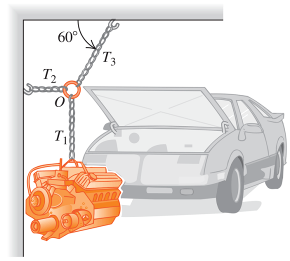
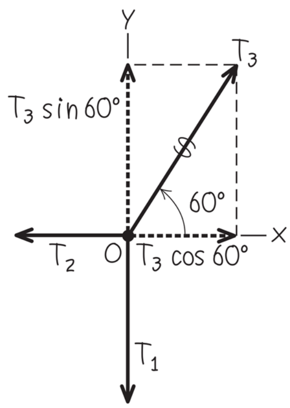
Figure 5.3

(a) Engine, chains, and ring



(b) Free-body diagram for engine

(c) Free-body diagram for ring O



Our sketches for this problem. Note that with our choice of axes, all but one of the forces lie along either the x-axis or the y-axis.

IDENTIFY and SET UP The target variables are the tension magnitudes T_1, T_2, and T_3 in the three chains (Fig. 5.3a). All the objects are in equilibrium, so we'll use Newton's first law. We need three independent equations, one for each target variable. However, applying Newton's first law in component form to just one object gives only *two* equations [the x- and y-equations in Eqs. (5.1)]. So we'll have to consider more than one object in equilibrium. We'll look at the engine (which is acted on by T_1) and the ring (which is attached to all three chains and so is acted on by all three tensions).

Figures 5.3b and 5.3c show our free-body diagrams and choice of coordinate axes. Two forces act on the engine: its weight w and the upward force T_1 exerted by the vertical chain. Three forces act on the ring: the tensions from the vertical chain (T_1), the horizontal chain (T_2), and the slanted chain (T_3). Because the vertical chain has negligible weight, it exerts forces of the same magnitude T_1 at both of its ends (see Example 5.1). (If the weight of this chain were not negligible, these two forces would have different magnitudes; see

Example 5.2.) The weight of the ring is also negligible, so it isn't included in Fig. 5.3c.

EXECUTE The forces acting on the engine are along the y-axis only, so Newton's first law [Eqs. (5.1)] says

$$\text{Engine}: \qquad \sum F_y = T_1 + (-w) = 0 \qquad \text{and} \qquad T_1 = w$$

The horizontal and slanted chains don't exert forces on the engine itself because they are not attached to it. These forces do appear when we apply Newton's first law to the ring, however. In the free-body diagram for the ring (Fig. 5.3c), remember that T_1, T_2, and T_3 are the *magnitudes* of the forces. We resolve the force with magnitude T_3 into its x- and y-components. Applying Newton's first law in component form to the ring gives us the two equations

$$\begin{aligned} \text{Ring}: \qquad & \sum F_x = T_3 \cos 60^\circ + (-T_2) = 0 \\ \text{Ring}: \qquad & \sum F_y = T_3 \sin 60^\circ + (-T_1) = 0 \end{aligned}$$

Because $T_1 = w$ (from the engine equation), we can rewrite the second ring equation as

$$T_3 = \frac{T_1}{\sin 60^\circ} = \frac{w}{\sin 60^\circ} = 1.2w$$

We can now use this result in the first ring equation:

$$T_2 = T_3 \cos 60^\circ = w\frac{\cos 60^\circ}{\sin 60^\circ} = 0.58w$$

EVALUATE The chain attached to the ceiling exerts a force on the ring with a *vertical* component equal to T_1, which in turn is equal to w. But this force also has a horizontal component, so its magnitude T_3 is somewhat greater than w. This chain is under the greatest tension and is the one most susceptible to breaking.

To get enough equations to solve this problem, we had to consider not only the forces on the engine but also the forces acting on a second object (the ring connecting the chains). Situations like this are fairly common in equilibrium problems, so keep this technique in mind.

KEY CONCEPT

In two-dimensional problems that involve forces, always write *two* force equations for each object: one for the x-components of the forces and one for the y-components of the forces.

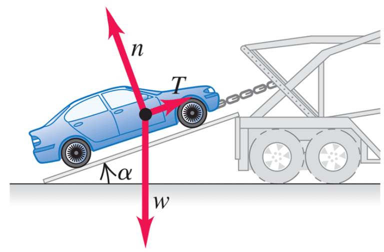
Video Tutor Solution: **Example 5.3**

Example 5.4 An inclined plane

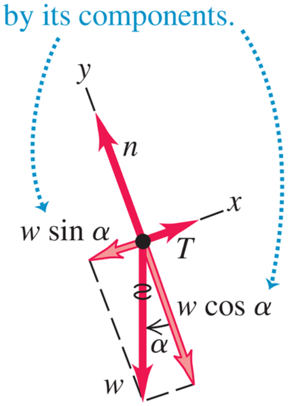
A car of weight w rests on a slanted ramp attached to a trailer (Fig. 5.4a). Only a cable running from the trailer to the car prevents the car from rolling off the ramp. (The car's brakes are off and its transmission is in neutral.) Find the tension in the cable and the force that the ramp exerts on the car's tires.

Figure 5.4

(a) Car on ramp



(b) Free-body diagram for car



A cable holds a car at rest on a ramp.

IDENTIFY The car is in equilibrium, so we use Newton's first law. The ramp exerts a separate force on each of the car's tires, but for simplicity we lump these forces into a single force. For a further simplification, we'll neglect any friction force the ramp exerts on the tires (see Fig. 4.2b). Hence the ramp exerts only a force on the car that is *perpendicular* to the ramp. As in Section 4.1, we call this force the *normal* force (see Fig. 4.2a). The two target variables are the magnitude T of the tension in the cable and the magnitude n of the normal force.

SET UP Figure 5.4 shows the three forces acting on the car: its weight (magnitude w), the tension in the cable (magnitude T), and the normal force (magnitude n). Note that the angle α between the ramp and the horizontal is equal to the angle α between the weight vector $\vec{w}$ and the downward normal to the plane of the ramp. Note also that we choose the x- and y-axes to be parallel and perpendicular to the ramp so that we need to resolve only one force (the weight) into x- and y-components. If we had chosen axes that were horizontal and vertical, we'd have to resolve both the normal force and the tension into components.

EXECUTE To write down the x- and y-components of Newton's first law, we must first find the components of the weight. One complication is that the angle α in Fig. 5.4b is *not* measured from the $+x$-axis toward the $+y$-axis. Hence we *cannot* use Eqs. (1.5) directly to find the components. (You may want to review Section 1.8 to make sure that you understand this important point.)

One way to find the components of $\vec{w}$ is to consider the right triangles in Fig. 5.4b. The sine of α is the magnitude of the x-component of $\vec{w}$ (that is, the side of the triangle opposite α) divided by the magnitude w (the hypotenuse of the triangle). Similarly, the cosine of α is the magnitude of the y-component (the side of the triangle adjacent to α) divided by w. Both components are negative, so $w_x = -w \sin \alpha$ and $w_y = -w \cos \alpha$.

Another approach is to recognize that one component of $\vec{w}$ must involve $\sin \alpha$ while the other component involves $\cos \alpha$. To decide which is which, draw the free-body diagram so that the angle α is noticeably smaller or larger than 45°. (You'll have to fight the natural tendency to draw such angles as being close to 45°.) We've drawn Fig. 5.4b so that α is smaller than 45°, so sin α is less than cos α. The figure shows that the x-component of $\vec{w}$ is smaller than the y-component, so the x-component must involve sin α and the y-component must involve cos α. We again find $w_x = -w \sin \alpha$ and $w_y = -w \cos \alpha$.

In Fig. 5.4b we draw a wiggly line through the original vector representing the weight to remind us not to count it twice. Newton's first law gives us

$$\sum F_x = T + (-w \sin \alpha) = 0$$
$$\sum F_y = n + (-w \cos \alpha) = 0$$

(Remember that T, w, and n are all *magnitudes* of vectors and are therefore all positive.) Solving these equations for T and n, we find

$$T = w \sin \alpha$$
$$n = w \cos \alpha$$

EVALUATE Our answers for T and n depend on the value of α. To check this dependence, let's look at some special cases. If the ramp is horizontal ($\alpha = 0$), we get $T = 0$ and $n = w$: No cable tension T is needed to hold the car, and the normal force n is equal in magnitude to the weight. If the ramp is vertical ($\alpha = 90^\circ$), we get $T = w$ and $n = 0$: The cable tension T supports all of the car's weight, and there's nothing pushing the car against the ramp.

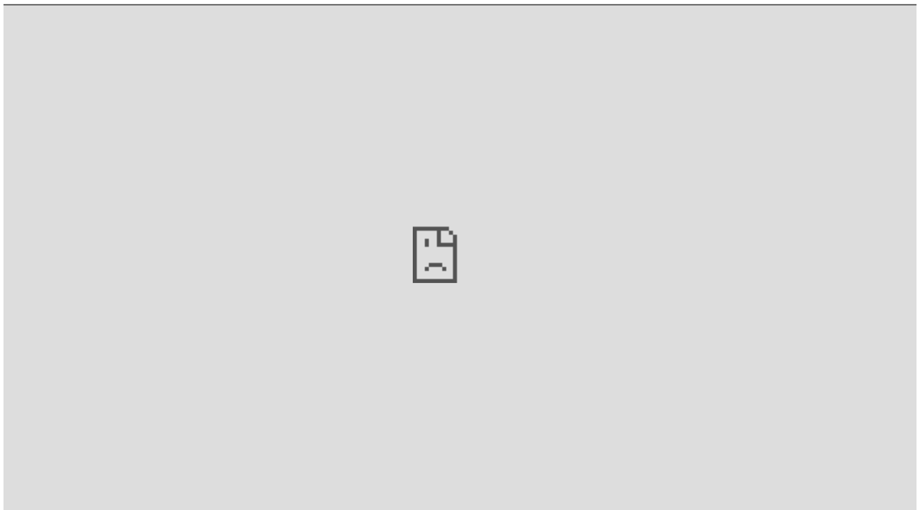
CAUTION **Normal force and weight may not be equal** It's a common error to assume that the normal-force magnitude n equals the weight w. Our result shows that this is *not* always the case. Always treat n as a variable and solve for its value, as we've done here.

? How would the answers for T and n be affected if the car were being pulled up the ramp at a constant speed? This, too, is an equilibrium situation, since the car's velocity is constant. So the calculation is the same, and T and n have the same values as when the car is at rest. (It's true that T must be greater than $w \sin \alpha$ to *start* the car moving up the ramp, but that's not what we asked.)

KEY CONCEPT

In two-dimensional equilibrium problems, choose the coordinate axes so that as many forces as possible lie along either the x-axis or the y-axis.

Video Tutor Solution: **Example 5.4**

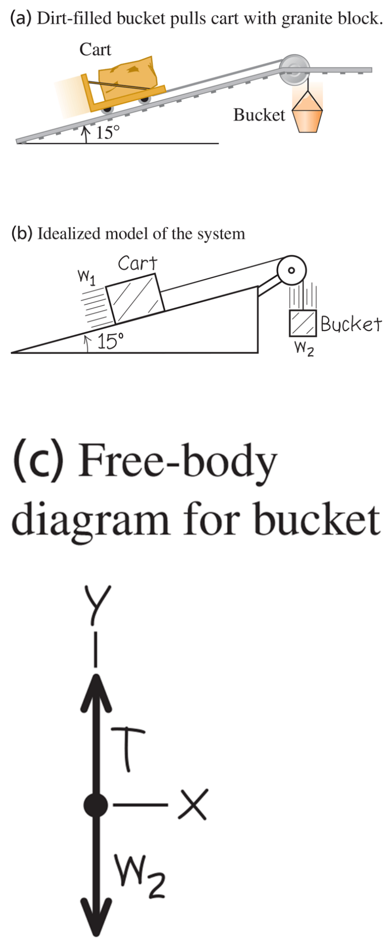


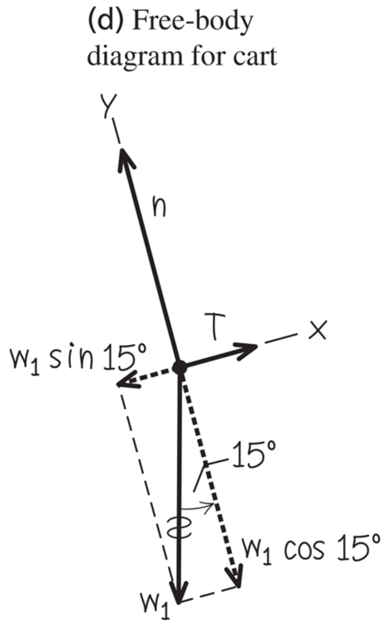
Example 5.5 Equilibrium of objects connected by cable and pulley

WITH VARIATION PROBLEMS

Your firm needs to haul granite blocks up a 15° slope out of a quarry and to lower dirt into the quarry to fill the holes. You design a system in which a granite block on a cart with steel wheels (weight w_1, including both block and cart) is pulled uphill on steel rails by a dirt-filled bucket (weight w_2, including both dirt and bucket) that descends vertically into the quarry (Fig. 5.5a). How must the weights w_1 and w_2 be related in order for the system to move with constant speed? Ignore friction in the pulley and wheels, and ignore the weight of the cable.

Figure 5.5





Our sketches for this problem.

IDENTIFY and SET UP The cart and bucket each move with a constant velocity (in a straight line at constant speed). Hence each object is in equilibrium, and we can apply Newton's first law to each. Our target is an expression relating the weights w_1 and w_2.

Figure 5.5b shows our idealized model for the system, and Figs. 5.5c and 5.5d show our free-body diagrams. The two forces on the bucket are its weight w_2 and an upward tension exerted by the cable. As for the car on the ramp in Example 5.4, three forces act on the cart: its weight w_1, a normal force of magnitude n exerted by the rails, and a tension force from the cable. Since we're assuming that the cable has negligible weight, the tension forces that the cable exerts on the cart and on the bucket have the same magnitude T. (We're ignoring friction, so we assume that the rails exert no force on the cart parallel to the incline.) Note that we are free to orient the axes differently for each object; the choices shown are the most convenient ones. We find the components of the weight force in the same way that we did in Example 5.4. (Compare Fig. 5.5d with Fig. 5.4b.)

EXECUTE Applying $\sum F_y = 0$ to the bucket in Fig. 5.5c, we find

$$\sum F_y = T + (-w_2) = 0 \qquad \text{so} \qquad T = w_2$$

Applying $\sum F_x = 0$ to the cart (and block) in Fig. 5.5d, we get

$$\sum F_x = T + (-w_1 \sin 15^\circ) = 0 \qquad \text{so} \qquad T = w_1 \sin 15^\circ$$

Equating the two expressions for T, we find

$$w_2 = w_1 \sin 15^\circ = 0.26 w_1$$

EVALUATE Our analysis doesn't depend at all on the direction in which the cart and bucket move. Hence the system can move with constant speed in *either* direction if the weight of the dirt and bucket is 26% of the weight of the granite block and cart. What would happen if w_2 were greater than $0.26w_1$? If it were less than $0.26w_1$?

Notice that we didn't need the equation $\sum F_y = 0$ for the cart and block. Can you use this to show that $n = w_1 \cos 15^\circ$?

KEY CONCEPT

If there's more than one object in a problem involving forces, you are free to choose different x- and y-axes for each object to make it easier to find force components.

Video Tutor Solution: **Example 5.5**

Test Your Understanding of Section 5.1

A traffic light of weight w hangs from two lightweight cables, one on each side of the light. Each cable hangs at a 45° angle from the horizontal. What is the tension in each cable? (i) $w/2$; (ii) $w/\sqrt{2}$; (iii) w; (iv) $w\sqrt{2}$; (v) $2w$.

5.2 Using Newton’s Second Law: Dynamics of Particles

We are now ready to discuss *dynamics* problems. In these problems, we apply Newton’s second law to objects on which the net force is *not* zero. These objects are *not* in equilibrium and hence are accelerating:

(5.2)

Newton’s second law: If the *net* force on an object is not zero ... $\sum \vec{F} = m\vec{a}$ ... the object has *acceleration* in the same direction as the net force.

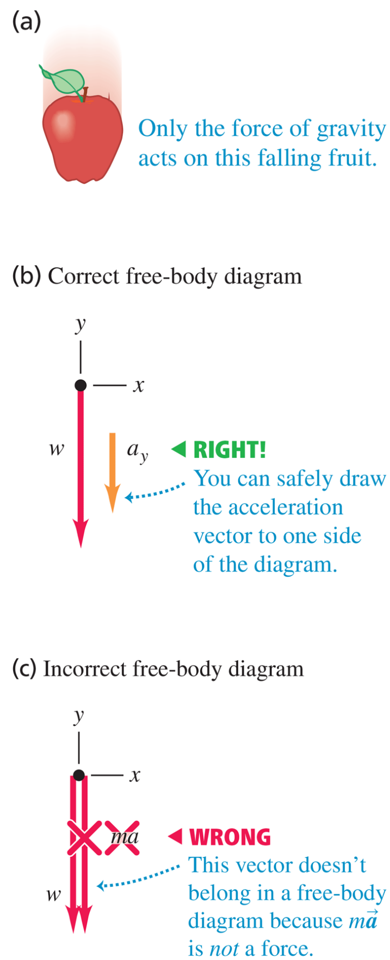
Mass of object

$$\sum \vec{F} = m\vec{a} \qquad (5.2)$$

Each component of the net force on the object ... $\sum F_x = ma_x \quad \sum F_y = ma_y$ ... equals the object’s mass times the corresponding acceleration component.

CAUTION $m\vec{a}$ **doesn’t belong in free-body diagrams** Remember that the quantity $m\vec{a}$ is the *result* of forces acting on an object, *not* a force itself. When you draw the free-body diagram for an accelerating object (like the fruit in Fig. 5.6a), *never* include the “$m\vec{a}$ force” because *there is no such force* (Fig. 5.6c). Review Section 4.3 if you’re not clear on this point. Sometimes we draw the acceleration vector $\vec{a}$ *alongside* a free-body diagram, as in Fig. 5.6b. But we *never* draw the acceleration vector with its tail touching the object (a position reserved exclusively for forces that act on the object).

Figure 5.6



Correct and incorrect free-body diagrams for a falling object.

The following problem-solving strategy is very similar to Problem-Solving Strategy 5.1 for equilibrium problems in Section 5.1. Study it carefully, watch how we apply it in our examples, and use it when you tackle the end-of-chapter problems. You can use this strategy to solve *any* dynamics problem.

Problem-Solving Strategy 5.2 Newton's Second Law: Dynamics of Particles

IDENTIFY *the relevant concepts*: You have to use Newton's second law, Eqs. (5.2), for *any* problem that involves forces acting on an accelerating object.

Identify the target variable—usually an acceleration or a force. If the target variable is something else, you'll need to select another concept to use. For example, suppose the target variable is how fast a sled is moving when it reaches the bottom of a hill. Newton's second law will let you find the sled's acceleration; you'll then use the constant-acceleration relationships from Section 2.4 to find velocity from acceleration.

SET UP *the problem* by using the following steps:

1. Draw a simple sketch of the situation that shows each moving object. For each object, draw a free-body diagram that shows all the forces acting *on* the object. [The sums in Eqs. (5.2) include the forces that act on the object, *not* the forces that it exerts on anything else.] Make sure you can answer the question "What other object is applying this force?" for each force in your diagram. Never include the quantity $m\vec{a}$ in your free-body diagram; it's not a force!
2. Label each force with an algebraic symbol for the force's *magnitude*. Usually, one of the forces will be the object's weight; it's usually best to label this as $w = mg$.
3. Choose your x- and y-coordinate axes for each object, and show them in its free-body diagram. Indicate the positive direction for each axis. If you know the direction of the acceleration, it usually simplifies things to take one positive axis along that direction. If your problem involves two or more objects that accelerate in different directions, you can use a different set of axes for each object.

4. In addition to Newton's second law, $\sum \vec{F} = m\vec{a}$, identify any other equations you might need. For example, you might need one or more of the equations for motion with constant acceleration. If more than one object is involved, there may be relationships among their motions; for example, they may be connected by a rope. Express any such relationships as equations relating the accelerations of the various objects.

EXECUTE *the solution* as follows:

1. For each object, determine the components of the forces along each of the object's coordinate axes. When you represent a force in terms of its components, draw a wiggly line through the original force vector to remind you not to include it twice.
2. List all of the known and unknown quantities. In your list, identify the target variable or variables.
3. For each object, write a separate equation for each component of Newton's second law, as in Eqs. (5.2). Write any additional equations that you identified in step 4 of "Set Up." (You need as many equations as there are target variables.)
4. Do the easy part—the math! Solve the equations to find the target variable(s).

EVALUATE *your answer*: Does your answer have the correct units? (When appropriate, use the conversion $1\ \text{N} = 1\ \text{kg} \cdot \text{m/s}^2$.) Does it have the correct algebraic sign? When possible, consider particular values or extreme cases of quantities and compare the results with your intuitive expectations. Ask, "Does this result make sense?"

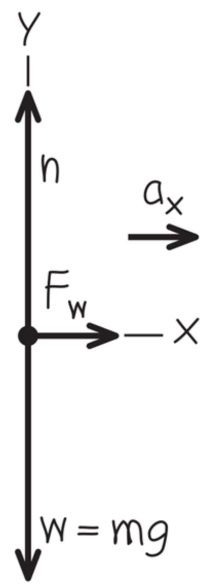
Example 5.6 Straight-line motion with a constant force

An iceboat is at rest on a frictionless horizontal surface (Fig. 5.7a). Due to the blowing wind, 4.0 s after the iceboat is released, it is moving to the right at 6.0 m/s (about 22 km/h, or 13 mi/h). What constant horizontal force F_{W} does the wind exert on the iceboat? The combined mass of iceboat and rider is 200 kg.

Figure 5.7

(a) Iceboat and rider on frictionless ice

(b) Free-body diagram for iceboat and rider



Our sketches for this problem.

IDENTIFY and SET UP Our target variable is one of the forces (F_W) acting on the accelerating iceboat, so we need to use Newton's second law. The forces acting on the iceboat and rider (considered as a unit) are the weight w, the normal force n exerted by the surface, and the horizontal force F_W. Figure 5.7b shows the free-body diagram. The net force and hence the acceleration are to the right, so we chose the positive x-axis in this direction. The acceleration isn't given; we'll need to find it. Since the wind is assumed to exert a constant force, the resulting acceleration is constant and we can use one of the constant-acceleration formulas from Section 2.4.

The iceboat starts at rest (its initial x-velocity is $v_{0x} = 0$) and it attains an x-velocity $v_x = 6.0 \text{ m/s}$ after an elapsed time $t = 4.0 \text{ s}$. To relate the x-acceleration a_x to these quantities we use Eq. (2.8), $v_x = v_{0x} + a_x t$. There is no vertical acceleration, so we expect that the normal force on the iceboat is equal in magnitude to the iceboat's weight.

EXECUTE The *known* quantities are the mass $m = 200$ kg, the initial and final x-velocities $v_{0x} = 0$ and $v_x = 6.0$ m/s, and the elapsed time $t = 4.0$ s. There are three *unknown* quantities: the acceleration a_x, the normal force n, and the horizontal force F_W. Hence we need three equations.

The first two equations are the x- and y-equations for Newton's second law, Eqs. (5.2). The force F_W is in the positive x-direction, while the forces n and $w = mg$ are in the positive and negative y-directions, respectively. Hence we have

$$\begin{aligned}\sum F_x &= F_W = ma_x \\ \sum F_y &= n + (-mg) = 0 \quad \text{so} \quad n = mg\end{aligned}$$

The third equation is Eq. (2.8) for constant acceleration:

$$v_x = v_{0x} + a_x t$$

To find F_W, we first solve this third equation for a_x and then substitute the result into the $\sum F_x$ equation:

$$\begin{aligned}a_x &= \frac{v_x - v_{0x}}{t} = \frac{6.0\ \text{m/s} - 0}{4.0\ \text{s}} = 1.5\ \text{m/s}^2 \\ F_w &= ma_x = (200\,\text{kg})\left(1.5\,\text{m/s}^2\right) = 300\,\text{kg}\cdot\text{m/s}^2\end{aligned}$$

Since $1\ \text{kg}\cdot\text{m/s}^2 = 1$ N, the final answer is

$$F_W = 300\ \text{N (about 67 lb)}$$

EVALUATE Our answers for F_W and n have the correct units for a force, and (as expected) the magnitude n of the normal force is equal to mg. Does it seem reasonable that the force F_W is substantially *less* than the weight of the boat, mg?

KEY CONCEPT

For problems in which an object is accelerating, it's usually best to choose one positive axis to be in the direction of the acceleration.

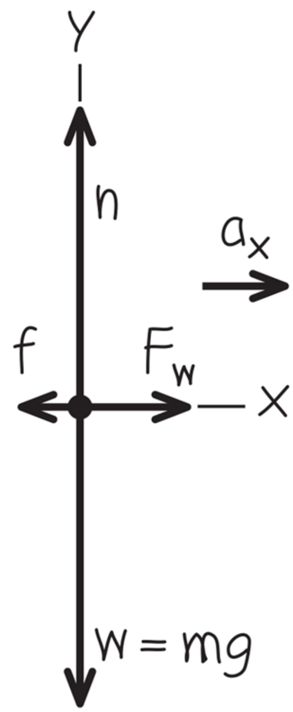
Video Tutor Solution: **Example 5.6**

Example 5.7 Straight-line motion with friction

Suppose a constant horizontal friction force with magnitude 100 N opposes the motion of the iceboat in Example 5.6. In this case, what constant force F_W must the wind exert on the iceboat to cause the same constant x-acceleration $a_x = 1.5\ \text{m/s}^2$?

IDENTIFY and SET UP Again the target variable is F_W. We are given the x-acceleration, so to find F_W all we need is Newton's second law. Figure 5.8 shows our new free-body diagram. The only difference from Fig. 5.7b is the addition of the friction force $\vec{f}$, which points in the negative x-direction (opposite the motion). Because the wind must now overcome the friction force to yield the same acceleration as in Example 5.6, we expect our answer for F_W to be greater than the 300 N we found there.

Figure 5.8



Our free-body diagram for the iceboat and rider with friction force $\vec{f}$ opposing the motion.

EXECUTE Two forces now have x-components: the force of the wind (x-component $+F_W$) and the friction force (x-component $-f$). The x-component of Newton's second law gives

$$\sum F_x = F_W + (-f) = ma_x$$
$$F_w = ma_x + f = (200\,\mathrm{kg})\left(1.5\,\mathrm{m/s^2}\right) + (100\,\mathrm{N}) = 400\,\mathrm{N}$$

EVALUATE The required value of F_W is 100 N greater than in Example 5.6 because the wind must now push against an additional 100 N friction force.

KEY CONCEPT

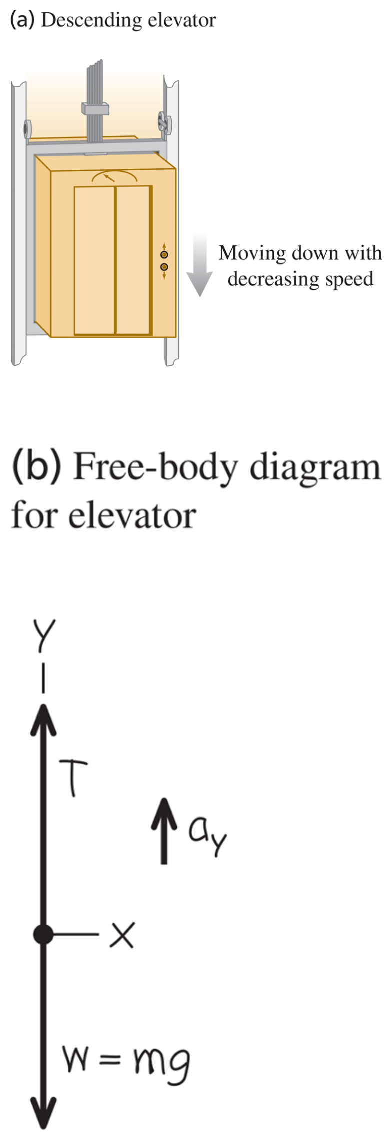
When friction is present, the friction force on an object is always in the direction that opposes sliding.

Video Tutor Solution: Example 5.7

Example 5.8 Tension in an elevator cable

An elevator and its load have a combined mass of 800 kg (Fig. 5.9a). The elevator is initially moving downward at 10.0 m/s; it slows to a stop with constant acceleration in a distance of 25.0 m. What is the tension T in the supporting cable while the elevator is being brought to rest?

Figure 5.9



Our sketches for this problem.

IDENTIFY and SET UP The target variable is the tension T, which we'll find by using Newton's second law. As in Example 5.6, we'll use a constant-acceleration formula to determine the acceleration. Our free-body diagram (Fig. 5.9b) shows two forces acting on the elevator: its weight w and the tension force T of the cable. The elevator is moving downward with decreasing speed, so its acceleration is upward; we chose the positive y-axis to be upward.

The elevator is moving in the negative y-direction, so both its initial y-velocity v_{0y} and its y-displacement $y - y_0$ are negative: $v_{0y} = -10.0 \text{ m/s}$ and $y - y_0 = -25.0 \text{ m}$. The final y-velocity is $v_y = 0$. To find the y-acceleration a_y from this information, we'll use Eq. (2.13) in the form $v^2{}_y = v^2{}_{0y} + 2a_y\,(y - y_0)$. Once we have a_y, we'll substitute it into the y-component of Newton's second law from Eqs. (5.2) and solve for T. The net force must be upward to give an upward acceleration, so we expect T to be greater than the weight $w = mg = (800 \text{ kg})(9.80 \text{ m/s}^2) = 7840 \text{ N}$.

EXECUTE First let's write out Newton's second law. The tension force acts upward and the weight acts downward, so

$$\sum F_y = T + (-w) = ma_y$$

We solve for the target variable T:

$$T = w + ma_y = mg + ma_y = m\,(g + a_y)$$

To determine a_y, we rewrite the constant-acceleration equation $v_y{}^2 = v_{0y}{}^2 + 2a_y\,(y - y_0)$:

$$a_y = \frac{v_y{}^2 - v_{0y}{}^2}{2\,(y - y_0)} = \frac{(0)^2 - (-10.0 \text{ m/s})^2}{2(-25.0 \text{ m})} = +2.00 \text{ m/s}^2$$

The acceleration is upward (positive), just as it should be.

Now we can substitute the acceleration into the equation for the tension:

$$T = m\,(g + a_y) = (800 \text{ kg})(9.80 \text{ m/s}^2 + 2.00 \text{ m/s}^2) = 9440 \text{ N}$$

EVALUATE The tension is greater than the weight, as expected. Can you see that we would get the same answers for a_y and T if the elevator were moving *upward* and *gaining* speed at a rate of 2.00 m/s^2?

KEY CONCEPT

If an object suspended from a cable (or rope or string) is accelerating vertically, the tension in the cable is *not* equal to the weight of the object.

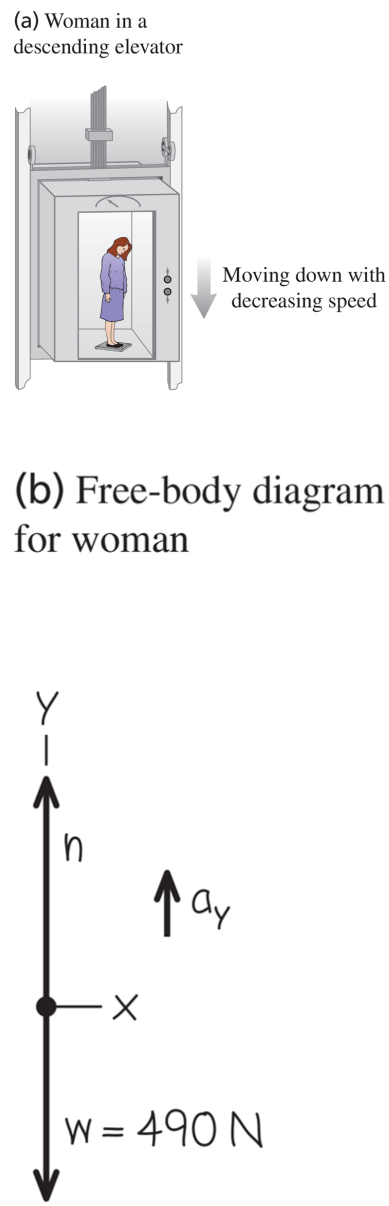
Video Tutor Solution: **Example 5.8**

Example 5.9 Apparent weight in an accelerating elevator

A 50.0 kg woman stands on a bathroom scale while riding in the elevator in Example 5.8. While the elevator is moving downward with decreasing speed, what is the reading on the scale?

IDENTIFY and SET UP The scale (Fig. 5.10a) reads the magnitude of the downward force exerted *by* the woman *on* the scale. By Newton's third law, this equals the magnitude of the upward normal force exerted *by* the scale *on* the woman. Hence our target variable is the magnitude n of the normal force. We'll find n by applying Newton's second law to the woman. We already know her acceleration; it's the same as the acceleration of the elevator, which we calculated in Example 5.8.

Figure 5.10



Our sketches for this problem.

Figure 5.10b shows our free-body diagram for the woman. The forces acting on her are the normal force n exerted by the scale and her weight $w = mg = (50.0 \text{ kg})(9.80 \text{ m/s}^2) = 490 \text{ N}$. (The tension force, which played a major role in Example 5.8, doesn't appear here because it doesn't act on the woman.) From Example 5.8, the y-acceleration of the elevator and of the woman is $a_y = +2.00 \text{ m/s}^2$. As in Example 5.8, the upward force on the object accelerating upward (in this case, the normal force on the woman) will have to be greater than the object's weight to produce the upward acceleration.

EXECUTE Newton's second law gives

$$\begin{aligned} \sum F_y &= n + (-mg) = ma_y \\ n &= mg + ma_y = m(g + a_y) \\ &= (50.0\,\text{kg})\left(9.80\,\text{m/s}^2 + 2.00\,\text{m/s}^2\right) = 590\,\text{N} \end{aligned}$$

EVALUATE Our answer for n means that while the elevator is stopping, the scale pushes up on the woman with a force of 590 N. By Newton's third law, she pushes down on the scale with the same force. So the scale reads 590 N, which is 100 N more than her actual weight. The scale reading is called the passenger's **apparent weight**. The woman *feels* the floor pushing up harder on her feet than when the elevator is stationary or moving with constant velocity.

What would the woman feel if the elevator were accelerating *downward*, so that $a_y = -2.00 \text{ m/s}^2$? This would be the case if the elevator were moving upward with decreasing speed or moving downward with increasing speed. To find the answer for this situation, we just insert the new value of a_y in our equation for n:

$$n = m(g + a_y) = (50.0 \text{ kg})[9.80 \text{ m/s}^2 + (-2.00 \text{ m/s}^2)] = 390 \text{ N}$$

Now the woman would feel as though she weighs only 390 N, or 100 N *less* than her actual weight w.

You can feel these effects yourself; try taking a few steps in an elevator that is coming to a stop after descending (when your apparent weight is greater than w) or coming to a stop after ascending (when your apparent weight is less than w).

KEY CONCEPT

When you are riding in an accelerating vehicle such as an elevator, your apparent weight (the normal force that the vehicle exerts on you) is in general *not* equal to your actual weight.

Video Tutor Solution: Example 5.9

Apparent Weight and Apparent Weightlessness

Let's generalize the result of Example 5.9. When a passenger with mass m rides in an elevator with y-acceleration a_y, a scale shows the passenger's apparent weight to be

$$n = m(g + a_y)$$

When the elevator is accelerating upward, a_y is positive and n is greater than the passenger's weight $w = mg$. When the elevator is accelerating downward, a_y is negative and n is less than the weight. If the passenger doesn't know the elevator is accelerating, she may feel as though her weight is changing; indeed, this is just what the scale shows.

The extreme case occurs when the elevator has a downward acceleration $a_y = -g$—that is, when it is in free fall. In that case $n = 0$ and the passenger *seems* to be weightless. Similarly, an astronaut orbiting the earth with a spacecraft experiences *apparent weightlessness* (Fig. 5.11). In each case, the person is not truly weightless because a gravitational force still acts. But the person's *sensations* in this free-fall condition are exactly the same as though the person were in outer space with no gravitational force at all. In both cases the person and the vehicle (elevator or spacecraft) fall together with the same acceleration g, so nothing pushes the person against the floor or walls of the vehicle.

Figure 5.11

Astronauts in orbit feel "weightless" because they have the same acceleration as their spacecraft. They are *not* outside the pull of the earth's gravity. (We'll discuss the motions of orbiting objects in detail in Chapter 12.)

Example 5.10 Acceleration down a hill

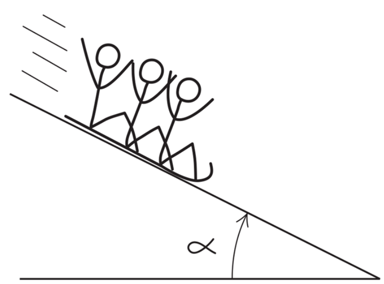
A toboggan loaded with students (total weight w) slides down a snow-covered hill that slopes at a constant angle α. The toboggan is well waxed, so there is virtually no friction. What is its acceleration?

IDENTIFY and SET UP Our target variable is the acceleration, which we'll find by using Newton's second law. There is no friction, so only two forces act on the toboggan: its weight w and the normal force n exerted by the hill.

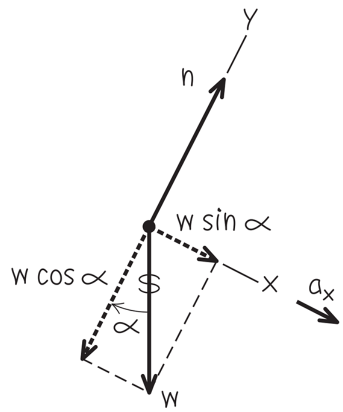
Figure 5.12 shows our sketch and free-body diagram. We take axes parallel and perpendicular to the surface of the hill, so that the acceleration (which is parallel to the hill) is along the positive x-direction.

Figure 5.12

(a) The situation



(b) Free-body diagram for toboggan



Our sketches for this problem.

EXECUTE The normal force has only a y-component, but the weight has both x- and y-components: $w_x = w \sin \alpha$ and $w_y = -w \cos \alpha$. (In Example 5.4 we had $w_x = -w \sin \alpha$. The difference is that the positive x-axis was uphill in Example 5.4 but is downhill in Fig. 5.12b.) The wiggly line in Fig. 5.12b reminds us that we have resolved the weight into its components. The acceleration is purely in the $+x$ − direction, so $a_y = 0$. Newton's second law in component form from Eqs. (5.2) then tells us that

$$\begin{aligned} \sum F_x &= w \sin \alpha = ma_x \\ \sum F_y &= n - w \cos \alpha = ma_y = 0 \end{aligned}$$

Since $w = mg$, the x-component equation gives $mg \sin \alpha = ma_x$, or

$$a_x = g \sin \alpha$$

Note that we didn't need the y-component equation to find the acceleration. That's part of the beauty of choosing the x-axis to lie along the acceleration direction! The y-equation tells us the magnitude of the normal force exerted by the hill on the toboggan:

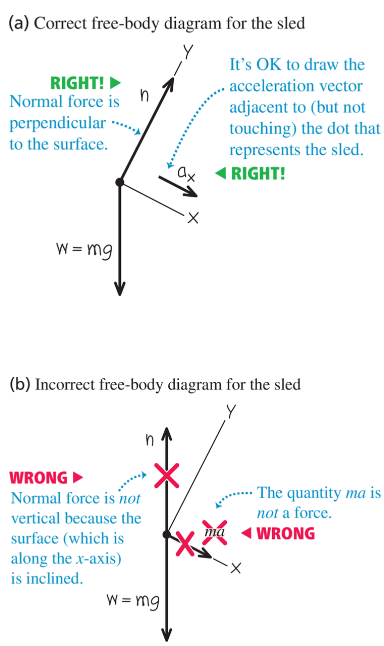
$$n = w \cos \alpha = mg \cos \alpha$$

EVALUATE Notice that the normal force n is not equal to the toboggan's weight (compare Example 5.4). Notice also that the mass m does not appear in our result for the acceleration. That's because the downhill force on the toboggan (a component of the weight) is proportional to m, so the mass cancels out when we use $\sum F_x = ma_x$ to calculate a_x. Hence *any* toboggan, regardless of its mass, slides down a frictionless hill with acceleration $g \sin \alpha$.

If the plane is horizontal, $\alpha = 0$ and $a_x = 0$ (the toboggan does not accelerate); if the plane is vertical, $\alpha = 90^\circ$ and $a_x = g$ (the toboggan is in free fall).

CAUTION Common free-body diagram errors Figure 5.13 shows both the correct way (Fig. 5.13a) and a common *incorrect* way (Fig. 5.13b) to draw the free-body diagram for the toboggan. The diagram in Fig. 5.13b is wrong for two reasons: The normal force must be drawn perpendicular to the surface (remember, "normal" means perpendicular), and there's no such thing as the "$m\vec{a}$ force."

