gamma^\beta = B$. To show that $\mathsf{U} = \mathsf{T}^{-1}$, observe that

$$[\mathsf{UT}]_\beta = [\mathsf{U}]_\gamma^\beta [\mathsf{T}]_\beta^\gamma = BA = I_n = [\mathsf{I}_\mathsf{V}]_\beta$$

by Theorem 2.11 (p. 89). So $\mathsf{UT} = \mathsf{I}_\mathsf{V}$, and similarly, $\mathsf{TU} = \mathsf{I}_\mathsf{W}$. ∎

Example 3

Let β and γ be the standard ordered bases of $\mathsf{P}_1(R)$ and R^2, respectively. For T as in Example 1, we have

$$[\mathsf{T}]_\beta^\gamma = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad [\mathsf{T}^{-1}]_\gamma^\beta = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

It can be verified by matrix multiplication that each matrix is the inverse of the other. $\blacklozenge$

Corollary 1. *Let* V *be a finite-dimensional vector space with an ordered basis* β, *and let* $\mathsf{T}\colon \mathsf{V} \to \mathsf{V}$ *be linear. Then* T *is invertible if and only if* $[\mathsf{T}]_\beta$ *is invertible. Furthermore,* $[\mathsf{T}^{-1}]_\beta = ([\mathsf{T}]_\beta)^{-1}$.

Proof. Exercise. ∎

Corollary 2. *Let* A *be an* $n \times n$ *matrix. Then* A *is invertible if and only if* L_A *is invertible. Furthermore,* $(\mathsf{L}_A)^{-1} = \mathsf{L}_{A^{-1}}$.

Proof. Exercise. ∎

The notion of invertibility may be used to formalize what may already have been observed by the reader, that is, that certain vector spaces strongly resemble one another except for the form of their vectors. For example, in the case of $\mathsf{M}_{2\times 2}(F)$ and F^4, if we associate to each matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the 4-tuple (a, b, c, d), we see that sums and scalar products associate in a similar manner; that is, in terms of the vector space structure, these two vector spaces may be considered identical or *isomorphic*.

Definitions. *Let* V *and* W *be vector spaces. We say that* V *is* ***isomorphic*** *to* W *if there exists a linear transformation* $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ *that is invertible. Such a linear transformation is called an* ***isomorphism*** *from* V *onto* W.

We leave as an exercise (see Exercise 13) the proof that "is isomorphic to" is an equivalence relation. (See Appendix A.) So we need only say that V and W are isomorphic.

Example 4

Define $\mathsf{T}\colon \mathsf{F}^2 \to \mathsf{P}_1(F)$ by $\mathsf{T}(a_1, a_2) = a_1 + a_2 x$. It is easily checked that T is an isomorphism; so F^2 is isomorphic to $\mathsf{P}_1(F)$. ♦

Example 5

Define

$$\mathsf{T}\colon \mathsf{P}_3(R) \to \mathsf{M}_{2\times 2}(R) \quad \text{by } \mathsf{T}(f) = \begin{pmatrix} f(1) & f(2) \\ f(3) & f(4) \end{pmatrix}.$$

It is easily verified that T is linear. By use of the Lagrange interpolation formula in Section 1.6, it can be shown (compare with Exercise 22) that $\mathsf{T}(f) = O$ only when f is the zero polynomial. Thus T is one-to-one (see Exercise 11). Moreover, because $\dim(\mathsf{P}_3(R)) = \dim(\mathsf{M}_{2\times 2}(R))$, it follows that T is invertible by Theorem 2.5 (p. 71). We conclude that $\mathsf{P}_3(R)$ is isomorphic to $\mathsf{M}_{2\times 2}(R)$. ♦

In each of Examples 4 and 5, the reader may have observed that isomorphic vector spaces have equal dimensions. As the next theorem shows, this is no coincidence.

Theorem 2.19. *Let* V *and* W *be finite-dimensional vector spaces (over the same field). Then* V *is isomorphic to* W *if and only if* $\dim(\mathsf{V}) = \dim(\mathsf{W})$.

Proof. Suppose that V is isomorphic to W and that $\mathsf{T}: \mathsf{V} \to \mathsf{W}$ is an isomorphism from V to W. By the lemma preceding Theorem 2.18, we have that $\dim(\mathsf{V}) = \dim(\mathsf{W})$.

Now suppose that $\dim(\mathsf{V}) = \dim(\mathsf{W})$, and let $\beta = \{v_1, v_2, \ldots, v_n\}$ and $\gamma = \{w_1, w_2, \ldots, w_n\}$ be bases for V and W, respectively. By Theorem 2.6 (p. 73), there exists $\mathsf{T}: \mathsf{V} \to \mathsf{W}$ such that T is linear and $\mathsf{T}(v_i) = w_i$ for $i = 1, 2, \ldots, n$. Using Theorem 2.2 (p. 68), we have

$$\mathsf{R}(\mathsf{T}) = \operatorname{span}(\mathsf{T}(\beta)) = \operatorname{span}(\gamma) = \mathsf{W}.$$

So T is onto. From Theorem 2.5 (p. 71), we have that T is also one-to-one. Hence T is an isomorphism. ∎

By the lemma to Theorem 2.18, if V and W are isomorphic, then either both of V and W are finite-dimensional or both are infinite-dimensional.

Corollary. *Let* V *be a vector space over* F. *Then* V *is isomorphic to* F^n *if and only if* $\dim(\mathsf{V}) = n$.

Up to this point, we have associated linear transformations with their matrix representations. We are now in a position to prove that, as a vector space, the collection of all linear transformations between two given vector spaces may be identified with the appropriate vector space of $m \times n$ matrices.

Theorem 2.20. *Let* V *and* W *be finite-dimensional vector spaces over* F *of dimensions* n *and* m*, respectively, and let* β *and* γ *be ordered bases for* V *and* W*, respectively. Then the function* $\Phi_\beta^\gamma: \mathcal{L}(\mathsf{V}, \mathsf{W}) \to \mathsf{M}_{m\times n}(F)$*, defined by* $\Phi_\beta^\gamma(\mathsf{T}) = [\mathsf{T}]_\beta^\gamma$ *for* $\mathsf{T} \in \mathcal{L}(\mathsf{V}, \mathsf{W})$*, is an isomorphism.*

Proof. By Theorem 2.8 (p. 83), Φ_β^γ is linear. Hence we must show that Φ_β^γ is one-to-one and onto. This is accomplished if we show that for every $m \times n$ matrix A, there exists a unique linear transformation $\mathsf{T}: \mathsf{V} \to \mathsf{W}$ such that $\Phi_\beta^\gamma(\mathsf{T}) = A$. Let $\beta = \{v_1, v_2, \ldots, v_n\}$, $\gamma = \{w_1, w_2, \ldots, w_m\}$, and let A be a given $m \times n$ matrix. By Theorem 2.6 (p. 73), there exists a unique linear transformation $\mathsf{T}: \mathsf{V} \to \mathsf{W}$ such that

$$\mathsf{T}(v_j) = \sum_{i=1}^{m} A_{ij} w_i \quad \text{for } 1 \leq j \leq n.$$

But this means that $[\mathsf{T}]_\beta^\gamma = A$, or $\Phi_\beta^\gamma(\mathsf{T}) = A$. So Φ_β^γ is an isomorphism. ∎

Corollary. *Let* V *and* W *be finite-dimensional vector spaces of dimensions* n *and* m*, respectively. Then* $\mathcal{L}(\mathsf{V},\mathsf{W})$ *is finite-dimensional of dimension* mn*.*

Proof. The proof follows from Theorems 2.20 and 2.19 and the fact that $\dim(\mathsf{M}_{m\times n}(F)) = mn$. ∎

We conclude this section with a result that allows us to see more clearly the relationship between linear transformations defined on abstract finite-dimensional vector spaces and linear transformations from F^n to F^m.

We begin by naming the transformation $x \to [x]_\beta$ introduced in Section 2.2.

Definition. *Let* β *be an ordered basis for an* n*-dimensional vector space* V *over the field* F*. The* **standard representation of** V **with respect to** β *is the function* $\phi_\beta\colon \mathsf{V} \to \mathsf{F}^n$ *defined by* $\phi_\beta(x) = [x]_\beta$ *for each* $x \in \mathsf{V}$*.*

Example 6

Let $\beta = \{(1,0),(0,1)\}$ and $\gamma = \{(1,2),(3,4)\}$. It is easily observed that β and γ are ordered bases for R^2. For $x = (1,-2)$, we have

$$\phi_\beta(x) = [x]_\beta = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \text{and} \quad \phi_\gamma(x) = [x]_\gamma = \begin{pmatrix} -5 \\ 2 \end{pmatrix}. \quad \blacklozenge$$

We observed earlier that ϕ_β is a linear transformation. The next theorem tells us much more.

Theorem 2.21. *For any finite-dimensional vector space* V *with ordered basis* β*,* ϕ_β *is an isomorphism.*

Proof. Exercise. ∎

This theorem provides us with an alternate proof that an n-dimensional vector space is isomorphic to F^n (see the corollary to Theorem 2.19).

Let V and W be vector spaces of dimension n and m, respectively, and let $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ be a linear transformation. Define $A = [\mathsf{T}]_\beta^\gamma$, where β and γ are arbitrary ordered bases of V and W, respectively. We are now able to use ϕ_β and ϕ_γ to study the relationship between the linear transformations T and $\mathsf{L}_A\colon \mathsf{F}^n \to \mathsf{F}^m$.

Let us first consider Figure 2.2. Notice that there are two composites of linear transformations that map V into F^m:

1. Map V into F^n with ϕ_β and follow this transformation with L_A; this yields the composite $\mathsf{L}_A\phi_\beta$.
2. Map V into W with T and follow it by ϕ_γ to obtain the composite $\phi_\gamma\mathsf{T}$.

$$\begin{tikzcd}
\mathsf{V} \arrow[r, "\mathsf{T}"] \arrow[d, "\phi_\beta"'] \arrow[dr, dashed, bend right, "(1)"] \arrow[dr, dashed, bend left, "(2)"] & \mathsf{W} \arrow[d, "\phi_\gamma"] \\
\mathsf{F}^n \arrow[r, "\mathsf{L}_A"'] & \mathsf{F}^m
\end{tikzcd}$$

Figure 2.2

These two composites are depicted by the dashed arrows in the diagram. By a simple reformulation of Theorem 2.14 (p. 92), we may conclude that

$$\mathsf{L}_A\phi_\beta = \phi_\gamma\mathsf{T};$$

that is, the diagram "commutes." Heuristically, this relationship indicates that after V and W are identified with F^n and F^m via ϕ_β and ϕ_γ, respectively, we may "identify" T with L_A. This diagram allows us to transfer operations on abstract vector spaces to ones on F^n and F^m.

Example 7

Recall the linear transformation $\mathsf{T}\colon \mathsf{P}_3(R) \to \mathsf{P}_2(R)$ defined in Example 4 of Section 2.2 ($\mathsf{T}(f(x)) = f'(x)$). Let β and γ be the standard ordered bases for $\mathsf{P}_3(R)$ and $\mathsf{P}_2(R)$, respectively, and let $\phi_\beta\colon \mathsf{P}_3(R) \to \mathsf{R}^4$ and $\phi_\gamma\colon \mathsf{P}_2(R) \to \mathsf{R}^3$ be the corresponding standard representations of $\mathsf{P}_3(R)$ and $\mathsf{P}_2(R)$. If $A = [\mathsf{T}]_\beta^\gamma$, then

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Consider the polynomial $p(x) = 2+x-3x^2+5x^3$. We show that $\mathsf{L}_A\phi_\beta(p(x)) = \phi_\gamma\mathsf{T}(p(x))$. Now

$$\mathsf{L}_A\phi_\beta(p(x)) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -6 \\ 15 \end{pmatrix}.$$

But since $\mathsf{T}(p(x)) = p'(x) = 1 - 6x + 15x^2$, we have

$$\phi_\gamma\mathsf{T}(p(x)) = \begin{pmatrix} 1 \\ -6 \\ 15 \end{pmatrix}.$$

So $\mathsf{L}_A\phi_\beta(p(x)) = \phi_\gamma\mathsf{T}(p(x))$. ◆

Try repeating Example 7 with different polynomials $p(x)$.

EXERCISES

1. Label the following statements as true or false. In each part, V and W are vector spaces with ordered (finite) bases α and β, respectively, $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ is linear, and A and B are matrices.

(a) $\left([\mathsf{T}]_\alpha^\beta\right)^{-1} = [\mathsf{T}^{-1}]_\alpha^\beta$.
(b) T is invertible if and only if T is one-to-one and onto.
(c) $\mathsf{T} = \mathsf{L}_A$, where $A = [\mathsf{T}]_\alpha^\beta$.
(d) $\mathsf{M}_{2\times 3}(F)$ is isomorphic to F^5.
(e) $\mathsf{P}_n(F)$ is isomorphic to $\mathsf{P}_m(F)$ if and only if $n = m$.
(f) $AB = I$ implies that A and B are invertible.
(g) If A is invertible, then $(A^{-1})^{-1} = A$.
(h) A is invertible if and only if L_A is invertible.
(i) A must be square in order to possess an inverse.

2. For each of the following linear transformations T, determine whether T is invertible and justify your answer.

(a) $\mathsf{T}\colon \mathsf{R}^2 \to \mathsf{R}^3$ defined by $\mathsf{T}(a_1, a_2) = (a_1 - 2a_2, a_2, 3a_1 + 4a_2)$.
(b) $\mathsf{T}\colon \mathsf{R}^2 \to \mathsf{R}^3$ defined by $\mathsf{T}(a_1, a_2) = (3a_1 - a_2, a_2, 4a_1)$.
(c) $\mathsf{T}\colon \mathsf{R}^3 \to \mathsf{R}^3$ defined by $\mathsf{T}(a_1, a_2, a_3) = (3a_1 - 2a_3, a_2, 3a_1 + 4a_2)$.
(d) $\mathsf{T}\colon \mathsf{P}_3(R) \to \mathsf{P}_2(R)$ defined by $\mathsf{T}(p(x)) = p'(x)$.
(e) $\mathsf{T}\colon \mathsf{M}_{2\times 2}(R) \to \mathsf{P}_2(R)$ defined by $\mathsf{T}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + 2bx + (c+d)x^2$.
(f) $\mathsf{T}\colon \mathsf{M}_{2\times 2}(R) \to \mathsf{M}_{2\times 2}(R)$ defined by $\mathsf{T}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & a \\ c & c+d \end{pmatrix}$.

3. Which of the following pairs of vector spaces are isomorphic? Justify your answers.

(a) F^3 and $\mathsf{P}_3(F)$.
(b) F^4 and $\mathsf{P}_3(F)$.
(c) $\mathsf{M}_{2\times 2}(R)$ and $\mathsf{P}_3(R)$.
(d) $\mathsf{V} = \{A \in \mathsf{M}_{2\times 2}(R)\colon \operatorname{tr}(A) = 0\}$ and R^4.

4.[†] Let A and B be $n \times n$ invertible matrices. Prove that AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

5.[†] Let A be invertible. Prove that A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$. Visit `goo.gl/suFm6V` for a solution.

6. Prove that if A is invertible and $AB = O$, then $B = O$.

7. Let A be an $n \times n$ matrix.

(a) Suppose that $A^2 = O$. Prove that A is not invertible.
(b) Suppose that $AB = O$ for some nonzero $n \times n$ matrix B. Could A be invertible? Explain.

8. Prove Corollaries 1 and 2 of Theorem 2.18.

9.† Let A and B be $n \times n$ matrices such that AB is invertible.

(a) Prove that A and B are invertible. *Hint:* See Exercise 12 of Section 2.3.
(b) Give an example to show that a product of nonsquare matrices can be invertible even though the factors, by definition, are not.

10.† Let A and B be $n \times n$ matrices such that $AB = I_n$.

(a) Use Exercise 9 to conclude that A and B are invertible.
(b) Prove $A = B^{-1}$ (and hence $B = A^{-1}$). (We are, in effect, saying that for square matrices, a "one-sided" inverse is a "two-sided" inverse.)
(c) State and prove analogous results for linear transformations defined on finite-dimensional vector spaces.

11. Verify that the transformation in Example 5 is one-to-one.

12. Prove Theorem 2.21.

13. Let $\sim$ mean "is isomorphic to." Prove that $\sim$ is an equivalence relation on the class of vector spaces over F.

14. Let

$$\mathsf{V} = \left\{ \begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} : a, b, c \in F \right\}.$$

Construct an isomorphism from V to F^3.

15. Let V and W be n-dimensional vector spaces, and let $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ be a linear transformation. Suppose that β is a basis for V. Prove that T is an isomorphism if and only if $\mathsf{T}(\beta)$ is a basis for W.

16. Let B be an $n \times n$ invertible matrix. Define $\Phi\colon \mathsf{M}_{n\times n}(F) \to \mathsf{M}_{n\times n}(F)$ by $\Phi(A) = B^{-1}AB$. Prove that Φ is an isomorphism.

17.† Let V and W be finite-dimensional vector spaces and $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ be an isomorphism. Let V_0 be a subspace of V.

(a) Prove that $\mathsf{T}(\mathsf{V}_0)$ is a subspace of W.
(b) Prove that $\dim(\mathsf{V}_0) = \dim(\mathsf{T}(\mathsf{V}_0))$.

18. Repeat Example 7 with the polynomial $p(x) = 1 + x + 2x^2 + x^3$.

19. In Example 5 of Section 2.1, the mapping $\mathsf{T}\colon \mathsf{M}_{2\times 2}(R) \to \mathsf{M}_{2\times 2}(R)$ defined by $\mathsf{T}(M) = M^t$ for each $M \in \mathsf{M}_{2\times 2}(R)$ is a linear transformation. Let $\beta = \{E^{11}, E^{12}, E^{21}, E^{22}\}$, which is a basis for $\mathsf{M}_{2\times 2}(R)$, as noted in Example 3 of Section 1.6.

(a) Compute $[\mathsf{T}]_\beta$.

(b) Verify that $\mathsf{L}_A\phi_\beta(M) = \phi_\beta\mathsf{T}(M)$ for $A = [\mathsf{T}]_\beta$ and

$$M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

20.[†] Let $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ be a linear transformation from an n-dimensional vector space V to an m-dimensional vector space W. Let β and γ be ordered bases for V and W, respectively. Prove that $\operatorname{rank}(\mathsf{T}) = \operatorname{rank}(\mathsf{L}_A)$ and that $\operatorname{nullity}(\mathsf{T}) = \operatorname{nullity}(\mathsf{L}_A)$, where $A = [\mathsf{T}]_\beta^\gamma$. *Hint:* Apply Exercise 17 to Figure 2.2.

21. Let V and W be finite-dimensional vector spaces with ordered bases $\beta = \{v_1, v_2, \ldots, v_n\}$ and $\gamma = \{w_1, w_2, \ldots, w_m\}$, respectively. By Theorem 2.6 (p. 73), there exist linear transformations $\mathsf{T}_{ij}\colon \mathsf{V} \to \mathsf{W}$ such that

$$\mathsf{T}_{ij}(v_k) = \begin{cases} w_i & \text{if } k = j \\ 0 & \text{if } k \neq j. \end{cases}$$

First prove that $\{\mathsf{T}_{ij}\colon 1 \le i \le m,\ 1 \le j \le n\}$ is a basis for $\mathcal{L}(\mathsf{V}, \mathsf{W})$. Then let M^{ij} be the $m \times n$ matrix with 1 in the ith row and jth column and 0 elsewhere, and prove that $[\mathsf{T}_{ij}]_\beta^\gamma = M^{ij}$. Again by Theorem 2.6, there exists a linear transformation $\Phi_\beta^\gamma\colon \mathcal{L}(\mathsf{V}, \mathsf{W}) \to \mathsf{M}_{m\times n}(F)$ such that $\Phi_\beta^\gamma(\mathsf{T}_{ij}) = M^{ij}$. Prove that Φ_β^γ is an isomorphism.

22. Let $c_0, c_1, \ldots, c_n$ be distinct scalars from an infinite field F. Define $\mathsf{T}\colon \mathsf{P}_n(F) \to \mathsf{F}^{n+1}$ by $\mathsf{T}(f) = (f(c_0), f(c_1), \ldots, f(c_n))$. Prove that T is an isomorphism. *Hint:* Use the Lagrange polynomials associated with $c_0, c_1, \ldots, c_n$.

23. Let W denote the vector space of all sequences in F that have only a finite number of nonzero terms (defined in Exercise 18 of Section 1.6), and let $\mathsf{Z} = \mathsf{P}(F)$. Define

$$\mathsf{T}\colon \mathsf{W} \to \mathsf{Z} \quad \text{by} \quad \mathsf{T}(\sigma) = \sum_{i=0}^{n} \sigma(i)x^i,$$

where n is the largest integer such that $\sigma(n) \neq 0$. Prove that T is an isomorphism.

The following exercise requires familiarity with the concept of *quotient space* defined in Exercise 31 of Section 1.3 and with Exercise 42 of Section 2.1.

24. Let V and Z be vector spaces and $T\colon V \to Z$ be a linear transformation that is onto. Define the mapping

$$\overline{T}\colon V/N(T) \to Z \quad \text{by} \quad \overline{T}(v + N(T)) = T(v)$$

for any coset $v + N(T)$ in $V/N(T)$.

(a) Prove that $\overline{T}$ is well-defined; that is, prove that if $v + N(T) = v' + N(T)$, then $T(v) = T(v')$.
(b) Prove that $\overline{T}$ is linear.
(c) Prove that $\overline{T}$ is an isomorphism.
(d) Prove that the diagram shown in Figure 2.3 commutes; that is, prove that $T = \overline{T}\eta$.

$$\begin{tikzcd} V \arrow[rr, dashed, "T"] \arrow[dr, dashed, "\eta"'] & & Z \\ & V/N(T) \arrow[ur, dashed, "\overline{T}"'] & \end{tikzcd}$$

Figure 2.3

25. Let V be a nonzero vector space over a field F, and suppose that S is a basis for V. (By the corollary to Theorem 1.13 (p. 61) in Section 1.7, every vector space has a basis.) Let $\mathcal{C}(S, F)$ denote the vector space of all functions $f \in \mathcal{F}(S, F)$ such that $f(s) = 0$ for all but a finite number of vectors in S. (See Exercise 14 of Section 1.3.) Let $\Psi\colon \mathcal{C}(S, F) \to V$ be defined by $\Psi(f) = \mathit{0}$ if f is the zero function, and

$$\Psi(f) = \sum_{s \in S,\, f(s) \neq 0} f(s)s,$$

otherwise. Prove that Ψ is an isomorphism. Thus every nonzero vector space can be viewed as a space of functions.

2.5 THE CHANGE OF COORDINATE MATRIX

In many areas of mathematics, a change of variable is used to simplify the appearance of an expression. For example, in calculus an antiderivative of $2xe^{x^2}$ can be found by making the change of variable $u = x^2$. The resulting expression is of such a simple form that an antiderivative is easily recognized:

$$\int 2xe^{x^2}\, dx = \int e^u\, du = e^u + c = e^{x^2} + c.$$

Similarly, in geometry the change of variable

$$x = \frac{2}{\sqrt{5}}x' - \frac{1}{\sqrt{5}}y'$$

$$y = \frac{1}{\sqrt{5}}x' + \frac{2}{\sqrt{5}}y'$$

can be used to transform the equation $2x^2 - 4xy + 5y^2 = 1$ into the simpler equation $(x')^2 + 6(y')^2 = 1$, in which form it is easily seen to be the equation of an ellipse. (See Figure 2.4.) We will see how this change of variable is determined in Section 6.5. Geometrically, the change of variable

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x' \\ y' \end{pmatrix}$$

is a change in the way that the position of a point P in the plane is described. This is done by introducing a new frame of reference, an $x'y'$-coordinate system with coordinate axes rotated from the original xy-coordinate axes. In this case, the new coordinate axes are chosen to lie in the direction of the axes of the ellipse. The unit vectors along the x'-axis and the y'-axis form an ordered basis

$$\beta' = \left\{ \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\}$$

for R^2, and the change of variable is actually a change from $[P]_\beta = \begin{pmatrix} x \\ y \end{pmatrix}$, the coordinate vector of P relative to the standard ordered basis $\beta = \{e_1, e_2\}$, to $[P]_{\beta'} = \begin{pmatrix} x' \\ y' \end{pmatrix}$, the coordinate vector of P relative to the new rotated basis β'.

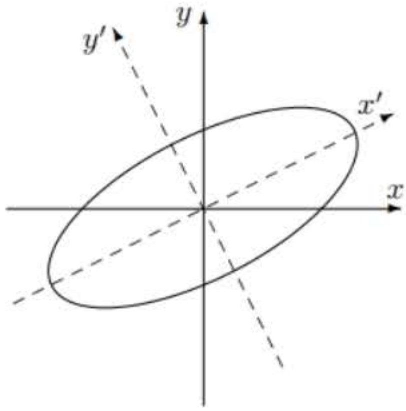


Figure 2.4

A natural question arises: How can a coordinate vector relative to one basis be changed into a coordinate vector relative to the other? Notice that the system of equations relating the new and old coordinates can be represented by the matrix equation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}.$$

Notice also that the matrix

$$Q = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

equals $[\mathsf{I}]_{\beta'}^{\beta}$, where I denotes the identity transformation on R^2. Thus $[v]_\beta = Q[v]_{\beta'}$ for all $v \in \mathsf{R}^2$. A similar result is true in general.

Theorem 2.22. *Let β and β' be two ordered bases for a finite-dimensional vector space V, and let $Q = [\mathsf{I}_\mathsf{V}]_{\beta'}^{\beta}$. Then*
(a) *Q is invertible.*
(b) *For any $v \in \mathsf{V}$, $[v]_\beta = Q[v]_{\beta'}$.*

Proof. (a) Since I_V is invertible, Q is invertible by Theorem 2.18 (p. 102).
(b) For any $v \in \mathsf{V}$,

$$[v]_\beta = [\mathsf{I}_\mathsf{V}(v)]_\beta = [\mathsf{I}_\mathsf{V}]_{\beta'}^{\beta}[v]_{\beta'} = Q[v]_{\beta'}$$

by Theorem 2.14 (p. 92). ∎

The matrix $Q = [\mathsf{I}_\mathsf{V}]_{\beta'}^{\beta}$ defined in Theorem 2.22 is called a **change of coordinate matrix**. Because of part (b) of the theorem, we say that Q **changes β'-coordinates into β-coordinates**. Observe that if $\beta = \{x_1, x_2, \ldots, x_n\}$ and $\beta' = \{x'_1, x'_2, \ldots, x'_n\}$, then

$$x'_j = \sum_{i=1}^{n} Q_{ij} x_i$$

for $j = 1, 2, \ldots, n$; that is, the jth column of Q is $[x'_j]_\beta$.

Notice that if Q changes β'-coordinates into β-coordinates, then Q^{-1} changes β-coordinates into β'-coordinates. (See Exercise 11.)

Example 1

In R^2, let $\beta = \{(1,1), (1,-1)\}$ and $\beta' = \{(2,4), (3,1)\}$. Since

$$(2,4) = 3(1,1) - 1(1,-1) \qquad \text{and} \qquad (3,1) = 2(1,1) + 1(1,-1),$$

the matrix that changes β'-coordinates into β-coordinates is

$$Q = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}.$$

Thus, for instance,

$$[(2,4)]_\beta = Q[(2,4)]_{\beta'} = Q\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}3\\-1\end{pmatrix}. \quad \blacklozenge$$

For the remainder of this section, we consider only linear transformations that map a vector space V into itself. Such a linear transformation is called a **linear operator** on V. Suppose now that T is a linear operator on a finite-dimensional vector space V and that β and β' are ordered bases for V. Then V can be represented by the matrices $[\mathsf{T}]_\beta$ and $[\mathsf{T}]_{\beta'}$. What is the relationship between these matrices? The next theorem provides a simple answer using a change of coordinate matrix.

Theorem 2.23. *Let T be a linear operator on a finite-dimensional vector space V, and let β and β' be ordered bases for V. Suppose that Q is the change of coordinate matrix that changes β'-coordinates into β-coordinates. Then*

$$[\mathsf{T}]_{\beta'} = Q^{-1}[\mathsf{T}]_\beta Q.$$

Proof. Let I be the identity transformation on V. Then $\mathsf{T} = \mathsf{IT} = \mathsf{TI}$; hence, by Theorem 2.11 (p. 89),

$$Q[\mathsf{T}]_{\beta'} = [\mathsf{I}]_{\beta'}^{\beta}[\mathsf{T}]_{\beta'}^{\beta'} = [\mathsf{IT}]_{\beta'}^{\beta} = [\mathsf{TI}]_{\beta'}^{\beta} = [\mathsf{T}]_{\beta}^{\beta}[\mathsf{I}]_{\beta'}^{\beta} = [\mathsf{T}]_\beta Q.$$

Therefore $[\mathsf{T}]_{\beta'} = Q^{-1}[\mathsf{T}]_\beta Q$. ∎

Example 2

Let T be the linear operator on R^2 defined by

$$\mathsf{T}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}3a-b\\a+3b\end{pmatrix},$$

and let β and β' be the ordered bases in Example 1. The reader should verify that

$$[\mathsf{T}]_\beta = \begin{pmatrix}3 & 1\\-1 & 3\end{pmatrix}.$$

In Example 1, we saw that the change of coordinate matrix that changes β'-coordinates into β-coordinates is

$$Q = \begin{pmatrix}3 & 2\\-1 & 1\end{pmatrix},$$

and it is easily verified that

$$Q^{-1} = \frac{1}{5}\begin{pmatrix}1 & -2\\1 & 3\end{pmatrix}.$$

Hence, by Theorem 2.23,

$$[\mathsf{T}]_{\beta'} = Q^{-1}[\mathsf{T}]_\beta Q = \begin{pmatrix} 4 & 1 \\ -2 & 2 \end{pmatrix}.$$

To show that this is the correct matrix, we can verify that the image under T of each vector of β' is the linear combination of the vectors of β' with the entries of the corresponding column as its coefficients. For example, the image of the second vector in β' is

$$\mathsf{T}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \end{pmatrix} = 1\begin{pmatrix} 2 \\ 4 \end{pmatrix} + 2\begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

Notice that the coefficients of the linear combination are the entries of the second column of $[\mathsf{T}]_{\beta'}$. ◆

It is often useful to apply Theorem 2.23 to compute $[\mathsf{T}]_\beta$, as the next example shows.

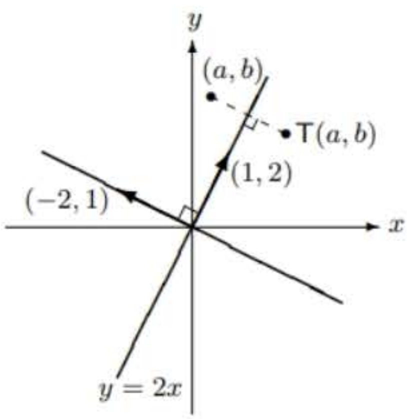


Figure 2.5

Example 3

Recall the reflection about the x-axis in Example 3 of Section 2.1. The rule $(x, y) \to (x, -y)$ is easy to obtain. We now derive the less obvious rule for the reflection T about the line $y = 2x$. (See Figure 2.5.) We wish to find an expression for $\mathsf{T}(a, b)$ for any (a, b) in R^2. Since T is linear, it is completely determined by its values on a basis for R^2. Clearly, $\mathsf{T}(1, 2) = (1, 2)$ and $\mathsf{T}(-2, 1) = -(-2, 1) = (2, -1)$. Therefore if we let

$$\beta' = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\},$$

then β' is an ordered basis for R^2 and

$$[\mathsf{T}]_{\beta'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let β be the standard ordered basis for R^2, and let Q be the matrix that changes β'-coordinates into β-coordinates. Then

$$Q = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

and $Q^{-1}[\mathsf{T}]_\beta Q = [\mathsf{T}]_{\beta'}$. We can solve this equation for $[\mathsf{T}]_\beta$ to obtain that $[\mathsf{T}]_\beta = Q[\mathsf{T}]_{\beta'}Q^{-1}$. Because

$$Q^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix},$$

the reader can verify that

$$[\mathsf{T}]_\beta = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}.$$

Since β is the standard ordered basis, it follows that T is left-multiplication by $[\mathsf{T}]_\beta$. Thus for any (a, b) in R^2, we have

$$\mathsf{T}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -3a + 4b \\ 4a + 3b \end{pmatrix}. \quad \blacklozenge$$

A useful special case of Theorem 2.23 is contained in the next corollary, whose proof is left as an exercise.

Corollary. *Let $A \in \mathsf{M}_{n\times n}(F)$, and let γ be an ordered basis for F^n. Then $[\mathsf{L}_A]_\gamma = Q^{-1}AQ$, where Q is the $n \times n$ matrix whose jth column is the jth vector of γ.*

Example 4

Let

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 3 \\ 0 & -1 & 0 \end{pmatrix},$$

and let

$$\gamma = \left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\},$$

which is an ordered basis for R^3. Let Q be the 3×3 matrix whose jth column is the jth vector of γ. Then

$$Q = \begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad Q^{-1} = \begin{pmatrix} -1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

So by the preceding corollary,

$$[\mathsf{L}_A]_\gamma = Q^{-1}AQ = \begin{pmatrix} 0 & 2 & 8 \\ -1 & 4 & 6 \\ 0 & -1 & -1 \end{pmatrix}. \quad \blacklozenge$$

The relationship between the matrices $[\mathsf{T}]_{\beta'}$ and $[\mathsf{T}]_\beta$ in Theorem 2.23 will be the subject of further study in Chapters 5, 6, and 7. At this time, however, we introduce the name for this relationship.

Definition. *Let A and B be matrices in $\mathsf{M}_{n\times n}(F)$. We say that B is* ***similar*** *to A if there exists an invertible matrix Q such that $B = Q^{-1}AQ$.*

Observe that the relation of similarity is an equivalence relation (see Exercise 9). So we need only say that A and B are similar.

Notice also that in this terminology Theorem 2.23 can be stated as follows: If T is a linear operator on a finite-dimensional vector space V, and if β and β' are any ordered bases for V, then $[\mathsf{T}]_{\beta'}$ is similar to $[\mathsf{T}]_\beta$.

Theorem 2.23 can be generalized to allow $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$, where V is distinct from W. In this case, we can change bases in V as well as in W (see Exercise 8).

EXERCISES

1. Label the following statements as true or false.

- **(a)** Suppose that $\beta = \{x_1, x_2, \ldots, x_n\}$ and $\beta' = \{x'_1, x'_2, \ldots, x'_n\}$ are ordered bases for a vector space and Q is the change of coordinate matrix that changes β'-coordinates into β-coordinates. Then the jth column of Q is $[x_j]_{\beta'}$.
- **(b)** Every change of coordinate matrix is invertible.
- **(c)** Let T be a linear operator on a finite-dimensional vector space V, let β and β' be ordered bases for V, and let Q be the change of coordinate matrix that changes β'-coordinates into β-coordinates. Then $[\mathsf{T}]_\beta = Q[\mathsf{T}]_{\beta'}Q^{-1}$.
- **(d)** The matrices $A, B \in \mathsf{M}_{n\times n}(F)$ are called similar if $B = Q^tAQ$ for some $Q \in \mathsf{M}_{n\times n}(F)$.
- **(e)** Let T be a linear operator on a finite-dimensional vector space V. Then for any ordered bases β and γ for V, $[\mathsf{T}]_\beta$ is similar to $[\mathsf{T}]_\gamma$.

2. For each of the following pairs of ordered bases β and β' for R^2, find the change of coordinate matrix that changes β'-coordinates into β-coordinates.

(a) $\beta = \{e_1, e_2\}$ and $\beta' = \{(a_1, a_2), (b_1, b_2)\}$
(b) $\beta = \{(-1, 3), (2, -1)\}$ and $\beta' = \{(0, 10), (5, 0)\}$
(c) $\beta = \{(2, 5), (-1, -3)\}$ and $\beta' = \{e_1, e_2\}$
(d) $\beta = \{(-4, 3), (2, -1)\}$ and $\beta' = \{(2, 1), (-4, 1)\}$

3. For each of the following pairs of ordered bases β and β' for $\mathsf{P}_2(R)$, find the change of coordinate matrix that changes β'-coordinates into β-coordinates.

(a) $\beta = \{x^2, x, 1\}$ and
$\beta' = \{a_2x^2 + a_1x + a_0, b_2x^2 + b_1x + b_0, c_2x^2 + c_1x + c_0\}$
(b) $\beta = \{1, x, x^2\}$ and
$\beta' = \{a_2x^2 + a_1x + a_0, b_2x^2 + b_1x + b_0, c_2x^2 + c_1x + c_0\}$
(c) $\beta = \{2x^2 - x, 3x^2 + 1, x^2\}$ and $\beta' = \{1, x, x^2\}$
(d) $\beta = \{x^2 - x + 1, x + 1, x^2 + 1\}$ and
$\beta' = \{x^2 + x + 4, 4x^2 - 3x + 2, 2x^2 + 3\}$
(e) $\beta = \{x^2 - x, x^2 + 1, x - 1\}$ and
$\beta' = \{5x^2 - 2x - 3, -2x^2 + 5x + 5, 2x^2 - x - 3\}$
(f) $\beta = \{2x^2 - x + 1, x^2 + 3x - 2, -x^2 + 2x + 1\}$ and
$\beta' = \{9x - 9, x^2 + 21x - 2, 3x^2 + 5x + 2\}$

4. Let T be the linear operator on R^2 defined by

$$\mathsf{T}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a + b \\ a - 3b \end{pmatrix},$$

let β be the standard ordered basis for R^2, and let

$$\beta' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}.$$

Use Theorem 2.23 and the fact that

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

to find $[\mathsf{T}]_{\beta'}$.

5. Let T be the linear operator on $\mathsf{P}_1(R)$ defined by $\mathsf{T}(p(x)) = p'(x)$, the derivative of $p(x)$. Let $\beta = \{1, x\}$ and $\beta' = \{1 + x, 1 - x\}$. Use Theorem 2.23 and the fact that

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

to find $[\mathsf{T}]_{\beta'}$.

6. For each matrix A and ordered basis β, find $[\mathsf{L}_A]_\beta$. Also, find an invertible matrix Q such that $[\mathsf{L}_A]_\beta = Q^{-1}AQ$.

(a) $A = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$

(b) $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

(c) $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$

(d) $A = \begin{pmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

7. In R^2, let L be the line $y = mx$, where $m \neq 0$. Find an expression for $\mathsf{T}(x, y)$, where

(a) T is the reflection of R^2 about L.

(b) T is the projection on L along the line perpendicular to L. (See the definition of *projection* in the exercises of Section 2.1.)

8. Prove the following generalization of Theorem 2.23. Let $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ be a linear transformation from a finite-dimensional vector space V to a finite-dimensional vector space W. Let β and β' be ordered bases for V, and let γ and γ' be ordered bases for W. Then $[\mathsf{T}]_{\beta'}^{\gamma'} = P^{-1}[\mathsf{T}]_\beta^\gamma Q$, where Q is the matrix that changes β'-coordinates into β-coordinates and P is the matrix that changes γ'-coordinates into γ-coordinates.

9. Prove that "is similar to" is an equivalence relation on $\mathsf{M}_{n\times n}(F)$.

10. **(a)** Prove that if A and B are similar $n \times n$ matrices, then $\operatorname{tr}(A) = \operatorname{tr}(B)$. *Hint:* Use Exercise 13 of Section 2.3.

(b) How would you define the trace of a linear operator on a finite-dimensional vector space? Justify that your definition is well-defined.

11. Let V be a finite-dimensional vector space with ordered bases α, β, and γ.

(a) Prove that if Q and R are the change of coordinate matrices that change α-coordinates into β-coordinates and β-coordinates into γ-coordinates, respectively, then RQ is the change of coordinate matrix that changes α-coordinates into γ-coordinates.

(b) Prove that if Q changes α-coordinates into β-coordinates, then Q^{-1} changes β-coordinates into α-coordinates.

12. Prove the corollary to Theorem 2.23.

13.[†] Let V be a finite-dimensional vector space over a field F, and let $\beta = \{x_1, x_2, \ldots, x_n\}$ be an ordered basis for V. Let Q be an $n \times n$ invertible matrix with entries from F. Define

$$x'_j = \sum_{i=1}^{n} Q_{ij} x_i \quad \text{for } 1 \le j \le n,$$

and set $\beta' = \{x'_1, x'_2, \ldots, x'_n\}$. Prove that β' is a basis for V and hence that Q is the change of coordinate matrix changing β'-coordinates into β-coordinates. Visit `goo.gl/vsxsGH` for a solution.

14. Prove the converse of Exercise 8: If A and B are each $m \times n$ matrices with entries from a field F, and if there exist invertible $m \times m$ and $n \times n$ matrices P and Q, respectively, such that $B = P^{-1}AQ$, then there exist an n-dimensional vector space V and an m-dimensional vector space W (both over F), ordered bases β and β' for V and γ and γ' for W, and a linear transformation $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ such that

$$A = [\mathsf{T}]_\beta^\gamma \quad \text{and} \quad B = [\mathsf{T}]_{\beta'}^{\gamma'}.$$

Hints: Let $\mathsf{V} = \mathsf{F}^n$, $\mathsf{W} = \mathsf{F}^m$, $\mathsf{T} = \mathsf{L}_A$, and β and γ be the standard ordered bases for F^n and F^m, respectively. Now apply the results of Exercise 13 to obtain ordered bases β' and γ' from β and γ via Q and P, respectively.

2.6* DUAL SPACES

In this section, we are concerned exclusively with linear transformations from a vector space V into its field of scalars F, which is itself a vector space of dimension 1 over F. Such a linear transformation is called a **linear functional** on V. We generally use the letters $\mathsf{f}, \mathsf{g}, \mathsf{h}, \ldots$ to denote linear functionals. As we see in Example 1, the definite integral provides us with one of the most important examples of a linear functional in mathematics.

Example 1

Let V be the vector space of continuous real-valued functions on the interval $[0, 2\pi]$. Fix a function $g \in \mathsf{V}$. The function $\mathsf{h}\colon \mathsf{V} \to R$ defined by

$$\mathsf{h}(x) = \frac{1}{2\pi} \int_0^{2\pi} x(t) g(t)\, dt$$

is a linear functional on V. In the cases that $g(t)$ equals $\sin nt$ or $\cos nt$, $h(x)$ is often called the ***n*th Fourier coefficient of x**. ♦

Example 2

Let $\mathsf{V} = \mathsf{M}_{n\times n}(F)$, and define $\mathsf{f}\colon \mathsf{V} \to F$ by $\mathsf{f}(A) = \operatorname{tr}(A)$, the trace of A. By Exercise 6 of Section 1.3, we have that f is a linear functional. ♦

Example 3

Let V be a finite-dimensional vector space, and let $\beta = \{x_1, x_2, \dots, x_n\}$ be an ordered basis for V. For each $i = 1, 2, \dots, n$, define $\mathsf{f}_i(x) = a_i$, where

$$[x]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

is the coordinate vector of x relative to β. Then f_i is a linear functional on V called the ***i*th coordinate function with respect to the basis** β. Note that $\mathsf{f}_i(x_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta. These linear functionals play an important role in the theory of dual spaces (see Theorem 2.24). ♦

Definition. *For a vector space* V *over* F*, we define the* ***dual space*** *of* V *to be the vector space* $\mathcal{L}(\mathsf{V}, F)$*, denoted by* V^*.

Thus V^* is the vector space consisting of all linear functionals on V with the operations of addition and scalar multiplication as defined in Section 2.2. Note that if V is finite-dimensional, then by the corollary to Theorem 2.20 (p. 105)

$$\dim(\mathsf{V}^*) = \dim(\mathcal{L}(\mathsf{V}, F)) = \dim(\mathsf{V}) \cdot \dim(F) = \dim(\mathsf{V}).$$

Hence by Theorem 2.19 (p. 104), V and V^* are isomorphic. We also define the **double dual** V^{**} of V to be the dual of V^*. In Theorem 2.26, we show, in fact, that there is a natural identification of V and V^{**} in the case that V is finite-dimensional.

Theorem 2.24. *Suppose that* V *is a finite-dimensional vector space with the ordered basis* $\beta = \{x_1, x_2, \dots, x_n\}$*. Let* f_i $(1 \le i \le n)$ *be the* i*th coordinate function with respect to* β *as just defined, and let* $\beta^* = \{\mathsf{f}_1, \mathsf{f}_2, \dots, \mathsf{f}_n\}$*. Then* β^* *is an ordered basis for* V^**, and, for any* $\mathsf{f} \in \mathsf{V}^*$*, we have*

$$\mathsf{f} = \sum_{i=1}^{n} \mathsf{f}(x_i)\mathsf{f}_i.$$

Proof. Let $\mathsf{f} \in \mathsf{V}^*$. Since $\dim(\mathsf{V}^*) = n$, we need only show that

$$\mathsf{f} = \sum_{i=1}^{n} \mathsf{f}(x_i)\mathsf{f}_i,$$

from which it follows that β^* generates V^*, and hence is a basis by Corollary 2(a) to the replacement theorem (p. 48). Let

$$\mathsf{g} = \sum_{i=1}^{n} \mathsf{f}(x_i)\mathsf{f}_i.$$

For $1 \le j \le n$, we have

$$\begin{aligned}\mathsf{g}(x_j) &= \left(\sum_{i=1}^{n} \mathsf{f}(x_i)\mathsf{f}_i\right)(x_j) = \sum_{i=1}^{n} \mathsf{f}(x_i)\mathsf{f}_i(x_j) \\ &= \sum_{i=1}^{n} \mathsf{f}(x_i)\delta_{ij} = \mathsf{f}(x_j).\end{aligned}$$

Therefore $\mathsf{f} = \mathsf{g}$ by the corollary to Theorem 2.6 (p. 73). ∎

Definition. *Using the notation of Theorem 2.24, we call the ordered basis* $\beta^* = \{\mathsf{f}_1, \mathsf{f}_2, \ldots, \mathsf{f}_n\}$ *of* V^* *that satisfies* $\mathsf{f}_i(x_j) = \delta_{ij}$ $(1 \le i, j \le n)$ *the* ***dual basis*** *of* β.

Example 4

Let $\beta = \{(2,1), (3,1)\}$ be an ordered basis for R^2. Suppose that the dual basis of β is given by $\beta^* = \{\mathsf{f}_1, \mathsf{f}_2\}$. To explicitly determine a formula for f_1, we need to consider the equations

$$\begin{aligned}1 &= \mathsf{f}_1(2,1) = \mathsf{f}_1(2e_1 + e_2) = 2\mathsf{f}_1(e_1) + \mathsf{f}_1(e_2) \\ 0 &= \mathsf{f}_1(3,1) = \mathsf{f}_1(3e_1 + e_2) = 3\mathsf{f}_1(e_1) + \mathsf{f}_1(e_2).\end{aligned}$$

Solving these equations, we obtain $\mathsf{f}_1(e_1) = -1$ and $\mathsf{f}_1(e_2) = 3$; that is, $\mathsf{f}_1(x,y) = -x + 3y$. Similarly, it can be shown that $\mathsf{f}_2(x,y) = x - 2y$. ♦

Now assume that V and W are n- and m-dimensional vector spaces over F with ordered bases β and γ, respectively. In Section 2.4, we proved that there is a one-to-one correspondence between linear transformations $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ and $m \times n$ matrices (over F) via the correspondence $\mathsf{T} \leftrightarrow [\mathsf{T}]_\beta^\gamma$. For a matrix of the form $A = [\mathsf{T}]_\beta^\gamma$, the question arises as to whether or not there exists a linear transformation U associated with T in some natural way such that U may be represented in some basis as A^t. Of course, if $m \neq n$, it would be impossible for U to be a linear transformation from V into W. We now answer this question by applying what we have already learned about dual spaces.

Theorem 2.25. *Let* V *and* W *be finite-dimensional vector spaces over* F *with ordered bases* β *and* γ*, respectively. For any linear transformation* $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$*, the mapping* $\mathsf{T}^t\colon \mathsf{W}^* \to \mathsf{V}^*$ *defined by* $\mathsf{T}^t(\mathsf{g}) = \mathsf{gT}$ *for all* $\mathsf{g} \in \mathsf{W}^*$ *is a linear transformation with the property that* $[\mathsf{T}^t]_{\gamma^*}^{\beta^*} = ([\mathsf{T}]_\beta^\gamma)^t$.

Proof. For $\mathsf{g} \in \mathsf{W}^*$, it is clear that $\mathsf{T}^t(\mathsf{g}) = \mathsf{gT}$ is a linear functional on V and hence is in V^*. Thus T^t maps W^* into V^*. We leave the proof that T^t is linear to the reader.

To complete the proof, let $\beta = \{x_1, x_2, \ldots, x_n\}$ and $\gamma = \{y_1, y_2, \ldots, y_m\}$ with dual bases $\beta^* = \{\mathsf{f}_1, \mathsf{f}_2, \ldots, \mathsf{f}_n\}$ and $\gamma^* = \{\mathsf{g}_1, \mathsf{g}_2, \ldots, \mathsf{g}_m\}$, respectively. For convenience, let $A = [\mathsf{T}]_\beta^\gamma$. To find the jth column of $[\mathsf{T}^t]_{\gamma^*}^{\beta^*}$, we begin by expressing $\mathsf{T}^t(\mathsf{g}_j)$ as a linear combination of the vectors of β^*. By Theorem 2.24, we have

$$\mathsf{T}^t(\mathsf{g}_j) = \mathsf{g}_j\mathsf{T} = \sum_{s=1}^{n} (\mathsf{g}_j\mathsf{T})(x_s)\mathsf{f}_s.$$

So the row i, column j entry of $[\mathsf{T}^t]_{\gamma^*}^{\beta^*}$ is

$$\begin{aligned}(\mathsf{g}_j\mathsf{T})(x_i) = \mathsf{g}_j(\mathsf{T}(x_i)) &= \mathsf{g}_j\left(\sum_{k=1}^{m} A_{ki}y_k\right) \\ &= \sum_{k=1}^{m} A_{ki}\mathsf{g}_j(y_k) = \sum_{k=1}^{m} A_{ki}\delta_{jk} = A_{ji}.\end{aligned}$$

Hence $[\mathsf{T}^t]_{\gamma^*}^{\beta^*} = A^t$. ∎

The linear transformation T^t defined in Theorem 2.25 is called the **transpose** of T. It is clear that T^t is the unique linear transformation U such that $[\mathsf{U}]_{\gamma^*}^{\beta^*} = ([\mathsf{T}]_\beta^\gamma)^t$.

We illustrate Theorem 2.25 with the next example.

Example 5

Define $\mathsf{T}\colon \mathsf{P}_1(R) \to \mathsf{R}^2$ by $\mathsf{T}(p(x)) = (p(0), p(2))$. Let β and γ be the standard ordered bases for $\mathsf{P}_1(R)$ and R^2, respectively. Clearly,

$$[\mathsf{T}]_\beta^\gamma = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}.$$

We compute $[\mathsf{T}^t]_{\gamma^*}^{\beta^*}$ directly from the definition. Let $\beta^* = \{\mathsf{f}_1, \mathsf{f}_2\}$ and $\gamma^* = \{\mathsf{g}_1, \mathsf{g}_2\}$. Suppose that $[\mathsf{T}^t]_{\gamma^*}^{\beta^*} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\mathsf{T}^t(\mathsf{g}_1) = a\mathsf{f}_1 + c\mathsf{f}_2$. So

$$\mathsf{T}^t(\mathsf{g}_1)(1) = (a\mathsf{f}_1 + c\mathsf{f}_2)(1) = a\mathsf{f}_1(1) + c\mathsf{f}_2(1) = a(1) + c(0) = a.$$

But also

$$(\mathsf{T}^t(\mathsf{g}_1))(1) = \mathsf{g}_1(\mathsf{T}(1)) = \mathsf{g}_1(1, 1) = 1.$$

So $a = 1$. Using similar computations, we obtain that $c = 0$, $b = 1$, and $d = 2$. Hence a direct computation yields

$$[\mathsf{T}^t]_{\gamma^*}^{\beta^*} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = \left([\mathsf{T}]_\beta^\gamma\right)^t,$$

as predicted by Theorem 2.25. ◆

We now concern ourselves with demonstrating that any finite-dimensional vector space V can be identified in a natural way with its double dual V^{**}. There is, in fact, an isomorphism between V and V^{**} that does not depend on any choice of bases for the two vector spaces.

For a vector $x \in \mathsf{V}$, we define $\widehat{x}\colon \mathsf{V}^* \to F$ by $\widehat{x}(\mathsf{f}) = \mathsf{f}(x)$ for every $\mathsf{f} \in \mathsf{V}^*$. It is easy to verify that $\widehat{x}$ is a linear functional on V^*, so $\widehat{x} \in \mathsf{V}^{**}$. The correspondence $x \leftrightarrow \widehat{x}$ allows us to define the desired isomorphism between V and V^{**}.

Lemma. *Let V be a finite-dimensional vector space, and let $x \in \mathsf{V}$. If $\widehat{x}(\mathsf{f}) = 0$ for all $\mathsf{f} \in \mathsf{V}^*$, then $x = 0$.*

Proof. Let $x \neq 0$. We show that there exists $\mathsf{f} \in \mathsf{V}^*$ such that $\widehat{x}(\mathsf{f}) \neq 0$. Choose an ordered basis $\beta = \{x_1, x_2, \ldots, x_n\}$ for V such that $x_1 = x$. Let $\{\mathsf{f}_1, \mathsf{f}_2, \ldots, \mathsf{f}_n\}$ be the dual basis of β. Then $\mathsf{f}_1(x_1) = 1 \neq 0$. Let $\mathsf{f} = \mathsf{f}_1$. ∎

Theorem 2.26. *Let V be a finite-dimensional vector space, and define $\psi\colon \mathsf{V} \to \mathsf{V}^{**}$ by $\psi(x) = \widehat{x}$. Then ψ is an isomorphism.*

Proof. (a) ψ is linear: Let $x, y \in \mathsf{V}$ and $c \in F$. For $\mathsf{f} \in \mathsf{V}^*$, we have

$$\begin{aligned}\psi(cx + y)(\mathsf{f}) &= \mathsf{f}(cx + y) = c\mathsf{f}(x) + \mathsf{f}(y) = c\widehat{x}(\mathsf{f}) + \widehat{y}(\mathsf{f}) \\ &= (c\widehat{x} + \widehat{y})(\mathsf{f}).\end{aligned}$$

Therefore

$$\psi(cx + y) = c\widehat{x} + \widehat{y} = c\psi(x) + \psi(y).$$

(b) ψ is one-to-one: Suppose that $\psi(x)$ is the zero functional on V^* for some $x \in \mathsf{V}$. Then $\widehat{x}(\mathsf{f}) = 0$ for every $\mathsf{f} \in \mathsf{V}^*$. By the previous lemma, we conclude that $x = 0$.

(c) ψ is an isomorphism: This follows from (b) and the fact that $\dim(\mathsf{V}) = \dim(\mathsf{V}^{**})$. ∎

Corollary. *Let V be a finite-dimensional vector space with dual space V^*. Then every ordered basis for V^* is the dual basis for some basis for V.*

Proof. Let $\{\mathsf{f}_1, \mathsf{f}_2, \ldots, \mathsf{f}_n\}$ be an ordered basis for V^*. We may combine Theorems 2.24 and 2.26 to conclude that for this basis for V^* there exists a dual basis $\{\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_n\}$ in V^{**}, that is, $\delta_{ij} = \widehat{x}_i(\mathsf{f}_j) = \mathsf{f}_j(x_i)$ for all i and j. Thus $\{\mathsf{f}_1, \mathsf{f}_2, \ldots, \mathsf{f}_n\}$ is the dual basis of $\{x_1, x_2, \ldots, x_n\}$. ∎

Although many of the ideas of this section (e.g., the existence of a dual space) can be extended to the case where V is not finite-dimensional, only a finite-dimensional vector space is isomorphic to its double dual via the map $x \to \widehat{x}$. In fact, for infinite-dimensional vector spaces, no two of V, V^*, and V^{**} are isomorphic.

EXERCISES

1. Label the following statements as true or false. Assume that all vector spaces are finite-dimensional.

- **(a)** Every linear transformation is a linear functional.
- **(b)** A linear functional defined on a field may be represented as a 1×1 matrix.
- **(c)** Every vector space is isomorphic to its dual space.
- **(d)** Every vector space is isomorphic to the dual of some vector space.
- **(e)** If T is an isomorphism from V onto V^* and β is a finite ordered basis for V, then $\mathsf{T}(\beta) = \beta^*$.
- **(f)** If T is a linear transformation from V to W, then the domain of $(\mathsf{T}^t)^t$ is V^{**}.
- **(g)** If V is isomorphic to W, then V^* is isomorphic to W^*.
- **(h)** The derivative of a function may be considered as a linear functional on the vector space of differentiable functions.

2. For the following functions f on a vector space V, determine which are linear functionals.

- **(a)** $\mathsf{V} = \mathsf{P}(R)$; $\mathsf{f}(p(x)) = 2p'(0) + p''(1)$, where $'$ denotes differentiation
- **(b)** $\mathsf{V} = \mathsf{R}^2$; $\mathsf{f}(x, y) = (2x, 4y)$
- **(c)** $\mathsf{V} = \mathsf{M}_{2\times 2}(F)$; $\mathsf{f}(A) = \operatorname{tr}(A)$
- **(d)** $\mathsf{V} = \mathsf{R}^3$; $\mathsf{f}(x, y, z) = x^2 + y^2 + z^2$
- **(e)** $\mathsf{V} = \mathsf{P}(R)$; $\mathsf{f}(p(x)) = \int_0^1 p(t)\,dt$
- **(f)** $\mathsf{V} = \mathsf{M}_{2\times 2}(F)$; $\mathsf{f}(A) = A_{11}$

3. For each of the following vector spaces V and bases β, find explicit formulas for vectors of the dual basis β^* for V^*, as in Example 4.

- **(a)** $\mathsf{V} = \mathsf{R}^3$; $\beta = \{(1, 0, 1), (1, 2, 1), (0, 0, 1)\}$
- **(b)** $\mathsf{V} = \mathsf{P}_2(R)$; $\beta = \{1, x, x^2\}$

4. Let $\mathsf{V} = \mathsf{R}^3$, and define $\mathsf{f}_1, \mathsf{f}_2, \mathsf{f}_3 \in \mathsf{V}^*$ as follows:

$$\mathsf{f}_1(x, y, z) = x - 2y, \quad \mathsf{f}_2(x, y, z) = x + y + z, \quad \mathsf{f}_3(x, y, z) = y - 3z.$$

Prove that $\{\mathsf{f}_1, \mathsf{f}_2, \mathsf{f}_3\}$ is a basis for V^*, and then find a basis for V for which it is the dual basis.

5. Let $\mathsf{V} = \mathsf{P}_1(R)$, and, for $p(x) \in \mathsf{V}$, define $\mathsf{f}_1, \mathsf{f}_2 \in \mathsf{V}^*$ by

$$\mathsf{f}_1(p(x)) = \int_0^1 p(t)\,dt \quad \text{and} \quad \mathsf{f}_2(p(x)) = \int_0^2 p(t)\,dt.$$

Prove that $\{\mathsf{f}_1, \mathsf{f}_2\}$ is a basis for V^*, and find a basis for V for which it is the dual basis.

6. Define $\mathsf{f} \in (\mathsf{R}^2)^*$ by $\mathsf{f}(x, y) = 2x + y$ and $\mathsf{T}\colon \mathsf{R}^2 \to \mathsf{R}^2$ by $\mathsf{T}(x, y) = (3x + 2y, x)$.

- **(a)** Compute $\mathsf{T}^t(\mathsf{f})$.
- **(b)** Compute $[\mathsf{T}^t]_{\beta^*}$, where β is the standard ordered basis for R^2 and $\beta^* = \{\mathsf{f}_1, \mathsf{f}_2\}$ is the dual basis, by finding scalars a, b, c, and d such that $\mathsf{T}^t(\mathsf{f}_1) = a\mathsf{f}_1 + c\mathsf{f}_2$ and $\mathsf{T}^t(\mathsf{f}_2) = b\mathsf{f}_1 + d\mathsf{f}_2$.
- **(c)** Compute $[\mathsf{T}]_\beta$ and $([\mathsf{T}]_\beta)^t$, and compare your results with (b).

7. Let $\mathsf{V} = \mathsf{P}_1(R)$ and $\mathsf{W} = \mathsf{R}^2$ with respective standard ordered bases β and γ. Define $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ by

$$\mathsf{T}(p(x)) = (p(0) - 2p(1), p(0) + p'(0)),$$

where $p'(x)$ is the derivative of $p(x)$.

- **(a)** For $\mathsf{f} \in \mathsf{W}^*$ defined by $\mathsf{f}(a, b) = a - 2b$, compute $\mathsf{T}^t(\mathsf{f})$.
- **(b)** Compute $[\mathsf{T}^t]_{\gamma^*}^{\beta^*}$ without appealing to Theorem 2.25.
- **(c)** Compute $[\mathsf{T}]_\beta^\gamma$ and its transpose, and compare your results with (b).

8. Let $\{u, v\}$ be a linearly independent set in R^3. Show that the plane $\{su + tv\colon s, t \in R\}$ through the origin in R^3 may be identified with the null space of a vector in $(\mathsf{R}^3)^*$.

9. Prove that a function $\mathsf{T}\colon \mathsf{F}^n \to \mathsf{F}^m$ is linear if and only if there exist $\mathsf{f}_1, \mathsf{f}_2, \ldots, \mathsf{f}_m \in (\mathsf{F}^n)^*$ such that $\mathsf{T}(x) = (\mathsf{f}_1(x), \mathsf{f}_2(x), \ldots, \mathsf{f}_m(x))$ for all $x \in \mathsf{F}^n$. *Hint:* If T is linear, define $\mathsf{f}_i(x) = (\mathsf{g}_i\mathsf{T})(x)$ for $x \in \mathsf{F}^n$; that is, $\mathsf{f}_i = \mathsf{T}^t(\mathsf{g}_i)$ for $1 \leq i \leq m$, where $\{\mathsf{g}_1, \mathsf{g}_2, \ldots, \mathsf{g}_m\}$ is the dual basis of the standard ordered basis for F^m.

10. Let $\mathsf{V} = \mathsf{P}_n(F)$, and let $c_0, c_1, \ldots, c_n$ be distinct scalars in F.

- **(a)** For $0 \leq i \leq n$, define $\mathsf{f}_i \in \mathsf{V}^*$ by $\mathsf{f}_i(p(x)) = p(c_i)$. Prove that $\{\mathsf{f}_0, \mathsf{f}_1, \ldots, \mathsf{f}_n\}$ is a basis for V^*. *Hint:* Apply any linear combination of this set that equals the zero transformation to $p(x) = (x - c_1)(x - c_2)\cdots(x - c_n)$, and deduce that the first coefficient is zero.

(b) Use the corollary to Theorem 2.26 and (a) to show that there exist unique polynomials $p_0(x), p_1(x), \ldots, p_n(x)$ such that $p_i(c_j) = \delta_{ij}$ for $0 \leq i \leq n$. These polynomials are the Lagrange polynomials defined in Section 1.6.

(c) For any scalars $a_0, a_1, \ldots, a_n$ (not necessarily distinct), deduce that there exists a unique polynomial $q(x)$ of degree at most n such that $q(c_i) = a_i$ for $0 \leq i \leq n$. In fact,

$$q(x) = \sum_{i=0}^{n} a_i p_i(x).$$

(d) Deduce the Lagrange interpolation formula:

$$p(x) = \sum_{i=0}^{n} p(c_i) p_i(x)$$

for any $p(x) \in \mathsf{V}$.

(e) Prove that

$$\int_a^b p(t)\, dt = \sum_{i=0}^{n} p(c_i) d_i,$$

where

$$d_i = \int_a^b p_i(t)\, dt.$$

Suppose now that

$$c_i = a + \frac{i(b-a)}{n} \quad \text{for } i = 0, 1, \ldots, n.$$

For $n = 1$, the preceding result yields the trapezoidal rule for evaluating the definite integral of a polynomial. For $n = 2$, this result yields Simpson's rule for evaluating the definite integral of a polynomial.

$$\begin{tikzcd} \mathsf{V} \arrow[r, "\mathsf{T}"] \arrow[d, "\psi_1"'] & \mathsf{W} \arrow[d, "\psi_2"] \\ \mathsf{V}^{**} \arrow[r, "\mathsf{T}^{tt}"] & \mathsf{W}^{**} \end{tikzcd}$$

Figure 2.6

11. Let V and W be finite-dimensional vector spaces over F, and let ψ_1 and ψ_2 be the isomorphisms between V and V^{**} and W and W^{**}, respectively, as defined in Theorem 2.26. Let $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ be linear, and define

$\mathsf{T}^{tt} = (\mathsf{T}^t)^t$. Prove that the diagram depicted in Figure 2.6 commutes (i.e., prove that $\psi_2\mathsf{T} = \mathsf{T}^{tt}\psi_1$). Visit `goo.gl/Lkd6XZ` for a solution.

12. Let V be a finite-dimensional vector space with the ordered basis β. Prove that $\psi(\beta) = \beta^{**}$, where ψ is defined in Theorem 2.26.

In Exercises 13 through 17, V denotes a finite-dimensional vector space over F. For every subset S of V, define the **annihilator** S^0 of S as

$$S^0 = \{\mathsf{f} \in \mathsf{V}^* : \mathsf{f}(x) = 0 \text{ for all } x \in S\}.$$

13. **(a)** Prove that S^0 is a subspace of V^*.
(b) If W is a subspace of V and $x \notin \mathsf{W}$, prove that there exists $\mathsf{f} \in \mathsf{W}^0$ such that $\mathsf{f}(x) \neq 0$.
(c) Prove that $(S^0)^0 = \operatorname{span}(\psi(S))$, where ψ is defined as in Theorem 2.26.
(d) For subspaces W_1 and W_2, prove that $\mathsf{W}_1 = \mathsf{W}_2$ if and only if $\mathsf{W}_1^0 = \mathsf{W}_2^0$.
(e) For subspaces W_1 and W_2, show that $(\mathsf{W}_1 + \mathsf{W}_2)^0 = \mathsf{W}_1^0 \cap \mathsf{W}_2^0$.

14. Prove that if W is a subspace of V, then $\dim(\mathsf{W}) + \dim(\mathsf{W}^0) = \dim(\mathsf{V})$. *Hint:* Extend an ordered basis $\{x_1, x_2, \ldots, x_k\}$ of W to an ordered basis $\beta = \{x_1, x_2, \ldots, x_n\}$ of V. Let $\beta^* = \{\mathsf{f}_1, \mathsf{f}_2, \ldots, \mathsf{f}_n\}$. Prove that $\{\mathsf{f}_{k+1}, \mathsf{f}_{k+2}, \ldots, \mathsf{f}_n\}$ is a basis for W^0.

15. Suppose that W is a finite-dimensional vector space and that $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ is linear. Prove that $\mathsf{N}(\mathsf{T}^t) = (\mathsf{R}(\mathsf{T}))^0$.

16. Use Exercises 14 and 15 to deduce that $\operatorname{rank}(\mathsf{L}_{A^t}) = \operatorname{rank}(\mathsf{L}_A)$ for any $A \in \mathsf{M}_{m\times n}(F)$.

In Exercises 17 through 20, assume that V and W are finite-dimensional vector spaces. (It can be shown, however, that these exercises are true for all vector spaces V and W.)

17. Let T be a linear operator on V, and let W be a subspace of V. Prove that W is T-invariant (as defined in the exercises of Section 2.1) if and only if W^0 is T^t-invariant.

18. Let V be a nonzero vector space over a field F, and let S be a basis for V. (By the corollary to Theorem 1.13 (p. 61) in Section 1.7, every vector space has a basis.) Let $\Phi\colon \mathsf{V}^* \to \mathcal{F}(S, F)$ be the mapping defined by $\Phi(\mathsf{f}) = \mathsf{f}_S$, the restriction of f to S. Prove that Φ is an isomorphism. *Hint:* Apply Exercise 35 of Section 2.1.

19. Let V be a nonzero vector space, and let W be a proper subspace of V (i.e., $\mathsf{W} \neq \mathsf{V}$).

(a) Let $\mathsf{g} \in \mathsf{W}^*$ and $v \in \mathsf{V}$ with $v \notin \mathsf{W}$. Prove that for any scalar a there exists a function $\mathsf{f} \in \mathsf{V}^*$ such that $\mathsf{f}(v) = a$ and $\mathsf{f}(x) = \mathsf{g}(x)$ for all x in W. *Hint:* For the infinite-dimensional case, use Exercise 4 of Section 1.7 and Exercise 35 of Section 2.1.

(b) Use (a) to prove there exists a nonzero linear functional $\mathsf{f} \in \mathsf{V}^*$ such that $\mathsf{f}(x) = \mathit{0}$ for all $x \in \mathsf{W}$.

20. Let V and W be nonzero vector spaces over the same field, and let $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ be a linear transformation.

(a) Prove that T is onto if and only if T^t is one-to-one.
(b) Prove that T^t is onto if and only if T is one-to-one.

Hint: In the infinite-dimensional case, use Exercise 19 for parts of the proof.

2.7* HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

As an introduction to this section, consider the following physical problem. A weight of mass m is attached to a vertically suspended spring that is allowed to stretch until the forces acting on the weight are in equilibrium. Suppose that the weight is now motionless and impose an xy-coordinate system with the weight at the origin and the spring lying on the positive y-axis (see Figure 2.7).

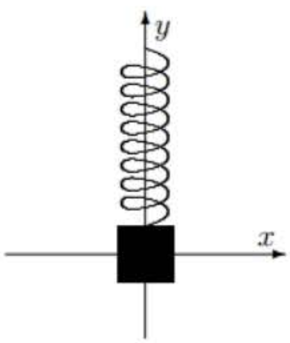


Figure 2.7

Suppose that at a certain time, say $t = 0$, the weight is lowered a distance s along the y-axis and released. The spring then begins to oscillate.

We describe the motion of the spring. At any time $t \geq 0$, let $F(t)$ denote the force acting on the weight and $y(t)$ denote the position of the weight along the y-axis. For example, $y(0) = -s$. The second derivative of y with respect to time, $y''(t)$, is the acceleration of the weight at time t; hence, by Newton's second law of motion,

$$F(t) = my''(t). \tag{1}$$

It is reasonable to assume that the force acting on the weight is due totally to the tension of the spring, and that this force satisfies Hooke's law: *The force acting on the weight is proportional to its displacement from the equilibrium position, but acts in the opposite direction.* If $k > 0$ is the proportionality constant, then Hooke's law states that

$$F(t) = -ky(t). \tag{2}$$

Combining (1) and (2), we obtain $my'' = -ky$ or

$$y'' + \frac{k}{m}y = 0. \tag{3}$$

The expression (3) is an example of a *differential equation.* A **differential equation** in an unknown function $y = y(t)$ is an equation involving y, t, and derivatives of y. If the differential equation is of the form

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y^{(1)} + a_0 y = f, \tag{4}$$

where $a_0, a_1, \ldots, a_n$ and f are functions of t and $y^{(k)}$ denotes the kth derivative of y, then the equation is said to be **linear**. The functions a_i are called the **coefficients** of the differential equation (4). Thus (3) is an example of a linear differential equation in which the coefficients are constants and the function f is identically zero. When f is identically zero, (4) is called **homogeneous**.

In this section, we apply the linear algebra we have studied to solve homogeneous linear differential equations with constant coefficients. If $a_n \neq 0$, we say that differential equation (4) is of **order** $\boldsymbol{n}$. In this case, we divide both sides by a_n to obtain a new, but equivalent, equation

$$y^{(n)} + b_{n-1}y^{(n-1)} + \cdots + b_1 y^{(1)} + b_0 y = 0,$$

where $b_i = a_i/a_n$ for $i = 0, 1, \ldots, n-1$. Because of this observation, we always assume that the coefficient a_n in (4) is 1.

A **solution** to (4) is a function that when substituted for y reduces (4) to an identity.

Example 1

The function $y(t) = \sin\sqrt{k/m}\,t$ is a solution to (3) since

$$y''(t) + \frac{k}{m}y(t) = -\frac{k}{m}\sin\sqrt{\frac{k}{m}}\,t + \frac{k}{m}\sin\sqrt{\frac{k}{m}}\,t = 0$$

for all t. Notice, however, that substituting $y(t) = t$ into (3) yields

$$y''(t) + \frac{k}{m}y(t) = \frac{k}{m}t,$$

which is not identically zero. Thus $y(t) = t$ is not a solution to (3). ◆

In our study of differential equations, it is useful to regard solutions as complex-valued functions of a real variable even though the solutions that are meaningful to us in a physical sense are real-valued. The convenience of this viewpoint will become clear later. Thus we are concerned with the vector space $\mathcal{F}(R, C)$ (as defined in Example 3 of Section 1.2). In order to consider complex-valued functions of a real variable as solutions to differential equations, we must define what it means to differentiate such functions. Given a complex-valued function $x \in \mathcal{F}(R, C)$ of a real variable t, there exist unique real-valued functions x_1 and x_2 of t, such that

$$x(t) = x_1(t) + ix_2(t) \quad \text{for} \quad t \in R,$$

where i is the imaginary number such that $i^2 = -1$. We call x_1 the **real part** and x_2 the **imaginary part** of x.

Definitions. *Given a function $x \in \mathcal{F}(R, C)$ with real part x_1 and imaginary part x_2, we say that x is* **differentiable** *if x_1 and x_2 are differentiable. If x is differentiable, we define the* **derivative** *x' of x by*

$$x' = x_1' + ix_2'.$$

We illustrate some computations with complex-valued functions in the following example.

Example 2

Suppose that $x(t) = \cos 2t + i \sin 2t$. Then

$$x'(t) = -2\sin 2t + 2i\cos 2t.$$

We next find the real and imaginary parts of x^2. Since

$$\begin{aligned} x^2(t) &= (\cos 2t + i\sin 2t)^2 = (\cos^2 2t - \sin^2 2t) + i(2\sin 2t \cos 2t) \\ &= \cos 4t + i \sin 4t, \end{aligned}$$

the real part of $x^2(t)$ is $\cos 4t$, and the imaginary part is $\sin 4t$. ◆

The next theorem indicates that we may limit our investigations to a vector space considerably smaller than $\mathcal{F}(R, C)$. Its proof, which is illustrated in Example 3, involves a simple induction argument, which we omit.

Theorem 2.27. *Any solution to a homogeneous linear differential equation with constant coefficients has derivatives of all orders; that is, if x is a solution to such an equation, then $x^{(k)}$ exists for every positive integer k.*

Example 3

To illustrate Theorem 2.27, consider the equation

$$y^{(2)} + 4y = 0.$$

Clearly, to qualify as a solution, a function y must have two derivatives. If y is a solution, however, then

$$y^{(2)} = -4y.$$

Thus since $y^{(2)}$ is a constant multiple of a function y that has two derivatives, $y^{(2)}$ must have two derivatives. Hence $y^{(4)}$ exists; in fact,

$$y^{(4)} = -4y^{(2)}.$$

Since $y^{(4)}$ is a constant multiple of a function that we have shown has at least two derivatives, it also has at least two derivatives; hence $y^{(6)}$ exists. Continuing in this manner, we can show that any solution has derivatives of all orders. ♦

Definition. *We use C^∞ to denote the set of all functions in $\mathcal{F}(R, C)$ that have derivatives of all orders.*

It is a simple exercise to show that C^∞ is a subspace of $\mathcal{F}(R, C)$ and hence a vector space over C. In view of Theorem 2.27, it is this vector space that is of interest to us. For $x \in \mathsf{C}^\infty$, the derivative x' of x also lies in C^∞. We can use the derivative operation to define a mapping $\mathsf{D}\colon \mathsf{C}^\infty \to \mathsf{C}^\infty$ by

$$\mathsf{D}(x) = x' \quad \text{for } x \in \mathsf{C}^\infty.$$

It is easy to show that D is a linear operator. More generally, consider any polynomial over C of the form

$$p(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0.$$

If we define

$$p(\mathsf{D}) = a_n \mathsf{D}^n + a_{n-1}\mathsf{D}^{n-1} + \cdots + a_1 \mathsf{D} + a_0 \mathsf{I},$$

then $p(\mathsf{D})$ is a linear operator on C^∞. (See Appendix E.)

Definitions. *For any polynomial $p(t)$ over C of positive degree, we call $p(\mathsf{D})$ a* ***differential operator with constant coefficients****, or, more simply, a* **differential operator**. *The* **order** *of the differential operator $p(\mathsf{D})$ is the degree of the polynomial $p(t)$.*

Differential operators are useful since they provide us with a means of reformulating a differential equation in the context of linear algebra. Any homogeneous linear differential equation with constant coefficients,

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y^{(1)} + a_0 y = 0,$$

can be rewritten using differential operators as

$$(\mathsf{D}^n + a_{n-1}\mathsf{D}^{n-1} + \cdots + a_1\mathsf{D} + a_0\mathsf{I})(y) = \mathit{0}.$$

Definition. *Given the differential equation above, the complex polynomial*

$$p(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$$

is called the **auxiliary polynomial** *associated with the equation.*

For example, (3) has the auxiliary polynomial

$$p(t) = t^2 + \frac{k}{m}.$$

Any homogeneous linear differential equation with constant coefficients can be rewritten as

$$p(\mathsf{D})(y) = \mathit{0},$$

where $p(t)$ is the auxiliary polynomial associated with the equation. Clearly, this equation implies the following theorem.

Theorem 2.28. *The set of all solutions to a homogeneous linear differential equation with constant coefficients coincides with the null space of $p(\mathsf{D})$, where $p(t)$ is the auxiliary polynomial associated with the equation.*

Proof. Exercise. ∎

Corollary. *The set of all solutions to a homogeneous linear differential equation with constant coefficients is a subspace of C^∞.*

In view of the preceding corollary, we call the set of solutions to a homogeneous linear differential equation with constant coefficients the **solution space** of the equation. A practical way of describing such a space is in terms of a basis. We now examine a certain class of functions that is of use in finding bases for these solution spaces.

For a real number s, we are familiar with the real number e^s, where e is the unique number whose natural logarithm is 1 (i.e., $\ln e = 1$). We know, for instance, certain properties of exponentiation, namely,

$$e^{s+t} = e^s e^t \quad \text{and} \quad e^{-t} = \frac{1}{e^t}$$

for any real numbers s and t. We now extend the definition of powers of e to include complex numbers in such a way that these properties are preserved.

Definition. *Let $c = a + ib$ be a complex number with real part a and imaginary part b. Define*

$$e^c = e^a(\cos b + i \sin b).$$

The special case

$$e^{ib} = \cos b + i \sin b$$

is called **Euler's formula**.

For example, for $c = 2 + i(\pi/3)$,

$$e^c = e^2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = e^2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right).$$

Clearly, if c is real ($b = 0$), then we obtain the usual result: $e^c = e^a$. Using the approach of Example 2, we can show by the use of trigonometric identities that

$$e^{c+d} = e^c e^d \quad \text{and} \quad e^{-c} = \frac{1}{e^c}$$

for any complex numbers c and d.

Definition. *A function $f\colon R \to C$ defined by $f(t) = e^{ct}$ for a fixed complex number c is called an* **exponential function**.

The derivative of an exponential function, as described in the next theorem, is consistent with the real version. The proof involves a straightforward computation, which we leave as an exercise.

Theorem 2.29. *For any exponential function $f(t) = e^{ct}$, $f'(t) = ce^{ct}$.*

Proof. Exercise. ∎

We can use exponential functions to describe all solutions to a homogeneous linear differential equation of order 1. Recall that the **order** of such an equation is the degree of its auxiliary polynomial. Thus an equation of order 1 is of the form

$$y' + a_0 y = 0. \tag{5}$$

Theorem 2.30. *The solution space for (5) is of dimension 1 and has $\{e^{-a_0 t}\}$ as a basis.*

Proof. Clearly (5) has $e^{-a_0 t}$ as a solution. Suppose that $x(t)$ is any solution to (5). Then

$$x'(t) = -a_0 x(t) \quad \text{for all } t \in R.$$

Define

$$z(t) = e^{a_0t}x(t).$$

Differentiating z yields

$$z'(t) = (e^{a_0t})'x(t) + e^{a_0t}x'(t) = a_0e^{a_0t}x(t) - a_0e^{a_0t}x(t) = 0.$$

(Notice that the familiar product rule for differentiation holds for complex-valued functions of a real variable. A justification of this involves a lengthy, although direct, computation.)

Since z' is identically zero, z is a constant function. (Again, this fact, well known for real-valued functions, is also true for complex-valued functions. The proof, which relies on the real case, involves looking separately at the real and imaginary parts of z.) Thus there exists a complex number k such that

$$z(t) = e^{a_0t}x(t) = k \quad \text{for all } t \in R.$$

So

$$x(t) = ke^{-a_0t}.$$

We conclude that any solution to (5) is a scalar multiple of e^{-a_0t}. ∎

Another way of stating Theorem 2.30 is as follows.

Corollary. *For any complex number c, the null space of the differential operator $\mathsf{D} - c\mathsf{I}$ has $\{e^{ct}\}$ as a basis.*

We next concern ourselves with differential equations of order greater than one. Given an nth order homogeneous linear differential equation with constant coefficients,

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y^{(1)} + a_0y = 0,$$

its auxiliary polynomial

$$p(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0$$

factors into a product of polynomials of degree 1, that is,

$$p(t) = (t - c_1)(t - c_2)\cdots(t - c_n),$$

where $c_1, c_2, \ldots, c_n$ are (not necessarily distinct) complex numbers. (This follows from the fundamental theorem of algebra in Appendix D.) Thus

$$p(\mathsf{D}) = (\mathsf{D} - c_1\mathsf{I})(\mathsf{D} - c_2\mathsf{I})\cdots(\mathsf{D} - c_n\mathsf{I}).$$

The operators $\mathsf{D} - c_i\mathsf{I}$ commute, and so, by Exercise 9, we have that

$$\mathsf{N}(\mathsf{D} - c_i\mathsf{I}) \subseteq \mathsf{N}(p(\mathsf{D})) \quad \text{for all } i.$$

Since $\mathsf{N}(p(\mathsf{D}))$ coincides with the solution space of the given differential equation, we can deduce the following result from the preceding corollary.

Theorem 2.31. *Let $p(t)$ be the auxiliary polynomial for a homogeneous linear differential equation with constant coefficients. For any complex number c, if c is a zero of $p(t)$, then e^{ct} is a solution to the differential equation.*

Example 4

Given the differential equation

$$y'' - 3y' + 2y = 0,$$

its auxiliary polynomial is

$$p(t) = t^2 - 3t + 2 = (t-1)(t-2).$$

Hence, by Theorem 2.31, e^t and e^{2t} are solutions to the differential equation because $c = 1$ and $c = 2$ are zeros of $p(t)$. Since the solution space of the differential equation is a subspace of C^∞, $\text{span}(\{e^t, e^{2t}\})$ lies in the solution space. It is a simple matter to show that $\{e^t, e^{2t}\}$ is linearly independent. Thus if we can show that the solution space is two-dimensional, we can conclude that $\{e^t, e^{2t}\}$ is a basis for the solution space. This result is a consequence of the next theorem. ◆

Theorem 2.32. *For any differential operator $p(\mathsf{D})$ of order n, the null space of $p(\mathsf{D})$ is an n-dimensional subspace of C^∞.*

As a preliminary to the proof of Theorem 2.32, we establish two lemmas.

Lemma 1. *The differential operator $\mathsf{D} - c\mathsf{I}\colon \mathsf{C}^\infty \to \mathsf{C}^\infty$ is onto for any complex number c.*

Proof. Let $v \in \mathsf{C}^\infty$. We wish to find a $u \in \mathsf{C}^\infty$ such that $(\mathsf{D} - c\mathsf{I})u = v$. Let $w(t) = v(t)e^{-ct}$ for $t \in R$. Clearly, $w \in \mathsf{C}^\infty$ because both v and e^{-ct} lie in C^∞. Let w_1 and w_2 be the real and imaginary parts of w. Then w_1 and w_2 are continuous because they are differentiable. Hence they have antiderivatives, say, W_1 and W_2, respectively. Let $W\colon R \to C$ be defined by

$$W(t) = W_1(t) + iW_2(t) \quad \text{for } t \in R.$$

Then $W \in \mathsf{C}^\infty$, and the real and imaginary parts of W are W_1 and W_2, respectively. Furthermore, $W' = w$. Finally, let $u\colon R \to C$ be defined by $u(t) = W(t)e^{ct}$ for $t \in R$. Clearly $u \in \mathsf{C}^\infty$, and since

$$\begin{aligned}(\mathsf{D} - c\mathsf{I})u(t) &= u'(t) - cu(t)\\ &= W'(t)e^{ct} + W(t)ce^{ct} - cW(t)e^{ct}\\ &= w(t)e^{ct}\\ &= v(t)e^{-ct}e^{ct}\\ &= v(t),\end{aligned}$$

we have $(\mathsf{D} - c\mathsf{I})u = v$. ∎

Lemma 2. *Let V be a vector space, and suppose that T and U are linear operators on V such that U is onto and the null spaces of T and U are finite-dimensional. Then the null space of TU is finite-dimensional, and*

$$\dim(\mathsf{N}(\mathsf{TU})) = \dim(\mathsf{N}(\mathsf{T})) + \dim(\mathsf{N}(\mathsf{U})).$$

Proof. Let $p = \dim(\mathsf{N}(\mathsf{T}))$, $q = \dim(\mathsf{N}(\mathsf{U}))$, and $\{u_1, u_2, \ldots, u_p\}$ and $\{v_1, v_2, \ldots, v_q\}$ be bases for $\mathsf{N}(\mathsf{T})$ and $\mathsf{N}(\mathsf{U})$, respectively. Since U is onto, we can choose for each i $(1 \le i \le p)$ a vector $w_i \in \mathsf{V}$ such that $\mathsf{U}(w_i) = u_i$. Note that the w_i's are distinct. Furthermore, for any i and j, $w_i \neq v_j$, for otherwise $u_i = \mathsf{U}(w_i) = \mathsf{U}(v_j) = \mathit{0}$—a contradiction. Hence the set

$$\beta = \{w_1, w_2, \ldots, w_p, v_1, v_2, \ldots, v_q\}$$

contains $p+q$ distinct vectors. To complete the proof of the lemma, it suffices to show that β is a basis for $\mathsf{N}(\mathsf{TU})$.

We first show that β generates $\mathsf{N}(\mathsf{TU})$. Since for any w_i and v_j in β, $\mathsf{TU}(w_i) = \mathsf{T}(u_i) = \mathit{0}$ and $\mathsf{TU}(v_j) = \mathsf{T}(\mathit{0}) = \mathit{0}$, it follows that $\beta \subseteq \mathsf{N}(\mathsf{TU})$. Now suppose that $v \in \mathsf{N}(\mathsf{TU})$. Then $\mathit{0} = \mathsf{TU}(v) = \mathsf{T}(\mathsf{U}(v))$. Thus $\mathsf{U}(v) \in \mathsf{N}(\mathsf{T})$. So there exist scalars $a_1, a_2, \ldots, a_p$ such that

$$\begin{aligned}\mathsf{U}(v) &= a_1u_1 + a_2u_2 + \cdots + a_pu_p \\ &= a_1U(w_1) + a_2U(w_2) + \cdots + a_pU(w_p) \\ &= \mathsf{U}(a_1w_1 + a_2w_2 + \cdots + a_pw_p).\end{aligned}$$

Hence

$$\mathsf{U}(v - (a_1w_1 + a_2w_2 + \cdots + a_pw_p)) = \mathit{0}.$$

Consequently, $v - (a_1w_1 + a_2w_2 + \cdots + a_pw_p)$ lies in $\mathsf{N}(\mathsf{U})$. It follows that there exist scalars $b_1, b_2, \ldots, b_q$ such that

$$v - (a_1w_1 + a_2w_2 + \cdots + a_pw_p) = b_1v_1 + b_2v_2 + \cdots + b_qv_q$$

or

$$v = a_1w_1 + a_2w_2 + \cdots + a_pw_p + b_1v_1 + b_2v_2 + \cdots + b_qv_q.$$

Therefore β spans $\mathsf{N}(\mathsf{TU})$.

To prove that β is linearly independent, let $a_1, a_2, \ldots, a_p, b_1, b_2, \ldots, b_q$ be any scalars such that

$$a_1w_1 + a_2w_2 + \cdots + a_pw_p + b_1v_1 + b_2v_2 + \cdots + b_qv_q = \mathit{0}. \tag{6}$$

Applying U to both sides of (6), we obtain

$$a_1u_1 + a_2u_2 + \cdots + a_pu_p = \mathit{0}.$$

Since $\{u_1, u_2, \ldots, u_p\}$ is linearly independent, the a_i's are all zero. Thus (6) reduces to

$$b_1v_1 + b_2v_2 + \cdots + b_qv_q = \mathit{0}.$$

Again, the linear independence of $\{v_1, v_2, \ldots, v_q\}$ implies that the b_i's are all zero. We conclude that β is a basis for $\mathsf{N}(\mathsf{TU})$. Hence $\mathsf{N}(\mathsf{TU})$ is finite-dimensional, and $\dim(\mathsf{N}(\mathsf{TU})) = p + q = \dim(\mathsf{N}(\mathsf{T})) + \dim(\mathsf{N}(\mathsf{U}))$. ∎

Proof of Theorem 2.32. The proof is by mathematical induction on the order of the differential operator $p(\mathsf{D})$. The first-order case coincides with Theorem 2.30. For some integer $n > 1$, suppose that Theorem 2.32 holds for any differential operator of order less than n, and consider a differential operator $p(\mathsf{D})$ of order n. The polynomial $p(t)$ can be factored into a product of two polynomials as follows:

$$p(t) = q(t)(t - c),$$

where $q(t)$ is a polynomial of degree $n - 1$ and c is a complex number. Thus the given differential operator may be rewritten as

$$p(\mathsf{D}) = q(\mathsf{D})(\mathsf{D} - c\mathsf{I}).$$

Now, by Lemma 1, $\mathsf{D} - c\mathsf{I}$ is onto, and by the corollary to Theorem 2.30, $\dim(\mathsf{N}(\mathsf{D} - c\mathsf{I})) = 1$. Also, by the induction hypothesis, $\dim(\mathsf{N}(q(\mathsf{D})) = n - 1$. Thus, by Lemma 2, we conclude that

$$\begin{aligned}\dim(\mathsf{N}(p(\mathsf{D}))) &= \dim(\mathsf{N}(q(\mathsf{D}))) + \dim(\mathsf{N}(\mathsf{D} - c\mathsf{I}))\\ &= (n-1) + 1 = n.\end{aligned}$$ ∎

Corollary. *The solution space of any nth-order homogeneous linear differential equation with constant coefficients is an n-dimensional subspace of C^∞.*

The corollary to Theorem 2.32 reduces the problem of finding all solutions to an nth-order homogeneous linear differential equation with constant coefficients to finding a set of n linearly independent solutions to the equation. By the results of Chapter 1, any such set must be a basis for the solution space. The next theorem enables us to find a basis quickly for many such equations. Hints for its proof are provided in the exercises.

Theorem 2.33. *Given n distinct complex numbers $c_1, c_2, \ldots, c_n$, the set of exponential functions $\{e^{c_1t}, e^{c_2t}, \ldots, e^{c_nt}\}$ is linearly independent.*

Proof. Exercise. (See Exercise 10.) ∎

Corollary. *For any nth-order homogeneous linear differential equation with constant coefficients, if the auxiliary polynomial has n distinct zeros $c_1, c_2, \ldots, c_n$, then $\{e^{c_1 t}, e^{c_2 t}, \ldots, e^{c_n t}\}$ is a basis for the solution space of the differential equation.*

Proof. Exercise. (See Exercise 10.) ∎

Example 5

We find all solutions to the differential equation

$$y'' + 5y' + 4y = 0.$$

Since the auxiliary polynomial factors as $(t+4)(t+1)$, it has two distinct zeros, -1 and -4. Thus $\{e^{-t}, e^{-4t}\}$ is a basis for the solution space. So any solution to the given equation is of the form

$$y(t) = b_1 e^{-t} + b_2 e^{-4t}$$

for unique scalars b_1 and b_2. ◆

Example 6

We find all solutions to the differential equation

$$y'' + 9y = 0.$$

The auxiliary polynomial t^2+9 factors as $(t-3i)(t+3i)$ and hence has distinct zeros $c_1 = 3i$ and $c_2 = -3i$. Thus $\{e^{3it}, e^{-3it}\}$ is a basis for the solution space. Since

$$\cos 3t = \frac{1}{2}(e^{3it} + e^{-3it}) \quad \text{and} \quad \sin 3t = \frac{1}{2i}(e^{3it} - e^{-3it}),$$

it follows from Exercise 7 that $\{\cos 3t, \sin 3t\}$ is also a basis for this solution space. This basis has an advantage over the original one because it consists of the familiar sine and cosine functions and makes no reference to the imaginary number i. Using this latter basis, we see that any solution to the given equation is of the form

$$y(t) = b_1 \cos 3t + b_2 \sin 3t$$

for unique scalars $b_1$and b_2. ◆

Next consider the differential equation

$$y'' + 2y' + y = 0,$$

for which the auxiliary polynomial is $(t+1)^2$. By Theorem 2.31, e^{-t} is a solution to this equation. By the corollary to Theorem 2.32, its solution space is two-dimensional. In order to obtain a basis for the solution space, we need a solution that is linearly independent of e^{-t}. The reader can verify that te^{-t} is a such a solution. The following lemma extends this result.

Lemma. *For a given complex number c and positive integer n, suppose that $(t-c)^n$ is the auxiliary polynomial of a homogeneous linear differential equation with constant coefficients. Then the set*

$$\beta = \{e^{ct}, te^{ct}, \ldots, t^{n-1}e^{ct}\}$$

is a basis for the solution space of the equation.

Proof. Since the solution space is n-dimensional, we need only show that β is linearly independent and lies in the solution space. First, observe that for any positive integer k,

$$\begin{aligned}(\mathsf{D} - c\mathsf{I})(t^k e^{ct}) &= kt^{k-1}e^{ct} + ct^k e^{ct} - ct^k e^{ct}\\ &= kt^{k-1}e^{ct}.\end{aligned}$$

Hence for $k < n$,

$$(\mathsf{D} - c\mathsf{I})^n(t^k e^{ct}) = 0.$$

It follows that β is a subset of the solution space.

We next show that β is linearly independent. Consider any linear combination of vectors in β such that

$$b_0 e^{ct} + b_1 te^{ct} + \cdots + b_{n-1}t^{n-1}e^{ct} = 0 \tag{7}$$

for some scalars $b_0, b_1, \ldots, b_{n-1}$. Dividing by e^{ct} in (7), we obtain

$$b_0 + b_1 t + \cdots + b_{n-1}t^{n-1} = 0. \tag{8}$$

Thus the left side of (8) must be the zero polynomial function. We conclude that the coefficients $b_0, b_1, \ldots, b_{n-1}$ are all zero. So β is linearly independent and hence is a basis for the solution space. ∎

Example 7

We find all solutions to the differential equation

$$y^{(4)} - 4y^{(3)} + 6y^{(2)} - 4y^{(1)} + y = 0.$$

Since the auxiliary polynomial is

$$t^4 - 4t^3 + 6t^2 - 4t + 1 = (t-1)^4,$$

we can immediately conclude by the preceding lemma that $\{e^t, te^t, t^2e^t, t^3e^t\}$ is a basis for the solution space. So any solution y to the given differential equation is of the form

$$y(t) = b_1 e^t + b_2 te^t + b_3 t^2 e^t + b_4 t^3 e^t$$

for unique scalars b_1, b_2, b_3, and b_4. ◆

The most general situation is stated in the following theorem.

Theorem 2.34. *Given a homogeneous linear differential equation with constant coefficients and auxiliary polynomial*

$$(t - c_1)^{n_1}(t - c_2)^{n_2} \cdots (t - c_k)^{n_k},$$

where $n_1, n_2, \ldots, n_k$ are positive integers and $c_1, c_2, \ldots, c_k$ are distinct complex numbers, the following set is a basis for the solution space of the equation:

$$\{e^{c_1 t}, te^{c_1 t}, \ldots, t^{n_1 - 1}e^{c_1 t}, \ldots, e^{c_k t}, te^{c_k t}, \ldots, t^{n_k - 1}e^{c_k t}\}.$$

Proof. Exercise. ∎

Example 8

The differential equation

$$y^{(3)} - 4y^{(2)} + 5y^{(1)} - 2y = 0$$

has the auxiliary polynomial

$$t^3 - 4t^2 + 5t - 2 = (t - 1)^2(t - 2).$$

By Theorem 2.34, $\{e^t, te^t, e^{2t}\}$ is a basis for the solution space of the differential equation. Thus any solution y has the form

$$y(t) = b_1 e^t + b_2 te^t + b_3 e^{2t}$$

for unique scalars b_1, b_2, and b_3. ◆

EXERCISES

1. Label the following statements as true or false.
   - **(a)** The set of solutions to an nth-order homogeneous linear differential equation with constant coefficients is an n-dimensional subspace of C^∞.
   - **(b)** The solution space of a homogeneous linear differential equation with constant coefficients is the null space of a differential operator.
   - **(c)** The auxiliary polynomial of a homogeneous linear differential equation with constant coefficients is a solution to the differential equation.
   - **(d)** Any solution to a homogeneous linear differential equation with constant coefficients is of the form ae^{ct} or $at^k e^{ct}$, where a and c are complex numbers and k is a positive integer.

(e) Any linear combination of solutions to a given homogeneous linear differential equation with constant coefficients is also a solution to the given equation.

(f) For any homogeneous linear differential equation with constant coefficients having auxiliary polynomial $p(t)$, if $c_1, c_2, \dots, c_k$ are the distinct zeros of $p(t)$, then $\{e^{c_1 t}, e^{c_2 t}, \dots, e^{c_k t}\}$ is a basis for the solution space of the given differential equation.

(g) Given any polynomial $p(t) \in \mathsf{P}(C)$, there exists a homogeneous linear differential equation with constant coefficients whose auxiliary polynomial is $p(t)$.

2. For each of the following parts, determine whether the statement is true or false. Justify your claim with either a proof or a counterexample, whichever is appropriate.

(a) Any finite-dimensional subspace of C^∞ is the solution space of a homogeneous linear differential equation with constant coefficients.

(b) There exists a homogeneous linear differential equation with constant coefficients whose solution space has the basis $\{t, t^2\}$.

(c) For any homogeneous linear differential equation with constant coefficients, if x is a solution to the equation, so is its derivative x'.

Given two polynomials $p(t)$ and $q(t)$ in $\mathsf{P}(C)$, if $x \in \mathsf{N}(p(\mathsf{D}))$ and $y \in \mathsf{N}(q(\mathsf{D}))$, then

(d) $x + y \in \mathsf{N}(p(\mathsf{D})q(\mathsf{D}))$.

(e) $xy \in \mathsf{N}(p(\mathsf{D})q(\mathsf{D}))$.

3. Find a basis for the solution space of each of the following differential equations.

(a) $y'' + 2y' + y = 0$

(b) $y''' = y'$

(c) $y^{(4)} - 2y^{(2)} + y = 0$

(d) $y'' + 2y' + y = 0$

(e) $y^{(3)} - y^{(2)} + 3y^{(1)} + 5y = 0$

4. Find a basis for each of the following subspaces of C^∞.

(a) $\mathsf{N}(\mathsf{D}^2 - \mathsf{D} - \mathsf{I})$

(b) $\mathsf{N}(\mathsf{D}^3 - 3\mathsf{D}^2 + 3\mathsf{D} - \mathsf{I})$

(c) $\mathsf{N}(\mathsf{D}^3 + 6\mathsf{D}^2 + 8\mathsf{D})$

5. Show that C^∞ is a subspace of $\mathcal{F}(R, C)$.

6. **(a)** Show that $\mathsf{D}\colon \mathsf{C}^\infty \to \mathsf{C}^\infty$ is a linear operator.

(b) Show that any differential operator is a linear operator on C^∞.

7. Prove that if $\{x, y\}$ is a basis for a vector space over C, then so is

$$\left\{\frac{1}{2}(x+y), \frac{1}{2i}(x-y)\right\}.$$

8. Consider a second-order homogeneous linear differential equation with constant coefficients in which the auxiliary polynomial has distinct conjugate complex roots $a+ib$ and $a-ib$, where $a, b \in R$. Show that $\{e^{at}\cos bt, e^{at}\sin bt\}$ is a basis for the solution space.

9. Suppose that $\{\mathsf{U}_1, \mathsf{U}_2, \ldots, \mathsf{U}_n\}$ is a collection of pairwise commutative linear operators on a vector space V (i.e., operators such that $\mathsf{U}_i\mathsf{U}_j = \mathsf{U}_j\mathsf{U}_i$ for all i, j). Prove that, for any i $(1 \le i \le n)$,

$$\mathsf{N}(\mathsf{U}_i) \subseteq \mathsf{N}(\mathsf{U}_1\mathsf{U}_2\cdots\mathsf{U}_n).$$

10. Prove Theorem 2.33 and its corollary. *Hints:* For Theorem 2.33, use mathematical induction on n. In the inductive step, let $a_1, a_2, \ldots, a_n$ be scalars such that $\sum_{i=1}^{n} a_i e^{c_i t} = 0$. Multiply both sides of this equation by $e^{-c_n t}$, and differentiate the resulting equation with respect to t. For the corollary, use Theorems 2.31, 2.33, and 2.32. Visit `goo.gl/oKTEbV` for a solution.

11. Prove Theorem 2.34. *Hint:* First verify that the alleged basis lies in the solution space. Then verify that this set is linearly independent by mathematical induction on k as follows. The case $k=1$ is the lemma to Theorem 2.34. Assuming that the theorem holds for $k-1$ distinct c_i's, apply the operator $(\mathsf{D} - c_k\mathsf{I})^{n_k}$ to any linear combination of the alleged basis that equals 0.

12. Let V be the solution space of an nth-order homogeneous linear differential equation with constant coefficients having auxiliary polynomial $p(t)$. Prove that if $p(t) = g(t)h(t)$, where $g(t)$ and $h(t)$ are polynomials of positive degree, then

$$\mathsf{N}(h(\mathsf{D})) = \mathsf{R}(g(\mathsf{D}_\mathsf{V})) = g(\mathsf{D})(\mathsf{V}),$$

where $\mathsf{D}_\mathsf{V}\colon \mathsf{V} \to \mathsf{V}$ is defined by $\mathsf{D}_\mathsf{V}(x) = x'$ for $x \in \mathsf{V}$. *Hint:* First prove $g(\mathsf{D})(\mathsf{V}) \subseteq \mathsf{N}(h(\mathsf{D}))$. Then prove that the two spaces have the same finite dimension.

13. A differential equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y^{(1)} + a_0 y = x$$

is called a **nonhomogeneous** linear differential equation with constant coefficients if the a_i's are constant and x is a function that is not identically zero.

(a) Prove that for any $x \in \mathsf{C}^\infty$ there exists $y \in \mathsf{C}^\infty$ such that y is a solution to the differential equation. *Hint:* Use Lemma 1 to Theorem 2.32 to show that for any polynomial $p(t)$, the linear operator $p(\mathsf{D})\colon \mathsf{C}^\infty \to \mathsf{C}^\infty$ is onto.

(b) Let V be the solution space for the homogeneous linear equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y^{(1)} + a_0 y = \mathit{0}.$$

Prove that if z is any solution to the associated nonhomogeneous linear differential equation, then the set of all solutions to the nonhomogeneous linear differential equation is

$$\{z + y\colon y \in \mathsf{V}\}.$$

14. Given any nth-order homogeneous linear differential equation with constant coefficients, prove that, for any solution x and any $t_0 \in R$, if $x(t_0) = x'(t_0) = \cdots = x^{(n-1)}(t_0) = 0$, then $x = \mathit{0}$ (the zero function). *Hint:* Use mathematical induction on n as follows. First prove the conclusion for the case $n = 1$. Next suppose that it is true for equations of order $n-1$, and consider an nth-order differential equation with auxiliary polynomial $p(t)$. Factor $p(t) = q(t)(t-c)$, and let $z = q((D))x$. Show that $z(t_0) = 0$ and $z' - cz = \mathit{0}$ to conclude that $z = \mathit{0}$. Now apply the induction hypothesis.

15. Let V be the solution space of an nth-order homogeneous linear differential equation with constant coefficients. Fix $t_0 \in R$, and define a mapping $\Phi\colon \mathsf{V} \to \mathsf{C}^n$ by

$$\Phi(x) = \begin{pmatrix} x(t_0) \\ x'(t_0) \\ \vdots \\ x^{(n-1)}(t_0) \end{pmatrix} \quad \text{for each } x \in \mathsf{V}.$$

(a) Prove that Φ is linear and its null space is the zero subspace of V. Deduce that Φ is an isomorphism. *Hint:* Use Exercise 14.

(b) Prove the following: For any nth-order homogeneous linear differential equation with constant coefficients, any $t_0 \in R$, and any complex numbers $c_0, c_1, \ldots, c_{n-1}$ (not necessarily distinct), there exists exactly one solution, x, to the given differential equation such that $x(t_0) = c_0$ and $x^{(k)}(t_0) = c_k$ for $k = 1, 2, \ldots, n-1$.

16. *Pendular Motion.* It is well known that the motion of a pendulum is approximated by the differential equation

$$\theta'' + \frac{g}{l}\theta = \mathit{0},$$

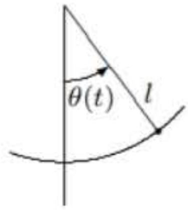


Figure 2.8

where $\theta(t)$ is the angle in radians that the pendulum makes with a vertical line at time t (see Figure 2.8), interpreted so that θ is positive if the pendulum is to the right and negative if the pendulum is to the left of the vertical line as viewed by the reader. Here l is the length of the pendulum and g is the magnitude of acceleration due to gravity. The variable t and constants l and g must be in compatible units (e.g., t in seconds, l in meters, and g in meters per second per second).

(a) Express an arbitrary solution to this equation as a linear combination of two real-valued solutions.

(b) Find the unique solution to the equation that satisfies the conditions

$$\theta(0) = \theta_0 > 0 \quad \text{and} \quad \theta'(0) = 0.$$

(The significance of these conditions is that at time $t = 0$ the pendulum is released from a position displaced from the vertical by θ_0.)

(c) Prove that it takes $2\pi\sqrt{l/g}$ units of time for the pendulum to make one circuit back and forth. (This time is called the **period** of the pendulum.)

17. *Periodic Motion of a Spring without Damping.* Find the general solution to (3), which describes the periodic motion of a spring, ignoring frictional forces.

18. *Periodic Motion of a Spring with Damping.* The ideal periodic motion described by solutions to (3) is due to the ignoring of frictional forces. In reality, however, there is a frictional force acting on the motion that is proportional to the speed of motion, but that acts in the opposite direction. The modification of (3) to account for the frictional force, called the *damping force*, is given by

$$my'' + ry' + ky = 0,$$

where $r > 0$ is the proportionality constant.

(a) Find the general solution to this equation.
(b) Find the unique solution in (a) that satisfies the initial conditions $y(0) = 0$ and $y'(0) = v_0$, the initial velocity.
(c) For $y(t)$ as in (b), show that the amplitude of the oscillation decreases to zero; that is, prove that $\lim_{t\to\infty} y(t) = 0$.

19. In our study of differential equations, we have regarded solutions as complex-valued functions even though functions that are useful in describing physical motion are real-valued. Justify this approach.

20. The following parts, which do not involve linear algebra, are included for the sake of completeness.

(a) Prove Theorem 2.27. *Hint:* Use mathematical induction on the number of derivatives possessed by a solution.
(b) For any $c, d \in C$, prove that

$$e^{c+d} = c^c e^d \quad \text{and} \quad e^{-c} = \frac{1}{e^c}.$$

(c) Prove Theorem 2.28.
(d) Prove Theorem 2.29.
(e) Prove the product rule for differentiating complex-valued functions of a real variable: For any differentiable functions x and y in $\mathcal{F}(R, C)$, the product xy is differentiable and

$$(xy)' = x'y + xy'.$$

Hint: Apply the rules of differentiation to the real and imaginary parts of xy.
(f) Prove that if $x \in \mathcal{F}(R, C)$ and $x' = \mathit{0}$, then x is a constant function.

INDEX OF DEFINITIONS FOR CHAPTER 2

3

Elementary Matrix Operations and Systems of Linear Equations

*T*his chapter is devoted to two related objectives:

1. the study of certain "rank-preserving" operations on matrices;
2. the application of these operations and the theory of linear transformations to the solution of systems of linear equations.

As a consequence of objective 1, we obtain a simple method for computing the rank of a linear transformation between finite-dimensional vector spaces by applying these rank-preserving matrix operations to a matrix that represents that transformation.

Solving a system of linear equations is probably the most important application of linear algebra. The familiar method of elimination for solving systems of linear equations, which was discussed in Section 1.4, involves the elimination of variables so that a simpler system can be obtained. The technique by which the variables are eliminated utilizes three types of operations:

1. interchanging any two equations in the system;
2. multiplying any equation in the system by a nonzero constant;
3. adding a multiple of one equation to another.

In Section 3.3, we express a system of linear equations as a single matrix equation. In this representation of the system, the three operations above are the "elementary row operations" for matrices. These operations provide a convenient computational method for determining all solutions to a system of linear equations.

3.1 ELEMENTARY MATRIX OPERATIONS AND ELEMENTARY MATRICES

In this section, we define the elementary operations that are used throughout the chapter. In subsequent sections, we use these operations to obtain simple computational methods for determining the rank of a linear transformation and the solution of a system of linear equations. There are two types of elementary matrix operations—row operations and column operations. As we will see, the row operations are more useful. They arise from the three operations that can be used to eliminate variables in a system of linear equations.

Definitions. *Let A be an $m \times n$ matrix. Any one of the following three operations on the rows [columns] of A is called an* ***elementary row [column] operation****:*

(1) *interchanging any two rows [columns] of A;*
(2) *multiplying any row [column] of A by a nonzero scalar;*
(3) *adding any scalar multiple of a row [column] of A to another row [column].*

Any of these three operations is called an ***elementary operation****. Elementary operations are of* ***type 1****,* ***type 2****, or* ***type 3*** *depending on whether they are obtained by (1), (2), or (3).*

Example 1

Let

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & -1 & 3 \\ 4 & 0 & 1 & 2 \end{pmatrix}.$$

Interchanging the second row of A with the first row is an example of an elementary row operation of type 1. The resulting matrix is

$$B = \begin{pmatrix} 2 & 1 & -1 & 3 \\ 1 & 2 & 3 & 4 \\ 4 & 0 & 1 & 2 \end{pmatrix}.$$

Multiplying the second column of A by 3 is an example of an elementary column operation of type 2. The resulting matrix is

$$C = \begin{pmatrix} 1 & 6 & 3 & 4 \\ 2 & 3 & -1 & 3 \\ 4 & 0 & 1 & 2 \end{pmatrix}.$$

Adding 4 times the third row of A to the first row is an example of an elementary row operation of type 3. In this case, the resulting matrix is

$$M = \begin{pmatrix} 17 & 2 & 7 & 12 \\ 2 & 1 & -1 & 3 \\ 4 & 0 & 1 & 2 \end{pmatrix}. \quad \blacklozenge$$

Notice that if a matrix Q can be obtained from a matrix P by means of an elementary row operation, then P can be obtained from Q by an elementary row operation of the same type. (See Exercise 8.) So, in Example 1, A can be obtained from M by adding -4 times the third row of M to the first row of M.

Definition. *An $n \times n$ **elementary matrix** is a matrix obtained by performing an elementary operation on I_n. The elementary matrix is said to be of **type 1**, **2**, or **3** according to whether the elementary operation performed on I_n is a type 1, 2, or 3 operation, respectively.*

For example, interchanging the first two rows of I_3 produces the elementary matrix

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that E can also be obtained by interchanging the first two columns of I_3. In fact, *any elementary matrix can be obtained in at least two ways*—either by performing an elementary row operation on I_n or by performing an elementary column operation on I_n. (See Exercise 4.) Similarly,

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is an elementary matrix since it can be obtained from I_3 by an elementary column operation of type 3 (adding -2 times the first column of I_3 to the third column) or by an elementary row operation of type 3 (adding -2 times the third row to the first row).

Our first theorem shows that performing an elementary row operation on a matrix is equivalent to multiplying the matrix by an elementary matrix.

Theorem 3.1. *Let $A \in \mathsf{M}_{m \times n}(F)$, and suppose that B is obtained from A by performing an elementary row [column] operation. Then there exists an $m \times m$ [$n \times n$] elementary matrix E such that $B = EA$ [$B = AE$]. In fact, E is obtained from I_m [I_n] by performing the same elementary row [column] operation as that which was performed on A to obtain B. Conversely, if E is an elementary $m \times m$ [$n \times n$] matrix, then EA [AE] is the matrix obtained from A by performing the same elementary row [column] operation as that which produces E from I_m [I_n].*

The proof, which we omit, requires verifying Theorem 3.1 for each type of elementary row operation. The proof for column operations can then be obtained by using the matrix transpose to transform a column operation into a row operation. The details are left as an exercise. (See Exercise 7.)

The next example illustrates the use of the theorem.

Example 2

Consider the matrices A and B in Example 1. In this case, B is obtained from A by interchanging the first two rows of A. Performing this same operation on I_3, we obtain the elementary matrix

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $EA = B$.

In the second part of Example 1, C is obtained from A by multiplying the second column of A by 3. Performing this same operation on I_4, we obtain the elementary matrix

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Observe that $AE = C$. ◆

It is a useful fact that the inverse of an elementary matrix is also an elementary matrix.

Theorem 3.2. *Elementary matrices are invertible, and the inverse of an elementary matrix is an elementary matrix of the same type.*

Proof. Let E be an elementary $n \times n$ matrix. Then E can be obtained by an elementary row operation on I_n. By reversing the steps used to transform I_n into E, we can transform E back into I_n. The result is that I_n can be obtained from E by an elementary row operation of the same type. By Theorem 3.1, there is an elementary matrix $\overline{E}$ such that $\overline{E}E = I_n$. Therefore, by Exercise 10 of Section 2.4, E is invertible and $E^{-1} = \overline{E}$. ∎

EXERCISES

1. Label the following statements as true or false.

- **(a)** An elementary matrix is always square.
- **(b)** The only entries of an elementary matrix are zeros and ones.
- **(c)** The $n \times n$ identity matrix is an elementary matrix.
- **(d)** The product of two $n \times n$ elementary matrices is an elementary matrix.
- **(e)** The inverse of an elementary matrix is an elementary matrix.
- **(f)** The sum of two $n \times n$ elementary matrices is an elementary matrix.
- **(g)** The transpose of an elementary matrix is an elementary matrix.

(h) If B is a matrix that can be obtained by performing an elementary row operation on a matrix A, then B can also be obtained by performing an elementary column operation on A.

(i) If B is a matrix that can be obtained by performing an elementary row operation on a matrix A, then A can be obtained by performing an elementary row operation on B.

2. Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -2 & 1 \\ 1 & -3 & 1 \end{pmatrix}, \text{ and } C = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & -2 \\ 1 & -3 & 1 \end{pmatrix}.$$

Find an elementary operation that transforms A into B and an elementary operation that transforms B into C. By means of several additional operations, transform C into I_3.

3. Use the proof of Theorem 3.2 to obtain the inverse of each of the following elementary matrices.

(a) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **(c)** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

4. Prove the assertion made on page 149: Any elementary $n \times n$ matrix can be obtained in at least two ways—either by performing an elementary row operation on I_n or by performing an elementary column operation on I_n.

5. Prove that E is an elementary matrix if and only if E^t is.

6. Let A be an $m \times n$ matrix. Prove that if B can be obtained from A by an elementary row [column] operation, then B^t can be obtained from A^t by the corresponding elementary column [row] operation.

7. Prove Theorem 3.1.

8. Prove that if a matrix Q can be obtained from a matrix P by an elementary row operation, then P can be obtained from Q by an elementary row operation of the same type. *Hint:* Treat each type of elementary row operation separately.

9. Prove that any elementary row [column] operation of type 1 can be obtained by a succession of three elementary row [column] operations of type 3 followed by one elementary row [column] operation of type 2. Visit `goo.gl/oNJBFz` for a solution.

10. Prove that any elementary row [column] operation of type 2 can be obtained by *dividing* some row [column] by a nonzero scalar.

11. Prove that any elementary row [column] operation of type 3 can be obtained by *subtracting* a multiple of some row [column] from another row [column].

12. Let A be an $m \times n$ matrix. Prove that there exists a sequence of elementary row operations of types 1 and 3 that transforms A into an upper triangular matrix.

3.2 THE RANK OF A MATRIX AND MATRIX INVERSES

In this section, we define the *rank* of a matrix. We then use elementary operations to compute the rank of a matrix and a linear transformation. The section concludes with a procedure for computing the inverse of an invertible matrix.

Definition. *If $A \in \mathsf{M}_{m\times n}(F)$, we define the* **rank** *of A, denoted* $\text{rank}(A)$, *to be the rank of the linear transformation $\mathsf{L}_A \colon \mathsf{F}^n \to \mathsf{F}^m$.*

Many results about the rank of a matrix follow immediately from the corresponding facts about a linear transformation. An important result of this type, which follows from Fact 3 (p. 101) and Corollary 2 to Theorem 2.18 (p. 103), is that *an $n \times n$ matrix is invertible if and only if its rank is n.*

Every matrix A is the matrix representation of the linear transformation L_A with respect to the appropriate standard ordered bases. Thus the rank of the linear transformation L_A is the same as the rank of one of its matrix representations, namely, A. The next theorem extends this fact to any matrix representation of any linear transformation defined on finite-dimensional vector spaces.

Theorem 3.3. *Let $\mathsf{T} \colon \mathsf{V} \to \mathsf{W}$ be a linear transformation between finite-dimensional vector spaces, and let β and γ be ordered bases for V and W, respectively. Then* $\text{rank}(\mathsf{T}) = \text{rank}([\mathsf{T}]_\beta^\gamma)$.

Proof. This is a restatement of Exercise 20 of Section 2.4. ∎

Now that the problem of finding the rank of a linear transformation has been reduced to the problem of finding the rank of a matrix, we need a result that allows us to perform rank-preserving operations on matrices. The next theorem and its corollary tell us how to do this.

Theorem 3.4. *Let A be an $m \times n$ matrix. If P and Q are invertible $m \times m$ and $n \times n$ matrices, respectively, then*

(a) $\text{rank}(AQ) = \text{rank}(A)$,

(b) $\text{rank}(PA) = \text{rank}(A)$,

and therefore,

(c) $\text{rank}(PAQ) = \text{rank}(A)$.

Proof. First observe that

$$\mathsf{R}(\mathsf{L}_{AQ}) = \mathsf{R}(\mathsf{L}_A\mathsf{L}_Q) = \mathsf{L}_A\mathsf{L}_Q(\mathsf{F}^n) = \mathsf{L}_A(\mathsf{L}_Q(\mathsf{F}^n)) = \mathsf{L}_A(\mathsf{F}^n) = \mathsf{R}(\mathsf{L}_A)$$

since L_Q is onto. Therefore

$$\text{rank}(AQ) = \dim(\mathsf{R}(\mathsf{L}_{AQ})) = \dim(\mathsf{R}(\mathsf{L}_A)) = \text{rank}(A).$$

This establishes (a). To establish (b), apply Exercise 17 of Section 2.4 to $\mathsf{T} = \mathsf{L}_P$. We omit the details. Finally, applying (a) and (b), we have

$$\text{rank}(PAQ) = \text{rank}(PA) = \text{rank}(A). \qquad \blacksquare$$

Corollary. *Elementary row and column operations on a matrix are rank-preserving.*

Proof. If B is obtained from a matrix A by an elementary row operation, then there exists an elementary matrix E such that $B = EA$. By Theorem 3.2 (p. 150), E is invertible, and hence $\text{rank}(B) = \text{rank}(A)$ by Theorem 3.4. The proof that elementary column operations are rank-preserving is left as an exercise. ∎

Now that we have a class of matrix operations that preserve rank, we need a way of examining a transformed matrix to ascertain its rank. The next theorem is the first of several in this direction.

Theorem 3.5. *The rank of any matrix equals the maximum number of its linearly independent columns; that is, the rank of a matrix is the dimension of the subspace generated by its columns.*

Proof. For any $A \in \mathsf{M}_{m\times n}(F)$,

$$\text{rank}(A) = \text{rank}(\mathsf{L}_A) = \dim(\mathsf{R}(\mathsf{L}_A)).$$

Let β be the standard ordered basis for F^n. Then β spans F^n and hence, by Theorem 2.2 (p. 68),

$$\mathsf{R}(\mathsf{L}_A) = \text{span}(\mathsf{L}_A(\beta)) = \text{span}\left(\{\mathsf{L}_A(e_1), \mathsf{L}_A(e_2), \ldots, \mathsf{L}_A(e_n)\}\right).$$

But, for any j, we have seen in Theorem 2.13(b) (p. 91) that $\mathsf{L}_A(e_j) = Ae_j = a_j$, where a_j is the jth column of A. Hence

$$\mathsf{R}(\mathsf{L}_A) = \text{span}\left(\{a_1, a_2, \ldots, a_n\}\right).$$

Thus

$$\text{rank}(A) = \dim(\mathsf{R}(\mathsf{L}_A)) = \dim(\text{span}\left(\{a_1, a_2, \ldots, a_n\}\right)). \qquad \blacksquare$$

Example 1

Let

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Observe that the first and second columns of A are linearly independent and that the third column is a linear combination of the first two. Thus

$$\operatorname{rank}(A) = \dim\left(\operatorname{span}\left(\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}\right)\right) = 2. \quad \blacklozenge$$

To compute the rank of a matrix A, it is frequently useful to postpone the use of Theorem 3.5 until A has been suitably modified by means of appropriate elementary row and column operations so that the number of linearly independent columns is obvious. The corollary to Theorem 3.4 guarantees that the rank of the modified matrix is the same as the rank of A. One such modification of A can be obtained by using elementary row and column operations to introduce zero entries. The next example illustrates this procedure.

Example 2

Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 1 & 1 & 2 \end{pmatrix}.$$

If we subtract the first row of A from rows 2 and 3 (type 3 elementary row operations), the result is

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -2 & 2 \\ 0 & -1 & 1 \end{pmatrix}.$$

If we now subtract twice the first column from the second and subtract the first column from the third (type 3 elementary column operations), we obtain

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & -1 & 1 \end{pmatrix}.$$

It is now obvious that the maximum number of linearly independent columns of this matrix is 2. Hence the rank of A is 2. $\blacklozenge$

The next theorem uses this process to transform a matrix into a particularly simple form. The power of this theorem can be seen in its corollaries.

Theorem 3.6. *Let A be an $m \times n$ matrix of rank r. Then $r \leq m$, $r \leq n$, and, by means of a finite number of elementary row and column operations, A can be transformed into the matrix*

$$D = \begin{pmatrix} I_r & O_1 \\ O_2 & O_3 \end{pmatrix},$$

where O_1, O_2, and O_3 are zero matrices. Thus $D_{ii} = 1$ for $i \leq r$ and $D_{ij} = 0$ otherwise.

Theorem 3.6 and its corollaries are quite important. Its proof, though easy to understand, is tedious to read. As an aid in following the proof, we first consider an example.

Example 3

Consider the matrix

$$A = \begin{pmatrix} 0 & 2 & 4 & 2 & 2 \\ 4 & 4 & 4 & 8 & 0 \\ 8 & 2 & 0 & 10 & 2 \\ 6 & 3 & 2 & 9 & 1 \end{pmatrix}.$$

By means of a succession of elementary row and column operations, we can transform A into a matrix D as in Theorem 3.6. We list many of the intermediate matrices, but on several occasions a matrix is transformed from the preceding one by means of several elementary operations. The number above each arrow indicates how many elementary operations are involved. Try to identify the nature of each elementary operation (row or column and type) in the following matrix transformations.

$$\begin{pmatrix} 0 & 2 & 4 & 2 & 2 \\ 4 & 4 & 4 & 8 & 0 \\ 8 & 2 & 0 & 10 & 2 \\ 6 & 3 & 2 & 9 & 1 \end{pmatrix} \xrightarrow{1} \begin{pmatrix} 4 & 4 & 4 & 8 & 0 \\ 0 & 2 & 4 & 2 & 2 \\ 8 & 2 & 0 & 10 & 2 \\ 6 & 3 & 2 & 9 & 1 \end{pmatrix} \xrightarrow{1} \begin{pmatrix} 1 & 1 & 1 & 2 & 0 \\ 0 & 2 & 4 & 2 & 2 \\ 8 & 2 & 0 & 10 & 2 \\ 6 & 3 & 2 & 9 & 1 \end{pmatrix} \xrightarrow{2}$$

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 0 \\ 0 & 2 & 4 & 2 & 2 \\ 0 & -6 & -8 & -6 & 2 \\ 0 & -3 & -4 & -3 & 1 \end{pmatrix} \xrightarrow{3} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 2 & 2 \\ 0 & -6 & -8 & -6 & 2 \\ 0 & -3 & -4 & -3 & 1 \end{pmatrix} \xrightarrow{1}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & -6 & -8 & -6 & 2 \\ 0 & -3 & -4 & -3 & 1 \end{pmatrix} \xrightarrow{2} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 4 & 0 & 8 \\ 0 & 0 & 2 & 0 & 4 \end{pmatrix} \xrightarrow{3} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 8 \\ 0 & 0 & 2 & 0 & 4 \end{pmatrix} \xrightarrow{1}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 0 & 4 \end{pmatrix} \xrightarrow{1} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{1} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = D$$

By the corollary to Theorem 3.4, $\text{rank}(A) = \text{rank}(D)$. Clearly, however, $\text{rank}(D) = 3$; so $\text{rank}(A) = 3$. ◆

Note that the first two elementary operations in Example 3 result in a 1 in the 1,1 position, and the next several operations (type 3) result in 0's everywhere in the first row and first column except for the 1,1 position. Subsequent elementary operations do not change the first row and first column. With this example in mind, we proceed with the proof of Theorem 3.6.

Proof of Theorem 3.6. If A is the zero matrix, $r = 0$ by Exercise 3. In this case, the conclusion follows with $D = A$.

Now suppose that $A \neq O$ and $r = \text{rank}(A)$; then $r > 0$. The proof is by mathematical induction on m, the number of rows of A.

Suppose that $m = 1$. By means of at most one type 1 column operation and at most one type 2 column operation, A can be transformed into a matrix with a 1 in the 1,1 position. By means of at most $n - 1$ type 3 column operations, this matrix can in turn be transformed into the matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}.$$

Note that there is one linearly independent column in D. So $\text{rank}(D) = \text{rank}(A) = 1$ by the corollary to Theorem 3.4 and by Theorem 3.5. Thus the theorem is established for $m = 1$.

Next assume that the theorem holds for any matrix with at most $m - 1$ rows (for some $m > 1$). We must prove that the theorem holds for any matrix with m rows.

Suppose that A is any $m \times n$ matrix. If $n = 1$, Theorem 3.6 can be established in a manner analogous to that for $m = 1$ (see Exercise 10).

We now suppose that $n > 1$. Since $A \neq O$, $A_{ij} \neq 0$ for some i, j. By means of at most one elementary row and at most one elementary column operation (each of type 1), we can move the nonzero entry to the 1,1 position (just as was done in Example 3). By means of at most one additional type 2 operation, we can assure a 1 in the 1,1 position. (Look at the second operation in Example 3.) By means of at most $m-1$ type 3 row operations and at most $n - 1$ type 3 column operations, we can eliminate all nonzero entries in the first row and the first column with the exception of the 1 in the 1,1 position. (In Example 3, we used two row and three column operations to do this.)

Thus, with a finite number of elementary operations, A can be transformed into a matrix

$$B = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & B' & \\ 0 & & & \end{array}\right),$$

where B' is an $(m-1) \times (n-1)$ matrix. In Example 3, for instance,

$$B' = \begin{pmatrix} 2 & 4 & 2 & 2 \\ -6 & -8 & -6 & 2 \\ -3 & -4 & -3 & 1 \end{pmatrix}.$$

By Exercise 11, B' has rank one less than B. Since $\text{rank}(A) = \text{rank}(B) = r$, $\text{rank}(B') = r - 1$. Therefore $r - 1 \leq m - 1$ and $r - 1 \leq n - 1$ by the induction hypothesis. Hence $r \leq m$ and $r \leq n$.

Also by the induction hypothesis, B' can be transformed by a finite number of elementary row and column operations into the $(m-1) \times (n-1)$ matrix D' such that

$$D' = \begin{pmatrix} I_{r-1} & O_4 \\ O_5 & O_6 \end{pmatrix},$$

where O_4, O_5, and O_6 are zero matrices. That is, D' consists of all zeros except for its first $r - 1$ diagonal entries, which are ones. Let

$$D = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & D' & \\ 0 & & & \end{array}\right).$$

We see that the theorem now follows once we show that D can be obtained from B by means of a finite number of elementary row and column operations. However this follows by repeated applications of Exercise 12.

Thus, since A can be transformed into B and B can be transformed into D, each by a finite number of elementary operations, A can be transformed into D by a finite number of elementary operations.

Finally, since D' contains ones as its first $r-1$ diagonal entries, D contains ones as its first r diagonal entries and zeros elsewhere. This establishes the theorem. ∎

Corollary 1. *Let A be an $m \times n$ matrix of rank r. Then there exist invertible matrices B and C of sizes $m \times m$ and $n \times n$, respectively, such that $D = BAC$, where*

$$D = \begin{pmatrix} I_r & O_1 \\ O_2 & O_3 \end{pmatrix}$$

is the $m \times n$ matrix in which O_1, O_2, and O_3 are zero matrices.

Proof. By Theorem 3.6, A can be transformed by means of a finite number of elementary row and column operations into the matrix D. We can appeal to Theorem 3.1 (p. 149) each time we perform an elementary operation. Thus there exist elementary $m \times m$ matrices $E_1, E_2, \ldots, E_p$ and elementary $n \times n$ matrices $G_1, G_2, \ldots, G_q$ such that

$$D = E_p E_{p-1} \cdots E_2 E_1 A G_1 G_2 \cdots G_q.$$

By Theorem 3.2 (p. 150), each E_j and G_j is invertible. Let $B = E_pE_{p-1}\cdots E_1$ and $C = G_1G_2\cdots G_q$. Then B and C are invertible by Exercise 4 of Section 2.4, and $D = BAC$. ∎

Corollary 2. *Let A be an $m \times n$ matrix. Then*

(a) $\operatorname{rank}(A^t) = \operatorname{rank}(A)$.

(b) *The rank of any matrix equals the maximum number of its linearly independent rows; that is, the rank of a matrix is the dimension of the subspace generated by its rows.*

(c) *The rows and columns of any matrix generate subspaces of the same dimension, numerically equal to the rank of the matrix.*

Proof. (a) By Corollary 1, there exist invertible matrices B and C such that $D = BAC$, where D satisfies the stated conditions of the corollary. Taking transposes, we have

$$D^t = (BAC)^t = C^tA^tB^t.$$

Since B and C are invertible, so are B^t and C^t by Exercise 4 of Section 2.4. Hence by Theorem 3.4,

$$\operatorname{rank}(A^t) = \operatorname{rank}(C^tA^tB^t) = \operatorname{rank}(D^t).$$

Suppose that $r = \operatorname{rank}(A)$. Then D^t is an $n \times m$ matrix with the form of the matrix D in Corollary 1, and hence $\operatorname{rank}(D^t) = r$ by Theorem 3.5. Thus

$$\operatorname{rank}(A^t) = \operatorname{rank}(D^t) = r = \operatorname{rank}(A).$$

This establishes (a).

The proofs of (b) and (c) are left as exercises. (See Exercise 13.) ∎

Corollary 3. *Every invertible matrix is a product of elementary matrices.*

Proof. If A is an invertible $n \times n$ matrix, then $\operatorname{rank}(A) = n$. Hence the matrix D in Corollary 1 equals I_n, and there exist invertible matrices B and C such that $I_n = BAC$.