Figure 5.13



Correct and incorrect free-body diagrams for a toboggan on a frictionless hill.

KEY CONCEPT

In problems with an object on an incline, it's usually best to take the positive x-direction for that object to be down the incline. The force of gravity will then have both an x-component and a y-component.

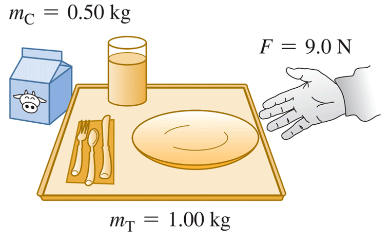
Video Tutor Solution: **Example 5.10**

Example 5.11 Two objects with the same acceleration

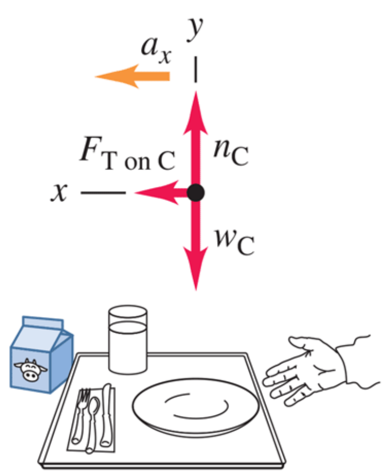
You push a 1.00 kg food tray through the cafeteria line with a constant 9.0 N force. The tray pushes a 0.50 kg milk carton (Fig. 5.14a). The tray and carton slide on a horizontal surface so greasy that friction can be ignored. Find the acceleration of the tray and carton and the horizontal force that the tray exerts on the carton.

Figure 5.14

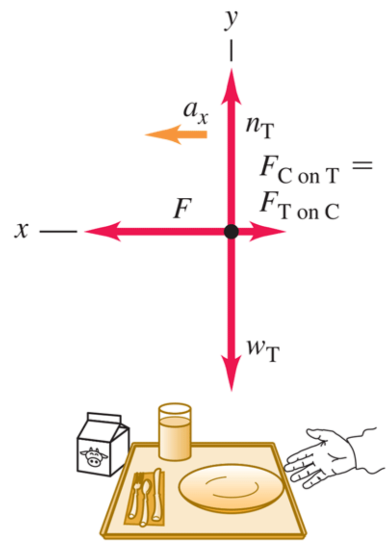
(a) A milk carton and a food tray



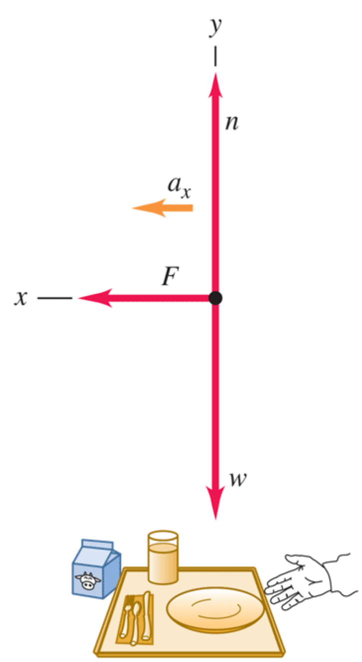
(b) Free-body diagram for milk carton



(c) Free-body diagram for food tray



(d) Free-body diagram for carton and tray as a composite object



Pushing a food tray and milk carton in the cafeteria line.

IDENTIFY and SET UP Our *two* target variables are the acceleration of the tray–carton system and the force of the tray on the carton. We'll use Newton's second law to get two equations, one for each target variable. We set up and solve the problem in two ways.

Method 1: We treat the carton (mass m_{C}) and tray (mass m_{T}) as separate objects, each with its own free-body diagram (Figs. 5.14b and 5.14c). The force F that you exert on the tray doesn't appear in the free-body diagram for the carton, which is accelerated by the force (of magnitude $F_{\mathrm{T\ on\ C}}$) exerted on it by the tray. By Newton's third law, the carton exerts a force of equal magnitude on the tray: $F_{\mathrm{C\ on\ T}} = F_{\mathrm{T\ on\ C}}$. We take the acceleration to be in the positive x-direction; both the tray and milk carton move with the same x-acceleration a_x.

Method 2: We treat the tray and milk carton as a composite object of mass $m = m_{\mathrm{T}} + m_{\mathrm{C}} = 1.50\ \mathrm{kg}$ (Fig. 5.14d). The only horizontal force acting on this object is the force F that you exert. The forces $F_{\mathrm{T\ on\ C}}$ and $F_{\mathrm{C\ on\ T}}$ don't come into play because they're *internal* to this composite object, and Newton's second law tells us that only *external* forces affect an object's acceleration (see Section 4.3). To find the magnitude $F_{\mathrm{T\ on\ C}}$ we'll again apply Newton's second law to the carton, as in Method 1.

EXECUTE *Method 1*: The x-component equations of Newton's second law are

$$\begin{aligned}\text{Tray}: \quad & \sum F_x = F - F_{\mathrm{C\ on\ T}} = F - F_{\mathrm{T\ on\ C}} = m_{\mathrm{T}} a_x \\ \text{Carton}: \quad & \sum F_x = F_{\mathrm{T\ on\ C}} = m_{\mathrm{C}} a_x\end{aligned}$$

These are two simultaneous equations for the two target variables a_x and $F_{\mathrm{T\ on\ C}}$. (Two equations are all we need, which means that the y-components don't play a role in this example.) An easy way to solve the two equations for a_x is to add them; this eliminates $F_{\text{T on C}}$, giving

$$F = m_\text{T}a_x + m_\text{C}a_x = (m_\text{T} + m_\text{C})a_x$$

We solve this equation for a_x :

$$a_x = \frac{F}{m_\text{T} + m_\text{C}} = \frac{9.0\ \text{N}}{1.00\ \text{kg} + 0.50\ \text{kg}} = 6.0\ \text{m/s}^2 = 0.61g$$

Substituting this value into the carton equation gives

$$F_{\text{T on C}} = m_\text{C}a_x = (0.50\ \text{kg})(6.0\ \text{m/s}^2) = 3.0\ \text{N}$$

Method 2: The x-component of Newton's second law for the composite object of mass m is

$$\sum F_x = F = ma_x$$

The acceleration of this composite object is

$$a_x = \frac{F}{m} = \frac{9.0\ \text{N}}{1.50\ \text{kg}} = 6.0\ \text{m/s}^2$$

Then, looking at the milk carton by itself, we see that to give it an acceleration of 6.0 m/s^2 requires that the tray exert a force

$$F_{\text{T on C}} = m_\text{C}a_x = (0.50\ \text{kg})(6.0\ \text{m/s}^2) = 3.0\ \text{N}$$

EVALUATE The answers are the same with both methods. To check the answers, note that there are different forces on the two sides of the tray: $F = 9.0\ \text{N}$ on the right and $F_{\text{C on T}} = 3.0\ \text{N}$ on the left. The net horizontal force on the tray is $F - F_{\text{C on T}} = 6.0\ \text{N}$, exactly enough to accelerate a 1.00 kg tray at 6.0 m/s^2.

Treating two objects as a single, composite object works *only* if the two objects have the same magnitude *and* direction of acceleration.

If the accelerations are different we must treat the two objects separately, as in the next example.

KEY CONCEPT

When two objects are touching each other, the free-body diagram for each object must include the force exerted on it by the other object.

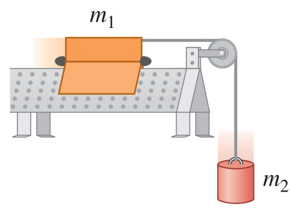
Video Tutor Solution: **Example 5.11**

Example 5.12 Two objects with the same magnitude of acceleration

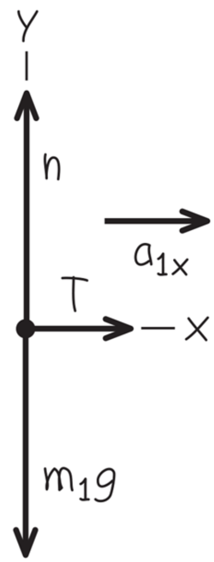
Figure 5.15a shows an air-track glider with mass m_1 moving on a level, frictionless air track in the physics lab. The glider is connected to a lab weight with mass m_2 by a light, flexible, nonstretching string that passes over a stationary, frictionless pulley. Find the acceleration of each object and the tension in the string.

Figure 5.15

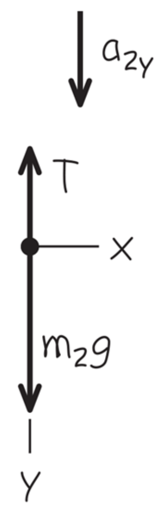
(a) Apparatus



(b) Free-body diagram for glider



(c) Free-body diagram for weight



Our sketches for this problem.

IDENTIFY and SET UP The glider and weight are accelerating, so again we must use Newton's second law. Our three target variables are the tension T in the string and the accelerations of the two objects.

The two objects move in different directions—one horizontal, one vertical—so we can't consider them to be a single unit as we did the objects in Example 5.11. Figures 5.15b and 5.15c show our free-body diagrams and coordinate systems. It's convenient to have both objects accelerate in the positive axis directions, so we chose the positive y-direction for the lab weight to be downward.

We consider the string to be massless and to slide over the pulley without friction, so the tension T in the string is the same throughout and it applies a force of the same magnitude T to each object. (You may want to review Conceptual Example 4.10, in which we discussed the tension force exerted by a massless rope.) The weights are $m_1 g$ and $m_2 g$.

While the *directions* of the two accelerations are different, their *magnitudes* are the same. (That's because the string doesn't stretch, so the two objects must move equal distances in equal times and their speeds at any instant must be equal. When the speeds change, they change at the same rate, so the accelerations of the two objects must have the same magnitude a.) We can express this relationship as $a_{1x} = a_{2y} = a$, which means that we have only *two* target variables: a and the tension T.

What results do we expect? If $m_1 = 0$ (or, approximately, for m_1 much less than m_2) the lab weight will fall freely with acceleration g, and the tension in the string will be zero. For $m_2 = 0$ (or, approximately, for m_2 much less than m_1) we expect zero acceleration and zero tension.

EXECUTE Newton's second law gives

$$\begin{aligned} \text{Glider:} \quad & \sum F_x = T = m_1 a_{1x} = m_1 a \\ \text{Glider:} \quad & \sum F_y = n + (-m_1 g) = m_1 a_{1y} = 0 \\ \text{Lab weight}: \quad & \sum F_y = m_2 g + (-T) = m_2 a_{2y} = m_2 a \end{aligned}$$

(There are no forces on the lab weight in the x-direction.) In these equations we've used $a_{1y} = 0$ (the glider doesn't accelerate vertically) and $a_{1x} = a_{2y} = a$.

The x-equation for the glider and the equation for the lab weight give us two simultaneous equations for T and a:

$$\begin{aligned} \text{Glider}: \qquad & T = m_1 a \\ \text{Lab weight}: \qquad & m_2 g - T = m_2 a \end{aligned}$$

We add the two equations to eliminate T, giving

$$m_2 g = m_1 a + m_2 a = (m_1 + m_2) a$$

and so the magnitude of each object's acceleration is

$$a = \frac{m_2}{m_1 + m_2} g$$

Substituting this back into the glider equation $T = m_1 a$, we get

$$T = \frac{m_1 m_2}{m_1 + m_2} g$$

EVALUATE The acceleration is in general less than g, as you might expect; the string tension keeps the lab weight from falling freely. The tension T is *not* equal to the weight $m_2 g$ of the lab weight, but is *less* by a factor of $m_1 / (m_1 + m_2)$. If T *were* equal to $m_2 g$, then the lab weight would be in equilibrium, and it isn't.

As predicted, the acceleration is equal to g for $m_1 = 0$ and equal to zero for $m_2 = 0$, and $T = 0$ for either $m_1 = 0$ or $m_2 = 0$.

CAUTION **Tension and weight may not be equal** It's a common mistake to assume that if an object is attached to a vertical string, the string tension must be equal to the object's weight. That was the case in Example 5.5, where the acceleration was zero, but it's not the case in this example! (Tension and weight were also not equal for the accelerating elevator in Example 5.8.) The only safe approach is *always* to treat the tension as a variable, as we did here.

KEY CONCEPT

If two objects are connected by a string under tension, both objects have the same magnitude of acceleration but may accelerate in different directions. Choose the positive x-direction for each object to be in the direction of its acceleration.

Video Tutor Solution: Example 5.12

Test Your Understanding of Section 5.2

Suppose you hold the glider in Example 5.12 so that it and the weight are initially at rest. You give the glider a push to the left in Fig. 5.15a and then release it. The string remains taut as the glider moves to the left, comes instantaneously to rest, then moves to the right. At the instant the glider has zero velocity, what is the tension in the string? (i) Greater than in Example 5.12; (ii) the same as in Example 5.12; (iii) less than in Example 5.12 but greater than zero; (iv) zero.

5.3 Friction Forces

We've seen several problems in which an object rests or slides on a surface that exerts forces on the object. Whenever two objects interact by direct contact (touching) of their surfaces, we describe the interaction in terms of *contact forces*. The normal force is one example of a contact force; in this section we'll look in detail at another contact force, the force of friction.

Friction is important in many aspects of everyday life. The oil in a car engine minimizes friction between moving parts, but without friction between the tires and the road we couldn't drive or turn the car. Air drag —the friction force exerted by the air on an object moving through it— decreases automotive fuel economy but makes parachutes work. Without friction, nails would pull out and most forms of animal locomotion would be impossible (Fig. 5.16).

Figure 5.16

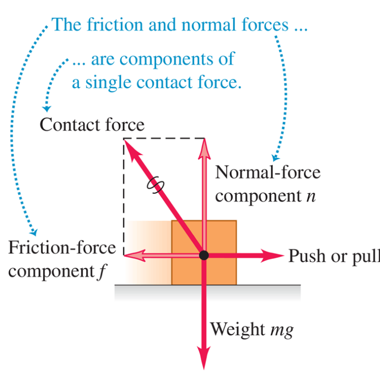
There is friction between the feet of this caterpillar (the larval stage of a butterfly of the family Papilionidae) and the surfaces over which it walks. Without friction, the caterpillar could not move forward or climb over obstacles.

Kinetic and Static Friction

When you try to slide a heavy box of books across the floor, the box doesn't move at all unless you push with a certain minimum force. Once the box starts moving, you can usually keep it moving with less force than you needed to get it started. If you take some of the books out, you need less force to get it started or keep it moving. What can we say in general about this behavior?

First, when an object rests or slides on a surface, we can think of the surface as exerting a single contact force on the object, with force components perpendicular and parallel to the surface (Fig. 5.17). The perpendicular component vector is the normal force, denoted by $\vec{n}$. The component vector parallel to the surface (and perpendicular to $\vec{n}$) is the **friction force**, denoted by $\vec{f}$. If the surface is frictionless, then $\vec{f}$ is zero but there is still a normal force. (Frictionless surfaces are an unattainable idealization, like a massless rope. But we can approximate a surface as frictionless if the effects of friction are negligibly small.) The direction of the friction force is always such as to oppose relative motion of the two surfaces.

Figure 5.17



When a block is pushed or pulled over a surface, the surface exerts a contact force on it.

The kind of friction that acts when an object slides over a surface is called a **kinetic friction force** $\vec{f}_k$. The adjective "kinetic" and the subscript "k" remind us that the two surfaces are moving relative to each other. The *magnitude* of the kinetic friction force usually increases when the normal force increases. This is why it takes more force to slide a full box of books across the floor than an empty one. Automotive brakes use the same principle: The harder the brake pads are squeezed against the rotating brake discs, the greater the braking effect. In many cases the magnitude of the kinetic friction force f_k is found experimentally to be approximately *proportional* to the magnitude n of the normal force:

(5.3)

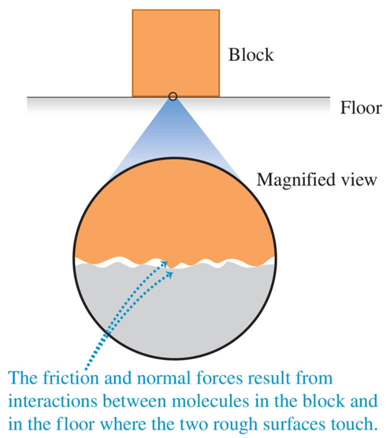
$$\underbrace{f_k}_{\text{Magnitude of kinetic friction force}} = \overbrace{\mu_k}^{\text{Coefficient of kinetic friction}} \underbrace{n}_{\text{Magnitude of normal force}} \qquad (5.3)$$

Here μ_k (pronounced "mu-sub-k") is a constant called the **coefficient of kinetic friction**. The more slippery the surface, the smaller this coefficient. Because it is a quotient of two force magnitudes, μ_k is a pure number without units.

CAUTION Friction and normal forces are always perpendicular Remember that Eq. (5.3) is *not* a vector equation because $\vec{f}_k$ and $\vec{n}$ are always perpendicular. Rather, it is a scalar relationship between the magnitudes of the two forces.

Equation (5.3) is only an approximate representation of a complex phenomenon. On a microscopic level, friction and normal forces result from the intermolecular forces (electrical in nature) between two rough surfaces at points where they come into contact (Fig. 5.18). As a box slides over the floor, bonds between the two surfaces form and break, and the total number of such bonds varies. Hence the kinetic friction force is not perfectly constant. Smoothing the surfaces can actually increase friction, since more molecules can interact and bond; bringing two smooth surfaces of the same metal together can cause a “cold weld.” Lubricating oils work because an oil film between two surfaces (such as the pistons and cylinder walls in a car engine) prevents them from coming into actual contact.

Figure 5.18



A microscopic view of the friction and normal forces.

Table 5.1 lists some representative values of μ_k. Although these values are given with two significant figures, they are only approximate, since friction forces can also depend on the speed of the object relative to the surface. For now we’ll ignore this effect and assume that μ_k and f_k are independent of speed, in order to concentrate on the simplest cases. Table 5.1 also lists coefficients of static friction; we’ll define these shortly.

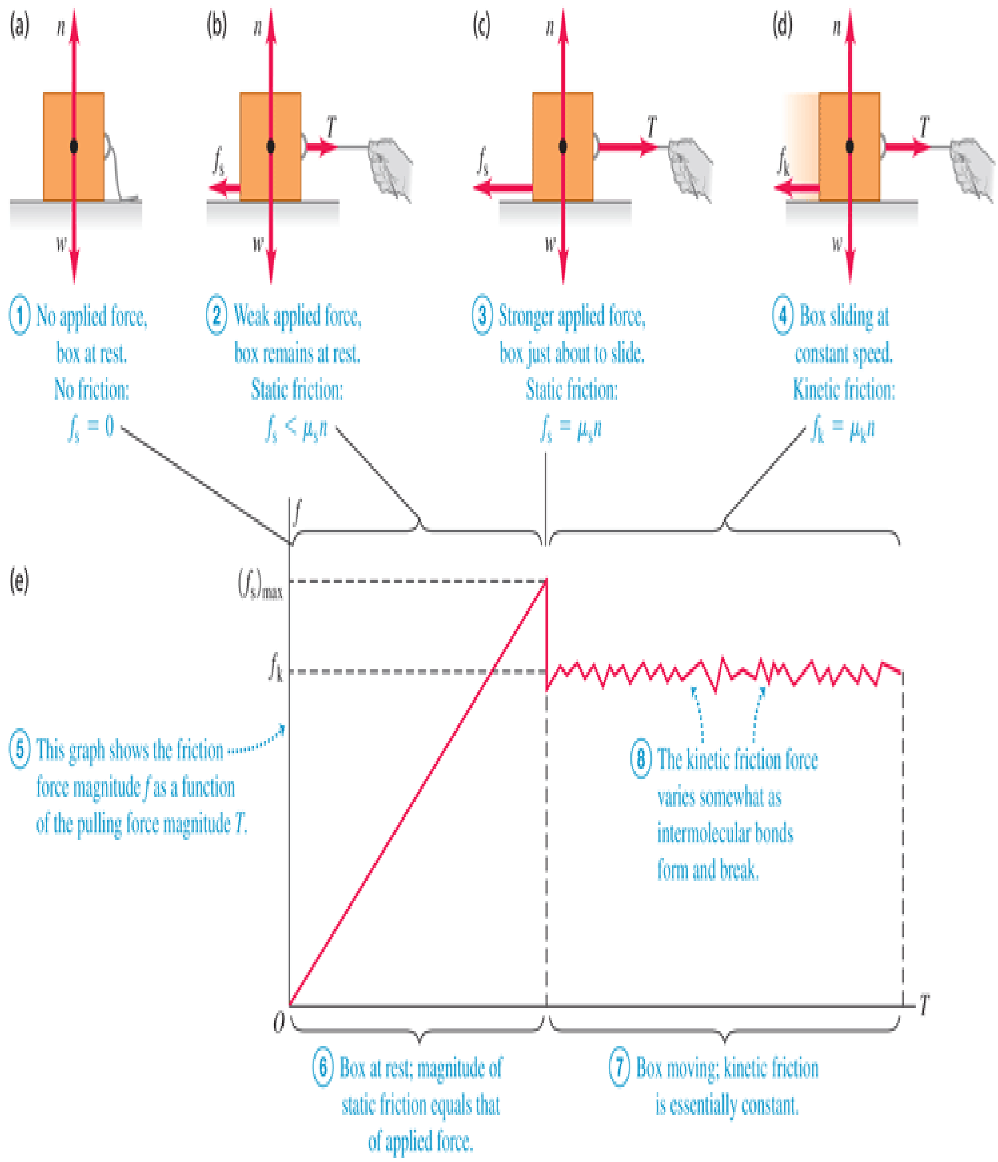
Table 5.1 **Approximate Coefficients of Friction**

Materials	Coefficient of Static Friction, μ_s	Coefficient of Kinetic Friction, μ_k
Steel on steel	0.74	0.57
Aluminum on steel	0.61	0.47
Copper on steel	0.53	0.36
Brass on steel	0.51	0.44
Zinc on cast iron	0.85	0.21
Copper on cast iron	1.05	0.29
Glass on glass	0.94	0.40
Copper on glass	0.68	0.53
Teflon on Teflon	0.04	0.04
Teflon on steel	0.04	0.04
Rubber on concrete (dry)	1.0	0.8
Rubber on concrete (wet)	0.30	0.25

Friction forces may also act when there is *no* relative motion. If you try to slide a box across the floor, the box may not move at all because the floor exerts an equal and opposite friction force on the box. This is called a **static friction force** $\vec{f}_s$.

In Fig. 5.19a, the box is at rest, in equilibrium, under the action of its weight $\vec{w}$ and the upward normal force $\vec{n}$. The normal force is equal in magnitude to the weight ($n = w$) and is exerted on the box by the floor. Now we tie a rope to the box (Fig. 5.19b) and gradually increase the tension T in the rope. At first the box remains at rest because the force of static friction f_s also increases and stays equal in magnitude to T.

Figure 5.19



When there is no relative motion, the magnitude of the static friction force f_s is less than or equal to $\mu_s n$. When there is relative motion, the magnitude of the kinetic friction force f_k equals $\mu_k n$.

At some point T becomes greater than the maximum static friction force f_s the surface can exert. Then the box "breaks loose" and starts to slide. Figure 5.19c shows the forces when T is at this critical value. For a given pair of surfaces the maximum value of f_s depends on the normal force. Experiment shows that in many cases this maximum value, called $(f_s)_{max}$, is approximately *proportional* to n; we call the proportionality factor μ_s

the **coefficient of static friction**. Table 5.1 lists some representative values of μ_s . In a particular situation, the actual force of static friction can have any magnitude between zero (when there is no other force parallel to the surface) and a maximum value given by $\mu_s n$:

(5.4)

Magnitude of static friction force

Coefficient of static friction

$$f_s \le (f_s)_{max} = \mu_s n \qquad (5.4)$$

Maximum static friction force

Magnitude of normal force

Application

Static Friction and Windshield Wipers

The squeak of windshield wipers on dry glass is a stick-slip phenomenon. The moving wiper blade sticks to the glass momentarily, then slides when the force applied to the blade by the wiper motor overcomes the maximum force of static friction. When the glass is wet from rain or windshield cleaning solution, friction is reduced and the wiper blade doesn’t stick.

Like Eq. (5.3), this is a relationship between magnitudes, *not* a vector relationship. The equality sign holds only when the applied force T has reached the critical value at which motion is about to start (Fig. 5.19c). When T is less than this value (Fig. 5.19b), the inequality sign holds. In that case we have to use the equilibrium conditions ($\sum \vec{F} = 0$) to find f_s. If there is no applied force ($T = 0$) as in Fig. 5.19a, then there is no static friction force either ($f_s = 0$).

As soon as the box starts to slide (Fig. 5.19d), the friction force usually *decreases* (Fig. 5.19e); it's easier to keep the box moving than to start it moving. Hence the coefficient of kinetic friction is usually *less* than the coefficient of static friction for any given pair of surfaces, as Table 5.1 shows.

In some situations the surfaces will alternately stick (static friction) and slip (kinetic friction). This is what causes the horrible sound made by chalk held at the wrong angle on a blackboard and the shriek of tires sliding on asphalt pavement. A more positive example is the motion of a violin bow against the string.

In the linear air tracks used in physics laboratories, gliders move with very little friction because they are supported on a layer of air. The friction force is velocity dependent, but at typical speeds the effective coefficient of friction is of the order of 0.001.

Example 5.13 Friction in horizontal motion

WITH VARIATION PROBLEMS

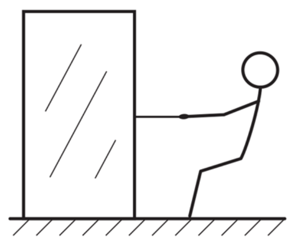
You want to move a 500 N crate across a level floor. To start the crate moving, you have to pull with a 230 N horizontal force. Once the crate starts to move, you can keep it moving at constant velocity with only 200 N. What are the coefficients of static and kinetic friction?

IDENTIFY and SET UP The crate is in equilibrium both when it is at rest and when it is moving at constant velocity, so we use Newton's first law, as expressed by Eqs. (5.1). We use Eqs. (5.3) and (5.4) to find the target variables μ_{s} and μ_{k}.

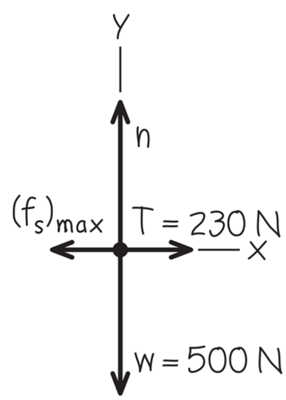
Figures 5.20a and 5.20b show our sketch and free-body diagram for the instant just before the crate starts to move, when the static friction force has its maximum possible value $(f_{\mathrm{s}})_{\mathrm{max}} = \mu_s n$. Once the crate is moving, the friction force changes to its kinetic form (Fig. 5.20c). In both situations, four forces act on the crate: the downward weight (magnitude $w = 500\ \mathrm{N}$), the upward normal force (magnitude n) exerted by the floor, a tension force (magnitude T) to the right exerted by the rope, and a friction force to the left exerted by the floor. Because the rope in Fig. 5.20a is in equilibrium, the tension is the same at both ends. Hence the tension force that the rope exerts on the crate has the same magnitude as the force you exert on the rope. Since it's easier to keep the crate moving than to start it moving, we expect that $\mu_{\mathrm{k}} < \mu_s$.

Figure 5.20

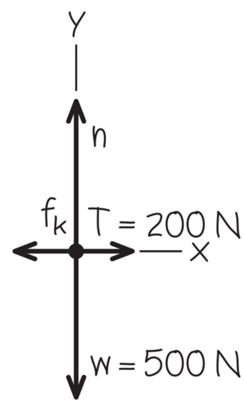
(a) Pulling a crate



(b) Free-body diagram for crate just before it starts to move



(c) Free-body diagram for crate moving at constant speed



Our sketches for this problem.

EXECUTE Just before the crate starts to move (Fig. 5.20b), we have from Eqs. (5.1)

$$\sum F_x = T + (-(f_s)_{\max}) = 0 \qquad \text{so} \qquad (f_s)_{\max} = T = 230\ \text{N}$$
$$\sum F_y = n + (-w) = 0 \qquad \text{so} \qquad n = w = 500\,\text{N}$$

Now we solve Eq. (5.4), $(f_s)_{\max} = \mu_s n$, for the value of μ_s :

$$\mu_s = \frac{(f_s)_{\max}}{n} = \frac{230\ \text{N}}{500\ \text{N}} = 0.46$$

After the crate starts to move (Fig. 5.20c) we have

$$\sum F_x = T + (-f_k) = 0 \quad \text{so} \quad f_k = T = 200\ \text{N}$$
$$\sum F_y = n + (-w) = 0 \quad \text{so} \quad n = w = 500\ \text{N}$$

Using $f_k = \mu_k n$ from Eq. (5.3), we find

$$\mu_k = \frac{f_k}{n} = \frac{200\ \text{N}}{500\ \text{N}} = 0.40$$

EVALUATE As expected, the coefficient of kinetic friction is less than the coefficient of static friction.

KEY CONCEPT

For any object, the *maximum* magnitude of the *static* friction force and the magnitude of the *kinetic* friction force are proportional to the magnitude of the normal force on that object.

Video Tutor Solution: **Example 5.13**

Example 5.14 Static friction can be less than the maximum

WITH VARIATION PROBLEMS

In Example 5.13, what is the friction force if the crate is at rest on the surface and a horizontal force of 50 N is applied to it?

IDENTIFY and SET UP The applied force is less than the maximum force of static friction, $(f_s)_{max} = 230\ \text{N}$. Hence the crate remains at rest and the net force acting on it is zero. The target variable is the magnitude f_s of the friction force. The free-body diagram is the same as in Fig. 5.20b, but with $(f_s)_{max}$ replaced by f_s and $T = 230$ N replaced by $T = 50$ N.

EXECUTE From the equilibrium conditions, Eqs. (5.1), we have

$$\sum F_x = T + (-f_s) = 0 \quad \text{so} \quad f_s = T = 50 \text{ N}$$

EVALUATE The friction force can prevent motion for any horizontal applied force up to $(f_s)_{max} = \mu_s n = 230$ N. Below that value, f_s has the same magnitude as the applied force.

KEY CONCEPT

The magnitude f_s of the static friction force on an object at rest does *not* have to equal the maximum magnitude $\mu_s n$. The actual value of f_s depends on the other forces acting on the object; you can find this value using Newton's first law.

Video Tutor Solution: **Example 5.14**

Example 5.15 Minimizing kinetic friction

WITH VARIATION PROBLEMS

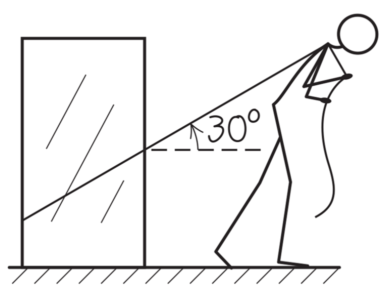
In Example 5.13, suppose you move the crate by pulling upward on the rope at an angle of 30° above the horizontal. How hard must you pull to keep it moving with constant velocity? Assume that $\mu_k = 0.40$.

IDENTIFY and SET UP The crate is in equilibrium because its velocity is constant, so we again apply Newton's first law. Since the crate is in motion, the floor exerts a *kinetic* friction force. The target variable is the magnitude T of the tension force.

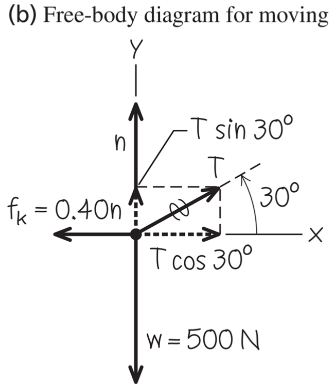
Figure 5.21 shows our sketch and free-body diagram. The kinetic friction force f_k is still equal to $\mu_k n$, but now the normal force n is *not* equal in magnitude to the crate's weight. The force exerted by the rope has a vertical component that tends to lift the crate off the floor; this *reduces* n and so reduces f_k.

Figure 5.21

(a) Pulling a crate at an angle



(b) Free-body diagram for moving crate



Our sketches for this problem.

EXECUTE From the equilibrium conditions and Eq. (5.3),

$f_{\mathrm{k}} = \mu_k n$, we have

$$\begin{aligned} \sum F_x &= T\cos 30^\circ + (-f_{\mathrm{k}}) = 0 \quad \text{so} \quad T\cos 30^\circ = \mu_{\mathrm{k}} n \\ \sum F_y &= T\sin 30^\circ + n + (-w) = 0 \quad \text{so} \quad n = w - T\sin 30^\circ \end{aligned}$$

These are two equations for the two unknown quantities T and n. One way to find T is to substitute the expression for n in the second equation into the first equation and then solve the resulting equation for T:

$$\begin{aligned} T\cos 30^\circ &= \mu_{\mathrm{k}}(w - T\sin 30^\circ) \\ T &= \frac{\mu_{\mathrm{k}} w}{\cos 30^\circ + \mu_{\mathrm{k}} \sin 30^\circ} = 188\,\mathrm{N} \end{aligned}$$

We can substitute this result into either of the original equations to obtain n. If we use the second equation, we get

$$n = w - T\sin 30^\circ = (500\ \mathrm{N}) - (188\ \mathrm{N})\sin 30^\circ = 406\ \mathrm{N}$$

EVALUATE As expected, the normal force is less than the 500 N weight of the box. It turns out that the tension required to keep the crate moving at constant speed is a little less than the 200 N force needed when you pulled horizontally in Example 5.13. Can you find an angle where the required pull is *minimum*?

KEY CONCEPT

In problems that involve kinetic friction, you'll always need at least three equations: two from Newton's first or second law in component form, and Eq. (5.3) for kinetic friction, $f_{\mathrm{k}} = \mu_k n$.

Video Tutor Solution: **Example 5.15**

Example 5.16 Toboggan ride with friction I

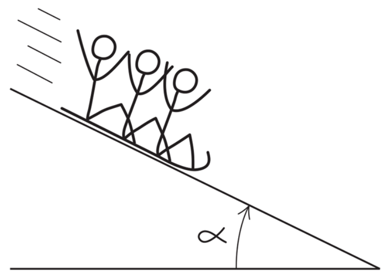
Let's go back to the toboggan we studied in Example 5.10. The wax has worn off, so there is now a nonzero coefficient of kinetic friction μ_{k}. The slope has just the right angle to make the toboggan slide with constant velocity. Find this angle in terms of w and μ_{k}.

IDENTIFY and SET UP Our target variable is the slope angle α. The toboggan is in equilibrium because its velocity is constant, so we use Newton's first law in the form of Eqs. (5.1).

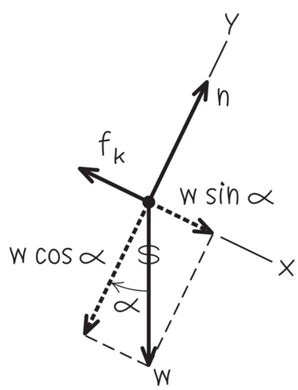
Three forces act on the toboggan: its weight, the normal force, and the kinetic friction force. The motion is downhill, so the friction force (which opposes the motion) is directed uphill. Figure 5.22 shows our sketch and free-body diagram (compare Fig. 5.12b in Example 5.10). From Eq. (5.3), the magnitude of the kinetic friction force is $f_{\mathrm{k}} = \mu_k n$. We expect that the greater the value of μ_{k}, the steeper will be the required slope.

Figure 5.22

(a) The situation



(b) Free-body diagram for toboggan



Our sketches for this problem.

EXECUTE The equilibrium conditions are

$$\sum F_x = w \sin \alpha + (-f_{\mathrm{k}}) = w \sin \alpha - \mu_k n = 0$$
$$\sum F_y = n + (-w \cos \alpha) = 0$$

Rearranging these two equations, we get

$$\mu_{\mathrm{k}} n = w \sin \alpha \quad \text{and} \quad n = w \cos \alpha$$

As in Example 5.10, the normal force is *not* equal to the weight. We eliminate n by dividing the first of these equations by the second, with the result

$$\mu_{\mathrm{k}} = \frac{\sin\alpha}{\cos\alpha} = \tan\ \alpha \quad \mathit{so} \quad \alpha = \arctan\mu_k$$

EVALUATE The weight w doesn't appear in this expression. *Any* toboggan, regardless of its weight, slides down an incline with constant speed if the coefficient of kinetic friction equals the tangent of the slope angle of the incline. The arctangent function increases as its argument increases, so it's indeed true that the slope angle α for constant speed increases as μ_{k} increases.

KEY CONCEPT

When kinetic friction of magnitude $f_{\mathrm{k}} = \mu_k n$ is present for an object on an incline, the magnitude n of the normal force on the object is *not* equal to the object's weight.

Video Tutor Solution: **Example 5.16**

Example 5.17 Toboggan ride with friction II

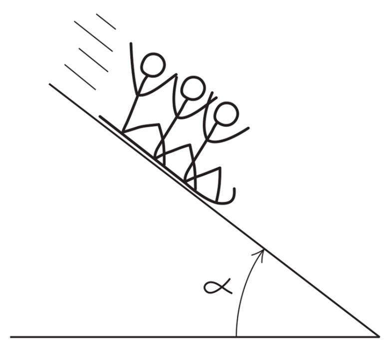
The same toboggan with the same coefficient of friction as in Example 5.16 *accelerates* down a steeper hill. Derive an expression for the acceleration in terms of g, α, μ_k, and w.

IDENTIFY and SET UP The toboggan is accelerating, so we must use Newton's second law as given in Eqs. (5.2). Our target variable is the downhill acceleration.

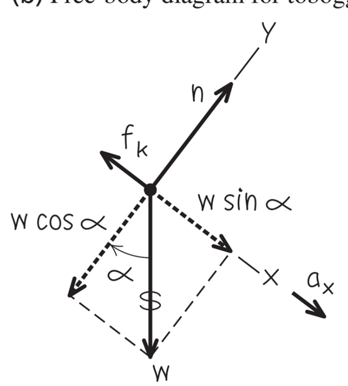
Our sketch and free-body diagram (Fig. 5.23) are almost the same as for Example 5.16. The toboggan's y-component of acceleration a_y is still zero but the x-component a_x is not, so we've drawn w sin α, the downhill component of weight, as a longer vector than the (uphill) friction force.

Figure 5.23

(a) The situation



(b) Free-body diagram for toboggan



Our sketches for this problem.

EXECUTE It's convenient to express the weight as $w = mg$. Then Newton's second law in component form says

$$\sum F_x = mg \sin \alpha + (-f_{\mathrm{k}}) = ma_x$$
$$\sum F_Y = n + (-mg \cos \alpha) = 0$$

From the second equation and Eq. (5.3) we get an expression for f_{k} :

$$n = mg \cos \alpha$$
$$f_{\mathrm{k}} = \mu_{\mathrm{k}} n = \mu_{\mathrm{k}} mg \cos \alpha$$

We substitute this into the x-component equation and solve for a_x:

$$mg \sin \alpha + (-\mu_k \, mg \cos \alpha) = ma_x$$
$$a_x = g \, (\sin \alpha - \mu_{\mathrm{k}} \cos \alpha)$$

EVALUATE As for the frictionless toboggan in Example 5.10, the acceleration doesn't depend on the mass m of the toboggan. That's because all of the forces that act on the toboggan (weight, normal force, and kinetic friction force) are proportional to m.

Let's check some special cases. If the hill is vertical $(\alpha = 90^\circ)$ so that $\sin \alpha = 1$ and $\cos \alpha = 0$, we have $a_x = g$ (the toboggan falls freely). For a certain value of α the acceleration is zero; this happens if

$$\sin \alpha = \mu_k \cos \alpha \qquad \text{and} \qquad \mu_k = \tan \alpha$$

This agrees with our result for the constant-velocity toboggan in Example 5.16. If the angle is even smaller, μ_{k} *cos* α is greater than $\sin \alpha$ and a_x is *negative;* if we give the toboggan an initial downhill push to start it moving, it will slow down and stop. Finally, if the hill is frictionless so that $\mu_k = 0$, we retrieve the result of Example 5.10: $a_x = g \sin \alpha$.

Notice that we started with a simple problem (Example 5.10) and extended it to more and more general situations. The general result we found in this example includes *all* the previous ones as special cases. Don't memorize this result, but do make sure you understand how we obtained it and what it means.

Suppose instead we give the toboggan an initial push *up* the hill. The direction of the kinetic friction force is now reversed, so the acceleration is different from the downhill value. It turns out that the expression for a_x is the same as for downhill motion except that the minus sign becomes plus. Can you show this?

KEY CONCEPT

The magnitude $f_{\mathrm{k}} = \mu_k n$ of the kinetic friction force is the same whether or not the object is accelerating.

Video Tutor Solution: **Example 5.17**

Rolling Friction

It's a lot easier to move a loaded filing cabinet across a horizontal floor by using a cart with wheels than by sliding it. How much easier? We can define a **coefficient of rolling friction** μ_{r}, which is the horizontal force needed for constant speed on a flat surface divided by the upward normal force exerted by the surface. Transportation engineers call μ_r the *tractive resistance*. Typical values of μ_r are 0.002 to 0.003 for steel wheels on steel rails and 0.01 to 0.02 for rubber tires on concrete. These values show one reason railroad trains are generally much more fuel efficient than highway trucks.