As in the proof of Corollary 1, note that $B = E_pE_{p-1}\cdots E_1$ and $C = G_1G_2\cdots G_q$, where the E_i's and G_i's are elementary matrices. Thus $A = B^{-1}I_nC^{-1} = B^{-1}C^{-1}$, so that

$$A = E_1^{-1}E_2^{-1}\cdots E_p^{-1}G_q^{-1}G_{q-1}^{-1}\cdots G_1^{-1}.$$

The inverses of elementary matrices are elementary matrices, however, and hence A is the product of elementary matrices. ∎

We now use Corollary 2 to relate the rank of a matrix product to the rank of each factor. Notice how the proof exploits the relationship between the rank of a matrix and the rank of a linear transformation.

Theorem 3.7. *Let $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ and $\mathsf{U}\colon \mathsf{W} \to \mathsf{Z}$ be linear transformations on finite-dimensional vector spaces V, W, and Z, and let A and B be matrices such that the product AB is defined. Then*

(a) $\operatorname{rank}(\mathsf{UT}) \le \operatorname{rank}(\mathsf{U})$.
(b) $\operatorname{rank}(\mathsf{UT}) \le \operatorname{rank}(\mathsf{T})$.
(c) $\operatorname{rank}(AB) \le \operatorname{rank}(A)$.
(d) $\operatorname{rank}(AB) \le \operatorname{rank}(B)$.

Proof. We prove these items in the order: (a), (c), (d), and (b).

(a) Clearly, $\mathsf{R}(\mathsf{T}) \subseteq \mathsf{W}$. Hence

$$\mathsf{R}(\mathsf{UT}) = \mathsf{UT}(\mathsf{V}) = \mathsf{U}(\mathsf{T}(\mathsf{V})) = \mathsf{U}(\mathsf{R}(\mathsf{T})) \subseteq \mathsf{U}(\mathsf{W}) = \mathsf{R}(\mathsf{U}).$$

Thus

$$\operatorname{rank}(\mathsf{UT}) = \dim(\mathsf{R}(\mathsf{UT})) \le \dim(\mathsf{R}(\mathsf{U})) = \operatorname{rank}(\mathsf{U}).$$

(c) By (a),

$$\operatorname{rank}(AB) = \operatorname{rank}(\mathsf{L}_{AB}) = \operatorname{rank}(\mathsf{L}_A\mathsf{L}_B) \le \operatorname{rank}(\mathsf{L}_A) = \operatorname{rank}(A).$$

(d) By (c) and Corollary 2 to Theorem 3.6,

$$\operatorname{rank}(AB) = \operatorname{rank}((AB)^t) = \operatorname{rank}(B^tA^t) \le \operatorname{rank}(B^t) = \operatorname{rank}(B).$$

(b) Let α, β, and γ be ordered bases for V, W, and Z, respectively, and let $A' = [\mathsf{U}]_\beta^\gamma$ and $B' = [\mathsf{T}]_\alpha^\beta$. Then $A'B' = [\mathsf{UT}]_\alpha^\gamma$ by Theorem 2.11 (p. 89). Hence, by Theorem 3.3 and (d),

$$\operatorname{rank}(\mathsf{UT}) = \operatorname{rank}(A'B') \le \operatorname{rank}(B') = \operatorname{rank}(\mathsf{T}). \qquad \blacksquare$$

It is important to be able to compute the rank of any matrix. We can use the corollary to Theorem 3.4, Theorems 3.5 and 3.6, and Corollary 2 to Theorem 3.6 to accomplish this goal.

The object is to perform elementary row and column operations on a matrix to "simplify" it (so that the transformed matrix has many zero entries) to the point where a simple observation enables us to determine how many linearly independent rows or columns the matrix has, and thus to determine its rank.

Example 4

(a) Let

$$A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{pmatrix}.$$

Note that the first and second rows of A are linearly independent since one is not a multiple of the other. Thus $\operatorname{rank}(A) = 2$.

(b) Let

$$A = \begin{pmatrix} 1 & 3 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 3 & 0 & 0 \end{pmatrix}.$$

In this case, there are several ways to proceed. Suppose that we begin with an elementary row operation to obtain a zero in the 2,1 position. Subtracting the first row from the second row, we obtain

$$\begin{pmatrix} 1 & 3 & 1 & 1 \\ 0 & -3 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{pmatrix}.$$

Now note that the third row is a multiple of the second row, and the first and second rows are linearly independent. Thus $\operatorname{rank}(A) = 2$.

As an alternative method, note that the first, third, and fourth columns of A are identical and that the first and second columns of A are linearly independent. Hence $\operatorname{rank}(A) = 2$.

(c) Let

$$A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & -1 & 1 & 0 \end{pmatrix}.$$

Using elementary row operations, we can transform A as follows:

$$A \longrightarrow \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -5 & -1 \\ 0 & -3 & -2 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -5 & -1 \\ 0 & 0 & 3 & 0 \end{pmatrix}.$$

It is clear that the last matrix has three linearly independent rows and hence has rank 3. ◆

In summary, perform row and column operations until the matrix is simplified enough so that the maximum number of linearly independent rows or columns is obvious.

The Inverse of a Matrix

We have remarked that an $n \times n$ matrix is invertible if and only if its rank is n. Since we know how to compute the rank of any matrix, we can always test a matrix to determine whether it is invertible. We now provide a simple technique for computing the inverse of a matrix that utilizes elementary row operations.

Definition. *Let A and B be $m \times n$ and $m \times p$ matrices, respectively. By the* **augmented matrix** $(A|B)$, *we mean the $m \times (n+p)$ matrix $(A\ B)$, that is, the matrix whose first n columns are the columns of A, and whose last p columns are the columns of B.*

Let A be an invertible $n \times n$ matrix, and consider the $n \times 2n$ augmented matrix $C = (A|I_n)$. By Exercise 15, we have

$$A^{-1}C = (A^{-1}A|A^{-1}I_n) = (I_n|A^{-1}). \tag{1}$$

By Corollary 3 to Theorem 3.6, A^{-1} is the product of elementary matrices, say $A^{-1} = E_pE_{p-1}\cdots E_1$. Thus (1) becomes

$$E_pE_{p-1}\cdots E_1(A|I_n) = A^{-1}C = (I_n|A^{-1}).$$

Because multiplying a matrix on the left by an elementary matrix transforms the matrix by an elementary row operation (Theorem 3.1 p. 149), we have the following result: *If A is an invertible $n \times n$ matrix, then it is possible to transform the matrix $(A|I_n)$ into the matrix $(I_n|A^{-1})$ by means of a finite number of elementary row operations.*

Conversely, suppose that A is invertible and that, for some $n \times n$ matrix B, the matrix $(A|I_n)$ can be transformed into the matrix $(I_n|B)$ by a finite number of elementary row operations. Let $E_1, E_2, \ldots, E_p$ be the elementary matrices associated with these elementary row operations as in Theorem 3.1; then

$$E_pE_{p-1}\cdots E_1(A|I_n) = (I_n|B). \tag{2}$$

Letting $M = E_pE_{p-1}\cdots E_1$, we have from (2) that

$$(MA|M) = M(A|I_n) = (I_n|B).$$

Hence $MA = I_n$ and $M = B$. It follows that $M = A^{-1}$. So $B = A^{-1}$. Thus we have the following result: *If A is an invertible $n \times n$ matrix, and the matrix $(A|I_n)$ is transformed into a matrix of the form $(I_n|B)$ by means of a finite number of elementary row operations, then $B = A^{-1}$.*

If, on the other hand, A is an $n \times n$ matrix that is not invertible, then $\operatorname{rank}(A) < n$. Hence any attempt to transform $(A|I_n)$ into a matrix of the form $(I_n|B)$ by means of elementary row operations must fail because otherwise A can be transformed into I_n using the same row operations. This is impossible, however, because elementary row operations preserve rank. In fact, A can be transformed into a matrix with a row containing only zero entries, yielding the following result: *If A is an $n \times n$ matrix that is not invertible, then any attempt to transform $(A|I_n)$ into a matrix of the form $(I_n|B)$ by elementary row operations produces a row whose first n entries are zeros.*

The next two examples demonstrate these comments.

Example 5

We determine whether the matrix

$$A = \begin{pmatrix} 0 & 2 & 4 \\ 2 & 4 & 2 \\ 3 & 3 & 1 \end{pmatrix}$$

is invertible, and if it is, we compute its inverse.

We attempt to use elementary row operations to transform

$$(A|I) = \left(\begin{array}{ccc|ccc} 0 & 2 & 4 & 1 & 0 & 0 \\ 2 & 4 & 2 & 0 & 1 & 0 \\ 3 & 3 & 1 & 0 & 0 & 1 \end{array}\right)$$

into a matrix of the form $(I|B)$. One method for accomplishing this transformation is to change each column of A successively, beginning with the first column, into the corresponding column of I. Since we need a nonzero entry in the 1,1 position, we begin by interchanging rows 1 and 2. The result is

$$\left(\begin{array}{ccc|ccc} 2 & 4 & 2 & 0 & 1 & 0 \\ 0 & 2 & 4 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 0 & 1 \end{array}\right).$$

In order to place a 1 in the 1,1 position, we must multiply the first row by $\frac{1}{2}$; this operation yields

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 2 & 4 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 0 & 1 \end{array}\right).$$

We now complete work in the first column by adding -3 times row 1 to row 3 to obtain

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 2 & 4 & 1 & 0 & 0 \\ 0 & -3 & -2 & 0 & -\frac{3}{2} & 1 \end{array}\right).$$

In order to change the second column of the preceding matrix into the second column of I, we multiply row 2 by $\frac{1}{2}$ to obtain a 1 in the 2,2 position. This operation produces

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & -3 & -2 & 0 & -\frac{3}{2} & 1 \end{array}\right).$$

We now complete our work on the second column by adding -2 times row 2 to row 1 and 3 times row 2 to row 3. The result is

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -3 & -1 & \frac{1}{2} & 0 \\ 0 & 1 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 4 & \frac{3}{2} & -\frac{3}{2} & 1 \end{array}\right).$$

Only the third column remains to be changed. In order to place a 1 in the 3,3 position, we multiply row 3 by $\frac{1}{4}$; this operation yields

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -3 & -1 & \frac{1}{2} & 0 \\ 0 & 1 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{8} & -\frac{3}{8} & \frac{1}{4} \end{array}\right).$$

Adding appropriate multiples of row 3 to rows 1 and 2 completes the process and gives

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{8} & -\frac{5}{8} & \frac{3}{4} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{8} & -\frac{3}{8} & \frac{1}{4} \end{array}\right).$$

Thus A is invertible, and

$$A^{-1} = \begin{pmatrix} \frac{1}{8} & -\frac{5}{8} & \frac{3}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} \\ \frac{3}{8} & -\frac{3}{8} & \frac{1}{4} \end{pmatrix}. \quad \blacklozenge$$

Example 6

We determine whether the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 1 & 5 & 4 \end{pmatrix}$$

is invertible, and if it is, we compute its inverse. Using a strategy similar to the one used in Example 5, we attempt to use elementary row operations to transform

$$(A|I) = \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 1 & 5 & 4 & 0 & 0 & 1 \end{array}\right)$$

into a matrix of the form $(I|B)$. We first add -2 times row 1 to row 2 and -1 times row 1 to row 3. We then add row 2 to row 3. The result,

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 1 & 5 & 4 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -3 & -3 & -2 & 1 & 0 \\ 0 & 3 & 3 & -1 & 0 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -3 & -3 & -2 & 1 & 0 \\ 0 & 0 & 0 & -3 & 1 & 1 \end{array}\right),$$

is a matrix with a row whose first 3 entries are zeros. Therefore A is not invertible. ◆

Being able to test for invertibility and compute the inverse of a matrix allows us, with the help of Theorem 2.18 (p. 102) and its corollaries, to test for invertibility and compute the inverse of a linear transformation. The next example demonstrates this technique.

Example 7

Let $\mathsf{T}\colon \mathsf{P}_2(R) \to \mathsf{P}_2(R)$ be defined by $\mathsf{T}(f(x)) = f(x) + f'(x) + f''(x)$, where $f'(x)$ and $f''(x)$ denote the first and second derivatives of $f(x)$. We use Corollary 1 of Theorem 2.18 (p. 103) to test T for invertibility and compute the inverse if T is invertible. Taking β to be the standard ordered basis of $\mathsf{P}_2(R)$, we have

$$[\mathsf{T}]_\beta = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Using the method of Examples 5 and 6, we can show that $[\mathsf{T}]_\beta$ is invertible with inverse

$$([\mathsf{T}]_\beta)^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus T is invertible, and $([\mathsf{T}]_\beta)^{-1} = [\mathsf{T}^{-1}]_\beta$. Hence by Theorem 2.14 (p. 92), we have

$$\begin{aligned} [\mathsf{T}^{-1}(a_0 + a_1x + a_2x^2)]_\beta &= \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \\ &= \begin{pmatrix} a_0 - a_1 \\ a_1 - 2a_2 \\ a_2 \end{pmatrix}. \end{aligned}$$

Therefore

$$\mathsf{T}^{-1}(a_0 + a_1x + a_2x^2) = (a_0 - a_1) + (a_1 - 2a_2)x + a_2x^2. \quad ◆$$

EXERCISES

1. Label the following statements as true or false.

- **(a)** The rank of a matrix is equal to the number of its nonzero columns.
- **(b)** The product of two matrices always has rank equal to the lesser of the ranks of the two matrices.

(c) The $m \times n$ zero matrix is the only $m \times n$ matrix having rank 0.
(d) Elementary row operations preserve rank.
(e) Elementary column operations do not necessarily preserve rank.
(f) The rank of a matrix is equal to the maximum number of linearly independent rows in the matrix.
(g) The inverse of a matrix can be computed exclusively by means of elementary row operations.
(h) The rank of an $n \times n$ matrix is at most n.
(i) An $n \times n$ matrix having rank n is invertible.

2. Find the rank of the following matrices.

(a) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ **(c)** $\begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{pmatrix}$ **(e)** $\begin{pmatrix} 1 & 2 & 3 & 1 & 1 \\ 1 & 4 & 0 & 1 & 2 \\ 0 & 2 & -3 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$

(f) $\begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 2 & 4 & 1 & 3 & 0 \\ 3 & 6 & 2 & 5 & 1 \\ -4 & -8 & 1 & -3 & 1 \end{pmatrix}$ **(g)** $\begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}$

3. Prove that for any $m \times n$ matrix A, $\text{rank}(A) = 0$ if and only if A is the zero matrix.

4. Use elementary row and column operations to transform each of the following matrices into a matrix D satisfying the conditions of Theorem 3.6, and then determine the rank of each matrix.

(a) $\begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 0 & -1 & 2 \\ 1 & 1 & 1 & 2 \end{pmatrix}$ **(b)** $\begin{pmatrix} 2 & 1 \\ -1 & 2 \\ 2 & 1 \end{pmatrix}$

5. For each of the following matrices, compute the rank and the inverse if it exists.

(a) $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ **(c)** $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \\ 2 & 3 & -1 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & -2 & 4 \\ 1 & 1 & -1 \\ 2 & 4 & -5 \end{pmatrix}$ **(e)** $\begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$ **(f)** $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

(g) $\begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 5 & 1 \\ -2 & -3 & 0 & 3 \\ 3 & 4 & -2 & -3 \end{pmatrix}$ **(h)** $\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 2 \\ 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & -3 \end{pmatrix}$

6. For each of the following linear transformations T, determine whether T is invertible, and compute T^{-1} if it exists.

(a) $\mathsf{T}\colon \mathsf{P}_2(R) \to \mathsf{P}_2(R)$ defined by $\mathsf{T}(f(x)) = f''(x) + 2f'(x) - f(x)$.
(b) $\mathsf{T}\colon \mathsf{P}_2(R) \to \mathsf{P}_2(R)$ defined by $\mathsf{T}(f(x)) = (x+1)f'(x)$.
(c) $\mathsf{T}\colon \mathsf{R}^3 \to \mathsf{R}^3$ defined by

$$\mathsf{T}(a_1, a_2, a_3) = (a_1 + 2a_2 + a_3, -a_1 + a_2 + 2a_3, a_1 + a_3).$$

(d) $\mathsf{T}\colon \mathsf{R}^3 \to \mathsf{P}_2(R)$ defined by

$$\mathsf{T}(a_1, a_2, a_3) = (a_1 + a_2 + a_3) + (a_1 - a_2 + a_3)x + a_1x^2.$$

(e) $\mathsf{T}\colon \mathsf{P}_2(R) \to \mathsf{R}^3$ defined by $\mathsf{T}(f(x)) = (f(-1), f(0), f(1))$.
(f) $\mathsf{T}\colon \mathsf{M}_{2\times 2}(R) \to \mathsf{R}^4$ defined by

$$\mathsf{T}(A) = (\mathrm{tr}(A), \mathrm{tr}(A^t), \mathrm{tr}(EA), \mathrm{tr}(AE)),$$

where

$$E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

7. Express the invertible matrix

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

as a product of elementary matrices.

8. Let A be an $m \times n$ matrix. Prove that if c is any nonzero scalar, then $\mathrm{rank}(cA) = \mathrm{rank}(A)$.

9. Complete the proof of the corollary to Theorem 3.4 by showing that elementary column operations preserve rank. Visit `goo.gl/7KgM6F` for a solution.

10. Prove Theorem 3.6 for the case that A is an $m \times 1$ matrix.

11. Let

$$B = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & B' & \\ 0 & & & \end{array}\right),$$

where B' is an $m \times n$ submatrix of B. Prove that if $\mathrm{rank}(B) = r$, then $\mathrm{rank}(B') = r - 1$.

12. Let B' and D' be $m \times n$ matrices, and let B and D be $(m+1) \times (n+1)$ matrices respectively defined by

$$B = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & B' & \\ 0 & & & \end{array}\right) \quad \text{and} \quad D = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & D' & \\ 0 & & & \end{array}\right).$$

Prove that if B' can be transformed into D' by an elementary row [column] operation, then B can be transformed into D by an elementary row [column] operation.

13. Prove (b) and (c) of Corollary 2 to Theorem 3.6.

14. Let $\mathsf{T}, \mathsf{U}\colon \mathsf{V} \to \mathsf{W}$ be linear transformations.

(a) Prove that $\mathsf{R}(\mathsf{T}+\mathsf{U}) \subseteq \mathsf{R}(\mathsf{T}) + \mathsf{R}(\mathsf{U})$. (See the definition of the sum of subsets of a vector space on page 22.)

(b) Prove that if W is finite-dimensional, then $\operatorname{rank}(\mathsf{T}+\mathsf{U}) \leq \operatorname{rank}(\mathsf{T}) + \operatorname{rank}(\mathsf{U})$.

(c) Deduce from (b) that $\operatorname{rank}(A+B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$ for any $m \times n$ matrices A and B.

15. Suppose that A and B are matrices having n rows. Prove that $M(A|B) = (MA|MB)$ for any $m \times n$ matrix M.

16. Supply the details to the proof of (b) of Theorem 3.4.

17. Prove that if B is a 3×1 matrix and C is a 1×3 matrix, then the 3×3 matrix BC has rank at most 1. Conversely, show that if A is any 3×3 matrix having rank 1, then there exist a 3×1 matrix B and a 1×3 matrix C such that $A = BC$.

18. Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Prove that AB can be written as a sum of n matrices of rank at most one.

19. Let A be an $m \times n$ matrix with rank m and B be an $n \times p$ matrix with rank n. Determine the rank of AB. Justify your answer.

20. Let

$$A = \begin{pmatrix} 1 & 0 & -1 & 2 & 1 \\ -1 & 1 & 3 & -1 & 0 \\ -2 & 1 & 4 & -1 & 3 \\ 3 & -1 & -5 & 1 & -6 \end{pmatrix}.$$

(a) Find a 5×5 matrix M with rank 2 such that $AM = O$, where O is the 4×5 zero matrix.

(b) Suppose that B is a 5×5 matrix such that $AB = O$. Prove that $\operatorname{rank}(B) \leq 2$.

21. Let A be an $m \times n$ matrix with rank m. Prove that there exists an $n \times m$ matrix B such that $AB = I_m$.

22. Let B be an $n \times m$ matrix with rank m. Prove that there exists an $m \times n$ matrix A such that $AB = I_m$.

3.3 SYSTEMS OF LINEAR EQUATIONS—THEORETICAL ASPECTS

This section and the next are devoted to the study of systems of linear equations, which arise naturally in both the physical and social sciences. In this section, we apply results from Chapter 2 to describe the solution sets of systems of linear equations as subsets of a vector space. In Section 3.4, we will use elementary row operations to provide a computational method for finding all solutions to such systems.

The system of equations

$$(S) \qquad \begin{aligned} a_{11}x_1 + \ a_{12}x_2 + \cdots + \ a_{1n}x_n &= b_1 \\ a_{21}x_1 + \ a_{22}x_2 + \cdots + \ a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned}$$

where a_{ij} and b_i ($1 \leq i \leq m$ and $1 \leq j \leq n$) are scalars in a field F and $x_1, x_2, \ldots, x_n$ are n variables taking values in F, is called a **system of m linear equations in n unknowns over the field F**.

The $m \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

is called the **coefficient matrix** of the system (S).

If we let

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},$$

then the system (S) may be rewritten as a single matrix equation

$$Ax = b.$$

To exploit the results that we have developed, we often consider a system of linear equations as a single matrix equation.

A **solution** to the system (S) is an n-tuple

$$s = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix} \in \mathsf{F}^n$$

such that $As = b$. The set of all solutions to the system (S) is called the **solution set** of the system. System (S) is called **consistent** if its solution set is nonempty; otherwise it is called **inconsistent**.

Example 1

(a) Consider the system

$$\begin{aligned} x_1 + x_2 &= 3 \\ x_1 - x_2 &= 1. \end{aligned}$$

By use of familiar techniques, we can solve the preceding system and conclude that there is only one solution: $x_1 = 2$, $x_2 = 1$; that is,

$$s = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

In matrix form, the system can be written

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix};$$

so

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

(b) Consider

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 1 \\ x_1 - x_2 + 2x_3 &= 6; \end{aligned}$$

that is,

$$\begin{pmatrix} 2 & 3 & 1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}.$$

This system has many solutions, such as

$$s = \begin{pmatrix} -6 \\ 2 \\ 7 \end{pmatrix} \quad \text{and} \quad s = \begin{pmatrix} 8 \\ -4 \\ -3 \end{pmatrix}.$$

(c) Consider

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_1 + x_2 &= 1; \end{aligned}$$

that is,

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

It is evident that this system has no solutions. Thus we see that a system of linear equations can have one, many, or no solutions. ◆

We must be able to recognize when a system has a solution and then be able to describe all its solutions. This section and the next are devoted to this end.

We begin our study of systems of linear equations by examining the class of *homogeneous* systems of linear equations. Our first result (Theorem 3.8) shows that the set of solutions to a homogeneous system of m linear equations in n unknowns forms a subspace of F^n. We can then apply the theory of vector spaces to this set of solutions. For example, a basis for the solution space can be found, and any solution can be expressed as a linear combination of the vectors in the basis.

Definitions. *A system $Ax = b$ of m linear equations in n unknowns is said to be* ***homogeneous*** *if $b = \mathit{0}$. Otherwise the system is said to be* ***nonhomogeneous****.*

Any homogeneous system has at least one solution, namely, the zero vector. The next result gives further information about the set of solutions to a homogeneous system.

Theorem 3.8. *Let $Ax = \mathit{0}$ be a homogeneous system of m linear equations in n unknowns over a field F. Let K denote the set of all solutions to $Ax = \mathit{0}$. Then $\mathsf{K} = \mathsf{N}(\mathsf{L}_A)$; hence K is a subspace of F^n of dimension $n - \operatorname{rank}(\mathsf{L}_A) = n - \operatorname{rank}(A)$.*

Proof. Clearly, $\mathsf{K} = \{s \in \mathsf{F}^n : As = \mathit{0}\} = \mathsf{N}(\mathsf{L}_A)$. The second part now follows from the dimension theorem (p. 70). ∎

Corollary. *If $m < n$, the system $Ax = \mathit{0}$ has a nonzero solution.*

Proof. Suppose that $m < n$. Then $\operatorname{rank}(A) = \operatorname{rank}(\mathsf{L}_A) \leq m$. Hence

$$\dim(\mathsf{K}) = n - \operatorname{rank}(\mathsf{L}_A) \geq n - m > 0,$$

where $\mathsf{K} = \mathsf{N}(\mathsf{L}_A)$. Since $\dim(\mathsf{K}) > 0$, $\mathsf{K} \neq \{\mathit{0}\}$. Thus there exists a nonzero vector $s \in \mathsf{K}$; so s is a nonzero solution to $Ax = \mathit{0}$. ∎

Example 2

(a) Consider the system

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ x_1 - x_2 - x_3 &= 0. \end{aligned}$$

Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & -1 \end{pmatrix}$$

be the coefficient matrix of this system. It is clear that $\operatorname{rank}(A) = 2$. If K is the solution set of this system, then $\dim(\mathsf{K}) = 3 - 2 = 1$. Thus any nonzero solution constitutes a basis for K. For example, since

$$\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

is a solution to the given system,

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \right\}$$

is a basis for K. Thus any vector in K is of the form

$$t \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} t \\ -2t \\ 3t \end{pmatrix},$$

where $t \in R$.

(b) Consider the system $x_1 - 2x_2 + x_3 = 0$ of one equation in three unknowns. If $A = \begin{pmatrix} 1 & -2 & 1 \end{pmatrix}$ is the coefficient matrix, then $\operatorname{rank}(A) = 1$. Hence if K is the solution set, then $\dim(\mathsf{K}) = 3 - 1 = 2$. Note that

$$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

are linearly independent vectors in K. Thus they constitute a basis for K, so that

$$\mathsf{K} = \left\{ t_1 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} : t_1, t_2 \in R \right\}. \quad \blacklozenge$$

In Section 3.4, explicit computational methods for finding a basis for the solution set of a homogeneous system are discussed.

We now turn to the study of nonhomogeneous systems. Our next result shows that the solution set of a nonhomogeneous system $Ax = b$ can be described in terms of the solution set of the homogeneous system $Ax = \mathit{0}$. We refer to the equation $Ax = \mathit{0}$ as the **homogeneous system corresponding to** $Ax = b$.

Theorem 3.9. *Let K be the solution set of a consistent system of linear equations $Ax = b$, and let $\mathsf{K_H}$ be the solution set of the corresponding homogeneous system $Ax = \mathit{0}$. Then for any solution s to $Ax = b$*

$$K = \{s\} + \mathsf{K_H} = \{s + k\colon k \in \mathsf{K_H}\}.$$

Proof. Let s be any solution to $Ax = b$. We must show that $K = \{s\} + \mathsf{K_H}$. If $w \in K$, then $Aw = b$. Hence

$$A(w - s) = Aw - As = b - b = \mathit{0}.$$

So $w - s \in \mathsf{K_H}$. Thus there exists $k \in \mathsf{K_H}$ such that $w - s = k$. It follows that $w = s + k \in \{s\} + \mathsf{K_H}$, and therefore

$$K \subseteq \{s\} + \mathsf{K_H}.$$

Conversely, suppose that $w \in \{s\} + \mathsf{K_H}$; then $w = s + k$ for some $k \in \mathsf{K_H}$. But then $Aw = A(s + k) = As + Ak = b + \mathit{0} = b$; so $w \in K$. Therefore $\{s\} + \mathsf{K_H} \subseteq K$, and thus $K = \{s\} + \mathsf{K_H}$. ∎

Example 3

(a) Consider the system

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 7 \\ x_1 - x_2 - x_3 &= -4. \end{aligned}$$

The corresponding homogeneous system is the system in Example 2(a). It is easily verified that

$$s = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

is a solution to the preceding nonhomogeneous system. So the solution set of the system is

$$\mathsf{K} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} : t \in R \right\}$$

by Theorem 3.9.

(b) Consider the system $x_1 - 2x_2 + x_3 = 4$. The corresponding homogeneous system is the system in Example 2(b). Since

$$s = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$$

is a solution to the given system, the solution set K can be written as

$$\mathsf{K} = \left\{ \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} : t_1, t_2 \in R \right\}. \quad \blacklozenge$$

The following theorem provides us with a means of computing solutions to certain systems of linear equations.

Theorem 3.10. *Let $Ax = b$ be a system of n linear equations in n unknowns. If A is invertible, then the system has exactly one solution, namely, $A^{-1}b$. Conversely, if the system has exactly one solution, then A is invertible.*

Proof. Suppose that A is invertible. Substituting $A^{-1}b$ into the system, we have $A(A^{-1}b) = (AA^{-1})b = b$. Thus $A^{-1}b$ is a solution. If s is an arbitrary solution, then $As = b$. Multiplying both sides by A^{-1} gives $s = A^{-1}b$. Thus the system has one and only one solution, namely, $A^{-1}b$.

Conversely, suppose that the system has exactly one solution s. Let $\mathsf{K_H}$ denote the solution set for the corresponding homogeneous system $Ax = \mathit{0}$. By Theorem 3.9, $\{s\} = \{s\} + \mathsf{K_H}$. But this is so only if $\mathsf{K_H} = \{\mathit{0}\}$. Thus $\mathsf{N}(\mathsf{L}_A) = \{\mathit{0}\}$, and hence A is invertible. ∎

Example 4

Consider the following system of three linear equations in three unknowns:

$$\begin{aligned} 2x_2 + 4x_3 &= 2 \\ 2x_1 + 4x_2 + 2x_3 &= 3 \\ 3x_1 + 3x_2 + x_3 &= 1. \end{aligned}$$

In Example 5 of Section 3.2, we computed the inverse of the coefficient matrix A of this system. Thus the system has exactly one solution, namely,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1}b = \begin{pmatrix} \frac{1}{8} & -\frac{5}{8} & \frac{3}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} \\ \frac{3}{8} & -\frac{3}{8} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{7}{8} \\ \frac{5}{4} \\ -\frac{1}{8} \end{pmatrix}. \quad \blacklozenge$$

We use this technique for solving systems of linear equations having invertible coefficient matrices in the application that concludes this section.

In Example 1(c), we saw a system of linear equations that has no solutions. We now establish a criterion for determining when a system has solutions. This criterion involves the rank of the coefficient matrix of the system $Ax = b$ and the rank of the matrix $(A|b)$. The matrix $(A|b)$ is called the **augmented matrix of the system** $Ax = b$.

Theorem 3.11. *Let $Ax = b$ be a system of linear equations. Then the system is consistent if and only if* $\text{rank}(A) = \text{rank}(A|b)$.

Proof. To say that $Ax = b$ has a solution is equivalent to saying that $b \in \mathsf{R}(\mathsf{L}_A)$. (See Exercise 8.) In the proof of Theorem 3.5 (p. 153), we saw that

$$\mathsf{R}(\mathsf{L}_A) = \text{span}(\{a_1, a_2, \ldots, a_n\}),$$

the span of the columns of A. Thus $Ax = b$ has a solution if and only if $b \in \text{span}(\{a_1, a_2, \ldots, a_n\})$. But $b \in \text{span}(\{a_1, a_2, \ldots, a_n\})$ if and only if $\text{span}(\{a_1, a_2, \ldots, a_n\}) = \text{span}(\{a_1, a_2, \ldots, a_n, b\})$. This last statement is equivalent to

$$\dim(\text{span}(\{a_1, a_2, \ldots, a_n\})) = \dim(\text{span}(\{a_1, a_2, \ldots, a_n, b\})).$$

So by Theorem 3.5, the preceding equation reduces to

$$\text{rank}(A) = \text{rank}(A|b). \qquad \blacksquare$$

Example 5

Recall the system of equations

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_1 + x_2 &= 1 \end{aligned}$$

in Example 1(c).

Since

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad (A|b) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix},$$

$\text{rank}(A) = 1$ and $\text{rank}(A|b) = 2$. Because the two ranks are unequal, the system has no solutions. ◆

Example 6

We can use Theorem 3.11 to determine whether $(3, 3, 2)$ is in the range of the linear transformation $\mathsf{T}\colon \mathsf{R}^3 \to \mathsf{R}^3$ defined by

$$\mathsf{T}(a_1, a_2, a_3) = (a_1 + a_2 + a_3, a_1 - a_2 + a_3, a_1 + a_3).$$

Now $(3, 3, 2) \in \mathsf{R}(\mathsf{T})$ if and only if there exists a vector $s = (x_1, x_2, x_3)$ in R^3 such that $\mathsf{T}(s) = (3, 3, 2)$. Such a vector s must be a solution to the system

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 - x_2 + x_3 &= 3 \\ x_1 \qquad + x_3 &= 2. \end{aligned}$$

Since the ranks of the coefficient matrix and the augmented matrix of this system are 2 and 3, respectively, it follows that this system has no solutions. Hence $(3, 3, 2) \notin \mathsf{R}(\mathsf{T})$. ♦

An Application

In 1973, Wassily Leontief won the Nobel prize in economics for his work in developing a mathematical model that can be used to describe various economic phenomena. We close this section by applying some of the ideas we have studied to illustrate two special cases of his work.

We begin by considering a simple society composed of three people (industries)—a farmer who grows all the food, a tailor who makes all the clothing, and a carpenter who builds all the housing. We assume that each person sells to and buys from a central pool and that everything produced is consumed. Since no commodities either enter or leave the system, this case is referred to as the **closed model**.

Each of these three individuals consumes all three of the commodities produced in the society. Suppose that the proportion of each of the commodities consumed by each person is given in the following table. Notice that each of the columns of the table must sum to 1.

	Food	Clothing	Housing
Farmer	0.40	0.20	0.20
Tailor	0.10	0.70	0.20
Carpenter	0.50	0.10	0.60

Let $p_1, p_2,$ and p_3 denote the incomes of the farmer, tailor, and carpenter, respectively. To ensure that this society survives, we require that the consumption of each individual equals his or her income. Note that the farmer consumes 20% of the clothing. Because the total cost of all clothing is p_2, the tailor's income, the amount spent by the farmer on clothing is $0.20p_2$. Moreover, the amount spent by the farmer on food, clothing, and housing must equal the farmer's income, and so we obtain the equation

$$0.40p_1 + 0.20p_2 + 0.20p_3 = p_1.$$

Similar equations describing the expenditures of the tailor and carpenter produce the following system of linear equations:

$$\begin{aligned}0.40p_1 + 0.20p_2 + 0.20p_3 &= p_1\\ 0.10p_1 + 0.70p_2 + 0.20p_3 &= p_2\\ 0.50p_1 + 0.10p_2 + 0.60p_3 &= p_3.\end{aligned}$$

This system can be written as $Ap = p$, where

$$p = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

and A is the coefficient matrix of the system. In this context, A is called the **input–output (or consumption) matrix**, and $Ap = p$ is called the **equilibrium condition**.

For vectors $b = (b_1, b_2, \ldots, b_n)$ and $c = (c_1, c_2, \ldots, c_n)$ in R^n, we use the notation $b \geq c$ $[b > c]$ to mean $b_i \geq c_i$ $[b_i > c_i]$ for all i. The vector b is called **nonnegative [positive]** if $b \geq \mathit{0}$ $[b > \mathit{0}]$.

At first, it may seem reasonable to replace the equilibrium condition by the inequality $Ap \leq p$, that is, the requirement that consumption not exceed production. But, in fact, $Ap \leq p$ implies that $Ap = p$ in the closed model. For otherwise, there exists a k for which

$$p_k > \sum_j A_{kj} p_j.$$

Hence, since the columns of A sum to 1,

$$\sum_i p_i > \sum_i \sum_j A_{ij} p_j = \sum_j \left(\sum_i A_{ij} \right) p_j = \sum_j p_j,$$

which is a contradiction.

One solution to the homogeneous system $(I - A)x = \mathit{0}$, which is equivalent to the equilibrium condition, is

$$p = \begin{pmatrix} 0.25 \\ 0.35 \\ 0.40 \end{pmatrix}.$$

We may interpret this to mean that the society survives if the farmer, tailor, and carpenter have incomes in the proportions $25 : 35 : 40$ (or $5 : 7 : 8$).

Notice that we are not simply interested in any nonzero solution to the system, but in one that is nonnegative. Thus we must consider the question of whether the system $(I - A)x = \mathit{0}$ has a nonnegative solution, where A is a matrix with nonnegative entries whose columns sum to 1. A useful theorem in this direction (whose proof may be found in "Applications of Matrices to Economic Models and Social Science Relationships," by Ben Noble, *Proceedings of the Summer Conference for College Teachers on Applied Mathematics*, 1971, CUPM, Berkeley, California) is stated below.

Theorem 3.12. *Let A be an $n \times n$ input–output matrix having the form*

$$A = \begin{pmatrix} B & C \\ D & E \end{pmatrix},$$

where D is a $1 \times (n-1)$ positive vector and C is an $(n-1) \times 1$ positive vector. Then $(I - A)x = 0$ has a one-dimensional solution set that is generated by a nonnegative vector.

Observe that any input–output matrix with all positive entries satisfies the hypothesis of this theorem. The following matrix does also:

$$\begin{pmatrix} 0.75 & 0.50 & 0.65 \\ 0 & 0.25 & 0.35 \\ 0.25 & 0.25 & 0 \end{pmatrix}.$$

In the **open model**, we assume that there is an outside demand for each of the commodities produced. Returning to our simple society, let x_1, x_2, and x_3 be the monetary values of food, clothing, and housing produced with respective outside demands d_1, d_2, and d_3. Let A be the 3×3 matrix such that A_{ij} represents the amount (in a fixed monetary unit such as the dollar) of commodity i required to produce one monetary unit of commodity j. Then the value of the surplus of food in the society is

$$x_1 - (A_{11}x_1 + A_{12}x_2 + A_{13}x_3),$$

that is, the value of food produced minus the value of food consumed while producing the three commodities. The assumption that everything produced is consumed gives us a similar equilibrium condition for the open model, namely, that the surplus of each of the three commodities must equal the corresponding outside demands. Hence

$$x_i - \sum_{j=1}^{3} A_{ij}x_j = d_i \quad \text{for } i = 1, 2, 3.$$

In general, we must find a nonnegative solution to $(I - A)x = d$, where A is a matrix with nonnegative entries such that the sum of the entries of each column of A does not exceed one, and $d \geq 0$. It is easy to see that if $(I - A)^{-1}$ exists and is nonnegative, then the desired solution is $(I - A)^{-1}d$.

Recall that for a real number a, the series $1 + a + a^2 + \cdots$ converges to $(1-a)^{-1}$ if $|a| < 1$. Similarly, it can be shown (using the concept of convergence of matrices developed in Section 5.3) that the series $I + A + A^2 + \cdots$ converges to $(I-A)^{-1}$ if $\{A^n\}$ converges to the zero matrix. In this case, $(I-A)^{-1}$ is nonnegative since the matrices $I, A, A^2, \ldots$ are nonnegative.

To illustrate the open model, suppose that 30 cents worth of food, 10 cents worth of clothing, and 30 cents worth of housing are required for the production of \$1 worth of food. Similarly, suppose that 20 cents worth of food, 40 cents worth of clothing, and 20 cents worth of housing are required for the production of \$1 of clothing. Finally, suppose that 30 cents worth of food, 10 cents worth of clothing, and 30 cents worth of housing are required for the production of \$1 worth of housing. Then the input–output matrix is

$$A = \begin{pmatrix} 0.30 & 0.20 & 0.30 \\ 0.10 & 0.40 & 0.10 \\ 0.30 & 0.20 & 0.30 \end{pmatrix};$$

so

$$I - A = \begin{pmatrix} 0.70 & -0.20 & -0.30 \\ -0.10 & 0.60 & -0.10 \\ -0.30 & -0.20 & 0.70 \end{pmatrix} \quad \text{and} \quad (I-A)^{-1} = \begin{pmatrix} 2.0 & 1.0 & 1.0 \\ 0.5 & 2.0 & 0.5 \\ 1.0 & 1.0 & 2.0 \end{pmatrix}.$$

Since $(I-A)^{-1}$ is nonnegative, we can find a (unique) nonnegative solution to $(I-A)x = d$ for any demand d. For example, suppose that there are outside demands for \$30 billion in food, \$20 billion in clothing, and \$10 billion in housing. If we set

$$d = \begin{pmatrix} 30 \\ 20 \\ 10 \end{pmatrix},$$

then

$$x = (I-A)^{-1}d = \begin{pmatrix} 90 \\ 60 \\ 70 \end{pmatrix}.$$

So a gross production of \$90 billion of food, \$60 billion of clothing, and \$70 billion of housing is necessary to meet the required demands.

EXERCISES

1. Label the following statements as true or false.

- **(a)** Any system of linear equations has at least one solution.
- **(b)** Any system of linear equations has at most one solution.
- **(c)** Any homogeneous system of linear equations has at least one solution.

(d) Any system of n linear equations in n unknowns has at most one solution.

(e) Any system of n linear equations in n unknowns has at least one solution.

(f) If the homogeneous system corresponding to a given system of linear equations has a solution, then the given system has a solution.

(g) If the coefficient matrix of a homogeneous system of n linear equations in n unknowns is invertible, then the system has no nonzero solutions.

(h) The solution set of any system of m linear equations in n unknowns is a subspace of F^n.

2. For each of the following homogeneous systems of linear equations, find the dimension of and a basis for the solution space.

(a) $$\begin{aligned} x_1 + 3x_2 &= 0 \\ 2x_1 + 6x_2 &= 0 \end{aligned}$$

(b) $$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ 4x_1 + x_2 - 2x_3 &= 0 \end{aligned}$$

(c) $$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ 2x_1 + x_2 + x_3 &= 0 \end{aligned}$$

(d) $$\begin{aligned} 2x_1 + x_2 - x_3 &= 0 \\ x_1 - x_2 + x_3 &= 0 \\ x_1 + 2x_2 - 2x_3 &= 0 \end{aligned}$$

(e) $x_1 + 2x_2 - 3x_3 + x_4 = 0$

(f) $$\begin{aligned} x_1 + 2x_2 &= 0 \\ x_1 - x_2 &= 0 \end{aligned}$$

(g) $$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 &= 0 \\ x_2 - x_3 + x_4 &= 0 \end{aligned}$$

3. Using the results of Exercise 2, find all solutions to the following systems.

(a) $$\begin{aligned} x_1 + 3x_2 &= 5 \\ 2x_1 + 6x_2 &= 10 \end{aligned}$$

(b) $$\begin{aligned} x_1 + x_2 - x_3 &= 1 \\ 4x_1 + x_2 - 2x_3 &= 3 \end{aligned}$$

(c) $$\begin{aligned} x_1 + 2x_2 - x_3 &= 3 \\ 2x_1 + x_2 + x_3 &= 6 \end{aligned}$$

(d) $$\begin{aligned} 2x_1 + x_2 - x_3 &= 5 \\ x_1 - x_2 + x_3 &= 1 \\ x_1 + 2x_2 - 2x_3 &= 4 \end{aligned}$$

(e) $x_1 + 2x_2 - 3x_3 + x_4 = 1$

(f) $$\begin{aligned} x_1 + 2x_2 &= 5 \\ x_1 - x_2 &= -1 \end{aligned}$$

(g) $$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 &= 1 \\ x_2 - x_3 + x_4 &= 1 \end{aligned}$$

4. For each system of linear equations with the invertible coefficient matrix A,

(1) Compute A^{-1}.

(2) Use A^{-1} to solve the system.

(a) $\begin{aligned} x_1 + 3x_2 &= 4 \\ 2x_1 + 5x_2 &= 3 \end{aligned}$ **(b)** $\begin{aligned} x_1 + 2x_2 - x_3 &= 5 \\ x_1 + x_2 + x_3 &= 1 \\ 2x_1 - 2x_2 + x_3 &= 4 \end{aligned}$

5. Give an example of a system of n linear equations in n unknowns with infinitely many solutions.

6. Let $\mathsf{T}\colon \mathsf{R}^3 \to \mathsf{R}^2$ be defined by $\mathsf{T}(a,b,c) = (a+b, 2a-c)$. Determine $\mathsf{T}^{-1}(1,11)$.

7. Determine which of the following systems of linear equations has a solution.

(a) $\begin{aligned} x_1 + x_2 - x_3 + 2x_4 &= 2 \\ x_1 + x_2 + 2x_3 &= 1 \\ 2x_1 + 2x_2 + x_3 + 2x_4 &= 4 \end{aligned}$ **(b)** $\begin{aligned} x_1 + x_2 - x_3 &= 1 \\ 2x_1 + x_2 + 3x_3 &= 2 \end{aligned}$

(c) $\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ x_1 + x_2 - x_3 &= 0 \\ x_1 + 2x_2 + x_3 &= 3 \end{aligned}$ **(d)** $\begin{aligned} x_1 + x_2 + 3x_3 - x_4 &= 0 \\ x_1 + x_2 + x_3 + x_4 &= 1 \\ x_1 - 2x_2 + x_3 - x_4 &= 1 \\ 4x_1 + x_2 + 8x_3 - x_4 &= 0 \end{aligned}$

(e) $\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ 2x_1 + x_2 + 2x_3 &= 3 \\ x_1 - 4x_2 + 7x_3 &= 4 \end{aligned}$

8. Let $\mathsf{T}\colon \mathsf{R}^3 \to \mathsf{R}^3$ be defined by $\mathsf{T}(a,b,c) = (a+b, b-2c, a+2c)$. For each vector v in R^3, determine whether $v \in \mathsf{R}(\mathsf{T})$.

(a) $v = (1, 3, -2)$ **(b)** $v = (2, 1, 1)$

9. Prove that the system of linear equations $Ax = b$ has a solution if and only if $b \in \mathsf{R}(\mathsf{L}_A)$. Visit `goo.gl/JfwjBa` for a solution.

10. Prove or give a counterexample to the following statement: If the coefficient matrix of a system of m linear equations in n unknowns has rank m, then the system has a solution.

11. In the closed model of Leontief with food, clothing, and housing as the basic industries, suppose that the input–output matrix is

$$A = \begin{pmatrix} \frac{7}{16} & \frac{1}{2} & \frac{3}{16} \\ \frac{5}{16} & \frac{1}{6} & \frac{5}{16} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}.$$

At what ratio must the farmer, tailor, and carpenter produce in order for equilibrium to be attained?

12. A certain economy consists of two sectors: goods and services. Suppose that 60% of all goods and 30% of all services are used in the production of goods. What proportion of the total economic output is used in the production of goods?

13. In the notation of the open model of Leontief, suppose that

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{5} \end{pmatrix} \quad \text{and} \quad d = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

are the input–output matrix and the demand vector, respectively. How much of each commodity must be produced to satisfy this demand?

14. A certain economy consisting of the two sectors of goods and services supports a defense system that consumes \$90 billion worth of goods and \$20 billion worth of services from the economy but does not contribute to economic production. Suppose that 50 cents worth of goods and 20 cents worth of services are required to produce \$1 worth of goods and that 30 cents worth of goods and 60 cents worth of services are required to produce \$1 worth of services. What must the total output of the economic system be to support this defense system?

3.4 SYSTEMS OF LINEAR EQUATIONS—COMPUTATIONAL ASPECTS

In Section 3.3, we obtained a necessary and sufficient condition for a system of linear equations to have solutions (Theorem 3.11 p. 174) and learned how to express the solutions to a nonhomogeneous system in terms of solutions to the corresponding homogeneous system (Theorem 3.9 p. 172). The latter result enables us to determine all the solutions to a given system if we can find one solution to the given system and a basis for the solution set of the corresponding homogeneous system. In this section, we use elementary row operations to accomplish these two objectives simultaneously. The essence of this technique is to transform a given system of linear equations into a system having the same solutions, but which is easier to solve (as in Section 1.4).

Definition. *Two systems of linear equations are called* ***equivalent*** *if they have the same solution set.*

The following theorem and corollary give a useful method for obtaining equivalent systems.

Theorem 3.13. *Let $Ax = b$ be a system of m linear equations in n unknowns, and let C be an invertible $m \times m$ matrix. Then the system $(CA)x = Cb$ is equivalent to $Ax = b$.*

Proof. Let K be the solution set for $Ax = b$ and K' the solution set for $(CA)x = Cb$. If $w \in K$, then $Aw = b$. So $(CA)w = Cb$, and hence $w \in K'$. Thus $K \subseteq K'$.

Conversely, if $w \in K'$, then $(CA)w = Cb$. Hence

$$Aw = C^{-1}(CAw) = C^{-1}(Cb) = b;$$

so $w \in K$. Thus $K' \subseteq K$, and therefore, $K = K'$. ∎

Corollary. *Let $Ax = b$ be a system of m linear equations in n unknowns. If $(A'|b')$ is obtained from $(A|b)$ by a finite number of elementary row operations, then the system $A'x = b'$ is equivalent to the original system.*

Proof. Suppose that $(A'|b')$ is obtained from $(A|b)$ by elementary row operations. These may be executed by multiplying $(A|b)$ by elementary $m \times m$ matrices $E_1, E_2, \ldots, E_p$. Let $C = E_p \cdots E_2 E_1$; then

$$(A'|b') = C(A|b) = (CA|Cb).$$

Since each E_i is invertible, so is C. Now $A' = CA$ and $b' = Cb$. Thus by Theorem 3.13, the system $A'x = b'$ is equivalent to the system $Ax = b$. ∎

We now describe a method for solving any system of linear equations. Consider, for example, the system of linear equations

$$\begin{aligned} 3x_1 + 2x_2 + 3x_3 - 2x_4 &= 1 \\ x_1 + x_2 + x_3 \phantom{{} - 2x_4} &= 3 \\ x_1 + 2x_2 + x_3 - x_4 &= 2. \end{aligned}$$

First, we form the augmented matrix

$$\left(\begin{array}{cccr|c} 3 & 2 & 3 & -2 & 1 \\ 1 & 1 & 1 & 0 & 3 \\ 1 & 2 & 1 & -1 & 2 \end{array}\right).$$

By using elementary row operations, we transform the augmented matrix into an upper triangular matrix in which the first nonzero entry of each row is 1, and it occurs in a column to the right of the first nonzero entry of each preceding row. (Recall that matrix A is upper triangular if $A_{ij} = 0$ whenever $i > j$.)

1. *In the leftmost nonzero column, create a 1 in the first row.* In our example, we can accomplish this step by interchanging the first and third rows. The resulting matrix is

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 2 \\ 1 & 1 & 1 & 0 & 3 \\ 3 & 2 & 3 & -2 & 1 \end{array}\right).$$

2. *By means of type 3 row operations, use the first row to obtain zeros in the remaining positions of the leftmost nonzero column.* In our example, we must add -1 times the first row to the second row and then add -3 times the first row to the third row to obtain

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 2 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & -4 & 0 & 1 & -5 \end{array}\right).$$

3. *Create a 1 in the next row in the leftmost possible column, without using previous row(s).* In our example, the second column is the leftmost possible column, and we can obtain a 1 in the second row, second column by multiplying the second row by -1. This operation produces

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 2 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & -4 & 0 & 1 & -5 \end{array}\right).$$

4. *Now use type 3 elementary row operations to obtain zeros below the 1 created in the preceding step.* In our example, we must add four times the second row to the third row. The resulting matrix is

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 2 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & -3 & -9 \end{array}\right).$$

5. *Repeat steps 3 and 4 on each succeeding row until no nonzero rows remain.* (This creates zeros below the first nonzero entry in each row.) In our example, this can be accomplished by multiplying the third row by $-\frac{1}{3}$. This operation produces

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 2 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right).$$

We have now obtained the desired matrix. To complete the simplification of the augmented matrix, we must make the first nonzero entry in each row the only nonzero entry in its column. (This corresponds to eliminating certain unknowns from all but one of the equations.)

6. *Work upward, beginning with the last nonzero row, and add multiples of each row to the rows above.* (This creates zeros above the first nonzero entry in each row.) In our example, the third row is the last nonzero row, and the first nonzero entry of this row lies in column 4. Hence we add the third row to the first and second rows to obtain zeros in row 1, column 4 and row 2, column 4. The resulting matrix is

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 5 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right).$$

7. *Repeat the process described in step 6 for each preceding row until it is performed with the second row, at which time the reduction process is complete.* In our example, we must add -2 times the second row to the first row in order to make the first row, second column entry become zero. This operation produces

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right).$$

We have now obtained the desired reduction of the augmented matrix. This matrix corresponds to the system of linear equations

$$\begin{aligned} x_1 + x_3 &= 1 \\ x_2 &= 2 \\ x_4 &= 3. \end{aligned}$$

Recall that, by the corollary to Theorem 3.13, this system is equivalent to the original system. But this system is easily solved. Obviously $x_2 = 2$ and $x_4 = 3$. Moreover, x_1 and x_3 can have any values provided their sum is 1. Letting $x_3 = t$, we then have $x_1 = 1 - t$. Thus an arbitrary solution to the original system has the form

$$\begin{pmatrix} 1-t \\ 2 \\ t \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 3 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Observe that

$$\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

is a basis for the homogeneous system of equations corresponding to the given system.

In the preceding example we performed elementary row operations on the augmented matrix of the system until we obtained the augmented matrix of a system having properties 1, 2, and 3 on page 28. Such a matrix has a special name.

Definition. *A matrix is said to be in* **reduced row echelon form** *if the following three conditions are satisfied.*

(a) *Any row containing a nonzero entry precedes any row in which all the entries are zero (if any).*

(b) *The first nonzero entry in each row is the only nonzero entry in its column.*

(c) *The first nonzero entry in each row is 1 and it occurs in a column to the right of the first nonzero entry in the preceding row.*

Example 1

(a) The matrix on page 184 is in reduced row echelon form. Note that the first nonzero entry of each row is 1 and that the column containing each such entry has all zeros otherwise. Also note that each time we move downward to a new row, we must move to the right one or more columns to find the first nonzero entry of the new row.

(b) The matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

is *not* in reduced row echelon form, because the first column, which contains the first nonzero entry in row 1, contains another nonzero entry. Similarly, the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

is not in reduced row echelon form, because the first nonzero entry of the second row is not to the right of the first nonzero entry of the first row. Finally, the matrix

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

is not in reduced row echelon form, because the first nonzero entry of the first row is not 1. ♦

It can be shown (see the corollary to Theorem 3.16) that the reduced row echelon form of a matrix is unique; that is, if different sequences of elementary row operations are used to transform a matrix into matrices Q and Q' in reduced row echelon form, then $Q = Q'$. Thus, although there are many different sequences of elementary row operations that can be used to transform a given matrix into reduced row echelon form, they all produce the same result.

The procedure described on pages 182–184 for reducing an augmented matrix to reduced row echelon form is called **Gaussian elimination**. It consists of two separate parts.

1. In the *forward pass* (steps 1–5), the augmented matrix is transformed into an upper triangular matrix in which the first nonzero entry of each row is 1, and it occurs in a column to the right of the first nonzero entry of each preceding row.
2. In the *backward pass* or *back-substitution* (steps 6–7), the upper triangular matrix is transformed into reduced row echelon form by making the first nonzero entry of each row the only nonzero entry of its column.

Of all the methods for transforming a matrix into its reduced row echelon form, Gaussian elimination requires the fewest arithmetic operations. (For large matrices, it requires approximately 50% fewer operations than the Gauss-Jordan method, in which the matrix is transformed into reduced row echelon form by using the first nonzero entry in each row to make zero all other entries in its column.) Because of this efficiency, Gaussian elimination is the preferred method when solving systems of linear equations on a computer. In this context, the Gaussian elimination procedure is usually modified in order to minimize roundoff errors. Since discussion of these techniques is inappropriate here, readers who are interested in such matters are referred to books on numerical analysis.

When a matrix is in reduced row echelon form, the corresponding system of linear equations is easy to solve. We present below a procedure for solving any system of linear equations for which the augmented matrix is in reduced row echelon form. First, however, we note that every matrix can be transformed into reduced row echelon form by Gaussian elimination. In the forward pass, we satisfy conditions (a) and (c) in the definition of reduced row echelon form and thereby make zero all entries below the first nonzero entry in each row. Then in the backward pass, we make zero all entries above the first nonzero entry in each row, thereby satisfying condition (b) in the definition of reduced row echelon form.

Theorem 3.14. *Gaussian elimination transforms any matrix into its reduced row echelon form.*

We now describe a method for solving a system in which the augmented matrix is in reduced row echelon form. To illustrate this procedure, we consider the system

$$\begin{aligned} 2x_1 + 3x_2 + x_3 + 4x_4 - 9x_5 &= 17 \\ x_1 + x_2 + x_3 + x_4 - 3x_5 &= 6 \\ x_1 + x_2 + x_3 + 2x_4 - 5x_5 &= 8 \\ 2x_1 + 2x_2 + 2x_3 + 3x_4 - 8x_5 &= 14, \end{aligned}$$

for which the augmented matrix is

$$\left(\begin{array}{ccccc|c} 2 & 3 & 1 & 4 & -9 & 17 \\ 1 & 1 & 1 & 1 & -3 & 6 \\ 1 & 1 & 1 & 2 & -5 & 8 \\ 2 & 2 & 2 & 3 & -8 & 14 \end{array}\right).$$

Applying Gaussian elimination to the augmented matrix of the system produces the following sequence of matrices.

$$\left(\begin{array}{ccccc|c} 2 & 3 & 1 & 4 & -9 & 17 \\ 1 & 1 & 1 & 1 & -3 & 6 \\ 1 & 1 & 1 & 2 & -5 & 8 \\ 2 & 2 & 2 & 3 & -8 & 14 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -3 & 6 \\ 2 & 3 & 1 & 4 & -9 & 17 \\ 1 & 1 & 1 & 2 & -5 & 8 \\ 2 & 2 & 2 & 3 & -8 & 14 \end{array}\right) \longrightarrow$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -3 & 6 \\ 0 & 1 & -1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 1 & -2 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -3 & 6 \\ 0 & 1 & -1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & -1 & 4 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -2 & 3 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The system of linear equations corresponding to this last matrix is

$$\begin{aligned} x_1 \qquad + 2x_3 \qquad - 2x_5 &= 3 \\ x_2 - \ x_3 \qquad + \ x_5 &= 1 \\ x_4 - 2x_5 &= 2. \end{aligned}$$

Notice that we have ignored the last row since it consists entirely of zeros.

To solve a system for which the augmented matrix is in reduced row echelon form, divide the variables into two sets. The first set consists of those variables that appear as leftmost variables in one of the equations of the system (in this case the set is $\{x_1, x_2, x_4\}$). The second set consists of all the remaining variables (in this case, $\{x_3, x_5\}$). To each variable in the second set, assign a parametric value $t_1, t_2, \ldots$ ($x_3 = t_1$, $x_5 = t_2$), and then solve for the variables of the first set in terms of those in the second set:

$$\begin{aligned} x_1 &= -2x_3 + 2x_5 + 3 = -2t_1 + 2t_2 + 3 \\ x_2 &= \quad x_3 - \ x_5 + 1 = \quad t_1 - \ t_2 + 1 \\ x_4 &= \qquad\quad 2x_5 + 2 = \qquad\quad 2t_2 + 2. \end{aligned}$$

Thus an arbitrary solution is of the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2t_1 + 2t_2 + 3 \\ t_1 - t_2 + 1 \\ t_1 \\ 2t_2 + 2 \\ t_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 2 \\ -1 \\ 0 \\ 2 \\ 1 \end{pmatrix},$$

where $t_1, t_2 \in R$. Notice that

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \right\}$$

is a basis for the solution set of the corresponding homogeneous system of equations and

$$\begin{pmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}$$

is a particular solution to the original system.

Therefore, in simplifying the augmented matrix of the system to reduced row echelon form, we are in effect simultaneously finding a particular solution to the original system and a basis for the solution set of the associated homogeneous system. Moreover, this procedure detects when a system is inconsistent, for by Exercise 3, solutions exist if and only if, in the reduction of the augmented matrix to reduced row echelon form, we do not obtain a row in which the only nonzero entry lies in the last column.

Thus to use this procedure for solving a system $Ax = b$ of m linear equations in n unknowns, we need only begin to transform the augmented matrix $(A|b)$ into its reduced row echelon form $(A'|b')$ by means of Gaussian elimination. If a row is obtained in which the only nonzero entry lies in the last column, then the original system is inconsistent. Otherwise, discard any zero rows from $(A'|b')$, and write the corresponding system of equations. Solve this system as described above to obtain an arbitrary solution of the form

$$s = s_0 + t_1u_1 + t_2u_2 + \cdots + t_{n-r}u_{n-r},$$

where r is the number of nonzero rows in A' ($r \leq m$). The preceding equation is called a **general solution** of the system $Ax = b$. It expresses an arbitrary solution s of $Ax = b$ in terms of $n - r$ parameters. The following theorem states that s cannot be expressed in fewer than $n - r$ parameters.

Theorem 3.15. *Let $Ax = b$ be a system of r nonzero equations in n unknowns. Suppose that $\operatorname{rank}(A) = \operatorname{rank}(A|b)$ and that $(A|b)$ is in reduced row echelon form. Then*

(a) $\operatorname{rank}(A) = r$.

(b) *If the general solution obtained by the procedure above is of the form*

$$s = s_0 + t_1u_1 + t_2u_2 + \cdots + t_{n-r}u_{n-r},$$

then $\{u_1, u_2, \ldots, u_{n-r}\}$ is a basis for the solution set of the corresponding homogeneous system, and s_0 is a solution to the original system.

Proof. Since $(A|b)$ is in reduced row echelon form, $(A|b)$ must have r nonzero rows. Clearly these rows are linearly independent by the definition of the reduced row echelon form, and so $\text{rank}(A|b) = r$. Thus $\text{rank}(A) = r$.

Let K be the solution set for $Ax = b$, and let $\mathsf{K_H}$ be the solution set for $Ax = \mathit{0}$. Setting $t_1 = t_2 = \cdots = t_{n-r} = 0$, we see that $s = s_0 \in K$. But by Theorem 3.9 (p. 172), $K = \{s_0\} + \mathsf{K_H}$. Hence

$$\mathsf{K_H} = \{-s_0\} + K = \text{span}(\{u_1, u_2, \ldots, u_{n-r}\}).$$

Because $\text{rank}(A) = r$, we have $\dim(\mathsf{K_H}) = n - r$. Thus since $\dim(\mathsf{K_H}) = n - r$ and $\mathsf{K_H}$ is generated by a set $\{u_1, u_2, \ldots, u_{n-r}\}$ containing at most $n - r$ vectors, we conclude that this set is a basis for $\mathsf{K_H}$. ∎

An Interpretation of the Reduced Row Echelon Form

Let A be an $m \times n$ matrix with columns $a_1, a_2, \ldots, a_n$, and let B be the reduced row echelon form of A. Denote the columns of B by $b_1, b_2, \ldots, b_n$. If the rank of A is r, then the rank of B is also r by the corollary to Theorem 3.4 (p. 152). Because B is in reduced row echelon form, no nonzero row of B can be a linear combination of the other rows of B. Hence B must have exactly r nonzero rows, and if $r \geq 1$, the vectors $e_1, e_2, \ldots, e_r$ must occur among the columns of B. For $i = 1, 2, \ldots, r$, let j_i denote a column number of B such that $b_{j_i} = e_i$. We claim that $a_{j_1}, a_{j_2}, \ldots, a_{j_r}$, the columns of A corresponding to these columns of B, are linearly independent. For suppose that there are scalars $c_1, c_2, \ldots, c_r$ such that

$$c_1 a_{j_1} + c_2 a_{j_2} + \cdots + c_r a_{j_r} = \mathit{0}.$$

Because B can be obtained from A by a sequence of elementary row operations, there exists (as in the proof of the corollary to Theorem 3.13) an invertible $m \times m$ matrix M such that $MA = B$. Multiplying the preceding equation by M yields

$$c_1 M a_{j_1} + c_2 M a_{j_2} + \cdots + c_r M a_{j_r} = \mathit{0}.$$

Since $M a_{j_i} = b_{j_i} = e_i$, it follows that

$$c_1 e_1 + c_2 e_2 + \cdots + c_r e_r = \mathit{0}.$$

Hence $c_1 = c_2 = \cdots = c_r = 0$, proving that the vectors $a_{j_1}, a_{j_2}, \ldots, a_{j_r}$ are linearly independent.

Because B has only r nonzero rows, every column of B has the form

$$\begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_r \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

for scalars $d_1, d_2, \ldots, d_r$. The corresponding column of A must be

$$\begin{aligned} M^{-1}(d_1e_1 + d_2e_2 + \cdots + d_re_r) &= d_1M^{-1}e_1 + d_2M^{-1}e_2 + \cdots + d_rM^{-1}e_r \\ &= d_1M^{-1}b_{j_1} + d_2M^{-1}b_{j_2} + \cdots + d_rM^{-1}b_{j_r} \\ &= d_1a_{j_1} + d_2a_{j_2} + \cdots + d_ra_{j_r}. \end{aligned}$$

The next theorem summarizes these results.

Theorem 3.16. *Let A be an $m \times n$ matrix of rank r, where $r > 0$, and let B be the reduced row echelon form of A. Then*

(a) *The number of nonzero rows in B is r.*
(b) *For each $i = 1, 2, \ldots, r$, there is a column b_{j_i} of B such that $b_{j_i} = e_i$.*
(c) *The columns of A numbered $j_1, j_2, \ldots, j_r$ are linearly independent.*
(d) *For each $k = 1, 2, \ldots, n$, if column k of B is $d_1e_1 + d_2e_2 + \cdots + d_re_r$, then column k of A is $d_1a_{j_1} + d_2a_{j_2} + \cdots + d_ra_{j_r}$.*

Corollary. *The reduced row echelon form of a matrix is unique.*

Proof. Exercise. (See Exercise 15.) ∎

Example 2

Let

$$A = \begin{pmatrix} 2 & 4 & 6 & 2 & 4 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 4 & 8 & 0 & 0 \\ 3 & 6 & 7 & 5 & 9 \end{pmatrix}.$$

The reduced row echelon form of A is

$$B = \begin{pmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since B has three nonzero rows, the rank of A is 3. The first, third, and fifth columns of B are e_1, e_2, and e_3; so Theorem 3.16(c) asserts that the first, third, and fifth columns of A are linearly independent.