Fluid Resistance and Terminal Speed

Sticking your hand out the window of a fast-moving car will convince you of the existence of **fluid resistance**, the force that a fluid (a gas or liquid) exerts on an object moving through it. The moving object exerts a force on the fluid to push it out of the way. By Newton's third law, the fluid pushes back on the object with an equal and opposite force.

The *direction* of the fluid resistance force acting on an object is always opposite the direction of the object's velocity relative to the fluid. The *magnitude* of the fluid resistance force usually increases with the speed of the object through the fluid. This is very different from the kinetic friction force between two surfaces in contact, which we can usually regard as independent of speed. For small objects moving at very low speeds, the magnitude f of the fluid resistance force is approximately proportional to the object's speed v:

(5.5)

$$f = kv \quad \text{(fluid resistance at low speed)}$$

where k is a proportionality constant that depends on the shape and size of the object and the properties of the fluid. Equation (5.5) is appropriate for dust particles falling in air or a ball bearing falling in oil. For larger objects moving through air at the speed of a tossed tennis ball or faster, the resisting force is approximately proportional to v^2 rather than to v. It is then called **air drag** or simply *drag*. Airplanes, falling raindrops, and bicyclists all experience air drag. In this case we replace Eq. (5.5) by

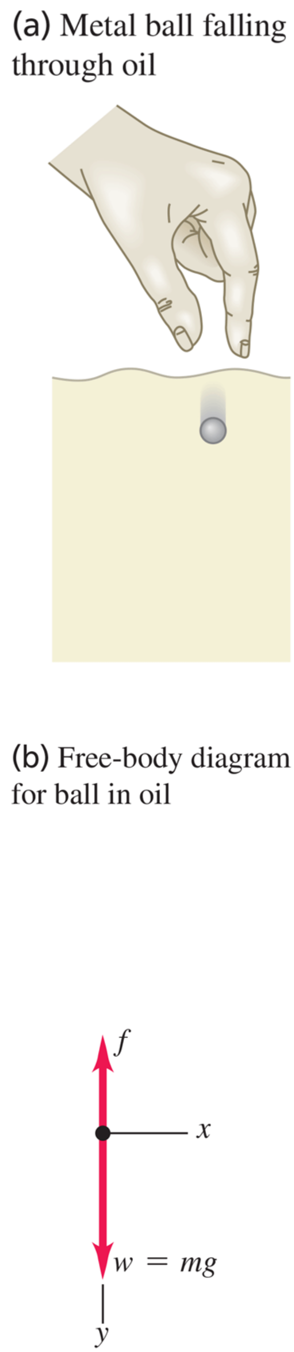
(5.6)

$$f = Dv^2 \text{ (fluid resistance at high speed)}$$

Because of the v^2 dependence, air drag increases rapidly with increasing speed. The air drag on a typical car is negligible at low speeds but comparable to or greater than rolling resistance at highway speeds. The value of D depends on the shape and size of the object and on the density of the air. You should verify that the units of the constant k in Eq. (5.5) are $\mathrm{N \cdot s/m}$ or $\mathrm{kg/s}$, and that the units of the constant D in Eq. (5.6) are $\mathrm{N \cdot s^2/m^2}$ or $\mathrm{kg/m}$.

Because of the effects of fluid resistance, an object falling in a fluid does *not* have a constant acceleration. To describe its motion, we can't use the constant-acceleration relationships from Chapter 2; instead, we have to start over with Newton's second law. As an example, suppose you drop a metal ball at the surface of a bucket of oil and let it fall to the bottom (Fig. 5.24a). The fluid resistance force in this situation is given by Eq. (5.5). What are the acceleration, velocity, and position of the metal ball as functions of time?

Figure 5.24



Motion with fluid resistance.

BIO Application

Pollen and Fluid Resistance

These spiky spheres are pollen grains from the ragweed flower (*Ambrosia artemisiifolia*) and a common cause of hay fever. Because of their small radius (about 10 μm $= 0.01$ mm), when they are released into the air the fluid resistance force on them is proportional to their speed. The terminal speed given by Eq. (5.8) is only about 1 cm/s. Hence even a moderate wind can keep pollen grains aloft and carry them substantial distances from their source.

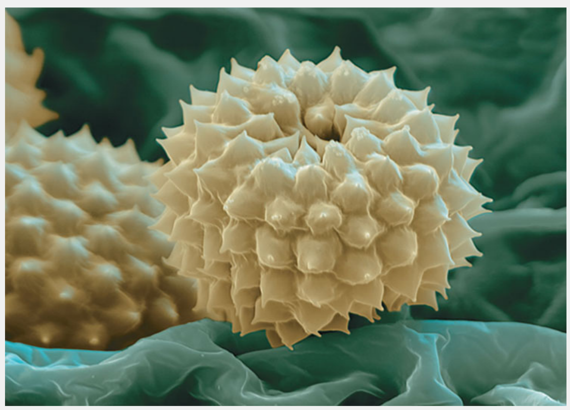


Figure 5.24b shows the free-body diagram. We take the positive y-direction to be downward and neglect any force associated with buoyancy in the oil. Since the ball is moving downward, its speed v is equal to its y-velocity v_y and the fluid resistance force is in the $-y$-direction. There are no x-components, so Newton's second law gives

(5.7)

$$\sum F_y = mg + (-kv_y) = ma_y$$

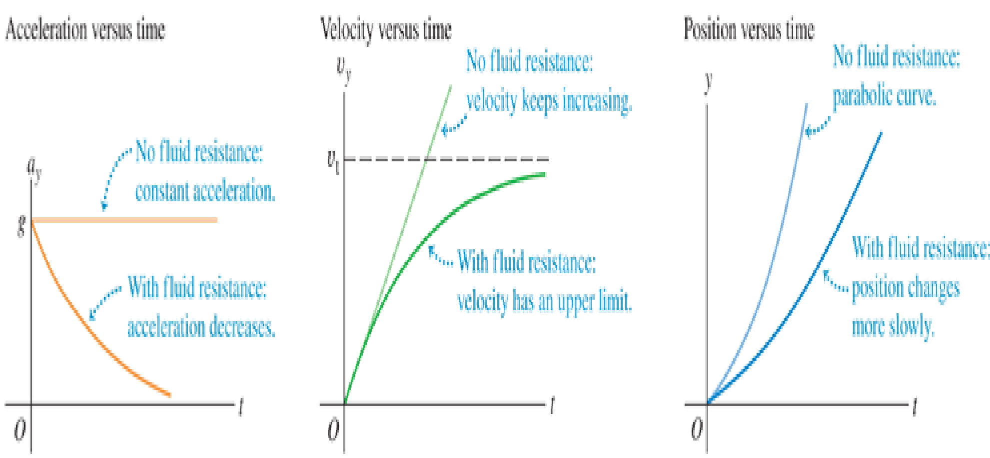
When the ball first starts to move, $v_y = 0$, the resisting force is zero and the initial acceleration is $a_y = g$. As the speed increases, the resisting force also increases, until finally it is equal in magnitude to the weight. At this time $mg - kv_y = 0$, the acceleration is zero, and there is no further increase in speed. The final speed v_t, called the **terminal speed**, is given by $mg - kv_t = 0$, or

(5.8)

$$v_t = \frac{mg}{k} \quad \text{(terminal speed, fluid resistance } f = kv\text{)}$$

Figure 5.25 shows how the acceleration, velocity, and position vary with time. As time goes by, the acceleration approaches zero and the velocity approaches v_t (remember that we chose the positive y-direction to be down). The slope of the graph of y versus T becomes constant as the velocity becomes constant.

Figure 5.25



Graphs of the motion of an object falling without fluid resistance and with fluid resistance proportional to the speed.

To see how the graphs in Fig. 5.25 are derived, we must find the relationship between velocity and time during the interval before the terminal speed is reached. We go back to Newton's second law for the falling ball, Eq. (5.7), which we rewrite with $a_y = dv_y/dt$:

$$m\frac{dv_y}{dt} = mg - kv_y$$

After rearranging terms and replacing mg/k by v_t , we integrate both sides, noting that $v_y = 0$ when $t = 0$:

$$\int_0^v \frac{dv_y}{v_y - v_\mathrm{t}} = -\frac{k}{m}\int_0^t dt$$

which integrates to

$$\ln\frac{v_\mathrm{t} - v_y}{v_\mathrm{t}} = -\frac{k}{m}t \qquad \text{or} \qquad 1 - \frac{v_y}{v_\mathrm{t}} = e^{-(k/m)t}$$

and finally

(5.9)

$$v_y = v_\mathrm{t}\left[1 - e^{-(k/m)t}\right]$$

Note that v_y becomes equal to the terminal speed v_t only in the limit that $t \to \infty$; the ball cannot attain terminal speed in any finite length of time.

The derivative of v_y in Eq. (5.9) gives a_y as a function of time, and the integral of v_y gives y as a function of time. We leave the derivations for you to complete; the results are

(5.10)

$$a_y = ge^{-(k/m)t}$$

(5.11)

$$y = v_t \left[t - \frac{m}{k} \left(1 - e^{-(k/m)t}\right) \right]$$

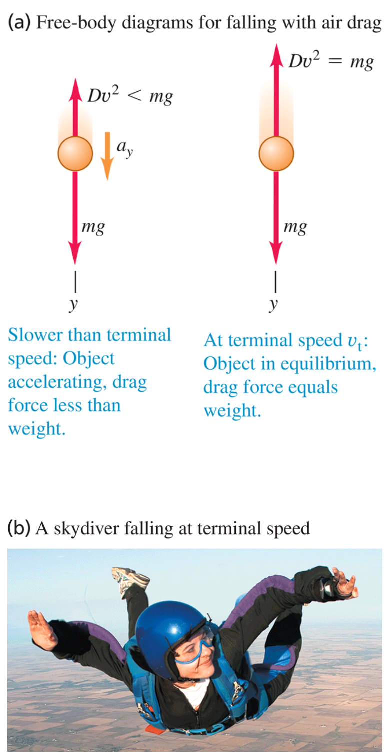
Now look again at Fig. 5.25, which shows graphs of these three relationships.

In deriving the terminal speed in Eq. (5.8), we assumed that the fluid resistance force is proportional to the speed. For an object falling through the air at high speeds, so that the fluid resistance is equal to Dv^2 as in Eq. (5.6), the terminal speed is reached when Dv^2 equals the weight mg (Fig. 5.26a). You can show that the terminal speed v_t is given by

(5.12)

$$v_t = \sqrt{\frac{mg}{D}} \quad \text{(terminal speed, fluid resistance } f = Dv^2\text{)}$$

Figure 5.26

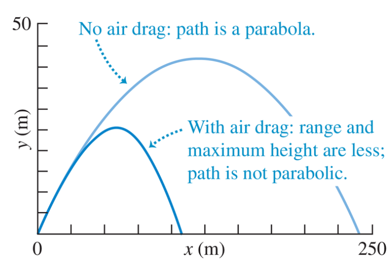


(a) Air drag and terminal speed. (b) By changing the positions of their arms and legs while falling, skydivers can change the value of the constant D in Eq. (5.6) and hence adjust the terminal speed of their fall [Eq. (5.12)].

This expression for terminal speed explains why heavy objects in air tend to fall faster than light objects. Two objects that have the same physical size but different mass (say, a table-tennis ball and a lead ball with the same radius) have the same value of D but different values of m. The more massive object has a higher terminal speed and falls faster. The same idea explains why a sheet of paper falls faster if you first crumple it into a ball; the mass m is the same, but the smaller size makes D smaller (less air drag for a given speed) and v_t larger. Skydivers use the same principle to control their descent (Fig. 5.26b).

Figure 5.27 shows the trajectories of a baseball with and without air drag, assuming a coefficient $D = 1.3 \times 10^{-3}$ kg/m (appropriate for a batted ball at sea level). Both the range of the baseball and the maximum height reached are substantially smaller than the zero-drag calculation would lead you to believe. Hence the baseball trajectory we calculated in Example 3.7 (Section 3.3) by ignoring air drag is unrealistic. Air drag is an important part of the game of baseball!

Figure 5.27



Computer-generated trajectories of a baseball launched at 50 m/s at 35° above the horizontal. Note that the scales are different on the horizontal and vertical axes.

Example 5.18 Terminal speed of a skydiver

For a human body falling through air in a spread-eagle position (Fig. 5.26b), the numerical value of the constant D in Eq. (5.6) is about 0.25 kg/m. Find the terminal speed for a 50 kg skydiver.

IDENTIFY and SET UP This example uses the relationship among terminal speed, mass, and drag coefficient. We use Eq. (5.12) to find the target variable v_t.

EXECUTE We find for $m = 50$ kg :

$$\begin{aligned} v_t &= \sqrt{\frac{mg}{D}} = \sqrt{\frac{(50 \text{ kg})(9.8 \text{ m/s}^2)}{0.25 \text{ kg/m}}} \\ &= 44 \text{ m/s } (\text{about } 160 \text{ km/h, or } 99 \text{ mi/h}) \end{aligned}$$

EVALUATE The terminal speed is proportional to the square root of the skydiver's mass. A skydiver with the same drag coefficient D but twice the mass would have a terminal speed $\sqrt{2} = 1.41$ times greater, or 63 m/s. (A more massive skydiver would also have more frontal area and hence a larger drag coefficient, so his terminal speed would be a bit less than 63 m/s.) Even the 50 kg skydiver's terminal speed is quite high, so skydives don't last very long. A drop from 2800 m (9200 ft) to the surface at the terminal speed takes only $(2800 \text{ m})/(44 \text{ m/s}) = 64$ s.

When the skydiver deploys the parachute, the value of D increases greatly. Hence the terminal speed of the skydiver with parachute decreases dramatically to a much lower value.

KEY CONCEPT

A falling object reaches its terminal speed when the upward force of fluid resistance equals the downward force of gravity. Depending on the object's speed, use either Eq. (5.8) or Eq. (5.12) to find the terminal speed.

Video Tutor Solution: **Example 5.18**

Test Your Understanding of Section 5.3

Consider a box that is placed on different surfaces. (a) In which situation(s) is *no* friction force acting on the box? (b) In which situation(s) is a *static* friction force acting on the box? (c) In which situation(s) is a *kinetic* friction force acting on the box? (i) The box is at rest on a rough horizontal surface. (ii) The box is at rest on a rough tilted surface. (iii) The box is on the rough-surfaced flat bed of a truck that is moving at a constant velocity on a straight, level road, and the box remains in place in the middle of the truck bed. (iv) The box is on the rough-surfaced flat bed of a truck that is speeding up on a straight, level road, and the box remains in place in the middle of the truck bed. (v) The box is on the rough-surfaced flat bed of a truck that is climbing a hill, and the box is sliding toward the back of the truck.

5.4 Dynamics of Circular Motion

Video Tutor Demo: Ball Leaves Circular Track

We talked about uniform circular motion in Section 3.4. We showed that when a particle moves in a circular path with constant speed, the particle's acceleration has a constant magnitude a_{rad} given by

(5.13)

Magnitude of acceleration of an object in **uniform circular motion**

$$a_{\text{rad}} = \frac{v^2}{R} \qquad (5.13)$$

Speed of object

Radius of object's circular path

The subscript "rad" is a reminder that at each point the acceleration points radially inward toward the center of the circle, perpendicular to the instantaneous velocity. We explained in Section 3.4 why this acceleration is often called *centripetal acceleration* or *radial acceleration*.

We can also express the centripetal acceleration a_{rad} in terms of the *period* T, the time for one revolution:

(5.14)

$$T = \frac{2\pi R}{v}$$

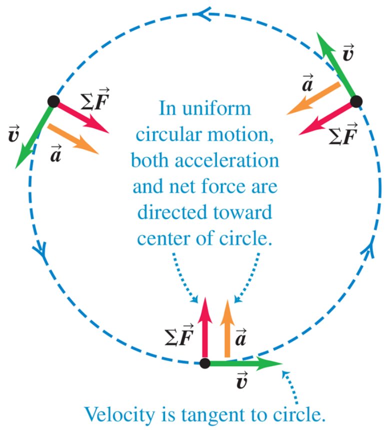
In terms of the period, a_{rad} is

(5.15)

$$\underbrace{a_{\mathrm{rad}}}_{\text{Magnitude of acceleration of an object in \textbf{uniform circular motion}}} = \frac{4\pi^2 \overset{\text{Radius of object's circular path}}{R}}{\underset{\text{Period of motion}}{T^2}} \qquad (5.15)$$

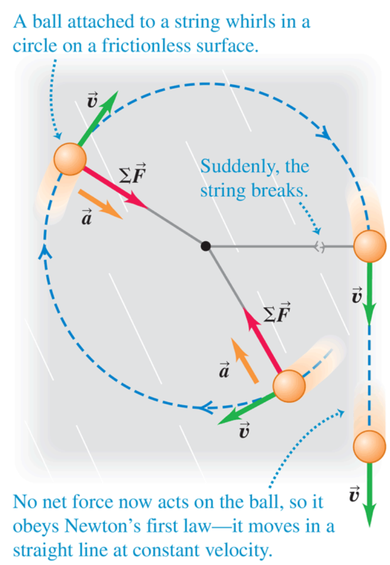
Uniform circular motion, like all other motion of a particle, is governed by Newton's second law. To make the particle accelerate toward the center of the circle, the net force $\sum \vec{F}$ on the particle must always be directed toward the center (Fig. 5.28). The magnitude of the acceleration is constant, so the magnitude F_{net} of the net force must also be constant. If the inward net force stops acting, the particle flies off in a straight line tangent to the circle (Fig. 5.29).

Figure 5.28



Net force, acceleration, and velocity in uniform circular motion.

Figure 5.29



What happens if the inward radial force suddenly ceases to act on an object in circular motion?

The magnitude of the radial acceleration is given by $a_{\text{rad}} = v^2/R$, so the magnitude F_{net} of the net force on a particle with mass m in uniform circular motion must be

(5.16)

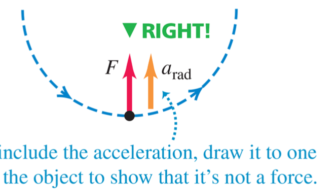
$$F_{\mathrm{net}} = ma_{\mathrm{rad}} = m\frac{v^2}{R} \text{ (uniform circular motion)}$$

Uniform circular motion can result from *any* combination of forces, just so the net force $\sum \vec{F}$ is always directed toward the center of the circle and has a constant magnitude. Note that the object need not move around a complete circle: Equation (5.16) is valid for *any* path that can be regarded as part of a circular arc.

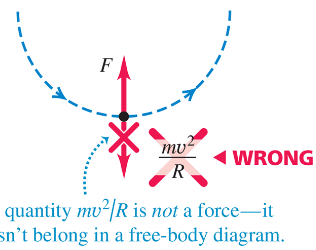
CAUTION Avoid using "centrifugal force" Figure 5.30 shows a correct free-body diagram for uniform circular motion (Fig. 5.30a) and an *incorrect* diagram (Fig. 5.30b). Figure 5.30b is incorrect because it includes an extra outward force of magnitude $m\left(v^2/R\right)$ to "keep the object out there" or to "keep it in equilibrium." There are three reasons not to include such an outward force, called *centrifugal force* ("centrifugal" means "fleeing from the center"). First, the object does *not* "stay out there": It is in constant motion around its circular path. Because its velocity is constantly changing in direction, the object accelerates and is *not* in equilibrium. Second, if there *were* an outward force that balanced the inward force, the net force would be zero and the object would move in a straight line, not a circle (Fig. 5.29). Third, the quantity $m\left(v^2/R\right)$ is *not* a force; it corresponds to the $m\vec{a}$ side of $\sum \vec{F} = m\vec{a}$ and does not appear in $\sum \vec{F}$ (Fig. 5.30a). It's true that when you ride in a car that goes around a circular path, you tend to slide to the outside of the turn as though there was a "centrifugal force." But we saw in Section 4.2 that what happens is that you tend to keep moving in a straight line, and the outer side of the car "runs into" you as the car turns (Fig. 4.10c). *In an inertial frame of reference there is no such thing as "centrifugal force."* We won't mention this term again, and we strongly advise you to avoid it.

Figure 5.30

(a) Correct free-body diagram



(b) Incorrect free-body diagram



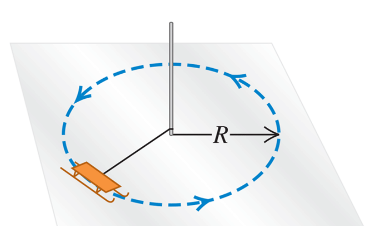
Right and wrong ways to depict uniform circular motion.

Example 5.19 Force in uniform circular motion

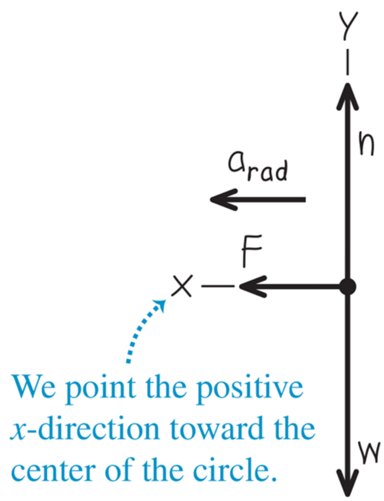
A sled with a mass of 25.0 kg rests on a horizontal sheet of essentially frictionless ice. It is attached by a 5.00 m rope to a post set in the ice. Once given a push, the sled revolves uniformly in a circle around the post (Fig. 5.31a). If the sled makes five complete revolutions every minute, find the force F exerted on it by the rope.

Figure 5.31

(a) A sled in uniform circular motion



(b) Free-body diagram for sled



(a) The situation. (b) Our free-body diagram.

IDENTIFY and SET UP The sled is in uniform circular motion, so it has a constant radial acceleration. We'll apply Newton's second law to the sled to find the magnitude F of the force exerted by the rope (our target variable).

Figure 5.31b shows our free-body diagram for the sled. The acceleration has only an x-component; this is toward the center of the circle, so we denote it as a_{rad}. The acceleration isn't given, so we'll need to determine its value by using Eq. (5.13) or Eq. (5.15).

EXECUTE The force F appears in Newton's second law for the x-direction:

$$\sum F_x = F = ma_{\text{rad}}$$

We can find the centripetal acceleration a_{rad} by using Eq. (5.15). The sled moves in a circle of radius $R = 5.00\ \text{m}$ with a period

$T = (60.0 \text{ s})/(5 \text{ rev}) = 12.0 \text{ s}$, so

$$a_{\text{rad}} = \frac{4\pi^2 R}{T^2} = \frac{4\pi^2 (5.00 \text{ m})}{(12.0 \text{ s})^2} = 1.37 \text{ m/s}^2$$

The magnitude F of the force exerted by the rope is then

$$F = ma_{\text{rad}} = (25.0 \text{ kg})(1.37 \text{ m/s}^2) = 34.3 \text{ kg} \cdot \text{m/s}^2 = 34.3 \text{ N}$$

EVALUATE You can check our value for a_{rad} by first using Eq. (5.14), $v = 2\pi R/T$, to find the speed and then using $a_{\text{rad}} = v^2/R$ from Eq. (5.13). Do you get the same result?

A greater force would be needed if the sled moved around the circle at a higher speed v. In fact, if v were doubled while R remained the same, F would be four times greater. Can you show this? How would F change if v remained the same but the radius R were doubled?

KEY CONCEPT

In problems that involve forces on an object in uniform circular motion, take the positive x-direction to be toward the center of the circle. The net force component in that direction is equal to the object's mass times its radial acceleration.

Video Tutor Solution: **Example 5.19**

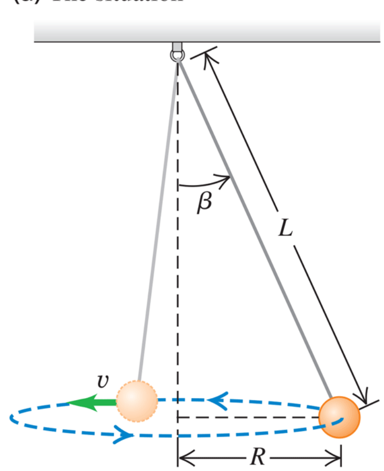
Example 5.20 A conical pendulum

WITH VARIATION PROBLEMS

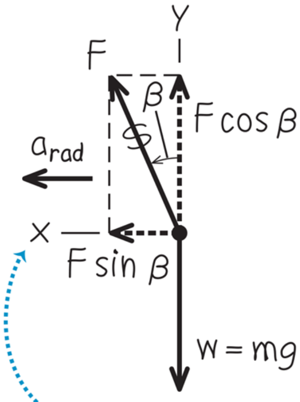
An inventor designs a pendulum clock using a bob with mass m at the end of a thin wire of length L. Instead of swinging back and forth, the bob is to move in a horizontal circle at constant speed v, with the wire making a fixed angle β with the vertical direction (Fig. 5.32a). This is called a *conical pendulum* because the suspending wire traces out a cone. Find the tension F in the wire and the period T (the time for one revolution of the bob).

Figure 5.32

(a) The situation



(b) Free-body diagram
for pendulum bob



(a) The situation. (b) Our free-body diagram.

IDENTIFY and SET UP To find our target variables, the tension F and period T, we need two equations. These will be the horizontal and vertical components of Newton's second law applied to the bob. We'll find the radial acceleration of the bob from one of the circular motion equations.

Figure 5.32b shows our free-body diagram and coordinate system for the bob at a particular instant. There are just two forces on the bob: the weight mg and the tension F in the wire. Note that the center of the circular path is in the same horizontal plane as the bob, *not* at the top end of the wire. The horizontal component of tension is the force that produces the radial acceleration a_{rad}.

EXECUTE The bob has zero vertical acceleration; the horizontal acceleration is toward the center of the circle, which is why we use the symbol a_{rad}. Newton's second law, Eqs. (5.2), says

$$\sum F_x = F \sin\beta = ma_{\mathrm{rad}}$$
$$\sum F_y = F\cos\beta + (-mg) = 0$$

These are two equations for the two unknowns F and β. The equation for $\sum F_y$ gives $F = mg/\cos\beta$; that's our target expression for F in terms of β. Substituting this result into the equation for $\sum F_x$ and using sin β/cos β $=$ tan β, we find

$$a_{\text{rad}} = g\ \tan\beta$$

To relate β to the period T, we use Eq. (5.15) for a_{rad}, solve for T, and insert $a_{\text{rad}} = g\ \tan\beta$:

$$a_{\text{rad}} = \frac{4\pi^2 R}{T^2} \quad \text{so} \quad T^2 = \frac{4\pi^2 R}{a_{\text{rad}}}$$
$$T = 2\pi\sqrt{\frac{R}{g\tan\beta}}$$

Figure 5.32a shows that $R = L\sin\beta$. We substitute this and use $\sin\beta/\tan\beta = \cos\beta$:

$$T = 2\pi\sqrt{\frac{L\cos\beta}{g}}$$

EVALUATE For a given length L, as the angle β increases, $\cos\beta$ decreases, the period T becomes smaller, and the tension $F = mg/\cos\beta$ increases. The angle can never be 90°, however; this would require that $T = 0$, $F = \infty$, and $v = \infty$. A conical pendulum would not make a very good clock because the period depends on the angle β in such a direct way.

KEY CONCEPT

In uniform circular motion, any kind of force (or component of a force) can produce the radial acceleration.

Video Tutor Solution: **Example 5.20**

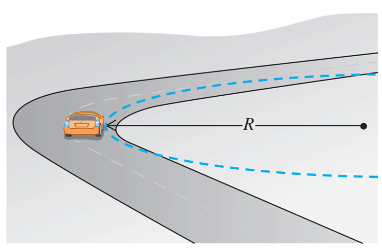
Example 5.21 Rounding a flat curve

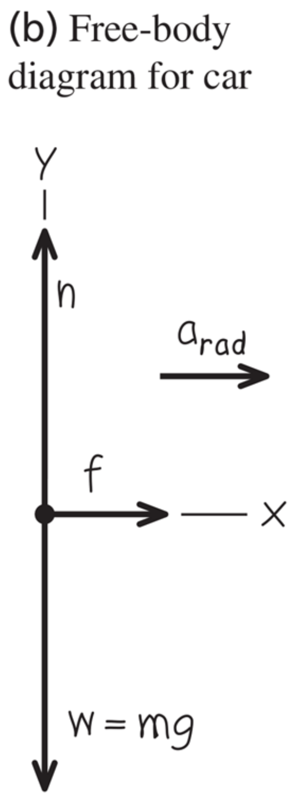
WITH VARIATION PROBLEMS

The sports car in Example 3.11 (Section 3.4) is rounding a flat, unbanked curve with radius R (Fig. 5.33a). If the coefficient of static friction between tires and road is μ_s, what is the maximum speed v_{max} at which the driver can take the curve without sliding?

Figure 5.33

(a) Car rounding flat curve





(a) The situation. (b) Our free-body diagram.

IDENTIFY and SET UP The car's acceleration as it rounds the curve has magnitude $a_{\text{rad}} = v^2/R$. Hence the maximum speed v_{max} (our target variable) corresponds to the maximum acceleration a_{rad} and to the maximum horizontal force on the car toward the center of its circular path. The only horizontal force acting on the car is the friction force exerted by the road. So to solve this problem we'll need Newton's second law, the equations of uniform circular motion, and our knowledge of the friction force from Section 5.3.

The free-body diagram in Fig. 5.33b includes the car's weight $w = mg$ and the two forces exerted by the road: the normal force n and the horizontal friction force f. The friction force must point toward the center of the circular path in order to cause the radial acceleration. The car doesn't slide toward or away from the center of the circle, so the friction force is *static* friction, with a maximum magnitude $f_{\max} = \mu_s n$ [see Eq. (5.4)].

EXECUTE The acceleration toward the center of the circular path is $a_{\text{rad}} = v^2/R$. There is no vertical acceleration. Thus

$$\sum F_x = f = ma_{\text{rad}} = m\frac{v^2}{R}$$
$$\sum F_y = n + (-mg) = 0$$

The second equation shows that $n = mg$. The first equation shows that the friction force *needed* to keep the car moving in its circular path increases with the car's speed. But the maximum friction force *available* is $f_{\max} = \mu_s n = \mu_{\text{s}} mg$, and this determines the car's maximum speed. Substituting $\mu_{\text{s}} mg$ for f and $v_{\max}$ for v in the first equation, we find

$$\mu_{\text{s}} mg = m\frac{{v_{\max}}^2}{R} \qquad \text{so} \qquad v_{\max} = \sqrt{\mu_{\text{s}}\, gR}$$

As an example, if $\mu_{\text{s}} = 0.96$ and $R = 230 \text{ m}$, we have

$$v_{\max} = \sqrt{(0.96)(9.8 \text{ m/s}^2)(230 \text{ m})} = 47 \text{ m/s}$$

or about $170 \text{ km}/h$ (100 mi/h). This is the maximum speed for this radius.

EVALUATE If the car's speed is slower than $v_{\max} = \sqrt{\mu_{\text{s}}\, gR}$, the required friction force is less than the maximum value $f_{\max} = \mu_s mg$, and the car can easily make the curve. If we try to take the curve going *faster* than $v_{\max}$, we'll skid. We could still go in a circle without skidding at this higher speed, but the radius would have to be larger.

The maximum centripetal acceleration (called the "lateral acceleration" in Example 3.11) is equal to $\mu_{\text{s}}\, g$. That's why it's best to take curves at less than the posted speed limit if the road is wet or icy, either of which can reduce the value of μ_s and hence $\mu_s g$.

KEY CONCEPT

For a vehicle following a curved path on a level road, the radial acceleration is produced by the static friction force exerted by the road on the vehicle.

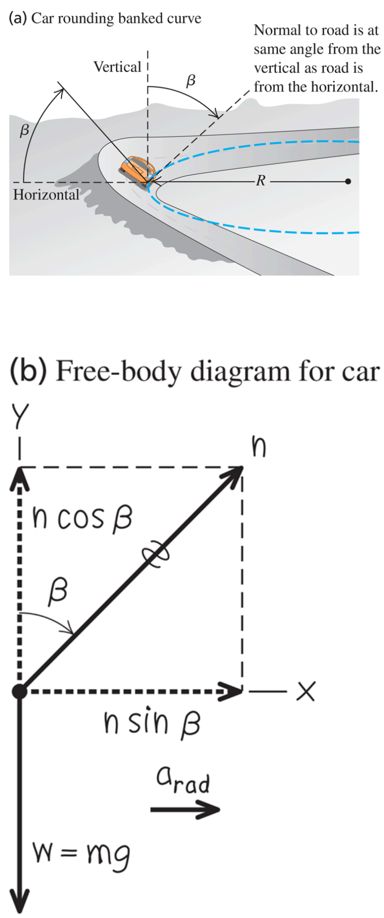
Video Tutor Solution: Example 5.21

Example 5.22 Rounding a banked curve

WITH VARIATION PROBLEMS

For a car traveling at a certain speed, it is possible to bank a curve at just the right angle so that no friction is needed to maintain the car's turning radius. Then a car can safely round the curve even on wet ice. (Bobsled racing depends on this idea.) Your engineering firm plans to rebuild the curve in Example 5.21 so that a car moving at a chosen speed v can safely make the turn even with no friction (Fig. 5.34a). At what angle β should the curve be banked?

Figure 5.34



(a) The situation. (b) Our free-body diagram.

IDENTIFY and SET UP With no friction, the only forces acting on the car are its weight and the normal force. Because the road is banked, the normal force (which acts perpendicular to the road surface) has a horizontal component. This component causes the car's horizontal acceleration toward the center of the car's circular path. We'll use Newton's second law to find the target variable β.

Our free-body diagram (Fig. 5.34b) is very similar to the diagram for the conical pendulum in Example 5.20 (Fig. 5.32b). The normal force acting on the car plays the role of the tension force exerted by the wire on the pendulum bob.

EXECUTE The normal force $\vec{n}$ is perpendicular to the roadway and is at an angle β with the vertical (Fig. 5.34b). Thus it has a vertical component $n \cos \beta$ and a horizontal component $n \sin \beta$. The acceleration in the x-direction is the centripetal acceleration $a_{\text{rad}} = v^2/R$; there is no acceleration in the y-direction. Thus the equations of Newton's second law are

$$\begin{aligned} \sum F_x &= n \sin \beta = m a_{\text{rad}} \\ \sum F_y &= n \cos \beta + (-mg) = 0 \end{aligned}$$

From the $\sum F_y$ equation, $n = mg/\cos \beta$. Substituting this into the $\sum F_x$ equation and using $a_{\text{rad}} = v^2/R$, we get an expression for the bank angle:

$$\tan \beta = \frac{a_{\text{rad}}}{g} = \frac{v^2}{gR} \qquad \text{so} \qquad \beta = \arctan \frac{v^2}{gR}$$

EVALUATE The bank angle depends on both the speed and the radius. For a given radius, no one angle is correct for all speeds. In the design of highways and railroads, curves are often banked for the average speed of the traffic over them. If $R = 230 \text{ m}$ and $v = 25 \text{ m/s}$ (equal to a highway speed of 88 km/h, or 55 mi/h), then

$$\beta = \arctan \frac{(25 \text{ m/s})^2}{(9.8 \text{ m/s}^2)(230 \text{ m})} = 15^\circ$$

This is within the range of bank angles actually used in highways.

KEY CONCEPT

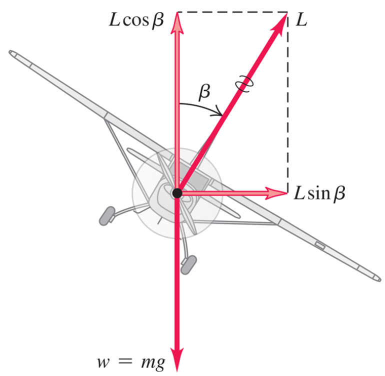
The normal force exerted by the road on a vehicle can provide the radial acceleration needed for the vehicle to follow a curved path, provided the road is banked at the correct angle for the vehicle's speed.

Video Tutor Solution: Example 5.22

Banked Curves and the Flight of Airplanes

The results of Example 5.22 also apply to an airplane when it makes a turn in level flight (Fig. 5.35). When an airplane is flying in a straight line at a constant speed and at a steady altitude, the airplane's weight is exactly balanced by the lift force $\vec{L}$ exerted by the air. (The upward lift force that the air exerts on the wings is a reaction to the downward push the wings exert on the air as they move through it.) To make the airplane turn, the pilot banks the airplane to one side so that the lift force has a horizontal component, as Fig. 5.35 shows. (The pilot also changes the angle at which the wings "bite" into the air so that the vertical component of lift continues to balance the weight.) The bank angle is related to the airplane's speed v and the radius R of the turn by the same expression as in Example 5.22: $\tan \beta = v^2/gR$. For an airplane to make a tight turn (small R) at high speed (large v), $\tan\beta$ must be large and the required bank angle β must approach 90°.

Figure 5.35



An airplane banks to one side in order to turn in that direction. The vertical component of the lift force $\vec{L}$ balances the force of gravity; the horizontal component of $\vec{L}$ causes the acceleration v^2/R.

We can also apply the results of Example 5.22 to the *pilot* of an airplane. The free-body diagram for the pilot of the airplane is exactly as shown in Fig. 5.34b; the normal force $n = mg/\cos\beta$ is exerted on the pilot by the seat. As in Example 5.9, n is equal to the apparent weight of the pilot, which is greater than the pilot's true weight mg. In a tight turn with a large bank angle β, the pilot's apparent weight can be tremendous: $n = 5.8mg$ at $\beta = 80^\circ$ and $n = 9.6mg$ at $\beta = 84^\circ$. Pilots black out in such tight turns because the apparent weight of their blood increases by the same factor, and the human heart isn't strong enough to pump such apparently "heavy" blood to the brain.

Motion in a Vertical Circle

In Examples 5.19, 5.20, 5.21, and 5.22 the object moved in a horizontal circle. Motion in a *vertical* circle is no different in principle, but the weight of the object has to be treated carefully. The following example shows what we mean.

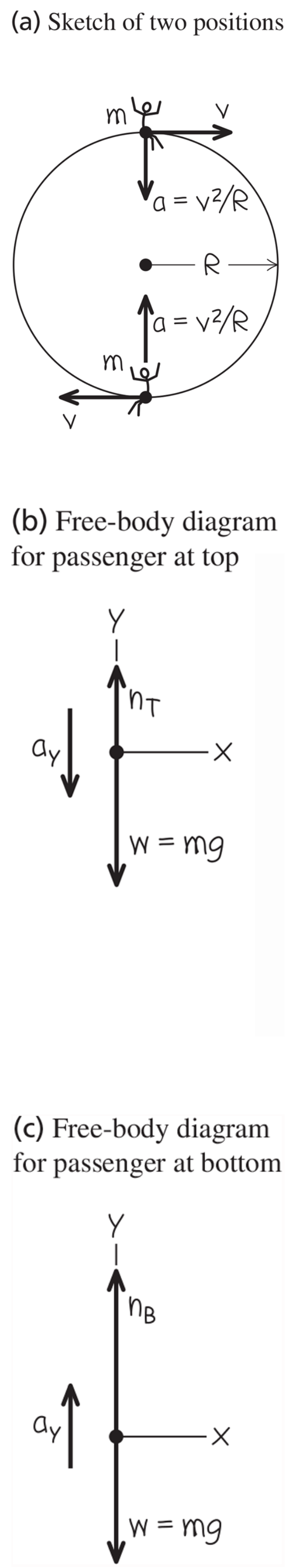
Example 5.23 Uniform circular motion in a vertical circle

A passenger on a carnival Ferris wheel moves in a vertical circle of radius R with constant speed v. The seat remains upright during the motion. Find expressions for the force the seat exerts on the passenger when at the top of the circle and when at the bottom.

IDENTIFY and SET UP The target variables are n_{T}, the upward normal force the seat applies to the passenger at the top of the circle, and n_{B}, the normal force at the bottom. We'll find these by using Newton's second law and the uniform circular motion equations.

Figure 5.36a shows the passenger's velocity and acceleration at the two positions. The acceleration always points toward the center of the circle—downward at the top of the circle and upward at the bottom of the circle. At each position the only forces acting are vertical: the upward normal force and the downward force of gravity. Hence we need only the vertical component of Newton's second law. Figures 5.36b and 5.36c show free-body diagrams for the two positions. We take the positive y-direction as upward in both cases (that is, *opposite* the direction of the acceleration at the top of the circle).

Figure 5.36



Our sketches for this problem.