Let the columns of A be denoted a_1, a_2, a_3, a_4, and a_5. Because the second column of B is $2e_1$, it follows from Theorem 3.16(d) that $a_2 = 2a_1$, as is easily checked. Moreover, since the fourth column of B is $4e_1 + (-1)e_2$, the same result shows that

$$a_4 = 4a_1 + (-1)a_3. \quad \blacklozenge$$

In Example 6 of Section 1.6, we extracted a basis for R^3 from the generating set

$$S = \{(2,-3,5), (8,-12,20), (1,0,-2), (0,2,-1), (7,2,0)\}.$$

The procedure described there can be streamlined by using Theorem 3.16. We begin by noting that if S were linearly independent, then S would be a basis for R^3. In this case, it is clear that S is linearly dependent because S contains more than $\dim(\mathsf{R}^3) = 3$ vectors. Nevertheless, it is instructive to consider the calculation that is needed to determine whether S is linearly dependent or linearly independent. Recall that S is linearly dependent if there are scalars c_1, c_2, c_3, c_4, and c_5, not all zero, such that

$$c_1(2,-3,5)+c_2(8,-12,20)+c_3(1,0,-2)+c_4(0,2,-1)+c_5(7,2,0) = (0,0,0).$$

Thus S is linearly dependent if and only if the system of linear equations

$$\begin{aligned} 2c_1 + 8c_2 + c_3 \phantom{{}- c_4} + 7c_5 &= 0 \\ -3c_1 - 12c_2 \phantom{{}+2c_3} + 2c_4 + 2c_5 &= 0 \\ 5c_1 + 20c_2 - 2c_3 - c_4 \phantom{{}+7c_5} &= 0 \end{aligned}$$

has a nonzero solution. The augmented matrix of this system of equations is

$$A = \begin{pmatrix} 2 & 8 & 1 & 0 & 7 & 0 \\ -3 & -12 & 0 & 2 & 2 & 0 \\ 5 & 20 & -2 & -1 & 0 & 0 \end{pmatrix},$$

and its reduced row echelon form is

$$B = \begin{pmatrix} 1 & 4 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \end{pmatrix}.$$

Using the technique described earlier in this section, we can find nonzero solutions of the preceding system, confirming that S is linearly dependent. However, Theorem 3.16(c) gives us additional information. Since the first, third, and fourth columns of B are e_1, e_2, and e_3, we conclude that the first, third, and fourth columns of A are linearly independent. But the columns of A other than the last column (which is the zero vector) are vectors in S. Hence

$$\beta = \{(2,-3,5), (1,0,-2), (0,2,-1)\}$$

is a linearly independent subset of S. If follows from (b) of Corollary 2 to the replacement theorem (p. 48) that β is a basis for R^3.

Because every finite-dimensional vector space over F is isomorphic to F^n for some n, a similar approach can be used to reduce any finite generating set to a basis. This technique is illustrated in the next example.

Example 3

The set

$$S = \{2+x+2x^2+3x^3, 4+2x+4x^2+6x^3, 6+3x+8x^2+7x^3, 2+x+5x^3, 4+x+9x^3\}$$

generates a subspace V of $\mathsf{P}_3(R)$. To find a subset of S that is a basis for V, we consider the subset

$$S' = \{(2,1,2,3), (4,2,4,6), (6,3,8,7), (2,1,0,5), (4,1,0,9)\}$$

consisting of the images of the polynomials in S under the standard representation of $\mathsf{P}_3(R)$ with respect to the standard ordered basis. Note that the 4×5 matrix in which the columns are the vectors in S' is the matrix A in Example 2. From the reduced row echelon form of A, which is the matrix B in Example 2, we see that the first, third, and fifth columns of A are linearly independent and the second and fourth columns of A are linear combinations of the first, third, and fifth columns. Hence

$$\{(2,1,2,3), (6,3,8,7), (4,1,0,9)\}$$

is a basis for the subspace of R^4 that is generated by S'. It follows that

$$\{2+x+2x^2+3x^3, 6+3x+8x^2+7x^3, 4+x+9x^3\}$$

is a basis for the subspace V of $\mathsf{P}_3(R)$. ◆

We conclude this section by describing a method for extending a linearly independent subset S of a finite-dimensional vector space V to a basis for V. Recall that this is always possible by (c) of Corollary 2 to the replacement theorem (p. 48). Our approach is based on the replacement theorem and assumes that we can find an explicit basis β for V. Let S' be the ordered set consisting of the vectors in S followed by those in β. Since $\beta \subseteq S'$, the set S' generates V. We can then apply the technique described above to reduce this generating set to a basis for V containing S.

Example 4

Let

$$V = \{(x_1, x_2, x_3, x_4, x_5) \in \mathsf{R}^5 \colon x_1 + 7x_2 + 5x_3 - 4x_4 + 2x_5 = 0\}.$$

It is easily verified that V is a subspace of R^5 and that

$$S = \{(-2,0,0,-1,-1), (1,1,-2,-1,-1), (-5,1,0,1,1)\}$$

is a linearly independent subset of V.

To extend S to a basis for V, we first obtain a basis β for V. To do so, we solve the system of linear equations that defines V. Since in this case V is defined by a single equation, we need only write the equation as

$$x_1 = -7x_2 - 5x_3 + 4x_4 - 2x_5$$

and assign parametric values to x_2, x_3, x_4, and x_5. If $x_2 = t_1$, $x_3 = t_2$, $x_4 = t_3$, and $x_5 = t_4$, then the vectors in V have the form

$$\begin{aligned}(x_1,x_2,x_3,x_4,x_5) &= (-7t_1 - 5t_2 + 4t_3 - 2t_4, t_1, t_2, t_3, t_4)\\ &= t_1(-7,1,0,0,0) + t_2(-5,0,1,0,0) + t_3(4,0,0,1,0) + t_4(-2,0,0,0,1).\end{aligned}$$

Hence

$$\beta = \{(-7,1,0,0,0), (-5,0,1,0,0), (4,0,0,1,0), (-2,0,0,0,1)\}$$

is a basis for V by Theorem 3.15.

The matrix whose columns consist of the vectors in S followed by those in β is

$$\begin{pmatrix} -2 & 1 & -5 & -7 & -5 & 4 & -2 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and its reduced row echelon form is

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & -.5 & 0 & 0 \\ 0 & 0 & 1 & 1 & .5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus

$$\{(-2,0,0,-1,-1), (1,1,-2,-1,-1), (-5,1,0,1,1), (4,0,0,1,0)\}$$

is a basis for V containing S. ◆

EXERCISES

1. Label the following statements as true or false.

(a) If $(A'|b')$ is obtained from $(A|b)$ by a finite sequence of elementary column operations, then the systems $Ax = b$ and $A'x = b'$ are equivalent.

(b) If $(A'|b')$ is obtained from $(A|b)$ by a finite sequence of elementary row operations, then the systems $Ax = b$ and $A'x = b'$ are equivalent.

(c) If A is an $n \times n$ matrix with rank n, then the reduced row echelon form of A is I_n.

(d) Any matrix can be put in reduced row echelon form by means of a finite sequence of elementary row operations.

(e) If $(A|b)$ is in reduced row echelon form, then the system $Ax = b$ is consistent.

(f) Let $Ax = b$ be a system of m linear equations in n unknowns for which the augmented matrix is in reduced row echelon form. If this system is consistent, then the dimension of the solution set of $Ax = \mathit{0}$ is $n - r$, where r equals the number of nonzero rows in A.

(g) If a matrix A is transformed by elementary row operations into a matrix A' in reduced row echelon form, then the number of nonzero rows in A' equals the rank of A.

2. Use Gaussian elimination to solve the following systems of linear equations.

(a)
$$\begin{aligned} x_1 + 2x_2 - x_3 &= -1 \\ 2x_1 + 2x_2 + x_3 &= 1 \\ 3x_1 + 5x_2 - 2x_3 &= -1 \end{aligned}$$

(b)
$$\begin{aligned} x_1 - 2x_2 - x_3 &= 1 \\ 2x_1 - 3x_2 + x_3 &= 6 \\ 3x_1 - 5x_2 \phantom{{}+x_3} &= 7 \\ x_1 \phantom{{}-5x_2} + 5x_3 &= 9 \end{aligned}$$

(c)
$$\begin{aligned} x_1 + 2x_2 \phantom{{}-x_3} + 2x_4 &= 6 \\ 3x_1 + 5x_2 - x_3 + 6x_4 &= 17 \\ 2x_1 + 4x_2 + x_3 + 2x_4 &= 12 \\ 2x_1 \phantom{{}+4x_2} - 7x_3 + 11x_4 &= 7 \end{aligned}$$

(d)
$$\begin{aligned} x_1 - x_2 - 2x_3 + 3x_4 &= -63 \\ 2x_1 - x_2 + 6x_3 + 6x_4 &= -2 \\ -2x_1 + x_2 - 4x_3 - 3x_4 &= 0 \\ 3x_1 - 2x_2 + 9x_3 + 10x_4 &= -5 \end{aligned}$$

(e)
$$\begin{aligned} x_1 - 4x_2 - x_3 + x_4 &= 3 \\ 2x_1 - 8x_2 + x_3 - 4x_4 &= 9 \\ -x_1 + 4x_2 - 2x_3 + 5x_4 &= -6 \end{aligned}$$

(f)
$$\begin{aligned} x_1 + 2x_2 - x_3 + 3x_4 &= 2 \\ 2x_1 + 4x_2 - x_3 + 6x_4 &= 5 \\ x_2 \phantom{{}-x_3} + 2x_4 &= 3 \end{aligned}$$

(g)
$$\begin{aligned} 2x_1 - 2x_2 - x_3 + 6x_4 - 2x_5 &= 1 \\ x_1 - x_2 + x_3 + 2x_4 - x_5 &= 2 \\ 4x_1 - 4x_2 + 5x_3 + 7x_4 - x_5 &= 6 \end{aligned}$$

(h)
$$\begin{aligned} 3x_1 - x_2 + x_3 - x_4 + 2x_5 &= 5 \\ x_1 - x_2 - x_3 - 2x_4 - x_5 &= 2 \\ 5x_1 - 2x_2 + x_3 - 3x_4 + 3x_5 &= 10 \\ 2x_1 - x_2 \qquad - 2x_4 + x_5 &= 5 \end{aligned}$$

(i)
$$\begin{aligned} 3x_1 - x_2 + 2x_3 + 4x_4 + x_5 &= 2 \\ x_1 - x_2 + 2x_3 + 3x_4 + x_5 &= -1 \\ 2x_1 - 3x_2 + 6x_3 + 9x_4 + 4x_5 &= -5 \\ 7x_1 - 2x_2 + 4x_3 + 8x_4 + x_5 &= 6 \end{aligned}$$

(j)
$$\begin{aligned} 2x_1 \qquad + 3x_3 \qquad - 4x_5 &= 5 \\ 3x_1 - 4x_2 + 8x_3 + 3x_4 \qquad &= 8 \\ x_1 - x_2 + 2x_3 + x_4 - x_5 &= 2 \\ -2x_1 + 5x_2 - 9x_3 - 3x_4 - 5x_5 &= -8 \end{aligned}$$

3. Suppose that the augmented matrix of a system $Ax = b$ is transformed into a matrix $(A'|b')$ in reduced row echelon form by a finite sequence of elementary row operations.

(a) Prove that $\text{rank}(A') \neq \text{rank}(A'|b')$ if and only if $(A'|b')$ contains a row in which the only nonzero entry lies in the last column.

(b) Deduce that $Ax = b$ is consistent if and only if $(A'|b')$ contains no row in which the only nonzero entry lies in the last column.

4. For each of the systems that follow, apply Exercise 3 to determine whether the system is consistent. If the system is consistent, find all solutions. Finally, find a basis for the solution set of the corresponding homogeneous system.

(a)
$$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 &= 2 \\ 2x_1 + x_2 + x_3 - x_4 &= 3 \\ x_1 + 2x_2 - 3x_3 + 2x_4 &= 2 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + x_2 - 3x_3 + x_4 &= -2 \\ x_1 + x_2 + x_3 - x_4 &= 2 \\ x_1 + x_2 - x_3 \qquad &= 0 \end{aligned}$$

(c)
$$\begin{aligned} x_1 + x_2 - 3x_3 + x_4 &= 1 \\ x_1 + x_2 + x_3 - x_4 &= 2 \\ x_1 + x_2 - x_3 \qquad &= 0 \end{aligned}$$

5. Let the reduced row echelon form of A be

$$\begin{pmatrix} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & -5 & 0 & -3 \\ 0 & 0 & 0 & 1 & 6 \end{pmatrix}.$$

Determine A if the first, second, and fourth columns of A are

$$\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix},$$

respectively.

6. Let the reduced row echelon form of A be

$$\begin{pmatrix} 1 & -3 & 0 & 4 & 0 & 5 \\ 0 & 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Determine A if the first, third, and sixth columns of A are

$$\begin{pmatrix} 1 \\ -2 \\ -1 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 1 \\ 2 \\ -4 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 3 \\ -9 \\ 2 \\ 5 \end{pmatrix},$$

respectively.

7. It can be shown that the vectors $u_1 = (2, -3, 1)$, $u_2 = (1, 4, -2)$, $u_3 = (-8, 12, -4)$, $u_4 = (1, 37, -17)$, and $u_5 = (-3, -5, 8)$ generate R^3. Find a subset of $\{u_1, u_2, u_3, u_4, u_5\}$ that is a basis for R^3.

8. Let W denote the subspace of R^5 consisting of all vectors having coordinates that sum to zero. The vectors

$$\begin{aligned} u_1 &= (2, -3, 4, -5, 2), & u_2 &= (-6, 9, -12, 15, -6), \\ u_3 &= (3, -2, 7, -9, 1), & u_4 &= (2, -8, 2, -2, 6), \\ u_5 &= (-1, 1, 2, 1, -3), & u_6 &= (0, -3, -18, 9, 12), \\ u_7 &= (1, 0, -2, 3, -2), \quad \text{and} & u_8 &= (2, -1, 1, -9, 7) \end{aligned}$$

generate W. Find a subset of $\{u_1, u_2, \ldots, u_8\}$ that is a basis for W.

9. Let W be the subspace of $\mathsf{M}_{2\times 2}(R)$ consisting of the symmetric 2×2 matrices. The set

$$S = \left\{ \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 9 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \right\}$$

generates W. Find a subset of S that is a basis for W.

10. Let

$$\mathsf{V} = \{(x_1, x_2, x_3, x_4, x_5) \in \mathsf{R}^5 \colon x_1 - 2x_2 + 3x_3 - x_4 + 2x_5 = 0\}.$$

(a) Show that $S = \{(0, 1, 1, 1, 0)\}$ is a linearly independent subset of V.

(b) Extend S to a basis for V.

11. Let V be as in Exercise 10.

(a) Show that $S = \{(1, 2, 1, 0, 0)\}$ is a linearly independent subset of V.

(b) Extend S to a basis for V.

12. Let V denote the set of all solutions to the system of linear equations

$$\begin{aligned} x_1 - x_2 \qquad\;\; + 2x_4 - 3x_5 + \;\, x_6 &= 0 \\ 2x_1 - x_2 - x_3 + 3x_4 - 4x_5 + 4x_6 &= 0. \end{aligned}$$

(a) Show that $S = \{(0, -1, 0, 1, 1, 0), (1, 0, 1, 1, 1, 0)\}$ is a linearly independent subset of V.

(b) Extend S to a basis for V.

13. Let V be as in Exercise 12.

(a) Show that $S = \{(1, 0, 1, 1, 1, 0), (0, 2, 1, 1, 0, 0)\}$ is a linearly independent subset of V.

(b) Extend S to a basis for V.

14. If $(A|b)$ is in reduced row echelon form, prove that A is also in reduced row echelon form.

15. Prove the corollary to Theorem 3.16: The reduced row echelon form of a matrix is unique. Visit `goo.gl/cZVzxM` for a solution.

INDEX OF DEFINITIONS FOR CHAPTER 3

4

Determinants

The determinant, which has played a prominent role in the theory of linear algebra, is a special scalar-valued function defined on the set of square matrices. Although it still has a place in the study of linear algebra and its applications, its role is less central than in former times. Yet no linear algebra book would be complete without a systematic treatment of the determinant, and we present one here. However, the main use of determinants in this book is to compute and establish the properties of eigenvalues, which we discuss in Chapter 5.

Although the determinant is not a linear transformation on $\mathsf{M}_{n\times n}(F)$ for $n > 1$, it does possess a kind of linearity (called *n-linearity*) as well as other properties that are examined in this chapter. In Section 4.1, we consider the determinant on the set of 2×2 matrices and derive its important properties and develop an efficient computational procedure. To illustrate the important role that determinants play in geometry, we also include optional material that explores the applications of the determinant to the study of area and orientation. In Sections 4.2 and 4.3, we extend the definition of the determinant to all square matrices and derive its important properties and develop an efficient computational procedure. For the reader who prefers to treat determinants lightly, Section 4.4 contains the essential properties that are needed in later chapters. Finally, Section 4.5, which is optional, offers an axiomatic approach to determinants by showing how to characterize the determinant in terms of three key properties.

4.1 DETERMINANTS OF ORDER 2

In this section, we define the determinant of a 2×2 matrix and investigate its geometric significance in terms of area and orientation.

Definition. *If*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a 2×2 *matrix with entries from a field* F, *then we define the* **determinant** *of* A, *denoted* $\det(A)$ *or* $|A|$, *to be the scalar* $ad - bc$.

Example 1

For the matrices

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}$$

in $\mathsf{M}_{2\times 2}(R)$, we have

$$\det(A) = 1\cdot 4 - 2\cdot 3 = -2 \quad \text{and} \quad \det(B) = 3\cdot 4 - 2\cdot 6 = 0. \quad \blacklozenge$$

For the matrices A and B in Example 1, we have

$$A + B = \begin{pmatrix} 4 & 4 \\ 9 & 8 \end{pmatrix},$$

and so

$$\det(A+B) = 4\cdot 8 - 4\cdot 9 = -4.$$

Since $\det(A + B) \neq \det(A) + \det(B)$, the function $\det\colon \mathsf{M}_{2\times 2}(R) \to R$ is *not* a linear transformation. Nevertheless, the determinant does possess an important linearity property, which is explained in the following theorem.

Theorem 4.1. *The function* $\det\colon \mathsf{M}_{2\times 2}(F) \to F$ *is a linear function of each row of a* 2×2 *matrix when the other row is held fixed. That is, if* u, v, *and* w *are in* F^2 *and* k *is a scalar, then*

$$\det\begin{pmatrix} u + kv \\ w \end{pmatrix} = \det\begin{pmatrix} u \\ w \end{pmatrix} + k\det\begin{pmatrix} v \\ w \end{pmatrix}$$

and

$$\det\begin{pmatrix} w \\ u + kv \end{pmatrix} = \det\begin{pmatrix} w \\ u \end{pmatrix} + k\det\begin{pmatrix} w \\ v \end{pmatrix}.$$

Proof. Let $u = (a_1, a_2)$, $v = (b_1, b_2)$, and $w = (c_1, c_2)$ be in F^2 and k be a scalar. Then

$$\begin{aligned} \det\begin{pmatrix} u \\ w \end{pmatrix} + k\det\begin{pmatrix} v \\ w \end{pmatrix} &= \det\begin{pmatrix} a_1 & a_2 \\ c_1 & c_2 \end{pmatrix} + k\det\begin{pmatrix} b_1 & b_2 \\ c_1 & c_2 \end{pmatrix} \\ &= (a_1c_2 - a_2c_1) + k(b_1c_2 - b_2c_1) \\ &= (a_1 + kb_1)c_2 - (a_2 + kb_2)c_1 \end{aligned}$$

$$= \det \begin{pmatrix} a_1 + kb_1 & a_2 + kb_2 \\ c_1 & c_2 \end{pmatrix}$$

$$= \det \begin{pmatrix} u + kv \\ w \end{pmatrix}.$$

A similar calculation shows that

$$\det \begin{pmatrix} w \\ u \end{pmatrix} + k \det \begin{pmatrix} w \\ v \end{pmatrix} = \det \begin{pmatrix} w \\ u + kv \end{pmatrix}. \qquad \blacksquare$$

For the 2×2 matrices A and B in Example 1, it is easily checked that A is invertible but B is not. Note that $\det(A) \neq 0$ but $\det(B) = 0$. We now show that this property is true in general.

Theorem 4.2. *Let $A \in \mathsf{M}_{2\times 2}(F)$. Then the determinant of A is nonzero if and only if A is invertible. Moreover, if A is invertible, then*

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}.$$

Proof. If $\det(A) \neq 0$, then we can define a matrix

$$M = \frac{1}{\det(A)} \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}.$$

A straightforward calculation shows that $AM = MA = I$, and so A is invertible and $M = A^{-1}$.

Conversely, suppose that A is invertible. A remark on page 152 shows that the rank of

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

must be 2. Hence $A_{11} \neq 0$ or $A_{21} \neq 0$. If $A_{11} \neq 0$, add $-A_{21}/A_{11}$ times row 1 of A to row 2 to obtain the matrix

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} - \dfrac{A_{12}A_{21}}{A_{11}} \end{pmatrix}.$$

Because elementary row operations are rank-preserving by the corollary to Theorem 3.4 (p. 152), it follows that

$$A_{22} - \frac{A_{12}A_{21}}{A_{11}} \neq 0.$$

Therefore $\det(A) = A_{11}A_{22} - A_{12}A_{21} \neq 0$. On the other hand, if $A_{21} \neq 0$, we see that $\det(A) \neq 0$ by adding $-A_{11}/A_{21}$ times row 2 of A to row 1 and applying a similar argument. Thus, in either case, $\det(A) \neq 0$. $\blacksquare$

In Sections 4.2 and 4.3, we extend the definition of the determinant to $n \times n$ matrices and show that Theorem 4.2 remains true in this more general context. In the remainder of this section, which can be omitted if desired, we explore the geometric significance of the determinant of a 2×2 matrix. In particular, we show the importance of the sign of the determinant in the study of orientation.

The Area of a Parallelogram

By the **angle** between two vectors in R^2, we mean the angle with measure θ $(0 \leq \theta < \pi)$ that is formed by the vectors having the same magnitude and direction as the given vectors but emanating from the origin. (See Figure 4.1.)

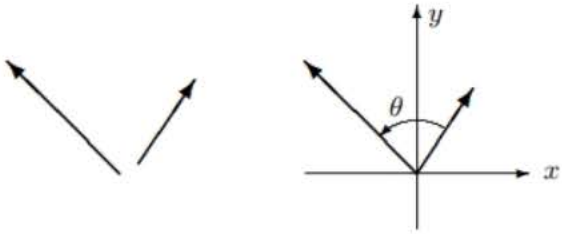


Figure 4.1: Angle between two vectors in R^2

If $\beta = \{u, v\}$ is an ordered basis for R^2, we define the **orientation** of β to be the real number

$$\mathcal{O}\begin{pmatrix} u \\ v \end{pmatrix} = \frac{\det \begin{pmatrix} u \\ v \end{pmatrix}}{\left|\det \begin{pmatrix} u \\ v \end{pmatrix}\right|}.$$

(The denominator of this fraction is nonzero by Theorem 4.2.) Clearly

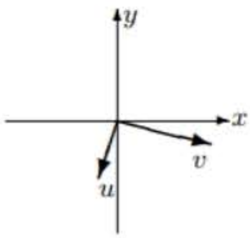
$$\mathcal{O}\begin{pmatrix} u \\ v \end{pmatrix} = \pm 1.$$

Notice that

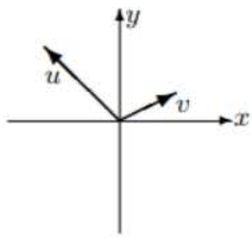
$$\mathcal{O}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = 1 \quad \text{and} \quad \mathcal{O}\begin{pmatrix} e_1 \\ -e_2 \end{pmatrix} = -1.$$

Recall that a coordinate system $\{u, v\}$ is called **right-handed** if u can be rotated in a counterclockwise direction through an angle θ $(0 < \theta < \pi)$ to coincide with v. Otherwise $\{u, v\}$ is called a **left-handed** system. (See Figure 4.2.) In general (see Exercise 12),

$$\mathcal{O}\begin{pmatrix} u \\ v \end{pmatrix} = 1$$



A right-handed coordinate system



A left-handed coordinate system

Figure 4.2

if and only if the ordered basis $\{u, v\}$ forms a right-handed coordinate system. For convenience, we also define

$$\mathcal{O}\begin{pmatrix} u \\ v \end{pmatrix} = 1$$

if $\{u, v\}$ is linearly dependent.

Any ordered set $\{u, v\}$ in R^2 determines a parallelogram in the following manner. Regarding u and v as arrows emanating from the origin of R^2, we call the parallelogram having u and v as adjacent sides the **parallelogram determined by u and v.** (See Figure 4.3.) Observe that if the set $\{u, v\}$

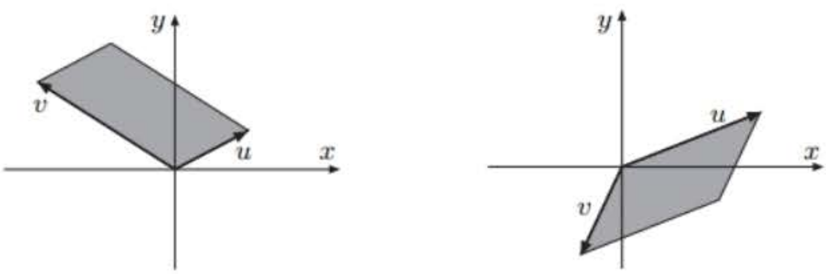


Figure 4.3: Parallelograms determined by u and v

is linearly dependent (i.e., if u and v are parallel), then the "parallelogram" determined by u and v is actually a line segment, which we consider to be a degenerate parallelogram having area zero.

There is an interesting relationship between

$$\mathcal{A}\begin{pmatrix} u \\ v \end{pmatrix},$$

the area of the parallelogram determined by u and v, and

$$\det\begin{pmatrix} u \\ v \end{pmatrix},$$

which we now investigate. Observe first, however, that since

$$\det \begin{pmatrix} u \\ v \end{pmatrix}$$

may be negative, we cannot expect that

$$\mathcal{A} \begin{pmatrix} u \\ v \end{pmatrix} = \det \begin{pmatrix} u \\ v \end{pmatrix}.$$

But we can prove that

$$\mathcal{A} \begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{O} \begin{pmatrix} u \\ v \end{pmatrix} \cdot \det \begin{pmatrix} u \\ v \end{pmatrix},$$

from which it follows that

$$\mathcal{A} \begin{pmatrix} u \\ v \end{pmatrix} = \left| \det \begin{pmatrix} u \\ v \end{pmatrix} \right|.$$

Our argument that

$$\mathcal{A} \begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{O} \begin{pmatrix} u \\ v \end{pmatrix} \cdot \det \begin{pmatrix} u \\ v \end{pmatrix}$$

employs a technique that, although somewhat indirect, can be generalized to R^n. First, since

$$\mathcal{O} \begin{pmatrix} u \\ v \end{pmatrix} = \pm 1,$$

we may multiply both sides of the desired equation by

$$\mathcal{O} \begin{pmatrix} u \\ v \end{pmatrix}$$

to obtain the equivalent form

$$\mathcal{O} \begin{pmatrix} u \\ v \end{pmatrix} \cdot \mathcal{A} \begin{pmatrix} u \\ v \end{pmatrix} = \det \begin{pmatrix} u \\ v \end{pmatrix}.$$

We establish this equation by verifying that the three conditions of Exercise 11 are satisfied by the function

$$\delta \begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{O} \begin{pmatrix} u \\ v \end{pmatrix} \cdot \mathcal{A} \begin{pmatrix} u \\ v \end{pmatrix}.$$

(a) We begin by showing that for any real number c

$$\delta \begin{pmatrix} u \\ cv \end{pmatrix} = c \cdot \delta \begin{pmatrix} u \\ v \end{pmatrix}.$$

Observe that this equation is valid if $c = 0$ because

$$\delta\begin{pmatrix} u \\ cv \end{pmatrix} = \mathcal{O}\begin{pmatrix} u \\ 0 \end{pmatrix} \cdot \mathcal{A}\begin{pmatrix} u \\ 0 \end{pmatrix} = 1 \cdot 0 = 0.$$

So assume that $c \neq 0$. Regarding cv as the base of the parallelogram determined by u and cv, we see that

$$\mathcal{A}\begin{pmatrix} u \\ cv \end{pmatrix} = \text{base} \times \text{altitude} = |c|(\text{length of } v)(\text{altitude}) = |c| \cdot \mathcal{A}\begin{pmatrix} u \\ v \end{pmatrix},$$

since the altitude h of the parallelogram determined by u and cv is the same as that in the parallelogram determined by u and v. (See Figure 4.4.) Hence

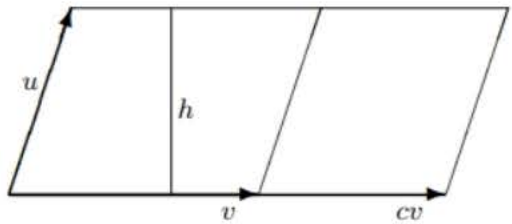


Figure 4.4

$$\delta\begin{pmatrix} u \\ cv \end{pmatrix} = \mathcal{O}\begin{pmatrix} u \\ cv \end{pmatrix} \cdot \mathcal{A}\begin{pmatrix} u \\ cv \end{pmatrix} = \left[\frac{c}{|c|} \cdot \mathcal{O}\begin{pmatrix} u \\ v \end{pmatrix}\right]\left[|c| \cdot \mathcal{A}\begin{pmatrix} u \\ v \end{pmatrix}\right]$$

$$= c \cdot \mathcal{O}\begin{pmatrix} u \\ v \end{pmatrix} \cdot \mathcal{A}\begin{pmatrix} u \\ v \end{pmatrix} = c \cdot \delta\begin{pmatrix} u \\ v \end{pmatrix}.$$

A similar argument shows that

$$\delta\begin{pmatrix} cu \\ v \end{pmatrix} = c \cdot \delta\begin{pmatrix} u \\ v \end{pmatrix}.$$

We next prove that

$$\delta\begin{pmatrix} u \\ au + bw \end{pmatrix} = b \cdot \delta\begin{pmatrix} u \\ w \end{pmatrix}$$

for any $u, w \in \mathsf{R}^2$ and any real numbers a and b. Because the parallelograms determined by u and w and by u and $u + w$ have a common base u and the same altitude (see Figure 4.5), it follows that

$$\mathcal{A}\begin{pmatrix} u \\ w \end{pmatrix} = \mathcal{A}\begin{pmatrix} u \\ u + w \end{pmatrix}.$$

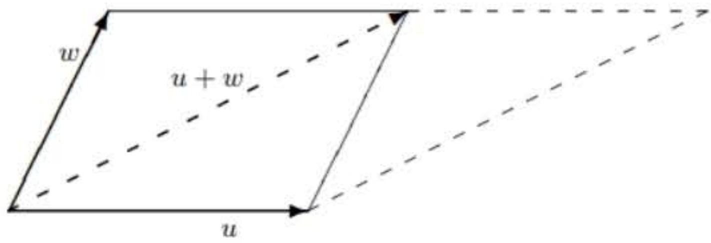


Figure 4.5

If $a = 0$, then

$$\delta\begin{pmatrix} u \\ au + bw \end{pmatrix} = \delta\begin{pmatrix} u \\ bw \end{pmatrix} = b\cdot\delta\begin{pmatrix} u \\ w \end{pmatrix}$$

by the first paragraph of (a). Otherwise, if $a \neq 0$, then

$$\delta\begin{pmatrix} u \\ au + bw \end{pmatrix} = a\cdot\delta\begin{pmatrix} u \\ u + \dfrac{b}{a}w \end{pmatrix} = a\cdot\delta\begin{pmatrix} u \\ \dfrac{b}{a}w \end{pmatrix} = b\cdot\delta\begin{pmatrix} u \\ w \end{pmatrix}.$$

So the desired conclusion is obtained in either case.

We are now able to show that

$$\delta\begin{pmatrix} u \\ v_1 + v_2 \end{pmatrix} = \delta\begin{pmatrix} u \\ v_1 \end{pmatrix} + \delta\begin{pmatrix} u \\ v_2 \end{pmatrix}$$

for all $u, v_1, v_2 \in \mathsf{R}^2$. Since the result is immediate if $u = 0$, we assume that $u \neq 0$. Choose any vector $w \in \mathsf{R}^2$ such that $\{u, w\}$ is linearly independent. Then for any vectors $v_1, v_2 \in \mathsf{R}^2$ there exist scalars a_i and b_i such that $v_i = a_i u + b_i w$ $(i = 1, 2)$. Thus

$$\delta\begin{pmatrix} u \\ v_1 + v_2 \end{pmatrix} = \delta\begin{pmatrix} u \\ (a_1 + a_2)u + (b_1 + b_2)w \end{pmatrix} = (b_1 + b_2)\delta\begin{pmatrix} u \\ w \end{pmatrix}$$

$$= \delta\begin{pmatrix} u \\ a_1 u + b_1 w \end{pmatrix} + \delta\begin{pmatrix} u \\ a_2 u + b_2 w \end{pmatrix} = \delta\begin{pmatrix} u \\ v_1 \end{pmatrix} + \delta\begin{pmatrix} u \\ v_2 \end{pmatrix}.$$

A similar argument shows that

$$\delta\begin{pmatrix} u_1 + u_2 \\ v \end{pmatrix} = \delta\begin{pmatrix} u_1 \\ v \end{pmatrix} + \delta\begin{pmatrix} u_2 \\ v \end{pmatrix}$$

for all $u_1, u_2, v \in \mathsf{R}^2$.

(b) Since

$$\mathcal{A}\begin{pmatrix} u \\ u \end{pmatrix} = 0, \quad \text{it follows that} \quad \delta\begin{pmatrix} u \\ u \end{pmatrix} = \mathcal{O}\begin{pmatrix} u \\ u \end{pmatrix}\cdot\mathcal{A}\begin{pmatrix} u \\ u \end{pmatrix} = 0$$

for any $u \in \mathsf{R}^2$.

(c) Because the parallelogram determined by e_1 and e_2 is the unit square,

$$\delta\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \mathcal{O}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \cdot \mathcal{A}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = 1 \cdot 1 = 1.$$

Therefore δ satisfies the three conditions of Exercise 11, and hence $\delta = \det$. So the area of the parallelogram determined by u and v equals

$$\mathcal{O}\begin{pmatrix} u \\ v \end{pmatrix} \cdot \det\begin{pmatrix} u \\ v \end{pmatrix}.$$

Thus we see, for example, that the area of the parallelogram determined by $u = (-1, 5)$ and $v = (4, -2)$ is

$$\left|\det\begin{pmatrix} u \\ v \end{pmatrix}\right| = \left|\det\begin{pmatrix} -1 & 5 \\ 4 & -2 \end{pmatrix}\right| = 18.$$

EXERCISES

1. Label the following statements as true or false.

(a) The function $\det\colon \mathsf{M}_{2\times 2}(F) \to F$ is a linear transformation.
(b) The determinant of a 2×2 matrix is a linear function of each row of the matrix when the other row is held fixed.
(c) If $A \in \mathsf{M}_{2\times 2}(F)$ and $\det(A) = 0$, then A is invertible.
(d) If u and v are vectors in R^2 emanating from the origin, then the area of the parallelogram having u and v as adjacent sides is

$$\det\begin{pmatrix} u \\ v \end{pmatrix}.$$

(e) A coordinate system is right-handed if and only if its orientation equals 1.

2. Compute the determinants of the following matrices in $\mathsf{M}_{2\times 2}(R)$.

(a) $\begin{pmatrix} 6 & -3 \\ 2 & 4 \end{pmatrix}$ **(b)** $\begin{pmatrix} -5 & 2 \\ 6 & 1 \end{pmatrix}$ **(c)** $\begin{pmatrix} 8 & 0 \\ 3 & -1 \end{pmatrix}$

3. Compute the determinants of the following matrices in $\mathsf{M}_{2\times 2}(C)$.

(a) $\begin{pmatrix} -1+i & 1-4i \\ 3+2i & 2-3i \end{pmatrix}$ **(b)** $\begin{pmatrix} 5-2i & 6+4i \\ -3+i & 7i \end{pmatrix}$ **(c)** $\begin{pmatrix} 2i & 3 \\ 4 & 6i \end{pmatrix}$

4. For each of the following pairs of vectors u and v in R^2, compute the area of the parallelogram determined by u and v.

(a) $u = (3, -2)$ and $v = (2, 5)$

(b) $u = (1, 3)$ and $v = (-3, 1)$
(c) $u = (4, -1)$ and $v = (-6, -2)$
(d) $u = (3, 4)$ and $v = (2, -6)$

5. Prove that if B is the matrix obtained by interchanging the rows of a 2×2 matrix A, then $\det(B) = -\det(A)$.

6. Prove that if the two columns of $A \in \mathsf{M}_{2\times 2}(F)$ are identical, then $\det(A) = 0$.

7. Prove that $\det(A^t) = \det(A)$ for any $A \in \mathsf{M}_{2\times 2}(F)$.

8. Prove that if $A \in \mathsf{M}_{2\times 2}(F)$ is upper triangular, then $\det(A)$ equals the product of the diagonal entries of A.

9. Prove that $\det(AB) = \det(A) \cdot \det(B)$ for any $A, B \in \mathsf{M}_{2\times 2}(F)$.

10. The **classical adjoint** of a 2×2 matrix $A \in \mathsf{M}_{2\times 2}(F)$ is the matrix

$$C = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}.$$

Prove that

(a) $CA = AC = [\det(A)]I$.
(b) $\det(C) = \det(A)$.
(c) The classical adjoint of A^t is C^t.
(d) If A is invertible, then $A^{-1} = [\det(A)]^{-1}C$.

11. Let $\delta\colon \mathsf{M}_{2\times 2}(F) \to F$ be a function with the following three properties.

(i) δ is a linear function of each row of the matrix when the other row is held fixed.

(ii) If the two rows of $A \in \mathsf{M}_{2\times 2}(F)$ are identical, then $\delta(A) = 0$.

(iii) If I is the 2×2 identity matrix, then $\delta(I) = 1$.

(a) Prove that $\delta(E) = \det(E)$ for all elementary matrices $E \in \mathsf{M}_{2\times 2}(F)$.
(b) Prove that $\delta(EA) = \delta(E)\delta(A)$ for all $A \in \mathsf{M}_{2\times 2}(F)$ and all elementary matrices $E \in \mathsf{M}_{2\times 2}(F)$.

12. Let $\delta\colon \mathsf{M}_{2\times 2}(F) \to F$ be a function with properties (i), (ii), and (iii) in Exercise 11. Use Exercise 11 to prove that $\delta(A) = \det(A)$ for all $A \in \mathsf{M}_{2\times 2}(F)$. (This result is generalized in Section 4.5.) Visit `goo.gl/ztwxWA` for a solution.

13. Let $\{u, v\}$ be an ordered basis for R^2. Prove that

$$\mathcal{O}\begin{pmatrix} u \\ v \end{pmatrix} = 1$$

if and only if $\{u, v\}$ forms a right-handed coordinate system. *Hint:* Recall the definition of a rotation given in Example 2 of Section 2.1.

4.2 DETERMINANTS OF ORDER n

In this section, we extend the definition of the determinant to $n \times n$ matrices for $n \geq 3$. For this definition, it is convenient to introduce the following notation: Given $A \in \mathsf{M}_{n\times n}(F)$, for $n \geq 2$, denote the $(n-1)\times(n-1)$ matrix obtained from A by deleting row i and column j by $\tilde{A}_{ij}$. Thus for

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \in \mathsf{M}_{3\times 3}(R),$$

we have

$$\tilde{A}_{11} = \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix}, \qquad \tilde{A}_{13} = \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}, \quad \text{and} \quad \tilde{A}_{32} = \begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix},$$

and for

$$B = \begin{pmatrix} 1 & -1 & 2 & -1 \\ -3 & 4 & 1 & -1 \\ 2 & -5 & -3 & 8 \\ -2 & 6 & -4 & 1 \end{pmatrix} \in \mathsf{M}_{4\times 4}(R),$$

we have

$$\tilde{B}_{23} = \begin{pmatrix} 1 & -1 & -1 \\ 2 & -5 & 8 \\ -2 & 6 & 1 \end{pmatrix} \quad \text{and} \quad \tilde{B}_{42} = \begin{pmatrix} 1 & 2 & -1 \\ -3 & 1 & -1 \\ 2 & -3 & 8 \end{pmatrix}.$$

Definitions. *Let $A \in \mathsf{M}_{n\times n}(F)$. If $n = 1$, so that $A = (A_{11})$, we define $\det(A) = A_{11}$. For $n \geq 2$, we define $\det(A)$ recursively as*

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \cdot \det(\tilde{A}_{1j}).$$

The scalar $\det(A)$ is called the **determinant** *of A and is also denoted by $|A|$. The scalar*

$$(-1)^{i+j} \det(\tilde{A}_{ij})$$

is called the **cofactor** *of the entry of A in row i, column j.*

Letting

$$c_{ij} = (-1)^{i+j} \det(\tilde{A}_{ij})$$

denote the cofactor of the row i, column j entry of A, we can express the formula for the determinant of A as

$$\det(A) = A_{11}c_{11} + A_{12}c_{12} + \cdots + A_{1n}c_{1n}.$$

Thus the determinant of A equals the sum of the products of each entry in row 1 of A multiplied by its cofactor. This formula is called **cofactor expansion along the first row** of A. Note that, for 2×2 matrices, this definition of the determinant of A agrees with the one given in Section 4.1 because

$$\det(A) = A_{11}(-1)^{1+1}\det(\tilde{A}_{11}) + A_{12}(-1)^{1+2}\det(\tilde{A}_{12}) = A_{11}A_{22} - A_{12}A_{21}.$$

Example 1

Let

$$A = \begin{pmatrix} 1 & 3 & -3 \\ -3 & -5 & 2 \\ -4 & 4 & -6 \end{pmatrix} \in \mathsf{M}_{3\times 3}(R).$$

Using cofactor expansion along the first row of A, we obtain

$$\begin{aligned}
\det(A) &= (-1)^{1+1}A_{11}\cdot\det(\tilde{A}_{11}) + (-1)^{1+2}A_{12}\cdot\det(\tilde{A}_{12}) \\
&\qquad + (-1)^{1+3}A_{13}\cdot\det(\tilde{A}_{13}) \\
&= (-1)^2(1)\cdot\det\begin{pmatrix} -5 & 2 \\ 4 & -6 \end{pmatrix} + (-1)^3(3)\cdot\det\begin{pmatrix} -3 & 2 \\ -4 & -6 \end{pmatrix} \\
&\qquad + (-1)^4(-3)\cdot\det\begin{pmatrix} -3 & -5 \\ -4 & 4 \end{pmatrix} \\
&= 1\left[-5(-6) - 2(4)\right] - 3\left[-3(-6) - 2(-4)\right] - 3\left[-3(4) - (-5)(-4)\right] \\
&= 1(22) - 3(26) - 3(-32) \\
&= 40. \quad \blacklozenge
\end{aligned}$$

Example 2

Let

$$B = \begin{pmatrix} 0 & 1 & 3 \\ -2 & -3 & -5 \\ 4 & -4 & 4 \end{pmatrix} \in \mathsf{M}_{3\times 3}(R).$$

Using cofactor expansion along the first row of B, we obtain

$$\begin{aligned}
\det(B) &= (-1)^{1+1}B_{11}\cdot \det(\tilde{B}_{11}) + (-1)^{1+2}B_{12}\cdot \det(\tilde{B}_{12}) \\
&\qquad + (-1)^{1+3}B_{13}\cdot \det(\tilde{B}_{13}) \\
&= (-1)^2(0)\cdot \det\begin{pmatrix} -3 & -5 \\ -4 & 4 \end{pmatrix} + (-1)^3(1)\cdot \det\begin{pmatrix} -2 & -5 \\ 4 & 4 \end{pmatrix} \\
&\qquad + (-1)^4(3)\cdot \det\begin{pmatrix} -2 & -3 \\ 4 & -4 \end{pmatrix} \\
&= 0 - 1\,[-2(4) - (-5)(4)] + 3\,[-2(-4) - (-3)(4)] \\
&= 0 - 1(12) + 3(20) \\
&= 48. \quad \blacklozenge
\end{aligned}$$

Example 3

Let

$$C = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & -3 \\ -2 & -3 & -5 & 2 \\ 4 & -4 & 4 & -6 \end{pmatrix} \in \mathsf{M}_{4\times 4}(R).$$

Using cofactor expansion along the first row of C and the results of Examples 1 and 2, we obtain

$$\begin{aligned}
\det(C) &= (-1)^2(2)\cdot \det(\tilde{C}_{11}) + (-1)^3(0)\cdot \det(\tilde{C}_{12}) \\
&\qquad + (-1)^4(0)\cdot \det(\tilde{C}_{13}) + (-1)^5(1)\cdot \det(\tilde{C}_{14}) \\
&= (-1)^2(2)\cdot \det\begin{pmatrix} 1 & 3 & -3 \\ -3 & -5 & 2 \\ -4 & 4 & -6 \end{pmatrix} + 0 + 0 \\
&\qquad + (-1)^5(1)\cdot \det\begin{pmatrix} 0 & 1 & 3 \\ -2 & -3 & -5 \\ 4 & -4 & 4 \end{pmatrix} \\
&= 2(40) + 0 + 0 - 1(48) \\
&= 32. \quad \blacklozenge
\end{aligned}$$

Example 4

The determinant of the $n \times n$ identity matrix is 1. We prove this assertion by mathematical induction on n. The result is clearly true for the 1×1 identity matrix. Assume that the determinant of the $(n-1) \times (n-1)$ identity matrix is 1 for some $n \geq 2$, and let I denote the $n \times n$ identity matrix. Using cofactor expansion along the first row of I, we obtain

$$\begin{aligned}\det(I) &= (-1)^2(1)\cdot\det(\tilde{I}_{11}) + (-1)^3(0)\cdot\det(\tilde{I}_{12}) + \cdots \\ &\qquad + (-1)^{1+n}(0)\cdot\det(\tilde{I}_{1n}) \\ &= 1(1) + 0 + \cdots + 0 \\ &= 1\end{aligned}$$

because $\tilde{I}_{11}$ is the $(n-1) \times (n-1)$ identity matrix. This shows that the determinant of the $n \times n$ identity matrix is 1, and so the determinant of any identity matrix is 1 by the principle of mathematical induction. ◆

As is illustrated in Example 3, the calculation of a determinant using the recursive definition is extremely tedious, even for matrices as small as 4×4. Later in this section, we present a more efficient method for evaluating determinants, but we must first learn more about them.

Recall from Theorem 4.1 (p. 200) that, although the determinant of a 2×2 matrix is *not* a linear transformation, it is a linear function of each row when the other row is held fixed. We now show that a similar property is true for determinants of any size.

Theorem 4.3. *The determinant of an $n \times n$ matrix is a linear function of each row when the remaining rows are held fixed. That is, for $1 \leq r \leq n$, we have*

$$\det\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u + kv \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} = \det\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} + k\det\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ v \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}$$

whenever k is a scalar and u, v, and each a_i are row vectors in F^n.

Proof. The proof is by mathematical induction on n. The result is immediate if $n = 1$. Assume that for some integer $n \geq 2$ the determinant of any $(n-1) \times (n-1)$ matrix is a linear function of each row when the remaining rows are held fixed. Let A be an $n \times n$ matrix with rows $a_1, a_2, \ldots, a_n$, respectively, and suppose that for some r $(1 \leq r \leq n)$, we have $a_r = u + kv$ for some

$u, v \in \mathsf{F}^n$ and some scalar k. Let $u = (b_1, b_2, \ldots, b_n)$ and $v = (c_1, c_2, \ldots, c_n)$, and let B and C be the matrices obtained from A by replacing row r of A by u and v, respectively. We must prove that $\det(A) = \det(B) + k\det(C)$. We leave the proof of this fact to the reader for the case $r = 1$. For $r > 1$ and $1 \leq j \leq n$, the rows of $\tilde{A}_{1j}$, $\tilde{B}_{1j}$, and $\tilde{C}_{1j}$ are the same except for row $r-1$. Moreover, row $r-1$ of $\tilde{A}_{1j}$ is

$$(b_1 + kc_1, \ldots, b_{j-1} + kc_{j-1}, b_{j+1} + kc_{j+1}, \ldots, b_n + kc_n),$$

which is the sum of row $r-1$ of $\tilde{B}_{1j}$ and k times row $r-1$ of $\tilde{C}_{1j}$. Since $\tilde{B}_{1j}$ and $\tilde{C}_{1j}$ are $(n-1) \times (n-1)$ matrices, we have

$$\det(\tilde{A}_{1j}) = \det(\tilde{B}_{1j}) + k\det(\tilde{C}_{1j})$$

by the induction hypothesis. Thus since $A_{1j} = B_{1j} = C_{1j}$, we have

$$\begin{aligned}
\det(A) &= \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \cdot \det(\tilde{A}_{1j}) \\
&= \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \cdot \left[\det(\tilde{B}_{1j}) + k\det(\tilde{C}_{1j})\right] \\
&= \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \cdot \det(\tilde{B}_{1j}) + k\sum_{j=1}^{n} (-1)^{1+j} A_{1j} \cdot \det(\tilde{C}_{1j}) \\
&= \det(B) + k\det(C).
\end{aligned}$$

This shows that the theorem is true for $n \times n$ matrices, and so the theorem is true for all square matrices by mathematical induction. ∎

Corollary. *If $A \in \mathsf{M}_{n\times n}(F)$ has a row consisting entirely of zeros, then $\det(A) = 0$.*

Proof. See Exercise 24. ∎

The definition of a determinant requires that the determinant of a matrix be evaluated by cofactor expansion along the first row. Our next theorem shows that the determinant of a square matrix can be evaluated by cofactor expansion along *any* row. Its proof requires the following technical result.

Lemma. *Let $B \in \mathsf{M}_{n\times n}(F)$, where $n \geq 2$. If row i of B equals e_k for some k $(1 \leq k \leq n)$, then $\det(B) = (-1)^{i+k}\det(\tilde{B}_{ik})$.*

Proof. The proof is by mathematical induction on n. The lemma is easily proved for $n = 2$. Assume that for some integer $n \geq 3$, the lemma is true for $(n-1) \times (n-1)$ matrices, and let B be an $n \times n$ matrix in which row i of B

equals e_k for some k $(1 \leq k \leq n)$. The result follows immediately from the definition of the determinant if $i = 1$. Suppose therefore that $1 < i \leq n$. For each $j \neq k$ $(1 \leq j \leq n)$, let C_{ij} denote the $(n-2) \times (n-2)$ matrix obtained from B by deleting rows 1 and i and columns j and k. For each j, row $i-1$ of $\tilde{B}_{1j}$ is the following vector in F^{n-1}:

$$\begin{cases} e_{k-1} & \text{if } j < k \\ \mathit{0} & \text{if } j = k \\ e_k & \text{if } j > k. \end{cases}$$

Hence by the induction hypothesis and the corollary to Theorem 4.3, we have

$$\det(\tilde{B}_{1j}) = \begin{cases} (-1)^{(i-1)+(k-1)} \det(C_{ij}) & \text{if } j < k \\ 0 & \text{if } j = k \\ (-1)^{(i-1)+k} \det(C_{ij}) & \text{if } j > k. \end{cases}$$

Therefore

$$\begin{aligned}
\det(B) &= \sum_{j=1}^{n} (-1)^{1+j} B_{1j} \cdot \det(\tilde{B}_{1j}) \\
&= \sum_{j<k} (-1)^{1+j} B_{1j} \cdot \det(\tilde{B}_{1j}) + \sum_{j>k} (-1)^{1+j} B_{1j} \cdot \det(\tilde{B}_{1j}) \\
&= \sum_{j<k} (-1)^{1+j} B_{1j} \cdot \left[(-1)^{(i-1)+(k-1)} \det(C_{ij})\right] \\
&\qquad + \sum_{j>k} (-1)^{1+j} B_{1j} \cdot \left[(-1)^{(i-1)+k} \det(C_{ij})\right] \\
&= (-1)^{i+k} \left[\sum_{j<k} (-1)^{1+j} B_{1j} \cdot \det(C_{ij}) \right. \\
&\qquad \left. + \sum_{j>k} (-1)^{1+(j-1)} B_{1j} \cdot \det(C_{ij})\right].
\end{aligned}$$

Because the expression inside the preceding bracket is the cofactor expansion of $\tilde{B}_{ik}$ along the first row, it follows that

$$\det(B) = (-1)^{i+k} \det(\tilde{B}_{ik}).$$

This shows that the lemma is true for $n \times n$ matrices, and so the lemma is true for all square matrices by mathematical induction. ∎

We are now able to prove that cofactor expansion along any row can be used to evaluate the determinant of a square matrix.

Theorem 4.4. *The determinant of a square matrix can be evaluated by cofactor expansion along any row. That is, if $A \in \mathsf{M}_{n\times n}(F)$, then for any integer i $(1 \le i \le n)$,*

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \cdot \det(\tilde{A}_{ij}).$$

Proof. Cofactor expansion along the first row of A gives the determinant of A by definition. So the result is true if $i = 1$. Fix $i > 1$. Row i of A can be written as $\sum_{j=1}^{n} A_{ij} e_j$. For $1 \le j \le n$, let B_j denote the matrix obtained from A by replacing row i of A by e_j. Then by Theorem 4.3 and the lemma, we have

$$\det(A) = \sum_{j=1}^{n} A_{ij} \cdot \det(B_j) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \cdot \det(\tilde{A}_{ij}). \qquad \blacksquare$$

Corollary. *If $A \in \mathsf{M}_{n\times n}(F)$ has two identical rows, then $\det(A) = 0$.*

Proof. The proof is by mathematical induction on n. We leave the proof of the result to the reader in the case that $n = 2$. Assume that for some integer $n \ge 3$, it is true for $(n-1) \times (n-1)$ matrices, and let rows r and s of $A \in \mathsf{M}_{n\times n}(F)$ be identical for $r \ne s$. Because $n \ge 3$, we can choose an integer i $(1 \le i \le n)$ other than r and s. Now

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \cdot \det(\tilde{A}_{ij})$$

by Theorem 4.4. Since each $\tilde{A}_{ij}$ is an $(n-1) \times (n-1)$ matrix with two identical rows, the induction hypothesis implies that each $\det(\tilde{A}_{ij}) = 0$, and hence $\det(A) = 0$. This completes the proof for $n \times n$ matrices, and so the lemma is true for all square matrices by mathematical induction. $\blacksquare$

It is possible to evaluate determinants more efficiently by combining cofactor expansion with the use of elementary row operations. Before such a process can be developed, we need to learn what happens to the determinant of a matrix if we perform an elementary row operation on that matrix. Theorem 4.3 provides this information for elementary row operations of type 2 (those in which a row is multiplied by a nonzero scalar). Next we turn our attention to elementary row operations of type 1 (those in which two rows are interchanged).

Theorem 4.5. *If $A \in \mathsf{M}_{n\times n}(F)$ and B is a matrix obtained from A by interchanging any two rows of A, then $\det(B) = -\det(A)$.*

Proof. Let the rows of $A \in \mathsf{M}_{n\times n}(F)$ be $a_1, a_2, \ldots, a_n$, and let B be the matrix obtained from A by interchanging rows r and s, where $r < s$. Thus

$$A = \begin{pmatrix} a_1 \\ \vdots \\ a_r \\ \vdots \\ a_s \\ \vdots \\ a_n \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} a_1 \\ \vdots \\ a_s \\ \vdots \\ a_r \\ \vdots \\ a_n \end{pmatrix}.$$

Consider the matrix obtained from A by replacing rows r and s by $a_r + a_s$. By the corollary to Theorem 4.4 and Theorem 4.3, we have

$$\begin{aligned}
0 = \det \begin{pmatrix} a_1 \\ \vdots \\ a_r + a_s \\ \vdots \\ a_r + a_s \\ \vdots \\ a_n \end{pmatrix} &= \det \begin{pmatrix} a_1 \\ \vdots \\ a_r \\ \vdots \\ a_r + a_s \\ \vdots \\ a_n \end{pmatrix} + \det \begin{pmatrix} a_1 \\ \vdots \\ a_s \\ \vdots \\ a_r + a_s \\ \vdots \\ a_n \end{pmatrix} \\
&= \det \begin{pmatrix} a_1 \\ \vdots \\ a_r \\ \vdots \\ a_r \\ \vdots \\ a_n \end{pmatrix} + \det \begin{pmatrix} a_1 \\ \vdots \\ a_r \\ \vdots \\ a_s \\ \vdots \\ a_n \end{pmatrix} + \det \begin{pmatrix} a_1 \\ \vdots \\ a_s \\ \vdots \\ a_r \\ \vdots \\ a_n \end{pmatrix} + \det \begin{pmatrix} a_1 \\ \vdots \\ a_s \\ \vdots \\ a_s \\ \vdots \\ a_n \end{pmatrix} \\
&= 0 + \det(A) + \det(B) + 0.
\end{aligned}$$

Therefore $\det(B) = -\det(A)$. ∎

We now complete our investigation of how an elementary row operation affects the determinant of a matrix by showing that elementary row operations of type 3 do not change the determinant of a matrix.

Theorem 4.6. *Let $A \in \mathsf{M}_{n\times n}(F)$, and let B be a matrix obtained by adding a multiple of one row of A to another row of A. Then $\det(B) = \det(A)$.*

Proof. Suppose that B is the $n \times n$ matrix obtained from A by adding k times row r to row s, where $r \neq s$. Let the rows of A be $a_1, a_2, \ldots, a_n$, and the rows of B be $b_1, b_2, \ldots, b_n$. Then $b_i = a_i$ for $i \neq s$ and $b_s = a_s + ka_r$. Let C be the matrix obtained from A by replacing row s with a_r. Applying Theorem 4.3 to row s of B, we obtain

$$\det(B) = \det(A) + k\det(C) = \det(A)$$

because $\det(C) = 0$ by the corollary to Theorem 4.4. ∎

In Theorem 4.2 (p. 201), we proved that a 2×2 matrix is invertible if and only if its determinant is nonzero. As a consequence of Theorem 4.6, we can prove half of the promised generalization of this result in the following corollary. The converse is proved in the corollary to Theorem 4.7.

Corollary. *If $A \in \mathsf{M}_{n\times n}(F)$ has rank less than n, then* $\det(A) = 0$.

Proof. If the rank of A is less than n, then the rows $a_1, a_2, \ldots, a_n$ of A are linearly dependent. By Exercise 14 of Section 1.5, some row of A, say, row r, is a linear combination of the other rows. So there exist scalars c_i such that

$$a_r = c_1a_1 + \cdots + c_{r-1}a_{r-1} + c_{r+1}a_{r+1} + \cdots + c_na_n.$$

Let B be the matrix obtained from A by adding $-c_i$ times row i to row r for each $i \neq r$. Then row r of B consists entirely of zeros, and so $\det(B) = 0$. But by Theorem 4.6, $\det(B) = \det(A)$. Hence $\det(A) = 0$. ∎

The following rules summarize the effect of an elementary row operation on the determinant of a matrix $A \in \mathsf{M}_{n\times n}(F)$.

(a) If B is a matrix obtained by interchanging any two rows of A, then $\det(B) = -\det(A)$.
(b) If B is a matrix obtained by multiplying a row of A by a nonzero scalar k, then $\det(B) = k\det(A)$.
(c) If B is a matrix obtained by adding a multiple of one row of A to another row of A, then $\det(B) = \det(A)$.

These facts can be used to simplify the evaluation of a determinant. Consider, for instance, the matrix in Example 1:

$$A = \begin{pmatrix} 1 & 3 & -3 \\ -3 & -5 & 2 \\ -4 & 4 & -6 \end{pmatrix}.$$

Adding 3 times row 1 of A to row 2 and 4 times row 1 to row 3, we obtain

$$M = \begin{pmatrix} 1 & 3 & -3 \\ 0 & 4 & -7 \\ 0 & 16 & -18 \end{pmatrix}.$$

Since M was obtained by performing two type 3 elementary row operations on A, we have $\det(A) = \det(M)$. The cofactor expansion of M along the first row gives

$$\det(M) = (-1)^{1+1}(1)\cdot\det(\tilde{M}_{11}) + (-1)^{1+2}(3)\cdot\det(\tilde{M}_{12}) + (-1)^{1+3}(-3)\cdot\det(\tilde{M}_{13}).$$

Both $\tilde{M}_{12}$ and $\tilde{M}_{13}$ have a column consisting entirely of zeros, and so $\det(\tilde{M}_{12}) = \det(\tilde{M}_{13}) = 0$ by the corollary to Theorem 4.6. Hence

$$\begin{aligned}\det(M) &= (-1)^{1+1}(1)\cdot\det(\tilde{M}_{11})\\ &= (-1)^{1+1}(1)\cdot\det\begin{pmatrix}4 & -7\\ 16 & -18\end{pmatrix}\\ &= 1[4(-18)-(-7)(16)] = 40.\end{aligned}$$

Thus with the use of two elementary row operations of type 3, we have reduced the computation of $\det(A)$ to the evaluation of one determinant of a 2×2 matrix.

But we can do even better. If we add -4 times row 2 of M to row 3 (another elementary row operation of type 3), we obtain

$$P = \begin{pmatrix}1 & 3 & -3\\ 0 & 4 & -7\\ 0 & 0 & 10\end{pmatrix}.$$

Evaluating $\det(P)$ by cofactor expansion along the first row, we have

$$\begin{aligned}\det(P) &= (-1)^{1+1}(1)\cdot\det(\tilde{P}_{11})\\ &= (-1)^{1+1}(1)\cdot\det\begin{pmatrix}4 & -7\\ 0 & 10\end{pmatrix} = 1\cdot 4\cdot 10 = 40,\end{aligned}$$

as described earlier. Since $\det(A) = \det(M) = \det(P)$, it follows that $\det(A) = 40$.

The preceding calculation of $\det(P)$ illustrates an important general fact. *The determinant of an upper triangular matrix is the product of its diagonal entries.* (See Exercise 23.) By using elementary row operations of types 1 and 3 only, we can transform any square matrix into an upper triangular matrix, and so we can easily evaluate the determinant of any square matrix. The next two examples illustrate this technique.

Example 5

To evaluate the determinant of the matrix

$$B = \begin{pmatrix}0 & 1 & 3\\ -2 & -3 & -5\\ 4 & -4 & 4\end{pmatrix}$$

in Example 2, we must begin with a row interchange. Interchanging rows 1 and 2 of B produces

$$C = \begin{pmatrix} -2 & -3 & -5 \\ 0 & 1 & 3 \\ 4 & -4 & 4 \end{pmatrix}.$$

By means of a sequence of elementary row operations of type 3, we can transform C into an upper triangular matrix:

$$\begin{pmatrix} -2 & -3 & -5 \\ 0 & 1 & 3 \\ 4 & -4 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} -2 & -3 & -5 \\ 0 & 1 & 3 \\ 0 & -10 & -6 \end{pmatrix} \longrightarrow \begin{pmatrix} -2 & -3 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 24 \end{pmatrix}.$$

Thus $\det(C) = -2 \cdot 1 \cdot 24 = -48$. Since C was obtained from B by an interchange of rows, it follows that

$$\det(B) = -\det(C) = 48. \quad \blacklozenge$$

Example 6

The technique in Example 5 can be used to evaluate the determinant of the matrix

$$C = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & -3 \\ -2 & -3 & -5 & 2 \\ 4 & -4 & 4 & -6 \end{pmatrix}$$

in Example 3. This matrix can be transformed into an upper triangular matrix by means of the following sequence of elementary row operations of type 3:

$$\begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & -3 \\ -2 & -3 & -5 & 2 \\ 4 & -4 & 4 & -6 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & -3 \\ 0 & -3 & -5 & 3 \\ 0 & -4 & 4 & -8 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 4 & -6 \\ 0 & 0 & 16 & -20 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

Thus $\det(C) = 2 \cdot 1 \cdot 4 \cdot 4 = 32$. $\blacklozenge$

Using elementary row operations to evaluate the determinant of a matrix, as illustrated in Example 6, is far more efficient than using cofactor expansion. Consider first the evaluation of a 2×2 matrix. Since

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc,$$

the evaluation of the determinant of a 2×2 matrix requires 2 multiplications (and 1 subtraction). For $n \geq 3$, evaluating the determinant of an $n \times n$ matrix by cofactor expansion along any row expresses the determinant as a sum of n products involving determinants of $(n-1) \times (n-1)$ matrices. Thus in all, the evaluation of the determinant of an $n \times n$ matrix by cofactor expansion along any row requires over $n!$ multiplications, whereas evaluating the determinant of an $n \times n$ matrix by elementary row operations as in Examples 5 and 6 can be shown to require only $(n^3 + 2n - 3)/3$ multiplications. To evaluate the determinant of a 20×20 matrix, which is not large by present standards, cofactor expansion along a row requires over $20! \approx 2.4 \times 10^{18}$ multiplications. Thus it would take a computer performing one billion multiplications per second over 77 years to evaluate the determinant of a 20×20 matrix by this method. By contrast, the method using elementary row operations requires only 2679 multiplications for this calculation and would take the same computer less than three-millionths of a second! It is easy to see why most computer programs for evaluating the determinant of an arbitrary matrix do not use cofactor expansion.

In this section, we have defined the determinant of a square matrix in terms of cofactor expansion along the first row. We then showed that the determinant of a square matrix can be evaluated using cofactor expansion along *any* row. In addition, we showed that the determinant possesses a number of special properties, including properties that enable us to calculate $\det(B)$ from $\det(A)$ whenever B is a matrix obtained from A by means of an elementary row operation. These properties enable us to evaluate determinants much more efficiently. In the next section, we continue this approach to discover additional properties of determinants.

EXERCISES

1. Label the following statements as true or false.

(a) The function $\det\colon \mathsf{M}_{n\times n}(F) \to F$ is a linear transformation.

(b) The determinant of a square matrix can be evaluated by cofactor expansion along any row.

(c) If two rows of a square matrix A are identical, then $\det(A) = 0$.

(d) If B is a matrix obtained from a square matrix A by interchanging any two rows, then $\det(B) = -\det(A)$.

(e) If B is a matrix obtained from a square matrix A by multiplying a row of A by a scalar, then $\det(B) = \det(A)$.

(f) If B is a matrix obtained from a square matrix A by adding k times row i to row j, then $\det(B) = k\det(A)$.

(g) If $A \in \mathsf{M}_{n\times n}(F)$ has rank n, then $\det(A) = 0$.

(h) The determinant of an upper triangular matrix equals the product of its diagonal entries.

2. Find the value of k that satisfies the following equation:

$$\det\begin{pmatrix} 3a_1 & 3a_2 & 3a_3 \\ 3b_1 & 3b_2 & 3b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{pmatrix} = k\det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

3. Find the value of k that satisfies the following equation:

$$\det\begin{pmatrix} 2a_1 & 2a_2 & 2a_3 \\ 3b_1+5c_1 & 3b_2+5c_2 & 3b_3+5c_3 \\ 7c_1 & 7c_2 & 7c_3 \end{pmatrix} = k\det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

4. Find the value of k that satisfies the following equation:

$$\det\begin{pmatrix} b_1+c_1 & b_2+c_2 & b_3+c_3 \\ a_1+c_1 & a_2+c_2 & a_3+c_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{pmatrix} = k\det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

In Exercises 5–12, evaluate the determinant of the given matrix by cofactor expansion along the indicated row.

5. $\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{pmatrix}$
along the first row

6. $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ -1 & 3 & 0 \end{pmatrix}$
along the first row

7. $\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{pmatrix}$
along the second row

8. $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ -1 & 3 & 0 \end{pmatrix}$
along the third row

9. $\begin{pmatrix} 0 & 1+i & 2 \\ -2i & 0 & 1-i \\ 3 & 4i & 0 \end{pmatrix}$
along the third row

10. $\begin{pmatrix} i & 2+i & 0 \\ -1 & 3 & 2i \\ 0 & -1 & 1-i \end{pmatrix}$
along the second row

11. $\begin{pmatrix} 0 & 2 & 1 & 3 \\ 1 & 0 & -2 & 2 \\ 3 & -1 & 0 & 1 \\ -1 & 1 & 2 & 0 \end{pmatrix}$
along the fourth row

12. $\begin{pmatrix} 1 & -1 & 2 & -1 \\ -3 & 4 & 1 & -1 \\ 2 & -5 & -3 & 8 \\ -2 & 6 & -4 & 1 \end{pmatrix}$
along the fourth row

In Exercises 13–22, evaluate the determinant of the given matrix by any legitimate method.

13. $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ **14.** $\begin{pmatrix} 2 & 3 & 4 \\ 5 & 6 & 0 \\ 7 & 0 & 0 \end{pmatrix}$

15. $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ **16.** $\begin{pmatrix} -1 & 3 & 2 \\ 4 & -8 & 1 \\ 2 & 2 & 5 \end{pmatrix}$

17. $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & -5 \\ 6 & -4 & 3 \end{pmatrix}$ **18.** $\begin{pmatrix} 1 & -2 & 3 \\ -1 & 2 & -5 \\ 3 & -1 & 2 \end{pmatrix}$

19. $\begin{pmatrix} i & 2 & -1 \\ 3 & 1+i & 2 \\ -2i & 1 & 4-i \end{pmatrix}$ **20.** $\begin{pmatrix} -1 & 2+i & 3 \\ 1-i & i & 1 \\ 3i & 2 & -1+i \end{pmatrix}$

21. $\begin{pmatrix} 1 & 0 & -2 & 3 \\ -3 & 1 & 1 & 2 \\ 0 & 4 & -1 & 1 \\ 2 & 3 & 0 & 1 \end{pmatrix}$ **22.** $\begin{pmatrix} 1 & -2 & 3 & -12 \\ -5 & 12 & -14 & 19 \\ -9 & 22 & -20 & 31 \\ -4 & 9 & -14 & 15 \end{pmatrix}$

23. Prove that the determinant of an upper triangular matrix is the product of its diagonal entries.

24. Prove the corollary to Theorem 4.3.

25. Prove that $\det(kA) = k^n \det(A)$ for any $A \in \mathsf{M}_{n\times n}(F)$.

26. Let $A \in \mathsf{M}_{n\times n}(F)$. Under what conditions is $\det(-A) = \det(A)$?

27. Prove that if $A \in \mathsf{M}_{n\times n}(F)$ has two identical columns, then $\det(A) = 0$.

28. Compute $\det(E_i)$ if E_i is an elementary matrix of type i.

29.† Prove that if E is an elementary matrix, then $\det(E^t) = \det(E)$. Visit `goo.gl/6ZoU5Z` for a solution.

30. Let the rows of $A \in \mathsf{M}_{n\times n}(F)$ be $a_1, a_2, \ldots, a_n$, and let B be the matrix in which the rows are $a_n, a_{n-1}, \ldots, a_1$. Calculate $\det(B)$ in terms of $\det(A)$.

4.3 PROPERTIES OF DETERMINANTS

In Theorem 3.1, we saw that performing an elementary row operation on a matrix can be accomplished by multiplying the matrix by an elementary matrix. This result is very useful in studying the effects on the determinant of applying a sequence of elementary row operations. Because the determinant of the $n \times n$ identity matrix is 1 (see Example 4 in Section 4.2), we can interpret the statements on page 217 as the following facts about the determinants of elementary matrices.

(a) If E is an elementary matrix obtained by interchanging any two rows of I, then $\det(E) = -1$.
(b) If E is an elementary matrix obtained by multiplying some row of I by the nonzero scalar k, then $\det(E) = k$.
(c) If E is an elementary matrix obtained by adding a multiple of some row of I to another row, then $\det(E) = 1$.

We now apply these facts about determinants of elementary matrices to prove that the determinant is a *multiplicative* function.

Theorem 4.7. *For any $A, B \in \mathsf{M}_{n\times n}(F)$, $\det(AB) = \det(A)\cdot\det(B)$.*

Proof. We begin by establishing the result when A is an elementary matrix. If A is an elementary matrix obtained by interchanging two rows of I, then $\det(A) = -1$. But by Theorem 3.1 (p. 149), AB is a matrix obtained by interchanging two rows of B. Hence by Theorem 4.5 (p. 215), $\det(AB) = -\det(B) = \det(A)\cdot\det(B)$. Similar arguments establish the result when A is an elementary matrix of type 2 or type 3. (See Exercise 18.)

If A is an $n \times n$ matrix with rank less than n, then $\det(A) = 0$ by the corollary to Theorem 4.6 (p. 216). Since $\operatorname{rank}(AB) \le \operatorname{rank}(A) < n$ by Theorem 3.7 (p. 159), we have $\det(AB) = 0$. Thus $\det(AB) = \det(A)\cdot\det(B)$ in this case.

On the other hand, if A has rank n, then A is invertible and hence is the product of elementary matrices (Corollary 3 to Theorem 3.6 p. 158), say, $A = E_m \cdots E_2E_1$. The first paragraph of this proof shows that

$$\begin{aligned}
\det(AB) &= \det(E_m \cdots E_2E_1B)\\
&= \det(E_m)\cdot\det(E_{m-1}\cdots E_2E_1B)\\
&\ \ \vdots\\
&= \det(E_m)\cdot\cdots\cdot\det(E_2)\cdot\det(E_1)\cdot\det(B)\\
&= \det(E_m\cdots E_2E_1)\cdot\det(B)\\
&= \det(A)\cdot\det(B).
\end{aligned}$$ ∎

Corollary. *A matrix $A \in \mathsf{M}_{n\times n}(F)$ is invertible if and only if $\det(A) \neq 0$. Furthermore, if A is invertible, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.*

Proof. If $A \in \mathsf{M}_{n\times n}(F)$ is not invertible, then the rank of A is less than n. So $\det(A) = 0$ by the corollary to Theorem 4.6 (p. 217). On the other hand, if $A \in \mathsf{M}_{n\times n}(F)$ is invertible, then

$$\det(A)\cdot\det(A^{-1}) = \det(AA^{-1}) = \det(I) = 1$$

by Theorem 4.7. Hence $\det(A) \neq 0$ and $\det(A^{-1}) = \dfrac{1}{\det(A)}$. ∎

In our discussion of determinants until now, we have used only the rows of a matrix. For example, the recursive definition of a determinant involved cofactor expansion along a *row*, and the more efficient method developed in Section 4.2 used elementary *row* operations. Our next result shows that the determinants of A and A^t are always equal. Since the rows of A are the columns of A^t, this fact enables us to translate any statement about determinants that involves the rows of a matrix into a corresponding statement that involves its columns.

Theorem 4.8. *For any $A \in \mathsf{M}_{n\times n}(F)$, $\det(A^t) = \det(A)$.*

Proof. If A is not invertible, then $\operatorname{rank}(A) < n$. But $\operatorname{rank}(A^t) = \operatorname{rank}(A)$ by Corollary 2 to Theorem 3.6 (p. 158), and so A^t is not invertible. Thus $\det(A^t) = 0 = \det(A)$ in this case.

On the other hand, if A is invertible, then A is a product of elementary matrices, say $A = E_m \cdots E_2E_1$. Since $\det(E_i) = \det(E_i^t)$ for every i by Exercise 29 of Section 4.2, by Theorem 4.7 we have

$$\begin{aligned}
\det(A^t) &= \det(E_1^tE_2^t\cdots E_m^t)\\
&= \det(E_1^t)\cdot\det(E_2^t)\cdot\cdots\cdot\det(E_m^t)\\
&= \det(E_1)\cdot\det(E_2)\cdot\cdots\cdot\det(E_m)\\
&= \det(E_m)\cdot\cdots\cdot\det(E_2)\cdot\det(E_1)\\
&= \det(E_m\cdots E_2E_1)\\
&= \det(A).
\end{aligned}$$

Thus, in either case, $\det(A^t) = \det(A)$. ∎

Among the many consequences of Theorem 4.8 are that determinants can be evaluated by cofactor expansion along a column, and that elementary column operations can be used as well as elementary row operations in evaluating a determinant. (The effect on the determinant of performing an elementary column operation is the same as the effect of performing the corresponding elementary row operation.) We conclude our discussion of determinant properties with a well-known result that relates determinants to the solutions of certain types of systems of linear equations.

Theorem 4.9 (Cramer's Rule). *Let $Ax = b$ be the matrix form of a system of n linear equations in n unknowns, where $x = (x_1, x_2, \ldots, x_n)^t$.*

If $\det(A) \neq 0$*, then this system has a unique solution, and for each* k *(*$k = 1, 2, \ldots, n$*),*

$$x_k = \frac{\det(M_k)}{\det(A)},$$

where M_k *is the* $n \times n$ *matrix obtained from* A *by replacing column* k *of* A *by* b.

Proof. If $\det(A) \neq 0$, then the system $Ax = b$ has a unique solution by the corollary to Theorem 4.7 and Theorem 3.10 (p. 173). For each integer k ($1 \leq k \leq n$), let a_k denote the kth column of A and X_k denote the matrix obtained from the $n \times n$ identity matrix by replacing column k by x. Then by Theorem 2.13 (p. 91), AX_k is the $n \times n$ matrix whose ith column is

$$Ae_i = a_i \text{ if } i \neq k \qquad \text{and} \qquad Ax = b \text{ if } i = k.$$

Thus $AX_k = M_k$. Evaluating X_k by cofactor expansion along row k produces

$$\det(X_k) = x_k \cdot \det(I_{n-1}) = x_k.$$

Hence by Theorem 4.7,

$$\det(M_k) = \det(AX_k) = \det(A) \cdot \det(X_k) = \det(A) \cdot x_k.$$

Therefore

$$x_k = [\det(A)]^{-1} \cdot \det(M_k). \qquad \blacksquare$$

Example 1

We illustrate Theorem 4.9 by using Cramer's rule to solve the following system of linear equations:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 2 \\ x_1 \qquad\quad + \ x_3 &= 3 \\ x_1 + \ x_2 - \ x_3 &= 1. \end{aligned}$$

The matrix form of this system of linear equations is $Ax = b$, where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}.$$

Because $\det(A) = 6 \neq 0$, Cramer's rule applies. Using the notation of Theorem 4.9, we have

$$x_1 = \frac{\det(M_1)}{\det(A)} = \frac{\det \begin{pmatrix} 2 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}}{\det(A)} = \frac{15}{6} = \frac{5}{2},$$

$$x_2 = \frac{\det(M_2)}{\det(A)} = \frac{\det \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 1 & -1 \end{pmatrix}}{\det(A)} = \frac{-6}{6} = -1,$$

and

$$x_3 = \frac{\det(M_3)}{\det(A)} = \frac{\det \begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 3 \\ 1 & 1 & 1 \end{pmatrix}}{\det(A)} = \frac{3}{6} = \frac{1}{2}.$$

Thus the unique solution to the given system of linear equations is

$$(x_1, x_2, x_3) = \left(\frac{5}{2}, -1, \frac{1}{2}\right). \quad \blacklozenge$$

In applications involving systems of linear equations, we sometimes need to know that there is a solution in which the unknowns are integers. In this situation, Cramer's rule can be useful because it implies that a system of linear equations with integral coefficients has an integral solution if the determinant of its coefficient matrix is ± 1. On the other hand, Cramer's rule is not useful for computation because it requires evaluating $n+1$ determinants of $n \times n$ matrices to solve a system of n linear equations in n unknowns. The amount of computation to do this is far greater than that required to solve the system by the method of Gaussian elimination, which was discussed in Section 3.4. Thus Cramer's rule is primarily of theoretical and aesthetic interest, rather than of computational value.

As in Section 4.1, it is possible to interpret the determinant of a matrix $A \in \mathsf{M}_{n\times n}(R)$ geometrically. If the rows of A are $a_1, a_2, \ldots, a_n$, respectively, then $|\det(A)|$ is the ***n*-dimensional volume** (the generalization of area in R^2 and volume in R^3) of the parallelepiped having the vectors $a_1, a_2, \ldots, a_n$ as adjacent sides. (For a proof of a more generalized result, see Jerrold E. Marsden and Michael J. Hoffman, *Elementary Classical Analysis*, W.H. Freeman and Company, New York, 1993, p. 524.)

Example 2

The volume of the parallelepiped having the vectors $a_1 = (1, -2, 1)$, $a_2 = (1, 0, -1)$, and $a_3 = (1, 1, 1)$ as adjacent sides is

$$\left| \det \begin{pmatrix} 1 & -2 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} \right| = 6.$$

Note that the object in question is a rectangular parallelepiped (see Figure 4.6) with sides of lengths $\sqrt{6}$, $\sqrt{2}$, and $\sqrt{3}$. Hence by the familiar formula for volume, its volume should be $\sqrt{6}\cdot\sqrt{2}\cdot\sqrt{3} = 6$, as the determinant calculation shows. $\blacklozenge$

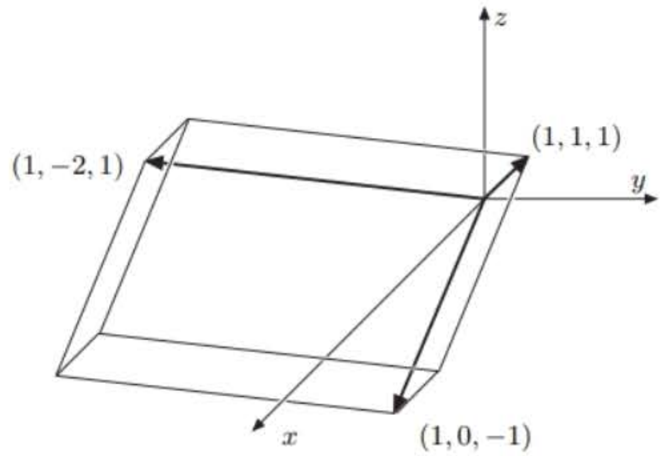


Figure 4.6: Parallelepiped determined by three vectors in R^3.

In our earlier discussion of the geometric significance of the determinant formed from the vectors in an ordered basis for R^2, we also saw that this determinant is positive if and only if the basis induces a right-handed coordinate system. A similar statement is true in R^n. Specifically, if γ is any ordered basis for R^n and β is the standard ordered basis for R^n, then γ induces a *right-handed coordinate system* if and only if $\det(Q) > 0$, where Q is the change of coordinate matrix changing γ-coordinates into β-coordinates. Thus, for instance,

$$\gamma = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

induces a left-handed coordinate system in R^3 because

$$\det \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -2 < 0,$$

whereas

$$\gamma' = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

induces a right-handed coordinate system in R^3 because

$$\det \begin{pmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 5 > 0.$$

More generally, if β and γ are two ordered bases for R^n, then the coordinate systems induced by β and γ have the same **orientation** (either both are right-handed or both are left-handed) if and only if $\det(Q) > 0$, where Q is the change of coordinate matrix changing γ-coordinates into β-coordinates.

EXERCISES

1. Label the following statements as true or false.

(a) If E is an elementary matrix, then $\det(E) = \pm 1$.
(b) For any $A, B \in \mathsf{M}_{n\times n}(F)$, $\det(AB) = \det(A) \cdot \det(B)$.
(c) A matrix $M \in \mathsf{M}_{n\times n}(F)$ is invertible if and only if $\det(M) = 0$.
(d) A matrix $M \in \mathsf{M}_{n\times n}(F)$ has rank n if and only if $\det(M) \neq 0$.
(e) For any $A \in \mathsf{M}_{n\times n}(F)$, $\det(A^t) = -\det(A)$.
(f) The determinant of a square matrix can be evaluated by cofactor expansion along any column.
(g) Every system of n linear equations in n unknowns can be solved by Cramer's rule.
(h) Let $Ax = b$ be the matrix form of a system of n linear equations in n unknowns, where $x = (x_1, x_2, \ldots, x_n)^t$. If $\det(A) \neq 0$ and if M_k is the $n \times n$ matrix obtained from A by replacing row k of A by b^t, then the unique solution of $Ax = b$ is

$$x_k = \frac{\det(M_k)}{\det(A)} \quad \text{for } k = 1, 2, \ldots, n.$$

In Exercises 2–7, use Cramer's rule to solve the given system of linear equations.

2. $$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$
where $a_{11}a_{22} - a_{12}a_{21} \neq 0$

3. $$\begin{aligned} 2x_1 + x_2 - 3x_3 &= 5 \\ x_1 - 2x_2 + x_3 &= 10 \\ 3x_1 + 4x_2 - 2x_3 &= 0 \end{aligned}$$

4. $$\begin{aligned} 2x_1 + x_2 - 3x_3 &= 1 \\ x_1 - 2x_2 + x_3 &= 0 \\ 3x_1 + 4x_2 - 2x_3 &= -5 \end{aligned}$$

5. $$\begin{aligned} x_1 - x_2 + 4x_3 &= -4 \\ -8x_1 + 3x_2 + x_3 &= 8 \\ 2x_1 - x_2 + x_3 &= 0 \end{aligned}$$

6. $$\begin{aligned} x_1 - x_2 + 4x_3 &= -2 \\ -8x_1 + 3x_2 + x_3 &= 0 \\ 2x_1 - x_2 + x_3 &= 6 \end{aligned}$$

7. $$\begin{aligned} 3x_1 + x_2 + x_3 &= 4 \\ -2x_1 - x_2 \phantom{{}+x_3} &= 12 \\ x_1 + 2x_2 + x_3 &= -8 \end{aligned}$$

8. Use Theorem 4.8 to prove a result analogous to Theorem 4.3 (p. 212), but for columns.

9. Prove that an upper triangular $n \times n$ matrix is invertible if and only if all its diagonal entries are nonzero.

10. A matrix $M \in \mathsf{M}_{n\times n}(F)$ is called **nilpotent** if, for some positive integer k, $M^k = O$, where O is the $n \times n$ zero matrix. Prove that if M is nilpotent, then $\det(M) = 0$.

11. A matrix $M \in \mathsf{M}_{n\times n}(C)$ is called **skew-symmetric** if $M^t = -M$. Prove that if M is skew-symmetric and n is odd, then M is not invertible. What happens if n is even?

12. A matrix $Q \in \mathsf{M}_{n\times n}(R)$ is called **orthogonal** if $QQ^t = I$. Prove that if Q is orthogonal, then $\det(Q) = \pm 1$.

13. For $M \in \mathsf{M}_{n\times n}(C)$, let $\overline{M}$ be the matrix such that $(\overline{M})_{ij} = \overline{M_{ij}}$ for all i, j, where $\overline{M_{ij}}$ is the complex conjugate of M_{ij}.

(**a**) Prove that $\det(\overline{M}) = \overline{\det(M)}$.

(**b**) A matrix $Q \in \mathsf{M}_{n\times n}(C)$ is called **unitary** if $QQ^* = I$, where $Q^* = \overline{Q^t}$. Prove that if Q is a unitary matrix, then $|\det(Q)| = 1$.

14. Let $\beta = \{u_1, u_2, \ldots, u_n\}$ be a subset of F^n containing n distinct vectors, and let B be the matrix in $\mathsf{M}_{n\times n}(F)$ having u_j as column j. Prove that β is a basis for F^n if and only if $\det(B) \neq 0$.

15.† Prove that if $A, B \in \mathsf{M}_{n\times n}(F)$ are similar, then $\det(A) = \det(B)$.

16. Use determinants to prove that if $A, B \in \mathsf{M}_{n\times n}(F)$ are such that $AB = I$, then A is invertible (and hence $B = A^{-1}$).

17. Let $A, B \in \mathsf{M}_{n\times n}(F)$ be such that $AB = -BA$. Prove that if n is odd and F is not a field of characteristic two, then A or B is not invertible.

18. Complete the proof of Theorem 4.7 by showing that if A is an elementary matrix of type 2 or type 3, then $\det(AB) = \det(A) \cdot \det(B)$.

19. A matrix $A \in \mathsf{M}_{n\times n}(F)$ is called **lower triangular** if $A_{ij} = 0$ for $1 \leq i < j \leq n$. Suppose that A is a lower triangular matrix. Describe $\det(A)$ in terms of the entries of A.

20. Suppose that $M \in \mathsf{M}_{n\times n}(F)$ can be written in the form

$$M = \begin{pmatrix} A & B \\ O & I \end{pmatrix},$$

where A is a square matrix. Prove that $\det(M) = \det(A)$.

21.† Prove that if $M \in \mathsf{M}_{n\times n}(F)$ can be written in the form

$$M = \begin{pmatrix} A & B \\ O & C \end{pmatrix},$$

where A and C are square matrices, then $\det(M) = \det(A) \cdot \det(C)$. Visit `goo.gl/4sG3iv` for a solution.

22. Let $\mathsf{T}\colon \mathsf{P}_n(F) \to \mathsf{F}^{n+1}$ be the linear transformation defined in Exercise 22 of Section 2.4 by $\mathsf{T}(f) = (f(c_0), f(c_1), \ldots, f(c_n))$, where $c_0, c_1, \ldots, c_n$ are distinct scalars in an infinite field F. Let β be the standard ordered basis for $\mathsf{P}_n(F)$ and γ be the standard ordered basis for F^{n+1}.

(a) Show that $M = [\mathsf{T}]_\beta^\gamma$ has the form

$$\begin{pmatrix} 1 & c_0 & c_0^2 & \cdots & c_0^n \\ 1 & c_1 & c_1^2 & \cdots & c_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & c_n & c_n^2 & \cdots & c_n^n \end{pmatrix}.$$

A matrix with this form is called a **Vandermonde matrix**.

(b) Use Exercise 22 of Section 2.4 to prove that $\det(M) \neq 0$.

(c) Prove that

$$\det(M) = \prod_{0 \le i < j \le n} (c_j - c_i),$$

the product of all terms of the form $c_j - c_i$ for $0 \le i < j \le n$.

23. Let $A \in \mathsf{M}_{n\times n}(F)$ be nonzero. For any m $(1 \le m \le n)$, an $m \times m$ **submatrix** is obtained by deleting any $n - m$ rows and any $n - m$ columns of A.

(a) Let k $(1 \le k \le n)$ denote the largest integer such that some $k \times k$ submatrix has a nonzero determinant. Prove that $\operatorname{rank}(A) = k$.

(b) Conversely, suppose that $\operatorname{rank}(A) = k$. Prove that there exists a $k \times k$ submatrix with a nonzero determinant.

24. Let $A \in \mathsf{M}_{n\times n}(F)$ have the form

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & a_0 \\ -1 & 0 & 0 & \cdots & 0 & a_1 \\ 0 & -1 & 0 & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & a_{n-1} \end{pmatrix}.$$

Compute $\det(A + tI)$, where I is the $n \times n$ identity matrix.

25. Let c_{jk} denote the cofactor of the row j, column k entry of the matrix $A \in \mathsf{M}_{n\times n}(F)$.

(a) Prove that if B is the matrix obtained from A by replacing column k by e_j, then $\det(B) = c_{jk}$.

(b) Show that for $1 \le j \le n$, we have

$$A\begin{pmatrix} c_{j1} \\ c_{j2} \\ \vdots \\ c_{jn} \end{pmatrix} = \det(A) \cdot e_j.$$

Hint: Apply Cramer's rule to $Ax = e_j$.

(c) Deduce that if C is the $n \times n$ matrix such that $C_{ij} = c_{ji}$, then $AC = [\det(A)]I$.

(d) Show that if $\det(A) \neq 0$, then $A^{-1} = [\det(A)]^{-1}C$.

The following definition is used in Exercises 26–27.

Definition. *The* ***classical adjoint*** *of a square matrix A is the transpose of the matrix whose ij-entry is the ij-cofactor of A.*

26. Find the classical adjoint of each of the following matrices.

(a) $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ **(b)** $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

(c) $\begin{pmatrix} -4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ **(d)** $\begin{pmatrix} 3 & 6 & 7 \\ 0 & 4 & 8 \\ 0 & 0 & 5 \end{pmatrix}$

(e) $\begin{pmatrix} 1-i & 0 & 0 \\ 4 & 3i & 0 \\ 2i & 1+4i & -1 \end{pmatrix}$ **(f)** $\begin{pmatrix} 7 & 1 & 4 \\ 6 & -3 & 0 \\ -3 & 5 & -2 \end{pmatrix}$

(g) $\begin{pmatrix} -1 & 2 & 5 \\ 8 & 0 & -3 \\ 4 & 6 & 1 \end{pmatrix}$ **(h)** $\begin{pmatrix} 3 & 2+i & 0 \\ -1+i & 0 & i \\ 0 & 1 & 3-2i \end{pmatrix}$

27. Let C be the classical adjoint of $A \in \mathsf{M}_{n\times n}(F)$. Prove the following statements.

(a) $\det(C) = [\det(A)]^{n-1}$.

(b) C^t is the classical adjoint of A^t.

(c) If A is an invertible upper triangular matrix, then C and A^{-1} are both upper triangular matrices.

28. Let $y_1, y_2, \ldots, y_n$ be linearly independent functions in C^∞. For each $y \in \mathsf{C}^\infty$, define $\mathsf{T}(y) \in \mathsf{C}^\infty$ by

$$[\mathsf{T}(y)](t) = \det\begin{pmatrix} y(t) & y_1(t) & y_2(t) & \cdots & y_n(t) \\ y'(t) & y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & \vdots & & \vdots \\ y^{(n)}(t) & y_1^{(n)}(t) & y_2^{(n)}(t) & \cdots & y_n^{(n)}(t) \end{pmatrix}.$$

The preceding determinant is called the **Wronskian** of $y, y_1, \ldots, y_n$.

(a) Prove that $\mathsf{T}\colon \mathsf{C}^\infty \to \mathsf{C}^\infty$ is a linear transformation.

(b) Prove that $\mathsf{N(T)}$ contains $\text{span}(\{y_1, y_2, \ldots, y_n\})$.

4.4 SUMMARY—IMPORTANT FACTS ABOUT DETERMINANTS

In this section, we summarize the important properties of the determinant needed for the remainder of the text. The results contained in this section have been derived in Sections 4.2 and 4.3; consequently, the facts presented here are stated without proofs.

The **determinant** of an $n \times n$ matrix A having entries from a field F is a scalar in F, denoted by $\det(A)$ or $|A|$, and can be computed as follows.

1. If A is 1×1, then $\det(A) = A_{11}$, the single entry of A.

2. If A is 2×2, then $\det(A) = A_{11}A_{22} - A_{12}A_{21}$. For example,

$$\det \begin{pmatrix} -1 & 2 \\ 5 & 3 \end{pmatrix} = (-1)(3) - (2)(5) = -13.$$

3. If A is $n \times n$ for $n > 2$, then, for each i, we can evaluate the determinant by *cofactor expansion along row* i as

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \cdot \det(\tilde{A}_{ij}),$$

or, for each j, we can evaluate the determinant *by cofactor expansion along column* j as

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} A_{ij} \cdot \det(\tilde{A}_{ij}),$$

where $\tilde{A}_{ij}$ is the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from A.

In the formulas above, the scalar $(-1)^{i+j} \det(\tilde{A}_{ij})$ is called the **cofactor** of the row i column j entry of A. In this language, the determinant of A is evaluated as the sum of terms obtained by multiplying each entry of some row or column of A by the cofactor of that entry. Thus $\det(A)$ is expressed in terms of n determinants of $(n-1) \times (n-1)$ matrices. These determinants are then evaluated in terms of determinants of $(n-2) \times (n-2)$ matrices, and so forth, until 2×2 matrices are obtained. The determinants of the 2×2 matrices are then evaluated as in item 2.

Let us consider two examples of this technique in evaluating the determinant of the 4×4 matrix

$$A = \begin{pmatrix} 2 & 1 & 1 & 5 \\ 1 & 1 & -4 & -1 \\ 2 & 0 & -3 & 1 \\ 3 & 6 & 1 & 2 \end{pmatrix}.$$

To evaluate the determinant of A by expanding along the fourth row, we must know the cofactors of each entry of that row. The cofactor of $A_{41} = 3$ is $(-1)^{4+1}\det(B)$, where

$$B = \begin{pmatrix} 1 & 1 & 5 \\ 1 & -4 & -1 \\ 0 & -3 & 1 \end{pmatrix}.$$

Let us evaluate this determinant by expanding along the first column. We have

$$\begin{aligned}
\det(B) &= (-1)^{1+1}(1)\det\begin{pmatrix} -4 & -1 \\ -3 & 1 \end{pmatrix} + (-1)^{2+1}(1)\det\begin{pmatrix} 1 & 5 \\ -3 & 1 \end{pmatrix} \\
&\quad + (-1)^{3+1}(0)\det\begin{pmatrix} 1 & 5 \\ -4 & -1 \end{pmatrix} \\
&= 1(1)[(-4)(1) - (-1)(-3)] + (-1)(1)[(1)(1) - (5)(-3)] + 0 \\
&= -7 - 16 + 0 = -23.
\end{aligned}$$

Thus the cofactor of A_{41} is $(-1)^5(-23) = 23$. Similarly, the cofactors of A_{42}, A_{43}, and A_{44} are 8, 11, and -13, respectively. We can now evaluate the determinant of A by multiplying each entry of the fourth row by its cofactor; this gives

$$\det(A) = 3(23) + 6(8) + 1(11) + 2(-13) = 102.$$

For the sake of comparison, let us also compute the determinant of A by expansion along the second column. The reader should verify that the cofactors of A_{12}, A_{22}, and A_{42} are -14, 40, and 8, respectively. Thus

$$\begin{aligned}
\det(A) &= (-1)^{1+2}(1)\det\begin{pmatrix} 1 & -4 & -1 \\ 2 & -3 & 1 \\ 3 & 1 & 2 \end{pmatrix} + (-1)^{2+2}(1)\det\begin{pmatrix} 2 & 1 & 5 \\ 2 & -3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \\
&\quad + (-1)^{3+2}(0)\det\begin{pmatrix} 2 & 1 & 5 \\ 1 & -4 & -1 \\ 3 & 1 & 2 \end{pmatrix} + (-1)^{4+2}(6)\det\begin{pmatrix} 2 & 1 & 5 \\ 1 & -4 & -1 \\ 2 & -3 & 1 \end{pmatrix} \\
&= 14 + 40 + 0 + 48 = 102.
\end{aligned}$$

Of course, the fact that the value 102 is obtained again is no surprise since the value of the determinant of A is independent of the choice of row or column used in the expansion.

Observe that the computation of $\det(A)$ is easier when expanded along the second column than when expanded along the fourth row. The difference is the presence of a zero in the second column, which makes it unnecessary to evaluate one of the cofactors (the cofactor of A_{32}). For this reason, it is beneficial to evaluate the determinant of a matrix by expanding along a row or column of the matrix that contains the largest number of zero entries. In fact, it is often helpful to introduce zeros into the matrix by means of elementary row operations before computing the determinant. This technique utilizes the first three properties of the determinant.

Properties of the Determinant

1. If B is a matrix obtained by interchanging any two rows or interchanging any two columns of an $n \times n$ matrix A, then $\det(B) = -\det(A)$.
2. If B is a matrix obtained by multiplying each entry of some row or column of an $n \times n$ matrix A by a scalar k, then $\det(B) = k \cdot \det(A)$.
3. If B is a matrix obtained from an $n \times n$ matrix A by adding a multiple of row i to row j or a multiple of column i to column j for $i \neq j$, then $\det(B) = \det(A)$.

As an example of the use of these three properties in evaluating determinants, let us compute the determinant of the 4×4 matrix A considered previously. Our procedure is to introduce zeros into the second column of A by employing property 3, and then to expand along that column. (The elementary row operations used here consist of adding multiples of row 1 to rows 2 and 4.) This procedure yields

$$\det(A) = \det\begin{pmatrix} 2 & 1 & 1 & 5 \\ 1 & 1 & -4 & -1 \\ 2 & 0 & -3 & 1 \\ 3 & 6 & 1 & 2 \end{pmatrix} = \det\begin{pmatrix} 2 & 1 & 1 & 5 \\ -1 & 0 & -5 & -6 \\ 2 & 0 & -3 & 1 \\ -9 & 0 & -5 & -28 \end{pmatrix}$$

$$= 1(-1)^{1+2}\det\begin{pmatrix} -1 & -5 & -6 \\ 2 & -3 & 1 \\ -9 & -5 & -28 \end{pmatrix}.$$

The resulting determinant of a 3×3 matrix can be evaluated in the same manner: Use type 3 elementary row operations to introduce two zeros into the first column, and then expand along that column. This results in the value -102. Therefore

$$\det(A) = 1(-1)^{1+2}(-102) = 102.$$

The reader should compare this calculation of $\det(A)$ with the preceding ones to see how much less work is required when properties 1, 2, and 3 are employed.

In the chapters that follow, we often have to evaluate the determinant of matrices having special forms. The next two properties of the determinant are useful in this regard:

4. The determinant of an upper triangular matrix is the product of its diagonal entries. In particular, $\det(I) = 1$.
5. If two rows (or columns) of a matrix are identical, then the determinant of the matrix is zero.

As an illustration of property 4, notice that

$$\det\begin{pmatrix} -3 & 1 & 2 \\ 0 & 4 & 5 \\ 0 & 0 & -6 \end{pmatrix} = (-3)(4)(-6) = 72.$$

Property 4 provides an efficient method for evaluating the determinant of a matrix:

(a) Use Gaussian elimination and properties 1, 2, and 3 above to reduce the matrix to an upper triangular matrix.

(b) Compute the product of the diagonal entries.

For instance,

$$\det\begin{pmatrix} 1 & -1 & 2 & 1 \\ 2 & -1 & -1 & 4 \\ -4 & 5 & -10 & -6 \\ 3 & -2 & 10 & -1 \end{pmatrix} = \det\begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -5 & 2 \\ 0 & 1 & -2 & -2 \\ 0 & 1 & 4 & -4 \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -5 & 2 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 9 & -6 \end{pmatrix} = \det\begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -5 & 2 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & 6 \end{pmatrix}$$
$$= 1 \cdot 1 \cdot 3 \cdot 6 = 18.$$

The next three properties of the determinant are used frequently in later chapters. Indeed, perhaps the most significant property of the determinant is that it provides a simple characterization of invertible matrices. (See property 7.)

6. For any $n \times n$ matrices A and B, $\det(AB) = \det(A) \cdot \det(B)$.

7. An $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$. Furthermore, if A is invertible, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

8. For any $n \times n$ matrix A, the determinants of A and A^t are equal.

For example, property 7 guarantees that the matrix A on page 233 is invertible because $\det(A) = 102$.

The final property, stated as Exercise 15 of Section 4.3, is used in Chapter 5. It is a simple consequence of properties 6 and 7.

9. If A and B are similar matrices, then $\det(A) = \det(B)$.

EXERCISES

1. Label the following statements as true or false.

(a) The determinant of a square matrix may be computed by expanding the matrix along any row or column.
(b) In evaluating the determinant of a matrix, it is wise to expand along a row or column containing the largest number of zero entries.
(c) If two rows or columns of A are identical, then $\det(A) = 0$.
(d) If B is a matrix obtained by interchanging two rows or two columns of A, then $\det(B) = \det(A)$.
(e) If B is a matrix obtained by multiplying each entry of some row or column of A by a scalar, then $\det(B) = \det(A)$.
(f) If B is a matrix obtained from A by adding a multiple of some row to a different row, then $\det(B) = \det(A)$.
(g) The determinant of an upper triangular $n \times n$ matrix is the product of its diagonal entries.
(h) For every $A \in \mathsf{M}_{n\times n}(F)$, $\det(A^t) = -\det(A)$.
(i) If $A, B \in \mathsf{M}_{n\times n}(F)$, then $\det(AB) = \det(A) \cdot \det(B)$.
(j) If Q is an invertible matrix, then $\det(Q^{-1}) = [\det(Q)]^{-1}$.
(k) A matrix Q is invertible if and only if $\det(Q) \neq 0$.

2. Evaluate the determinant of the following 2×2 matrices.

(a) $\begin{pmatrix} 4 & -5 \\ 2 & 3 \end{pmatrix}$ **(b)** $\begin{pmatrix} -1 & 7 \\ 3 & 8 \end{pmatrix}$

(c) $\begin{pmatrix} 2+i & -1+3i \\ 1-2i & 3-i \end{pmatrix}$ **(d)** $\begin{pmatrix} 3 & 4i \\ -6i & 2i \end{pmatrix}$

3. Evaluate the determinant of the following matrices in the manner indicated.

(a) $\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{pmatrix}$ along the first row

(b) $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ -1 & 3 & 0 \end{pmatrix}$ along the first column

(c) $\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{pmatrix}$ along the second column

(d) $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ -1 & 3 & 0 \end{pmatrix}$ along the third row

(e) $\begin{pmatrix} 0 & 1+i & 2 \\ -2i & 0 & 1-i \\ 3 & 4i & 0 \end{pmatrix}$ along the third row

(f) $\begin{pmatrix} i & 2+i & 0 \\ -1 & 3 & 2i \\ 0 & -1 & 1-i \end{pmatrix}$ along the third column

(g) $\begin{pmatrix} 0 & 2 & 1 & 3 \\ 1 & 0 & -2 & 2 \\ 3 & -1 & 0 & 1 \\ -1 & 1 & 2 & 0 \end{pmatrix}$ along the fourth column

(h) $\begin{pmatrix} 1 & -1 & 2 & -1 \\ -3 & 4 & 1 & -1 \\ 2 & -5 & -3 & 8 \\ -2 & 6 & -4 & 1 \end{pmatrix}$ along the fourth row

4. Evaluate the determinant of the following matrices by any legitimate method.

(a) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

(b) $\begin{pmatrix} -1 & 3 & 2 \\ 4 & -8 & 1 \\ 2 & 2 & 5 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & -5 \\ 6 & -4 & 3 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & -2 & 3 \\ -1 & 2 & -5 \\ 3 & -1 & 2 \end{pmatrix}$

(e) $\begin{pmatrix} i & 2 & -1 \\ 3 & 1+i & 2 \\ -2i & 1 & 4-i \end{pmatrix}$

(f) $\begin{pmatrix} -1 & 2+i & 3 \\ 1-i & i & 1 \\ 3i & 2 & -1+i \end{pmatrix}$

(g) $\begin{pmatrix} 1 & 0 & -2 & 3 \\ -3 & 1 & 1 & 2 \\ 0 & 4 & -1 & 1 \\ 2 & 3 & 0 & 1 \end{pmatrix}$

(h) $\begin{pmatrix} 1 & -2 & 3 & -12 \\ -5 & 12 & -14 & 19 \\ -9 & 22 & -20 & 31 \\ -4 & 9 & -14 & 15 \end{pmatrix}$

5. Suppose that $M \in \mathsf{M}_{n\times n}(F)$ can be written in the form

$$M = \begin{pmatrix} A & B \\ O & I \end{pmatrix},$$

where A is a square matrix. Prove that $\det(M) = \det(A)$.

6.[†] Prove that if $M \in \mathsf{M}_{n\times n}(F)$ can be written in the form

$$M = \begin{pmatrix} A & B \\ O & C \end{pmatrix},$$

where A and C are square matrices, then $\det(M) = \det(A)\cdot\det(C)$. Visit `goo.gl/pGMdpX` for a solution.

4.5* A CHARACTERIZATION OF THE DETERMINANT

In Sections 4.2 and 4.3, we showed that the determinant possesses a number of properties. In this section, we show that three of these properties completely characterize the determinant; that is, the only function $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ having these three properties is the determinant. This characterization of the determinant is the one used in Section 4.1 to establish the relationship between $\det\begin{pmatrix} u \\ v \end{pmatrix}$ and the area of the parallelogram determined by u and v. The first of these properties that characterize the determinant is the one described in Theorem 4.3 (p. 212).

Definition. *A function $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ is called an **n-linear function** if it is a linear function of each row of an $n \times n$ matrix when the remaining $n-1$ rows are held fixed, that is, δ is n-linear if, for every $r = 1, 2, \ldots, n$, we have*

$$\delta\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u+kv \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} = \delta\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} + k\delta\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ v \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}$$

whenever k is a scalar and u, v, and each a_i are vectors in F^n.

Example 1

The function $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ defined by $\delta(A) = 0$ for each $A \in \mathsf{M}_{n\times n}(F)$ is an n-linear function. ♦

Example 2

For $1 \le j \le n$, define $\delta_j\colon \mathsf{M}_{n\times n}(F) \to F$ by $\delta_j(A) = A_{1j}A_{2j}\cdots A_{nj}$ for each $A \in \mathsf{M}_{n\times n}(F)$; that is, $\delta_j(A)$ equals the product of the entries of column j of

A. Let $A \in \mathsf{M}_{n\times n}(F)$, $a_i = (A_{i1}, A_{i2}, \ldots, A_{in})$, and $v = (b_1, b_2, \ldots, b_n) \in \mathsf{F}^n$. Then each δ_j is an n-linear function because, for any scalar k, we have

$$\delta\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ a_r + kv \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} = A_{1j}\cdots A_{(r-1)j}(A_{rj} + kb_j)A_{(r+1)j}\cdots A_{nj}$$

$$\begin{aligned} &= A_{1j}\cdots A_{(r-1)j}A_{rj}A_{(r+1)j}\cdots A_{nj} \\ &\qquad + A_{1j}\cdots A_{(r-1)j}(kb_j)A_{(r+1)j}\cdots A_{nj} \\ &= A_{1j}\cdots A_{(r-1)j}A_{rj}A_{(r+1)j}\cdots A_{nj} \\ &\qquad + k(A_{1j}\cdots A_{(r-1)j}b_jA_{(r+1)j}\cdots A_{nj}) \\ &= \delta\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ a_r \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} + k\delta\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ v \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}. \quad \blacklozenge \end{aligned}$$

Example 3

The function $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ defined for each $A \in \mathsf{M}_{n\times n}(F)$ by $\delta(A) = A_{11}A_{22}\cdots A_{nn}$ (i.e., $\delta(A)$ equals the product of the diagonal entries of A) is an n-linear function. ◆

Example 4

The function $\delta\colon \mathsf{M}_{n\times n}(R) \to R$ defined for each $A \in \mathsf{M}_{n\times n}(R)$ by $\delta(A) = \operatorname{tr}(A)$ is not an n-linear function for $n \geq 2$. For if I is the $n \times n$ identity matrix and A is the matrix obtained by multiplying the first row of I by 2, then $\delta(A) = n + 1 \neq 2n = 2\cdot\delta(I)$. ◆

Theorem 4.3 (p. 212) asserts that the determinant is an n-linear function. For our purposes this is the most important example of an n-linear function. Now we introduce the second of the properties used in the characterization of the determinant.

Definition. *An n-linear function $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ is called* **alternating** *if, for each $A \in \mathsf{M}_{n\times n}(F)$, we have $\delta(A) = 0$ whenever two adjacent rows of A are identical.*

Theorem 4.10. *Let $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ be an alternating n-linear function.*

(a) *If $A \in \mathsf{M}_{n\times n}(F)$ and B is a matrix obtained from A by interchanging any two rows of A, then $\delta(B) = -\delta(A)$.*

(b) *If $A \in \mathsf{M}_{n\times n}(F)$ has two identical rows, then $\delta(A) = 0$.*

Proof. (a) Let $A \in \mathsf{M}_{n\times n}(F)$, and let B be the matrix obtained from A by interchanging rows r and s, where $r < s$. We first establish the result in the case that $s = r + 1$. Because $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ is an n-linear function that is alternating, we have

$$\begin{aligned}
0 = \delta\begin{pmatrix} a_1 \\ \vdots \\ a_r + a_{r+1} \\ a_r + a_{r+1} \\ \vdots \\ a_n \end{pmatrix} &= \delta\begin{pmatrix} a_1 \\ \vdots \\ a_r \\ a_r + a_{r+1} \\ \vdots \\ a_n \end{pmatrix} + \delta\begin{pmatrix} a_1 \\ \vdots \\ a_{r+1} \\ a_r + a_{r+1} \\ \vdots \\ a_n \end{pmatrix} \\
&= \delta\begin{pmatrix} a_1 \\ \vdots \\ a_r \\ a_r \\ \vdots \\ a_n \end{pmatrix} + \delta\begin{pmatrix} a_1 \\ \vdots \\ a_r \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} + \delta\begin{pmatrix} a_1 \\ \vdots \\ a_{r+1} \\ a_r \\ \vdots \\ a_n \end{pmatrix} + \delta\begin{pmatrix} a_1 \\ \vdots \\ a_{r+1} \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} \\
&= 0 + \delta(A) + \delta(B) + 0.
\end{aligned}$$

Thus $\delta(B) = -\delta(A)$.

Next suppose that $s > r + 1$, and let the rows of A be $a_1, a_2, \ldots, a_n$. Beginning with a_r and a_{r+1}, successively interchange a_r with the row that follows it until the rows are in the sequence

$$a_1, a_2, \ldots, a_{r-1}, a_{r+1}, \ldots, a_s, a_r, a_{s+1}, \ldots, a_n.$$

In all, $s - r$ interchanges of adjacent rows are needed to produce this sequence. Then successively interchange a_s with the row that precedes it until the rows are in the order

$$a_1, a_2, \ldots, a_{r-1}, a_s, a_{r+1}, \ldots, a_{s-1}, a_r, a_{s+1}, \ldots, a_n.$$

This process requires an additional $s - r - 1$ interchanges of adjacent rows and produces the matrix B. It follows from the preceding paragraph that

$$\delta(B) = (-1)^{(s-r)+(s-r-1)}\delta(A) = -\delta(A).$$

(b) Suppose that rows r and s of $A \in \mathsf{M}_{n\times n}(F)$ are identical, where $r < s$. If $s = r + 1$, then $\delta(A) = 0$ because δ is alternating and two adjacent rows of A are identical. If $s > r+1$, let B be the matrix obtained from A by interchanging rows $r+1$ and s. Then $\delta(B) = 0$ because two adjacent rows of B are identical. But $\delta(B) = -\delta(A)$ by (a). Hence $\delta(A) = 0$. ∎

Corollary 1. *Let $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ be an alternating n-linear function. If B is a matrix obtained from $A \in \mathsf{M}_{n\times n}(F)$ by adding a multiple of some row of A to another row, then $\delta(B) = \delta(A)$.*

Proof. Let B be obtained from $A \in \mathsf{M}_{n\times n}(F)$ by adding k times row i of A to row j, where $j \neq i$, and let C be obtained from A by replacing row j of A by row i of A. Then the rows of A, B, and C are identical except for row j. Moreover, row j of B is the sum of row j of A and k times row j of C. Since δ is an n-linear function and C has two identical rows, it follows that

$$\delta(B) = \delta(A) + k\delta(C) = \delta(A) + k\cdot 0 = \delta(A).$$ ∎

The next result now follows as in the proof of the corollary to Theorem 4.6 (p. 216). (See Exercise 11.)

Corollary 2. *Let $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ be an alternating n-linear function. If $M \in \mathsf{M}_{n\times n}(F)$ has rank less than n, then $\delta(M) = 0$.*

Proof. Exercise. ∎

Corollary 3. *Let $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ be an alternating n-linear function, and let E_1, E_2, and E_3 in $\mathsf{M}_{n\times n}(F)$ be elementary matrices of types 1, 2, and 3, respectively. Suppose that E_2 is obtained by multiplying some row of I by the nonzero scalar k. Then $\delta(E_1) = -\delta(I)$, $\delta(E_2) = k\cdot\delta(I)$, and $\delta(E_3) = \delta(I)$.*

Proof. Exercise. ∎

We wish to show that under certain circumstances, the only alternating n-linear function $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ is the determinant, that is, $\delta(A) = \det(A)$ for all $A \in \mathsf{M}_{n\times n}(F)$. Because any scalar multiple of an alternating n-linear function is also an alternating n-linear function, we need a condition that distinguishes the determinant among its scalar multiples. Hence the third condition that is used in the characterization of the determinant is that the determinant of the $n \times n$ identity matrix is 1. Before we can establish the desired characterization of the determinant, we must first prove a result similar to Theorem 4.7 (p. 223). The proof of this result is also similar to that of Theorem 4.7, and so it is omitted. (See Exercise 12.)

Theorem 4.11. *Let $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ be an alternating n-linear function such that $\delta(I) = 1$. For any $A, B \in \mathsf{M}_{n\times n}(F)$, we have $\delta(AB) = \det(A)\cdot\delta(B)$.*

Proof. Exercise. ∎

Theorem 4.12. *If $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ is an alternating n-linear function such that $\delta(I) = 1$, then $\delta(A) = \det(A)$ for every $A \in \mathsf{M}_{n\times n}(F)$.*

Proof. Let $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ be an alternating n-linear function such that $\delta(I) = 1$, and let $A \in \mathsf{M}_{n\times n}(F)$. If A has rank less than n, then by Corollary 2 to Theorem 4.10, $\delta(A) = 0$. Since the corollary to Theorem 4.6 (p. 217) gives $\det(A) = 0$, we have $\delta(A) = \det(A)$ in this case. If, on the other hand, A has rank n, then A is invertible and hence is the product of elementary matrices (Corollary 3 to Theorem 3.6 p. 158), say $A = E_m \cdots E_2E_1$. Since $\delta(I) = 1$, it follows from Corollary 3 to Theorem 4.10 and the facts on page 223 that $\delta(E) = \det(E)$ for every elementary matrix E. Hence by Theorems 4.11 and 4.7 (p. 223), we have

$$\begin{aligned}
\delta(A) &= \delta(E_m \cdots E_2E_1)\\
&= \det(E_m)\delta(E_{m-1}\cdot \cdots \cdot E_2\cdot E_1)\\
&= \cdots\\
&= \det(E_m)\cdot \cdots \cdot \det(E_2)\cdot \det(E_1)\\
&= \det(E_m \cdots E_2E_1)\\
&= \det(A).
\end{aligned}$$ ∎

Theorem 4.12 provides the desired characterization of the determinant: It is the unique function $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ that is n-linear, is alternating, and has the property that $\delta(I) = 1$.

EXERCISES

1. Label the following statements as true or false.

(a) Any n-linear function $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ is a linear transformation.
(b) Any n-linear function $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ is a linear function of each row of an $n \times n$ matrix when the other $n-1$ rows are held fixed.
(c) If $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ is an alternating n-linear function and the matrix $A \in \mathsf{M}_{n\times n}(F)$ has two identical rows, then $\delta(A) = 0$.
(d) If $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ is an alternating n-linear function and B is obtained from $A \in \mathsf{M}_{n\times n}(F)$ by interchanging two rows of A, then $\delta(B) = \delta(A)$.
(e) There is a unique alternating n-linear function $\delta\colon \mathsf{M}_{n\times n}(F) \to F$.
(f) The function $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ defined by $\delta(A) = 0$ for every $A \in \mathsf{M}_{n\times n}(F)$ is an alternating n-linear function.

2. Determine all the 1-linear functions $\delta\colon \mathsf{M}_{1\times 1}(F) \to F$.

Determine which of the functions $\delta\colon \mathsf{M}_{3\times 3}(F) \to F$ in Exercises 3–10 are 3-linear functions. Justify each answer.

3. $\delta(A) = k$, where k is any nonzero scalar

4. $\delta(A) = A_{22}$

5. $\delta(A) = A_{11}A_{23}A_{32}$

6. $\delta(A) = A_{11} + A_{23} + A_{32}$

7. $\delta(A) = A_{11}A_{21}A_{32}$

8. $\delta(A) = A_{11}A_{31}A_{32}$

9. $\delta(A) = A_{11}^2 A_{22}^2 A_{33}^2$

10. $\delta(A) = A_{11}A_{22}A_{33} - A_{11}A_{21}A_{32}$

11. Prove Corollaries 2 and 3 of Theorem 4.10. Visit `goo.gl/FKcuqu` for a solution.

12. Prove Theorem 4.11.

13. Prove that $\det\colon \mathsf{M}_{2\times 2}(F) \to F$ is a 2-linear function of the *columns* of a matrix.

14. Let $a, b, c, d \in F$. Prove that the function $\delta\colon \mathsf{M}_{2\times 2}(F) \to F$ defined by $\delta(A) = A_{11}A_{22}a + A_{11}A_{21}b + A_{12}A_{22}c + A_{12}A_{21}d$ is a 2-linear function.

15. Prove that $\delta\colon \mathsf{M}_{2\times 2}(F) \to F$ is a 2-linear function if and only if it has the form

$$\delta(A) = A_{11}A_{22}a + A_{11}A_{21}b + A_{12}A_{22}c + A_{12}A_{21}d$$

for some scalars $a, b, c, d \in F$.

16. Prove that if $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ is an alternating n-linear function, then there exists a scalar k such that $\delta(A) = k\det(A)$ for all $A \in \mathsf{M}_{n\times n}(F)$.

17. Prove that a linear combination of two n-linear functions is an n-linear function, where the sum and scalar product of n-linear functions are as defined in Example 3 of Section 1.2 (p. 9).

18. Prove that the set of all n-linear functions over a field F is a vector space over F under the operations of function addition and scalar multiplication as defined in Example 3 of Section 1.2 (p. 9).

19. Let $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ be an n-linear function and F a field that does not have characteristic two. Prove that if $\delta(B) = -\delta(A)$ whenever B is obtained from $A \in \mathsf{M}_{n\times n}(F)$ by interchanging any two rows of A, then $\delta(M) = 0$ whenever $M \in \mathsf{M}_{n\times n}(F)$ has two identical rows.

20. Give an example to show that the implication in Exercise 19 need not hold if F has characteristic two.

INDEX OF DEFINITIONS FOR CHAPTER 4

5

Diagonalization

This chapter is concerned with the so-called *diagonalization problem*. For a given linear operator T on a finite-dimensional vector space V, we seek answers to the following questions.

1. Does there exist an ordered basis β for V such that $[\mathsf{T}]_\beta$ is a diagonal matrix?
2. If such a basis exists, how can it be found?

Since computations involving diagonal matrices are simple, an affirmative answer to question 1 leads us to a clearer understanding of how the operator T acts on V, and an answer to question 2 enables us to obtain easy solutions to many practical problems that can be formulated in a linear algebra context. We consider some of these problems and their solutions in this chapter; see, for example, Section 5.3.

We begin this chapter with the study of *eigenvalues* and *eigenvectors*, which not only play an essential role in the diagonalization problem, but also are important in their own right. For example, in the design of mechanical systems such as cars, bridges, and power plants, it is necessary to control unwanted vibrations to prevent failure of the system or some of its components. Spectacular instances of such failures occurred in the 1850 collapse of the Angers Bridge over the Maine River in France and the 1940 collapse of the Tacoma Narrows Bridge (then the third-longest suspension bridge in the world) into Puget Sound. These types of vibration problems can be analyzed by the study of the eigenvalues and eigenvectors of the differential equations that model the system. Other uses of eigenvalues and eigenvectors occur in face recognition, fingerprint authentication, and the statistical process called *principal component analysis*.

5.1 EIGENVALUES AND EIGENVECTORS

In Example 3 of Section 2.5, we were able to obtain a formula for the reflection of R^2 about the line $y = 2x$. The key to our success was to find a basis β' for which $[\mathsf{T}]_{\beta'}$ is a diagonal matrix. We now introduce the name for an operator or matrix that has such a basis.

Definitions. *A linear operator T on a finite-dimensional vector space V is called* **diagonalizable** *if there is an ordered basis β for V such that $[\mathsf{T}]_\beta$ is a diagonal matrix. A square matrix A is called* **diagonalizable** *if L_A is diagonalizable.*

We want to determine when a linear operator T on a finite-dimensional vector space V is diagonalizable and, if so, how to obtain an ordered basis $\beta = \{v_1, v_2, \ldots, v_n\}$ for V such that $[\mathsf{T}]_\beta$ is a diagonal matrix. Note that, if $D = [\mathsf{T}]_\beta$ is a diagonal matrix, then for each vector $v_j \in \beta$, we have

$$\mathsf{T}(v_j) = \sum_{i=1}^{n} D_{ij} v_i = D_{jj} v_j = \lambda_j v_j,$$

where $\lambda_j = D_{jj}$.

Conversely, if $\beta = \{v_1, v_2, \ldots, v_n\}$ is an ordered basis for V such that $\mathsf{T}(v_j) = \lambda_j v_j$ for some scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$, then clearly

$$[\mathsf{T}]_\beta = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

In the preceding paragraph, each vector v in the basis β satisfies the condition that $\mathsf{T}(v) = \lambda v$ for some scalar λ. Moreover, because v lies in a basis, v is nonzero. These computations motivate the following definitions.

Definitions. *Let T be a linear operator on a vector space V. A nonzero vector $v \in \mathsf{V}$ is called an* **eigenvector** *of T if there exists a scalar λ such that $\mathsf{T}(v) = \lambda v$. The scalar λ is called the* **eigenvalue** *corresponding to the eigenvector v.*

Let A be in $\mathsf{M}_{n\times n}(F)$. A nonzero vector $v \in \mathsf{F}^n$ is called an **eigenvector** *of A if v is an eigenvector of L_A; that is, if $Av = \lambda v$ for some scalar λ. The scalar λ is called the* **eigenvalue** *of A corresponding to the eigenvector v.*

The words *characteristic vector* and *proper vector* are also used in place of *eigenvector*. The corresponding terms for *eigenvalue* are *characteristic value* and *proper value.*

Note that a vector is an eigenvector of a matrix A if and only if it is an eigenvector of L_A. Likewise, a scalar λ is an eigenvalue of A if and only if it is an eigenvalue of L_A. Using the terminology of eigenvectors and eigenvalues, we can summarize the preceding discussion as follows.

Theorem 5.1. *A linear operator* T *on a finite-dimensional vector space* V *is diagonalizable if and only if there exists an ordered basis* β *for* V *consisting of eigenvectors of* T. *Furthermore, if* T *is diagonalizable,* $\beta = \{v_1, v_2, \ldots, v_n\}$ *is an ordered basis of eigenvectors of* T, *and* $D = [\mathsf{T}]_\beta$, *then* D *is a diagonal matrix and* D_{jj} *is the eigenvalue corresponding to* v_j *for* $1 \leq j \leq n$.

Corollary. *A matrix* $A \in \mathsf{M}_{n\times n}(F)$ *is diagonalizable if and only if there exists an ordered basis for* F^n *consisting of eigenvectors of* A. *Furthermore, if* $\{v_1, v_2, \ldots, v_n\}$ *is an ordered basis for* F^n *consisting of eigenvectors of* A *and* Q *is the* $n \times n$ *matrix whose* j*th column is* v_j *for* $j = 1, 2, \ldots, n$, *then* $D = Q^{-1}AQ$ *is a diagonal matrix such that* D_{jj} *is the eigenvalue of* A *corresponding to* v_j. *Hence* A *is diagonalizable if and only if it is similar to a diagonal matrix.*

To *diagonalize* a matrix or a linear operator is to find a basis of eigenvectors and the corresponding eigenvalues.

Before continuing our study of the diagonalization problem, we consider three examples of eigenvalues and eigenvectors.

Example 1

Let

$$A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \text{and} \quad v_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

Since

$$\mathsf{L}_A(v_1) = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} = -2\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -2v_1,$$

v_1 is an eigenvector of L_A, and hence of A. Here $\lambda_1 = -2$ is the eigenvalue corresponding to v_1. Furthermore,

$$\mathsf{L}_A(v_2) = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 15 \\ 20 \end{pmatrix} = 5\begin{pmatrix} 3 \\ 4 \end{pmatrix} = 5v_2,$$

and so v_2 is an eigenvector of L_A, and hence of A, with the corresponding eigenvalue $\lambda_2 = 5$. Note that $\beta = \{v_1, v_2\}$ is an ordered basis for R^2 consisting of eigenvectors of both A and L_A, and therefore A and L_A are diagonalizable. Moreover, by Theorem 5.1 and its corollary, if

$$Q = \begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix},$$

then

$$Q^{-1}AQ = [\mathsf{L}_A]_\beta = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}. \quad \blacklozenge$$

Example 2

Let T be the linear operator on R^2 that rotates each vector in the plane through an angle of $\pi/2$. It is clear geometrically that for any nonzero vector v, the vector $\mathsf{T}(v)$ does not lie on the line through $\mathit{0}$ determined by v; hence $\mathsf{T}(v)$ is not a multiple of v. Therefore T has no eigenvectors and, consequently, no eigenvalues. Thus there exist operators (and matrices) with no eigenvalues or eigenvectors. Of course, such operators and matrices are not diagonalizable. ◆

Example 3

Let $\mathsf{C}^\infty(R)$ denote the set of all functions $f\colon R \to R$ having derivatives of all orders. (Thus $\mathsf{C}^\infty(R)$ includes the polynomial functions, the sine and cosine functions, the exponential functions, etc.) Clearly, $\mathsf{C}^\infty(R)$ is a subspace of the vector space $\mathcal{F}(R, R)$ of all functions from R to R as defined in Section 1.2. Let $\mathsf{T}\colon \mathsf{C}^\infty(R) \to \mathsf{C}^\infty(R)$ be the function defined by $\mathsf{T}(f) = f'$, the derivative of f. It is easily verified that T is a linear operator on $\mathsf{C}^\infty(R)$. We determine the eigenvalues and eigenvectors of T.

Suppose that f is an eigenvector of T with corresponding eigenvalue λ. Then $f' = \mathsf{T}(f) = \lambda f$. This is a first-order differential equation whose solutions are of the form $f(t) = ce^{\lambda t}$ for some constant c. Consequently, every real number λ is an eigenvalue of T, and λ corresponds to eigenvectors of the form $ce^{\lambda t}$ for $c \neq 0$. Note that for $\lambda = 0$, the eigenvectors are the nonzero constant functions. ◆

In order to obtain a basis of eigenvectors for a matrix (or a linear operator), we need to be able to determine its eigenvalues and eigenvectors. The following theorem gives us a method for computing eigenvalues.

Theorem 5.2. *Let $A \in \mathsf{M}_{n\times n}(F)$. Then a scalar λ is an eigenvalue of A if and only if $\det(A - \lambda I_n) = 0$.*

Proof. A scalar λ is an eigenvalue of A if and only if there exists a nonzero vector $v \in \mathsf{F}^n$ such that $Av = \lambda v$, that is, $(A - \lambda I_n)(v) = \mathit{0}$. By Theorem 2.5 (p. 71), this is true if and only if $A - \lambda I_n$ is not invertible. However, this result is equivalent to the statement that $\det(A - \lambda I_n) = 0$. ∎

Definition. *Let $A \in \mathsf{M}_{n\times n}(F)$. The polynomial $f(t) = \det(A - tI_n)$ is called the* ***characteristic polynomial*** [1] *of A.*

[1] Although the entries of the matrix $A - tI_n$ are not scalars in the field F, they are scalars in another field $F(t)$, the field of quotients of polynomials in t with coefficients from F. Consequently, any results proved about determinants in Chapter 4 remain valid in this context.

Theorem 5.2 states that the eigenvalues of a matrix are the zeros of its characteristic polynomial. When determining the eigenvalues of a matrix or a linear operator, we normally compute its characteristic polynomial, as in the next example.

Example 4

To find the eigenvalues of

$$A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \in \mathsf{M}_{2\times 2}(R),$$

we compute its characteristic polynomial:

$$\det(A - tI_2) = \det \begin{pmatrix} 1-t & 1 \\ 4 & 1-t \end{pmatrix} = t^2 - 2t - 3 = (t-3)(t+1).$$

It follows from Theorem 5.2 that the only eigenvalues of A are 3 and -1. ◆

It is easily shown that similar matrices have the same determinant and the same characteristic polynomial (see Exercise 13). This fact enables us to define the determinant and the characteristic polynomial of a linear operator as follows.

Definitions. *Let T be a linear operator on a finite-dimensional vector space V. Choose any ordered basis β for V. We define the* ***determinant*** *of T, denoted $\det(\mathsf{T})$, to be the determinant of $A = [\mathsf{T}]_\beta$, and the* ***characteristic polynomial*** *$f(t)$ of T to be the characteristic polynomial of A. That is,*

$$f(t) = \det(A - tI_n).$$

The remark preceding these definitions shows that they are independent of the choice of ordered basis β. Thus if T is a linear operator on a finite-dimensional vector space V and β is an ordered basis for V, then λ is an eigenvalue of T if and only if λ is an eigenvalue of $[\mathsf{T}]_\beta$.

Example 5

Let T be the linear operator on $\mathsf{P}_2(R)$ defined by $\mathsf{T}(f(x)) = f(x)+(x+1)f'(x)$, let β be the standard ordered basis for $\mathsf{P}_2(R)$, and let $A = [\mathsf{T}]_\beta$. Then

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}.$$

The characteristic polynomial of T is

$$\det(A - tI_3) = \det \begin{pmatrix} 1-t & 1 & 0 \\ 0 & 2-t & 2 \\ 0 & 0 & 3-t \end{pmatrix}$$

$$= (1-t)(2-t)(3-t)$$
$$= -(t-1)(t-2)(t-3).$$

Hence λ is an eigenvalue of T (or A) if and only if $\lambda = 1, 2$, or 3. ◆

Examples 4 and 5 suggest that the characteristic polynomial of an $n \times n$ matrix A is a polynomial of degree n. The next theorem tells us even more. It can be proved by a straightforward induction argument.