EXECUTE At the top the acceleration has magnitude v^2/R, but its vertical component is negative because its direction is downward. Hence $a_y = -v^2/R$ and Newton's second law tells us that

$$\text{Top}: \quad \sum F_y = n_{\mathrm{T}} + (-mg) = -m\frac{v^2}{R} \quad \text{or}$$
$$n_{\mathrm{T}} = mg\left(1 - \frac{v^2}{gR}\right)$$

At the bottom the acceleration is upward, so $a_y = +v^2/R$ and Newton's second law says

$$\text{Bottom}: \quad \sum F_y = n_{\mathrm{B}} + (-mg) = +m\frac{v^2}{R} \quad \text{or}$$
$$n_{\mathrm{B}} = mg\left(1 + \frac{v^2}{gR}\right)$$

EVALUATE Our result for n_{T} tells us that at the top of the Ferris wheel, the upward force the seat applies to the passenger is *smaller* in magnitude than the passenger's weight $w = mg$. If the ride goes fast enough that $g - v^2/R$ becomes zero, the seat applies *no* force, and the passenger is about to become airborne. If v becomes still larger, n_{T} becomes negative; this means that a *downward* force (such as from a seat belt) is needed to keep the passenger in the seat. By contrast, the normal force n_{B} at the bottom is always *greater* than the passenger's weight. You feel the seat pushing up on you more firmly than when you are at rest. You can see that n_{T} and n_{B} are the values of the passenger's *apparent weight* at the top and bottom of the circle (see Section 5.2).

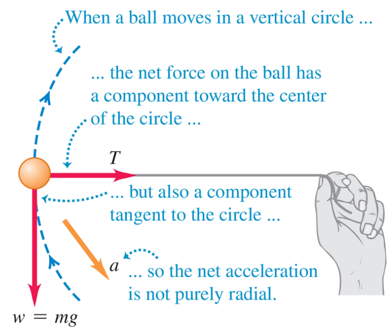
KEY CONCEPT

Even when an object moves with varying speed along a circular path, at any point along the path the net force component toward the center of the circle equals the object's mass times its radial acceleration.

Video Tutor Solution: **Example 5.23**

When we tie a string to an object and whirl it in a vertical circle, the analysis in Example 5.23 isn't directly applicable. The reason is that v is *not* constant in this case; except at the top and bottom of the circle, the net force (and hence the acceleration) does *not* point toward the center of the circle (Fig. 5.37). So both $\sum \vec{F}$ and $\vec{a}$ have a component tangent to the circle, which means that the speed changes. Hence this is a case of *nonuniform* circular motion (see Section 3.4). Even worse, we can't use the constant-acceleration formulas to relate the speeds at various points because *neither* the magnitude nor the direction of the acceleration is constant. The speed relationships we need are best obtained by using the concept of energy. We'll consider such problems in Chapter 7.

Figure 5.37



A ball moving in a vertical circle.

BIO Application

Circular Motion in a Centrifuge

An important tool in medicine and biological research is the ultracentrifuge, a device that makes use of the dynamics of circular motion. A tube is filled with a solvent that contains various small particles (for example, blood containing platelets and white and red blood cells). The tube is inserted into the centrifuge, which then spins at thousands of revolutions per minute. The solvent provides the inward force that keeps the particles in circular motion. The particles slowly drift away from the rotation axis within the solvent. Because the drift rate depends on the particle size and density, particles of different types become separated in the tube, making analysis much easier.

Test Your Understanding of Section 5.4

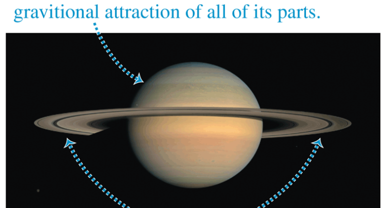
Satellites are held in orbit by the force of our planet's gravitational attraction. A satellite in a small-radius orbit moves at a higher speed than a satellite in an orbit of large radius. Based on this information, what can you conclude about the earth's gravitational attraction for the satellite? (i) It increases with increasing distance from the earth. (ii) It is the same at all distances from the earth. (iii) It decreases with increasing distance from the earth. (iv) This information by itself isn't enough to answer the question.

5.5 The Fundamental Forces of Nature

We have discussed several kinds of forces—including weight, tension, friction, fluid resistance, and the normal force—and we'll encounter others as we continue our study of physics. How many kinds of forces are there? Our best understanding is that all forces are expressions of just four distinct classes of *fundamental* forces, or interactions between particles (Fig. 5.38). Two are familiar in everyday experience. The other two involve interactions between subatomic particles that we cannot observe with the unaided senses.

Figure 5.38

(a) The gravitational interaction



(b) The electromagnetic interaction

(c) The strong interaction

The nucleus of a gold atom has 79 protons and 118 neutrons.

The strong interaction holds the protons and neutrons together and overcomes the electric repulsion of the protons.

(d) The weak interaction

Scientists find the age of this ancient skull by measuring its carbon-14—a form of carbon that is radioactive thanks to the weak interaction.

Examples of the fundamental interactions in nature.

Gravitational interactions include the familiar force of your *weight*, which results from the earth's gravitational attraction acting on you. The mutual gravitational attraction of various parts of the earth for each other holds our planet together, and likewise for the other planets (Fig. 5.38a). Newton recognized that the sun's gravitational attraction for the earth keeps our planet in its nearly circular orbit around the sun. In Chapter 13 we'll study gravitational interactions in more detail, including their vital role in the motions of planets and satellites.

The second familiar class of forces, **electromagnetic interactions**, includes electric and magnetic forces. If you run a comb through your hair, the comb ends up with an electric charge; you can use the electric force exerted by this charge to pick up bits of paper. All atoms contain positive and negative electric charge, so atoms and molecules can exert electric forces on one another. Contact forces, including the normal force, friction, and fluid resistance, are the result of electrical interactions between atoms on the surface of an object and atoms in its surroundings (Fig. 5.38b). *Magnetic* forces, such as those between magnets or between a magnet and a piece of iron, are actually the result of electric charges in motion. For example, an electromagnet causes magnetic interactions because electric charges move through its wires. We'll study electromagnetic interactions in detail in the second half of this book.

On the atomic or molecular scale, gravitational forces play no role because electric forces are enormously stronger: The electrical repulsion between two protons is stronger than their gravitational attraction by a factor of about 10^{35}. But in objects of astronomical size, positive and negative charges are usually present in nearly equal amounts, and the resulting electrical interactions nearly cancel out. Gravitational interactions are thus the dominant influence in the motion of planets and in the internal structure of stars.

The other two classes of interactions are less familiar. One, the **strong interaction**, is responsible for holding the nucleus of an atom together (Fig. 5.38c). Nuclei contain electrically neutral neutrons and positively charged protons. The electric force between charged protons tries to push them apart; the strong attractive force between nuclear particles counteracts this repulsion and makes the nucleus stable. In this context the strong interaction is also called the *strong nuclear force*. It has much shorter range than electrical interactions, but within its range it is much stronger. Without the strong interaction, the nuclei of atoms essential to life, such as carbon (six protons, six neutrons) and oxygen (eight protons, eight neutrons), would not exist and you would not be reading these words!

Finally, there is the **weak interaction**. Its range is so short that it plays a role only on the scale of the nucleus or smaller. The weak interaction is responsible for a common form of radioactivity called beta decay, in which a neutron in a radioactive nucleus is transformed into a proton while ejecting an electron and a nearly massless particle called an antineutrino. The weak interaction between the antineutrino and ordinary matter is so feeble that an antineutrino could easily penetrate a wall of lead a million kilometers thick!

An important application of the weak interaction is *radiocarbon dating*, a technique that enables scientists to determine the ages of many biological specimens (Fig. 5.38d). Naturally occurring carbon includes atoms of both carbon-12 (with six protons and six neutrons in the nucleus) and carbon-14 (with two additional neutrons). Living organisms take in carbon atoms of both kinds from their environment but stop doing so when they die. The weak interaction makes carbon-14 nuclei unstable—one of the neutrons changes to a proton, an electron, and an antineutrino—and these nuclei decay at a known rate. By measuring the fraction of carbon-14 that is left in an organism's remains, scientists can determine how long ago the organism died.

In the 1960s physicists developed a theory that described the electromagnetic and weak interactions as aspects of a single *electroweak* interaction. This theory has passed every experimental test to which it has been put. Encouraged by this success, physicists have made similar attempts to describe the strong, electromagnetic, and weak interactions in terms of a single *grand unified theory* (GUT) and have taken steps toward a possible unification of all interactions into a *theory of everything* (TOE). Such theories are still speculative, and there are many unanswered questions in this very active field of current research.

Chapter 5 Summary

Using Newton's first law: When an object is in equilibrium in an inertial frame of reference—that is, either at rest or moving with constant velocity—the vector sum of forces acting on it must be zero (Newton's first law). Free-body diagrams are essential in identifying the forces that act on the object being considered.

Newton's third law (action and reaction) is also frequently needed in equilibrium problems. The two forces in an action–reaction pair *never* act on the same object. (See Examples 5.1, 5.2, 5.3, 5.4 and 5.5.)

The normal force exerted on an object by a surface is *not* always equal to the object's weight. (See Example 5.4.)

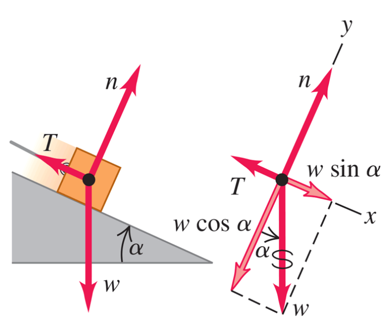
Vector form:

(5.1)

$$\sum \vec{\boldsymbol{F}} = \mathbf{0}$$

Component form:

$$\sum F_x = 0 \quad \sum F_y = 0$$



Using Newton's second law: If the vector sum of forces on an object is *not* zero, the object accelerates. The acceleration is related to the net force by Newton's second law.

Just as for equilibrium problems, free-body diagrams are essential for solving problems involving Newton's second law, and the normal force exerted on an object is not always equal to its weight. (See Examples 5.6, 5.7, 5.8, 5.9, 5.10, 5.11 and 5.12.)

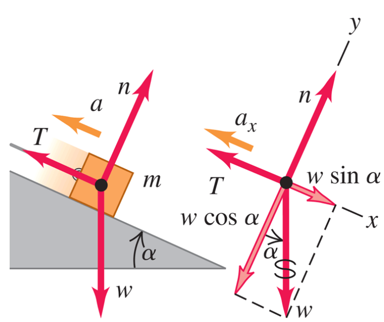
Vector form:

(5.2)

$$\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$$

Component form:

$$\sum F_x = ma_x \qquad \sum F_y = ma_y$$



Friction and fluid resistance: The contact force between two objects can always be represented in terms of a normal force $\vec{\boldsymbol{n}}$ perpendicular to the surface of contact and a friction force $\vec{\boldsymbol{f}}$ parallel to the surface.

When an object is sliding over the surface, the friction force is called *kinetic* friction. Its magnitude f_k is approximately equal to the normal force magnitude n multiplied by the coefficient of kinetic friction μ_k .

Magnitude of kinetic friction force:

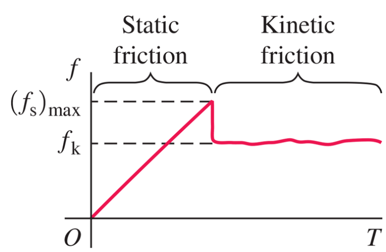
(5.3)

$$f_k = \mu_k n$$

Magnitude of static friction force:

(5.4)

$$f_s \leq (f_s)_{max} = \mu_s n$$



When an object is *not* moving relative to a surface, the friction force is called *static* friction. The *maximum* possible static friction force is approximately equal to the magnitude n of the normal force multiplied by the coefficient of static friction μ_s . The *actual* static friction force may be anything from zero to this maximum value, depending on the situation. Usually μ_s is greater than μ_k for a given pair of surfaces in contact. (See Examples 5.13, 5.14, 5.15, 5.16 and 5.17.)

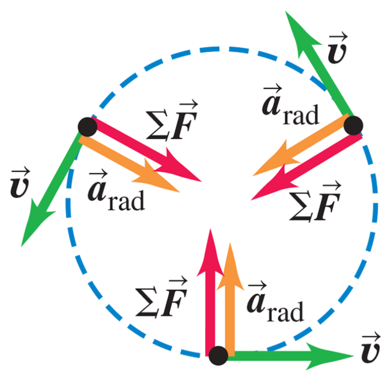
Rolling friction is similar to kinetic friction, but the force of fluid resistance depends on the speed of an object through a fluid. (See Example 5.18.)

Forces in circular motion: In uniform circular motion, the acceleration vector is directed toward the center of the circle. The motion is governed by Newton's second law, $\sum \vec{F} = m\vec{a}$. (See Examples 5.19, 5.20, 5.21, 5.22 and 5.23.)

Acceleration in uniform circular motion:

(5.13), (5.15)

$$a_{\text{rad}} = \frac{v^2}{R} = \frac{4\pi^2 R}{T^2}$$



Guided Practice: Applying Newton's Laws

For assigned homework and other learning materials, go to
Mastering Physics.

Key Example Variation Problems

Be sure to review Example 5.5 (Section 5.1) before attempting these problems. In all problems, ignore air resistance.

VP5.5.1 In a modified version of the cart and bucket in Fig. 5.5a, the angle of the slope is and the bucket weighs 255 N. The cart moves up the incline and the bucket moves downward, both at constant speed. The cable has negligible mass, and there is no friction. (a) What is the weight of the cart? (b) What is the tension in the cable?

VP5.5.2 You increase the angle of the slope in Fig. 5.5a to and use a different cart and a different bucket. You observe that the cart and bucket remain at rest when released and that the tension in the cable of negligible mass is 155 N. There is no friction. (a) What is the weight of the cart? (b) What is the *combined* weight of the cart and bucket?

VP5.5.3 You construct a version of the cart and bucket in Fig. 5.5a, but with a slope whose angle can be adjusted. You use a cart of mass 175 kg and a bucket of mass 65.0 kg. The cable has negligible mass, and there is no friction. (a) What must be the angle of the slope so that the cart moves downhill at a constant speed and the bucket moves upward at the same constant speed? (b) With this choice of angle, what will be the tension in the cable?

VP5.5.4 In the situation shown in Fig. 5.5a, let be the angle of the slope and suppose there *is* friction between the cart and the track. You find that if the cart and bucket each have the same weight , they remain at rest when released. In this case, what is the magnitude of the friction force on the cart? Is it less than, greater than, or equal to ?

Be sure to review Examples 5.13, 5.14, and 5.15 (Section 5.3) before attempting these problems.

VP5.15.1 You pull on a crate using a rope as in Fig. 5.21a, except the rope is at an angle of above the horizontal. The weight of the crate is 325 N, and the coefficient of kinetic friction between the crate and the floor is 0.250. (a) What must be the tension in the rope to make the crate move at a constant velocity? (b) What is the normal force that the floor exerts on the crate?

VP5.15.2 You pull on a large box using a rope as in Fig. 5.21a, except the rope is at an angle of *below* the horizontal. The weight of the box is 325 N, and the coefficient of kinetic friction between the box and the floor is 0.250. (a) What must be the tension in the rope to make the box move at a constant velocity? (b) What is the normal force that the floor exerts on the box?

VP5.15.3 You are using a lightweight rope to pull a sled along level ground. The sled weighs 475 N, the coefficient of kinetic friction between the sled and the ground is 0.200, the rope is at an angle of above the horizontal, and you pull on the rope with a force of 125 N. (a) Find the normal force that the ground exerts on the sled. (b) Find the acceleration of the sled. Is the sled speeding up or slowing down?

VP5.15.4 A large box of mass sits on a horizontal floor. You attach a lightweight rope to this box, hold the rope at an angle above the horizontal, and pull. You find that the minimum tension you can apply to the rope in order to make the box start moving is . Find the coefficient of static friction between the floor and the box.

Be sure to review Examples 5.20, 5.21, and 5.22 (Section 5.4) before attempting these problems.

VP5.22.1 You make a conical pendulum (see Fig. 5.32a) using a string of length 0.800 m and a bob of mass 0.250 kg. When the bob is moving in a circle at a constant speed, the string is at an angle of from the vertical. (a) What is the radius of the circle around which the bob moves? (b) How much time does it take the bob to complete one circle? (c) What is the tension in the string?

VP5.22.2 A competition cyclist rides at a constant 12.5 around a curve that is banked at . The cyclist and her bicycle have a combined mass of 64.0 kg. (a) What must be the radius of her turn if there is to be no friction force pushing her either up or down the banked curve? (b) What is the magnitude of her acceleration? (c) What is the magnitude of the normal force that the surface of the banked curve exerts on the bicycle?

VP5.22.3 An aerobatic airplane flying at a constant 80.0 makes a horizontal turn of radius 175 m. The pilot has mass 80.0 kg.

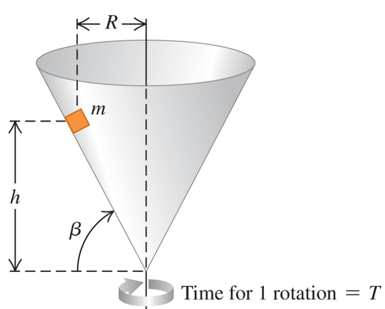
(a) What is the bank angle of the airplane? (b) What is the pilot's apparent weight during the turn? How many times greater than his actual weight is this?

VP5.22.4 A sports car moves around a banked curve at just the right constant speed so that no friction is needed to make the turn. During the turn, the driver (mass) feels as though she weighs times her actual weight. (a) Find the magnitude of the *net* force on the driver during the turn in terms of , , and . (b) Find the radius of the turn in terms of , , and .

Bridging Problem: In a Rotating Cone

A small block with mass is placed inside an inverted cone that is rotating about a vertical axis such that the time for one revolution of the cone is (Fig. 5.39). The walls of the cone make an angle with the horizontal. The coefficient of static friction between the block and the cone is If the block is to remain at a constant height above the apex of the cone, what are (a) the maximum value of and (b) the minimum value of ? (That is, find expressions for and in terms of and .)

Figure 5.39



A block inside a spinning cone.

Solution Guide

IDENTIFY AND SET UP

1. Although we want the block not to slide up or down on the inside of the cone, this is *not* an equilibrium problem. The block rotates with the cone and is in uniform circular motion, so it has an acceleration directed toward the center of its circular path.
2. Identify the forces on the block. What is the direction of the friction force when the cone is rotating as slowly as possible, so has its maximum value What is the direction of the friction force when the cone is rotating as rapidly as possible, so has its minimum value In these situations does the static friction force have its *maximum* magnitude? Why or why not?
3. Draw a free-body diagram for the block when the cone is rotating with and a free-body diagram when the cone is rotating with Choose coordinate axes, and remember that it's usually easiest to choose one of the axes to be in the direction of the acceleration.
4. What is the radius of the circular path that the block follows? Express this in terms of and .
5. List the unknown quantities, and decide which of these are the target variables.

EXECUTE

6. Write Newton's second law in component form for the case in which the cone is rotating with Write the acceleration in terms of , and h, and write the static friction force in terms of the normal force .
7. Solve these equations for the target variable

8. Repeat steps 6 and 7 for the case in which the cone is rotating with and solve for the target variable

EVALUATE

9. You'll end up with some fairly complicated expressions for and so check them over carefully. Do they have the correct units? Is the minimum time less than the maximum time as it must be?
10. What do your expressions for and become if ? Check your results by comparing them with Example 5.22 in Section 5.4.

Video Tutor Solution: Chapter 5 Bridging Problem

Questions/Exercises/Problems: Applying Newton's Laws

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

[Always assume that pulleys are frictionless and massless and that strings and cords are massless, unless otherwise noted.]

Discussion Questions

Q5.1 A man sits in a seat that is hanging from a rope. The rope passes over a pulley suspended from the ceiling, and the man holds the other end of the rope in his hands. What is the tension in the rope, and what force does the seat exert on him? Draw a free-body force diagram for the man.

Q5.2 "In general, the normal force is not equal to the weight." Give an example in which these two forces are equal in magnitude, and at least two examples in which they are not.

Q5.3 A clothesline hangs between two poles. No matter how tightly the line is stretched, it sags a little at the center. Explain why.

Q5.4 You drive a car up a steep hill at constant speed. Discuss all of the forces that act on the car. What pushes it up the hill?

Q5.5 For medical reasons, astronauts in outer space must determine their body mass at regular intervals. Devise a scheme for measuring body mass in an apparently weightless environment.

Q5.6 To push a box up a ramp, which requires less force: pushing horizontally or pushing parallel to the ramp? Why?

Q5.7 A woman in an elevator lets go of her briefcase, but it does not fall to the floor. How is the elevator moving?

Q5.8 A block rests on an inclined plane with enough friction to prevent it from sliding down. To start the block moving, is it easier to push it up the plane or down the plane? Why?

Q5.9 A crate slides up an inclined ramp and then slides down the ramp after momentarily stopping near the top. There is kinetic friction between the surface of the ramp and the crate. Which is greater? (i) The crate′s acceleration going up the ramp; (ii) the crate′s acceleration going down the ramp; (iii) both are the same. Explain.

Q5.10 A crate of books rests on a level floor. To move it along the floor at a constant velocity, why do you exert less force if you pull it at an angle θ above the horizontal than if you push it at the same angle below the horizontal?

Q5.11 In a world without friction, which of the following activities could you do (or not do)? Explain your reasoning. (a) Drive around an unbanked highway curve; (b) jump into the air; (c) start walking on a horizontal sidewalk; (d) climb a vertical ladder; (e) change lanes while you drive.

Q5.12 When you stand with bare feet in a wet bathtub, the grip feels fairly secure, and yet a catastrophic slip is quite possible. Explain this in terms of the two coefficients of friction.

Q5.13 You are pushing a large crate from the back of a freight elevator to the front as the elevator is moving to the next floor. In which situation is the force you must apply to move the crate the least, and in which is it the greatest: when the elevator is accelerating upward, when it is accelerating downward, or when it is traveling at constant speed? Explain.

Q5.14 It is often said that "friction always opposes motion." Give at least one example in which (a) static friction *causes* motion, and (b) kinetic friction *causes* motion.

Q5.15 If there is a net force on a particle in uniform circular motion, why doesn't the particle's speed change?

Q5.16 A curve in a road has a bank angle calculated and posted for 80 km/h. However, the road is covered with ice, so you cautiously plan to drive slower than this limit. What might happen to your car? Why?

Q5.17 You swing a ball on the end of a lightweight string in a horizontal circle at constant speed. Can the string ever be truly horizontal? If not, would it slope above the horizontal or below the horizontal? Why?

Q5.18 The centrifugal force is not included in the free-body diagrams of Figs. 5.34b and 5.35. Explain why not.

Q5.19 A professor swings a rubber stopper in a horizontal circle on the end of a string in front of his class. He tells Caroline, in the front row, that he is going to let the string go when the stopper is directly in front of her face. Should Caroline worry?

Q5.20 To keep the forces on the riders within allowable limits, many loop-the-loop roller coaster rides are designed so that the loop is not a perfect circle but instead has a larger radius of curvature at the bottom than at the top. Explain.

Q5.21 A tennis ball drops from rest at the top of a tall glass cylinder —first with the air pumped out of the cylinder so that there is no air resistance, and again after the air has been readmitted to the cylinder. You examine multiflash photographs of the two drops. Can you tell which photo belongs to which drop? If so, how?

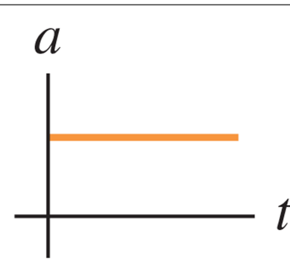
Q5.22 You throw a baseball straight upward with speed v_0. When the ball returns to the point from where you threw it, how does its speed compare to v_0 (a) in the absence of air resistance and (b) in the presence of air resistance? Explain.

Q5.23 You throw a baseball straight upward. If you do *not* ignore air resistance, how does the time required for the ball to reach its maximum height compare to the time required for it to fall from its maximum height back down to the height from which you threw it? Explain.

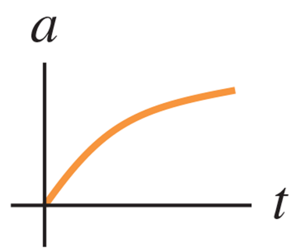
Q5.24 You have two identical tennis balls and fill one with water. You release both balls simultaneously from the top of a tall building. If air resistance is negligible, which ball will strike the ground first? Explain. What if air resistance is *not* negligible?

Q5.25 A ball is dropped from rest and feels air resistance as it falls. Which of the graphs in Fig. Q5.25 best represents its acceleration as a function of time?

Figure Q5.25



(a)



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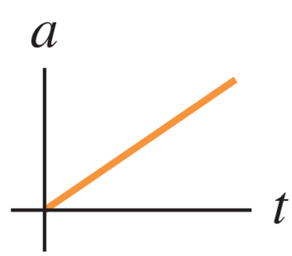
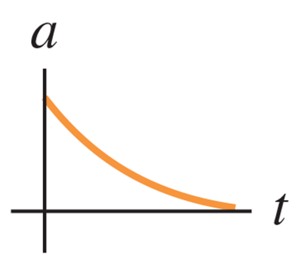
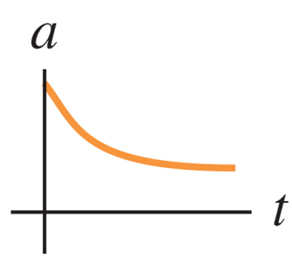


(c)



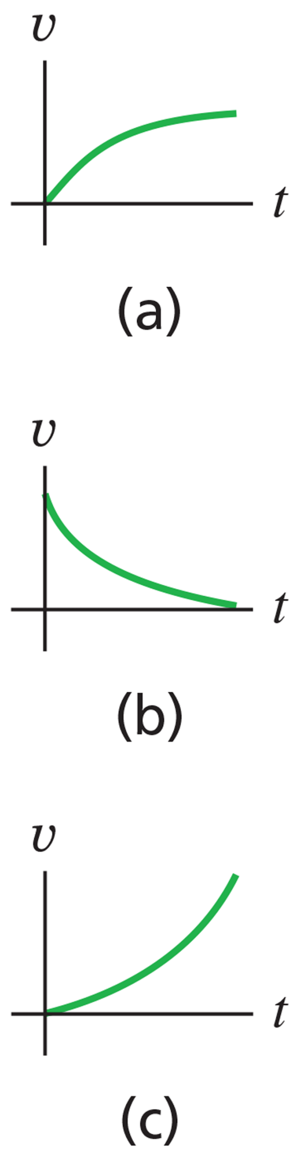
(d)

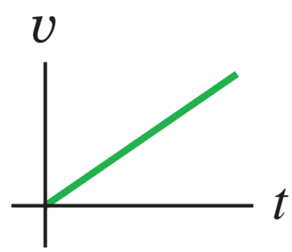


(e)

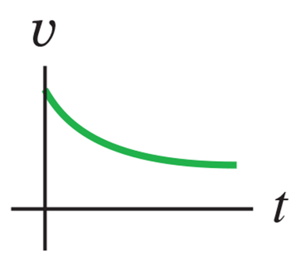
Q5.26 A ball is dropped from rest and feels air resistance as it falls. Which of the graphs in Fig. Q5.26 best represents its vertical velocity component as a function of time, if the +y-direction is taken to be downward?

Figure Q5.26





(d)



(e)

Q5.27 When a batted baseball moves with air drag, when does the ball travel a greater horizontal distance? (i) While climbing to its maximum height; (ii) while descending from its maximum height back to the ground; (iii) the same for both. Explain in terms of the forces acting on the ball.

Q5.28 "A ball is thrown from the edge of a high cliff. Regardless of the angle at which it is thrown, due to air resistance, the ball will eventually end up moving vertically downward." Justify this statement.

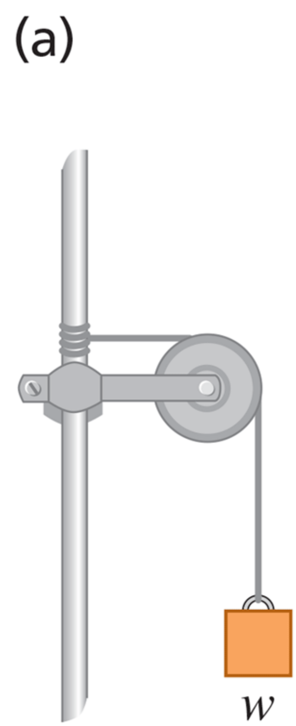
Exercises

Section 5.1 Using Newton's First Law: Particles in Equilibrium

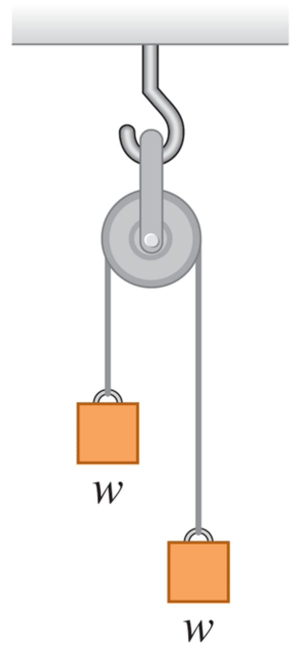
5.1 • Two 25.0 N weights are suspended at opposite ends of a rope that passes over a light, frictionless pulley. The pulley is attached to a chain from the ceiling. (a) What is the tension in the rope? (b) What is the tension in the chain?

5.2 • In Fig. E5.2 each of the suspended blocks has weight w. The pulleys are frictionless, and the ropes have negligible weight. In each case, draw a free-body diagram and calculate the tension T in the rope in terms of w.

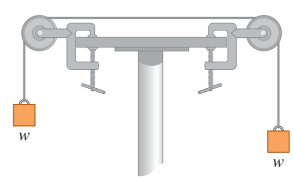
Figure E5.2



(b)



(c)

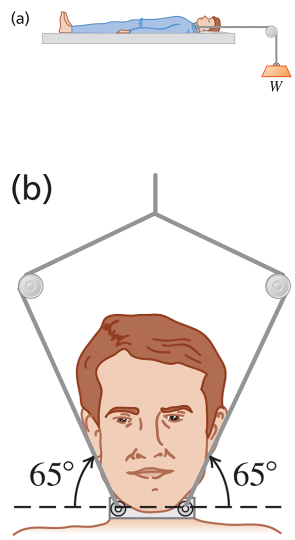


5.3 • A 75.0 kg wrecking ball hangs from a uniform, heavy-duty chain of mass 26.0 kg. (a) Find the maximum and minimum tensions in the chain. (b) What is the tension at a point three-fourths of the way up from the bottom of the chain?

5.4 •• BIO **Injuries to the Spinal Column.** In the treatment of spine injuries, it is often necessary to provide tension along the spinal column to stretch the backbone. One device for doing this is the Stryker frame (Fig. E5.4a). A weight w is attached to the patient (sometimes around a neck collar, Fig.

E5.4b), and friction between the person's body and the bed prevents sliding. (a) If the coefficient of static friction between a 78.5 kg patient's body and the bed is 0.75, what is the maximum traction force along the spinal column that w can provide without causing the patient to slide? (b) Under the conditions of maximum traction, what is the tension in each cable attached to the neck collar?

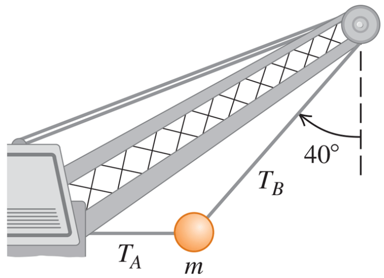
Figure E5.4



5.5 •• A picture frame hung against a wall is suspended by two wires attached to its upper corners. If the two wires make the same angle with the vertical, what must this angle be if the tension in each wire is equal to 0.75 of the weight of the frame? (Ignore any friction between the wall and the picture frame.)

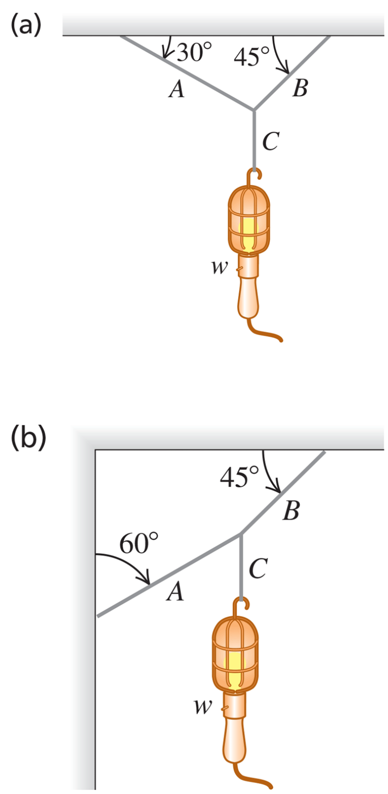
5.6 •• A large wrecking ball is held in place by two light steel cables (Fig. E5.6). If the mass m of the wrecking ball is 3620 kg, what are (a) the tension T_B in the cable that makes an angle of 40° with the vertical and (b) the tension T_A in the horizontal cable?

Figure E5.6



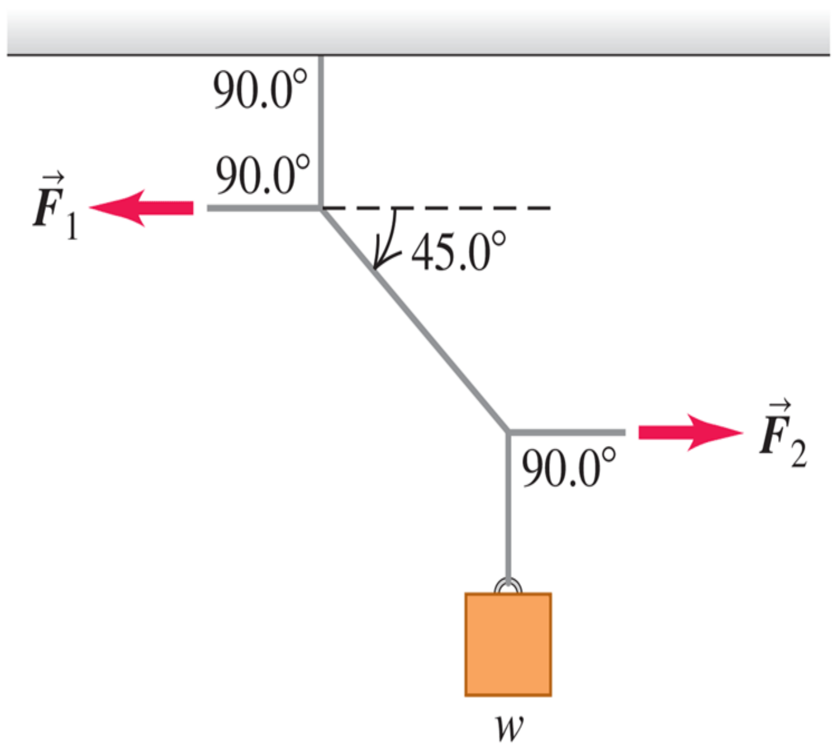
5.7 •• Find the tension in each cord in Fig. E5.7 if the weight of the suspended object is w.

Figure E5.7



5.8 •• In Fig. E5.8 the weight w is 60.0 N. (a) What is the tension in the diagonal string? (b) Find the magnitudes of the horizontal forces $\vec{F}_1$ and $\vec{F}_2$ that must be applied to hold the system in the position shown.

Figure E5.8



5.9 •• A man pushes on a piano with mass 180 kg; it slides at constant velocity down a ramp that is inclined at 19.0° above the horizontal floor. Neglect any friction acting on the piano. Calculate the magnitude of the force applied by the man if he pushes (a) parallel to the incline and (b) parallel to the floor.

Section 5.2 Using Newton's Second Law: Dynamics of Particles

5.10 •• **Apparent Weight**. A 550 N physics student stands on a bathroom scale in an elevator that is supported by a cable. The combined mass of student plus elevator is 850 kg. As the elevator starts moving, the scale reads 450 N. (a) Find the acceleration of the elevator (magnitude and direction). (b) What is the acceleration if the scale reads 670 N? (c) If the scale reads zero, should the student worry? Explain. (d) What is the tension in the cable in parts (a) and (c)?

5.11 •• BIO **Stay Awake!** An astronaut is inside a 2.25×10^6 kg rocket that is blasting off vertically from the launch pad. You want this rocket to reach the speed of sound (331 m/s) as quickly as possible, but astronauts are in danger of blacking out at an acceleration greater than $4g$. (a) What is the maximum initial thrust this rocket's engines can have but just barely avoid blackout? Start with a free-body diagram of the rocket. (b) What force, in terms of the astronaut's weight w, does the rocket exert on her? Start with a free-body diagram of the astronaut. (c) What is the shortest time it can take the rocket to reach the speed of sound?

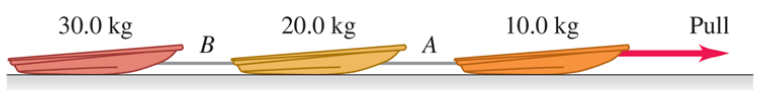
5.12 •• A rocket of initial mass 125 kg (including all the contents) has an engine that produces a constant vertical force (the *thrust*) of 1720 N. Inside this rocket, a 15.5 N electric power supply rests on the floor. (a) Find the initial acceleration of the rocket. (b) When the rocket initially accelerates, how hard does the floor push on the power supply? (*Hint*: Start with a free-body diagram for the power supply.)

5.13 •• CP ***Genesis* Crash**. On September 8, 2004, the *Genesis* spacecraft crashed in the Utah desert because its parachute did not open. The 210 kg capsule hit the ground at 311 km/h

and penetrated the soil to a depth of 81.0 cm. (a) What was its acceleration (in $\mathrm{m/s^2}$ and in g's), assumed to be constant, during the crash? (b) What force did the ground exert on the capsule during the crash? Express the force in newtons and as a multiple of the capsule's weight. (c) How long did this force last?

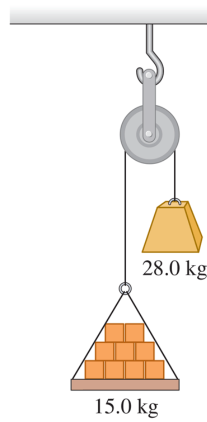
5.14 • Three sleds are being pulled horizontally on frictionless horizontal ice using horizontal ropes (Fig. E5.14). The pull is of magnitude 190 N. Find (a) the acceleration of the system and (b) the tension in ropes A and B.

Figure E5.14



5.15 •• **Atwood's Machine.** A 15.0 kg load of bricks hangs from one end of a rope that passes over a small, frictionless pulley. A 28.0 kg counterweight is suspended from the other end of the rope (Fig. E5.15). The system is released from rest. (a) Draw two free-body diagrams, one for the load of bricks and one for the counterweight. (b) What is the magnitude of the upward acceleration of the load of bricks? (c) What is the tension in the rope while the load is moving? How does the tension compare to the weight of the load of bricks? To the weight of the counterweight?

Figure E5.15



5.16 •• CP An 8.00 kg block of ice, released from rest at the top of a 1.50-m-long frictionless ramp, slides downhill, reaching a speed of 2.50 m/s at the bottom. (a) What is the angle between the ramp and the horizontal? (b) What would be the speed of the ice at the bottom if the motion were opposed by a constant friction force of 10.0 N parallel to the surface of the ramp?

5.17 •• A light rope is attached to a block with mass 4.00 kg that rests on a frictionless, horizontal surface. The horizontal rope passes over a frictionless, massless pulley, and a block with mass m is suspended from the other end. When the blocks are released, the tension in the rope is 15.0 N. (a) Draw two free-body diagrams: one for each block. (b) What is the acceleration of either block? (c) Find m. (d) How does the tension compare to the weight of the hanging block?

5.18 •• CP **Runway Design**. A transport plane takes off from a level landing field with two gliders in tow, one behind the other. The mass of each glider is 700 kg, and the total resistance (air drag plus friction with the runway) on each may be assumed constant and equal to 2500 N. The tension in the towrope between the transport plane and the first glider is not to exceed 12,000 N. (a) If a speed of 40 m/s is required for takeoff, what minimum length of runway is needed? (b) What is the tension in the towrope between the two gliders while they are accelerating for the takeoff?

5.19 •• CP BIO **Force During a Jump**. When jumping straight up from a crouched position, an average person can reach a maximum height of about 60 cm. During the jump, the person's body from the knees up typically rises a distance of around 50 cm. To keep the calculations simple and yet get a reasonable result, assume that the *entire body* rises this much during the jump. (a) With what initial speed does the person leave the ground to reach a height of 60 cm? (b) Draw a free-body diagram of the person during the jump. (c) In terms of this jumper's weight w, what force does the ground exert on him or her during the jump?

5.20 CP CALC A 2540 kg test rocket is launched vertically from the launch pad. Its fuel (of negligible mass) provides a thrust force such that its vertical velocity as a function of time is given by $v(t) = At + Bt^2$, where A and B are constants and time is measured from the instant the fuel is ignited. The rocket has an upward acceleration of $1.50\ \mathrm{m/s^2}$ at the instant of ignition and, 1.00 s later, an upward velocity of 2.00 m/s. (a) Determine A and B, including their SI units. (b) At 4.00 s after fuel ignition, what is the acceleration of the rocket, and (c) what thrust force does the burning fuel exert on it, assuming no air resistance? Express the thrust in newtons and as a multiple of the rocket's weight. (d) What was the initial thrust due to the fuel?