Theorem 5.3. *Let $A \in \mathsf{M}_{n\times n}(F)$.*

(a) *The characteristic polynomial of A is a polynomial of degree n with leading coefficient $(-1)^n$.*
(b) *A has at most n distinct eigenvalues.*

Proof. Exercise. ■

Theorem 5.2 enables us to determine all the eigenvalues of a matrix or a linear operator on a finite-dimensional vector space provided that we can compute the zeros of the characteristic polynomial. Our next result gives us a procedure for determining the eigenvectors corresponding to a given eigenvalue.

Theorem 5.4. *Let $A \in \mathsf{M}_{n\times n}(F)$, and let λ be an eigenvalue of A. Vector $v \in \mathsf{F}^n$ is an eigenvector of A corresponding to λ if and only if $v \neq \mathit{0}$ and $(A - \lambda I)v = \mathit{0}$.*

Proof. Exercise. ■

Example 6

To find all the eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$$

in Example 4, recall that A has two eigenvalues, $\lambda_1 = 3$ and $\lambda_2 = -1$. We begin by finding all the eigenvectors corresponding to $\lambda_1 = 3$. Let

$$B_1 = A - \lambda_1 I = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}.$$

Then

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathsf{R}^2$$

is an eigenvector corresponding to $\lambda_1 = 3$ if and only if $x \neq \mathit{0}$ and $x \in \mathsf{N}(\mathsf{L}_{B_1})$; that is, $x \neq \mathit{0}$ and

$$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2x_1 + x_2 \\ 4x_1 - 2x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Clearly the set of all solutions to this equation is

$$\left\{ t \begin{pmatrix} 1 \\ 2 \end{pmatrix} : t \in R \right\}.$$

Hence x is an eigenvector corresponding to $\lambda_1 = 3$ if and only if

$$x = t \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \text{for some } t \neq 0.$$

Now suppose that x is an eigenvector of A corresponding to $\lambda_2 = -1$. Let

$$B_2 = A - \lambda_2 I = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}.$$

Then

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathsf{N}(\mathsf{L}_{B_2})$$

if and only if x is a solution to the system

$$\begin{aligned} 2x_1 + x_2 &= 0 \\ 4x_1 + 2x_2 &= 0. \end{aligned}$$

Hence

$$\mathsf{N}(\mathsf{L}_{B_2}) = \left\{ t \begin{pmatrix} 1 \\ -2 \end{pmatrix} : t \in R \right\}.$$

Thus x is an eigenvector corresponding to $\lambda_2 = -1$ if and only if

$$x = t \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad \text{for some } t \neq 0.$$

Observe that

$$\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$$

is a basis for R^2 consisting of eigenvectors of A. Thus L_A, and hence A, is diagonalizable. ◆

Suppose that β is a basis for F^n consisting of eigenvectors of A. The corollary to Theorem 2.23, (p. 113), assures us that if Q is the $n \times n$ matrix whose columns are the vectors in β, then $Q^{-1}AQ$ is a diagonal matrix. In Example 6, for instance, if

$$Q = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix},$$

$$\begin{tikzcd} \mathsf{V} \arrow[r, "\mathsf{T}"] \arrow[d, "\phi_\beta"'] & \mathsf{V} \arrow[d, "\phi_\beta"] \\ \mathsf{F}^n \arrow[r, "\mathsf{L}_A"] & \mathsf{F}^n \end{tikzcd}$$

Figure 5.1

then

$$Q^{-1}AQ = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}.$$

Of course, the diagonal entries of this matrix are the eigenvalues of A that correspond to the respective columns of Q.

To find the eigenvectors of a linear operator T on an n-dimensional vector space, select an ordered basis β for V and let $A = [\mathsf{T}]_\beta$. Figure 5.1 is the special case of Figure 2.2 in Section 2.4 in which $\mathsf{V} = \mathsf{W}$ and $\beta = \gamma$. Recall that for $v \in \mathsf{V}$, $\phi_\beta(v) = [v]_\beta$, the coordinate vector of v relative to β. We show that $v \in \mathsf{V}$ is an eigenvector of T corresponding to λ if and only if $\phi_\beta(v)$ is an eigenvector of A corresponding to λ. Suppose that v is an eigenvector of T corresponding to λ. Then $\mathsf{T}(v) = \lambda v$. Hence

$$A\phi_\beta(v) = \mathsf{L}_A\phi_\beta(v) = \phi_\beta\mathsf{T}(v) = \phi_\beta(\lambda v) = \lambda\phi_\beta(v).$$

Now $\phi_\beta(v) \neq \mathit{0}$, since ϕ_β is an isomorphism; hence $\phi_\beta(v)$ is an eigenvector of A. This argument is reversible, and so we can establish that if $\phi_\beta(v)$ is an eigenvector of A corresponding to λ, then v is an eigenvector of T corresponding to λ. (See Exercise 14.)

An equivalent formulation of the result discussed in the preceding paragraph is that for an eigenvalue λ of A (and hence of T), a vector $y \in \mathsf{F}^n$ is an eigenvector of A corresponding to λ if and only if $\phi_\beta^{-1}(y)$ is an eigenvector of T corresponding to λ.

Thus we have reduced the problem of finding the eigenvectors of a linear operator on a finite-dimensional vector space to the problem of finding the eigenvectors of a matrix. The next example illustrates this procedure.

Example 7

Let $\mathsf{T}(f(x)) = f(x) + (x+1)f'(x)$ be the linear operator on $\mathsf{P}_2(R)$ defined in Example 5, and let β be the standard ordered basis for $\mathsf{P}_2(R)$. Recall that T has eigenvalues 1, 2, and 3 and that

$$A = [\mathsf{T}]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}.$$

We consider each eigenvalue separately.

Let $\lambda_1 = 1$, and define

$$B_1 = A - \lambda_1 I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$$

Then

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathsf{R}^3$$

is an eigenvector corresponding to $\lambda_1 = 1$ if and only if $x \neq \mathit{0}$ and $x \in \mathsf{N}(\mathsf{L}_{B_1})$; that is, x is a nonzero solution to the system

$$\begin{aligned} x_2 \qquad &= 0 \\ x_2 + 2x_3 &= 0 \\ 2x_3 &= 0. \end{aligned}$$

Notice that this system has three unknowns, x_1, x_2, and x_3, but one of these, x_1, does not actually appear in the system. Since the values of x_1 do not affect the system, we assign x_1 a parametric value, say $x_1 = a$, and solve the system for x_2 and x_3. Clearly, $x_2 = x_3 = 0$, and so the eigenvectors of A corresponding to $\lambda_1 = 1$ are of the form

$$a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = ae_1$$

for $a \neq 0$. Consequently, the eigenvectors of T corresponding to $\lambda_1 = 1$ are of the form

$$\phi_\beta^{-1}(ae_1) = a\phi_\beta^{-1}(e_1) = a \cdot 1 = a$$

for any $a \neq 0$. Hence the nonzero constant polynomials are the eigenvectors of T corresponding to $\lambda_1 = 1$.

Next let $\lambda_2 = 2$, and define

$$B_2 = A - \lambda_2 I = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is easily verified that

$$\mathsf{N}(\mathsf{L}_{B_2}) = \left\{ a \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} : a \in R \right\},$$

and hence the eigenvectors of T corresponding to $\lambda_2 = 2$ are of the form

$$\phi_\beta^{-1}\left(a\begin{pmatrix}1\\1\\0\end{pmatrix}\right) = a\phi_\beta^{-1}(e_1 + e_2) = a(1+x)$$

for $a \neq 0$.

Finally, consider $\lambda_3 = 3$ and

$$B_3 = A - \lambda_3 I = \begin{pmatrix}-2 & 1 & 0\\ 0 & -1 & 2\\ 0 & 0 & 0\end{pmatrix}.$$

Since

$$\mathsf{N}(\mathsf{L}_{B_3}) = \left\{a\begin{pmatrix}1\\2\\1\end{pmatrix} : a \in R\right\},$$

the eigenvectors of T corresponding to $\lambda_3 = 3$ are of the form

$$\phi_\beta^{-1}\left(a\begin{pmatrix}1\\2\\1\end{pmatrix}\right) = a\phi_\beta^{-1}(e_1 + 2e_2 + e_3) = a(1+2x+x^2)$$

for $a \neq 0$.

For each eigenvalue, select the corresponding eigenvector with $a = 1$ in the preceding descriptions to obtain $\gamma = \{1, 1+x, 1+2x+x^2\}$, which is an ordered basis for $\mathsf{P}_2(R)$ consisting of eigenvectors of T. Thus T is diagonalizable, and

$$[\mathsf{T}]_\gamma = \begin{pmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{pmatrix}. \quad \blacklozenge$$

We close this section with a geometric description of how a linear operator T acts on an eigenvector in the context of a vector space V over R. Let v be an eigenvector of T and λ be the corresponding eigenvalue. We can think of $\mathsf{W} = \text{span}(\{v\})$, the one-dimensional subspace of V spanned by v, as a line in V that passes through $\mathit{0}$ and v. For any $w \in \mathsf{W}$, $w = cv$ for some scalar c, and hence

$$\mathsf{T}(w) = \mathsf{T}(cv) = c\mathsf{T}(v) = c\lambda v = \lambda w;$$

so T acts on the vectors in W by multiplying each such vector by λ. There are several possible ways for T to act on the vectors in W, depending on the value of λ. We consider several cases. (See Figure 5.2.)

CASE 1. If $\lambda > 1$, then T moves vectors in W farther from $\mathit{0}$ by a factor of λ.

CASE 2. If $\lambda = 1$, then T acts as the identity operator on W.

CASE 3. If $0 < \lambda < 1$, then T moves vectors in W closer to $\mathit{0}$ by a factor of λ.

CASE 4. If $\lambda = 0$, then T acts as the zero transformation on W.

CASE 5. If $\lambda < 0$, then T reverses the orientation of W; that is, T moves vectors in W from one side of $\mathit{0}$ to the other.

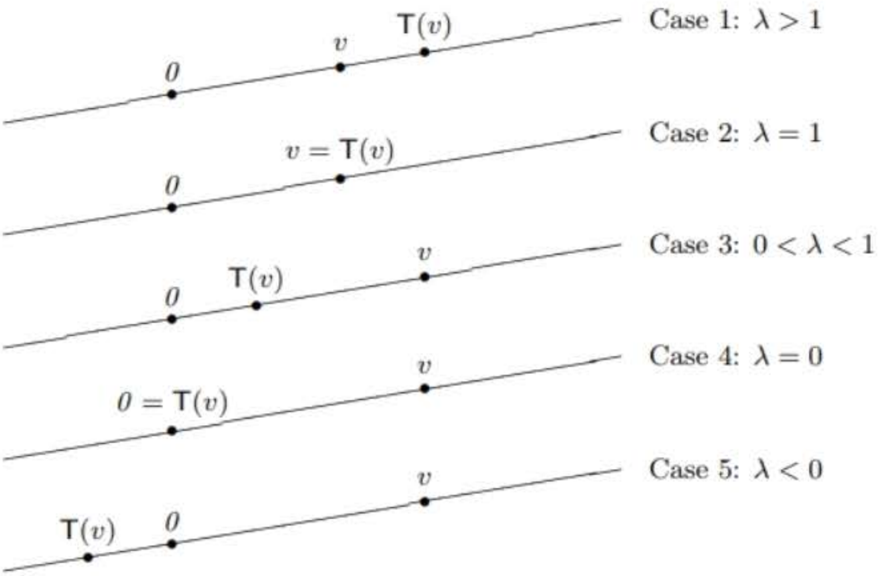


Figure 5.2: The action of T on $\mathsf{W} = \mathrm{span}(\{v\})$ when v is an eigenvector of T.

To illustrate these ideas, we consider the linear operators in Examples 3, 4, and 2 of Section 2.1.

For the operator T on R^2 defined by $\mathsf{T}(a_1, a_2) = (a_1, -a_2)$, the reflection about the x-axis, e_1 and e_2 are eigenvectors of T with corresponding eigenvalues 1 and -1, respectively. Since e_1 and e_2 span the x-axis and the y-axis, respectively, T acts as the identity on the x-axis and reverses the orientation of the y-axis.

For the operator T on R^2 defined by $\mathsf{T}(a_1, a_2) = (a_1, 0)$, the projection on the x-axis, e_1 and e_2 are eigenvectors of T with corresponding eigenvalues 1 and 0, respectively. Thus, T acts as the identity on the x-axis and as the zero operator on the y-axis.

Finally, we generalize Example 2 of this section by considering the operator that rotates the plane through the angle θ, which is defined by

$$\mathsf{T}_\theta(a_1, a_2) = (a_1 \cos\theta - a_2 \sin\theta, a_1 \sin\theta + a_2 \cos\theta).$$

Suppose that $0 < \theta < \pi$. Then for any nonzero vector v, the vectors v and $\mathsf{T}_\theta(v)$ are not collinear, and hence T_θ maps no one-dimensional subspace of R^2 into itself. But this implies that T_θ has no eigenvectors and therefore no eigenvalues. To confirm this conclusion, let β be the standard ordered basis for R^2, and note that the characteristic polynomial of T_θ is

$$\det([\mathsf{T}_\theta]_\beta - tI_2) = \det\begin{pmatrix} \cos\theta - t & -\sin\theta \\ \sin\theta & \cos\theta - t \end{pmatrix} = t^2 - (2\cos\theta)t + 1,$$

which has no real zeros because, for $0 < \theta < \pi$, the discriminant $4\cos^2\theta - 4$ is negative.

EXERCISES

1. Label the following statements as true or false.

- **(a)** Every linear operator on an n-dimensional vector space has n distinct eigenvalues.
- **(b)** If a real matrix has one eigenvector, then it has an infinite number of eigenvectors.
- **(c)** There exists a square matrix with no eigenvectors.
- **(d)** Eigenvalues must be nonzero scalars.
- **(e)** Any two eigenvectors are linearly independent.
- **(f)** The sum of two eigenvalues of a linear operator T is also an eigenvalue of T.
- **(g)** Linear operators on infinite-dimensional vector spaces never have eigenvalues.
- **(h)** An $n \times n$ matrix A with entries from a field F is similar to a diagonal matrix if and only if there is a basis for F^n consisting of eigenvectors of A.
- **(i)** Similar matrices always have the same eigenvalues.
- **(j)** Similar matrices always have the same eigenvectors.
- **(k)** The sum of two eigenvectors of an operator T is always an eigenvector of T.

2. For each of the following linear operators T on a vector space V, compute the determinant of T and the characteristic polynomial of T.

- **(a)** $\mathsf{V} = \mathsf{R}^2$, $\mathsf{T}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a - b \\ 5a + 3b \end{pmatrix}$
- **(b)** $\mathsf{V} = \mathsf{R}^3$, $\mathsf{T}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a - 3b + 2c \\ -2a + b + c \\ 4a - c \end{pmatrix}$

(c) $\mathsf{V} = \mathsf{P}_3(R)$,
$\mathsf{T}(a+bx+cx^2+dx^3) = (a-c)+(-a+b+d)x+(a+b-d)x^2-cx^3$

(d) $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$, $\mathsf{T}(A) = 2A^t - A$

3. For each of the following linear operators T on a vector space V and ordered bases β, compute $[\mathsf{T}]_\beta$, and determine whether β is a basis consisting of eigenvectors of T.

(a) $\mathsf{V} = \mathsf{R}^2$, $\mathsf{T}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 10a - 6b \\ 17a - 10b \end{pmatrix}$, and $\beta = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$

(b) $\mathsf{V} = \mathsf{P}_1(R)$, $\mathsf{T}(a+bx) = (6a-6b)+(12a-11b)x$, and $\beta = \{3+4x, 2+3x\}$

(c) $\mathsf{V} = \mathsf{R}^3$, $\mathsf{T}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3a+2b-2c \\ -4a-3b+2c \\ -c \end{pmatrix}$, and

$$\beta = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\}$$

(d) $\mathsf{V} = \mathsf{P}_2(R)$, $\mathsf{T}(a+bx+cx^2) =$

$$(-4a+2b-2c)-(7a+3b+7c)x+(7a+b+5c)x^2,$$

and $\beta = \{x - x^2, -1+x^2, -1-x+x^2\}$

(e) $\mathsf{V} = \mathsf{P}_3(R)$, $\mathsf{T}(a+bx+cx^2+dx^3) =$

$$-d+(-c+d)x+(a+b-2c)x^2+(-b+c-2d)x^3,$$

and $\beta = \{1-x+x^3, 1+x^2, 1, x+x^2\}$

(f) $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$, $\mathsf{T}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -7a-4b+4c-4d & b \\ -8a-4b+5c-4d & d \end{pmatrix}$, and

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \right\}$$

4. For each of the following matrices $A \in \mathsf{M}_{n\times n}(F)$,

(i) Determine all the eigenvalues of A.

(ii) For each eigenvalue λ of A, find the set of eigenvectors corresponding to λ.

(iii) If possible, find a basis for F^n consisting of eigenvectors of A.

(iv) If successful in finding such a basis, determine an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

(a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ for $F = R$

(b) $A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix}$ for $F = R$

(c) $A = \begin{pmatrix} i & 1 \\ 2 & -i \end{pmatrix}$ for $F = C$

(d) $A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix}$ for $F = R$

5. For each linear operator T on V, find the eigenvalues of T and an ordered basis β for V such that $[\mathsf{T}]_\beta$ is a diagonal matrix.

(a) $\mathsf{V} = \mathsf{R}^2$ and $\mathsf{T}(a, b) = (-2a + 3b, -10a + 9b)$
(b) $\mathsf{V} = \mathsf{R}^3$ and $\mathsf{T}(a, b, c) = (7a - 4b + 10c, 4a - 3b + 8c, -2a + b - 2c)$
(c) $\mathsf{V} = \mathsf{R}^3$ and $\mathsf{T}(a, b, c) = (-4a + 3b - 6c, 6a - 7b + 12c, 6a - 6b + 11c)$
(d) $\mathsf{V} = \mathsf{P}_1(R)$ and $\mathsf{T}(ax + b) = (-6a + 2b)x + (-6a + b)$
(e) $\mathsf{V} = \mathsf{P}_2(R)$ and $\mathsf{T}(f(x)) = xf'(x) + f(2)x + f(3)$
(f) $\mathsf{V} = \mathsf{P}_3(R)$ and $\mathsf{T}(f(x)) = f(x) + f(2)x$
(g) $\mathsf{V} = \mathsf{P}_3(R)$ and $\mathsf{T}(f(x)) = xf'(x) + f''(x) - f(2)$
(h) $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$ and $\mathsf{T}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$
(i) $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$ and $\mathsf{T}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$
(j) $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$ and $\mathsf{T}(A) = A^t + 2 \cdot \mathrm{tr}(A) \cdot I_2$

6. Prove Theorem 5.4.

7. Let T be a linear operator on a finite-dimensional vector space V, and let β be an ordered basis for V. Prove that λ is an eigenvalue of T if and only if λ is an eigenvalue of $[\mathsf{T}]_\beta$.

8. Let T be a linear operator on a finite-dimensional vector space V. Refer to the definition of the determinant of T on page 249 to prove the following results.

(a) Prove that this definition is independent of the choice of an ordered basis for V. That is, prove that if β and γ are two ordered bases for V, then $\det([\mathsf{T}]_\beta) = \det([\mathsf{T}]_\gamma)$.
(b) Prove that T is invertible if and only if $\det(\mathsf{T}) \neq 0$.
(c) Prove that if T is invertible, then $\det(\mathsf{T}^{-1}) = [\det(\mathsf{T})]^{-1}$.
(d) Prove that if U is also a linear operator on V, then $\det(\mathsf{TU}) = \det(\mathsf{T}) \cdot \det(\mathsf{U})$.

(e) Prove that $\det(\mathsf{T} - \lambda\mathsf{I_V}) = \det([\mathsf{T}]_\beta - \lambda I)$ for any scalar λ and any ordered basis β for V.

9. (a) Prove that a linear operator T on a finite-dimensional vector space is invertible if and only if zero is not an eigenvalue of T.
(b) Let T be an invertible linear operator. Prove that a scalar λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}.
(c) State and prove results analogous to (a) and (b) for matrices.

10. Prove that the eigenvalues of an upper triangular matrix M are the diagonal entries of M.

11. Let V be a finite-dimensional vector space, and let λ be any scalar.
(a) For any ordered basis β for V, prove that $[\lambda\mathsf{I_V}]_\beta = \lambda I$.
(b) Compute the characteristic polynomial of $\lambda\mathsf{I_V}$.
(c) Show that $\lambda\mathsf{I_V}$ is diagonalizable and has only one eigenvalue.

12. A **scalar matrix** is a square matrix of the form λI for some scalar λ; that is, a scalar matrix is a diagonal matrix in which all the diagonal entries are equal.
(a) Prove that if a square matrix A is similar to a scalar matrix λI, then $A = \lambda I$.
(b) Show that a diagonalizable matrix having only one eigenvalue is a scalar matrix.
(c) Prove that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.

13. (a) Prove that similar matrices have the same characteristic polynomial.
(b) Show that the definition of the characteristic polynomial of a linear operator on a finite-dimensional vector space V is independent of the choice of basis for V.

14. Let T be a linear operator on a finite-dimensional vector space V over a field F, let β be an ordered basis for V, and let $A = [\mathsf{T}]_\beta$. In reference to Figure 5.1, prove the following.
(a) If $v \in \mathsf{V}$ and $\phi_\beta(v)$ is an eigenvector of A corresponding to the eigenvalue λ, then v is an eigenvector of T corresponding to λ.
(b) If λ is an eigenvalue of A (and hence of T), then a vector $y \in \mathsf{F}^n$ is an eigenvector of A corresponding to λ if and only if $\phi_\beta^{-1}(y)$ is an eigenvector of T corresponding to λ.

15.† For any square matrix A, prove that A and A^t have the same characteristic polynomial (and hence the same eigenvalues). Visit `goo.gl/7Qss2u` for a solution.

16.† **(a)** Let T be a linear operator on a vector space V, and let x be an eigenvector of T corresponding to the eigenvalue λ. For any positive integer m, prove that x is an eigenvector of T^m corresponding to the eigenvalue λ^m.

(b) State and prove the analogous result for matrices.

17. Let T be a linear operator on a finite-dimensional vector space V, and let c be any scalar.

(a) Determine the relationship between the eigenvalues and eigenvectors of T (if any) and the eigenvalues and eigenvectors of $\mathsf{U} = \mathsf{T} - c\mathsf{I}$. Justify your answers.

(b) Prove that T is diagonalizable if and only if U is diagonalizable.

18. Let T be the linear operator on $\mathsf{M}_{n\times n}(R)$ defined by $\mathsf{T}(A) = A^t$.

(a) Show that ± 1 are the only eigenvalues of T.

(b) Describe the eigenvectors corresponding to each eigenvalue of T.

(c) Find an ordered basis β for $\mathsf{M}_{2\times 2}(R)$ such that $[\mathsf{T}]_\beta$ is a diagonal matrix.

(d) Find an ordered basis β for $\mathsf{M}_{n\times n}(R)$ such that $[\mathsf{T}]_\beta$ is a diagonal matrix for $n > 2$.

19. Let $A, B \in \mathsf{M}_{n\times n}(C)$.

(a) Prove that if B is invertible, then there exists a scalar $c \in C$ such that $A + cB$ is not invertible. *Hint:* Examine $\det(A + cB)$.

(b) Find nonzero 2×2 matrices A and B such that both A and $A+cB$ are invertible for all $c \in C$.

20. Let A be an $n \times n$ matrix with characteristic polynomial

$$f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0.$$

Prove that $f(0) = a_0 = \det(A)$. Deduce that A is invertible if and only if $a_0 \neq 0$.

21. Let A and $f(t)$ be as in Exercise 20.

(a) Prove that $f(t) = (A_{11} - t)(A_{22} - t)\cdots(A_{nn} - t) + q(t)$, where $q(t)$ is a polynomial of degree at most $n-2$. *Hint:* Apply mathematical induction to n.

(b) Show that $\operatorname{tr}(A) = (-1)^{n-1}a_{n-1}$.

22.† **(a)** Let T be a linear operator on a vector space V over the field F, and let $g(t)$ be a polynomial with coefficients from F. Prove that if x is an eigenvector of T with corresponding eigenvalue λ, then $g(\mathsf{T})(x) = g(\lambda)x$. That is, x is an eigenvector of $g(\mathsf{T})$ with corresponding eigenvalue $g(\lambda)$.

(b) State and prove a comparable result for matrices.

(c) Verify (b) for the matrix A in Exercise 4(a) with polynomial $g(t) = 2t^2 - t + 1$, eigenvector $x = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, and corresponding eigenvalue $\lambda = 4$.

23. Use Exercise 22 to prove that if $f(t)$ is the characteristic polynomial of a diagonalizable linear operator T, then $f(\mathsf{T}) = \mathsf{T}_0$, the zero operator. (In Section 5.4 we prove that this result does not depend on the diagonalizability of T.)

24. Use Exercise 21(a) to prove Theorem 5.3.

25. Determine the number of distinct characteristic polynomials of matrices in $\mathsf{M}_{2\times 2}(Z_2)$.

5.2 DIAGONALIZABILITY

In Section 5.1, we presented the diagonalization problem and observed that not all linear operators or matrices are diagonalizable. Although we are able to diagonalize operators and matrices and even obtain a necessary and sufficient condition for diagonalizability (Theorem 5.1 p. 247), we have not yet solved the diagonalization problem. What is still needed is a simple test to determine whether an operator or a matrix can be diagonalized, as well as a method for actually finding a basis of eigenvectors. In this section, we develop such a test and method.

In Example 6 of Section 5.1, we obtained a basis of eigenvectors by choosing one eigenvector corresponding to each eigenvalue. In general, such a procedure does not yield a basis, but the following theorem shows that any set constructed in this manner is linearly independent.

Theorem 5.5. *Let T be a linear operator on a vector space, and let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of T. For each $i = 1, 2, \ldots, k$, let S_i be a finite set of eigenvectors of T corresponding to λ_i. If each S_i $(i = 1, 2, \ldots, k)$, is linearly independent, then $S_1 \cup S_2 \cup \cdots \cup S_k$ is linearly independent.*

Proof. The proof is by mathematical induction on k. If $k = 1$, there is nothing to prove. So assume that the theorem holds for $k-1$ distinct eigenvalues, where $k-1 \geq 1$, and that we have k distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ of T. For each $i = 1, 2, \ldots, k$, let $S_i = \{v_{i1}, v_{i2}, \ldots, v_{in_i}\}$ be a linearly independent set of eigenvectors of T corresponding to λ_i. We wish to show that $S = S_1 \cup S_2 \cup \cdots \cup S_k$ is linearly independent.

Consider any scalars $\{a_{ij}\}$, where $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots n_i$, such that

$$\sum_{i=1}^{k} \sum_{j=1}^{n_i} a_{ij} v_{ij} = \mathit{0}. \tag{1}$$

Because v_{ij} is an eigenvector of T corresponding to λ_i, applying $\mathsf{T} - \lambda_k \mathsf{I}$ to both sides of (1) yields

$$\sum_{i=1}^{k-1} \sum_{j=1}^{n_i} a_{ij}(\lambda_i - \lambda_k)v_{ij} = \mathit{0}. \tag{2}$$

But $S_1 \cup S_2 \cup \cdots \cup S_{k-1}$ is linearly independent by the induction hypothesis, so that (2) implies $a_{ij}(\lambda_i - \lambda_k) = 0$ for $i = 1, 2, \ldots, k-1$ and $j = 1, 2, \ldots n_i$. Since $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct, it follows that $\lambda_i - \lambda_k \neq 0$ for $1 \leq i \leq k-1$. Hence $a_{ij} = 0$ for $i = 1, 2, \ldots, k-1$ and $j = 1, 2, \ldots n_i$, and therefore (1) reduces to $\sum_{j=1}^{n_k} a_{kj} v_{kj} = \mathit{0}$. But S_k is also linearly independent, and so $a_{kj} = 0$ for $j = 1, 2, \ldots n_k$. Consequently $a_{ij} = 0$ for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots n_i$, proving that S is linearly independent. ∎

Corollary. *Let* T *be a linear operator on an* n*-dimensional vector space* V. *If* T *has* n *distinct eigenvalues, then* T *is diagonalizable.*

Proof. Suppose that T has n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. For each i choose an eigenvector v_i corresponding to λ_i. By Theorem 5.5, $\{v_1, \ldots, v_n\}$ is linearly independent, and since $\dim(\mathsf{V}) = n$, this set is a basis for V. Thus, by Theorem 5.1 (p. 247), T is diagonalizable. ∎

Example 1

Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \mathsf{M}_{2\times 2}(R).$$

The characteristic polynomial of A (and hence of L_A) is

$$\det(A - tI) = \det \begin{pmatrix} 1-t & 1 \\ 1 & 1-t \end{pmatrix} = t(t-2),$$

and thus the eigenvalues of L_A are 0 and 2. Since L_A is a linear operator on the two-dimensional vector space R^2, we conclude from the preceding corollary that L_A (and hence A) is diagonalizable. ◆

The converse of the corollary to Theorem 5.5 is false. That is, it is not true that if T is diagonalizable, then it has n distinct eigenvalues. For example, the identity operator is diagonalizable even though it has only one eigenvalue, namely, $\lambda = 1$.

We have seen that diagonalizability requires the existence of eigenvalues. Actually, diagonalizability imposes a stronger condition on the characteristic polynomial.

Definition. *A polynomial $f(t)$ in $\mathsf{P}(F)$* **splits over** *F if there are scalars $c, a_1, \ldots, a_n$ (not necessarily distinct) in F such that*

$$f(t) = c(t - a_1)(t - a_2)\cdots(t - a_n).$$

For example, $t^2 - 1 = (t+1)(t-1)$ splits over R, but $(t^2+1)(t-2)$ does not split over R because t^2+1 cannot be factored into a product of linear factors. However, $(t^2+1)(t-2)$ does split over C because it factors into the product $(t+i)(t-i)(t-2)$. If $f(t)$ is the characteristic polynomial of a linear operator or a matrix over a field F, then the statement that $f(t)$ splits is understood to mean that it splits over F.

Theorem 5.6. *The characteristic polynomial of any diagonalizable linear operator on a vector space V over a field F splits over F.*

Proof. Let T be a diagonalizable linear operator on the n-dimensional vector space V, and let β be an ordered basis for V such that $[\mathsf{T}]_\beta = D$ is a diagonal matrix. Suppose that

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

and let $f(t)$ be the characteristic polynomial of T. Then

$$f(t) = \det(D - tI) = \det \begin{pmatrix} \lambda_1 - t & 0 & \cdots & 0 \\ 0 & \lambda_2 - t & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n - t \end{pmatrix}$$

$$= (\lambda_1 - t)(\lambda_2 - t)\cdots(\lambda_n - t) = (-1)^n(t - \lambda_1)(t - \lambda_2)\cdots(t - \lambda_n). \quad \blacksquare$$

From this theorem, it is clear that if T is a diagonalizable linear operator on an n-dimensional vector space that fails to have n distinct eigenvalues, then the characteristic polynomial of T must have repeated zeros.

The converse of Theorem 5.6 is false; that is, the characteristic polynomial of T may split, but T need not be diagonalizable. (See Example 3, which follows.) The following concept helps us determine when an operator whose characteristic polynomial splits is diagonalizable.

Definition. *Let λ be an eigenvalue of a linear operator or matrix with characteristic polynomial $f(t)$. The* **multiplicity** *(sometimes called the* **algebraic multiplicity***) of λ is the largest positive integer k for which $(t - \lambda)^k$ is a factor of $f(t)$.*

Example 2

Let

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{pmatrix},$$

which has characteristic polynomial $f(t) = -(t-3)^2(t-4)$. Hence $\lambda = 3$ is an eigenvalue of A with multiplicity 2, and $\lambda = 4$ is an eigenvalue of A with multiplicity 1. ♦

If T is a diagonalizable linear operator on a finite-dimensional vector space V, then there is an ordered basis β for V consisting of eigenvectors of T. We know from Theorem 5.1 (p. 247) that $[\mathsf{T}]_\beta$ is a diagonal matrix in which the diagonal entries are the eigenvalues of T. Since the characteristic polynomial of T is $\det([\mathsf{T}]_\beta - tI)$, it is easily seen that each eigenvalue of T must occur as a diagonal entry of $[\mathsf{T}]_\beta$ exactly as many times as its multiplicity. Hence β contains as many (linearly independent) eigenvectors corresponding to an eigenvalue as the multiplicity of that eigenvalue. So the number of linearly independent eigenvectors corresponding to a given eigenvalue is of interest in determining whether an operator can be diagonalized. Recalling from Theorem 5.4 (p. 250) that the eigenvectors of T corresponding to the eigenvalue λ are the nonzero vectors in the null space of $\mathsf{T} - \lambda\mathsf{I}$, we are led naturally to the study of this set.

Definition. *Let T be a linear operator on a vector space V, and let λ be an eigenvalue of T. Define $\mathsf{E}_\lambda = \{x \in \mathsf{V} \colon \mathsf{T}(x) = \lambda x\} = \mathsf{N}(\mathsf{T} - \lambda\mathsf{I}_\mathsf{V})$. The set E_λ is called the* **eigenspace** *of T corresponding to the eigenvalue λ. Analogously, we define the* **eigenspace** *of a square matrix A corresponding to the eigenvalue λ to be the eigenspace of L_A corresponding to λ.*

Clearly, E_λ is a subspace of V consisting of the zero vector and the eigenvectors of T corresponding to the eigenvalue λ. The maximum number of linearly independent eigenvectors of T corresponding to the eigenvalue λ is therefore the dimension of E_λ. Our next result relates this dimension to the multiplicity of λ.

Theorem 5.7. *Let T be a linear operator on a finite-dimensional vector space V, and let λ be an eigenvalue of T having multiplicity m. Then $1 \le \dim(\mathsf{E}_\lambda) \le m$.*

Proof. Choose an ordered basis $\{v_1, v_2, \ldots, v_p\}$ for E_λ, extend it to an ordered basis $\beta = \{v_1, v_2, \ldots, v_p, v_{p+1}, \ldots, v_n\}$ for V, and let $A = [\mathsf{T}]_\beta$. Observe that v_i $(1 \le i \le p)$ is an eigenvector of T corresponding to λ, and therefore

$$A = \begin{pmatrix} \lambda I_p & B \\ O & C \end{pmatrix}.$$

By Exercise 21 of Section 4.3, the characteristic polynomial of T is

$$\begin{aligned} f(t) = \det(A - tI_n) &= \det \begin{pmatrix} (\lambda - t)I_p & B \\ O & C - tI_{n-p} \end{pmatrix} \\ &= \det((\lambda - t)I_p) \cdot \det(C - tI_{n-p}) \\ &= (\lambda - t)^p g(t), \end{aligned}$$

where $g(t)$ is a polynomial. Thus $(\lambda - t)^p$ is a factor of $f(t)$, and hence the multiplicity of λ is at least p. But $\dim(\mathsf{E}_\lambda) = p$, and so $\dim(\mathsf{E}_\lambda) \leq m$. ∎

Example 3

Let T be the linear operator on $\mathsf{P}_2(R)$ defined by $\mathsf{T}(f(x)) = f'(x)$. The matrix representation of T with respect to the standard ordered basis β for $\mathsf{P}_2(R)$ is

$$[\mathsf{T}]_\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Consequently, the characteristic polynomial of T is

$$\det([\mathsf{T}]_\beta - tI) = \det \begin{pmatrix} -t & 1 & 0 \\ 0 & -t & 2 \\ 0 & 0 & -t \end{pmatrix} = -t^3.$$

Thus T has only one eigenvalue ($\lambda = 0$) with multiplicity 3. Solving $\mathsf{T}(f(x)) = f'(x) = \mathit{0}$ shows that $\mathsf{E}_\lambda = \mathsf{N}(\mathsf{T} - \lambda \mathsf{I}) = \mathsf{N}(\mathsf{T})$ is the subspace of $\mathsf{P}_2(R)$ consisting of the constant polynomials. So $\{1\}$ is a basis for E_λ, and therefore $\dim(\mathsf{E}_\lambda) = 1$. Consequently, there is no basis for $\mathsf{P}_2(R)$ consisting of eigenvectors of T, and therefore T is not diagonalizable. Even though T is not diagonalizable, we will see in Chapter 7 that its eigenvalue and eigenvectors are still useful for describing the behavior of T. ♦

Example 4

Let T be the linear operator on R^3 defined by

$$\mathsf{T}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 4a_1 + a_3 \\ 2a_1 + 3a_2 + 2a_3 \\ a_1 + 4a_3 \end{pmatrix}.$$

We determine the eigenspace of T corresponding to each eigenvalue. Let β be the standard ordered basis for R^3. Then

$$[\mathsf{T}]_\beta = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix},$$

and hence the characteristic polynomial of T is

$$\det([\mathsf{T}]_\beta - tI) = \det \begin{pmatrix} 4-t & 0 & 1 \\ 2 & 3-t & 2 \\ 1 & 0 & 4-t \end{pmatrix} = -(t-5)(t-3)^2.$$

So the eigenvalues of T are $\lambda_1 = 5$ and $\lambda_2 = 3$ with multiplicities 1 and 2, respectively.

Since

$$\mathsf{E}_{\lambda_1} = \mathsf{N}(\mathsf{T} - \lambda_1 \mathsf{I}) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathsf{R}^3 \colon \begin{pmatrix} -1 & 0 & 1 \\ 2 & -2 & 2 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\},$$

E_{λ_1} is the solution space of the system of linear equations

$$\begin{aligned} -x_1 \qquad\quad + \ x_3 &= 0 \\ 2x_1 - 2x_2 + 2x_3 &= 0 \\ x_1 \qquad\quad - \ x_3 &= 0. \end{aligned}$$

It is easily seen (using the techniques of Chapter 3) that

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$$

is a basis for E_{λ_1}. Hence $\dim(\mathsf{E}_{\lambda_1}) = 1$.

Similarly, $\mathsf{E}_{\lambda_2} = \mathsf{N}(\mathsf{T} - \lambda_2 \mathsf{I})$ is the solution space of the system

$$\begin{aligned} x_1 + \ x_3 &= 0 \\ 2x_1 + 2x_3 &= 0 \\ x_1 + \ x_3 &= 0. \end{aligned}$$

Since the unknown x_2 does not appear in this system, we assign it a parametric value, say, $x_2 = s$, and solve the system for x_1 and x_3, introducing another parameter t. The result is the general solution to the system

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = s \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \text{ for } s, t \in R.$$

It follows that

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for E_{λ_2}, and $\dim(\mathsf{E}_{\lambda_2}) = 2$.

In this case, the multiplicity of each eigenvalue λ_i is equal to the dimension of the corresponding eigenspace E_{λ_i}. Observe that the union of the two bases just derived, namely,

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\},$$

is linearly independent by Theorem 5.5 and hence is a basis for R^3 consisting of eigenvectors of T. Consequently, T is diagonalizable. ◆

Examples 3 and 4 suggest that an operator on V whose characteristic polynomial splits is diagonalizable if and only if the dimension of each eigenspace is equal to the multiplicity of the corresponding eigenvalue. This is indeed true, as our next theorem shows. Moreover, when the operator is diagonalizable, we can use Theorem 5.5 to construct a basis for V consisting of eigenvectors of the operator by collecting bases for the individual eigenspaces.

Theorem 5.8. *Let T be a linear operator on a finite-dimensional vector space V such that the characteristic polynomial of T splits. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct eigenvalues of T. Then*

(a) *T is diagonalizable if and only if the multiplicity of λ_i is equal to $\dim(\mathsf{E}_{\lambda_i})$ for all i.*

(b) *If T is diagonalizable and β_i is an ordered basis for E_{λ_i} for each i, then $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ is an ordered basis[2] for V consisting of eigenvectors of T.*

Proof. For each i, let m_i denote the multiplicity of λ_i, $d_i = \dim(\mathsf{E}_{\lambda_i})$, and $n = \dim(\mathsf{V})$.

First, suppose that T is diagonalizable. Let β be a basis for V consisting of eigenvectors of T. For each i, let $\beta_i = \beta \cap \mathsf{E}_{\lambda_i}$, the set of vectors in β that are eigenvectors corresponding to λ_i, and let n_i denote the number of vectors in β_i. Then $n_i \leq d_i$ for each i because β_i is a linearly independent subset of a subspace of dimension d_i, and $d_i \leq m_i$ by Theorem 5.7. The n_i's sum to n because β contains n vectors. The m_i's also sum to n because the degree of the characteristic polynomial of T is equal to the sum of the multiplicities of the eigenvalues. Thus

$$n = \sum_{i=1}^{k} n_i \leq \sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} m_i = n.$$

It follows that

$$\sum_{i=1}^{k} (m_i - d_i) = 0.$$

[2]We regard $\beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ as an ordered basis in the natural way—the vectors in β_1 are listed first (in the same order as in β_1), then the vectors in β_2 (in the same order as in β_2), etc.

Since $(m_i - d_i) \geq 0$ for all i, we conclude that $m_i = d_i$ for all i.

Conversely, suppose that $m_i = d_i$ for all i. We simultaneously show that T is diagonalizable and prove (b). For each i, let β_i be an ordered basis for E_{λ_i}, and let $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$. By Theorem 5.5, β is linearly independent. Furthermore, since $d_i = m_i$ for all i, β contains

$$\sum_{i=1}^{k} d_i = \sum_{i=1}^{k} m_i = n$$

vectors. Therefore β is an ordered basis for V consisting of eigenvectors of V, and we conclude that T is diagonalizable. ∎

This theorem completes our study of the diagonalization problem. We summarize our results.

Test for Diagonalizability

Let T be a linear operator on an n-dimensional vector space V. Then T is diagonalizable if and only if both of the following conditions hold.

1. The characteristic polynomial of T splits.
2. For each eigenvalue λ of T, the multiplicity of λ equals nullity$(\mathsf{T} - \lambda\mathsf{I})$, that is, the multiplicity of λ equals $n - \text{rank}(\mathsf{T} - \lambda\mathsf{I})$.

These same conditions can be used to test if a square matrix A is diagonalizable because diagonalizability of A is equivalent to diagonalizability of the operator L_A.

If T is a diagonalizable operator and $\beta_1, \beta_2, \ldots, \beta_k$ are ordered bases for the eigenspaces of T, then the union $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ is an ordered basis for V consisting of eigenvectors of T, and hence $[\mathsf{T}]_\beta$ is a diagonal matrix.

When testing T for diagonalizability, it is usually easiest to choose a convenient basis α for V and work with $B = [\mathsf{T}]_\alpha$. If the characteristic polynomial of B splits, then use condition 2 above to check if the multiplicity of each *repeated* eigenvalue of B equals $n - \text{rank}(B - \lambda I)$. (By Theorem 5.7, condition 2 is automatically satisfied for eigenvalues of multiplicity 1.) If so, then B, and hence T, is diagonalizable.

If T is diagonalizable and a basis β for V consisting of eigenvectors of T is desired, then we first find a basis for each eigenspace of B. The union of these bases is a basis γ for F^n consisting of eigenvectors of B. Each vector in γ is the coordinate vector relative to α of an eigenvector of T. The set consisting of these n eigenvectors of T is the desired basis β.

Furthermore, if A is an $n \times n$ diagonalizable matrix, we can use the corollary to Theorem 2.23 (p. 115) to find an invertible $n \times n$ matrix Q and a diagonal $n \times n$ matrix D such that $Q^{-1}AQ = D$. The matrix Q has as its columns the vectors in a basis of eigenvectors of A, and D has as its jth diagonal entry the eigenvalue of A corresponding to the jth column of Q.

We now consider some examples illustrating the preceding ideas.

Example 5

We test the matrix

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \in \mathsf{M}_{3\times 3}(R)$$

for diagonalizability.

The characteristic polynomial of A is $\det(A-tI) = -(t-4)(t-3)^2$, which splits, and so condition 1 of the test for diagonalization is satisfied. Also A has eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 3$ with multiplicities 1 and 2, respectively. Since λ_1 has multiplicity 1, condition 2 is satisfied for λ_1. Thus we need only test condition 2 for λ_2. Because

$$A - \lambda_2 I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

has rank 2, we see that $3 - \operatorname{rank}(A - \lambda_2 I) = 1$, which is not the multiplicity of λ_2. Thus condition 2 fails for λ_2, and A is therefore not diagonalizable. ◆

Example 6

Let T be the linear operator on $\mathsf{P}_2(R)$ defined by

$$\mathsf{T}(f(x)) = f(1) + f'(0)x + (f'(0) + f''(0))x^2.$$

We first test T for diagonalizability. Let α denote the standard ordered basis for $\mathsf{P}_2(R)$ and $B = [\mathsf{T}]_\alpha$. Then

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

The characteristic polynomial of B, and hence of T, is $-(t-1)^2(t-2)$, which splits. Hence condition 1 of the test for diagonalization is satisfied. Also B has the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$ with multiplicities 2 and 1, respectively. Condition 2 is satisfied for λ_2 because it has multiplicity 1. So we need only verify condition 2 for $\lambda_1 = 1$. For this case,

$$3 - \operatorname{rank}(B - \lambda_1 I) = 3 - \operatorname{rank}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = 3 - 1 = 2,$$

which is equal to the multiplicity of λ_1. Therefore T is diagonalizable.

We now find an ordered basis γ for R^3 consisting of eigenvectors of B. We consider each eigenvalue separately.

The eigenspace corresponding to $\lambda_1 = 1$ is

$$\mathsf{E}_{\lambda_1} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathsf{R}^3 \colon \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathit{0} \right\},$$

which is the solution space for the system

$$x_2 + x_3 = 0,$$

and has

$$\gamma_1 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

as a basis.

The eigenspace corresponding to $\lambda_2 = 2$ is

$$\mathsf{E}_{\lambda_2} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathsf{R}^3 \colon \begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathit{0} \right\},$$

which is the solution space for the system

$$\begin{aligned} -x_1 + x_2 + x_3 &= 0 \\ x_2 \qquad &= 0, \end{aligned}$$

and has

$$\gamma_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

as a basis.

Let

$$\gamma = \gamma_1 \cup \gamma_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Then γ is an ordered basis for R^3 consisting of eigenvectors of B.

Finally, observe that the vectors in γ are the coordinate vectors relative to α of the vectors in the set

$$\beta = \{1, -x + x^2, 1 + x^2\},$$

which is an ordered basis for $\mathsf{P}_2(R)$ consisting of eigenvectors of T. Thus

$$[\mathsf{T}]_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \quad \blacklozenge$$

Our next example is an application of diagonalization that is of interest in Section 5.3.

Example 7

Let

$$A = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}.$$

We show that A is diagonalizable and find a 2×2 matrix Q such that $Q^{-1}AQ$ is a diagonal matrix. We then show how to use this result to compute A^n for any positive integer n.

First observe that the characteristic polynomial of A is $(t-1)(t-2)$, and hence A has two distinct eigenvalues, $\lambda_1 = 1$ and $\lambda_2 = 2$. By applying the corollary to Theorem 5.5 to the operator L_A, we see that A is diagonalizable. Moreover,

$$\gamma_1 = \left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad \gamma_2 = \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

are bases for the eigenspaces E_{λ_1} and E_{λ_2}, respectively. Therefore

$$\gamma = \gamma_1 \cup \gamma_2 = \left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

is an ordered basis for R^2 consisting of eigenvectors of A. Let

$$Q = \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix},$$

the matrix whose columns are the vectors in γ. Then, by the corollary to Theorem 2.23 (p. 115),

$$D = Q^{-1}AQ = [\mathsf{L}_A]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

To find A^n for any positive integer n, observe that $A = QDQ^{-1}$. Therefore

$$\begin{aligned} A^n &= (QDQ^{-1})^n \\ &= (QDQ^{-1})(QDQ^{-1}) \cdots (QDQ^{-1}) \\ &= QD^nQ^{-1} \\ &= Q \begin{pmatrix} 1^n & 0 \\ 0 & 2^n \end{pmatrix} Q^{-1} \\ &= \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2^n \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2-2^n & 2-2^{n+1} \\ -1+2^n & -1+2^{n+1} \end{pmatrix}. \quad \blacklozenge \end{aligned}$$

We now consider an application that uses diagonalization to solve a system of differential equations.

Systems of Differential Equations

Consider the system of differential equations

$$\begin{aligned} x_1' &= 3x_1 + x_2 + x_3 \\ x_2' &= 2x_1 + 4x_2 + 2x_3 \\ x_3' &= -x_1 - x_2 + x_3, \end{aligned}$$

where, for each i, $x_i = x_i(t)$ is a differentiable real-valued function of the real variable t. Clearly, this system has a solution, namely, the solution in which each $x_i(t)$ is the zero function. We determine all of the solutions to this system.

Let $x\colon R \to \mathsf{R}^3$ be the function defined by

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}.$$

The **derivative** of x, denoted x', is defined by

$$x'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$$

be the coefficient matrix of the given system, so that we can rewrite the system as the matrix equation $x' = Ax$.

It can be verified that for

$$Q = \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & -2 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix},$$

we have $Q^{-1}AQ = D$. Substitute $A = QDQ^{-1}$ into $x' = Ax$ to obtain $x' = QDQ^{-1}x$ or, equivalently, $Q^{-1}x' = DQ^{-1}x$. The function $y\colon R \to \mathsf{R}^3$ defined by $y(t) = Q^{-1}x(t)$ can be shown to be differentiable, and $y' = Q^{-1}x'$ (see Exercise 17). Hence the original system can be written as $y' = Dy$.

Since D is a diagonal matrix, the system $y' = Dy$ is easy to solve. Setting

$$y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix},$$

we can rewrite $y' = Dy$ as

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} = \begin{pmatrix} 2y_1(t) \\ 2y_2(t) \\ 4y_3(t) \end{pmatrix}.$$

The three equations

$$\begin{aligned} y_1' &= 2y_1 \\ y_2' &= 2y_2 \\ y_3' &= 4y_3 \end{aligned}$$

are independent of each other, and thus can be solved individually. It is easily seen (as in Example 3 of Section 5.1) that the general solution to these equations is $y_1(t) = c_1e^{2t}$, $y_2(t) = c_2e^{2t}$, and $y_3(t) = c_3e^{4t}$, where c_1, c_2, and c_3 are arbitrary constants. Finally,

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = x(t) = Qy(t) = \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & -2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1e^{2t} \\ c_2e^{2t} \\ c_3e^{4t} \end{pmatrix}$$

$$= \begin{pmatrix} -c_1e^{2t} & & - \;\; c_3e^{4t} \\ & - c_2e^{2t} & - 2c_3e^{4t} \\ c_1e^{2t} & + c_2e^{2t} & + \;\; c_3e^{4t} \end{pmatrix}$$

yields the general solution of the original system. Note that this solution can be written as

$$x(t) = e^{2t}\left[c_1\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right] + e^{4t}\left[c_3\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}\right].$$

The expressions in brackets are arbitrary vectors in E_{λ_1} and E_{λ_2}, respectively, where $\lambda_1 = 2$ and $\lambda_2 = 4$. Thus the general solution of the original system is $x(t) = e^{2t}z_1 + e^{4t}z_2$, where $z_1 \in \mathsf{E}_{\lambda_1}$ and $z_2 \in \mathsf{E}_{\lambda_2}$. This result is generalized in Exercise 16.

Direct Sums*

Let T be a linear operator on a finite-dimensional vector space V. There is a way of decomposing V into simpler subspaces that offers insight into the behavior of T. This approach is especially useful in Chapter 7, where we study nondiagonalizable linear operators. In the case of diagonalizable operators, the simpler subspaces are the eigenspaces of the operator.

Definition. *Let $\mathsf{W}_1, \mathsf{W}_2, \ldots, \mathsf{W}_k$ be subspaces of a vector space V. We define the* **sum** *of these subspaces to be the set*

$$\{v_1 + v_2 + \cdots + v_k \colon v_i \in \mathsf{W}_i \text{ for } 1 \le i \le k\},$$

which we denote by $\mathsf{W}_1 + \mathsf{W}_2 + \cdots + \mathsf{W}_k$ or $\sum_{i=1}^{k} \mathsf{W}_i$.

It is a simple exercise to show that the sum of subspaces of a vector space is also a subspace.

Example 8

Let $\mathsf{V} = \mathsf{R}^3$, let W_1 denote the xy-plane, and let W_2 denote the yz-plane. Then $\mathsf{R}^3 = \mathsf{W}_1 + \mathsf{W}_2$ because, for any vector $(a, b, c) \in \mathsf{R}^3$, we have

$$(a, b, c) = (a, 0, 0) + (0, b, c),$$

where $(a, 0, 0) \in \mathsf{W}_1$ and $(0, b, c) \in \mathsf{W}_2$. ◆

Notice that in Example 8 the representation of (a, b, c) as a sum of vectors in W_1 and W_2 is not unique. For example, $(a, b, c) = (a, b, 0) + (0, 0, c)$ is another representation. Because we are often interested in sums for which representations are unique, we introduce a condition that assures this outcome. The definition of *direct sum* that follows is a generalization of the definition given in the exercises of Section 1.3.

Definition. *Let $\mathsf{W}, \mathsf{W}_1, \mathsf{W}_2, \ldots, \mathsf{W}_k$ be subspaces of a vector space V such that $\mathsf{W}_i \subseteq \mathsf{W}$ for $i = 1, 2, \ldots, k$. We call W the* **direct sum** *of the subspaces $\mathsf{W}_1, \mathsf{W}_2, \ldots, \mathsf{W}_k$, and write $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \cdots \oplus \mathsf{W}_k$, if*

$$\mathsf{V} = \sum_{i=1}^{k} \mathsf{W}_i \qquad \text{and} \qquad \mathsf{W}_j \cap \sum_{i \ne j} \mathsf{W}_i = \{\mathit{0}\} \quad \text{for each } j \ (1 \le j \le k).$$

Example 9

In R^5, let $\mathsf{W} = \{(x_1, x_2, x_3, x_4, x_5) \colon x_5 = 0\}$, $\mathsf{W}_1 = \{(a, b, 0, 0, 0) \colon a, b \in R\}$, $\mathsf{W}_2 = \{(0, 0, c, 0, 0) \colon c \in R\}$, and $\mathsf{W}_3 = \{(0, 0, 0, d, 0) \colon d \in R\}$. For any $(a, b, c, d, 0) \in \mathsf{W}$,

$$(a, b, c, d, 0) = (a, b, 0, 0, 0) + (0, 0, c, 0, 0) + (0, 0, 0, d, 0) \in \mathsf{W}_1 + \mathsf{W}_2 + \mathsf{W}_3.$$

Thus

$$\mathsf{W} = \sum_{i=1}^{3} \mathsf{W}_i.$$

To show that W is the direct sum of W_1, W_2, and W_3, we must prove that $\mathsf{W}_1 \cap (\mathsf{W}_2 + \mathsf{W}_3) = \mathsf{W}_2 \cap (\mathsf{W}_1 + \mathsf{W}_3) = \mathsf{W}_3 \cap (\mathsf{W}_1 + \mathsf{W}_2) = \{\mathit{0}\}$. But these equalities are obvious, and so $\mathsf{W} = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \mathsf{W}_3$. ◆

Our next result contains several conditions that are equivalent to the definition of a direct sum.

Theorem 5.9. *Let $\mathsf{W}_1, \mathsf{W}_2, \ldots, \mathsf{W}_k$ be subspaces of a finite-dimensional vector space V. The following conditions are equivalent.*

(a) $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \cdots \oplus \mathsf{W}_k$.

(b) $\mathsf{V} = \sum_{i=1}^{k} \mathsf{W}_i$ *and, for any vectors $v_1, v_2, \ldots, v_k$ such that $v_i \in \mathsf{W}_i$ $(1 \le i \le k)$, if $v_1 + v_2 + \cdots + v_k = \mathit{0}$, then $v_i = \mathit{0}$ for all i.*

(c) *Each vector $v \in \mathsf{V}$ can be uniquely written as $v = v_1 + v_2 + \cdots + v_k$, where $v_i \in \mathsf{W}_i$.*

(d) *If γ_i is an ordered basis for W_i $(1 \le i \le k)$, then $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$ is an ordered basis for V.*

(e) *For each $i = 1, 2, \ldots, k$, there exists an ordered basis γ_i for W_i such that $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$ is an ordered basis for V.*

Proof. Assume (a). We prove (b). Clearly

$$\mathsf{V} = \sum_{i=1}^{k} \mathsf{W}_i.$$

Now suppose that $v_1, v_2, \ldots, v_k$ are vectors such that $v_i \in \mathsf{W}_i$ for all i and $v_1 + v_2 + \cdots + v_k = \mathit{0}$. Then for any j

$$-v_j = \sum_{i \ne j} v_i \in \sum_{i \ne j} \mathsf{W}_i.$$

But $-v_j \in \mathsf{W}_j$ and hence

$$-v_j \in \mathsf{W}_j \cap \sum_{i \ne j} \mathsf{W}_i = \{\mathit{0}\}.$$

So $v_j = \mathit{0}$, proving (b).

Now assume (b). We prove (c). Let $v \in \mathsf{V}$. By (b), there exist vectors $v_1, v_2, \ldots, v_k$ such that $v_i \in \mathsf{W}_i$ and $v = v_1 + v_2 + \cdots + v_k$. We must show that this representation is unique. Suppose also that $v = w_1 + w_2 + \cdots + w_k$, where $w_i \in \mathsf{W}_i$ for all i. Then

$$(v_1 - w_1) + (v_2 - w_2) + \cdots + (v_k - w_k) = \mathit{0}.$$

But $v_i - w_i \in \mathsf{W}_i$ for all i, and therefore $v_i - w_i = \mathit{0}$ for all i by (b). Thus $v_i = w_i$ for all i, proving the uniqueness of the representation.

Now assume (c). We prove (d). For each i, let γ_i be an ordered basis for W_i. Since

$$\mathsf{V} = \sum_{i=1}^{k} \mathsf{W}_i$$

by (c), it follows that $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$ generates V. To show that this set is linearly independent, consider vectors $v_{ij} \in \gamma_i$ ($j = 1, 2, \ldots, m_i$ and $i = 1, 2, \ldots, k$) and scalars a_{ij} such that

$$\sum_{i,j} a_{ij} v_{ij} = \mathit{0}.$$

For each i, set

$$w_i = \sum_{j=1}^{m_i} a_{ij} v_{ij}.$$

Then for each i, $w_i \in \operatorname{span}(\gamma_i) = \mathsf{W}_i$ and

$$w_1 + w_2 + \cdots + w_k = \sum_{i,j} a_{ij} v_{ij} = \mathit{0}.$$

Since $\mathit{0} \in \mathsf{W}_i$ for each i and $\mathit{0} + \mathit{0} + \cdots + \mathit{0} = w_1 + w_2 + \cdots + w_k$, (c) implies that $w_i = \mathit{0}$ for all i. Thus

$$\mathit{0} = w_i = \sum_{j=1}^{m_i} a_{ij} v_{ij}$$

for each i. But each γ_i is linearly independent, and hence $a_{ij} = 0$ for all i and j. Consequently $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$ is linearly independent and therefore is a basis for V.

Clearly (e) follows immediately from (d).

Finally, we assume (e) and prove (a). For each i, let γ_i be an ordered basis for W_i such that $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$ is an ordered basis for V. Then

$$\begin{aligned}\mathsf{V} &= \operatorname{span}(\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k)\\ &= \operatorname{span}(\gamma_1) + \operatorname{span}(\gamma_2) + \cdots + \operatorname{span}(\gamma_k) = \sum_{i=1}^{k} \mathsf{W}_i\end{aligned}$$

by repeated applications of Exercise 14 of Section 1.4. Fix j ($1 \leq j \leq k$), and suppose that, for some nonzero vector $v \in \mathsf{V}$,

$$v \in \mathsf{W}_j \cap \sum_{i \neq j} \mathsf{W}_i.$$

Then

$$v \in \mathsf{W}_j = \operatorname{span}(\gamma_j) \quad \text{and} \quad v \in \sum_{i \neq j} \mathsf{W}_i = \operatorname{span}\left(\bigcup_{i \neq j} \gamma_i\right).$$

Hence v is a nontrivial linear combination of both γ_j and $\displaystyle\bigcup_{i \neq j} \gamma_i$, so that v can be expressed as a linear combination of $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$ in more than one way. But these representations contradict Theorem 1.8 (p. 44), and so we conclude that

$$\mathsf{W}_j \cap \sum_{i \neq j} \mathsf{W}_i = \{0\},$$

proving (a). ∎

With the aid of Theorem 5.9, we are able to characterize diagonalizability in terms of direct sums.

Theorem 5.10. *A linear operator* T *on a finite-dimensional vector space* V *is diagonalizable if and only if* V *is the direct sum of the eigenspaces of* T.

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct eigenvalues of T.

First suppose that T is diagonalizable, and for each i choose an ordered basis γ_i for the eigenspace E_{λ_i}. By Theorem 5.8, $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$ is a basis for V, and hence V is a direct sum of the E_{λ_i}'s by Theorem 5.9.

Conversely, suppose that V is a direct sum of the eigenspaces of T. For each i, choose an ordered basis γ_i of E_{λ_i}. By Theorem 5.9, the union $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$ is a basis for V. Since this basis consists of eigenvectors of T, we conclude that T is diagonalizable. ∎

Example 10

Let T be the linear operator on R^4 defined by

$$\mathsf{T}(a, b, c, d) = (a, b, 2c, 3d).$$

It is easily seen that T is diagonalizable with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$. Furthermore, the corresponding eigenspaces coincide with the subspaces W_1, W_2, and W_3 of Example 9. Thus Theorem 5.10 provides us with another proof that $\mathsf{R}^4 = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \mathsf{W}_3$. ◆

EXERCISES

1. Label the following statements as true or false.

(a) Any linear operator on an n-dimensional vector space that has fewer than n distinct eigenvalues is not diagonalizable.

(b) Two distinct eigenvectors corresponding to the same eigenvalue are always linearly dependent.

(c) If λ is an eigenvalue of a linear operator T, then each vector in E_λ is an eigenvector of T.

(d) If λ_1 and λ_2 are distinct eigenvalues of a linear operator T, then $\mathsf{E}_{\lambda_1} \cap \mathsf{E}_{\lambda_2} = \{0\}$.

(e) Let $A \in \mathsf{M}_{n\times n}(F)$ and $\beta = \{v_1, v_2, \dots, v_n\}$ be an ordered basis for F^n consisting of eigenvectors of A. If Q is the $n \times n$ matrix whose jth column is v_j $(1 \le j \le n)$, then $Q^{-1}AQ$ is a diagonal matrix.

(f) A linear operator T on a finite-dimensional vector space is diagonalizable if and only if the multiplicity of each eigenvalue λ equals the dimension of E_λ.

(g) Every diagonalizable linear operator on a nonzero vector space has at least one eigenvalue.

The following two items relate to the optional subsection on direct sums.

(h) If a vector space is the direct sum of subspaces $\mathsf{W}_1, \mathsf{W}_2, \dots, \mathsf{W}_k$, then $\mathsf{W}_i \cap \mathsf{W}_j = \{0\}$ for $i \neq j$.

(i) If

$$\mathsf{V} = \sum_{i=1}^{k} \mathsf{W}_i \quad \text{and} \quad \mathsf{W}_i \cap \mathsf{W}_j = \{0\} \quad \text{for } i \neq j,$$

then $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \cdots \oplus \mathsf{W}_k$.

2. For each of the following matrices $A \in \mathsf{M}_{n\times n}(R)$, test A for diagonalizability, and if A is diagonalizable, find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

(a) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ **(c)** $\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$

(d) $\begin{pmatrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{pmatrix}$ **(e)** $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$ **(f)** $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

(g) $\begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$

3. For each of the following linear operators T on a vector space V, test T for diagonalizability, and if T is diagonalizable, find a basis β for V such that $[\mathsf{T}]_\beta$ is a diagonal matrix.

(a) $\mathsf{V} = \mathsf{P}_3(R)$ and T is defined by $\mathsf{T}(f(x)) = f'(x) + f''(x)$.

(b) $\mathsf{V} = \mathsf{P}_2(R)$ and T is defined by $\mathsf{T}(ax^2 + bx + c) = cx^2 + bx + a$.

(c) $\mathsf{V} = \mathsf{R}^3$ and T is defined by

$$\mathsf{T}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_2 \\ -a_1 \\ 2a_3 \end{pmatrix}.$$

(d) $\mathsf{V} = \mathsf{P}_2(R)$ and T is defined by $\mathsf{T}(f(x)) = f(0) + f(1)(x + x^2)$.

(e) $\mathsf{V} = \mathsf{C}^2$ and T is defined by $\mathsf{T}(z, w) = (z + iw, iz + w)$.

(f) $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$ and T is defined by $\mathsf{T}(A) = A^t$.

4. Prove the matrix version of the corollary to Theorem 5.5: If $A \in \mathsf{M}_{n\times n}(F)$ has n distinct eigenvalues, then A is diagonalizable.

5. State and prove the matrix version of Theorem 5.6.

6. **(a)** Justify the test for diagonalizability and the method for diagonalization stated in this section.

(b) Formulate the results in (a) for matrices.

7. For

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \in \mathsf{M}_{2\times 2}(R),$$

find an expression for A^n, where n is an arbitrary positive integer.

8. Suppose that $A \in \mathsf{M}_{n\times n}(F)$ has two distinct eigenvalues, λ_1 and λ_2, and that $\dim(\mathsf{E}_{\lambda_1}) = n - 1$. Prove that A is diagonalizable.

9. Let T be a linear operator on a finite-dimensional vector space V, and suppose there exists an ordered basis β for V such that $[\mathsf{T}]_\beta$ is an upper triangular matrix.

(a) Prove that the characteristic polynomial for T splits.

(b) State and prove an analogous result for matrices.

The converse of (a) is treated in Exercise 12(b).

10. Let T be a linear operator on a finite-dimensional vector space V with the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ and corresponding multiplicities $m_1, m_2, \ldots, m_k$. Suppose that β is a basis for V such that $[\mathsf{T}]_\beta$ is an upper triangular matrix. Prove that the diagonal entries of $[\mathsf{T}]_\beta$ are $\lambda_1, \lambda_2, \ldots, \lambda_k$ and that each λ_i occurs m_i times $(1 \le i \le k)$.

11. Let A be an $n \times n$ matrix that is similar to an upper triangular matrix and has the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ with corresponding multiplicities $m_1, m_2, \ldots, m_k$. Prove the following statements.

(a) $\operatorname{tr}(A) = \displaystyle\sum_{i=1}^{k} m_i\lambda_i$

(b) $\det(A) = (\lambda_1)^{m_1}(\lambda_2)^{m_2}\cdots(\lambda_k)^{m_k}$.

12. **(a)** Prove that if $A \in \mathsf{M}_{n\times n}(F)$ and the characteristic polynomial of A splits, then A is similar to an upper triangular matrix. (This proves the converse of Exercise 9(b).) *Hint:* Use mathematical induction on n. For the general case, let v_1 be an eigenvector of A, and extend $\{v_1\}$ to a basis $\{v_1, v_2, \ldots, v_n\}$ for F^n. Let P be the $n \times n$ matrix whose jth column is v_j, and consider $P^{-1}AP$. Exercise 13(a) in Section 5.1 and Exercise 21 in Section 4.3 can be helpful.

(b) Prove the converse of Exercise 9(a).

Visit `goo.gl/gJSjRU` for a solution.

13. Let T be an invertible linear operator on a finite-dimensional vector space V.

(a) Recall that for any eigenvalue λ of T, λ^{-1} is an eigenvalue of T^{-1} (Exercise 9 of Section 5.1). Prove that the eigenspace of T corresponding to λ is the same as the eigenspace of T^{-1} corresponding to λ^{-1}.

(b) Prove that if T is diagonalizable, then T^{-1} is diagonalizable.

14. Let $A \in \mathsf{M}_{n\times n}(F)$. Recall from Exercise 15 of Section 5.1 that A and A^t have the same characteristic polynomial and hence share the same eigenvalues with the same multiplicities. For any eigenvalue λ of A and A^t, let E_λ and E'_λ denote the corresponding eigenspaces for A and A^t, respectively.

(a) Show by way of example that for a given common eigenvalue, these two eigenspaces need not be the same.

(b) Prove that for any eigenvalue λ, $\dim(\mathsf{E}_\lambda) = \dim(\mathsf{E}'_\lambda)$.

(c) Prove that if A is diagonalizable, then A^t is also diagonalizable.

15. Find the general solution to each system of differential equations.

(a) $\begin{aligned} x' &= x + y \\ y' &= 3x - y \end{aligned}$ **(b)** $\begin{aligned} x_1' &= 8x_1 + 10x_2 \\ x_2' &= -5x_1 - 7x_2 \end{aligned}$

(c) $\begin{aligned} x_1' &= x_1 + x_3 \\ x_2' &= x_2 + x_3 \\ x_3' &= 2x_3 \end{aligned}$

16. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

be the coefficient matrix of the system of differential equations

$$\begin{aligned}
x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
&\vdots \\
x_n' &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n.
\end{aligned}$$

Suppose that A is diagonalizable and that the distinct eigenvalues of A are $\lambda_1, \lambda_2, \ldots, \lambda_k$. Prove that a differentiable function $x\colon R \to \mathsf{R}^n$ is a solution to the system if and only if x is of the form

$$x(t) = e^{\lambda_1 t} z_1 + e^{\lambda_2 t} z_2 + \cdots + e^{\lambda_k t} z_k,$$

where $z_i \in \mathsf{E}_{\lambda_i}$ for $i = 1, 2, \ldots, k$. Use this result to prove that the set of solutions to the system is an n-dimensional real vector space.

17. Let $C \in \mathsf{M}_{m\times n}(R)$, and let Y be an $n \times p$ matrix of differentiable functions. Prove $(CY)' = CY'$, where $(Y')_{ij} = Y'_{ij}$ for all i, j.

Exercises 18 through 20 are concerned with *simultaneous diagonalization.*

Definitions. *Two linear operators* T *and* U *on a finite-dimensional vector space* V *are called* ***simultaneously diagonalizable*** *if there exists an ordered basis* β *for* V *such that both* $[\mathsf{T}]_\beta$ *and* $[\mathsf{U}]_\beta$ *are diagonal matrices. Similarly,* $A, B \in \mathsf{M}_{n\times n}(F)$ *are called* ***simultaneously diagonalizable*** *if there exists an invertible matrix* $Q \in \mathsf{M}_{n\times n}(F)$ *such that both* $Q^{-1}AQ$ *and* $Q^{-1}BQ$ *are diagonal matrices.*

18. **(a)** Prove that if T and U are simultaneously diagonalizable linear operators on a finite-dimensional vector space V, then the matrices $[\mathsf{T}]_\beta$ and $[\mathsf{U}]_\beta$ are simultaneously diagonalizable for any ordered basis β.

(b) Prove that if A and B are simultaneously diagonalizable matrices, then L_A and L_B are simultaneously diagonalizable linear operators.

19. **(a)** Prove that if T and U are simultaneously diagonalizable operators, then T and U commute (i.e., $\mathsf{TU} = \mathsf{UT}$).

(b) Show that if A and B are simultaneously diagonalizable matrices, then A and B commute.

The converses of (a) and (b) are established in Exercise 25 of Section 5.4.

20. Let T be a diagonalizable linear operator on a finite-dimensional vector space, and let m be any positive integer. Prove that T and T^m are simultaneously diagonalizable.

Exercises 21 through 24 are concerned with direct sums.

21. Let $\mathsf{W}_1, \mathsf{W}_2, \ldots, \mathsf{W}_k$ be subspaces of a finite-dimensional vector space V such that

$$\sum_{i=1}^{k} \mathsf{W}_i = \mathsf{V}.$$

Prove that V is the direct sum of $\mathsf{W}_1, \mathsf{W}_2, \ldots, \mathsf{W}_k$ if and only if

$$\dim(\mathsf{V}) = \sum_{i=1}^{k} \dim(\mathsf{W}_i).$$

22. Let V be a finite-dimensional vector space with a basis β, and let $\beta_1, \beta_2, \ldots, \beta_k$ be a partition of β (i.e., $\beta_1, \beta_2, \ldots, \beta_k$ are subsets of β such that $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ and $\beta_i \cap \beta_j = \varnothing$ if $i \neq j$). Prove that $\mathsf{V} = \operatorname{span}(\beta_1) \oplus \operatorname{span}(\beta_2) \oplus \cdots \oplus \operatorname{span}(\beta_k)$.

23. Let T be a linear operator on a finite-dimensional vector space V, and suppose that the distinct eigenvalues of T are $\lambda_1, \lambda_2, \ldots, \lambda_k$. Prove that

$$\operatorname{span}(\{x \in \mathsf{V} \colon x \text{ is an eigenvector of } \mathsf{T}\}) = \mathsf{E}_{\lambda_1} \oplus \mathsf{E}_{\lambda_2} \oplus \cdots \oplus \mathsf{E}_{\lambda_k}.$$

24. Let $\mathsf{W}_1, \mathsf{W}_2, \mathsf{K}_1, \mathsf{K}_2, \ldots, \mathsf{K}_p, \mathsf{M}_1, \mathsf{M}_2, \ldots, \mathsf{M}_q$ be subspaces of a vector space V such that $\mathsf{W}_1 = \mathsf{K}_1 \oplus \mathsf{K}_2 \oplus \cdots \oplus \mathsf{K}_p$ and $\mathsf{W}_2 = \mathsf{M}_1 \oplus \mathsf{M}_2 \oplus \cdots \oplus \mathsf{M}_q$. Prove that if $\mathsf{W}_1 \cap \mathsf{W}_2 = \{0\}$, then

$$\mathsf{W}_1 + \mathsf{W}_2 = \mathsf{W}_1 \oplus \mathsf{W}_2 = \mathsf{K}_1 \oplus \mathsf{K}_2 \oplus \cdots \oplus \mathsf{K}_p \oplus \mathsf{M}_1 \oplus \mathsf{M}_2 \oplus \cdots \oplus \mathsf{M}_q.$$

5.3* MATRIX LIMITS AND MARKOV CHAINS

In this section, we apply what we have learned thus far in Chapter 5 to study the *limit* of a sequence of powers $A, A^2, \ldots, A^n, \ldots$, where A is a square matrix with complex entries. Such sequences and their limits have practical applications in the natural and social sciences.

We assume familiarity with limits of sequences of real numbers. The limit of a sequence of complex numbers $\{z_m \colon m = 1, 2, \ldots\}$ can be defined in terms of the limits of the sequences of the real and imaginary parts: If $z_m = r_m + is_m$, where r_m and s_m are real numbers, and i is the imaginary number such that $i^2 = -1$, then

$$\lim_{m\to\infty} z_m = \lim_{m\to\infty} r_m + i \lim_{m\to\infty} s_m,$$

provided that $\lim\limits_{m\to\infty} r_m$ and $\lim\limits_{m\to\infty} s_m$ exist.

Definition. *Let $L, A_1, A_2, \ldots$ be $n \times p$ matrices having complex entries. The sequence $A_1, A_2, \ldots$ is said to* **converge** *to the $n \times p$ matrix L, called the* **limit** *of the sequence, if*

$$\lim_{m\to\infty} (A_m)_{ij} = L_{ij}$$

for all $1 \le i \le n$ and $1 \le j \le p$. To denote that L is the limit of the sequence, we write

$$\lim_{m\to\infty} A_m = L.$$

Example 1

If

$$A_m = \begin{pmatrix} 1-\frac{1}{m} & \left(-\frac{3}{4}\right)^m & \frac{3m^2}{m^2+1} + i\left(\frac{2m+1}{m-1}\right) \\ \left(\frac{i}{2}\right)^m & 2 & \left(1+\frac{1}{m}\right)^m \end{pmatrix},$$

then

$$\lim_{m\to\infty} A_m = \begin{pmatrix} 1 & 0 & 3+2i \\ 0 & 2 & e \end{pmatrix},$$

where e is the base of the natural logarithm. ◆

A simple, but important, property of matrix limits is contained in the next theorem. Note the analogy with the familiar property of limits of sequences of real numbers that asserts that if $\lim\limits_{m\to\infty} a_m$ exists, then

$$\lim_{m\to\infty} ca_m = c\left(\lim_{m\to\infty} a_m\right).$$

Theorem 5.11. *Let $A_1, A_2, \ldots$ be a sequence of $n \times p$ matrices with complex entries that converges to the matrix L. Then for any $P \in \mathsf{M}_{r\times n}(C)$ and $Q \in \mathsf{M}_{p\times s}(C)$,*

$$\lim_{m\to\infty} PA_m = PL \quad \text{and} \quad \lim_{m\to\infty} A_mQ = LQ.$$

Proof. For any i $(1 \le i \le r)$ and j $(1 \le j \le p)$,

$$\begin{aligned}\lim_{m\to\infty} (PA_m)_{ij} &= \lim_{m\to\infty} \sum_{k=1}^{n} P_{ik}(A_m)_{kj} \\ &= \sum_{k=1}^{n} P_{ik} \cdot \lim_{m\to\infty} (A_m)_{kj} = \sum_{k=1}^{n} P_{ik}L_{kj} = (PL)_{ij}.\end{aligned}$$

Hence $\lim\limits_{m\to\infty} PA_m = PL$. The proof that $\lim\limits_{m\to\infty} A_mQ = LQ$ is similar. ∎

Corollary. *Let $A \in \mathsf{M}_{n\times n}(C)$ be such that $\lim_{m\to\infty} A^m = L$. Then for any invertible matrix $Q \in \mathsf{M}_{n\times n}(C)$,*

$$\lim_{m\to\infty}(QAQ^{-1})^m = QLQ^{-1}.$$

Proof. Since

$$(QAQ^{-1})^m = (QAQ^{-1})(QAQ^{-1})\cdots(QAQ^{-1}) = QA^mQ^{-1},$$

we have

$$\lim_{m\to\infty}(QAQ^{-1})^m = \lim_{m\to\infty} QA^mQ^{-1} = Q\left(\lim_{m\to\infty} A^m\right)Q^{-1} = QLQ^{-1}$$

by applying Theorem 5.11 twice. ∎

In the discussion that follows, we frequently encounter the set

$$S = \{\lambda \in C\colon |\lambda| < 1 \text{ or } \lambda = 1\}.$$

Geometrically, this set consists of the complex number 1 and the interior of the unit disk (the disk of radius 1 centered at the origin). This set is of interest because if λ is a complex number, then $\lim_{m\to\infty} \lambda^m$ exists if and only $\lambda \in S$. This fact, which is obviously true if λ is real, can be shown to be true for complex numbers also.

The following important result gives necessary and sufficient conditions for the existence of the type of limit under consideration.

Theorem 5.12. *Let A be a square matrix with complex entries. Then $\lim_{m\to\infty} A^m$ exists if and only if both of the following conditions hold.*

(a) *Every eigenvalue of A is contained in S.*
(b) *If 1 is an eigenvalue of A, then the dimension of the eigenspace corresponding to 1 equals the multiplicity of 1 as an eigenvalue of A.*

One proof of this theorem, which relies on the theory of Jordan canonical forms (Section 7.2), can be found in Exercise 19 of Section 7.2. A second proof, which makes use of Schur's theorem (Theorem 6.14 of Section 6.4), can be found in the article by S. H. Friedberg and A. J. Insel, "Convergence of matrix powers," *Int. J. Math. Educ. Sci. Technol.*, 1992, Vol. 23, no. 5, pp. 765-769.

The necessity of condition (a) is easily justified. For suppose that λ is an eigenvalue of A such that $\lambda \notin S$. Let v be an eigenvector of A corresponding to λ. Regarding v as an $n \times 1$ matrix, we see that

$$\lim_{m\to\infty}(A^m v) = \left(\lim_{m\to\infty} A^m\right)v = Lv$$

by Theorem 5.11, where $L = \lim_{m\to\infty} A^m$. But $\lim_{m\to\infty}(A^m v) = \lim_{m\to\infty}(\lambda^m v)$ diverges because $\lim_{m\to\infty}\lambda^m$ does not exist. Hence if $\lim_{m\to\infty} A^m$ exists, then condition (a) of Theorem 5.12 must hold.

Although we are unable to prove the necessity of condition (b) here, we consider an example for which this condition fails. Observe that the characteristic polynomial for the matrix

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is $(t-1)^2$, and hence B has eigenvalue $\lambda = 1$ with multiplicity 2. It can easily be verified that $\dim(\mathsf{E}_\lambda) = 1$, so that condition (b) of Theorem 5.12 is violated. A simple mathematical induction argument can be used to show that

$$B^m = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix},$$

and therefore that $\lim_{m\to\infty} B^m$ does not exist. We see in Chapter 7 that if A is a matrix for which condition (b) fails, then A is similar to a matrix whose upper left 2×2 submatrix is precisely this matrix B.

In most of the applications involving matrix limits, the matrix is diagonalizable, and so condition (b) of Theorem 5.12 is automatically satisfied. In this case, Theorem 5.12 reduces to the following theorem, which can be proved using our previous results.

Theorem 5.13. *Let $A \in \mathsf{M}_{n\times n}(C)$ satisfy the following two conditions.*

(i) *Every eigenvalue of A is contained in S.*

(ii) *A is diagonalizable.*

Then $\lim_{m\to\infty} A^m$ exists.

Proof. Since A is diagonalizable, there exists an invertible matrix Q such that $Q^{-1}AQ = D$ is a diagonal matrix. Suppose that

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

Because $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A, condition (i) requires that for each i, either $\lambda_i = 1$ or $|\lambda_i| < 1$. Thus

$$\lim_{m\to\infty} {\lambda_i}^m = \begin{cases} 1 & \text{if } \lambda_i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

But since

$$D^m = \begin{pmatrix} {\lambda_1}^m & 0 & \cdots & 0 \\ 0 & {\lambda_2}^m & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & {\lambda_n}^m \end{pmatrix},$$

the sequence $D, D^2, \ldots$ converges to a limit L. Hence

$$\lim_{m\to\infty} A^m = \lim_{m\to\infty} (QDQ^{-1})^m = QLQ^{-1}$$

by the corollary to Theorem 5.11. ∎

The technique for computing $\lim_{m\to\infty} A^m$ used in the proof of Theorem 5.13 can be employed in actual computations, as we now illustrate. Let

$$A = \begin{pmatrix} \frac{7}{4} & -\frac{9}{4} & -\frac{15}{4} \\ \frac{3}{4} & \frac{7}{4} & \frac{3}{4} \\ \frac{3}{4} & -\frac{9}{4} & -\frac{11}{4} \end{pmatrix}.$$

Using the methods in Sections 5.1 and 5.2, we obtain

$$Q = \begin{pmatrix} 1 & 3 & -1 \\ -3 & -2 & 1 \\ 2 & 3 & -1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$$

such that $Q^{-1}AQ = D$. Hence

$$\lim_{m\to\infty} A^m = \lim_{m\to\infty} (QDQ^{-1})^m = \lim_{m\to\infty} QD^mQ^{-1} = Q\left(\lim_{m\to\infty} D^m\right)Q^{-1}$$

$$= \begin{pmatrix} 1 & 3 & -1 \\ -3 & -2 & 1 \\ 2 & 3 & -1 \end{pmatrix} \left[\lim_{m\to\infty} \begin{pmatrix} 1 & 0 & 0 \\ 0 & (-\frac{1}{2})^m & 0 \\ 0 & 0 & (\frac{1}{4})^m \end{pmatrix}\right] \begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & 2 \\ -5 & 3 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 & -1 \\ -3 & -2 & 1 \\ 2 & 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & 2 \\ -5 & 3 & 7 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ -2 & 0 & 2 \end{pmatrix}.$$

Next, we consider an application that uses the limit of powers of a matrix. Suppose that the population of a certain metropolitan area remains constant but there is a continual movement of people between the city and the suburbs. Specifically, let the entries of the following matrix A represent the probabilities that someone living in the city or in the suburbs on January 1 will be living in each region on January 1 of the next year.

$$\begin{array}{lcc} & \text{Currently living in the city} & \text{Currently living in the suburbs} \\ \text{Living next year in the city} & & \\ \text{Living next year in the suburbs} & & \end{array}$$

$$\begin{array}{l} \text{Living next year in the city} \\ \text{Living next year in the suburbs} \end{array} \begin{pmatrix} 0.90 & 0.02 \\ 0.10 & 0.98 \end{pmatrix} = A$$

For instance, the probability that someone living in the city (on January 1) will be living in the suburbs next year (on January 1) is 0.10. Notice that since the entries of A are probabilities, they are nonnegative. Moreover, the assumption of a constant population in the metropolitan area requires that the sum of the entries of each column of A be 1.

Any square matrix having these two properties (nonnegative entries and columns that sum to 1) is called a **transition matrix** or a **stochastic matrix**. For an arbitrary $n \times n$ transition matrix M, the rows and columns correspond to n **states**, and the entry M_{ij} represents the probability of moving from state j to state i in one **stage**.

In our example, there are two states (residing in the city and residing in the suburbs). So, for example, A_{21} is the probability of moving from the city to the suburbs in one stage, that is, in one year. We now determine the

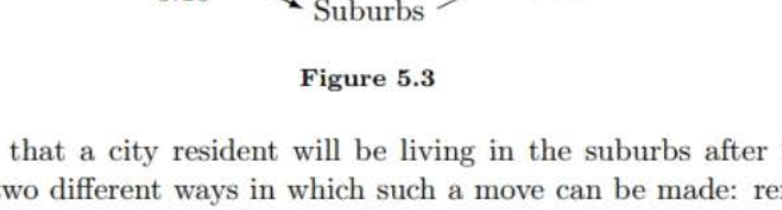


Figure 5.3

probability that a city resident will be living in the suburbs after 2 years. There are two different ways in which such a move can be made: remaining in the city for 1 year and then moving to the suburbs, or moving to the suburbs during the first year and remaining there the second year. (See Figure 5.3.) The probability that a city dweller remains in the city for the first year is 0.90, whereas the probability that the city dweller moves to the suburbs during the first year is 0.10. Hence the probability that a city dweller stays in the city for the first year and then moves to the suburbs during the second year is the product $(0.90)(0.10)$. Likewise, the probability that a city dweller moves to the suburbs in the first year and remains in the suburbs during the second year is the product $(0.10)(0.98)$. Thus the probability that a city dweller will be living in the suburbs after 2 years is the sum of these products, $(0.90)(0.10) + (0.10)(0.98) = 0.188$. Observe that this number is obtained by the same calculation as that which produces $(A^2)_{21}$, and hence $(A^2)_{21}$ represents the probability that a city dweller will be living in the suburbs after 2 years. In general, for any transition matrix M, the entry $(M^m)_{ij}$ represents the probability of moving from state j to state i in m stages.

Suppose additionally that in year 2000, 70% of the population of the metropolitan area lived in the city and 30% lived in the suburbs. We record these data as a column vector:

$$\begin{array}{l} \text{Proportion of city dwellers} \\ \text{Proportion of suburb residents} \end{array} \quad \begin{pmatrix} 0.70 \\ 0.30 \end{pmatrix} = P.$$

Notice that the rows of P correspond to the states of residing in the city and residing in the suburbs, respectively, and that these states are listed in the same order as the listing in the transition matrix A. Observe also that the column vector P contains nonnegative entries that sum to 1; such a vector is called a **probability vector**. In this terminology, each column of a transition matrix is a probability vector. It is often convenient to regard the entries of a transition matrix or a probability vector as proportions or percentages instead of probabilities, as we have already done with the probability vector P.

In the vector AP, the first coordinate is the sum $(0.90)(0.70)+(0.02)(0.30)$. The first term of this sum, $(0.90)(0.70)$, represents the proportion of the 2000 metropolitan population that remained in the city during the next year, and the second term, $(0.02)(0.30)$, represents the proportion of the 2000 metropolitan population that moved into the city during the next year. Hence the first coordinate of AP represents the proportion of the metropolitan population that was living in the city in 2001. Similarly, the second coordinate of

$$AP = \begin{pmatrix} 0.636 \\ 0.364 \end{pmatrix}$$

represents the proportion of the metropolitan population that was living in the suburbs in 2001. This argument can be easily extended to show that the coordinates of

$$A^2P = A(AP) = \begin{pmatrix} 0.57968 \\ 0.42032 \end{pmatrix}$$

represent the proportions of the metropolitan population that were living in each location in 2002. In general, the coordinates of A^mP represent the proportion of the metropolitan population that will be living in the city and suburbs, respectively, after m stages (m years after 2000).

Will the city eventually be depleted if this trend continues? In view of the preceding discussion, it is natural to define the eventual proportion of the city dwellers and suburbanites to be the first and second coordinates, respectively, of $\lim_{m\to\infty} A^m P$. We now compute this limit. It is easily shown that A is diagonalizable, and so there is an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$. In fact,

$$Q = \begin{pmatrix} \frac{1}{6} & -\frac{1}{6} \\ \frac{5}{6} & \frac{1}{6} \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 0.88 \end{pmatrix}.$$

Therefore

$$L = \lim_{m\to\infty} A^m = \lim_{m\to\infty} QD^mQ^{-1} = Q\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}Q^{-1} = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} \\ \frac{5}{6} & \frac{5}{6} \end{pmatrix}.$$

Consequently

$$\lim_{m\to\infty} A^m P = LP = \begin{pmatrix} \frac{1}{6} \\ \frac{5}{6} \end{pmatrix}.$$

Thus, eventually, $\frac{1}{6}$ of the population will live in the city and $\frac{5}{6}$ will live in the suburbs each year. Note that the vector LP satisfies $A(LP) = LP$. Hence LP is both a probability vector and an eigenvector of A corresponding to the eigenvalue 1. Since the eigenspace of A corresponding to the eigenvalue 1 is one-dimensional, there is only one such vector, and LP is independent of the initial choice of probability vector P. (See Exercise 15.) For example, had the 2000 metropolitan population consisted entirely of city dwellers, the limiting outcome would be the same.

In analyzing the city–suburb problem, we gave probabilistic interpretations of A^2 and AP, showing that A^2 is a transition matrix and AP is a probability vector. In fact, the product of any two transition matrices is a transition matrix, and the product of any transition matrix and probability vector is a probability vector. A proof of these facts is a simple corollary of the next theorem, which characterizes transition matrices and probability vectors.

Theorem 5.14. *Let M be an $n\times n$ matrix having real nonnegative entries, let v be a column vector in R^n having nonnegative coordinates, and let $u \in \mathsf{R}^n$ be the column vector in which each coordinate equals 1. Then*

(a) *M is a transition matrix if and only if $u^tM = u^t$;*

(b) *v is a probability vector if and only if $u^tv = (1)$.*

Proof. Exercise. ∎

Corollary.

(a) *The product of two $n \times n$ transition matrices is an $n \times n$ transition matrix. In particular, any power of a transition matrix is a transition matrix.*

(b) *The product of a transition matrix and a probability vector is a probability vector.*

Proof. Exercise. ∎

The city–suburb problem is an example of a process in which elements of a set are each classified as being in one of several fixed states that can switch over time. In general, such a process is called a **stochastic process**. The switching to a particular state is described by a probability, and in general this probability depends on such factors as the state in question, the time in question, some or all of the previous states in which the object has been (including the current state), and the states that other objects are in or have been in.

For instance, the object could be an American voter, and the state of the object could be his or her preference of political party; or the object could be a molecule of $\mathbf{H_2O}$, and the states could be the three physical states in which $\mathbf{H_2O}$ can exist (solid, liquid, and gas). In these examples, all four of the factors mentioned above influence the probability that an object is in a particular state at a particular time.

If, however, the probability that an object in one state changes to a different state in a fixed interval of time depends only on the two states (and not on the time, earlier states, or other factors), then the stochastic process is called a **Markov process**. If, in addition, the number of possible states is finite, then the Markov process is called a **Markov chain**. We treated the city–suburb example as a two-state Markov chain. Of course, a Markov process is usually only an idealization of reality because the probabilities involved are almost never constant over time.

With this in mind, we consider another Markov chain. A certain community college would like to obtain information about the likelihood that students in various categories will graduate. The school classifies a student as a sophomore or a freshman depending on the number of credits that the student has earned. Data from the school indicate that, from one fall semester to the next, 40% of the sophomores will graduate, 30% will remain sophomores, and 30% will quit permanently. For freshmen, the data show that 10% will graduate by next fall, 50% will become sophomores, 20% will remain freshmen, and 20% will quit permanently. During the present year, 50% of the students at the school are sophomores and 50% are freshmen. Assuming that the trend indicated by the data continues indefinitely, the school would like to know

1. the percentage of the present students who will graduate, the percentage who will be sophomores, the percentage who will be freshmen, and the percentage who will quit school permanently by next fall;
2. the same percentages as in item 1 for the fall semester two years hence; and
3. the percentage of its present students who will eventually graduate.

The preceding paragraph describes a four-state Markov chain with the following states:

1. having graduated
2. being a sophomore
3. being a freshman
4. having quit permanently.

The given data provide us with the transition matrix

$$A = \begin{pmatrix} 1 & 0.4 & 0.1 & 0 \\ 0 & 0.3 & 0.5 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0.3 & 0.2 & 1 \end{pmatrix}$$

of the Markov chain. (Notice that students who have graduated or have quit permanently are assumed to remain indefinitely in those respective states. Thus a freshman who quits the school and returns during a later semester is not regarded as having changed states—the student is assumed to have remained in the state of being a freshman during the time he or she was not enrolled.) Moreover, we are told that the present distribution of students is half in each of states 2 and 3 and none in states 1 and 4. The vector

$$P = \begin{pmatrix} 0 \\ 0.5 \\ 0.5 \\ 0 \end{pmatrix}$$

that describes the initial probability of being in each state is called the **initial probability vector** for the Markov chain.

To answer question 1, we must determine the probabilities that a present student will be in each state by next fall. As we have seen, these probabilities are the coordinates of the vector

$$AP = \begin{pmatrix} 1 & 0.4 & 0.1 & 0 \\ 0 & 0.3 & 0.5 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0.3 & 0.2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0.5 \\ 0.5 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.25 \\ 0.40 \\ 0.10 \\ 0.25 \end{pmatrix}.$$

Hence by next fall, 25% of the present students will graduate, 40% will be sophomores, 10% will be freshmen, and 25% will quit the school permanently. Similarly,

$$A^2P = A(AP) = \begin{pmatrix} 1 & 0.4 & 0.1 & 0 \\ 0 & 0.3 & 0.5 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0.3 & 0.2 & 1 \end{pmatrix} \begin{pmatrix} 0.25 \\ 0.40 \\ 0.10 \\ 0.25 \end{pmatrix} = \begin{pmatrix} 0.42 \\ 0.17 \\ 0.02 \\ 0.39 \end{pmatrix}$$

provides the information needed to answer question 2: within two years 42% of the present students will graduate, 17% will be sophomores, 2% will be freshmen, and 39% will quit school.

Finally, the answer to question 3 is provided by the vector LP, where $L = \lim_{m\to\infty} A^m$. For the matrices

$$Q = \begin{pmatrix} 1 & 4 & 19 & 0 \\ 0 & -7 & -40 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 3 & 13 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

we have $Q^{-1}AQ = D$. Thus

$$\begin{aligned} L &= \lim_{m\to\infty} A^m = Q\left(\lim_{m\to\infty} D^m\right)Q^{-1} \\ &= \begin{pmatrix} 1 & 4 & 19 & 0 \\ 0 & -7 & -40 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 3 & 13 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{4}{7} & \frac{27}{56} & 0 \\ 0 & -\frac{1}{7} & -\frac{5}{7} & 0 \\ 0 & 0 & \frac{1}{8} & 0 \\ 0 & \frac{3}{7} & \frac{29}{56} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \frac{4}{7} & \frac{27}{56} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{3}{7} & \frac{29}{56} & 1 \end{pmatrix}. \end{aligned}$$

So

$$LP = \begin{pmatrix} 1 & \frac{4}{7} & \frac{27}{56} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{3}{7} & \frac{29}{56} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0.5 \\ 0.5 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{59}{112} \\ 0 \\ 0 \\ \frac{53}{112} \end{pmatrix},$$

and hence the probability that one of the present students will graduate is $\frac{59}{112}$.

In the preceding two examples, we saw that $\lim_{m\to\infty} A^mP$, where A is the transition matrix and P is the initial probability vector of the Markov chain, gives the eventual proportions in each state. In general, however, the limit of powers of a transition matrix need not exist. For example, if

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then $\lim_{m\to\infty} M^m$ does not exist because odd powers of M equal M and even powers of M equal I. The reason that the limit fails to exist is that condition (a) of Theorem 5.12 does not hold for M (-1 is an eigenvalue). In fact, it can be shown (see Exercise 20 of Section 7.2) that the only transition matrices A such that $\lim_{m\to\infty} A^m$ does not exist are precisely those matrices for which condition (a) of Theorem 5.12 fails to hold.

But even if the limit of powers of the transition matrix exists, the computation of the limit may be quite difficult. (The reader is encouraged to work Exercise 6 to appreciate the truth of the last sentence.) Fortunately, there is a large and important class of transition matrices for which this limit exists and is easily computed—this is the class of *regular* transition matrices.

Definition. *A transition matrix is called* **regular** *if some power of the matrix contains only nonzero (i.e., positive) entries.*

Example 2

The transition matrix

$$\begin{pmatrix} 0.90 & 0.02 \\ 0.10 & 0.98 \end{pmatrix}$$

of the Markov chain used in the city–suburb problem is clearly regular because each entry is positive. On the other hand, the transition matrix

$$A = \begin{pmatrix} 1 & 0.4 & 0.1 & 0 \\ 0 & 0.3 & 0.5 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0.3 & 0.2 & 1 \end{pmatrix}$$

of the Markov chain describing community college enrollments is not regular because the first column of A^m is

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

for any power m.

Observe that a regular transition matrix may contain zero entries. For example,

$$M = \begin{pmatrix} 0.9 & 0.5 & 0 \\ 0 & 0.5 & 0.4 \\ 0.1 & 0 & 0.6 \end{pmatrix}$$

is regular because every entry of M^2 is positive. ◆

The remainder of this section is devoted to proving that, for a regular transition matrix A, the limit of the sequence of powers of A exists and has identical columns. From this fact, it is easy to compute this limit. In the course of proving this result, we obtain some interesting bounds for the magnitudes of eigenvalues of any square matrix. These bounds are given in terms of the sum of the absolute values of the rows and columns of the matrix. The necessary terminology is introduced in the definitions that follow.

Definitions. *Let $A \in \mathsf{M}_{n\times n}(C)$. For $1 \le i, j \le n$, define $\rho_i(A)$ to be the sum of the absolute values of the entries of row i of A, and define $\nu_j(A)$ to be equal to the sum of the absolute values of the entries of column j of A. Thus*

$$\rho_i(A) = \sum_{j=1}^{n} |A_{ij}| \quad \text{for } i = 1, 2, \ldots, n$$

and

$$\nu_j(A) = \sum_{i=1}^{n} |A_{ij}| \quad \text{for } j = 1, 2, \ldots, n.$$

The **row sum** *of A, denoted $\rho(A)$, and the* **column sum** *of A, denoted $\nu(A)$, are defined as*

$$\rho(A) = \max\{\rho_i(A) \colon 1 \le i \le n\} \quad \text{and} \quad \nu(A) = \max\{\nu_j(A) \colon 1 \le j \le n\}.$$

Example 3

For the matrix

$$A = \begin{pmatrix} 1 & -i & 3-4i \\ -2+i & 0 & 6 \\ 3 & 2 & i \end{pmatrix},$$

$\rho_1(A) = 7$, $\rho_2(A) = 6+\sqrt{5}$, $\rho_3(A) = 6$, $\nu_1(A) = 4+\sqrt{5}$, $\nu_2(A) = 3$, and $\nu_3(A) = 12$. Hence $\rho(A) = 6+\sqrt{5}$ and $\nu(A) = 12$. ◆

Our next results show that the smaller of $\rho(A)$ and $\nu(A)$ is an upper bound for the absolute values of eigenvalues of A. In the preceding example, for instance, A has no eigenvalue with absolute value greater than $6+\sqrt{5}$.

To obtain a geometric view of the following theorem, we introduce some terminology. For an $n\times n$ matrix A, we define the ith **Gershgorin disk** C_i to be the disk in the complex plane with center A_{ii} and radius $r_i = \rho_i(A) - |A_{ii}|$; that is,

$$C_i = \{z \in C \colon |z - A_{ii}| \le r_i\}.$$

For example, consider the matrix

$$A = \begin{pmatrix} 1+2i & 1 \\ 2i & -3 \end{pmatrix}.$$

For this matrix, C_1 is the disk with center $1+2i$ and radius 1, and C_2 is the disk with center -3 and radius 2. (See Figure 5.4.)

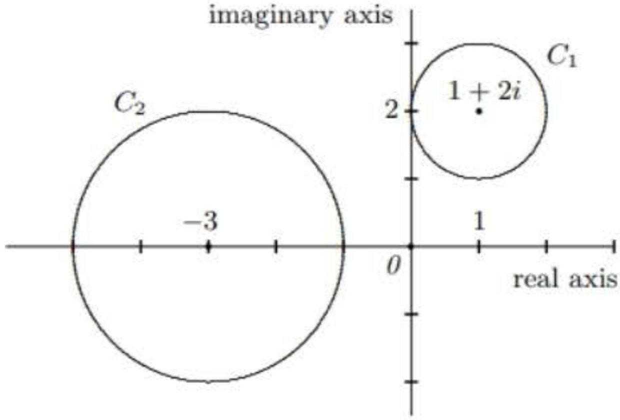


Figure 5.4

Gershgorin's circle theorem, stated below, tells us that all the eigenvalues of A are located within these two disks. In particular, we see that 0 is *not* an eigenvalue, and hence by Exercise 9(c) of Section 5.1, A is invertible.

Theorem 5.15 (Gershgorin's Circle Theorem). *Let $A \in \mathsf{M}_{n\times n}(C)$. Then every eigenvalue of A is contained in a Gershgorin disk.*

Proof. Let λ be an eigenvalue of A with the corresponding eigenvector

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}.$$

Then v satisfies the matrix equation $Av = \lambda v$, which can be written

$$\sum_{j=1}^{n} A_{ij}v_j = \lambda v_i \quad (i = 1, 2, \dots, n). \tag{3}$$

Suppose that v_k is a coordinate of v having the largest absolute value; note that $v_k \neq 0$ because v is an eigenvector of A.

We show that λ lies in C_k, that is, $|\lambda - A_{kk}| \leq r_k$. For $i = k$, it follows from (3) that

$$|\lambda v_k - A_{kk}v_k| = \left|\sum_{j=1}^{n} A_{kj}v_j - A_{kk}v_k\right| = \left|\sum_{j\neq k} A_{kj}v_j\right|$$

$$\begin{aligned} &\leq \sum_{j \neq k} |A_{kj}||v_j| \leq \sum_{j \neq k} |A_{kj}||v_k| \\ &= |v_k| \sum_{j \neq k} |A_{kj}| = |v_k| r_k. \end{aligned}$$

Thus

$$|v_k||\lambda - A_{kk}| \leq |v_k| r_k;$$

so

$$|\lambda - A_{kk}| \leq r_k$$

because $|v_k| > 0$. ∎

Corollary 1. *Let λ be any eigenvalue of $A \in \mathsf{M}_{n \times n}(C)$. Then $|\lambda| \leq \rho(A)$.*

Proof. By Gershgorin's circle theorem, $|\lambda - A_{kk}| \leq r_k$ for some k. Hence

$$\begin{aligned} |\lambda| &= |(\lambda - A_{kk}) + A_{kk}| \leq |\lambda - A_{kk}| + |A_{kk}| \\ &\leq r_k + |A_{kk}| = \rho_k(A) \leq \rho(A). \end{aligned}$$ ∎

Corollary 2. *Let λ be any eigenvalue of $A \in \mathsf{M}_{n \times n}(C)$. Then*

$$|\lambda| \leq \min\{\rho(A), \nu(A)\}.$$

Proof. Since $|\lambda| \leq \rho(A)$ by Corollary 1, it suffices to show that $|\lambda| \leq \nu(A)$. By Exercise 15 of Section 5.1, λ is an eigenvalue of A^t, and so $|\lambda| \leq \rho(A^t)$ by Corollary 1. But the rows of A^t are the columns of A; consequently $\rho(A^t) = \nu(A)$. Therefore $|\lambda| \leq \nu(A)$. ∎

The next corollary is immediate from Corollary 2.

Corollary 3. *If λ is an eigenvalue of a transition matrix, then $|\lambda| \leq 1$.*

The next result asserts that the upper bound in Corollary 3 is attained.

Theorem 5.16. *Every transition matrix has 1 as an eigenvalue.*

Proof. Let A be an $n \times n$ transition matrix, and let $u \in \mathsf{R}^n$ be the column vector in which each coordinate is 1. Then $A^t u = u$ by Theorem 5.14, and hence u is an eigenvector of A^t corresponding to the eigenvalue 1. But since A and A^t have the same eigenvalues, it follows that 1 is also an eigenvalue of A. ∎

Suppose that A is a transition matrix for which some eigenvector corresponding to the eigenvalue 1 has only nonnegative coordinates. Then some multiple of this vector is a probability vector P as well as an eigenvector of A corresponding to eigenvalue 1. It is interesting to observe that if P is the initial probability vector of a Markov chain having A as its transition matrix, then the Markov chain is completely static. For in this situation, $A^m P = P$ for every positive integer m; hence the probability of being in each state never changes. Consider, for instance, the city–suburb problem with

$$P = \begin{pmatrix} \frac{1}{6} \\ \frac{5}{6} \end{pmatrix}.$$

Theorem 5.17. *Let $A \in \mathsf{M}_{n\times n}(C)$ be a matrix in which each entry is a positive real number, and let λ be a complex eigenvalue of A such that $|\lambda| = \rho(A)$. Then $\lambda = \rho(A)$ and $\{u\}$ is a basis for E_λ, where $u \in \mathsf{C}^n$ is the column vector in which each coordinate equals 1.*

Proof. Let v be an eigenvector of A corresponding to λ, with coordinates $v_1, v_2, \ldots, v_n$. Suppose that v_k is the coordinate of v having the largest absolute value, and let $b = |v_k|$. Then

$$\begin{aligned} |\lambda| b = |\lambda||v_k| = |\lambda v_k| &= \left| \sum_{j=1}^{n} A_{kj} v_j \right| \leq \sum_{j=1}^{n} |A_{kj} v_j| \\ &= \sum_{j=1}^{n} |A_{kj}||v_j| \leq \sum_{j=1}^{n} |A_{kj}| b = \rho_k(A) b \leq \rho(A) b. \end{aligned} \tag{4}$$

Since $|\lambda| = \rho(A)$, the three inequalities in (4) are actually equalities; that is,

(a) $\left| \sum_{j=1}^{n} A_{kj} v_j \right| = \sum_{j=1}^{n} |A_{kj} v_j|$,

(b) $\sum_{j=1}^{n} |A_{kj}||v_j| = \sum_{j=1}^{n} |A_{kj}| b$, and

(c) $\rho_k(A) = \rho(A)$.

We will see in Exercise 15(b) of Section 6.1 that (a) holds if and only if all the terms $A_{kj} v_j$ $(j = 1, 2, \ldots, n)$ are obtained by multiplying some nonzero complex number z by nonnegative real numbers. Without loss of generality, we assume that $|z| = 1$. Thus there exist nonnegative real numbers $c_1, c_2, \ldots, c_n$ such that

$$A_{kj} v_j = c_j z. \tag{5}$$

By (b) and the assumption that $A_{kj} \neq 0$ for all k and j, we have

$$|v_j| = b \quad \text{for } j = 1, 2, \ldots, n. \tag{6}$$

Combining (5) and (6), we obtain

$$b = |v_j| = \left|\frac{c_j}{A_{kj}} z\right| = \frac{c_j}{A_{kj}} \quad \text{for } j = 1, 2, \ldots, n,$$

and therefore by (5), we have $v_j = bz$ for all j. So

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} bz \\ bz \\ \vdots \\ bz \end{pmatrix} = bzu,$$

and hence $\{u\}$ is a basis for E_λ.

Finally, observe that all of the entries of Au are positive because the same is true for the entries of both A and u. But $Au = \lambda u$, and hence $\lambda > 0$. Therefore, $\lambda = |\lambda| = \rho(A)$. ∎

Corollary 1. *Let $A \in \mathsf{M}_{n\times n}(C)$ be a matrix in which each entry is positive, and let λ be an eigenvalue of A such that $|\lambda| = \nu(A)$. Then $\lambda = \nu(A)$, and the dimension of E_λ equals 1.*

Proof. Exercise. ∎

Corollary 2. *Let $A \in \mathsf{M}_{n\times n}(C)$ be a transition matrix in which each entry is positive, and let λ be any eigenvalue of A other than 1. Then $|\lambda| < 1$. Moreover, the eigenspace corresponding to the eigenvalue 1 has dimension 1.*

Proof. Exercise. ∎

Our next result extends Corollary 2 to regular transition matrices and thus shows that regular transition matrices satisfy condition (a) of Theorems 5.12 and 5.13.

Theorem 5.18. *Let A be a regular transition matrix, and let λ be an eigenvalue of A. Then*

(a) $|\lambda| \le 1$.
(b) *If $|\lambda| = 1$, then $\lambda = 1$, and* $\dim(\mathsf{E}_\lambda) = 1$.

Proof. Statement (a) was proved as Corollary 3 to Theorem 5.15.

(b) Since A is regular, there exists a positive integer s such that A^s has only positive entries. Because A is a transition matrix and the entries of A^s are positive, the entries of $A^{s+1} = A^s(A)$ are positive. Suppose that $|\lambda| = 1$. Then λ^s and λ^{s+1} are eigenvalues of A^s and A^{s+1}, respectively, having absolute value 1. So by Corollary 2 to Theorem 5.17, $\lambda^s = \lambda^{s+1} = 1$. Thus $\lambda = 1$. Let E_λ and E'_λ denote the eigenspaces of A and A^s, respectively, corresponding to $\lambda = 1$. Then $\mathsf{E}_\lambda \subseteq \mathsf{E}'_\lambda$ and, by Corollary 2 to Theorem 5.17, $\dim(\mathsf{E}'_\lambda) = 1$. Hence $\mathsf{E}_\lambda = \mathsf{E}'_\lambda$, and $\dim(\mathsf{E}_\lambda) = 1$. ∎

Corollary. *Let A be a regular transition matrix that is diagonalizable. Then $\lim_{m\to\infty} A^m$ exists.*

The preceding corollary, which follows immediately from Theorems 5.18 and 5.13, is not the best possible result. In fact, it can be shown that if A is a regular transition matrix, then the multiplicity of 1 as an eigenvalue of A is 1. Thus, by Theorem 5.7 (p. 264), condition (b) of Theorem 5.12 is satisfied. So if A is a regular transition matrix, $\lim_{m\to\infty} A^m$ exists regardless of whether A is or is not diagonalizable. As with Theorem 5.12, however, the fact that the multiplicity of 1 as an eigenvalue of A is 1 cannot be proved at this time. Nevertheless, we state this result here (leaving the proof until Exercise 21 of Section 7.2) and deduce further facts about $\lim_{m\to\infty} A^m$ when A is a regular transition matrix.

Theorem 5.19. *Let A be a regular transition matrix. Then*

(a) *The multiplicity of 1 as an eigenvalue of A is 1.*
(b) *$\lim_{m\to\infty} A^m$ exists.*
(c) *$L = \lim_{m\to\infty} A^m$ is a transition matrix.*
(d) *$AL = LA = L$.*
(e) *The columns of L are identical. In fact, each column of L is equal to the unique probability vector v that is also an eigenvector of A corresponding to the eigenvalue 1.*
(f) *For any probability vector w, $\lim_{m\to\infty} (A^m w) = v$.*

Proof. (a) See Exercise 21 of Section 7.2.

(b) This follows from (a) and Theorems 5.18 and 5.12.

(c) By Theorem 5.14, we must show that $u^t L = u^t$. Now A^m is a transition matrix by the corollary to Theorem 5.14, so

$$u^t L = u^t \lim_{m\to\infty} A^m = \lim_{m\to\infty} u^t A^m = \lim_{m\to\infty} u^t = u^t,$$

and it follows that L is a transition matrix.

(d) By Theorem 5.11,

$$AL = A \lim_{m\to\infty} A^m = \lim_{m\to\infty} AA^m = \lim_{m\to\infty} A^{m+1} = L.$$

Similarly, $LA = L$.

(e) Since $AL = L$ by (d), each column of L is an eigenvector of A corresponding to the eigenvalue 1. Moreover, by (c), each column of L is a probability vector. Thus, by (a), each column of L is equal to the unique probability vector v corresponding to the eigenvalue 1 of A.

(f) Let w be any probability vector, and set $y = \lim_{m\to\infty} A^m w = Lw$. Then y is a probability vector by the corollary to Theorem 5.14, and also $Ay = ALw = Lw = y$ by (d). Hence y is also an eigenvector corresponding to the eigenvalue 1 of A. So $y = v$ by (e). ∎

Definition. *The vector v in Theorem 5.19(e) is called the* **fixed probability vector** *or* **stationary vector** *of the regular transition matrix A.*

Theorem 5.19 can be used to deduce information about the eventual distribution in each state of a Markov chain having a regular transition matrix.

Example 4

Because of an expected El Niño in the coming fall, the National Weather Service predicted that, during the following winter, the probability of above-average, average, and below-average snowfall for a particular region will be 0.5, 0.3, and 0.2, respectively. Historical weather records for this region show that if there is above-average snowfall in a given winter, then the probabilities of above-average, average, and below-average snowfall for the next winter are 0.4, 0.2, and 0.2, respectively. In addition, if there is average snowfall in a given winter, then the probabilities of above-average, average, and below-average snowfall for the next winter are 0.1, 0.7, and 0.2, respectively. Finally, if there is below-average snowfall in a given winter, then the probabilities of above-average, average, and below-average snowfall for the next winter are 0.5, 0.1, and 0.6, respectively.

Assuming that this historical weather trend continues, the situation described in the preceding paragraph is a three-state Markov chain in which the states are above-average, average, and below-average winter snowfall. Here the probability vector that gives the initial probability of being in each state during the coming winter is

$$P = \begin{pmatrix} 0.50 \\ 0.30 \\ 0.20 \end{pmatrix},$$

and the transition matrix is

$$A = \begin{pmatrix} 0.40 & 0.20 & 0.20 \\ 0.10 & 0.70 & 0.20 \\ 0.50 & 0.10 & 0.60 \end{pmatrix}.$$

The probabilities of being in each state m winters after the original National Weather Service prediction are the coordinates of the vector $A^m P$. The reader may check that

$$AP = \begin{pmatrix} 0.30 \\ 0.30 \\ 0.40 \end{pmatrix}, \quad A^2P = \begin{pmatrix} 0.26 \\ 0.32 \\ 0.42 \end{pmatrix}, \quad A^3P = \begin{pmatrix} 0.252 \\ 0.334 \\ 0.414 \end{pmatrix}, \quad \text{and } A^4P = \begin{pmatrix} 0.2504 \\ 0.3418 \\ 0.4078 \end{pmatrix}.$$

Note the apparent convergence of $A^m P$.

Since A is regular, the long-range prediction for the region's winter snowfall can be found by computing the fixed probability vector for A. This vector is the unique probability vector v such that $(A - I)v = \mathit{0}$. Letting

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

we see that the matrix equation $(A - I)v = \mathit{0}$ yields the following system of linear equations:

$$\begin{aligned} -0.60v_1 + 0.20v_2 + 0.20v_3 &= 0 \\ 0.10v_1 - 0.30v_2 + 0.20v_3 &= 0 \\ 0.50v_1 + 0.10v_2 - 0.40v_3 &= 0\,. \end{aligned}$$

It is easily shown that

$$\begin{pmatrix} 5 \\ 7 \\ 8 \end{pmatrix}$$

is a basis for the solution space of this system. Hence the unique fixed probability vector for A is

$$\begin{pmatrix} \frac{5}{5+7+8} \\ \frac{7}{5+7+8} \\ \frac{8}{5+7+8} \end{pmatrix} = \begin{pmatrix} 0.25 \\ 0.35 \\ 0.40 \end{pmatrix}.$$

Therefore, in the long run, the probabilities of above-average, average, and below-average winter snowfall for this region are 0.25, 0.35, and 0.40, respectively.

Note that if

$$Q = \begin{pmatrix} 5 & 0 & -3 \\ 7 & -1 & -1 \\ 8 & 1 & 4 \end{pmatrix},$$

then

$$Q^{-1}AQ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.2 \end{pmatrix}.$$

So

$$\lim_{m\to\infty} A^m = Q\left[\lim_{m\to\infty} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.2 \end{pmatrix}^m\right] Q^{-1} = Q \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} Q^{-1}$$

$$= \begin{pmatrix} 0.25 & 0.25 & 0.25 \\ 0.35 & 0.35 & 0.35 \\ 0.40 & 0.40 & 0.40 \end{pmatrix}. \quad \blacklozenge$$

Example 5

Farmers in Lamron plant one crop per year—either corn, soybeans, or wheat. Because they believe in the necessity of rotating their crops, these farmers do not plant the same crop in successive years. In fact, of the total acreage on which a particular crop is planted, exactly half is planted with each of the other two crops during the succeeding year. This year, 300 acres of corn, 200 acres of soybeans, and 100 acres of wheat were planted.

The situation just described is another three-state Markov chain in which the three states correspond to the planting of corn, soybeans, and wheat, respectively. In this problem, however, the amount of land devoted to each crop, rather than the percentage of the total acreage (600 acres), is given. By converting these amounts into fractions of the total acreage, we see that the transition matrix A and the initial probability vector P of the Markov chain are

$$A = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} \frac{300}{600} \\ \frac{200}{600} \\ \frac{100}{600} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{6} \end{pmatrix}.$$

The fraction of the total acreage devoted to each crop in m years is given by the coordinates of $A^m P$, and the eventual proportions of the total acreage used for each crop are the coordinates of $\lim\limits_{m\to\infty} A^m P$. Thus the eventual amounts of land devoted to each crop are found by multiplying this limit by the total acreage; that is, the eventual amounts of land used for each crop are the coordinates of $600 \cdot \lim\limits_{m\to\infty} A^m P$.

Since A is a regular transition matrix, Theorem 5.19 shows that $\lim\limits_{m\to\infty} A^m$ is a matrix L in which each column equals the unique fixed probability vector for A. It is easily seen that the fixed probability vector for A is

$$\begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}.$$

Hence

$$L = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix};$$

so

$$600 \cdot \lim_{m\to\infty} A^m P = 600LP = \begin{pmatrix} 200 \\ 200 \\ 200 \end{pmatrix}.$$

Thus, in the long run, we expect 200 acres of each crop to be planted each year. (For a direct computation of $600 \cdot \lim_{m\to\infty} A^m P$, see Exercise 14.) ◆

In this section, we have concentrated primarily on the theory of regular transition matrices. There is another interesting class of transition matrices that can be represented in the form

$$\begin{pmatrix} I & B \\ O & C \end{pmatrix},$$

where I is an identity matrix and O is a zero matrix. (Such transition matrices are not regular since the lower left block remains O in any power of the matrix.) The states corresponding to the identity submatrix are called **absorbing states** because such a state is never left once it is entered. A Markov chain is called an **absorbing Markov chain** if it is possible to go from an arbitrary state into an absorbing state in a finite number of stages. Observe that the Markov chain that describes the enrollment pattern in a community college is an absorbing Markov chain with states 1 and 4 as its absorbing states. (See page 291.) Readers interested in learning more about absorbing Markov chains are referred to *Introduction to Finite Mathematics* (third edition) by J. Kemeny, J. Snell, and G. Thompson (Prentice-Hall, Inc., Englewood Cliffs, N. J., 1974) or *Discrete Mathematical Models* by Fred S. Roberts (Prentice-Hall, Inc., Englewood Cliffs, N. J., 1976).

Applications*

Visit `goo.gl/cKTyHi` for an application of Markov chains to internet searches.

In species that reproduce sexually, the characteristics of an offspring with respect to a particular genetic trait are determined by a pair of genes, one inherited from each parent. The genes for a particular trait are of two types, which are denoted by G and g. The gene G represents the dominant characteristic, and g represents the recessive characteristic. Offspring with genotypes GG or Gg exhibit the dominant characteristic, whereas offspring with genotype gg exhibit the recessive characteristic. For example, in humans, brown eyes are a dominant characteristic and blue eyes are the corresponding recessive characteristic; thus the offspring with genotypes GG or Gg are brown-eyed, whereas those of type gg are blue-eyed.

Let us consider the probability of offspring of each genotype for a male fruit fly of genotype Gg. (We assume that the population under consideration is large, that mating is random with respect to genotype, and that the distribution of each genotype within the population is independent of sex and life expectancy.) Let

$$P = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

denote the present proportion of the fruit fly population with genotypes GG, Gg, and gg, respectively. This situation can by described by a three-state Markov chain with the following transition matrix:

$$\begin{array}{lll} & & \text{Genotype of female parent} \\ & & \begin{array}{ccc} \text{GG} & \text{Gg} & \text{gg} \end{array} \\ \begin{array}{l} \text{Genotype} \\ \text{of} \\ \text{offspring} \end{array} & \begin{array}{l} \text{GG} \\ \text{Gg} \\ \text{gg} \end{array} & \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix} = B. \end{array}$$

It is easily checked that B^2 contains only positive entries; so B is regular. Thus, if only males of genotype Gg survive until reproductive maturity, the proportion of offspring in the population having a certain genotype will stabilize at the fixed probability vector for B, which is

$$\begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}.$$

Now suppose that similar experiments are to be performed with males of genotypes GG and gg. As already mentioned, these experiments are three-state Markov chains with transition matrices

$$A = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & 0 & 0 \\ 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \end{pmatrix},$$

respectively. In order to consider the case where all male genotypes are permitted to reproduce, we must form the transition matrix $M = pA + qB + rC$, which is the linear combination of A, B, and C weighted by the proportion of males of each genotype. Thus

$$M = \begin{pmatrix} p + \frac{1}{2}q & \frac{1}{2}p + \frac{1}{4}q & 0 \\ \frac{1}{2}q + r & \frac{1}{2}p + \frac{1}{2}q + \frac{1}{2}r & p + \frac{1}{2}q \\ 0 & \frac{1}{4}q + \frac{1}{2}r & \frac{1}{2}q + r \end{pmatrix}.$$

To simplify the notation, let $a = p + \frac{1}{2}q$ and $b = \frac{1}{2}q + r$. (The numbers a and b represent the proportions of G and g genes, respectively, in the population.) Then

$$M = \begin{pmatrix} a & \frac{1}{2}a & 0 \\ b & \frac{1}{2} & a \\ 0 & \frac{1}{2}b & b \end{pmatrix},$$

where $a + b = p + q + r = 1$.

Let p', q', and r' denote the proportions of the first-generation offspring having genotypes GG, Gg, and gg, respectively. Then

$$\begin{pmatrix} p' \\ q' \\ r' \end{pmatrix} = MP = \begin{pmatrix} ap + \frac{1}{2}aq \\ bp + \frac{1}{2}q + ar \\ \frac{1}{2}bq + br \end{pmatrix} = \begin{pmatrix} a^2 \\ 2ab \\ b^2 \end{pmatrix}.$$

In order to consider the effects of unrestricted matings among the first-generation offspring, a new transition matrix $\widetilde{M}$ must be determined based upon the distribution of first-generation genotypes. As before, we find that

$$\widetilde{M} = \begin{pmatrix} p' + \frac{1}{2}q' & \frac{1}{2}p' + \frac{1}{4}q' & 0 \\ \frac{1}{2}q' + r' & \frac{1}{2}p' + \frac{1}{2}q' + \frac{1}{2}r' & p' + \frac{1}{2}q' \\ 0 & \frac{1}{4}q' + \frac{1}{2}r' & \frac{1}{2}q' + r' \end{pmatrix} = \begin{pmatrix} a' & \frac{1}{2}a' & 0 \\ b' & \frac{1}{2} & a' \\ 0 & \frac{1}{2}b' & b' \end{pmatrix},$$

where $a' = p' + \frac{1}{2}q'$ and $b' = \frac{1}{2}q' + r'$. However

$$a' = a^2 + \frac{1}{2}(2ab) = a(a+b) = a \quad \text{and} \quad b' = \frac{1}{2}(2ab) + b^2 = b(a+b) = b.$$

Thus $\widetilde{M} = M$; so the distribution of second-generation offspring among the three genotypes is

$$\begin{aligned} \widetilde{M}(MP) = M^2P &= \begin{pmatrix} a^3 + a^2b \\ a^2b + ab + ab^2 \\ ab^2 + b^3 \end{pmatrix} = \begin{pmatrix} a^2(a+b) \\ ab(a+1+b) \\ b^2(a+b) \end{pmatrix} = \begin{pmatrix} a^2 \\ 2ab \\ b^2 \end{pmatrix} \\ &= MP, \end{aligned}$$

the same as the first-generation offspring. In other words, MP is the fixed probability vector for M, and genetic equilibrium is achieved in the population after only one generation. (This result is called the *Hardy–Weinberg law.*) Notice that in the important special case that $a = b$ (or equivalently, that $p = r$), the distribution at equilibrium is

$$MP = \begin{pmatrix} a^2 \\ 2ab \\ b^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}.$$

EXERCISES

1. Label the following statements as true or false.

(a) If $A \in \mathsf{M}_{n\times n}(C)$ and $\lim_{m\to\infty} A^m = L$, then, for any invertible matrix $Q \in \mathsf{M}_{n\times n}(C)$, we have $\lim_{m\to\infty} QA^mQ^{-1} = QLQ^{-1}$.

(b) If 2 is an eigenvalue of $A \in \mathsf{M}_{n\times n}(C)$, then $\lim_{m\to\infty} A^m$ does not exist.

(c) Any vector

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathsf{R}^n$$

such that $x_1 + x_2 + \cdots + x_n = 1$ is a probability vector.

(d) The sum of the entries of each row of a transition matrix equals 1.

(e) The product of a transition matrix and a probability vector is a probability vector.

(f) Let z be any complex number such that $|z| < 1$. Then the matrix

$$\begin{pmatrix} 1 & z & -1 \\ z & 1 & 1 \\ -1 & 1 & z \end{pmatrix}$$

does not have 3 as an eigenvalue.

(g) Every transition matrix has 1 as an eigenvalue.

(h) No transition matrix can have -1 as an eigenvalue.

(i) If A is a transition matrix, then $\lim_{m\to\infty} A^m$ exists.

(j) If A is a regular transition matrix, then $\lim_{m\to\infty} A^m$ exists and has rank 1.

2. Determine whether $\lim_{m\to\infty} A^m$ exists for each of the following matrices A, and compute the limit if it exists.

(a) $\begin{pmatrix} 0.1 & 0.7 \\ 0.7 & 0.1 \end{pmatrix}$ **(b)** $\begin{pmatrix} -1.4 & 0.8 \\ -2.4 & 1.8 \end{pmatrix}$ **(c)** $\begin{pmatrix} 0.4 & 0.7 \\ 0.6 & 0.3 \end{pmatrix}$

(d) $\begin{pmatrix} -1.8 & 4.8 \\ -0.8 & 2.2 \end{pmatrix}$ **(e)** $\begin{pmatrix} -2 & -1 \\ 4 & 3 \end{pmatrix}$ **(f)** $\begin{pmatrix} 2.0 & -0.5 \\ 3.0 & -0.5 \end{pmatrix}$

(g) $\begin{pmatrix} -1.8 & 0 & -1.4 \\ -5.6 & 1 & -2.8 \\ 2.8 & 0 & 2.4 \end{pmatrix}$ **(h)** $\begin{pmatrix} 3.4 & -0.2 & 0.8 \\ 3.9 & 1.8 & 1.3 \\ -16.5 & -2.0 & -4.5 \end{pmatrix}$

(i) $\begin{pmatrix} -\frac{1}{2}-2i & 4i & \frac{1}{2}+5i \\ 1+2i & -3i & -1-4i \\ -1-2i & 4i & 1+5i \end{pmatrix}$

(j) $\begin{pmatrix} \dfrac{-26+i}{3} & \dfrac{-28-4i}{3} & 28 \\ \dfrac{-7+2i}{3} & \dfrac{-5+i}{3} & 7-2i \\ \dfrac{-13+6i}{6} & \dfrac{-5+6i}{6} & \dfrac{35-20i}{6} \end{pmatrix}$

3. Prove that if $A_1, A_2, \dots$ is a sequence of $n \times p$ matrices with complex entries such that $\lim_{m\to\infty} A_m = L$, then $\lim_{m\to\infty} (A_m)^t = L^t$.

4. Prove that if $A \in \mathsf{M}_{n\times n}(C)$ is diagonalizable and $L = \lim_{m\to\infty} A^m$ exists, then either $L = I_n$ or $\operatorname{rank}(L) < n$.

5. Find 2×2 matrices A and B having real entries such that $\lim_{m\to\infty} A^m$, $\lim_{m\to\infty} B^m$, and $\lim_{m\to\infty} (AB)^m$ all exist, but

$$\lim_{m\to\infty} (AB)^m \neq (\lim_{m\to\infty} A^m)(\lim_{m\to\infty} B^m).$$

6. In the week beginning June 1, 30% of the patients who arrived by helicopter at a hospital trauma unit were ambulatory and 70% were bedridden. One week after arrival, 60% of the ambulatory patients had been released, 20% remained ambulatory, and 20% had become bedridden. After the same amount of time, 10% of the bedridden patients had been released, 20% had become ambulatory, 50% remained bedridden, and 20% had died. Determine the percentages of helicopter arrivals during the week of June 1 who were in each of the four states one week after arrival. Assuming that the given percentages continue in the future, also determine the percentages of patients who eventually end up in each of the four states.

7. A player begins a game of chance by placing a marker in box 2, marked *Start*. (See Figure 5.5.) A die is rolled, and the marker is moved one square to the left if a 1 or a 2 is rolled and one square to the right if a 3, 4, 5, or 6 is rolled. This process continues until the marker lands in square 1, in which case the player wins the game, or in square 4, in which case the player loses the game. What is the probability of winning this game? *Hint:* Instead of diagonalizing the appropriate transition matrix A, it is easier to represent e_2 as a linear combination of eigenvectors of A and then apply A^n to the result.

Win	Start		Lose
1	2	3	4

Figure 5.5

8. Which of the following transition matrices are regular?

(a) $\begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.2 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix}$ **(b)** $\begin{pmatrix} 0.5 & 0 & 1 \\ 0.5 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ **(c)** $\begin{pmatrix} 0.5 & 0 & 0 \\ 0.5 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 0.5 & 0 & 1 \\ 0.5 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ **(e)** $\begin{pmatrix} \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{1}{3} & 0 & 1 \end{pmatrix}$ **(f)** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.7 & 0.2 \\ 0 & 0.3 & 0.8 \end{pmatrix}$

(g) $\begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 1 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 1 \end{pmatrix}$ **(h)** $\begin{pmatrix} \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 1 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 1 \end{pmatrix}$

9. Compute $\lim_{m\to\infty} A^m$ if it exists, for each matrix A in Exercise 8.

10. Each of the matrices that follow is a regular transition matrix for a three-state Markov chain. In all cases, the initial probability vector is

$$P = \begin{pmatrix} 0.3 \\ 0.3 \\ 0.4 \end{pmatrix}.$$

For each transition matrix, compute the proportions of objects in each state after two stages and the eventual proportions of objects in each state by determining the fixed probability vector.

(a) $\begin{pmatrix} 0.6 & 0.1 & 0.1 \\ 0.1 & 0.9 & 0.2 \\ 0.3 & 0 & 0.7 \end{pmatrix}$ **(b)** $\begin{pmatrix} 0.8 & 0.1 & 0.2 \\ 0.1 & 0.8 & 0.2 \\ 0.1 & 0.1 & 0.6 \end{pmatrix}$ **(c)** $\begin{pmatrix} 0.9 & 0.1 & 0.1 \\ 0.1 & 0.6 & 0.1 \\ 0 & 0.3 & 0.8 \end{pmatrix}$

(d) $\begin{pmatrix} 0.4 & 0.2 & 0.2 \\ 0.1 & 0.7 & 0.2 \\ 0.5 & 0.1 & 0.6 \end{pmatrix}$ **(e)** $\begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.2 & 0.5 & 0.3 \\ 0.3 & 0.2 & 0.5 \end{pmatrix}$ **(f)** $\begin{pmatrix} 0.6 & 0 & 0.4 \\ 0.2 & 0.8 & 0.2 \\ 0.2 & 0.2 & 0.4 \end{pmatrix}$

11. In 1940, a county land-use survey showed that 10% of the county land was urban, 50% was unused, and 40% was agricultural. Five years later, a follow-up survey revealed that 70% of the urban land had remained urban, 10% had become unused, and 20% had become agricultural. Likewise, 20% of the unused land had become urban, 60% had remained unused, and 20% had become agricultural. Finally, the 1945 survey showed that 20% of the agricultural land had become unused while 80% remained agricultural. Assuming that the trends indicated by the 1945 survey continue, compute the percentages of urban, unused, and agricultural land in the county in 1950 and the corresponding eventual percentages.