5.21 •• CP CALC A 2.00 kg box is moving to the right with speed 9.00 m/s on a horizontal, frictionless surface. At $t = 0$ a horizontal force is applied to the box. The force is directed to the left and has magnitude $F(t) = (6.00\ \mathrm{N/s^2})t^2$. (a) What distance does the box move from its position at $t = 0$ before its speed is reduced to zero? (b) If the force continues to be applied, what is the speed of the box at $t = 3.00$ s?

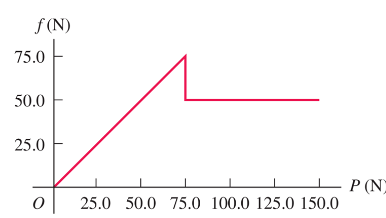
5.22 •• **CP CALC** A 5.00 kg crate is suspended from the end of a short vertical rope of negligible mass. An upward force $F(t)$ is applied to the end of the rope, and the height of the crate above its initial position is given by $y(t) = (2.80\ \mathrm{m/s})t + (0.610\ \mathrm{m/s^3})t^3$. What is the magnitude of F when $t = 4.00$ s?

Section 5.3 Friction Forces

5.23 • BIO **The Trendelenburg Position.** After emergencies with major blood loss, a patient is placed in the Trendelenburg position, in which the foot of the bed is raised to get maximum blood flow to the brain. If the coefficient of static friction between a typical patient and the bedsheets is 1.20, what is the maximum angle at which the bed can be tilted with respect to the floor before the patient begins to slide?

5.24 • In a laboratory experiment on friction, a 135 N block resting on a rough horizontal table is pulled by a horizontal wire. The pull gradually increases until the block begins to move and continues to increase thereafter. Figure E5.24 shows a graph of the friction force on this block as a function of the pull. (a) Identify the regions of the graph where static friction and kinetic friction occur. (b) Find the coefficients of static friction and kinetic friction between the block and the table. (c) Why does the graph slant upward at first but then level out? (d) What would the graph look like if a 135 N brick were placed on the block, and what would the coefficients of friction be?

Figure E5.24



5.25 • • CP A stockroom worker pushes a box with mass 16.8 kg on a horizontal surface with a constant speed of 3.50 m/s. The coefficient of kinetic friction between the box and the surface is 0.20. (a) What horizontal force must the worker apply to maintain the motion? (b) If the force calculated in part (a) is removed, how far does the box slide before coming to rest?

5.26 •• A box of bananas weighing 40.0 N rests on a horizontal surface. The coefficient of static friction between the box and the surface is 0.40, and the coefficient of kinetic friction is 0.20. (a) If no horizontal force is applied to the box and the box is at rest, how large is the friction force exerted on it? (b) What is the magnitude of the friction force if a monkey applies a horizontal force of 6.0 N to the box and the box is initially at rest? (c) What minimum horizontal force must the monkey apply to start the box in motion? (d) What minimum horizontal force must the monkey apply to keep the box moving at constant velocity once it has been started? (e) If the monkey applies a horizontal force of 18.0 N, what is the magnitude of the friction force and what is the box's acceleration?

5.27 •• A 45.0 kg crate of tools rests on a horizontal floor. You exert a gradually increasing horizontal push on it, and the crate just begins to move when your force exceeds 313 N. Then you must reduce your push to 208 N to keep it moving at a steady 25.0 cm/s. (a) What are the coefficients of static and kinetic friction between the crate and the floor? (b) What push must you exert to give it an acceleration of 1.10 m/s^2? (c) Suppose you were performing the same experiment on the moon, where the acceleration due to gravity is 1.62 m/s^2. (i) What magnitude push would cause it to move? (ii) What would its acceleration be if you maintained the push in part (b)?

5.28 • Consider the heaviest box that you can push at constant speed across a level floor, where the coefficient of kinetic friction is 0.50, and estimate the maximum horizontal force that you can apply to the box. A box sits on a ramp that is inclined at an angle of 60˚ above the horizontal. The coefficient of kinetic friction between the box and the ramp is 0.50. If you apply the same magnitude force, now parallel to the ramp, that you applied to the box on the floor, what is the heaviest box (in pounds) that you can push up the ramp at constant speed? (In both cases assume you can give enough extra push to get the box started moving.)

5.29 • **CP** Estimate the height of a typical playground slide and the angle its surface makes with the horizontal. (a) Some children like to slide down while sitting on a sheet of wax paper. This makes the friction force exerted by the slide very small. If a child starts from rest and we take the friction force to be zero, what is the speed of the child when he reaches the bottom of the slide? (b) If the child doesn't use the wax paper, his speed at the bottom is half the value calculated in part (a). What is the coefficient of kinetic friction between the child and the slide when wax paper isn't used? (c) A child wearing a different sort of clothing than the first child climbs the ladder to the top of the slide, sits on the slide, lets go of the handrail, and remains at rest. What is the minimum possible value for the coefficient of static friction between this child and the surface of the slide?

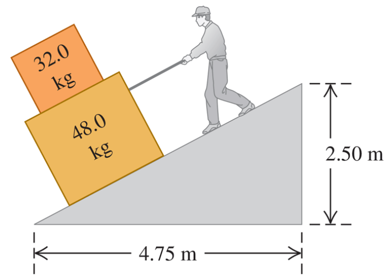
5.30 •• Some sliding rocks approach the base of a hill with a speed of 12 m/s. The hill rises at 36° above the horizontal and has coefficients of kinetic friction and static friction of 0.45 and 0.65, respectively, with these rocks. (a) Find the acceleration of the rocks as they slide up the hill. (b) Once a rock reaches its highest point, will it stay there or slide down the hill? If it stays, show why. If it slides, find its acceleration on the way down.

5.31 •• A box with mass 10.0 kg moves on a ramp that is inclined at an angle of 55.0° above the horizontal. The coefficient of kinetic friction between the box and the ramp surface is $\mu_k = 0.300$. Calculate the magnitude of the acceleration of the box if you push on the box with a constant force $F = 120.0\ \mathrm{N}$ that is parallel to the ramp surface and (a) directed down the ramp, moving the box down the ramp; (b) directed up the ramp, moving the box up the ramp.

5.32 •• A pickup truck is carrying a toolbox, but the rear gate of the truck is missing. The toolbox will slide out if it is set moving. The coefficients of kinetic friction and static friction between the box and the level bed of the truck are 0.355 and 0.650, respectively. Starting from rest, what is the shortest time this truck could accelerate uniformly to 30.0 m/s without causing the box to slide? Draw a free-body diagram of the toolbox.

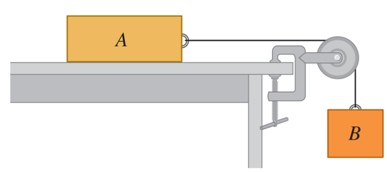
5.33 •• You are lowering two boxes, one on top of the other, down a ramp by pulling on a rope parallel to the surface of the ramp (Fig. E5.33). Both boxes move together at a constant speed of 15.0 cm/s. The coefficient of kinetic friction between the ramp and the lower box is 0.444, and the coefficient of static friction between the two boxes is 0.800. (a) What force do you need to exert to accomplish this? (b) What are the magnitude and direction of the friction force on the upper box?

Figure E5.33



5.34 •• Consider the system shown in Fig. E5.34. Block A weighs 45.0 N, and block B weighs 25.0 N. Once block B is set into downward motion, it descends at a constant speed. (a) Calculate the coefficient of kinetic friction between block A and the tabletop. (b) A cat, also of weight 45.0 N, falls asleep on top of block A. If block B is now set into downward motion, what is its acceleration (magnitude and direction)?

Figure E5.34

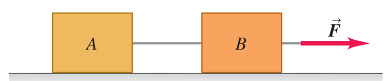


5.35 •• CP **Stopping Distance.** (a) If the coefficient of kinetic friction between tires and dry pavement is 0.80, what is the shortest distance in which you can stop a car by locking the brakes when the car is traveling at 28.7 m/s (about 65 mi/h) ? (b) On wet pavement the coefficient of kinetic friction may be only 0.25. How fast should you drive on wet pavement to be able to stop in the same distance as in part (a)? (*Note*: Locking the brakes is *not* the safest way to stop.)

5.36 •• CP A 25.0 kg box of textbooks rests on a loading ramp that makes an angle α with the horizontal. The coefficient of kinetic friction is 0.25, and the coefficient of static friction is 0.35. (a) As α is increased, find the minimum angle at which the box starts to slip. (b) At this angle, find the acceleration once the box has begun to move. (c) At this angle, how fast will the box be moving after it has slid 5.0 m along the loading ramp?

5.37 • Two crates connected by a rope lie on a horizontal surface (Fig. E5.37). Crate A has mass m_A, and crate B has mass m_B. The coefficient of kinetic friction between each crate and the surface is μ_k. The crates are pulled to the right at constant velocity by a horizontal force $\vec{F}$. Draw one or more free-body diagrams to calculate the following in terms of m_A, m_B, and μ_k: (a) the magnitude of $\vec{F}$ and (b) the tension in the rope connecting the blocks.

Figure E5.37



5.38 •• A box with mass m is dragged across a level floor with coefficient of kinetic friction μ_k by a rope that is pulled upward at an angle θ above the horizontal with a force of magnitude F. (a) In terms of m, μ_k, θ, and g, obtain an expression for the magnitude of the force required to move the box with constant speed. (b) Knowing that you are studying physics, a CPR instructor asks you how much force it would take to slide a 90 kg patient across a floor at constant speed by pulling on him at an angle of 25° above the horizontal. By dragging weights wrapped in an old pair of pants down the hall with a spring balance, you find that $\mu_k = 0.35$. Use the result of part (a) to answer the instructor's question.

5.39 •• **CP** As shown in Fig. E5.34, block A (mass 2.25 kg) rests on a tabletop. It is connected by a horizontal cord passing over a light, frictionless pulley to a hanging block B (mass 1.30 kg). The coefficient of kinetic friction between block A and the tabletop is 0.450. The blocks are released then from rest. Draw one or more free-body diagrams to find (a) the speed of each block after they move 3.00 cm and (b) the tension in the cord.

5.40 •• You throw a baseball straight upward. The drag force is proportional to v^2. In terms of g, what is the y-component of the ball's acceleration when the ball's speed is half its terminal speed and (a) it is moving up? (b) It is moving back down?

5.41 •• A large crate with mass m rests on a horizontal floor. The coefficients of friction between the crate and the floor are μ_s and μ_k. A woman pushes downward with a force $\vec{F}$ on the crate at an angle θ below the horizontal. (a) What magnitude of force $\vec{F}$ is required to keep the crate moving at constant velocity? (b) If μ_s is greater than some critical value, the woman cannot start the crate moving no matter how hard she pushes. Calculate this critical value of μ_s.

Section 5.4 Dynamics of Circular Motion

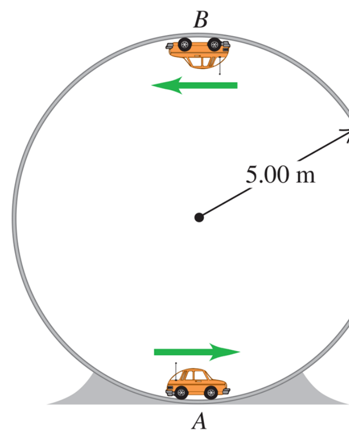
5.42 • You are sitting on the edge of a horizontal disk (for example, a playground merry-go-round) that has radius 3.00 m and is rotating at a constant rate about a vertical axis. (a) If the coefficient of static friction between you and the surface of the disk is 0.400, what is the minimum time for one revolution of the disk if you are not to slide off? (b) Your friend's weight is half yours. If the coefficient of static friction for him is the same as for you, what is the minimum time for one revolution if he is not to slide off?

5.43 • A stone with mass 0.80 kg is attached to one end of a string 0.90 m long. The string will break if its tension exceeds 60.0 N. The stone is whirled in a horizontal circle on a frictionless tabletop; the other end of the string remains fixed. (a) Draw a free-body diagram of the stone. (b) Find the maximum speed the stone can attain without the string breaking.

5.44 • BIO **Force on a Skater's Wrist.** A 52 kg ice skater spins about a vertical axis through her body with her arms horizontally outstretched; she makes 2.0 turns each second. The distance from one hand to the other is 1.50 m. Biometric measurements indicate that each hand typically makes up about 1.25% of body weight. (a) Draw a free-body diagram of one of the skater's hands. (b) What horizontal force must her wrist exert on her hand? (c) Express the force in part (b) as a multiple of the weight of her hand.

5.45 •• A small remote-controlled car with mass 1.60 kg moves at a constant speed of $v = 12.0$ m/s in a track formed by a vertical circle inside a hollow metal cylinder that has a radius of 5.00 m (Fig. E5.45). What is the magnitude of the normal force exerted on the car by the walls of the cylinder at (a) point A (bottom of the track) and (b) point B (top of the track)?

Figure E5.45



5.46 •• A small car with mass 0.800 kg travels at constant speed on the inside of a track that is a vertical circle with radius 5.00 m (Fig. E5.45). If the normal force exerted by the track on the car when it is at the top of the track (point B) is 6.00 N, what is the normal force on the car when it is at the bottom of the track (point A)?

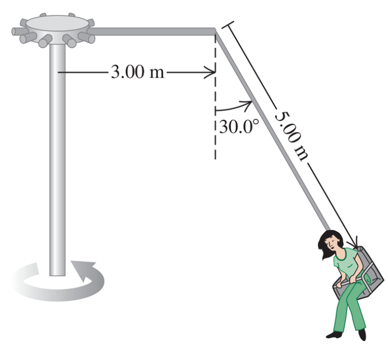
5.47 • A small model car with mass m travels at constant speed on the inside of a track that is a vertical circle with radius 5.00 m (Fig. E5.45). If the normal force exerted by the track on the car when it is at the bottom of the track (point A) is equal to $2.50mg$, how much time does it take the car to complete one revolution around the track?

5.48 • A flat (unbanked) curve on a highway has a radius of 170.0 m. A car rounds the curve at a speed of 25.0 m/s. (a) What is the minimum coefficient of static friction that will prevent sliding? (b) Suppose that the highway is icy and the coefficient of static friction between the tires and pavement is only one-third of what you found in part (a). What should be the maximum speed of the car so that it can round the curve safely?

5.49 •• A 1125 kg car and a 2250 kg pickup truck approach a curve on a highway that has a radius of 225 m. (a) At what angle should the highway engineer bank this curve so that vehicles traveling at 65.0 mi/h can safely round it regardless of the condition of their tires? Should the heavy truck go slower than the lighter car? (b) As the car and truck round the curve at 65.0 mi/h, find the normal force on each one due to the highway surface.

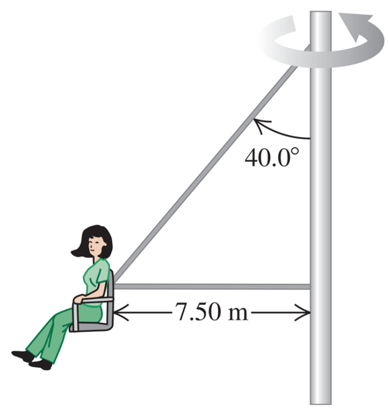
5.50 •• The "Giant Swing" at a county fair consists of a vertical central shaft with a number of horizontal arms attached at its upper end. Each arm supports a seat suspended from a cable 5.00 m long, and the upper end of the cable is fastened to the arm at a point 3.00 m from the central shaft (Fig. E5.50). (a) Find the time of one revolution of the swing if the cable supporting a seat makes an angle of 30.0° with the vertical. (b) Does the angle depend on the weight of the passenger for a given rate of revolution?

Figure E5.50



5.51 •• In another version of the "Giant Swing" (see Exercise 5.50), the seat is connected to two cables, one of which is horizontal (Fig. E5.51). The seat swings in a horizontal circle at a rate of 28.0 rpm (rev/min). If the seat weighs 255 N and an 825 N person is sitting in it, find the tension in each cable.

Figure E5.51



5.52 • A steel ball with mass m is suspended from the ceiling at the bottom end of a light, 15.0-m-long rope. The ball swings back and forth like a pendulum. When the ball is at its lowest point and the rope is vertical, the tension in the rope is three times the weight of the ball, so $T = 3mg$. (a) What is the speed of the ball as it swings through this point? (b) What is the speed of the ball if $T = mg$ at this point, where the rope is vertical?

5.53 •• **Rotating Space Stations**. One problem for humans living in outer space is that they are apparently weightless. One way around this problem is to design a space station that spins about its center at a constant rate. This creates "artificial gravity" at the outside rim of the station. (a) If the diameter of the space station is 800 m, how many revolutions per minute are needed for the "artificial gravity" acceleration to be $9.80\ \mathrm{m/s^2}$? (b) If the space station is a waiting area for travelers going to Mars, it might be desirable to simulate the acceleration due to gravity on the Martian surface ($3.70\ \mathrm{m/s^2}$). How many revolutions per minute are needed in this case?

5.54 • The Cosmo Clock 21 Ferris wheel in Yokohama, Japan, has a diameter of 100 m. Its name comes from its 60 arms, each of which can function as a second hand (so that it makes one revolution every 60.0 s). (a) Find the speed of the passengers when the Ferris wheel is rotating at this rate. (b) A passenger weighs 882 N at the weight-guessing booth on the ground. What is his apparent weight at the highest and at the lowest point on the Ferris wheel? (c) What would be the time for one revolution if the passenger's apparent weight at the highest point were zero? (d) What then would be the passenger's apparent weight at the lowest point?

5.55 •• A small rock with mass m is attached to a light string of length L and whirled in a vertical circle of radius R. (a) What is the minimum speed v at the rock's highest point for which it stays in a circular path? (b) If the speed at the rock's lowest point in its circular path is twice the value found in part (a), what is the tension in the string when the rock is at this point?

5.56 •• A 50.0 kg stunt pilot who has been diving her airplane vertically pulls out of the dive by changing her course to a circle in a vertical plane. (a) If the plane's speed at the lowest point of the circle is 95.0 m/s, what is the minimum radius of the circle so that the acceleration at this point will not exceed $4.00g$? (b) What is the apparent weight of the pilot at the lowest point of the pullout?

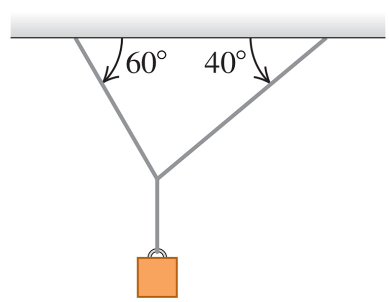
5.57 • **Stay Dry!** You tie a cord to a pail of water and swing the pail in a vertical circle of radius 0.600 m. What minimum speed must you give the pail at the highest point of the circle to avoid spilling water?

5.58 •• A bowling ball weighing 71.2 N (16.0 lb) is attached to the ceiling by a 3.80 m rope. The ball is pulled to one side and released; it then swings back and forth as a pendulum. As the rope swings through the vertical, the speed of the bowling ball is 4.20 m/s. At this instant, what are (a) the acceleration of the bowling ball, in magnitude and direction, and (b) the tension in the rope?

Problems

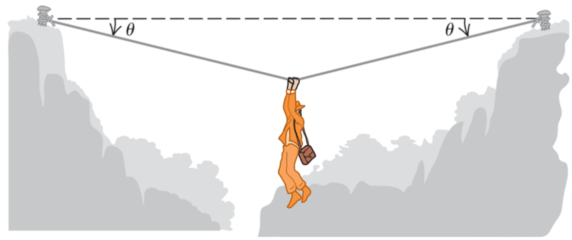
5.59 ••• Two ropes are connected to a steel cable that supports a hanging weight (Fig. P5.59). (a) Draw a free-body diagram showing all of the forces acting at the knot that connects the two ropes to the steel cable. Based on your diagram, which of the two ropes will have the greater tension? (b) If the maximum tension either rope can sustain without breaking is 5000 N, determine the maximum value of the hanging weight that these ropes can safely support. Ignore the weight of the ropes and of the steel cable.

Figure P5.59



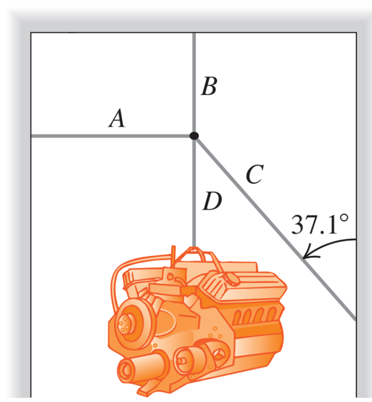
5.60 •• An adventurous archaeologist crosses between two rock cliffs by slowly going hand over hand along a rope stretched between the cliffs. He stops to rest at the middle of the rope (Fig. P5.60). The rope will break if the tension in it exceeds 2.50×10^4 N, and our hero's mass is 90.0 kg. (a) If the angle θ is 10.0°, what is the tension in the rope? (b) What is the smallest value θ can have if the rope is not to break?

Figure P5.60



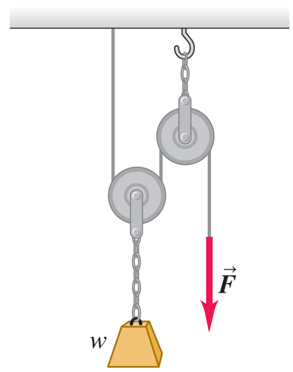
5.61 •• In a repair shop a truck engine that has mass 409 kg is held in place by four light cables (Fig. P5.61). Cable A is horizontal, cables B and D are vertical, and cable C makes an angle of 37.1° with a vertical wall. If the tension in cable A is 722 N, what are the tensions in cables B and C?

Figure P5.61



5.62 •• In Fig. P5.62 a worker lifts a weight w by pulling down on a rope with a force $\vec{F}$. The upper pulley is attached to the ceiling by a chain, and the lower pulley is attached to the weight by another chain. Draw one or more free-body diagrams to find the tension in each chain and the magnitude of $\vec{F}$, in terms of w, if the weight is lifted at constant speed. Assume that the rope, pulleys, and chains have negligible weights.

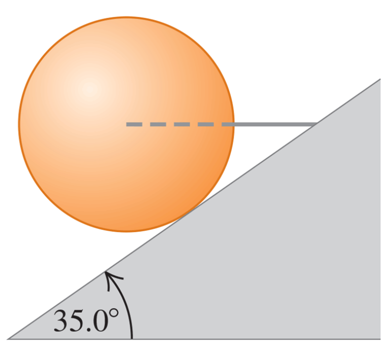
Figure P5.62



5.63 •• CP A small block sits at one end of a flat board that is 3.00 m long. The coefficients of friction between the block and the board are $\mu_s = 0.600$ and $\mu_k = 0.400$. The end of the board where the block sits is slowly raised until the angle the board makes with the horizontal is α_0, and then the block starts to slide down the board. If the angle is kept equal to α_0 as the block slides, what is the speed of the block when it reaches the bottom of the board?

5.64 ••• A horizontal wire holds a solid uniform ball of mass m in place on a tilted ramp that rises 35.0° above the horizontal. The surface of this ramp is perfectly smooth, and the wire is directed away from the center of the ball (Fig. P5.64). (a) Draw a free-body diagram of the ball. (b) How hard does the surface of the ramp push on the ball? (c) What is the tension in the wire?

Figure P5.64



5.65 •• CP A box of mass 12.0 kg sits at rest on a horizontal surface. The coefficient of kinetic friction between the surface and the box is 0.300. The box is initially at rest, and then a constant force of magnitude F and direction 37.0° below the horizontal is applied to the box; the box slides along the surface. (a) What is F if the box has a speed of 6.00 m/s after traveling a distance of 8.00 m? (b) What is F if the surface is frictionless and all the other quantities are the same? (c) What is F if all the quantities are the same as in part (a) but the force applied to the box is horizontal?

5.66 •• CP A box is sliding with a constant speed of 4.00 m/s in the $+x$-direction on a horizontal, frictionless surface. At $x = 0$ the box encounters a rough patch of the surface, and then the surface becomes even rougher. Between $x = 0$ and $x = 2.00$ m, the coefficient of kinetic friction between the box and the surface is 0.200; between $x = 2.00$ m and $x = 4.00$ m, it is 0.400. (a) What is the x-coordinate of the point where the box comes to rest? (b) How much time does it take the box to come to rest after it first encounters the rough patch at $x = 0$?

5.67 •• CP BIO **Forces During Chin-ups**. When you do a chin-up, you raise your chin just over a bar (the chinning bar), supporting yourself with only your arms. Typically, the body below the arms is raised by about 30 cm in a time of 1.0 s, starting from rest. Assume that the entire body of a 680 N person doing chin-ups is raised by 30 cm, and that half the 1.0 s is spent accelerating upward and the other half accelerating downward, uniformly in both cases. Draw a free-body diagram of the person's body, and use it to find the force his arms must exert on him during the accelerating part of the chin-up.

5.68 •• CP CALC A 2.00 kg box is suspended from the end of a light vertical rope. A time-dependent force is applied to the upper end of the rope, and the box moves upward with a velocity magnitude that varies in time according to $v(t) = (2.00 \text{ m/s}^2)t + (0.600 \text{ m/s}^3)t^2$. What is the tension in the rope when the velocity of the box is 9.00 m/s?

5.69 •••**CALC** A 3.00 kg box that is several hundred meters above the earth's surface is suspended from the end of a short vertical rope of negligible mass. A time-dependent upward force is applied to the upper end of the rope and results in a tension in the rope of $T(t) = (36.0 \text{ N/s})t$. The box is at rest at $t = 0$. The only forces on the box are the tension in the rope and gravity. (a) What is the velocity of the box at (i) $t = 1.00$ s and (ii) $t = 3.00$ s? (b) What is the maximum distance that the box descends below its initial position? (c) At what value of T does the box return to its initial position?

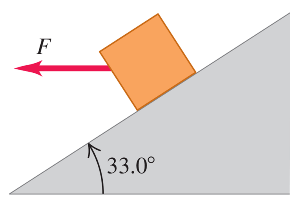
5.70 ••**CP** A 5.00 kg box sits at rest at the bottom of a ramp that is 8.00 m long and is inclined at 30.0° above the horizontal. The coefficient of kinetic friction is $\mu_k = 0.40$, and the coefficient of static friction is $\mu_s = 0.43$. What constant force F, applied parallel to the surface of the ramp, is required to push the box to the top of the ramp in a time of 6.00 s?

5.71 •• When a crate with mass 25.0 kg is placed on a ramp that is inclined at an angle α below the horizontal, it slides down the ramp with an acceleration of 4.9 m/s^2. The ramp is not frictionless. To increase the acceleration of the crate, a downward vertical force $\vec{F}$ is applied to the top of the crate. What must F be in order to increase the acceleration of the crate so that it is 9.8 m/s^2? How does the value of F that you calculate compare to the weight of the crate?

5.72 •• A large crate is at rest on a horizontal floor. The coefficient of static friction between the crate and the floor is 0.400. A force $\vec{F}$ is applied to the crate in a direction 30.0° above the horizontal. The minimum value of F required to get the crate to start sliding is 380 N. What is the mass of the crate?

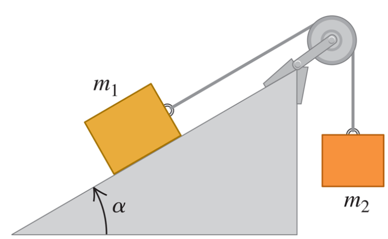
5.73 •• CP An 8.00 kg box sits on a ramp that is inclined at 33.0° above the horizontal. The coefficient of kinetic friction between the box and the surface of the ramp is $\mu_k = 0.300$. A constant *horizontal* force $F = 26.0$ N is applied to the box (Fig. P5.73), and the box moves down the ramp. If the box is initially at rest, what is its speed 2.00 s after the force is applied?

Figure P5.73



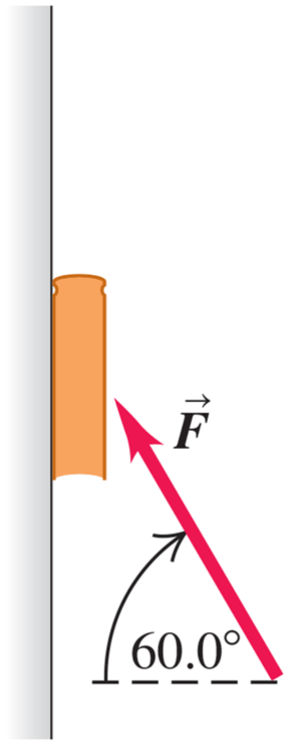
5.74 •• CP In Fig. P5.74, $m_1 = 20.0$ kg and $\alpha = 53.1°$. The coefficient of kinetic friction between the block of mass m_1 and the incline is $\mu_k = 0.40$. What must be the mass m_2 of the hanging block if it is to descend 12.0 m in the first 3.00 s after the system is released from rest?

Figure P5.74



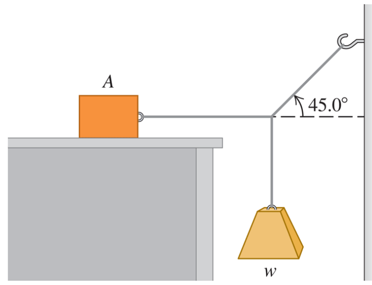
5.75 •• CP You place a book of mass 5.00 kg against a vertical wall. You apply a constant force $\vec{F}$ to the book, where $F = 96.0$ N and the force is at an angle of 60.0° above the horizontal (Fig. P5.75). The coefficient of kinetic friction between the book and the wall is 0.300. If the book is initially at rest, what is its speed after it has traveled 0.400 m up the wall?

Figure P5.75



5.76 •• Block A in Fig. P5.76 weighs 60.0 N. The coefficient of static friction between the block and the surface on which it rests is 0.25. The weight w is 12.0 N, and the system is in equilibrium. (a) Find the friction force exerted on block A. (b) Find the maximum weight w for which the system will remain in equilibrium.

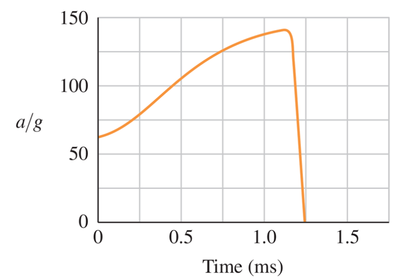
Figure P5.76



5.77 •• A block with mass m_1 is placed on an inclined plane with slope angle α and is connected to a hanging block with mass m_2 by a cord passing over a small, frictionless pulley (Fig. P5.74). The coefficient of static friction is μ_s, and the coefficient of kinetic friction is μ_k. (a) Find the value of m_2 for which the block of mass m_1 moves up the plane at constant speed once it is set in motion. (b) Find the value of m_2 for which the block of mass m_1 moves down the plane at constant speed once it is set in motion. (c) For what range of values of m_2 will the blocks remain at rest if they are released from rest?

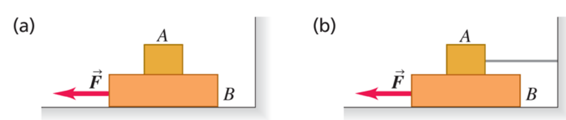
5.78 •• DATA BIO **The Flying Leap of a Flea.** High-speed motion pictures (3500 frames/second) of a jumping 210 μg flea yielded the data to plot the flea's acceleration as a function of time, as shown in Fig. P5.78. (See "The Flying Leap of the Flea," by M. Rothschild et al., *Scientific American*, November 1973.) This flea was about 2 mm long and jumped at a nearly vertical takeoff angle. Using the graph, (a) find the *initial* net external force on the flea. How does it compare to the flea's weight? (b) Find the *maximum* net external force on this jumping flea. When does this maximum force occur? (c) Use the graph to find the flea's maximum speed.

Figure P5.78



5.79 •• Block A in Fig. P5.79 weighs 1.20 N, and block B weighs 3.60 N. The coefficient of kinetic friction between all surfaces is 0.300. Find the magnitude of the horizontal force $\vec{F}$ necessary to drag block B to the left at constant speed (a) if A rests on B and moves with it (Fig. P5.79a), (b) if A is held at rest (Fig. P5.79b).

Figure P5.79



5.80 ••• CP **Elevator Design**. You are designing an elevator for a hospital. The force exerted on a passenger by the floor of the elevator is not to exceed 1.60 times the passenger's weight. The elevator accelerates upward with constant acceleration for a distance of 3.0 m and then starts to slow down. What is the maximum speed of the elevator?

5.81 ••• CP CALC You are standing on a bathroom scale in an elevator in a tall building. Your mass is 64 kg. The elevator starts from rest and travels upward with a speed that varies with time according to $v(t) = (3.0\ \mathrm{m/s^2})t + (0.20\ \mathrm{m/s^3})t^2$. When $t = 4.0$ s, what is the reading on the bathroom scale?

5.82 •• CP Consider the system shown in Fig. E5.34. As in Exercise 5.34, block A weighs 45.0 N and block B weighs

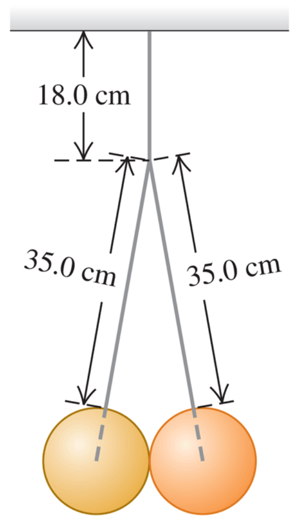
25.0 N. The system is released from rest, and block B has speed 3.30 m/s after it has descended 2.00 m. (a) While the blocks are moving, what is the tension in the rope connecting the blocks? (b) What is the coefficient of kinetic friction between block A and the tabletop?

5.83 •• **CP** Two blocks are suspended from opposite ends of a light rope that passes over a light, frictionless pulley. One block has mass m_1 and the other has mass m_2, where $m_2 > m_1$. The two blocks are released from rest, and the block with mass m_2 moves downward 5.00 m in 2.00 s after being released. While the blocks are moving, the tension in the rope is 16.0 N. Calculate m_1 and m_2.

5.84 ••• If the coefficient of static friction between a table and a uniform, massive rope is μ_s, what fraction of the rope can hang over the edge of the table without the rope sliding?

5.85 ••• Two identical 15.0 kg balls, each 25.0 cm in diameter, are suspended by two 35.0 cm wires (Fig. P5.85). The entire apparatus is supported by a single 18.0 cm wire, and the surfaces of the balls are perfectly smooth. (a) Find the tension in each of the three wires. (b) How hard does each ball push on the other one?

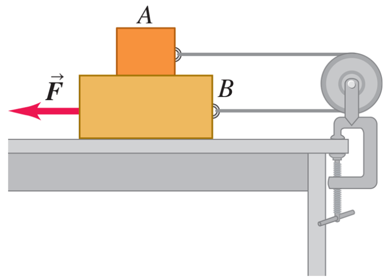
Figure P5.85



5.86 • CP **Traffic Court**. You are called as an expert witness in a trial for a traffic violation. The facts are these: A driver slammed on his brakes and came to a stop with constant acceleration. Measurements of his tires and the skid marks on the pavement indicate that he locked his car's wheels, the car traveled 192 ft before stopping, and the coefficient of kinetic friction between the road and his tires was 0.750. He was charged with speeding in a 45 mi/h zone but pleads innocent. What is your conclusion: guilty or innocent? How fast was he going when he hit his brakes?

5.87 ••• Block A in Fig. P5.87 weighs 1.90 N, and block B weighs 4.20 N. The coefficient of kinetic friction between all surfaces is 0.30. Find the magnitude of the horizontal force $\vec{F}$ necessary to drag block B to the left at constant speed if A and B are connected by a light, flexible cord passing around a fixed, frictionless pulley.

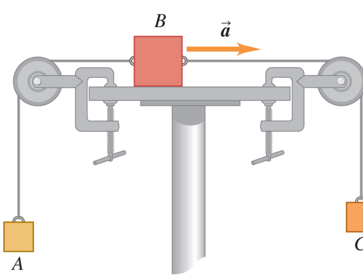
Figure P5.87



5.88 •• Block B has mass 5.00 kg and sits at rest on a horizontal, frictionless surface. Block A has mass 2.00 kg and sits at rest on top of block B. The coefficient of static friction between the two blocks is 0.400. A horizontal force $\vec{P}$ is then applied to block A. What is the largest value P can have and the blocks move together with equal accelerations?

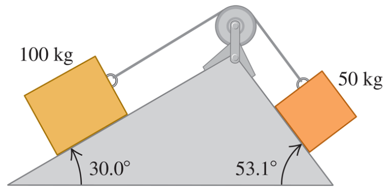
5.89 •• Block A in Fig. P5.89 has mass 4.00 kg, and block B has mass 12.0 kg. The coefficient of kinetic friction between block B and the horizontal surface is 0.25. (a) What is the mass of block C if block B is moving to the right and speeding up with an acceleration of 2.00 m/s^2? (b) What is the tension in each cord when block B has this acceleration?

Figure P5.89



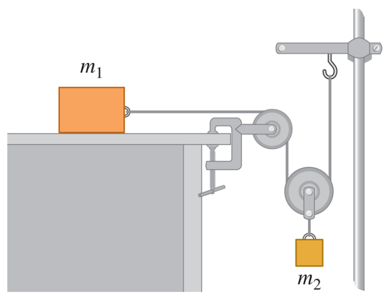
5.90 •• Two blocks connected by a cord passing over a small, frictionless pulley rest on frictionless planes (Fig. P5.90). (a) Which way will the system move when the blocks are released from rest? (b) What is the acceleration of the blocks? (c) What is the tension in the cord?

Figure P5.90



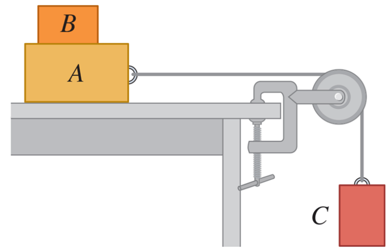
5.91 •• In terms of m_1, m_2, and g, find the acceleration of each block in Fig. P5.91. There is no friction anywhere in the system.

Figure P5.91



5.92 ••• Block B, with mass 5.00 kg, rests on block A, with mass 8.00 kg, which in turn is on a horizontal tabletop (Fig. P5.92). There is no friction between block A and the tabletop, but the coefficient of static friction between blocks A and B is 0.750. A light string attached to block A passes over a frictionless, massless pulley, and block C is suspended from the other end of the string. What is the largest mass that block C can have so that blocks A and B still slide together when the system is released from rest?

Figure P5.92

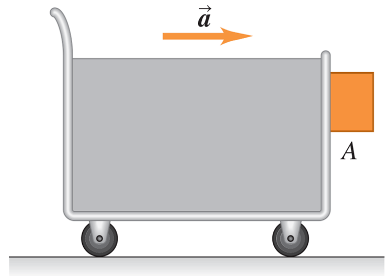


5.93 •• Consider two blocks connected as shown in Fig. P5.87. Block A has mass 2.00 kg, and block B has mass 5.00 kg. The table on which B sits is frictionless, the cord connecting the blocks is light and flexible, and the pulley is light and frictionless. The horizontal force $\vec{F}$ has magnitude $F = 20.0 \text{ N}$, and block B moves to the left with an acceleration of 1.50 m/s^2. (a) What is the tension in the cord that connects the two blocks? (b) What is the coefficient of kinetic friction that one block exerts on the other?

5.94 •• **Friction in an Elevator**. You are riding in an elevator on the way to the 18th floor of your dormitory. The elevator is accelerating upward with $a = 1.90 \text{ m}/s^2$. Beside you is the box containing your new computer; the box and its contents have a total mass of 36.0 kg. While the elevator is accelerating upward, you push horizontally on the box to slide it at constant speed toward the elevator door. If the coefficient of kinetic friction between the box and the elevator floor is $\mu_\text{k} = 0.32$, what magnitude of force must you apply?

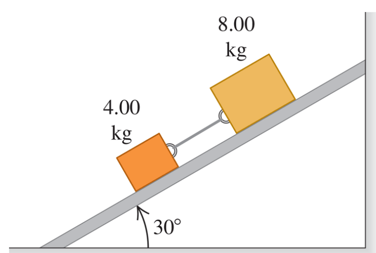
5.95 • A block is placed against the vertical front of a cart (Fig. P5.95). What acceleration must the cart have so that block A does not fall? The coefficient of static friction between the block and the cart is μ_s. How would an observer on the cart describe the behavior of the block?

Figure P5.95



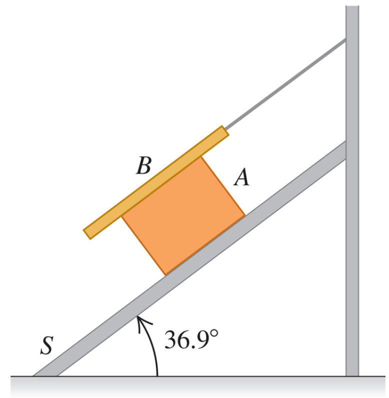
5.96 ••• Two blocks, with masses 4.00 kg and 8.00 kg, are connected by a string and slide down a 30.0° inclined plane (Fig. P5.96). The coefficient of kinetic friction between the 4.00 kg block and the plane is 0.25; that between the 8.00 kg block and the plane is 0.35. Calculate (a) the acceleration of each block and (b) the tension in the string. (c) What happens if the positions of the blocks are reversed, so that the 4.00 kg block is uphill from the 8.00 kg block?

Figure P5.96



5.97 ••• Block A, with weight $3w$, slides down an inclined plane S of slope angle 36.9° at a constant speed while plank B, with weight w, rests on top of A. The plank is attached by a cord to the wall (Fig. P5.97). (a) Draw a diagram of all the forces acting on block A. (b) If the coefficient of kinetic friction is the same between A and B and between S and A, determine its value.

Figure P5.97



5.98 •• Jack sits in the chair of a Ferris wheel that is rotating at a constant 0.100 rev/s. As Jack passes through the highest point of his circular path, the upward force that the chair exerts on him is equal to one-fourth of his weight. What is the radius of the circle in which Jack travels? Treat him as a point mass.

5.99 •• On the ride "Spindletop" at the amusement park Six Flags Over Texas, people stood against the inner wall of a hollow vertical cylinder with radius 2.5 m. The cylinder started to rotate, and when it reached a constant rotation rate of 0.60 rev/s, the floor dropped about 0.5 m. The people remained pinned against the wall without touching the floor. (a) Draw a force diagram for a person on this ride after the floor has dropped. (b) What minimum coefficient of static friction was required for the person not to slide downward to the new position of the floor? (c) Does your answer in part (b) depend on the person's mass? (*Note*: When such a ride is over, the cylinder is slowly brought to rest. As it slows down, people slide down the wall to the floor.)

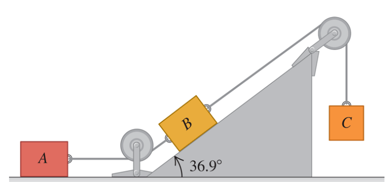
5.100 •• A large piece of ice breaks loose from the roof of a chalet at a ski resort and slides across the snow-covered terrain. It passes over the top of a hill in a path that has the shape of the arc of a circle of radius R. What is the speed v of the piece of ice as it passes over the top of the hill if the normal force exerted on it at this point is half its weight?

5.101 ••• A racetrack curve has radius 90.0 m and is banked at an angle of 18.0°. The coefficient of static friction between the tires and the roadway is 0.400. A race car with mass 1200 kg rounds the curve with the maximum speed to avoid skidding. (a) As the car rounds the curve, what is the normal force exerted on it by the road? What are the car's (b) radial acceleration and (c) speed?

5.102 ••• A racetrack curve has radius 120.0 m and is banked at an angle of 18.0°. The coefficient of static friction between the tires and the roadway is 0.300. A race car with mass 900 kg rounds the curve with the minimum speed needed to not slide down the banking. (a) As the car rounds the curve, what is the normal force exerted on it by the road? (b) What is the car's speed?

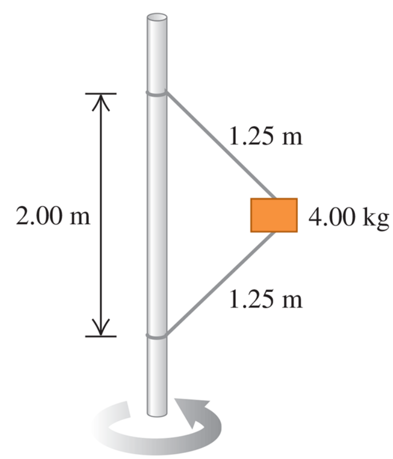
5.103 ••• Blocks A, B, and C are placed as in Fig. P5.103 and connected by ropes of negligible mass. Both A and B weigh 25.0 N each, and the coefficient of kinetic friction between each block and the surface is 0.35. Block C descends with constant velocity. (a) Draw separate free-body diagrams showing the forces acting on A and on B. (b) Find the tension in the rope connecting blocks A and B. (c) What is the weight of block C? (d) If the rope connecting A and B were cut, what would be the acceleration of C?

Figure P5.103



5.104 ••• A 4.00 kg block is attached to a vertical rod by means of two strings. When the system rotates about the axis of the rod, the strings are extended as shown in Fig. P5.104 and the tension in the upper string is 80.0 N. (a) What is the tension in the lower cord? (b) How many revolutions per minute does the system make? (c) Find the number of revolutions per minute at which the lower cord just goes slack. (d) Explain what happens if the number of revolutions per minute is less than that in part (c).

Figure P5.104

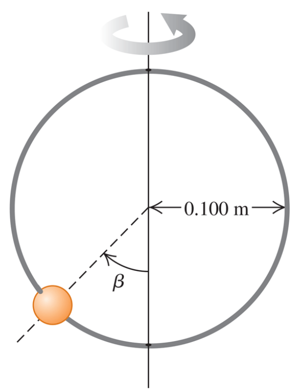


5.105 •• CALC You throw a rock downward into water with a speed of $3mg/k$, where k is the coefficient in Eq. (5.5). Assume that the relationship between fluid resistance and speed is as given in Eq. (5.5), and calculate the speed of the rock as a function of time.

5.106 ••• A box with mass m sits at the bottom of a long ramp that is sloped upward at an angle α above the horizontal. You give the box a quick shove, and after it leaves your hands it is moving up the ramp with an initial speed v_0. The box travels a distance d up the ramp and then slides back down. When it returns to its starting point, the speed of the box is half the speed it started with; it has speed $v_0/2$. What is the coefficient of kinetic friction between the box and the ramp? (Your answer should depend on only α.)

5.107 •• A small bead can slide without friction on a circular hoop that is in a vertical plane and has a radius of 0.100 m. The hoop rotates at a constant rate of 4.00 rev/s about a vertical diameter (Fig. P5.107). (a) Find the angle β at which the bead is in vertical equilibrium. (It has a radial acceleration toward the axis.) (b) Is it possible for the bead to "ride" at the same elevation as the center of the hoop? (c) What will happen if the hoop rotates at 1.00 rev/s?

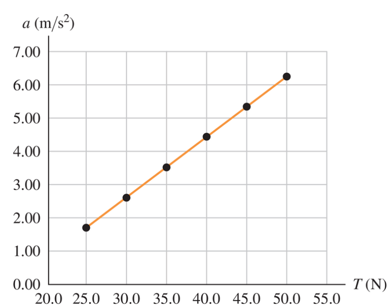
Figure P5.107



5.108 •• A physics major is working to pay her college tuition by performing in a traveling carnival. She rides a motorcycle inside a hollow, transparent plastic sphere. After gaining sufficient speed, she travels in a vertical circle with radius 13.0 m. She has mass 70.0 kg, and her motorcycle has mass 40.0 kg. (a) What minimum speed must she have at the top of the circle for the motorcycle tires to remain in contact with the sphere? (b) At the bottom of the circle, her speed is twice the value calculated in part (a). What is the magnitude of the normal force exerted on the motorcycle by the sphere at this point?

5.109 •• **DATA** In your physics lab, a block of mass m is at rest on a horizontal surface. You attach a light cord to the block and apply a horizontal force to the free end of the cord. You find that the block remains at rest until the tension T in the cord exceeds 20.0 N. For $T > 20.0$ N, you measure the acceleration of the block when T is maintained at a constant value, and you plot the results (Fig. P5.109). The equation for the straight line that best fits your data is $a = \ 0.182\ \mathrm{m/(N{\cdot}s^2)}\ T - 2.842\ \mathrm{m/s^2}$. For this block and surface, what are (a) the coefficient of static friction and (b) the coefficient of kinetic friction? (c) If the experiment were done on the earth's moon, where g is much smaller than on the earth, would the graph of a versus T still be fit well by a straight line? If so, how would the slope and intercept of the line differ from the values in Fig. P5.109? Or, would each of them be the same?

Figure P5.109



5.110 •• **DATA** A road heading due east passes over a small hill. You drive a car of mass m at constant speed v over the top of the hill, where the shape of the roadway is well approximated as an arc of a circle with radius R. Sensors have been placed on the road surface there to measure the downward force that cars exert on the surface at various speeds. The table gives values of this force versus speed for your car:

Speed (m/s)	6.00	8.00	10.0	12.0	14.0	16.0
Force (N)	8100	7690	7050	6100	5200	4200

Treat the car as a particle. (a) Plot the values in such a way that they are well fitted by a straight line. You might need to raise the speed, the force, or both to some power. (b) Use your graph from part (a) to calculate m and R. (c) What maximum speed can the car have at the top of the hill and still not lose contact with the road?

5.111 •• **DATA** You are an engineer working for a manufacturing company. You are designing a mechanism that uses a cable to drag heavy metal blocks a distance of 8.00 m along a ramp that is sloped at 40.0° above the horizontal. The coefficient of kinetic friction between these blocks and the incline is $\mu_{\mathrm{k}} = 0.350$. Each block has a mass of 2170 kg. The block will be placed on the bottom of the ramp, the cable will be attached, and the block will then be given just enough of a momentary push to overcome static friction. The block is then to accelerate at a constant rate to move the 8.00 m in 4.20 s. The cable is made of wire rope and is parallel to the ramp surface. The table gives the breaking strength of the cable as a function of its diameter; the safe load tension, which is 20% of the breaking strength; and the mass per meter of the cable:

Cable Diameter (in.)	Breaking Strength (kN)	Safe Load (kN)	Mass per Meter (kg/m)
$\frac{1}{4}$	24.4	4.89	0.16
$\frac{3}{8}$	54.3	10.9	0.36
$\frac{1}{2}$	95.2	19.0	0.63
$\frac{5}{8}$	149	29.7	0.98
$\frac{3}{4}$	212	42.3	1.41
$\frac{7}{8}$	286	57.4	1.92
1	372	74.3	2.50

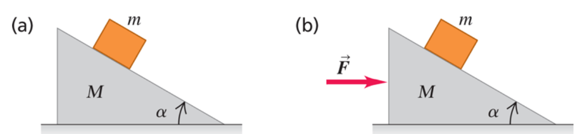
Source: www.engineeringtoolbox.com

(a) What is the minimum diameter of the cable that can be used to pull a block up the ramp without exceeding the safe load value of the tension in the cable? Ignore the mass of the cable, and select the diameter from those listed in the table. (b) You need to know safe load values for diameters that aren't in the table, so you hypothesize that the breaking strength and safe load limit are proportional to the cross-sectional area of the cable. Draw a graph that tests this hypothesis, and discuss its accuracy. What is your estimate of the safe load value for a cable with diameter $\frac{9}{16}$ in.? (c) The coefficient of static friction between the crate and the ramp is $\mu_s = 0.620$, which is nearly twice the value of the coefficient of kinetic friction. If the machinery jams and the block stops in the middle of the ramp, what is the tension in the cable? Is it larger or smaller than the value when the block is moving? (d) Is the actual tension in the cable, at its upper end, larger or smaller than the value calculated when you ignore the mass of the cable? If the cable is 9.00 m long, how accurate is it to ignore the cable's mass?

Challenge Problems

5.112 •••**Moving Wedge.** A wedge with mass M rests on a frictionless, horizontal tabletop. A block with mass m is placed on the wedge (Fig. P5.112a). There is no friction between the block and the wedge. The system is released from rest. (a) Calculate the acceleration of the wedge and the horizontal and vertical components of the acceleration of the block. (b) Do your answers to part (a) reduce to the correct results when M is very large? (c) As seen by a stationary observer, what is the shape of the trajectory of the block?

Figure P5.112

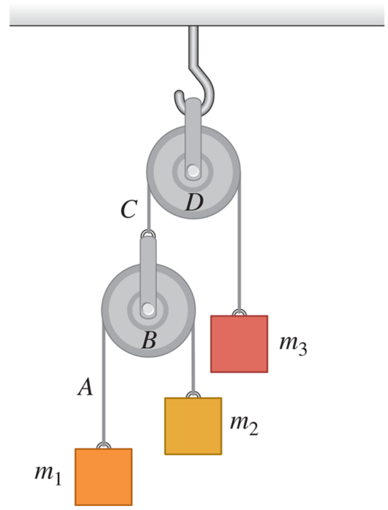


5.113 ••• A wedge with mass m rests on a frictionless, horizontal tabletop. A block with mass m is placed on the wedge, and a horizontal force $\vec{F}$ is applied to the wedge (Fig. P5.112b). What must the magnitude of $\vec{F}$ be if the block is to remain at a constant height above the tabletop?

5.114 ••• **Double Atwood's Machine.** In Fig. P5.114 masses m_1 and m_2 are connected by a light string A over a light, frictionless pulley B. The axle of pulley B is connected by a light string C over a light, frictionless pulley D to a mass m_3 . Pulley D is suspended from the ceiling by an attachment to its axle. The system is released from rest. In terms of m_1, m_2, m_3, and g, what are (a) the acceleration of block m_3; (b) the acceleration of pulley B; (c) the acceleration of block m_1; (d) the acceleration of block m_2;

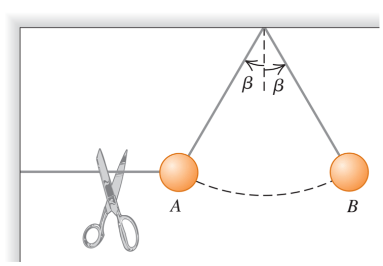
(e) the tension in string A; (f) the tension in string C? (g) What do your expressions give for the special case of $m_1 = m_2$ and $m_3 = m_1 + m_2$? Is this reasonable?

Figure P5.114



5.115 ••• A ball is held at rest at position A in Fig. P5.115 by two light strings. The horizontal string is cut, and the ball starts swinging as a pendulum. Position B is the farthest to the right that the ball can go as it swings back and forth. What is the ratio of the tension in the supporting string at B to its value at A before the string was cut?

Figure P5.115



MCAT-Style Passage Problems

Friction and Climbing Shoes. Shoes made for the sports of bouldering and rock climbing are designed to provide a great deal of friction between the foot and the surface of the ground. Such shoes on smooth rock might have a coefficient of static friction of 1.2 and a coefficient of kinetic friction of 0.90.

5.116 For a person wearing these shoes, what's the maximum angle (with respect to the horizontal) of a smooth rock that can be walked on without slipping? (a) 42°; (b) 50°; (c) 64°; (d) larger than 90°.

5.117 If the person steps onto a smooth rock surface that's inclined at an angle large enough that these shoes begin to slip, what will happen? (a) She will slide a short distance and stop; (b) she will accelerate down the surface; (c) she will slide down the surface at constant speed; (d) we can't tell what will happen without knowing her mass.

5.118 A person wearing these shoes stands on a smooth, horizontal rock. She pushes against the ground to begin running. What is the maximum horizontal acceleration she can have without slipping? (a) $0.20g$; (b) $0.75g$; (c) $0.90g$; (d) $1.2g$.

Answers: Applying Newton's Laws

Chapter Opening Question ?

(iii) The upward force exerted by the air has the same magnitude as the force of gravity. Although the seed and pappus are descending, their vertical velocity is constant, so their vertical acceleration is zero. According to Newton's first law, the net vertical force on the seed and pappus must also be zero. The individual vertical forces must balance.

Test Your Understanding

5.1 **(ii)** The two cables are arranged symmetrically, so the tension in either cable has the same magnitude T. The vertical component of the tension from each cable is $T \sin 45^\circ$ (or, equivalently, $T \cos 45^\circ$), so Newton's first law applied to the vertical forces tells us that $2T \sin 45^\circ - w = 0$. Hence $T = w/(2 \sin 45^\circ) = w/\sqrt{2} = 0.71w$. Each cable supports half of the weight of the traffic light, but the tension is greater than $w/2$ because only the vertical component of the tension counteracts the weight.

5.2 **(ii)** No matter what the instantaneous velocity of the glider, its acceleration is constant and has the value found in Example 5.12. In the same way, the acceleration of an object in free fall is the same whether it is ascending, descending, or at the high point of its motion (see Section 2.5).

5.3 (a): **(i)**, **(iii)**; (b): **(ii)**, **(iv)**; (c): **(v)** In situations (i) and (iii) the box is not accelerating (so the net force on it must be zero) and no other force is acting parallel to the horizontal surface; hence no friction force is needed to prevent sliding. In situations (ii) and (iv) the box would start to slide over the surface if no friction were present, so a static friction force must act to prevent this. In situation (v) the box is sliding over a rough surface, so a kinetic friction force acts on it.

5.4 **(iii)** A satellite of mass m orbiting the earth at speed v in an orbit of radius r has an acceleration of magnitude v^2/r, so the net force acting on it from the earth's gravity has magnitude $F = mv^2/r$. The farther the satellite is from the earth, the greater the value of r, the smaller the value of v, and hence the smaller the values of v^2/r and of F. In other words, the earth's gravitational force decreases with increasing distance.

Key Example Variation Problems

VP5.5.1 **a.** 425 N

b. 255 N

VP5.5.2 **a.** 367 N

b. 522 N

VP5.5.3 **a.** 21.8°

b. 637 N

VP5.5.4 $w(1 - \sin\theta)$, less than w

VP5.15.1 **a.** 79.3 N

b. 298 N

VP5.15.2 **a.** 90.2 N

b. 348 N

VP5.15.3 **a.** 449 N

b. 0.670 m/s^2, speeding up

VP5.15.4 $\mu_s = (T_{\text{min}} \cos\theta)/(mg - T_{\text{min}} \sin\theta)$

VP5.22.1 **a.** 0.274 m

b. 1.74 s

c. 2.61 N

VP5.22.2 **a.** 19.0 m

b. 8.22 m/s^2

c. 819 N

VP5.22.3 **a.** 75.0°

b. 3.03×10^3 N, 3.86 times greater

VP5.22.4 **a.** $mg\sqrt{x^2-1}$

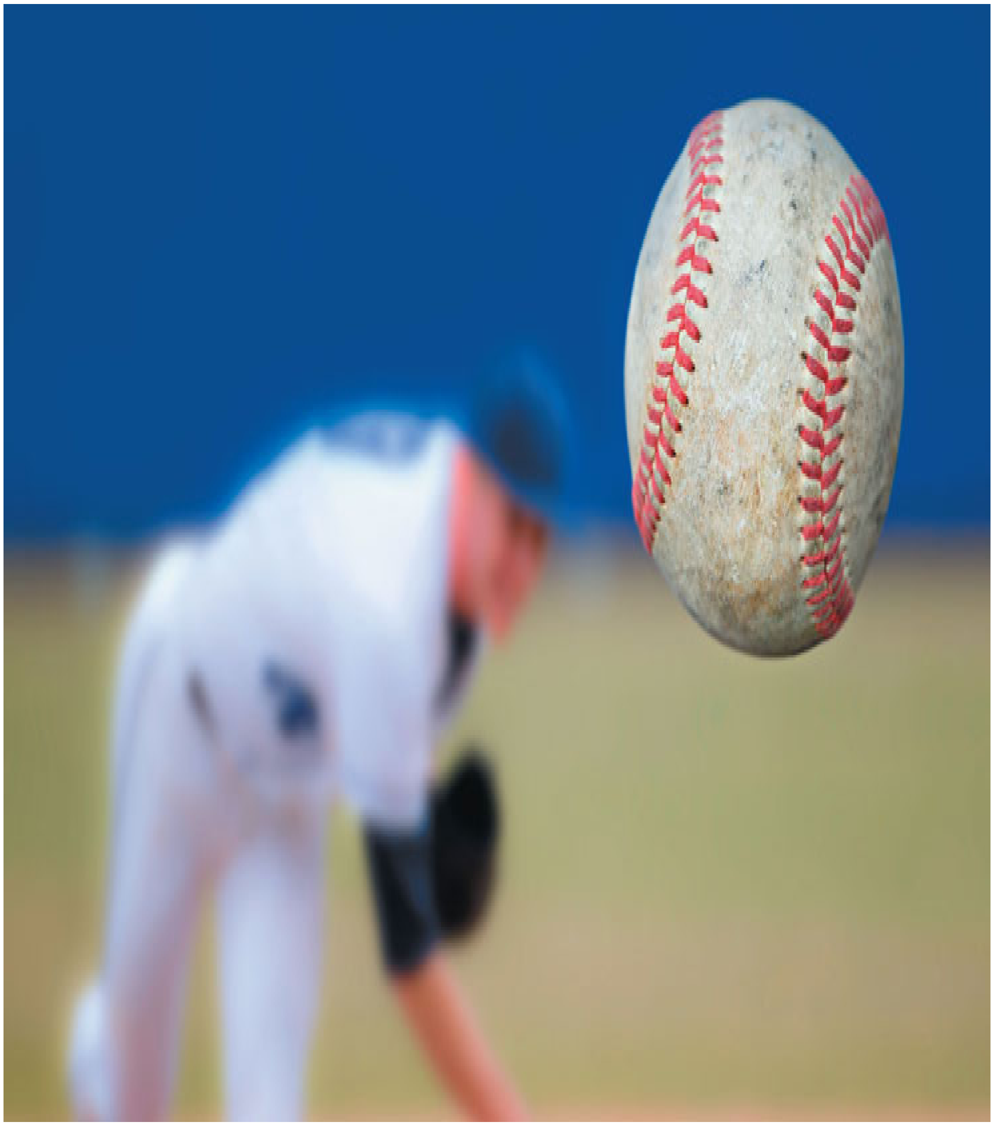
b. $\dfrac{v^2}{g\sqrt{x^2-1}}$

Bridging Problem

(a) $T_{\text{max}} = 2\pi\sqrt{\frac{h(\cos\beta+\mu_s\sin\beta)}{g\tan\beta(\sin\beta-\mu_s\cos\beta)}}$

(b) $T_{\text{min}} = 2\pi\sqrt{\frac{h(\cos\beta+\mu_s\sin\beta)}{g\tan\beta(\sin\beta+\mu_s\cos\beta)}}$

Chapter 6
Work and Kinetic Energy



? A baseball pitcher does work with his throwing arm to give the ball a property called kinetic energy, which depends on the ball's mass and speed. Which has the greatest kinetic energy? (i) A ball of mass 0.145 kg moving at 20.0 m/s; (ii) a smaller ball of mass 0.0145 kg moving at 200 m/s; (iii) a larger ball of mass 1.45 kg moving at 2.00 m/s; (iv) all three balls have the same kinetic energy; (v) it depends on the directions in which the balls move.

Learning Outcomes

In this chapter, you'll learn...

- 6.1 What it means for a force to do work on an object, and how to calculate the amount of work done.
- 6.2 The definition of the kinetic energy (energy of motion) of an object, and how the total work done on an object changes the object's kinetic energy.
- 6.3 How to use the relationship between total work and change in kinetic energy when the forces are not constant, the object follows a curved path, or both.
- 6.4 How to solve problems involving power (the rate of doing work).

You'll need to review...

- 1.10 The scalar product (or dot product) of two vectors.
- 2.4 Straight-line motion with constant acceleration.
- 4.3 Newton's second law.
- 4.5 Newton's third law.
- 5.1, 5.2 Using components to find the net force.

Suppose you try to find the speed of an arrow that has been shot from a bow. You apply Newton's laws and all the problem-solving techniques that we've learned, but you run across a major stumbling block: After the archer releases the arrow, the bow string exerts a *varying* force that depends on the arrow's position. As a result, the simple methods that we've learned aren't enough to calculate the speed. Never fear; we aren't by any means finished with mechanics, and there are other methods for dealing with such problems.

The new method that we're about to introduce uses the ideas of *work* and *energy*. The importance of the energy idea stems from the *principle of conservation of energy*: Energy is a quantity that can be converted from one form to another but cannot be created or destroyed. In an automobile engine, chemical energy stored in the fuel is converted partially to the energy of the automobile's motion and partially to thermal energy. In a microwave oven, electromagnetic energy obtained from your power company is converted to thermal energy of the food being cooked. In these and all other processes, the *total* energy—the sum of all energy present in all different forms—remains the same. No exception has ever been found.

We'll use the energy idea throughout the rest of this book to study a tremendous range of physical phenomena. This idea will help you understand how automotive engines work, how a camera's flash unit can produce a short burst of light, and the meaning of Einstein's famous equation $E = mc^2$.

In this chapter, though, our concentration will be on mechanics. We'll learn about one important form of energy called *kinetic energy*, or energy of motion, and how it relates to the concept of *work*. We'll also consider *power*, which is the time rate of doing work. In Chapter 7 we'll expand these ideas into a deeper understanding of the concepts of energy and the conservation of energy.

6.1 Work

You'd probably agree that it's hard work to pull a heavy sofa across the room, to lift a stack of encyclopedias from the floor to a high shelf, or to push a stalled car off the road. Indeed, all of these examples agree with the everyday meaning of "work"—any activity that requires muscular or mental effort.

In physics, work has a much more precise definition. By making use of this definition we'll find that in any motion, no matter how complicated, the total work done on a particle by all forces that act on it equals the change in its *kinetic energy*—a quantity that's related to the particle's mass and speed. This relationship holds even when the forces acting on the particle aren't constant, a situation that can be difficult or impossible to handle with the techniques you learned in Chapters 4 and 5. The ideas of work and kinetic energy enable us to solve problems in mechanics that we could not have attempted before.

In this section we'll see how work is defined and how to calculate work in a variety of situations involving *constant* forces. Later in this chapter we'll relate work and kinetic energy, and then apply these ideas to problems in which the forces are *not* constant.

The three examples of work described above—pulling a sofa, lifting encyclopedias, and pushing a car—have something in common. In each case you do work by exerting a *force* on an object while that object *moves* from one place to another—that is, undergoes a *displacement* (Fig. 6.1). You do more work if the force is greater (you push harder on the car) or if the displacement is greater (you push the car farther down the road).

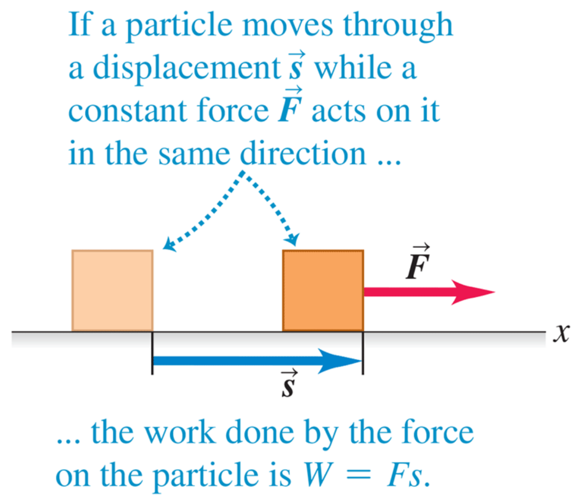
Figure 6.1

These people are doing work as they push on the car because they exert a force on the car as it moves.

The physicist's definition of work is based on these observations. Consider an object that undergoes a displacement of magnitude along a straight line. (For now, we'll assume that any object we discuss can be treated as a particle so that we can ignore any rotation or changes in shape of the object.) While the object moves, a constant force acts on it in the same direction as the displacement (Fig. 6.2). We define the **work** done by this constant force under these circumstances as the product of the force magnitude and the displacement magnitude

(6.1)

Figure 6.2



The work done by a constant force acting in the same direction as the displacement.

The work done on the object is greater if either the force or the displacement is greater, in agreement with our observations above.

CAUTION Don't confuse uppercase (work) with lowercase (weight). Though the symbols are similar, work and weight are different quantities.

The SI unit of work is the **joule** (abbreviated J, pronounced "jool," and named in honor of the 19th-century English physicist James Prescott Joule). From Eq. (6.1) we see that in any system of units, the unit of work is the unit of force multiplied by the unit of distance. In SI units the unit of force is the newton and the unit of distance is the meter, so 1 joule is equivalent to 1 *newton-meter*

If you lift an object with a weight of 1 N (about the weight of a medium-sized apple) a distance of 1 m at a constant speed, you exert a 1 N force on the object in the same direction as its 1 m displacement and so do 1 J of work on it.

As an illustration of Eq. (6.1), think of a person pushing a stalled car. If he pushes the car through a displacement with a constant force in the direction of motion, the amount of work he does on the car is given by Eq. (6.1): But what if the person pushes at an angle to the car's displacement (Fig. 6.3)? Then has a component in the direction of the displacement and a component that acts perpendicular to (Other forces must act on the car so that it moves along not in the direction of We're interested in only the work that the person does, however, so we'll consider only the force he exerts.) Only the parallel component is effective in moving the car, so we define the work as the product of this force component and the magnitude of the displacement. Hence or

(6.2)

Work done on a particle by constant force $\vec{F}$ during straight-line displacement $\vec{s}$

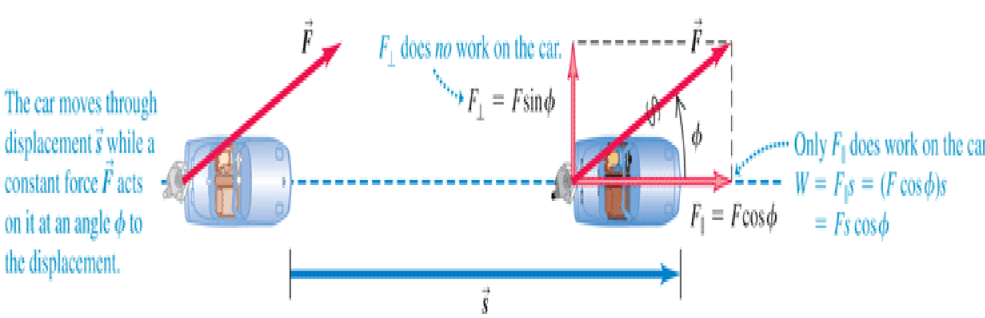
$$W = Fs\cos\phi \qquad (6.2)$$

Magnitude of $\vec{F}$

Angle between $\vec{F}$ and $\vec{s}$

Magnitude of $\vec{s}$

Figure 6.3



The work done by a constant force acting at an angle to the displacement.

BIO Application

Work and Muscle Fibers

Our ability to do work with our bodies comes from our skeletal muscles. The fiberlike cells of skeletal muscle, shown in this micrograph, can shorten, causing the muscle as a whole to contract and to exert force on the tendons to which it attaches. Muscle can exert a force of about 0.3 N per square millimeter of cross-sectional area: The greater the cross-sectional area, the more fibers the muscle has and the more force it can exert when it contracts.

If so that and are in the same direction, then and we are back to Eq. (6.1).

Equation (6.2) has the form of the *scalar product* of two vectors, which we introduced in Section 1.10: You may want to review that definition. Hence we can write Equation (6.2) more compactly as

(6.3)

Work done on a particle by constant force $\vec{F}$ during straight-line displacement $\vec{s}$

$$W = \underbrace{\vec{F} \cdot \vec{s}}_{\text{Scalar product (dot product) of vectors } \vec{F} \text{ and } \vec{s}} \tag{6.3}$$

CAUTION **Work is a scalar** An essential point: Work is a *scalar* quantity, even though it's calculated from two vector quantities (force and displacement). A 5 N force toward the east acting on an object that moves 6 m to the east does the same amount of work as a 5 N force toward the north acting on an object that moves 6 m to the north.

Example 6.1 Work done by a constant force

WITH VARIATION PROBLEMS

(a) Steve exerts a steady force of magnitude 210 N (about 47 lb) on the stalled car in Fig. 6.3 as he pushes it a distance of 18 m. The car also has a flat tire, so to make the car track straight Steve must push at an angle of to the direction of motion. How much work does Steve do? (b) In a helpful mood, Steve pushes a second stalled car with a steady force The displacement of the car is How much work does Steve do in this case?

IDENTIFY and SET UP In both parts (a) and (b), the target variable is the work done by Steve. In each case the force is constant and the displacement is along a straight line, so we can use Eq. (6.2) or (6.3). The angle between and is given in part (a), so we can apply Eq. (6.2) directly. In part (b) both and are given in terms of components, so it's best to calculate the scalar product by using Eq. (1.19):

EXECUTE

a. From Eq. (6.2),

b. The components of are and and the components of are and (There are no *z*-components for either vector.) Hence, using Eqs. (1.19) and (6.3), we have

EVALUATE In each case the work that Steve does is more than 1000 J. This shows that 1 joule is a rather small amount of work.

KEY CONCEPT

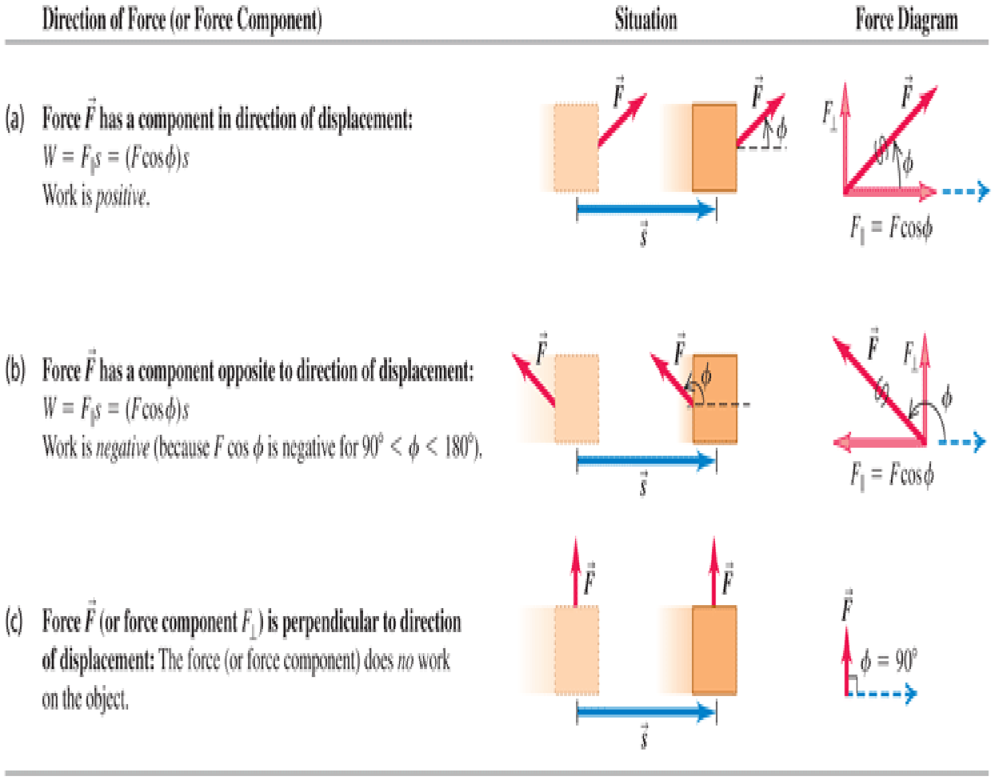
To find the work done by a constant force acting on an object that undergoes a straight-line displacement calculate the scalar product of these two vectors:

Video Tutor Solution: **Example 6.1**

Work: Positive, Negative, or Zero

In Example 6.1 the work done in pushing the cars was positive. But it's important to understand that work can also be negative or zero. This is the essential way in which work as defined in physics differs from the "everyday" definition of work. When the force has a component in the *same direction* as the displacement between and in Eq. (6.2) is positive and the work is *positive* (Fig. 6.4a). When the force has a component *opposite* to the displacement is negative and the work is *negative* (Fig. 6.4b). When the force is *perpendicular* to the displacement, and the work done by the force is *zero* (Fig. 6.4c). The cases of zero work and negative work bear closer examination, so let's look at some examples.

Figure 6.4

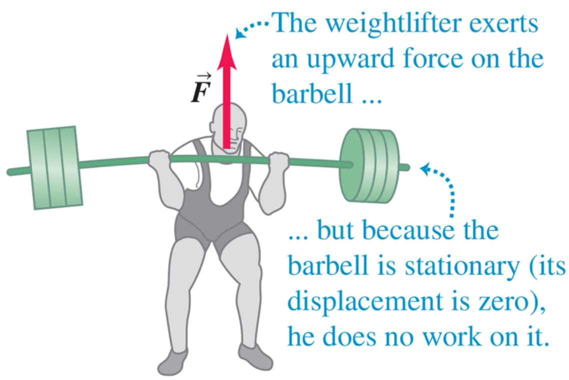


A constant force can do positive, negative, or zero work depending on the angle between and the displacement

---

There are many situations in which forces act but do zero work. You might think it's "hard work" to hold a barbell motionless in the air for 5 minutes (Fig. 6.5). But in fact, you aren't doing any work on the barbell because there is no displacement. (Holding the barbell requires you to keep the muscles of your arms contracted, and this consumes energy stored in carbohydrates and fat within your body. As these energy stores are used up, your muscles feel fatigued even though you do no work on the barbell.) Even when you carry a book while you walk with constant velocity on a level floor, you do no work on the book. It has a displacement, but the (vertical) supporting force that you exert on the book has no component in the direction of the (horizontal) motion. Then in Eq. (6.2), and When an object slides along a surface, the work done on the object by the normal force is zero; and when a ball on a string moves in uniform circular motion, the work done on the ball by the tension in the string is also zero. In both cases the work is zero because the force has no component in the direction of motion.

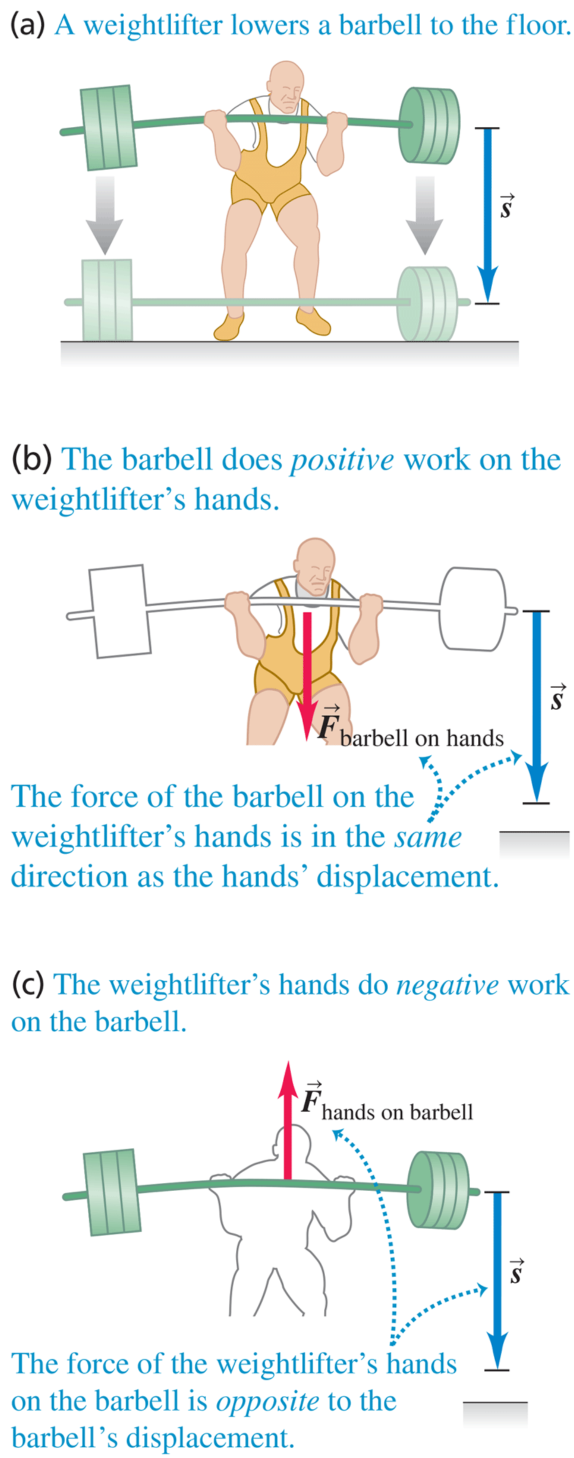
Figure 6.5



A weightlifter does no work on a barbell as long as he holds it stationary.

What does it mean to do *negative* work? The answer comes from Newton's third law of motion. When a weightlifter lowers a barbell as in Fig. 6.6a, his hands and the barbell move together with the same displacement The barbell exerts a force on his hands in the same direction as the hands' displacement, so the work done by the *barbell* on his *hands* is positive (Fig. 6.6b). But by Newton's third law the weightlifter's hands exert an equal and opposite force on the barbell (Fig. 6.6c). This force, which keeps the barbell from crashing to the floor, acts opposite to the barbell's displacement. Thus the work done by his *hands* on the *barbell* is negative. Because the weightlifter's hands and the barbell have the same displacement, the work that his hands do on the barbell is just the negative of the work that the barbell does on his hands. In general, when one object does negative work on a second object, the second object does an equal amount of *positive* work on the first object.

Figure 6.6



This weightlifter's hands do negative work on a barbell as the barbell does positive work on his hands.

As a final note, you should review Fig. 6.4 to help remember when work is positive, when it is zero, and when it is negative.

CAUTION Keep track of who's doing the work We always speak of work done *on* a particular object *by* a specific force. Always specify exactly what force is doing the work. When you lift a book, you exert an upward force on it and the book's displacement is upward, so the work done by the lifting force on the book is positive. But the work done by the *gravitational* force (weight) on a book being lifted is *negative* because the downward gravitational force is opposite to the upward displacement.