12. A diaper liner is placed in each diaper worn by a baby. If, after a diaper change, the liner is soiled, then it is replaced by a new liner. Otherwise, the liner is washed with the diapers and reused, except that each liner is replaced by a new liner after its second use, even if it has never been soiled. The probability that the baby will soil any diaper liner is one-third. If there are only new diaper liners at first, eventually what proportions of the diaper liners being used will be new, once-used, and twice-used?

13. In 1975, the automobile industry determined that 40% of American car owners drove large cars, 20% drove intermediate-sized cars, and 40% drove small cars. A second survey in 1985 showed that 70% of the large-car owners in 1975 still owned large cars in 1985, but 30% had changed to an intermediate-sized car. Of those who owned intermediate-sized cars in 1975, 10% had switched to large cars, 70% continued to drive intermediate-sized cars, and 20% had changed to small cars in 1985. Finally, of the small-car owners in 1975, 10% owned intermediate-sized cars and 90% owned small cars in 1985. Assuming that these trends continue, determine the percentages of Americans who own cars of each size in 1995 and the corresponding eventual percentages.

14. Show that if A and P are as in Example 5, then

$$A^m = \begin{pmatrix} r_m & r_{m+1} & r_{m+1} \\ r_{m+1} & r_m & r_{m+1} \\ r_{m+1} & r_{m+1} & r_m \end{pmatrix},$$

where

$$r_m = \frac{1}{3}\left[1 + \frac{(-1)^m}{2^{m-1}}\right].$$

Deduce that

$$600(A^mP) = A^m \begin{pmatrix} 300 \\ 200 \\ 100 \end{pmatrix} = \begin{pmatrix} 200 + \dfrac{(-1)^m}{2^m}(100) \\ 200 \\ 200 + \dfrac{(-1)^{m+1}}{2^m}(100) \end{pmatrix}.$$

15. Prove that if a 1-dimensional subspace W of R^n contains a nonzero vector with all nonnegative entries, then W contains a unique probability vector.

16. Prove Theorem 5.14 and its corollary.

17. Prove the two corollaries of Theorem 5.17. Visit `goo.gl/V5Hsou` for a solution.

18. Prove the corollary of Theorem 5.18.

19. Suppose that M and M' are $n \times n$ transition matrices.

(a) Prove that if M is regular, N is any $n \times n$ transition matrix, and c is a real number such that $0 < c \leq 1$, then $cM + (1-c)N$ is a regular transition matrix.

(b) Suppose that for all i, j, we have that $M'_{ij} > 0$ whenever $M_{ij} > 0$. Prove that there exists a transition matrix N and a real number c with $0 < c \leq 1$ such that $M' = cM + (1-c)N$.

(c) Deduce that if the nonzero entries of M and M' occur in the same positions, then M is regular if and only if M' is regular.

The following definition is used in Exercises 20–24.

Definition. *For $A \in \mathsf{M}_{n\times n}(C)$, define $e^A = \lim_{m\to\infty} B_m$, where*

$$B_m = I + A + \frac{A^2}{2!} + \cdots + \frac{A^m}{m!}.$$

This limit exists by Exercise 22 of Section 7.2. Thus e^A is the sum of the infinite series

$$I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots,$$

and B_m is the mth partial sum of this series. (Note the analogy with the power series

$$e^a = 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \cdots,$$

which is valid for all complex numbers a.)

20. Compute e^O and e^I, where O and I denote the $n \times n$ zero and identity matrices, respectively.

21. Let $P^{-1}AP = D$ be a diagonal matrix. Prove that $e^A = Pe^DP^{-1}$.

22. Let $A \in \mathsf{M}_{n\times n}(C)$ be diagonalizable. Use the result of Exercise 21 to show that e^A exists. (Exercise 22 of Section 7.2 shows that e^A exists for every $A \in \mathsf{M}_{n\times n}(C)$.)

23. Find $A, B \in \mathsf{M}_{2\times 2}(R)$ such that $e^Ae^B \neq e^{A+B}$.

24. Prove that a differentiable function $x\colon R \to \mathsf{R}^n$ is a solution to the system of differential equations defined in Exercise 16 of Section 5.2 if and only if $x(t) = e^{tA}v$ for some $v \in \mathsf{R}^n$, where A is defined in that exercise.

5.4 INVARIANT SUBSPACES AND THE CAYLEY–HAMILTON THEOREM

In Section 5.1, we observed that if v is an eigenvector of a linear operator T, then T maps the span of $\{v\}$ into itself. Subspaces that are mapped into themselves are of great importance in the study of linear operators (see, e.g., Exercises 29–33 of Section 2.1).

Definition. *Let T be a linear operator on a vector space V. A subspace W of V is called a* **T-invariant subspace** *of V if $\mathsf{T}(\mathsf{W}) \subseteq \mathsf{W}$, that is, if $\mathsf{T}(v) \in \mathsf{W}$ for all $v \in \mathsf{W}$.*

Example 1

Suppose that T is a linear operator on a vector space V. Then the following subspaces of V are T-invariant:

1. $\{0\}$
2. V
3. $\mathsf{R}(\mathsf{T})$
4. $\mathsf{N}(\mathsf{T})$
5. E_λ, for any eigenvalue λ of T.

The proofs that these subspaces are T-invariant are left as exercises. (See Exercise 3.) ◆

Example 2

Let T be the linear operator on R^3 defined by

$$\mathsf{T}(a, b, c) = (a + b, b + c, 0).$$

Then the xy-plane $= \{(x, y, 0)\colon x, y \in R\}$ and the x-axis $= \{(x, 0, 0)\colon x \in R\}$ are T-invariant subspaces of R^3. ◆

Let T be a linear operator on a vector space V, and let x be a nonzero vector in V. The subspace

$$\mathsf{W} = \operatorname{span}(\{x, \mathsf{T}(x), \mathsf{T}^2(x), \ldots\})$$

is called the **T-cyclic subspace** of V **generated by** x. It is a simple matter to show that W is T-invariant. In fact, W is the "smallest" T-invariant subspace of V containing x. That is, any T-invariant subspace of V containing x must also contain W (see Exercise 11). Cyclic subspaces have various uses. We apply them in this section to establish the Cayley–Hamilton theorem. In Exercise 31, we outline a method for using cyclic subspaces to compute the characteristic polynomial of a linear operator without resorting to determinants. Cyclic subspaces also play an important role in Chapter 7, where we study matrix representations of nondiagonalizable linear operators.

Example 3

Let T be the linear operator on R^3 defined by

$$\mathsf{T}(a, b, c) = (-b + c, a + c, 3c).$$

We determine the T-cyclic subspace generated by $e_1 = (1, 0, 0)$. Since

$$\mathsf{T}(e_1) = \mathsf{T}(1, 0, 0) = (0, 1, 0) = e_2$$

and

$$\mathsf{T}^2(e_1) = \mathsf{T}(\mathsf{T}(e_1)) = \mathsf{T}(e_2) = (-1, 0, 0) = -e_1,$$

it follows that

$$\operatorname{span}(\{e_1, \mathsf{T}(e_1), \mathsf{T}^2(e_1), \ldots\}) = \operatorname{span}(\{e_1, e_2\}) = \{(s, t, 0) \colon s, t \in R\}. \quad \blacklozenge$$

Example 4

Let T be the linear operator on $\mathsf{P}(R)$ defined by $\mathsf{T}(f(x)) = f'(x)$. Then the T-cyclic subspace generated by x^2 is $\operatorname{span}(\{x^2, 2x, 2\}) = \mathsf{P}_2(R)$. $\blacklozenge$

The existence of a T-invariant subspace provides the opportunity to define a new linear operator whose domain is this subspace. If T is a linear operator on V and W is a T-invariant subspace of V, then the restriction T_W of T to W (see Appendix B) is a mapping from W to W, and it follows that T_W is a linear operator on W (see Exercise 7). As a linear operator, T_W inherits certain properties from its parent operator T. The following result illustrates one way in which the two operators are linked.

Theorem 5.20. *Let T be a linear operator on a finite-dimensional vector space V, and let W be a T-invariant subspace of V. Then the characteristic polynomial of T_W divides the characteristic polynomial of T.*

Proof. Choose an ordered basis $\gamma = \{v_1, v_2, \ldots, v_k\}$ for W, and extend it to an ordered basis $\beta = \{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ for V. Let $A = [\mathsf{T}]_\beta$ and $B_1 = [\mathsf{T_W}]_\gamma$. Then, by Exercise 12, A can be written in the form

$$A = \begin{pmatrix} B_1 & B_2 \\ O & B_3 \end{pmatrix}.$$

Let $f(t)$ be the characteristic polynomial of T and $g(t)$ the characteristic polynomial of $\mathsf{T_W}$. Then

$$f(t) = \det(A - tI_n) = \det \begin{pmatrix} B_1 - tI_k & B_2 \\ O & B_3 - tI_{n-k} \end{pmatrix} = g(t) \cdot \det(B_3 - tI_{n-k})$$

by Exercise 21 of Section 4.3. Thus $g(t)$ divides $f(t)$. ∎

Example 5

Let T be the linear operator on R^4 defined by

$$\mathsf{T}(a, b, c, d) = (a + b + 2c - d, b + d, 2c - d, c + d),$$

and let $\mathsf{W} = \{(t, s, 0, 0) \colon t, s \in R\}$. Observe that W is a T-invariant subspace of R^4 because, for any vector $(a, b, 0, 0) \in \mathsf{W}$,

$$\mathsf{T}(a, b, 0, 0) = (a + b, b, 0, 0) \in \mathsf{W}.$$

Let $\gamma = \{e_1, e_2\}$, which is an ordered basis for W. Extend γ to the standard ordered basis β for R^4. Then

$$B_1 = [\mathsf{T_W}]_\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A = [\mathsf{T}]_\beta = \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

in the notation of Theorem 5.20. Let $f(t)$ be the characteristic polynomial of T and $g(t)$ be the characteristic polynomial of $\mathsf{T_W}$. Then

$$\begin{aligned} f(t) = \det(A - tI_4) &= \det \begin{pmatrix} 1-t & 1 & 2 & -1 \\ 0 & 1-t & 0 & 1 \\ 0 & 0 & 2-t & -1 \\ 0 & 0 & 1 & 1-t \end{pmatrix} \\ &= \det \begin{pmatrix} 1-t & 1 \\ 0 & 1-t \end{pmatrix} \cdot \det \begin{pmatrix} 2-t & -1 \\ 1 & 1-t \end{pmatrix} \\ &= g(t) \cdot \det \begin{pmatrix} 2-t & -1 \\ 1 & 1-t \end{pmatrix}. \quad \blacklozenge \end{aligned}$$

In view of Theorem 5.20, we may use the characteristic polynomial of $\mathsf{T_W}$ to gain information about the characteristic polynomial of T itself. In this regard, cyclic subspaces are useful because the characteristic polynomial of the restriction of a linear operator T to a cyclic subspace is readily computable.

Theorem 5.21. *Let T be a linear operator on a finite-dimensional vector space V, and let W denote the T-cyclic subspace of V generated by a nonzero vector $v \in \mathsf{V}$. Let $k = \dim(\mathsf{W})$. Then*

(a) $\{v, \mathsf{T}(v), \mathsf{T}^2(v), \ldots, \mathsf{T}^{k-1}(v)\}$ *is a basis for* W.

(b) *If* $a_0v + a_1\mathsf{T}(v) + \cdots + a_{k-1}\mathsf{T}^{k-1}(v) + \mathsf{T}^k(v) = 0$, *then the characteristic polynomial of* $\mathsf{T_W}$ *is* $f(t) = (-1)^k(a_0 + a_1t + \cdots + a_{k-1}t^{k-1} + t^k)$.

Proof. (a) Since $v \neq 0$, the set $\{v\}$ is linearly independent. Let j be the largest positive integer for which

$$\beta = \{v, \mathsf{T}(v), \ldots, \mathsf{T}^{j-1}(v)\}$$

is linearly independent. Such a j must exist because V is finite-dimensional. Let $\mathsf{Z} = \operatorname{span}(\beta)$. Then β is a basis for Z. Furthermore, $\mathsf{T}^j(v) \in \mathsf{Z}$ by Theorem 1.7 (p. 40). We use this information to show that Z is a T-invariant subspace of V. Let $w \in \mathsf{Z}$. Since w is a linear combination of the vectors of β, there exist scalars $b_0, b_1, \ldots, b_{j-1}$ such that

$$w = b_0v + b_1\mathsf{T}(v) + \cdots + b_{j-1}\mathsf{T}^{j-1}(v),$$

and hence

$$\mathsf{T}(w) = b_0\mathsf{T}(v) + b_1\mathsf{T}^2(v) + \cdots + b_{j-1}\mathsf{T}^j(v).$$

Thus $\mathsf{T}(w)$ is a linear combination of vectors in Z, and hence belongs to Z. So Z is T-invariant. Furthermore, $v \in \mathsf{Z}$. By Exercise 11, W is the smallest T-invariant subspace of V that contains v, so that $\mathsf{W} \subseteq \mathsf{Z}$. Clearly, $\mathsf{Z} \subseteq \mathsf{W}$, and so we conclude that $\mathsf{Z} = \mathsf{W}$. It follows that β is a basis for W, and therefore $\dim(\mathsf{W}) = j$. Thus $j = k$. This proves (a).

(b) Now view β (from (a)) as an ordered basis for W. Let $a_0, a_1, \ldots, a_{k-1}$ be the scalars such that

$$a_0v + a_1\mathsf{T}(v) + \cdots + a_{k-1}\mathsf{T}^{k-1}(v) + \mathsf{T}^k(v) = 0.$$

Observe that

$$[\mathsf{T_W}]_\beta = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{k-1} \end{pmatrix},$$

which has the characteristic polynomial

$$f(t) = (-1)^k(a_0 + a_1t + \cdots + a_{k-1}t^{k-1} + t^k)$$

by Exercise 19. Thus $f(t)$ is the characteristic polynomial of $\mathsf{T_W}$, proving (b). ∎

Example 6

Let T be the linear operator of Example 3, and let $\mathsf{W} = \text{span}(\{e_1, e_2\})$, the T-cyclic subspace generated by e_1. We compute the characteristic polynomial $f(t)$ of $\mathsf{T_W}$ in two ways: by means of Theorem 5.21 and by means of determinants.

(a) *By means of Theorem 5.21.* From Example 3, we have that $\{e_1, e_2\}$ is a cycle that generates W, and that $\mathsf{T}^2(e_1) = -e_1$. Hence

$$1e_1 + 0\mathsf{T}(e_1) + \mathsf{T}^2(e_1) = 0.$$

Therefore, by Theorem 5.21(b),

$$f(t) = (-1)^2(1 + 0t + t^2) = t^2 + 1.$$

(b) *By means of determinants.* Let $\beta = \{e_1, e_2\}$, which is an ordered basis for W. Since $\mathsf{T}(e_1) = e_2$ and $\mathsf{T}(e_2) = -e_1$, we have

$$[\mathsf{T_W}]_\beta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and therefore,

$$f(t) = \det \begin{pmatrix} -t & -1 \\ 1 & -t \end{pmatrix} = t^2 + 1. \quad \blacklozenge$$

The Cayley–Hamilton Theorem

As an illustration of the importance of Theorem 5.21, we prove a well-known result that is used in Chapter 7. The reader should refer to Appendix E for the definition of $f(\mathsf{T})$, where T is a linear operator and $f(x)$ is a polynomial.

Theorem 5.22 (Cayley–Hamilton). *Let T be a linear operator on a finite-dimensional vector space V, and let $f(t)$ be the characteristic polynomial of T. Then $f(\mathsf{T}) = \mathsf{T}_0$, the zero transformation. That is, T "satisfies" its characteristic equation.*

Proof. We show that $f(\mathsf{T})(v) = 0$ for all $v \in \mathsf{V}$. This is obvious if $v = 0$ because $f(\mathsf{T})$ is linear; so suppose that $v \neq 0$. Let W be the T-cyclic subspace generated by v, and suppose that $\dim(\mathsf{W}) = k$. By Theorem 5.21(a), there exist scalars $a_0, a_1, \ldots, a_{k-1}$ such that

$$a_0 v + a_1 \mathsf{T}(v) + \cdots + a_{k-1}\mathsf{T}^{k-1}(v) + \mathsf{T}^k(v) = 0.$$

Hence Theorem 5.21(b) implies that

$$g(t) = (-1)^k(a_0 + a_1t + \cdots + a_{k-1}t^{k-1} + t^k)$$

is the characteristic polynomial of $\mathsf{T_W}$. Combining these two equations yields

$$g(\mathsf{T})(v) = (-1)^k(a_0\mathsf{I} + a_1\mathsf{T} + \cdots + a_{k-1}\mathsf{T}^{k-1} + \mathsf{T}^k)(v) = \mathit{0}.$$

By Theorem 5.20, $g(t)$ divides $f(t)$; hence there exists a polynomial $q(t)$ such that $f(t) = q(t)g(t)$. So

$$f(\mathsf{T})(v) = q(\mathsf{T})g(\mathsf{T})(v) = q(\mathsf{T})(g(\mathsf{T})(v)) = q(\mathsf{T})(\mathit{0}) = \mathit{0}. \qquad \blacksquare$$

Example 7

Let T be the linear operator on R^2 defined by $\mathsf{T}(a, b) = (a + 2b, -2a + b)$, and let $\beta = \{e_1, e_2\}$. Then

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix},$$

where $A = [\mathsf{T}]_\beta$. The characteristic polynomial of T is, therefore,

$$f(t) = \det(A - tI) = \det\begin{pmatrix} 1-t & 2 \\ -2 & 1-t \end{pmatrix} = t^2 - 2t + 5.$$

It is easily verified that $\mathsf{T}_0 = f(\mathsf{T}) = \mathsf{T}^2 - 2\mathsf{T} + 5\mathsf{I}$. Similarly,

$$f(A) = A^2 - 2A + 5I = \begin{pmatrix} -3 & 4 \\ -4 & -3 \end{pmatrix} + \begin{pmatrix} -2 & -4 \\ 4 & -2 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \quad \blacklozenge$$

Example 7 suggests the following result.

Corollary (Cayley–Hamilton Theorem for Matrices). *Let A be an $n \times n$ matrix, and let $f(t)$ be the characteristic polynomial of A. Then $f(A) = O$, the $n \times n$ zero matrix.*

Proof. See Exercise 15. $\blacksquare$

Invariant Subspaces and Direct Sums*[3]

It is useful to decompose a finite-dimensional vector space V into a direct sum of as many T-invariant subspaces as possible because the behavior of T on V can be inferred from its behavior on the direct summands. For example,

[3]This subsection uses optional material on direct sums from Section 5.2.

T is diagonalizable if and only if V can be decomposed into a direct sum of one-dimensional T-invariant subspaces (see Exercise 35). In Chapter 7, we consider alternate ways of decomposing V into direct sums of T-invariant subspaces if T is not diagonalizable. We proceed to gather a few facts about direct sums of T-invariant subspaces that are used in Section 7.4. The first of these facts is about characteristic polynomials.

Theorem 5.23. *Let T be a linear operator on a finite-dimensional vector space V, and suppose that $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \cdots \oplus \mathsf{W}_k$, where W_i is a T-invariant subspace of V for each i $(1 \leq i \leq k)$. Suppose that $f_i(t)$ is the characteristic polynomial of $\mathsf{T}_{\mathsf{W}_i}$ $(1 \leq i \leq k)$. Then $f_1(t) \cdot f_2(t) \cdot \cdots \cdot f_k(t)$ is the characteristic polynomial of T.*

Proof. The proof is by mathematical induction on k. In what follows, $f(t)$ denotes the characteristic polynomial of T. Suppose first that $k = 2$. Let β_1 be an ordered basis for W_1, β_2 an ordered basis for W_2, and $\beta = \beta_1 \cup \beta_2$. Then β is an ordered basis for V by Theorem 5.9(d) (p. 275). Let $A = [\mathsf{T}]_\beta$, $B_1 = [\mathsf{T}_{\mathsf{W}_1}]_{\beta_1}$, and $B_2 = [\mathsf{T}_{\mathsf{W}_2}]_{\beta_2}$. By Exercise 33, it follows that

$$A = \begin{pmatrix} B_1 & O \\ O' & B_2 \end{pmatrix},$$

where O and O' are zero matrices of the appropriate sizes. Then

$$f(t) = \det(A - tI) = \det(B_1 - tI) \cdot \det(B_2 - tI) = f_1(t) \cdot f_2(t)$$

as in the proof of Theorem 5.20, proving the result for $k = 2$.

Now assume that the theorem is valid for $k-1$ summands, where $k-1 \geq 2$, and suppose that V is a direct sum of k subspaces, say,

$$\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \cdots \oplus \mathsf{W}_k.$$

Let $\mathsf{W} = \mathsf{W}_1 + \mathsf{W}_2 + \cdots + \mathsf{W}_{k-1}$. It is easily verified that W is T-invariant and that $\mathsf{V} = \mathsf{W} \oplus \mathsf{W}_k$. So by the case for $k = 2$, $f(t) = g(t) \cdot f_k(t)$, where $g(t)$ is the characteristic polynomial of T_W. Clearly $\mathsf{W} = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \cdots \oplus \mathsf{W}_{k-1}$, and therefore $g(t) = f_1(t) \cdot f_2(t) \cdot \cdots \cdot f_{k-1}(t)$ by the induction hypothesis. We conclude that $f(t) = g(t) \cdot f_k(t) = f_1(t) \cdot f_2(t) \cdot \cdots \cdot f_k(t)$. ∎

As an illustration of this result, suppose that T is a diagonalizable linear operator on a finite-dimensional vector space V with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. By Theorem 5.10 (p. 277), V is a direct sum of the eigenspaces of T. Since each eigenspace is T-invariant, we may view this situation in the context of Theorem 5.23. For each eigenvalue λ_i, the restriction of T to E_{λ_i} has characteristic polynomial $(\lambda_i - t)^{m_i}$, where m_i is the dimension of E_{λ_i}. By Theorem 5.23, the characteristic polynomial $f(t)$ of T is the product

$$f(t) = (\lambda_1 - t)^{m_1} (\lambda_2 - t)^{m_2} \cdots (\lambda_k - t)^{m_k}.$$

It follows that the multiplicity of each eigenvalue is equal to the dimension of the corresponding eigenspace, as expected.

Example 8

Let T be the linear operator on R^4 defined by

$$\mathsf{T}(a,b,c,d) = (2a-b, a+b, c-d, c+d),$$

and let $\mathsf{W}_1 = \{(s,t,0,0)\colon s,t \in R\}$ and $\mathsf{W}_2 = \{(0,0,s,t)\colon s,t \in R\}$. Notice that W_1 and W_2 are each T-invariant and that $\mathsf{R}^4 = \mathsf{W}_1 \oplus \mathsf{W}_2$. Let $\beta_1 = \{e_1, e_2\}$, $\beta_2 = \{e_3, e_4\}$, and $\beta = \beta_1 \cup \beta_2 = \{e_1, e_2, e_3, e_4\}$. Then β_1 is an ordered basis for W_1, β_2 is an ordered basis for W_2, and β is an ordered basis for R^4. Let $A = [\mathsf{T}]_\beta$, $B_1 = [\mathsf{T}_{\mathsf{W}_1}]_{\beta_1}$, and $B_2 = [\mathsf{T}_{\mathsf{W}_2}]_{\beta_2}$. Then

$$B_1 = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

and

$$A = \begin{pmatrix} B_1 & O \\ O & B_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Let $f(t)$, $f_1(t)$, and $f_2(t)$ denote the characteristic polynomials of T, $\mathsf{T}_{\mathsf{W}_1}$, and $\mathsf{T}_{\mathsf{W}_2}$, respectively. Then

$$f(t) = \det(A - tI) = \det(B_1 - tI)\cdot\det(B_2 - tI) = f_1(t)\cdot f_2(t). \quad \blacklozenge$$

The matrix A in Example 8 can be obtained by joining the matrices B_1 and B_2 in the manner explained in the next definition.

Definition. *Let $B_1 \in \mathsf{M}_{m\times m}(F)$, and let $B_2 \in \mathsf{M}_{n\times n}(F)$. We define the* ***direct sum*** *of B_1 and B_2, denoted $B_1 \oplus B_2$, as the $(m+n)\times(m+n)$ matrix A such that*

$$A_{ij} = \begin{cases} (B_1)_{ij} & \text{for } 1 \le i,j \le m \\ (B_2)_{(i-m),(j-m)} & \text{for } m+1 \le i,j \le n+m \\ 0 & \text{otherwise.} \end{cases}$$

If $B_1, B_2, \ldots, B_k$ are square matrices with entries from F, then we define the ***direct sum*** *of $B_1, B_2, \ldots, B_k$ recursively by*

$$B_1 \oplus B_2 \oplus \cdots \oplus B_k = (B_1 \oplus B_2 \oplus \cdots \oplus B_{k-1}) \oplus B_k.$$

If $A = B_1 \oplus B_2 \oplus \cdots \oplus B_k$, then we often write

$$A = \begin{pmatrix} B_1 & O & \cdots & O \\ O & B_2 & \cdots & O \\ \vdots & \vdots & & \vdots \\ O & O & \cdots & B_k \end{pmatrix}.$$

Example 9

Let

$$B_1 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \quad B_2 = (3), \quad \text{and} \quad B_3 = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}.$$

Then

$$B_1 \oplus B_2 \oplus B_3 = \left(\begin{array}{cc|c|ccc} 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 3 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right). \quad \blacklozenge$$

The final result of this section relates direct sums of matrices to direct sums of invariant subspaces. It is an extension of Exercise 33 to the case $k \geq 2$.

Theorem 5.24. *Let T be a linear operator on a finite-dimensional vector space V, and let $\mathsf{W}_1, \mathsf{W}_2, \ldots, \mathsf{W}_k$ be T-invariant subspaces of V such that $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \cdots \oplus \mathsf{W}_k$. For each i, let β_i be an ordered basis for W_i, and let $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$. Let $A = [\mathsf{T}]_\beta$ and $B_i = [\mathsf{T}_{\mathsf{W}_i}]_{\beta_i}$ for $i = 1, 2, \ldots, k$. Then $A = B_1 \oplus B_2 \oplus \cdots \oplus B_k$.*

Proof. See Exercise 34. ∎

EXERCISES

1. Label the following statements as true or false.
   - **(a)** There exists a linear operator T with no T-invariant subspace.
   - **(b)** If T is a linear operator on a finite-dimensional vector space V and W is a T-invariant subspace of V, then the characteristic polynomial of T_W divides the characteristic polynomial of T.
   - **(c)** Let T be a linear operator on a finite-dimensional vector space V, and let v and w be in V. If W is the T-cyclic subspace generated by v, W' is the T-cyclic subspace generated by w, and $\mathsf{W} = \mathsf{W}'$, then $v = w$.
   - **(d)** If T is a linear operator on a finite-dimensional vector space V, then for any $v \in \mathsf{V}$ the T-cyclic subspace generated by v is the same as the T-cyclic subspace generated by $\mathsf{T}(v)$.
   - **(e)** Let T be a linear operator on an n-dimensional vector space. Then there exists a polynomial $g(t)$ of degree n such that $g(\mathsf{T}) = \mathsf{T}_0$.
   - **(f)** Any polynomial of degree n with leading coefficient $(-1)^n$ is the characteristic polynomial of some linear operator.

(g) If T is a linear operator on a finite-dimensional vector space V, and if V is the direct sum of k T-invariant subspaces, then there is an ordered basis β for V such that $[\mathsf{T}]_\beta$ is a direct sum of k matrices.

2. For each of the following linear operators T on the vector space V, determine whether the given subspace W is a T-invariant subspace of V.

(a) $\mathsf{V} = \mathsf{P}_3(R)$, $\mathsf{T}(f(x)) = f'(x)$, and $\mathsf{W} = \mathsf{P}_2(R)$

(b) $\mathsf{V} = \mathsf{P}(R)$, $\mathsf{T}(f(x)) = xf(x)$, and $\mathsf{W} = \mathsf{P}_2(R)$

(c) $\mathsf{V} = \mathsf{R}^3$, $\mathsf{T}(a, b, c) = (a+b+c, a+b+c, a+b+c)$, and $\mathsf{W} = \{(t, t, t)\colon t \in R\}$

(d) $\mathsf{V} = \mathsf{C}([0, 1])$, $\mathsf{T}(f(t)) = \left[\int_0^1 f(x)\,dx\right] t$, and $\mathsf{W} = \{f \in \mathsf{V}\colon f(t) = at + b \text{ for some } a \text{ and } b\}$

(e) $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$, $\mathsf{T}(A) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A$, and $\mathsf{W} = \{A \in \mathsf{V}\colon A^t = A\}$

3. Let T be a linear operator on a finite-dimensional vector space V. Prove that the following subspaces are T-invariant.

(a) $\{\mathit{0}\}$ and V

(b) $\mathsf{N}(\mathsf{T})$ and $\mathsf{R}(\mathsf{T})$

(c) E_λ, for any eigenvalue λ of T

4. Let T be a linear operator on a vector space V, and let W be a T-invariant subspace of V. Prove that W is $g(\mathsf{T})$-invariant for any polynomial $g(t)$.

5. Let T be a linear operator on a vector space V. Prove that the intersection of any collection of T-invariant subspaces of V is a T-invariant subspace of V.

6. For each linear operator T on the vector space V, find an ordered basis for the T-cyclic subspace generated by the vector z.

(a) $\mathsf{V} = \mathsf{R}^4$, $\mathsf{T}(a, b, c, d) = (a+b, b-c, a+c, a+d)$, and $z = e_1$.

(b) $\mathsf{V} = \mathsf{P}_3(R)$, $\mathsf{T}(f(x)) = f''(x)$, and $z = x^3$.

(c) $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$, $\mathsf{T}(A) = A^t$, and $z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(d) $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$, $\mathsf{T}(A) = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} A$, and $z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

7. Prove that the restriction of a linear operator T to a T-invariant subspace is a linear operator on that subspace.

8. Let T be a linear operator on a vector space with a T-invariant subspace W. Prove that if v is an eigenvector of T_W with corresponding eigenvalue λ, then v is also an eigenvector of T with corresponding eigenvalue λ.

9. For each linear operator T and cyclic subspace W in Exercise 6, compute the characteristic polynomial of $\mathsf{T_W}$ in two ways, as in Example 6.

10. For each linear operator in Exercise 6, find the characteristic polynomial $f(t)$ of T, and verify that the characteristic polynomial of $\mathsf{T_W}$ (computed in Exercise 9) divides $f(t)$.

11. Let T be a linear operator on a vector space V, let v be a nonzero vector in V, and let W be the T-cyclic subspace of V generated by v. Prove that

(a) W is T-invariant.

(b) Any T-invariant subspace of V containing v also contains W.

12. Prove that $A = \begin{pmatrix} B_1 & B_2 \\ O & B_3 \end{pmatrix}$ in the proof of Theorem 5.20.

13. Let T be a linear operator on a vector space V, let v be a nonzero vector in V, and let W be the T-cyclic subspace of V generated by v. For any $w \in \mathsf{V}$, prove that $w \in \mathsf{W}$ if and only if there exists a polynomial $g(t)$ such that $w = g(\mathsf{T})(v)$.

14. Prove that the polynomial $g(t)$ of Exercise 13 can always be chosen so that its degree is less than $\dim(\mathsf{W})$.

15. Use the Cayley–Hamilton theorem (Theorem 5.22) to prove its corollary for matrices. *Warning:* If $f(t) = \det(A - tI)$ is the characteristic polynomial of A, it is tempting to "prove" that $f(A) = O$ by saying "$f(A) = \det(A - AI) = \det(O) = 0$." Why is this argument incorrect? Visit `goo.gl/ZMVn9i` for a solution.

16. Let T be a linear operator on a finite-dimensional vector space V.

(a) Prove that if the characteristic polynomial of T splits, then so does the characteristic polynomial of the restriction of T to any T-invariant subspace of V.

(b) Deduce that if the characteristic polynomial of T splits, then any nontrivial T-invariant subspace of V contains an eigenvector of T.

17. Let A be an $n \times n$ matrix. Prove that

$$\dim(\operatorname{span}(\{I_n, A, A^2, \ldots\})) \leq n.$$

18. Let A be an $n \times n$ matrix with characteristic polynomial

$$f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0.$$

(a) Prove that A is invertible if and only if $a_0 \neq 0$.

(b) Prove that if A is invertible, then

$$A^{-1} = (-1/a_0)[(-1)^n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I_n].$$

(c) Use (b) to compute A^{-1} for

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{pmatrix}.$$

19. Let A denote the $k \times k$ matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{k-1} \end{pmatrix},$$

where $a_0, a_1, \ldots, a_{k-1}$ are arbitrary scalars. Prove that the characteristic polynomial of A is

$$(-1)^k(a_0 + a_1 t + \cdots + a_{k-1}t^{k-1} + t^k).$$

Hint: Use mathematical induction on k, computing the determinant by cofactor expansion along the first row.

20. Let T be a linear operator on a vector space V, and suppose that V is a T-cyclic subspace of itself. Prove that if U is a linear operator on V, then $\mathsf{UT} = \mathsf{TU}$ if and only if $\mathsf{U} = g(\mathsf{T})$ for some polynomial $g(t)$. *Hint:* Suppose that V is generated by v. Choose $g(t)$ according to Exercise 13 so that $g(\mathsf{T})(v) = \mathsf{U}(v)$.

21. Let T be a linear operator on a two-dimensional vector space V. Prove that either V is a T-cyclic subspace of itself or $\mathsf{T} = c\mathsf{I}$ for some scalar c.

22. Let T be a linear operator on a two-dimensional vector space V and suppose that $\mathsf{T} \neq c\mathsf{I}$ for any scalar c. Show that if U is any linear operator on V such that $\mathsf{UT} = \mathsf{TU}$, then $\mathsf{U} = g(\mathsf{T})$ for some polynomial $g(t)$.

23. Let T be a linear operator on a finite-dimensional vector space V, and let W be a T-invariant subspace of V. Suppose that $v_1, v_2, \ldots, v_k$ are eigenvectors of T corresponding to distinct eigenvalues. Prove that if $v_1 + v_2 + \cdots + v_k$ is in W, then $v_i \in \mathsf{W}$ for all i. *Hint:* Use mathematical induction on k.

24. Prove that the restriction of a diagonalizable linear operator T to any nontrivial T-invariant subspace is also diagonalizable. *Hint:* Use the result of Exercise 23.

25. **(a)** Prove the converse to Exercise 19(a) of Section 5.2: If T and U are diagonalizable linear operators on a finite-dimensional vector space V such that $\mathsf{UT} = \mathsf{TU}$, then T and U are simultaneously diagonalizable. (See the definitions in the exercises of Section 5.2.) *Hint:* For any eigenvalue λ of T, show that E_λ is U-invariant, and apply Exercise 24 to obtain a basis for E_λ of eigenvectors of U.

(b) State and prove a matrix version of (a).

26. Let T be a linear operator on an n-dimensional vector space V such that T has n distinct eigenvalues. Prove that V is a T-cyclic subspace of itself. *Hint:* Use Exercise 23 to find a vector v such that $\{v, \mathsf{T}(v), \ldots, \mathsf{T}^{n-1}(v)\}$ is linearly independent.

Exercises 27 through 31 require familiarity with quotient spaces as defined in Exercise 31 of Section 1.3. Before attempting these exercises, the reader should first review the other exercises treating quotient spaces: Exercise 35 of Section 1.6, Exercise 42 of Section 2.1, and Exercise 24 of Section 2.4.

For the purposes of Exercises 27 through 31, T is a fixed linear operator on a finite-dimensional vector space V, and W is a nonzero T-invariant subspace of V. We require the following definition.

Definition. *Let T be a linear operator on a vector space V, and let W be a T-invariant subspace of V. Define $\overline{\mathsf{T}}\colon \mathsf{V}/\mathsf{W} \to \mathsf{V}/\mathsf{W}$ by*

$$\overline{\mathsf{T}}(v + \mathsf{W}) = \mathsf{T}(v) + \mathsf{W} \quad \textit{for any } v + \mathsf{W} \in \mathsf{V}/\mathsf{W}.$$

27. **(a)** Prove that $\overline{\mathsf{T}}$ is well defined. That is, show that $\overline{\mathsf{T}}(v + \mathsf{W}) = \overline{\mathsf{T}}(v' + \mathsf{W})$ whenever $v + \mathsf{W} = v' + \mathsf{W}$.

(b) Prove that $\overline{\mathsf{T}}$ is a linear operator on V/W.

(c) Let $\eta\colon \mathsf{V} \to \mathsf{V}/\mathsf{W}$ be the linear transformation defined in Exercise 42 of Section 2.1 by $\eta(v) = v + \mathsf{W}$. Show that the diagram of Figure 5.6 commutes; that is, prove that $\eta\mathsf{T} = \overline{\mathsf{T}}\eta$. (This exercise does not require the assumption that V is finite-dimensional.)

$$\begin{tikzcd} \mathsf{V} \arrow[r, "\mathsf{T}"] \arrow[d, "\eta"'] & \mathsf{V} \arrow[d, "\eta"] \\ \mathsf{V}/\mathsf{W} \arrow[r, "\overline{\mathsf{T}}"] & \mathsf{V}/\mathsf{W} \end{tikzcd}$$

Figure 5.6

28. Let $f(t)$, $g(t)$, and $h(t)$ be the characteristic polynomials of T, T_W, and $\overline{\mathsf{T}}$, respectively. Prove that $f(t) = g(t)h(t)$. *Hint:* Extend an ordered basis $\gamma = \{v_1, v_2, \ldots, v_k\}$ for W to an ordered basis $\beta = \{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ for V. Then show that the collection of cosets $\alpha = \{v_{k+1} + \mathsf{W}, v_{k+2} + \mathsf{W}, \ldots, v_n + \mathsf{W}\}$ is an ordered basis for V/W, and prove that

$$[\mathsf{T}]_\beta = \begin{pmatrix} B_1 & B_2 \\ O & B_3 \end{pmatrix},$$

where $B_1 = [\mathsf{T}]_\gamma$ and $B_3 = [\overline{\mathsf{T}}]_\alpha$.

29. Use the hint in Exercise 28 to prove that if T is diagonalizable, then so is $\overline{\mathsf{T}}$.

30. Prove that if both T_W and $\overline{\mathsf{T}}$ are diagonalizable and have no common eigenvalues, then T is diagonalizable.

The results of Theorem 5.21 and Exercise 28 are useful in devising methods for computing characteristic polynomials without the use of determinants. This is illustrated in the next exercise.

31. Let $A = \begin{pmatrix} 1 & 1 & -3 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{pmatrix}$, let $\mathsf{T} = \mathsf{L}_A$, and let W be the cyclic subspace of R^3 generated by e_1.

(a) Use Theorem 5.21 to compute the characteristic polynomial of T_W.
(b) Show that $\{e_2 + \mathsf{W}\}$ is a basis for R^3/W, and use this fact to compute the characteristic polynomial of $\overline{\mathsf{T}}$.
(c) Use the results of (a) and (b) to find the characteristic polynomial of A.

Exercises 32 through 39 are concerned with direct sums.

32. Let T be a linear operator on a vector space V, and let $\mathsf{W}_1, \mathsf{W}_2, \ldots, \mathsf{W}_k$ be T-invariant subspaces of V. Prove that $\mathsf{W}_1 + \mathsf{W}_2 + \cdots + \mathsf{W}_k$ is also a T-invariant subspace of V.

33. Give a direct proof of Theorem 5.24 for the case $k = 2$. (This result is used in the proof of Theorem 5.23.)

34. Prove Theorem 5.24. *Hint:* Begin with Exercise 33 and extend it using mathematical induction on k, the number of subspaces.

35. Let T be a linear operator on a finite-dimensional vector space V. Prove that T is diagonalizable if and only if V is the direct sum of one-dimensional T-invariant subspaces.

36. Let T be a linear operator on a finite-dimensional vector space V, and let $\mathsf{W}_1, \mathsf{W}_2, \ldots, \mathsf{W}_k$ be T-invariant subspaces of V such that $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \cdots \oplus \mathsf{W}_k$. Prove that

$$\det(\mathsf{T}) = \det(\mathsf{T}_{\mathsf{W}_1}) \cdot \det(\mathsf{T}_{\mathsf{W}_2}) \cdot \cdots \cdot \det(\mathsf{T}_{\mathsf{W}_k}).$$

37. Let T be a linear operator on a finite-dimensional vector space V, and let $\mathsf{W}_1, \mathsf{W}_2, \ldots, \mathsf{W}_k$ be T-invariant subspaces of V such that $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \cdots \oplus \mathsf{W}_k$. Prove that T is diagonalizable if and only if $\mathsf{T}_{\mathsf{W}_i}$ is diagonalizable for all i.

38. Let $\mathcal{C}$ be a collection of diagonalizable linear operators on a finite-dimensional vector space V. Prove that there is an ordered basis β such that $[\mathsf{T}]_\beta$ is a diagonal matrix for all $\mathsf{T} \in \mathcal{C}$ if and only if the operators of $\mathcal{C}$ commute under composition. (This is an extension of Exercise 25.) *Hints for the case that the operators commute:* The result is trivial if each operator has only one eigenvalue. Otherwise, establish the general result by mathematical induction on $\dim(\mathsf{V})$, using the fact that V is the direct sum of the eigenspaces of some operator in $\mathcal{C}$ that has more than one eigenvalue.

39. Let $B_1, B_2, \ldots, B_k$ be square matrices with entries in the same field, and let $A = B_1 \oplus B_2 \oplus \cdots \oplus B_k$. Prove that the characteristic polynomial of A is the product of the characteristic polynomials of the B_i's.

40. Let

$$A = \begin{pmatrix} 1 & 2 & \cdots & n \\ n+1 & n+2 & \cdots & 2n \\ \vdots & \vdots & & \vdots \\ n^2-n+1 & n^2-n+2 & \cdots & n^2 \end{pmatrix}.$$

Find the characteristic polynomial of A. *Hint:* First prove that A has rank 2 and that $\operatorname{span}(\{(1, 1, \ldots, 1), (1, 2, \ldots, n)\})$ is L_A-invariant.

41. Let $A \in \mathsf{M}_{n\times n}(R)$ be the matrix defined by $A_{ij} = 1$ for all i and j. Find the characteristic polynomial of A.

INDEX OF DEFINITIONS FOR CHAPTER 5

6

Inner Product Spaces

*M*ost applications of mathematics are involved with the concept of measurement and hence of the magnitude or relative size of various quantities. So it is not surprising that the fields of real and complex numbers, which have a built-in notion of distance, should play a special role. Except for Section 6.8, *in this chapter we assume that all vector spaces are over either the field of real numbers or the field of complex numbers.* See Appendix D for properties of complex numbers.

We introduce the idea of distance or length into vector spaces via a much richer structure, the so-called *inner product space* structure. This added structure provides applications to geometry (Sections 6.5 and 6.11), physics (Section 6.9), conditioning in systems of linear equations (Section 6.10), least squares (Section 6.3), and quadratic forms (Section 6.8).

6.1 INNER PRODUCTS AND NORMS

Many geometric notions such as angle, length, and perpendicularity in R^2 and R^3 may be extended to more general real and complex vector spaces. All of these ideas are related to the concept of *inner product*.

Definition. *Let* V *be a vector space over* F. *An* **inner product** *on* V *is a function that assigns, to every ordered pair of vectors* x *and* y *in* V, *a* *scalar in F, denoted $\langle x, y\rangle$, such that for all x, y, and z in V and all c in F, the following hold:*

(a) $\langle x+z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$.
(b) $\langle cx, y\rangle = c\,\langle x, y\rangle$.
(c) $\overline{\langle x, y\rangle} = \langle y, x\rangle$, *where the bar denotes complex conjugation.*
(d) *If $x \neq 0$, then $\langle x, x\rangle$ is a positive real number.*

Note that (c) reduces to $\langle x, y\rangle = \langle y, x\rangle$ if $F = R$. Conditions (a) and (b) simply require that the inner product be linear in the first component.

It is easily shown that if $a_1, a_2, \ldots, a_n \in F$ and $y, v_1, v_2, \ldots, v_n \in \mathsf{V}$, then

$$\left\langle \sum_{i=1}^{n} a_i v_i, y \right\rangle = \sum_{i=1}^{n} a_i \langle v_i, y\rangle .$$

Example 1

For $x = (a_1, a_2, \ldots, a_n)$ and $y = (b_1, b_2, \ldots, b_n)$ in F^n, define

$$\langle x, y\rangle = \sum_{i=1}^{n} a_i \overline{b}_i.$$

The verification that $\langle \cdot, \cdot \rangle$ satisfies conditions (a) through (d) is easy. For example, if $z = (c_1, c_2, \ldots, c_n)$, we have for (a)

$$\begin{aligned}\langle x+z, y\rangle &= \sum_{i=1}^{n}(a_i + c_i)\overline{b}_i = \sum_{i=1}^{n} a_i\overline{b}_i + \sum_{i=1}^{n} c_i\overline{b}_i \\ &= \langle x, y\rangle + \langle z, y\rangle .\end{aligned}$$

Thus, for $x = (1+i, 4)$ and $y = (2-3i, 4+5i)$ in C^2,

$$\langle x, y\rangle = (1+i)(2+3i) + 4(4-5i) = 15 - 15i. \quad \blacklozenge$$

The inner product in Example 1 is called the **standard inner product** on F^n. When $F = R$, the conjugations are not needed, and in early courses this standard inner product is usually called the *dot product* and is denoted by $x \cdot y$ instead of $\langle x, y\rangle$.

Example 2

If $\langle x, y\rangle$ is any inner product on a vector space V and $r > 0$, we may define another inner product by the rule $\langle x, y\rangle' = r\,\langle x, y\rangle$. If $r \leq 0$, then (d) would not hold. $\blacklozenge$

Example 3

Let $\mathsf{V} = \mathsf{C}([0,1])$, the vector space of real-valued continuous functions on $[0,1]$. For $f, g \in \mathsf{V}$, define $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$. Since the preceding integral is linear in f, (a) and (b) are immediate, and (c) is trivial. If $f \neq 0$, then f^2 is bounded away from zero on some subinterval of $[0,1]$ (continuity is used here), and hence $\langle f, f\rangle = \int_0^1 [f(t)]^2\,dt > 0$. ◆

Definition. *Let $A \in \mathsf{M}_{m\times n}(F)$. We define the **conjugate transpose** or **adjoint** of A to be the $n \times m$ matrix A^* such that $(A^*)_{ij} = \overline{A}_{ji}$ for all i, j.*

Example 4

Let

$$A = \begin{pmatrix} i & 1+2i \\ 2 & 3+4i \end{pmatrix}.$$

Then

$$A^* = \begin{pmatrix} -i & 2 \\ 1-2i & 3-4i \end{pmatrix}. \quad ◆$$

Notice that if x and y are viewed as column vectors in F^n, then $\langle x, y\rangle = y^*x$.

The conjugate transpose of a matrix plays a very important role in the remainder of this chapter. In the case that A has real entries, A^* is simply the transpose of A.

Example 5

Let $\mathsf{V} = \mathsf{M}_{n\times n}(F)$, and define $\langle A, B\rangle = \operatorname{tr}(B^*A)$ for $A, B \in \mathsf{V}$. (Recall that the trace of a matrix A is defined by $\operatorname{tr}(A) = \sum_{i=1}^n A_{ii}$.) We verify that (a) and (d) of the definition of inner product hold and leave (b) and (c) to the reader. For this purpose, let $A, B, C \in \mathsf{V}$. Then (using Exercise 6 of Section 1.3)

$$\begin{aligned}\langle A+B, C\rangle &= \operatorname{tr}(C^*(A+B)) = \operatorname{tr}(C^*A + C^*B)\\ &= \operatorname{tr}(C^*A) + \operatorname{tr}(C^*B) = \langle A, C\rangle + \langle B, C\rangle.\end{aligned}$$

Also

$$\begin{aligned}\langle A, A\rangle = \operatorname{tr}(A^*A) &= \sum_{i=1}^n (A^*A)_{ii} = \sum_{i=1}^n \sum_{k=1}^n (A^*)_{ik}A_{ki}\\ &= \sum_{i=1}^n \sum_{k=1}^n \overline{A}_{ki}A_{ki} = \sum_{i=1}^n \sum_{k=1}^n |A_{ki}|^2.\end{aligned}$$

Now if $A \neq O$, then $A_{ki} \neq 0$ for some k and i. So $\langle A, A\rangle > 0$. ◆

The inner product on $\mathsf{M}_{n\times n}(F)$ in Example 5 is called the **Frobenius inner product**.

A vector space V over F endowed with a specific inner product is called an **inner product space**. If $F = C$, we call V a **complex inner product space**, whereas if $F = R$, we call V a **real inner product space**.

It is clear that if V has an inner product $\langle x, y\rangle$ and W is a subspace of V, then W is also an inner product space when the same function $\langle x, y\rangle$ is restricted to the vectors $x, y \in \mathsf{W}$.

Thus Examples 1, 3, and 5 also provide examples of inner product spaces. *For the remainder of this chapter, F^n denotes the inner product space with the standard inner product as defined in Example 1. Likewise, $\mathsf{M}_{n\times n}(F)$ denotes the inner product space with the Frobenius inner product as defined in Example 5.* The reader is cautioned that two distinct inner products on a given vector space yield two distinct inner product spaces. For instance, it can be shown that both

$$\langle f(x), g(x)\rangle_1 = \int_0^1 f(t)g(t)\,dt \quad \text{and} \quad \langle f(x), g(x)\rangle_2 = \int_{-1}^1 f(t)g(t)\,dt$$

are inner products on the vector space $\mathsf{P}(R)$. Even though the underlying vector space is the same, however, these two inner products yield two different inner product spaces. For example, the polynomials $f(x) = x$ and $g(x) = x^2$ are orthogonal in the second inner product space, but not in the first.

A very important inner product space that resembles $\mathsf{C}([0,1])$ is the space H of continuous complex-valued functions defined on the interval $[0, 2\pi]$ with the inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}\,dt.$$

The reason for the constant $1/2\pi$ will become evident later. This inner product space, which arises often in the context of physical situations, is examined more closely in later sections.

At this point, we mention a few facts about integration of complex-valued functions. First, the imaginary number i can be treated as a constant under the integration sign. Second, every complex-valued function f may be written as $f = f_1 + if_2$, where f_1 and f_2 are real-valued functions. Thus we have

$$\int f\,dt = \int f_1\,dt + i\int f_2\,dt \quad \text{and} \quad \overline{\int f\,dt} = \int \overline{f}\,dt.$$

From these properties, as well as the assumption of continuity, it follows that H is an inner product space (see Exercise 16(a)).

Some properties that follow easily from the definition of an inner product are contained in the next theorem.

Theorem 6.1. *Let V be an inner product space. Then for $x, y, z \in \mathsf{V}$ and $c \in F$, the following statements are true.*

(a) $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$.
(b) $\langle x, cy \rangle = \overline{c}\,\langle x, y \rangle$.
(c) $\langle x, \mathit{0} \rangle = \langle \mathit{0}, x \rangle = 0$.
(d) $\langle x, x \rangle = 0$ *if and only if* $x = \mathit{0}$.
(e) *If* $\langle x, y \rangle = \langle x, z \rangle$ *for all* $x \in \mathsf{V}$, *then* $y = z$.

Proof. (a) We have

$$\begin{aligned}\langle x, y+z \rangle &= \overline{\langle y+z, x \rangle} = \overline{\langle y, x \rangle + \langle z, x \rangle} \\ &= \overline{\langle y, x \rangle} + \overline{\langle z, x \rangle} = \langle x, y \rangle + \langle x, z \rangle\,.\end{aligned}$$

The proofs of (b), (c), (d), and (e) are left as exercises. ∎

The reader should observe that (a) and (b) of Theorem 6.1 show that the inner product is **conjugate linear** in the second component.

In order to generalize the notion of length in R^3 to arbitrary inner product spaces, we need only observe that the length of $x = (a, b, c) \in \mathsf{R}^3$ is given by $\sqrt{a^2+b^2+c^2} = \sqrt{\langle x, x \rangle}$. This leads to the following definition.

Definition. *Let V be an inner product space. For $x \in \mathsf{V}$, we define the* ***norm*** *or* ***length*** *of x by $\|x\| = \sqrt{\langle x, x \rangle}$.*

Example 6

Let $\mathsf{V} = \mathsf{F}^n$. If $x = (a_1, a_2 \ldots, a_n)$, then

$$\|x\| = \|(a_1, a_2 \ldots, a_n)\| = \left[\sum_{i=1}^{n} |a_i|^2\right]^{1/2}$$

is the Euclidean definition of length. Note that if $n = 1$, we have $\|a\| = |a|$. ◆

As we might expect, the well-known properties of Euclidean length in R^3 hold in general, as shown next.

Theorem 6.2. *Let V be an inner product space over F. Then for all $x, y \in \mathsf{V}$ and $c \in F$, the following statements are true.*

(a) $\|cx\| = |c| \cdot \|x\|$.
(b) $\|x\| = 0$ *if and only if* $x = \mathit{0}$. *In any case,* $\|x\| \geq 0$.
(c) *(Cauchy–Schwarz Inequality)* $|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$.
(d) *(Triangle Inequality)* $\|x+y\| \leq \|x\| + \|y\|$.

Proof. We leave the proofs of (a) and (b) as exercises.

(c) If $y = \mathit{0}$, then the result is immediate. So assume that $y \neq \mathit{0}$. For any $c \in F$, we have

$$\begin{aligned} 0 \leq \langle x - cy, x - cy \rangle &= \langle x, x - cy \rangle - c \langle y, x - cy \rangle \\ &= \langle x, x \rangle - \overline{c} \langle x, y \rangle - c \langle y, x \rangle + c\overline{c} \langle y, y \rangle . \end{aligned}$$

In particular, if we set

$$c = \frac{\langle x, y \rangle}{\langle y, y \rangle},$$

then each of $\overline{c} \langle x, y \rangle$, $c \langle y, x \rangle$, and $c\overline{c} \langle y, y \rangle$ equals $\dfrac{\langle x, y \rangle \langle y, x \rangle}{\langle y, y \rangle} = \dfrac{|\langle x, y \rangle|^2}{\|y\|^2}$. So the preceding inequality becomes

$$0 \leq \langle x, x \rangle - \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle} = \|x\|^2 - \frac{|\langle x, y \rangle|^2}{\|y\|^2},$$

from which (c) follows.

(d) We have

$$\begin{aligned} \|x + y\|^2 &= \langle x + y, x + y \rangle = \langle x, x \rangle + \langle y, x \rangle + \langle x, y \rangle + \langle y, y \rangle \\ &= \|x\|^2 + 2\Re \langle x, y \rangle + \|y\|^2 \\ &\leq \|x\|^2 + 2 |\langle x, y \rangle| + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\| \cdot \|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2, \end{aligned}$$

where $\Re \langle x, y \rangle$ denotes the real part of the complex number $\langle x, y \rangle$. Note that we used (c) to prove (d). ∎

The case when equality results in (c) and (d) is considered in Exercise 15.

Example 7

For F^n, we may apply (c) and (d) of Theorem 6.2 to the standard inner product to obtain the following well-known inequalities:

$$\left| \sum_{i=1}^{n} a_i \overline{b}_i \right| \leq \left[\sum_{i=1}^{n} |a_i|^2 \right]^{1/2} \left[\sum_{i=1}^{n} |b_i|^2 \right]^{1/2}$$

and

$$\left[\sum_{i=1}^{n} |a_i + b_i|^2 \right]^{1/2} \leq \left[\sum_{i=1}^{n} |a_i|^2 \right]^{1/2} + \left[\sum_{i=1}^{n} |b_i|^2 \right]^{1/2}. \quad \blacklozenge$$

The reader may recall from earlier courses that, for x and y in R^3 or R^2, we have that $\langle x, y\rangle = \|x\| \cdot \|y\| \cos\theta$, where θ $(0 \le \theta \le \pi)$ denotes the angle between x and y. This equation implies (c) immediately since $|\cos\theta| \le 1$. Notice also that nonzero vectors x and y are perpendicular if and only if $\cos\theta = 0$, that is, if and only if $\langle x, y\rangle = 0$.

We are now at the point where we can generalize the notion of perpendicularity to arbitrary inner product spaces.

Definitions. *Let* V *be an inner product space. Vectors* x *and* y *in* V *are* ***orthogonal*** *(or* ***perpendicular****) if* $\langle x, y\rangle = 0$. *A subset* S *of* V *is* ***orthogonal*** *if any two distinct vectors in* S *are orthogonal. A vector* x *in* V *is a* ***unit vector*** *if* $\|x\| = 1$. *Finally, a subset* S *of* V *is* ***orthonormal*** *if* S *is orthogonal and consists entirely of unit vectors.*

Note that if $S = \{v_1, v_2, \ldots\}$, then S is orthonormal if and only if $\langle v_i, v_j\rangle = \delta_{ij}$, where δ_{ij} denotes the Kronecker delta. Also, observe that multiplying vectors by nonzero scalars does not affect their orthogonality and that if x is any nonzero vector, then $(1/\|x\|)x$ is a unit vector. The process of multiplying a nonzero vector by the reciprocal of its length is called **normalizing**.

Example 8

In F^3, $\{(1,1,0), (1,-1,1), (-1,1,2)\}$ is an orthogonal set of nonzero vectors, but it is not orthonormal; however, if we normalize the vectors in the set, we obtain the orthonormal set

$$\left\{\frac{1}{\sqrt{2}}(1,1,0), \frac{1}{\sqrt{3}}(1,-1,1), \frac{1}{\sqrt{6}}(-1,1,2)\right\}. \quad \blacklozenge$$

Our next example is of an infinite orthonormal set that is important in analysis. This set is used in later examples in this chapter.

Example 9

Recall the inner product space H (defined on page 330). We introduce an important orthonormal subset S of H. For what follows, i is the imaginary number such that $i^2 = -1$. For any integer n, let $f_n(t) = e^{int}$, where $0 \le t \le 2\pi$. (Recall that $e^{int} = \cos nt + i \sin nt$.) Now define $S = \{f_n : n \text{ is an integer}\}$. Clearly S is a subset of H. Using the property that $\overline{e^{it}} = e^{-it}$ for every real number t, we have, for $m \ne n$,

$$\begin{aligned}\langle f_m, f_n\rangle &= \frac{1}{2\pi}\int_0^{2\pi} e^{imt}\overline{e^{int}}\,dt = \frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)t}\,dt \\ &= \frac{1}{2\pi i(m-n)} e^{i(m-n)t}\Big|_0^{2\pi} = 0.\end{aligned}$$

Also,

$$\langle f_n, f_n \rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{i(n-n)t}\,dt = \frac{1}{2\pi}\int_0^{2\pi} 1\,dt = 1.$$

In other words, $\langle f_m, f_n \rangle = \delta_{mn}$. ◆

EXERCISES

1. Label the following statements as true or false.

(a) An inner product is a scalar-valued function on the set of ordered pairs of vectors.

(b) An inner product space must be over the field of real or complex numbers.

(c) An inner product is linear in both components.

(d) There is exactly one inner product on the vector space R^n.

(e) The triangle inequality only holds in finite-dimensional inner product spaces.

(f) Only square matrices have a conjugate-transpose.

(g) If x, y, and z are vectors in an inner product space such that $\langle x, y \rangle = \langle x, z \rangle$, then $y = z$.

(h) If $\langle x, y \rangle = 0$ for all x in an inner product space, then $y = \mathit{0}$.

2. Let $x = (2, 1+i, i)$ and $y = (2-i, 2, 1+2i)$ be vectors in C^3. Compute $\langle x, y \rangle$, $\|x\|$, $\|y\|$, and $\|x+y\|$. Then verify both the Cauchy–Schwarz inequality and the triangle inequality.

3. In $\mathsf{C}([0,1])$, let $f(t) = t$ and $g(t) = e^t$. Compute $\langle f, g \rangle$ (as defined in Example 3), $\|f\|$, $\|g\|$, and $\|f+g\|$. Then verify both the Cauchy–Schwarz inequality and the triangle inequality.

4. **(a)** Complete the proof in Example 5 that $\langle \cdot, \cdot \rangle$ is an inner product (the Frobenius inner product) on $\mathsf{M}_{n\times n}(F)$.

(b) Use the Frobenius inner product to compute $\|A\|$, $\|B\|$, and $\langle A, B \rangle$ for

$$A = \begin{pmatrix} 1 & 2+i \\ 3 & i \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1+i & 0 \\ i & -i \end{pmatrix}.$$

5. In C^2, show that $\langle x, y \rangle = xAy^*$ is an inner product, where

$$A = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}.$$

Compute $\langle x, y \rangle$ for $x = (1-i, 2+3i)$ and $y = (2+i, 3-2i)$.

6. Complete the proof of Theorem 6.1.

7. Complete the proof of Theorem 6.2.

8. Provide reasons why each of the following is not an inner product on the given vector spaces.

(a) $\langle (a, b), (c, d) \rangle = ac - bd$ on R^2.
(b) $\langle A, B \rangle = \mathrm{tr}(A + B)$ on $\mathsf{M}_{2\times 2}(R)$.
(c) $\langle f(x), g(x) \rangle = \int_0^1 f'(t)g(t)\,dt$ on $\mathsf{P}(R)$, where $'$ denotes differentiation.

9. Let β be a basis for a finite-dimensional inner product space.

(a) Prove that if $\langle x, z \rangle = 0$ for all $z \in \beta$, then $x = \mathit{0}$.
(b) Prove that if $\langle x, z \rangle = \langle y, z \rangle$ for all $z \in \beta$, then $x = y$.

10.† Let V be an inner product space, and suppose that x and y are orthogonal vectors in V. Prove that $\|x + y\|^2 = \|x\|^2 + \|y\|^2$. Deduce the Pythagorean theorem in R^2. Visit `goo.gl/1iTZzC` for a solution.

11. Prove the *parallelogram law* on an inner product space V; that is, show that

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \text{for all } x, y \in \mathsf{V}.$$

What does this equation state about parallelograms in R^2?

12.† Let $\{v_1, v_2, \dots, v_k\}$ be an orthogonal set in V, and let $a_1, a_2, \dots, a_k$ be scalars. Prove that

$$\left\| \sum_{i=1}^{k} a_i v_i \right\|^2 = \sum_{i=1}^{k} |a_i|^2 \|v_i\|^2.$$

13. Suppose that $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ are two inner products on a vector space V. Prove that $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_1 + \langle \cdot, \cdot \rangle_2$ is another inner product on V.

14. Let A and B be $n \times n$ matrices, and let c be a scalar. Prove that $(A + cB)^* = A^* + \bar{c}B^*$.

15. **(a)** Prove that if V is an inner product space, then $|\langle x, y \rangle| = \|x\| \cdot \|y\|$ if and only if one of the vectors x or y is a multiple of the other. *Hint:* If the identity holds and $y \neq \mathit{0}$, let

$$a = \frac{\langle x, y \rangle}{\|y\|^2},$$

and let $z = x - ay$. Prove that y and z are orthogonal and

$$|a| = \frac{\|x\|}{\|y\|}.$$

Then apply Exercise 10 to $\|x\|^2 = \|ay + z\|^2$ to obtain $\|z\| = 0$.

(b) Derive a similar result for the equality $\|x+y\| = \|x\| + \|y\|$, and generalize it to the case of n vectors.

16. (a) Show that the vector space H with $\langle \cdot, \cdot \rangle$ defined on page 330 is an inner product space.

(b) Let $\mathsf{V} = \mathsf{C}([0,1])$, and define

$$\langle f, g \rangle = \int_0^{1/2} f(t)g(t)\, dt.$$

Is this an inner product on V?

17. Let T be a linear operator on an inner product space V, and suppose that $\|\mathsf{T}(x)\| = \|x\|$ for all x. Prove that T is one-to-one.

18. Let V be a vector space over F, where $F = R$ or $F = C$, and let W be an inner product space over F with inner product $\langle \cdot, \cdot \rangle$. If $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ is linear, prove that $\langle x, y \rangle' = \langle \mathsf{T}(x), \mathsf{T}(y) \rangle$ defines an inner product on V if and only if T is one-to-one.

19. Let V be an inner product space. Prove that

(a) $\|x \pm y\|^2 = \|x\|^2 \pm 2\Re \langle x, y \rangle + \|y\|^2$ for all $x, y \in \mathsf{V}$, where $\Re \langle x, y \rangle$ denotes the real part of the complex number $\langle x, y \rangle$.

(b) $|\, \|x\| - \|y\| \,| \leq \|x - y\|$ for all $x, y \in \mathsf{V}$.

20. Let V be an inner product space over F. Prove the *polar identities*: For all $x, y \in \mathsf{V}$,

(a) $\langle x, y \rangle = \frac{1}{4}\|x+y\|^2 - \frac{1}{4}\|x-y\|^2 \quad$ if $F = R$;

(b) $\langle x, y \rangle = \frac{1}{4}\sum_{k=1}^{4} i^k \|x + i^k y\|^2 \quad$ if $F = C$, where $i^2 = -1$.

21. Let A be an $n \times n$ matrix. Define

$$A_1 = \frac{1}{2}(A + A^*) \quad \text{and} \quad A_2 = \frac{1}{2i}(A - A^*).$$

(a) Prove that $A_1^* = A_1$, $A_2^* = A_2$, and $A = A_1 + iA_2$. Would it be reasonable to define A_1 and A_2 to be the real and imaginary parts, respectively, of the matrix A?

(b) Let A be an $n \times n$ matrix. Prove that the representation in (a) is unique. That is, prove that if $A = B_1 + iB_2$, where $B_1^* = B_1$ and $B_2^* = B_2$, then $B_1 = A_1$ and $B_2 = A_2$.

22. Let V be a real or complex vector space (possibly infinite-dimensional), and let β be a basis for V. For $x, y \in \mathsf{V}$ there exist $v_1, v_2, \ldots, v_n \in \beta$ such that

$$x = \sum_{i=1}^{n} a_i v_i \quad \text{and} \quad y = \sum_{i=1}^{n} b_i v_i.$$