Total Work

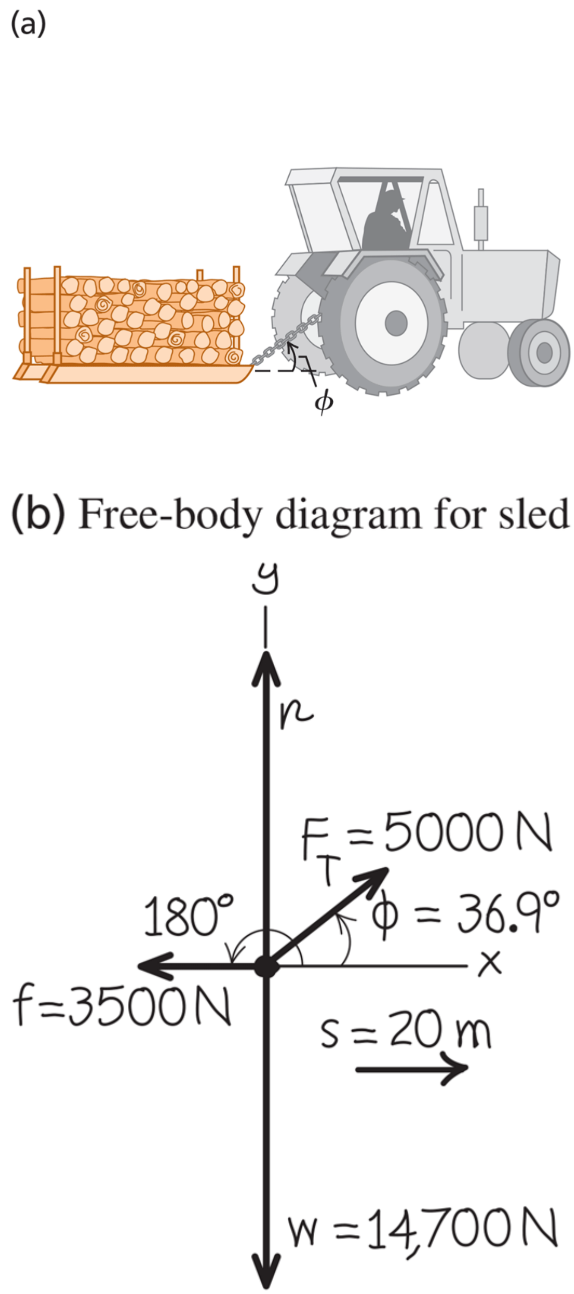
How do we calculate work when *several* forces act on an object? One way is to use Eq. (6.2) or (6.3) to compute the work done by each separate force. Then, because work is a scalar quantity, the *total* work done on the object by all the forces is the algebraic sum of the quantities of work done by the individual forces. An alternative way to find the total work is to compute the vector sum of the forces (that is, the net force) and then use this vector sum as in Eq. (6.2) or (6.3). The following example illustrates both of these techniques.

Example 6.2 Work done by several forces

WITH VARIATION PROBLEMS

A farmer hitches her tractor to a sled loaded with firewood and pulls it a distance of 20 m along level ground (Fig. 6.7a). The total weight of sled and load is 14,700 N. The tractor exerts a constant 5000 N force at an angle of above the horizontal. A 3500 N friction force opposes the sled's motion. Find the work done by each force acting on the sled and the total work done by all the forces.

Figure 6.7



Calculating the work done on a sled of firewood being pulled by a tractor.

IDENTIFY and SET UP We'll find the total work in two ways: (1) by adding the work done on the sled by each force and (2) by finding the work done by the net force on the sled. We first draw a free-body diagram showing all of the forces acting on the sled, and we choose a coordinate system (Fig. 6.7b). Each of these forces—weight, normal force, force of the tractor, and friction force—is constant, the sled's displacement is along a straight line, and we know the angle between the displacement (in the positive *x*-direction) and each force. Hence we can use Eq. (6.2) to calculate the work each force does.

As in Chapter 5, we'll find the net force by adding the components of the four forces. Newton's second law tells us that because the sled's motion is purely horizontal, the net force can have only a horizontal component.

EXECUTE

1. The work done by the weight is zero because its direction is perpendicular to the displacement (compare Fig. 6.4c). For the same reason, the work done by the normal force is also zero. (Note that we don't need to calculate the magnitude to conclude this.) So

   That leaves the work done by the force exerted by the tractor and the work done by the friction force From Eq. (6.2),

   The friction force is opposite to the displacement, so for this force and Again from Eq. (6.2), The total work done on the sled by all forces is the *algebraic* sum of the work done by the individual forces:

2. In the second approach, we first find the *vector* sum of all the forces (the net force) and then use it to compute the total work. It's easiest to find the net force by using components. From Fig. 6.7b,

We don't need the second equation; we know that the y-component of force is perpendicular to the displacement, so it does no work. Besides, there is no y-component of acceleration, so must be zero anyway. The total work is therefore the work done by the total x-component:

EVALUATE We get the same result for with either method, as we should. Note that the net force in the x-direction is *not* zero, and so the sled must accelerate as it moves. In Section 6.2 we'll return to this example and see how to use the concept of work to explore the sled's changes of speed.

KEY CONCEPT

To find the total work done on a moving object, calculate the sum of the amounts of work done by each force that acts on the object. The total work also equals the work done by the *net* force on the object.

Video Tutor Solution: **Example 6.2**

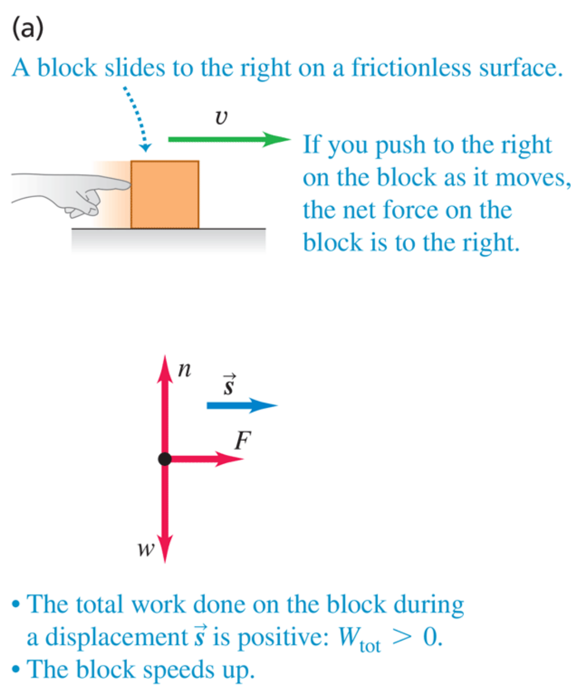
Test Your Understanding of Section 6.1

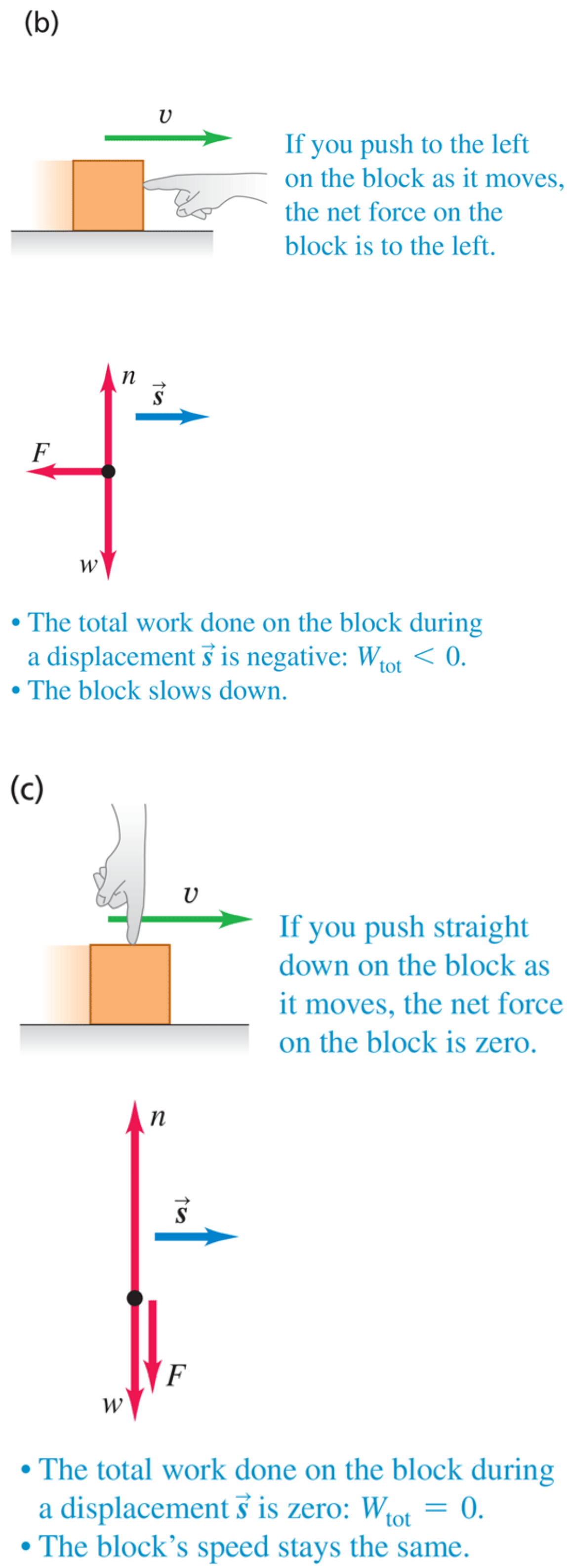
An electron moves in a straight line toward the east with a constant speed of It has electric, magnetic, and gravitational forces acting on it. During a 1 m displacement, the total work done on the electron is (i) positive; (ii) negative; (iii) zero; (iv) not enough information is given.

6.2 Kinetic Energy and the Work–Energy Theorem

The total work done on an object by external forces is related to the object's displacement—that is, to changes in its position. But the total work is also related to changes in the *speed* of the object. To see this, consider Fig. 6.8, which shows a block sliding on a frictionless table. The forces acting on the block are its weight $\vec{w}$, the normal force $\vec{n}$, and the force $\vec{F}$ exerted on it by the hand.

Figure 6.8





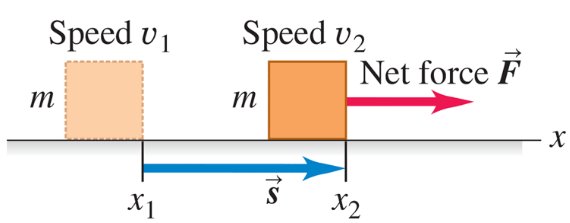
The relationship between the total work done on an object and how the object's speed changes.

In Fig. 6.8a the net force on the block is in the direction of its motion. From Newton's second law, this means that the block speeds up; from Eq. (6.1), this also means that the total work W_{tot} done on the block is positive. The total work is *negative* in Fig. 6.8b because the net force opposes the displacement; in this case the block slows down. The net force is zero in Fig. 6.8c, so the speed of the block stays the same and the total work done on the block is zero. We can conclude that *when a particle undergoes a displacement, it speeds up if* $W_{\text{tot}} > 0$, *slows down if* $W_{\text{tot}} < 0$, *and maintains the same speed if* $W_{\text{tot}} = 0$.

Let's make this more quantitative. In Fig. 6.9 a particle with mass m moves along the x-axis under the action of a constant net force with magnitude F that points in the positive x-direction. The particle's acceleration is constant and given by Newton's second law (Section 4.3): $F = ma_x$. As the particle moves from point x_1 to x_2, it undergoes a displacement $s = x_2 - x_1$ and its speed changes from v_1 to v_2. Using a constant-acceleration equation from Section 2.4, Eq. (2.13), and replacing v_{0x} by v_1, v_x by v_2, and $(x - x_0)$ by s, we have

$$\begin{aligned} v_2^2 &= v_1^2 + 2a_x s \\ &= \frac{v_2^2 - v_1^2}{2s} \end{aligned}$$

Figure 6.9



A constant net force $\vec{F}$ does work on a moving object.

When we multiply this equation by m and equate ma_x to the net force F, we find

(6.4)

$$\begin{aligned} F &= ma_x = m\frac{v_2^2 - v_1^2}{2s} \qquad \text{and} \\ Fs &= \tfrac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \end{aligned}$$

In Eq. (6.4) the product Fs is the work done by the net force F and thus is equal to the total work W_{tot} done by all the forces acting on the particle. The quantity $\frac{1}{2}mv^2$ is called the **kinetic energy** K of the particle:

(6.5)

Kinetic energy of a particle

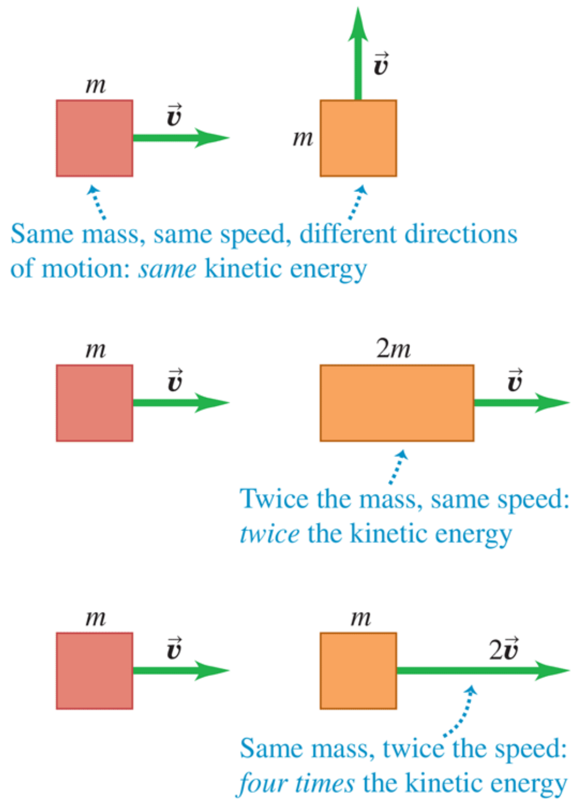
$$K = \frac{1}{2}mv^2 \tag{6.5}$$

Mass of particle

Speed of particle

? Like work, the kinetic energy of a particle is a scalar quantity; it depends on only the particle's mass and speed, not its direction of motion (Fig. 6.10). Kinetic energy can never be negative, and it is zero only when the particle is at rest.

Figure 6.10



Comparing the kinetic energy $K = \frac{1}{2}mv^2$ of different objects.

We can now interpret Eq. (6.4) in terms of work and kinetic energy. The first term on the right side of Eq. (6.4) is $K_2 = \frac{1}{2}mv_2^2$, the final kinetic energy of the particle (that is, after the displacement). The second term is the initial kinetic energy, $K_1 = \frac{1}{2}mv_1^2$, and the difference between these terms is the *change* in kinetic energy. So Eq. (6.4) says:

(6.6)

Work–energy theorem: Work done by the net force on a particle equals the change in the particle's kinetic energy.

$$\underbrace{W_{\text{tot}}}_{\substack{\text{Total work done on particle =} \\ \text{work done by net force}}} = \underbrace{K_2}_{\text{Final kinetic energy}} - \underbrace{K_1}_{\text{Initial kinetic energy}} = \underbrace{\Delta K}_{\text{Change in kinetic energy}} \tag{6.6}$$

This **work–energy theorem** agrees with our observations about the block in Fig. 6.8. When W_{tot} is *positive*, the kinetic energy *increases* (the final kinetic energy K_2 is greater than the initial kinetic energy K_1) and the particle is going faster at the end of the displacement than at the beginning. When W_{tot} is *negative*, the kinetic energy *decreases* (K_2 is less than K_1) and the speed is less after the displacement. When $W_{\text{tot}} = 0$, the kinetic energy stays the same ($K_1 = K_2$) and the speed is unchanged. Note that the work–energy theorem by itself tells us only about changes in *speed*, not velocity, since the kinetic energy doesn't depend on the direction of motion.

From Eq. (6.4) or Eq. (6.6), kinetic energy and work must have the same units. Hence the joule is the SI unit of both work and kinetic energy (and, as we'll see later, of all kinds of energy). To verify this, note that in SI the quantity $K = \frac{1}{2}mv^2$ has units $\text{kg}\cdot(\text{m/s})^2$ or $\text{kg}\cdot\text{m}^2/\text{s}^2$; we recall that $1\ \text{N} = 1\ \text{kg}\cdot\text{m/s}^2$, so

$$1\ \text{J} = 1\ \text{N}\cdot\text{m} = 1\ (\text{kg}\cdot\text{m/s}^2)\cdot\text{m} = 1\ \text{kg}\cdot\text{m}^2/\text{s}^2$$

Because we used Newton's laws in deriving the work–energy theorem, we can use this theorem only in an inertial frame of reference. Note that the work–energy theorem is valid in *any* inertial frame, but the values of W_{tot} and $K_2 - K_1$ may differ from one inertial frame to another (because the displacement and speed of an object may be different in different frames).

We've derived the work–energy theorem for the special case of straight-line motion with constant forces, and in the following examples we'll apply it to this special case only. We'll find in the next section that the theorem is valid even when the forces are not constant and the particle's trajectory is curved.

Problem-Solving Strategy 6.1 Work and Kinetic Energy

IDENTIFY *the relevant concepts*: The work–energy theorem, $W_{\text{tot}} = K_2 - K_1$, is extremely useful when you want to relate an object's speed v_1 at one point in its motion to its speed v_2 at a different point. (It's less useful for problems that involve the *time* it takes an object to go from point 1 to point 2 because the work–energy theorem doesn't involve time at all. For such problems it's usually best to use the relationships among time, position, velocity, and acceleration described in Chapters 2 and 3.)

SET UP *the problem* using the following steps:

1. Identify the initial and final positions of the object, and draw a free-body diagram showing all the forces that act on the object.
2. Choose a coordinate system. (If the motion is along a straight line, it's usually easiest to have both the initial and final positions lie along one of the axes.)
3. List the unknown and known quantities, and decide which unknowns are your target variables. The target variable may

be the object's initial or final speed, the magnitude of one of the forces acting on the object, or the object's displacement.

EXECUTE *the solution*: Calculate the work W done by each force. If the force is constant and the displacement is a straight line, you can use Eq. (6.2) or Eq. (6.3). (Later in this chapter we'll see how to handle varying forces and curved trajectories.) Be sure to check signs; W must be positive if the force has a component in the direction of the displacement, negative if the force has a component opposite to the displacement, and zero if the force and displacement are perpendicular.

Add the amounts of work done by each force to find the total work W_{tot}. Sometimes it's easier to calculate the vector sum of the forces (the net force) and then find the work done by the net force; this value is also equal to W_{tot}.

Write expressions for the initial and final kinetic energies, K_1 and K_2. Note that kinetic energy involves *mass*, not *weight*; if you are given the object's weight, use $w = mg$ to find the mass.

Finally, use Eq. (6.6), $W_{\text{tot}} = K_2 - K_1$, and Eq. (6.5), $K = \frac{1}{2}mv^2$, to solve for the target variable. Remember that the right-hand side of Eq. (6.6) represents the change of the object's kinetic energy between points 1 and 2; that is, it is the *final* kinetic energy minus the *initial* kinetic energy, never the other way around. (If you can predict the sign of W_{tot}, you can predict whether the object speeds up or slows down.)

EVALUATE *your answer*: Check whether your answer makes sense. Remember that kinetic energy $K = \frac{1}{2}mv^2$ can never be negative. If you come up with a negative value of K, perhaps you interchanged the initial and final kinetic energies in $W_{\text{tot}} = K_2 - K_1$ or made a sign error in one of the work calculations.

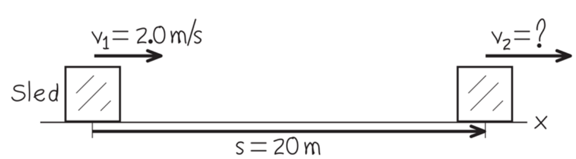
Example 6.3 Using work and energy to calculate speed

WITH VARIATION PROBLEMS

Let's look again at the sled in Fig. 6.7 and our results from Example 6.2. Suppose the sled's initial speed v_1 is 2.0 m/s. What is the speed of the sled after it moves 20 m?

IDENTIFY and SET UP We'll use the work–energy theorem, Eq. (6.6), $W_{\text{tot}} = K_2 - K_1$, since we are given the initial speed $v_1 = 2.0\ \text{m}/s$ and want to find the final speed v_2. Figure 6.11 shows our sketch of the situation. The motion is in the positive x-direction. In Example 6.2 we calculated the total work done by all the forces: $W_{\text{tot}} = 10\ \text{kJ}$. Hence the kinetic energy of the sled and its load must increase by 10 kJ, and the speed of the sled must also increase.

Figure 6.11



Our sketch for this problem.

EXECUTE To write expressions for the initial and final kinetic energies, we need the mass of the sled and load. The combined *weight* is 14,700 N, so the mass is

$$m = \frac{w}{g} = \frac{14,700\ \text{N}}{9.8\ \text{m/s}^2} = 1500\ \text{kg}$$

Then the initial kinetic energy K_1 is

$$K_1 = \frac{1}{2}mv_1^2 = \frac{1}{2}(1500\ \text{kg})(2.0\ \text{m/s})^2 = 3000\ \text{kg·m}^2/\text{s}^2 = 3000\ \text{J}$$

The final kinetic energy K_2 is

$$K_2 = \frac{1}{2}mv_2^2 = \frac{1}{2}(1500\ \text{kg})v_2^2$$

The work–energy theorem, Eq. (6.6), gives

$$K_2 = K_1 + W_{\text{tot}} = 3000\ \text{J} + 10,000\ \text{J} = 13,000\ \text{J}$$

Setting these two expressions for K_2 equal, substituting $1\ \text{J} = 1\ \text{kg·m}^2/\text{s}^2$, and solving for the final speed v_2, we find

$$v_2 = 4.2\ \text{m/s}$$

EVALUATE The total work is positive, so the kinetic energy increases ($K_2 > K_1$) and the speed increases ($v_2 > v_1$).

This problem can also be solved without the work–energy theorem. We can find the acceleration from $\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ and then use the equations of motion for constant acceleration to find v_2. Since the acceleration is along the x-axis,

$$a = a_x = \frac{\sum F_x}{m} = \frac{500\ \text{N}}{1500\ \text{kg}} = 0.333\ \text{m}/s^2$$

Then, using Eq. (2.13), we have

$$\begin{aligned} v_2^2 &= v_1^2 + 2as = (2.0\ \text{m/s})^2 + 2(0.333\ \text{m/s}^2)(20\ \text{m}) = 17.3\ \text{m}^2/\text{s}^2 \\ v_2 &= 4.2\,\text{m/s} \end{aligned}$$

This is the same result we obtained with the work–energy approach, but there we avoided the intermediate step of finding the acceleration. You'll find several other examples in this chapter and the next that *can* be done without using energy considerations but that are easier when energy methods are used. When a problem can be done by two methods, doing it by both methods (as we did here) is a good way to check your work.

KEY CONCEPT

You can use the work–energy theorem to easily relate the initial and final speeds of an object that moves while being acted on by constant forces.

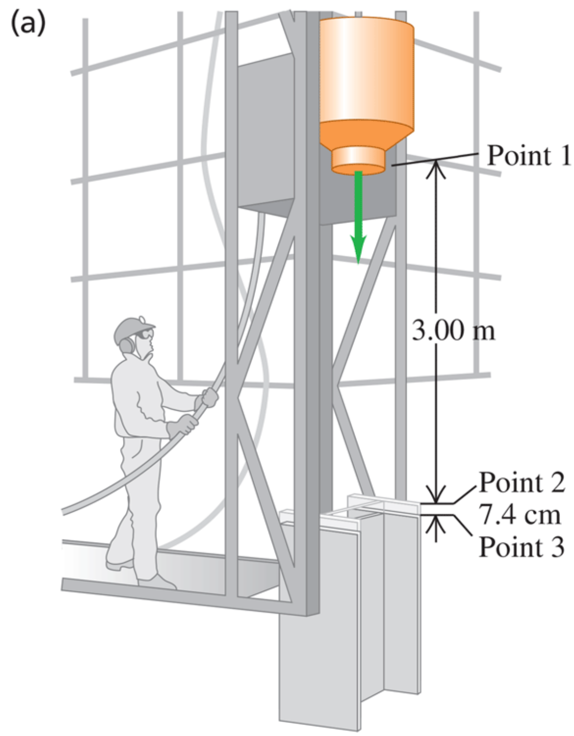
Video Tutor Solution: **Example 6.3**

Example 6.4 Forces on a hammerhead

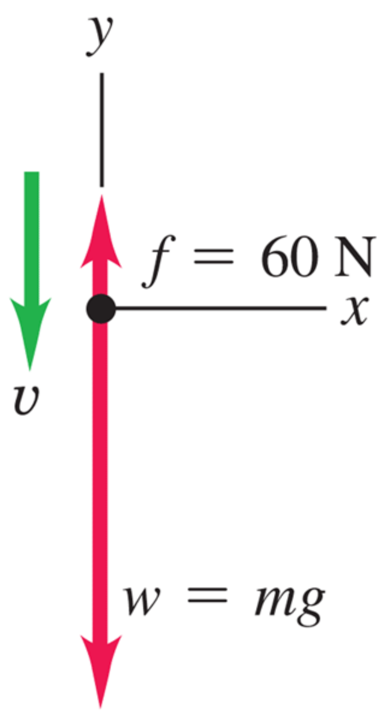
WITH VARIATION PROBLEMS

The 200 kg steel hammerhead of a pile driver is lifted 3.00 m above the top of a vertical I-beam being driven into the ground (Fig. 6.12a). The hammerhead is then dropped, driving the I-beam 7.4 cm deeper into the ground. The vertical guide rails exert a constant 60 N friction force on the hammerhead. Use the work–energy theorem to find (a) the speed of the hammerhead just as it hits the I-beam and (b) the average force the hammerhead exerts on the I-beam. Ignore the effects of the air.

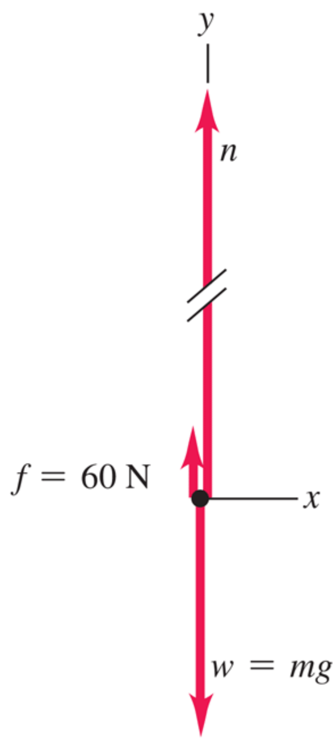
Figure 6.12



(b) Free-body diagram for falling hammerhead



(c) Free-body diagram for hammerhead when pushing I-beam



(a) A pile driver pounds an I-beam into the ground. (b), (c) Free-body diagrams. Vector lengths are not to scale.

IDENTIFY We'll use the work–energy theorem to relate the hammerhead's speed at different locations and the forces acting on it. There are *three* locations of interest: point 1, where the hammerhead starts from rest; point 2, where it first contacts the I-beam; and point 3, where the hammerhead and I-beam come to a halt (Fig. 6.12). The two target variables are the hammerhead's speed at point 2 and the average force the hammerhead exerts between points 2 and 3. Hence we'll apply the work–energy theorem twice: once for the motion from 1 to 2, and once for the motion from 2 to 3.

SET UP Figure 6.12b shows the vertical forces on the hammerhead as it falls from point 1 to point 2. (We can ignore any horizontal forces that may be present because they do no work as the hammerhead moves vertically.) For this part of the motion, our target variable is the hammerhead's final speed v_2.

Figure 6.12c shows the vertical forces on the hammerhead during the motion from point 2 to point 3. In addition to the forces shown in Fig. 6.12b, the I-beam exerts an upward normal force of magnitude n on the hammerhead. This force actually varies as the hammerhead comes to a halt, but for simplicity we'll treat n as a constant. Hence n represents the *average* value of this upward force during the motion. Our target variable for this part of the motion is the force that the *hammerhead* exerts on the I-beam; it is the reaction force to the normal force exerted by the I-beam, so by Newton's third law its magnitude is also n.

EXECUTE

a. From point 1 to point 2, the vertical forces are the downward weight $w = mg = (200\text{ kg})(9.8\text{ m/s}^2) = 1960\text{ N}$ and the upward friction force $f = 60\text{ N}$. Thus the net downward force is $w - f = 1900\text{ N}$. The displacement of the hammerhead from point 1 to point 2 is downward and equal to $s_{12} = 3.00\text{ m}$. The total work done on the hammerhead between point 1 and point 2 is then

$$W_{\text{tot}} = (w - f)s_{12} = (1900\text{ N})(3.00\text{ m}) = 5700\text{ J}$$

At point 1 the hammerhead is at rest, so its initial kinetic energy K_1 is zero. Hence the kinetic energy K_2 at point 2 equals the total work done on the hammerhead between points 1 and 2:

$$\begin{aligned} W_{\text{tot}} &= K_2 - K_1 = K_2 - 0 = \frac{1}{2}mv_2^2 - 0 \\ v_2 &= \sqrt{\frac{2W_{\text{tot}}}{m}} = \sqrt{\frac{2\,(5700\,\text{J})}{200\,\text{kg}}} = 7.55\,\text{m/s} \end{aligned}$$

This is the hammerhead's speed at point 2, just as it hits the I-beam.

b. As the hammerhead moves downward from point 2 to point 3, its displacement is $s_{23} = 7.4\ cm = 0.074\ m$ and the net downward force acting on it is $w - f - n$ (Fig. 6.12c). The total work done on the hammerhead during this displacement is

$$W_{\text{tot}} = (w - f - n)s_{23}$$

The initial kinetic energy for this part of the motion is K_2, which from part (a) equals 5700 J. The final kinetic energy is $K_3 = 0$ (the hammerhead ends at rest). From the work–energy theorem,

$$\begin{aligned} W_{\text{tot}} &= (w - f - n)s_{23} = K_3 - K_2 \\ n &= w - f - \frac{K_3 - K_2}{s_{23}} \\ &= 1960\,\text{N} - 60\,\text{N} - \frac{0\,\text{J} - 5700\,\text{J}}{0.074\,\text{m}} = 79,000\,\text{N} \end{aligned}$$

The downward force that the hammerhead exerts on the I-beam has this same magnitude, 79,000 N (about 9 tons)—more than 40 times the weight of the hammerhead.

EVALUATE The net change in the hammerhead's kinetic energy from point 1 to point 3 is zero; a relatively small net force does positive work over a large distance, and then a much larger net force does negative work over a much smaller distance. The same thing happens if you speed up your car gradually and then drive it into a brick wall. The very large force needed to reduce the kinetic energy to zero over a short distance is what does the damage to your car—and possibly to you.

KEY CONCEPT

If you know the initial and final speeds of an object that moves over a given straight-line distance, the work–energy theorem lets you calculate the net force that causes the change in speed.

Video Tutor Solution: Example 6.4

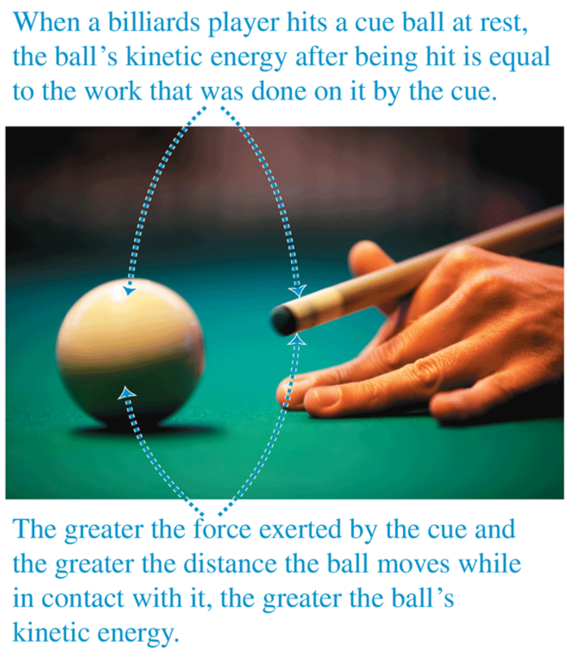
The Meaning of Kinetic Energy

Example 6.4 gives insight into the physical meaning of kinetic energy. The hammerhead is dropped from rest, and its kinetic energy when it hits the I-beam equals the total work done on it up to that point by the net force. This result is true in general: To accelerate a particle of mass m from rest (zero kinetic energy) up to a speed v, the total work done on it must equal the change in kinetic energy from zero to $K = \frac{1}{2}mv^2$:

$$W_{\text{tot}} = K - 0 = K$$

So *the kinetic energy of a particle is equal to the total work that was done to accelerate it from rest to its present speed* (Fig. 6.13). The definition $K = \frac{1}{2}mv^2$, Eq. (6.5), wasn't chosen at random; it's the *only* definition that agrees with this interpretation of kinetic energy.

Figure 6.13



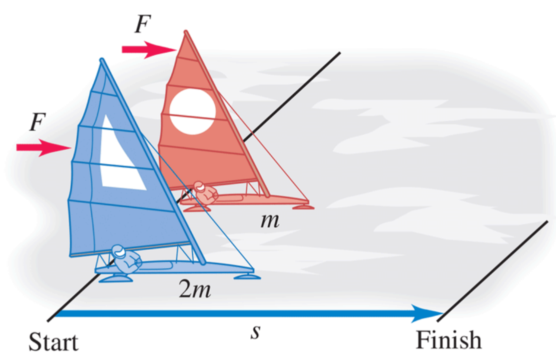
Imparting kinetic energy to a cue ball.

In the second part of Example 6.4 the kinetic energy of the hammerhead did work on the I-beam and drove it into the ground. This gives us another interpretation of kinetic energy: *The kinetic energy of a particle is equal to the total work that particle can do in the process of being brought to rest*. This is why you pull your hand and arm backward when you catch a ball. As the ball comes to rest, it does an amount of work (force times distance) on your hand equal to the ball's initial kinetic energy. By pulling your hand back, you maximize the distance over which the force acts and so minimize the force on your hand.

Conceptual Example 6.5 Comparing kinetic energies

Two iceboats like the one in Example 5.6 (Section 5.2) hold a race on a frictionless horizontal lake (Fig. 6.14). The two iceboats have masses m and $2m$. The iceboats have identical sails, so the wind exerts the same constant force $\vec{F}$ on each iceboat. They start from rest and cross the finish line a distance s away. Which iceboat crosses the finish line with greater kinetic energy?

Figure 6.14



A race between iceboats.

SOLUTION If you use the definition of kinetic energy, $K = \frac{1}{2}mv^2$, Eq. (6.5), the answer to this problem isn't obvious. The iceboat of mass $2m$ has greater mass, so you might guess that it has greater kinetic energy at the finish line. But the lighter iceboat, of mass m, has greater acceleration and crosses the finish line with a greater speed, so you might guess that *this* iceboat has the greater kinetic energy. How can we decide?

The key is to remember that *the kinetic energy of a particle is equal to the total work done to accelerate it from rest*. Both iceboats travel the same distance s from rest, and only the horizontal force F in the direction of motion does work on either iceboat. Hence the total work done between the starting line and the finish line is the *same* for each iceboat, $W_{\text{tot}} = Fs$. At the finish line, each iceboat has a kinetic energy equal to the work W_{tot} done on it, because each iceboat started from rest. So both iceboats have the *same* kinetic energy at the finish line!

You might think this is a "trick" question, but it isn't. If you really understand the meanings of quantities such as kinetic energy, you can solve problems more easily and with better insight.

Notice that we didn't need to know anything about how much time each iceboat took to reach the finish line. This is because the work–energy theorem makes no direct reference to time, only to displacement. In fact the iceboat of mass m has greater acceleration and so takes less time to reach the finish line than does the iceboat of mass $2m$.

KEY CONCEPT

The kinetic energy of an object with speed v equals the amount of work you must do to accelerate it from rest to speed v.

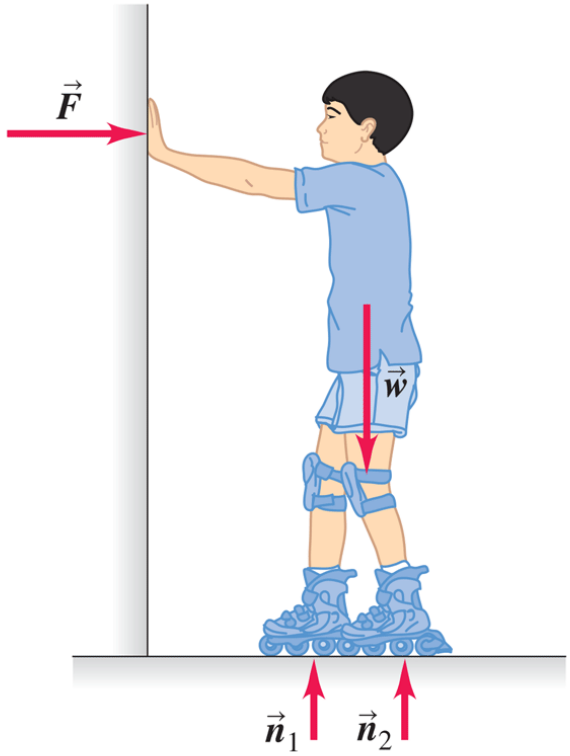
Video Tutor Solution: **Example 6.5**

Work and Kinetic Energy in Composite Systems

In this section we've been careful to apply the work–energy theorem only to objects that we can represent as *particles*—that is, as moving point masses. New subtleties appear for more complex systems that have to be represented as many particles with different motions. We can't go into these subtleties in detail in this chapter, but here's an example.

Suppose a boy stands on frictionless roller skates on a level surface, facing a rigid wall (Fig. 6.15). He pushes against the wall, which makes him move to the right. The forces acting on him are his weight $\vec{w}$, the upward normal forces $\vec{n}_1$ and $\vec{n}_2$ exerted by the ground on his skates, and the horizontal force $\vec{F}$ exerted on him by the wall. There is no vertical displacement, so $\vec{w}$, $\vec{n}_1$, and $\vec{n}_2$ do no work. Force $\vec{F}$ accelerates him to the right, but the parts of his body where that force is applied (the boy's hands) do not move while the force acts. Thus the force $\vec{F}$ also does no work. Where, then, does the boy's kinetic energy come from?

Figure 6.15



The external forces acting on a skater pushing off a wall. The work done by these forces is zero, but the skater's kinetic energy changes nonetheless.

The explanation is that it's not adequate to represent the boy as a single point mass. Different parts of the boy's body have different motions; his hands remain stationary against the wall while his torso is moving away from the wall. The various parts of his body interact with each other, and one part can exert forces and do work on another part. Therefore the *total* kinetic energy of this *composite* system of body parts can change, even though no work is done by forces applied by objects (such as the wall) that are outside the system. In Chapter 8 we'll consider further the motion of a collection of particles that interact with each other. We'll discover that just as for the boy in this example, the total kinetic energy of such a system can change even when no work is done on any part of the system by anything outside it.

Test Your Understanding of Section 6.2

Rank the following objects in order of their kinetic energy, from least to greatest. (i) A 2.0 kg object moving at 5.0 m/s; (ii) a 1.0 kg object that initially was at rest and then had 30 J of work done on it; (iii) a 1.0 kg object that initially was moving at 4.0 m/s and then had 20 J of work done on it; (iv) a 2.0 kg object that initially was moving at 10 m/s and then did 80 J of work on another object.

6.3 Work and Energy with Varying Forces

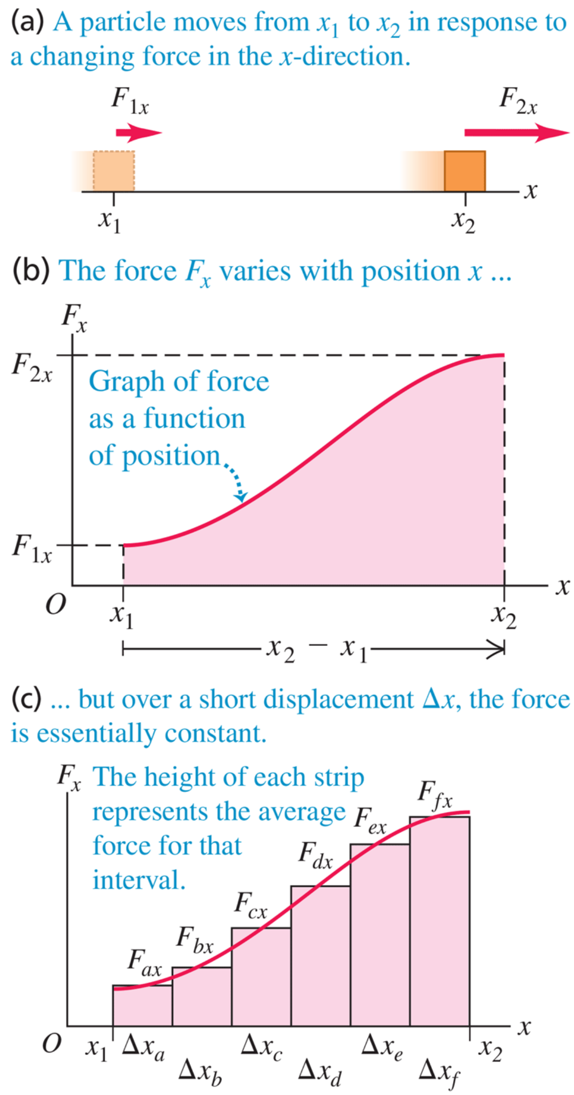
So far we've considered work done by *constant forces* only. But what happens when you stretch a spring? The more you stretch it, the harder you have to pull, so the force you exert is *not* constant as the spring is stretched. We've also restricted our discussion to *straight-line* motion. There are many situations in which an object moves along a curved path and is acted on by a force that varies in magnitude, direction, or both. We need to be able to compute the work done by the force in these more general cases. Fortunately, the work–energy theorem holds true even when forces are varying and when the object's path is not straight.

Work Done by a Varying Force, Straight-Line Motion

To add only one complication at a time, let's consider straight-line motion along the *x*-axis with a force whose *x*-component F_x may change as the object moves. (A real-life example is driving a car along a straight road with stop signs, so the driver has to alternately step on the gas and apply the brakes.) Suppose a particle moves along the *x*-axis from point x_1 to x_2 (Fig. 6.16a). Figure 6.16b is a graph of the *x*-component of force as a function of the particle's coordinate x. To find the work done by this force, we divide the total displacement into narrow segments Δx_a, Δx_b, and so on (Fig. 6.16c). We approximate the work done by the force during segment Δx_a as the average *x*-component of force F_{ax} in that segment multiplied by the *x*-displacement Δx_a. We do this for each segment and then add the results for all the segments. The work done by the force in the total displacement from x_1 to x_2 is approximately

$$W = F_{ax}\Delta x_a + F_{bx}\Delta x_b + \cdots$$

Figure 6.16



Calculating the work done by a varying force F_x in the x-direction as a particle moves from x_1 to x_2.

In the limit that the number of segments becomes very large and the width of each becomes very small, this sum becomes the *integral* of F_x from x_1 to x_2 :

(6.7)

Work done on a particle by a **varying x-component of** force F_x during **straight-line displacement** along x-axis

Upper limit = final position

$$W = \int_{x_1}^{x_2} F_x \, dx \qquad (6.7)$$

Integral of x-component of force

Lower limit = initial position

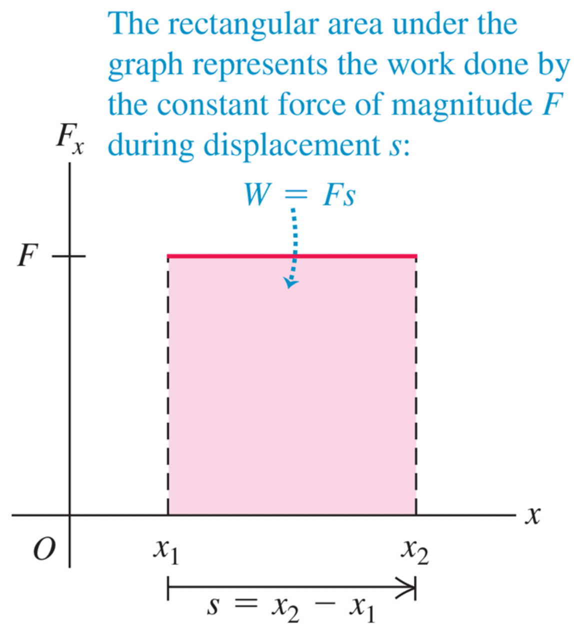
Note that $F_{ax}\ \Delta x_a$ represents the *area* of the first vertical strip in Fig. 6.16c and that the integral in Eq. (6.7) represents the area under the curve of Fig. 6.16b between x_1 and x_2. *On such a graph of force as a function of position, the total work done by the force is represented by the area under the curve between the initial and final positions*. Alternatively, the work W equals the average force that acts over the entire displacement, multiplied by the displacement.

In the special case that F_x , the x-component of the force, is constant, we can take it outside the integral in Eq. (6.7):

$$W = \int_{x_1}^{x_2} F_x dx = F_x \int_{x_1}^{x_2} dx = F_x \left(x_2 - x_1\right) \quad \text{(constant force)}$$

But $x_2 - x_1 = s$, the total displacement of the particle. So in the case of a constant force F, Eq. (6.7) says that $W = Fs$, in agreement with Eq. (6.1). The interpretation of work as the area under the curve of F_x as a function of x also holds for a constant force: $W = Fs$ is the area of a rectangle of height F and width s (Fig. 6.17).

Figure 6.17



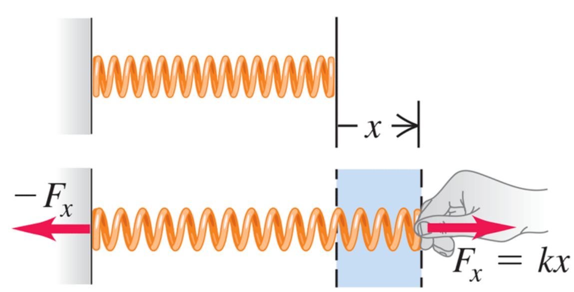
The work done by a constant force F in the x-direction as a particle moves from x_1 to x_2.

Now let's apply these ideas to the stretched spring. To keep a spring stretched beyond its unstretched length by an amount x, we have to apply a force of equal magnitude at each end (Fig. 6.18). If the elongation x is not too great, the force we apply to the right-hand end has an x-component directly proportional to x :

(6.8)

$$F_x = kx \quad \text{(force required to stretch a spring)}$$

Figure 6.18



The force needed to stretch an ideal spring is proportional to the spring's elongation: $F_x = kx$.

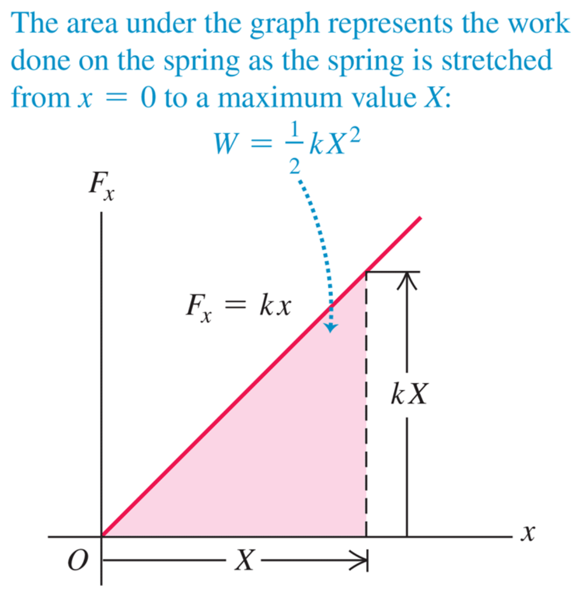
where k is a constant called the **force constant** (or spring constant) of the spring. The units of k are force divided by distance: N/m in SI units. A floppy toy spring such as a Slinky™ has a force constant of about 1 N/m; for the much stiffer springs in an automobile's suspension, k is about 10^5 N/m. The observation that force is directly proportional to elongation for elongations that are not too great was made by Robert Hooke in 1678 and is known as **Hooke's law**. It really shouldn't be called a "law," since it's a statement about a specific device and not a fundamental law of nature. Real springs don't always obey Eq. (6.8) precisely, but it's still a useful idealized model. We'll discuss Hooke's law more fully in Chapter 11.

To stretch a spring, we must do work. We apply equal and opposite forces to the ends of the spring and gradually increase the forces. We hold the left end stationary, so the force we apply at this end does no work. The force at the moving end *does* do work. Figure 6.19 is a graph of F_x as a function of x, the elongation of the spring. The work done by this force when the elongation goes from zero to a maximum value X is

(6.9)

$$W = \int_0^X F_x \, dx = \int_0^X kx \, dx = \frac{1}{2} kX^2$$

Figure 6.19



Calculating the work done to stretch a spring by a length X.

We can also obtain this result graphically. The area of the shaded triangle in Fig. 6.19, representing the total work done by the force, is equal to half the product of the base and altitude, or

$$W = \frac{1}{2}(X)(kX) = \frac{1}{2}kX^2$$

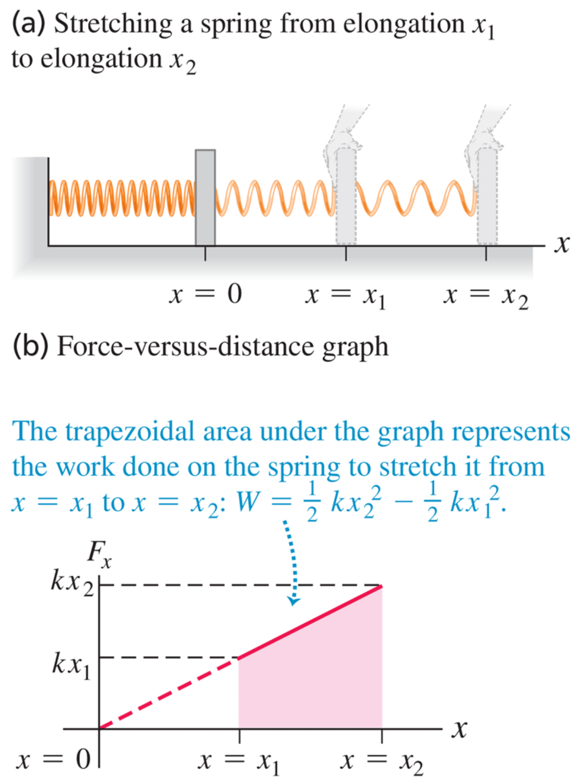
This equation also says that the work is the *average* force $kX/2$ multiplied by the total displacement X. We see that the total work is proportional to the *square* of the final elongation X. To stretch an ideal spring by 2 cm, you must do four times as much work as is needed to stretch it by 1 cm.

Equation (6.9) assumes that the spring was originally unstretched. If initially the spring is already stretched a distance x_1, the work we must do to stretch it to a greater elongation x_2 (Fig. 6.20a) is

(6.10)

$$W = \int_{x_1}^{x_2} F_x\,dx = \int_{x_1}^{x_2} kx\,dx = \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2$$

Figure 6.20



Calculating the work done to stretch a spring from one elongation to a greater one.

Use your knowledge of geometry to convince yourself that the trapezoidal area under the graph in Fig. 6.20b is given by the expression in Eq. (6.10).

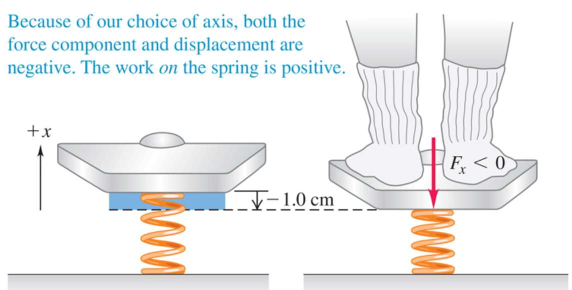
If the spring has spaces between the coils when it is unstretched, then it can also be compressed, and Hooke's law holds for compression as well as stretching. In this case the force and displacement are in the opposite directions from those shown in Fig. 6.18, so both F_x and x in Eq. (6.8) are negative. Since both F_x and x are reversed, the force again is in the same direction as the displacement, and the work done by F_x is again positive. So the total work is still given by Eq. (6.9) or (6.10), even when X is negative or either or both of x_1 and x_2 are negative.

CAUTION Work done *on* a spring vs. work done *by* a spring Equation (6.10) gives the work that *you* must do *on* a spring to change its length. If you stretch a spring that's originally relaxed, then $x_1 = 0,\ x_2 > 0$, and $W > 0$: The force you apply to one end of the spring is in the same direction as the displacement, and the work you do is positive. By contrast, the work that the *spring* does on whatever it's attached to is given by the *negative* of Eq. (6.10). Thus, as you pull on the spring, the spring does negative work on you.

Example 6.6 Work done on a spring scale

A woman weighing 600 N steps on a bathroom scale that contains a stiff spring (Fig. 6.21). In equilibrium, the spring is compressed 1.0 cm under her weight. Find the force constant of the spring and the total work done on it during the compression.

Figure 6.21



Compressing a spring in a bathroom scale.

IDENTIFY and SET UP In equilibrium the upward force exerted by the spring balances the downward force of the woman's weight. We'll use this principle and Eq. (6.8) to determine the force constant k, and we'll use Eq. (6.10) to calculate the work W that the woman does on the spring to compress it. We take positive values of x to correspond to elongation (upward in Fig. 6.21), so that both the displacement of the end of the spring (x) and the x-component of the force that the woman exerts on it (F_x) are negative. The applied force and the displacement are in the same direction, so the work done on the spring will be positive.

Execute The top of the spring is displaced by $x = -1.0\ \mathrm{cm}\ = -0.010\ \mathrm{m}$, and the woman exerts a force $F_x = -600\ \mathrm{N}$ on the spring. From Eq. (6.8) the force constant is then

$$k = \frac{F_x}{x} = \frac{-600\ \mathrm{N}}{-0.010\ \mathrm{m}} = 6.0 \times 10^4\ \mathrm{N/m}$$

Then, using $x_1 = 0$ and $x_2 = -0.010\ \mathrm{m}$ in Eq. (6.10), we have

$$\begin{aligned} W &= \tfrac{1}{2}kx_2^2 - \tfrac{1}{2}kx_1^1 \\ &= \tfrac{1}{2}\left(6.0 \times 10^4\,\mathrm{N/m}\right)\left(-0.010\,\mathrm{m}\right)^2 - 0 = 3.0\,\mathrm{J} \end{aligned}$$

EVALUATE The work done is positive, as expected. Our arbitrary choice of the positive direction has no effect on the answer for W. You can test this by taking the positive x-direction to be downward, corresponding to compression. Do you get the same values for k and W as we found here?

KEY CONCEPT

You must use Eq. (6.10) to calculate the work done by the nonconstant force that a spring exerts.

Video Tutor Solution: **Example 6.6**

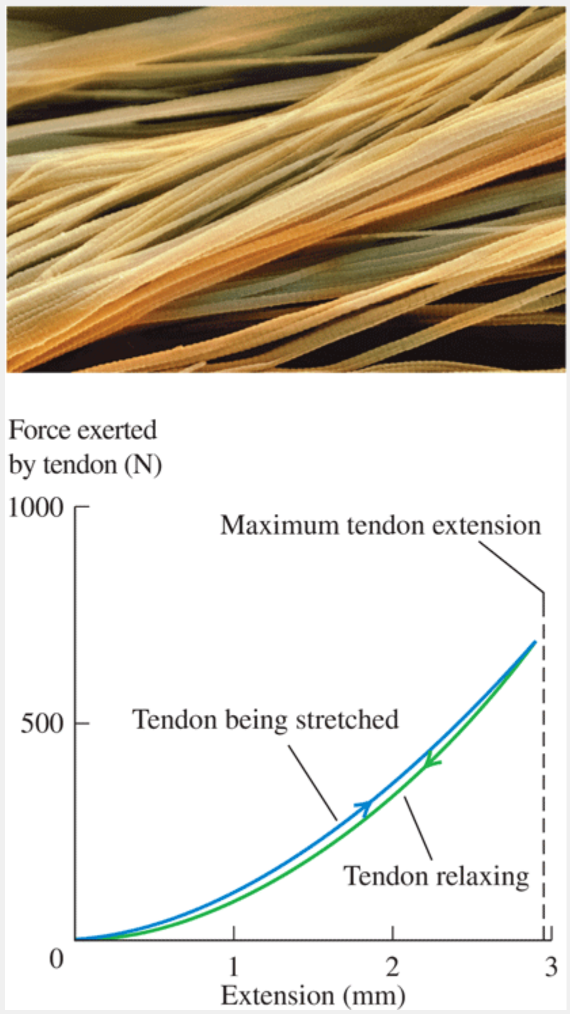
Work–Energy Theorem for Straight-Line Motion, Varying Forces

In Section 6.2 we derived the work–energy theorem, $W_{\text{tot}} = K_2 - K_1$, for the special case of straight-line motion with a constant net force. We can now prove that this theorem is true even when the force varies with position. As in Section 6.2, let's consider a particle that undergoes a displacement x while being acted on by a net force with x-component F_x, which we now allow to vary. Just as in Fig. 6.16, we divide the total displacement x into a large number of small segments Δx. We can apply the work–energy theorem, Eq. (6.6), to each segment because the value of F_x in each small segment is approximately constant. The change in kinetic energy in segment Δx_a is equal to the work $F_{ax}\,\Delta x_a$, and so on. The total change of kinetic energy is the sum of the changes in the individual segments, and thus is equal to the total work done on the particle during the entire displacement. So $W_{\text{tot}} = \Delta K$ holds for varying forces as well as for constant ones.

BIO Application

Tendons Are Nonideal Springs

Muscles exert forces via the tendons that attach them to bones. A tendon consists of long, stiff, elastic collagen fibers. The graph shows how the tendon from the hind leg of a wallaby (a small kangaroo-like marsupial) stretches in response to an applied force. The tendon does not exhibit the simple, straight-line behavior of an ideal spring, so the work it does has to be found by integration [Eq. (6.7)]. The tendon exerts less force while relaxing than while stretching. As a result, the relaxing tendon does only about 93% of the work that was done to stretch it.



Here's an alternative derivation of the work–energy theorem for a force that may vary with position. It involves making a change of variable from

x to v_x in the work integral. Note first that the acceleration a of the particle can be expressed in various ways, using $a_x = dv_x/dt$, $v_x = dx/dt$, and the chain rule for derivatives:

(6.11)

$$a_x = \frac{dv_x}{dt} = \frac{dv_x}{dx}\frac{dx}{dt} = v_x\frac{dv_x}{dx}$$

From this result, Eq. (6.7) tells us that the total work done by the *net* force F_x is

(6.12)

$$W_{\text{tot}} = \int_{x_1}^{x_2} F_x\, dx = \int_{x_1}^{x_2} ma_x\, dx = \int_{x_1}^{x_2} mv_x \frac{dv_x}{dx} dx$$

Now $(dv_x/dx)dx$ is the change in velocity dv_x during the displacement dx, so we can make that substitution in Eq. (6.12). This changes the integration variable from x to v_x, so we change the limits from x_1 and x_2 to the corresponding *x*-velocities v_1 and v_2 :

$$W_{\text{tot}} = \int_{v_1}^{v_2} mv_x\, dv_x$$

The integral of $v_x\, dv_x$ is just $v_x^2/2$. Substituting the upper and lower limits, we finally find

(6.13)

$$W_{\text{tot}} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

This is the same as Eq. (6.6), so the work–energy theorem is valid even without the assumption that the net force is constant.

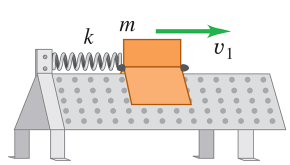
Example 6.7 Motion with a varying force

WITH VARIATION PROBLEMS

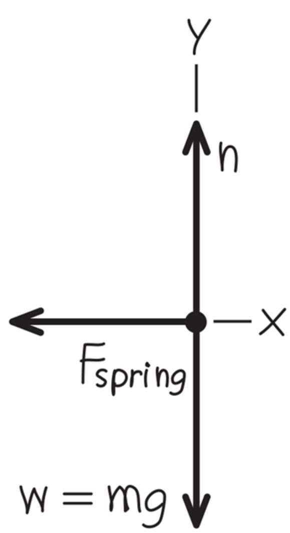
An air-track glider of mass 0.100 kg is attached to the end of a horizontal air track by a spring with force constant 20.0 N/m (Fig. 6.22a). Initially the spring is unstretched and the glider is moving at 1.50 m/s to the right. Find the maximum distance d that the glider moves to the right (a) if the air track is turned on, so that there is no friction, and (b) if the air is turned off, so that there is kinetic friction with coefficient $\mu_k = 0.47$.

Figure 6.22

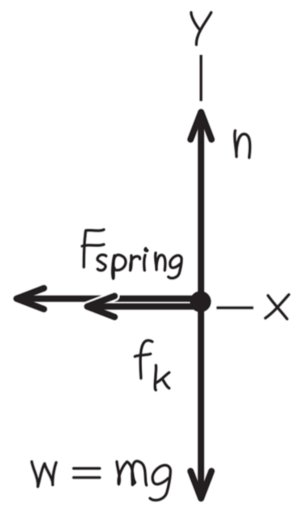
(a)



(b) No friction



(c) With friction



(a) A glider attached to an air track by a spring. (b), (c) Our free-body diagrams.

IDENTIFY and SET UP The force exerted by the spring is not constant, so we *cannot* use the constant-acceleration formulas of Chapter 2 to solve this problem. Instead, we'll use the work–energy theorem, since the total work done involves the distance moved (our target variable). In Figs. 6.22b and 6.22c we choose the positive x-direction to be to the right (in the direction of the glider's motion). We take $x = 0$ at the glider's initial position (where the spring is unstretched) and $x = d$ (the target variable) at the position where the glider stops. The motion is purely horizontal, so only the horizontal forces do work. Note that Eq. (6.10) gives the work done by the *glider* on the *spring* as it stretches; to use the work–energy theorem we need the work done by the *spring* on the *glider*, which is the negative of Eq. (6.10). We expect the glider to move farther without friction than with friction.

EXECUTE

a. Equation (6.10) says that as the glider moves from $x_1 = 0$ to $x_2 = d$, it does an amount of work $W = \frac{1}{2}kd^2 - \frac{1}{2}k(0)^2 = \frac{1}{2}kd^2$ on the spring. The amount of work that the *spring* does on the *glider* is the negative of this, $-\frac{1}{2}kd^2$. The spring stretches until the glider comes instantaneously to rest, so the final kinetic energy K_2 is zero. The initial kinetic energy is $\frac{1}{2}mv_1^2$, where $v_1 = 1.50$ m/s is the glider's initial speed. From the work–energy theorem,

$$-\frac{1}{2}kd^2 = 0 - \frac{1}{2}mv_1^2$$

We solve for the distance d the glider moves:

$$d = v_1\sqrt{\frac{m}{k}} = (1.50\ \mathrm{m/s})\sqrt{\frac{0.100\ \mathrm{kg}}{20.0\ \mathrm{N/m}}} = 0.106\ \mathrm{m} = 10.6\ \mathrm{cm}$$

The stretched spring subsequently pulls the glider back to the left, so the glider is at rest only instantaneously.

b. If the air is turned off, we must include the work done by the kinetic friction force. The normal force n is equal in magnitude to the weight of the glider, since the track is horizontal and there are no other vertical forces. Hence the kinetic friction force has constant magnitude $f_\mathrm{k} = \mu_\mathrm{k}mg$. The friction force is directed opposite to the displacement, so the work done by friction is

$$W_\mathrm{fric} = f_\mathrm{k}d\ \cos 180^\circ = -f_\mathrm{k}d = -\mu_\mathrm{k}mgd$$

The total work is the sum of W_fric and the work done by the spring, $-\frac{1}{2}kd^2$. The work–energy theorem then says that

$$-\mu_k mgd - \tfrac{1}{2}kd^2 = 0 - \tfrac{1}{2}mv_1^2 \quad \text{or}$$
$$\tfrac{1}{2}kd^2 + \mu_k mgd - \tfrac{1}{2}mv_1^2 = 0$$

This is a quadratic equation for d The solutions are

$$d = -\frac{\mu_k mg}{k} \pm \sqrt{\left(\frac{\mu_k mg}{k}\right)^2 + \frac{mv_1^2}{k}}$$

We have

$$\begin{aligned}\frac{\mu_k mg}{k} &= \frac{(0.47)(0.100\ \text{kg})(9.80\ \text{m/s}^2)}{20.0\ \text{N/m}} = 0.02303\ \text{m}\\ \frac{mv_1^2}{k} &= \frac{(0.100\,\text{kg})(1.50\,\text{m/s})^2}{20.0\,\text{N/m}} = 0.01125\,\text{m}^2\end{aligned}$$

so

$$\begin{aligned} d &= -(0.02303\ \text{m}) \pm \sqrt{(0.02303\ \text{m})^2 + 0.01125\ \text{m}^2}\\ &= 0.086\,\text{m} \quad \text{or} \quad -0.132\,\text{m}\end{aligned}$$

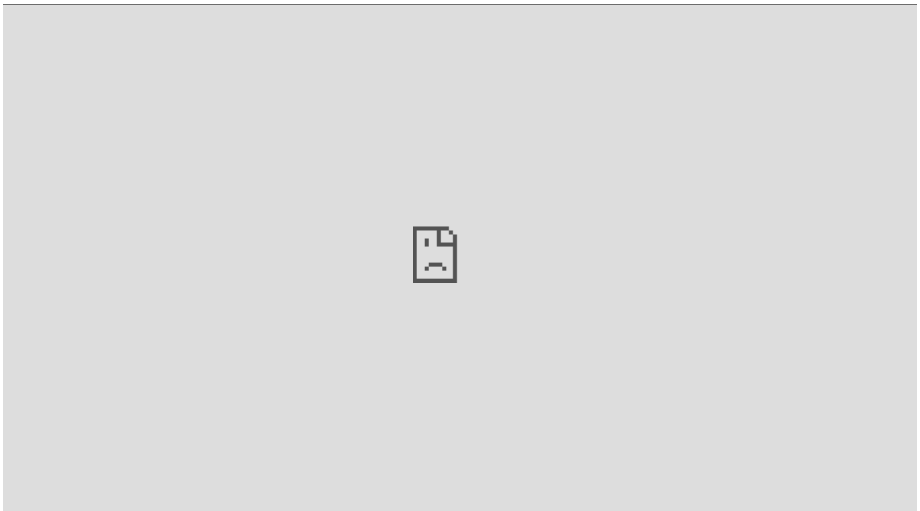
The quantity d is a positive displacement, so only the positive value of d makes sense. Thus with friction the glider moves a distance $d = 0.086$ m $= 8.6$ cm.

EVALUATE If we set $\mu_k = 0$, our algebraic solution for d in part (b) reduces to $d = v_1\sqrt{m/k}$, the zero-friction result from part (a). With friction, the glider goes a shorter distance. Again the glider stops instantaneously, and again the spring force pulls it toward the left; whether it moves or not depends on how great the *static* friction force is. How large would the coefficient of static friction μ_s have to be to keep the glider from springing back to the left?

KEY CONCEPT

The work–energy theorem also allows you to solve problems with *varying* forces, such as the force exerted by a spring.

Video Tutor Solution: Example 6.7

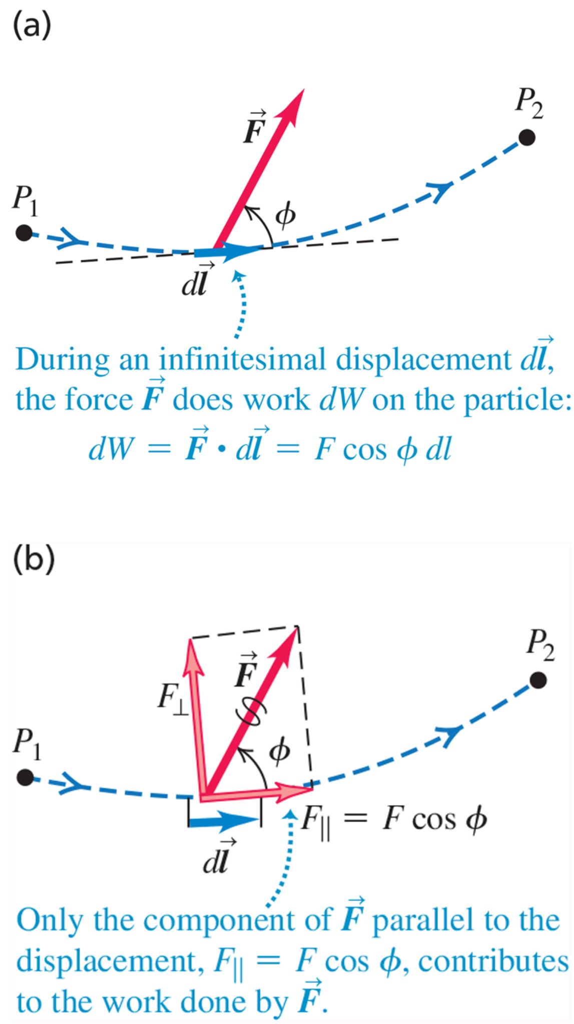


Work–Energy Theorem for Motion Along a Curve

We can generalize our definition of work further to include a force that varies in direction as well as magnitude, and a displacement that lies along a curved path. Figure 6.23a shows a particle moving from P_1 to P_2 along a curve. We divide the curve between these points into many infinitesimal vector displacements, and we call a typical one of these $d\vec{l}$. Each $d\vec{l}$ is tangent to the path at its position. Let $\vec{F}$ be the force at a typical point along the path, and let ϕ be the angle between $\vec{F}$ and $d\vec{l}$ at this point. Then the small element of work dW done on the particle during the displacement $d\vec{l}$ may be written as

$$dW = \vec{F} \cdot d\vec{l} = F \cos \phi \, dl = F_{\parallel} \, dl$$

Figure 6.23



A particle moves along a curved path from point P_1 to P_2, acted on by a force $\vec{F}$ that varies in magnitude and direction.

where $F_\parallel = F \ \cos \phi$ is the component of $\vec{F}$ in the direction parallel to $d\vec{l}$ (Fig. 6.23b). The work done by $\vec{F}$ on the particle as it moves from P_1 to P_2 is

(6.14)

Upper limit = final position

Scalar product (dot product) of $\vec{F}$ and displacement $d\vec{l}$

Work done on a particle by a **varying force** $\vec{F}$ along a **curved path**

$$W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{l} = \int_{P_1}^{P_2} F \cos\phi \, dl = \int_{P_1}^{P_2} F_{\parallel} \, dl \qquad (6.14)$$

Lower limit = initial position

Angle between $\vec{F}$ and $d\vec{l}$

Component of $\vec{F}$ parallel to $d\vec{l}$

The integral in Eq. (6.14) (shown in three versions) is called a *line integral*. We'll see shortly how to evaluate an integral of this kind.

We can now show that the work–energy theorem, Eq. (6.6), holds true even with varying forces and a displacement along a curved path. The force $\vec{\boldsymbol{F}}$ is essentially constant over any given infinitesimal segment $d\vec{\boldsymbol{l}}$ of the path, so we can apply the work–energy theorem for straight-line motion to that segment. Thus the change in the particle's kinetic energy K over that segment equals the work $dW = F_{\parallel}\ dl = \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{l}}$ done on the particle. Adding up these infinitesimal quantities of work from all the segments along the whole path gives the total work done, Eq. (6.14), which equals the total change in kinetic energy over the whole path. So $W_{\text{tot}} = \Delta K = K_2 - K_1$ is true *in general*, no matter what the path and no matter what the character of the forces. This can be proved more rigorously by using steps like those in Eq. (6.11), (6.12) and (6.13).

Note that only the component of the net force parallel to the path, $F_{\parallel}$, does work on the particle, so only this component can change the speed and kinetic energy of the particle. The component perpendicular to the path, $F_{\perp} = F\ \sin\phi$, has no effect on the particle's speed; it acts only to change the particle's direction.

To evaluate the line integral in Eq. (6.14) in a specific problem, we need some sort of detailed description of the path and of the way in which $\vec{F}$ varies along the path. We usually express the line integral in terms of some scalar variable, as in the following example.

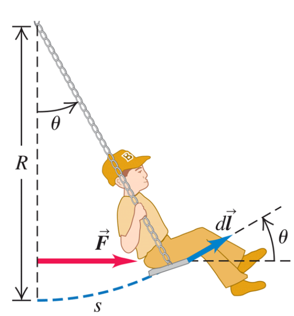
Example 6.8 Motion on a curved path

WITH VARIATION PROBLEMS

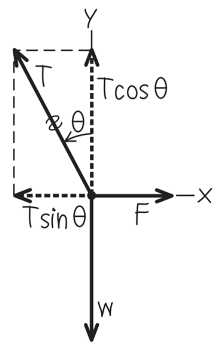
At a family picnic you are appointed to push your obnoxious cousin Throckmorton in a swing (Fig. 6.24a). His weight is w, the length of the chains is R, and you push Throcky until the chains make an angle θ_0 with the vertical. To do this, you exert a varying horizontal force $\vec{F}$ that starts at zero and gradually increases just enough that Throcky and the swing move very slowly and remain very nearly in equilibrium throughout the process. (a) What is the total work done on Throcky by all forces? (b) What is the work done by the tension T in the chains? (c) What is the work you do by exerting force $\vec{F}$? (Ignore the weight of the chains and seat.)

Figure 6.24

(a)



(b) Free-body diagram for Throckmorton (neglecting the weight of the chains and seat)



(a) Pushing cousin Throckmorton in a swing. (b) Our free-body diagram.

IDENTIFY and SET UP The motion is along a curve, so we'll use Eq. (6.14) to calculate the work done by the net force, by the tension force, and by the force $\vec{F}$. Figure 6.24b shows our free-body diagram and coordinate system for some arbitrary point in Throcky's motion. We have replaced the sum of the tensions in the two chains with a single tension T.

EXECUTE

a. There are two ways to find the total work done during the motion: (1) by calculating the work done by each force and then adding those quantities, and (2) by calculating the work done by the net force. The second approach is far easier here because Throcky is nearly in equilibrium at every point. Hence the net force on him is zero, the integral of the net force in Eq. (6.14) is zero, and the total work done on him is zero.

b. It's also easy to find the work done by the chain tension T because this force is perpendicular to the direction of motion at all points along the path. Hence at all points the angle between the chain tension and the displacement vector $d\vec{l}$ is 90° and the scalar product in Eq. (6.14) is zero. Thus the chain tension does zero work.

c. To compute the work done by $\vec{F}$, we need to calculate the line integral in Eq. (6.14). Inside the integral is the quantity $F\ \cos\phi\, dl$; let's see how to express each term in this quantity.

Figure 6.24a shows that the angle between $\vec{F}$ and $d\vec{l}$ is θ, so we replace ϕ in Eq. (6.14) with θ. The value of θ changes as Throcky moves.

To find the magnitude F of force $\vec{F}$, note that the net force on Throcky is zero (he is nearly in equilibrium at all points), so $\sum F_x = 0$ and $\sum F_y = 0$. From Fig. 6.24b,

$$\sum F_x = F + (-T\ \sin\theta) = 0 \quad \sum F_y = T\ \cos\theta + (-w) = 0$$

If you eliminate T from these two equations, you can show that $F = w\ \tan\theta$. As the angle θ increases, the tangent increases and F increases (you have to push harder).

To find the magnitude dl of the infinitesimal displacement $d\vec{l}$, note that Throcky moves through a circular arc of radius R (Fig. 6.24a). The arc length s equals the radius R multiplied by the length θ (in radians): $s = R\theta$. Therefore the displacement $d\vec{l}$ corresponding to a small change of angle $d\theta$ has a magnitude $dl = ds = R\, d\theta$.

When we put all the pieces together, the integral in Eq. (6.14) becomes

$$W = \int_{P_1}^{P_2} F\ \cos\phi\, dl = \int_0^{\theta_0} (w\tan\theta)\cos\theta\,(R\, d\theta) = \int_0^{\theta_0} wR\ \sin\theta\, d\theta$$

(Recall that tan $\tan\theta = \sin\theta/cos\ \theta$, so $\tan\theta\ cos\ \theta = sin\ \theta$.)

We've converted the *line* integral into an *ordinary* integral in terms of the angle θ. The limits of integration are from the starting position at $\theta = 0$ to the final position at $\theta = \theta_0$. The final result is

$$\begin{aligned} W &= wR\int_0^{\theta_0} \sin\theta\, d\theta = -wR\cos\theta\big|_0^{\theta_0} = -wR(\cos\theta_0 - 1) \\ &= wR\,(1 - \cos\,\theta_0) \end{aligned}$$

EVALUATE If $\theta_0 = 0$, there is no displacement; then $\cos\,\theta_0 = 1$ and $W = 0$, as we should expect. As θ_0 increases, $\cos\,\theta_0$ decreases and $W = wR(1 - \cos\,\theta_0)$ increases. So the farther along the arc you push Throcky, the more work you do. You can confirm that the quantity $R(1 - \cos\,\theta_0)$ is equal to h, the increase in Throcky's height during the displacement. So the work that you do to raise Throcky is just equal to his weight multiplied by the height that you raise him.

We can check our results by calculating the work done by the force of gravity $\vec{w}$. From part (a) the total work done on Throcky is zero, and from part (b) the work done by tension is zero. So gravity must do a negative amount of work that just balances the positive work done by the force $\vec{\boldsymbol{F}}$ that we calculated in part (c).

For variety, let's calculate the work done by gravity by using the form of Eq. (6.14) that involves the quantity $\vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{l}}$, and express the force $\vec{w}$ and displacement $d\vec{\boldsymbol{l}}$ in terms of their x- and y-components. The force of gravity has zero x-component and a y-component of $-w$. Figure 6.24a shows that $d\vec{\boldsymbol{l}}$ has a magnitude of ds, an x-component of $ds\ \cos\theta$, and a y-component of $ds\sin\theta$.

So

$$\begin{aligned} \vec{\boldsymbol{w}} &= \hat{\boldsymbol{j}}\,(-w) \\ d\vec{\boldsymbol{l}} &= \hat{\boldsymbol{\imath}}\,(ds\ \cos\,\theta) + \hat{\boldsymbol{j}}\,(ds\ \sin\,\theta) \end{aligned}$$

Use Eq. (1.19) to calculate the scalar product $\vec{w} \cdot d\vec{\boldsymbol{l}}$:

$$\vec{w} \cdot d\vec{\boldsymbol{l}} = (-w)(ds\ \sin\theta) = -w\ \sin\theta\, ds$$

Using $ds = R\,d\theta$, we find the work done by the force of gravity:

$$\int_{P_1}^{P_2} \vec{w} \cdot d\vec{\boldsymbol{l}} = \int_0^{\theta_0} (-w \sin\theta) R\,d\theta = -wR \int_0^{\theta_0} \sin\theta\,d\theta$$

The work done by gravity is indeed the negative of the work done by force $\vec{\boldsymbol{F}}$ that we calculated in part (c). Gravity does negative work because the force pulls downward while Throcky moves upward.

As we saw earlier, $R(1 - \cos\theta_0)$ is equal to h, the increase in Throcky's height during the displacement. So the work done by gravity along the curved path is $-mgh$, the *same* work that gravity would have done if Throcky had moved *straight upward* a distance h. This is an example of a more general result that we'll prove in Section 7.1.

KEY CONCEPT

The work–energy theorem can help you solve problems in which an object follows a curved path. Take care in calculating the work done on such a path.

Video Tutor Solution: **Example 6.8**

Test Your Understanding of Section 6.3

In Example 5.20 (Section 5.4) we examined a conical pendulum. The speed of the pendulum bob remains constant as it travels around the circle shown in Fig. 5.32a. (a) Over one complete circle, how much work does the tension force F do on the bob? (i) A positive amount; (ii) a negative amount; (iii) zero. (b) Over one complete circle, how much work does the weight do on the bob? (i) A positive amount; (ii) a negative amount; (iii) zero.

6.4 Power

The definition of work makes no reference to the passage of time. If you lift a barbell weighing 100 N through a vertical distance of 1.0 m at constant velocity, you do $(100\ \text{N})(1.0\ \text{m}) = 100\ \text{J}$ of work whether it takes you 1 second, 1 hour, or 1 year to do it. But often we need to know how quickly work is done. We describe this in terms of *power*. In ordinary conversation the word "power" is often synonymous with "energy" or "force." In physics we use a much more precise definition: **Power** is the time *rate* at which work is done. Like work and energy, power is a scalar quantity.

The average work done per unit time, or **average power** P_{av}, is defined to be

(6.15)

$$\underbrace{P_{\text{av}}}_{\text{Average power during time interval } \Delta t} = \frac{\overbrace{\Delta W}^{\text{Work done during time interval}}}{\underbrace{\Delta t}_{\text{Duration of time interval}}} \tag{6.15}$$

The rate at which work is done might not be constant. We define **instantaneous power** P as the quotient in Eq. (6.15) as Δt approaches zero:

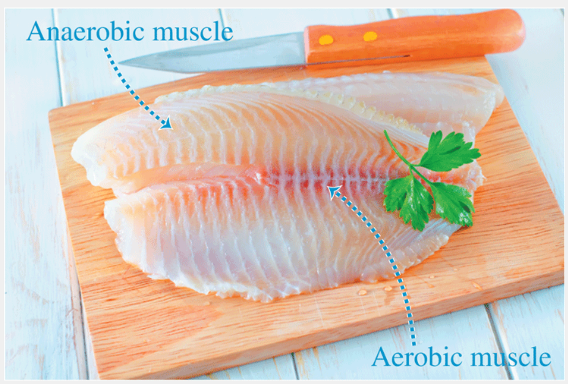
(6.16)

$$\underbrace{P}_{\text{Instantaneous power}} = \underbrace{\lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t}}_{\text{Average power over infinitesimally short time interval}} = \underbrace{\frac{dW}{dt}}_{\text{Time rate of doing work}} \tag{6.16}$$

BIO Application

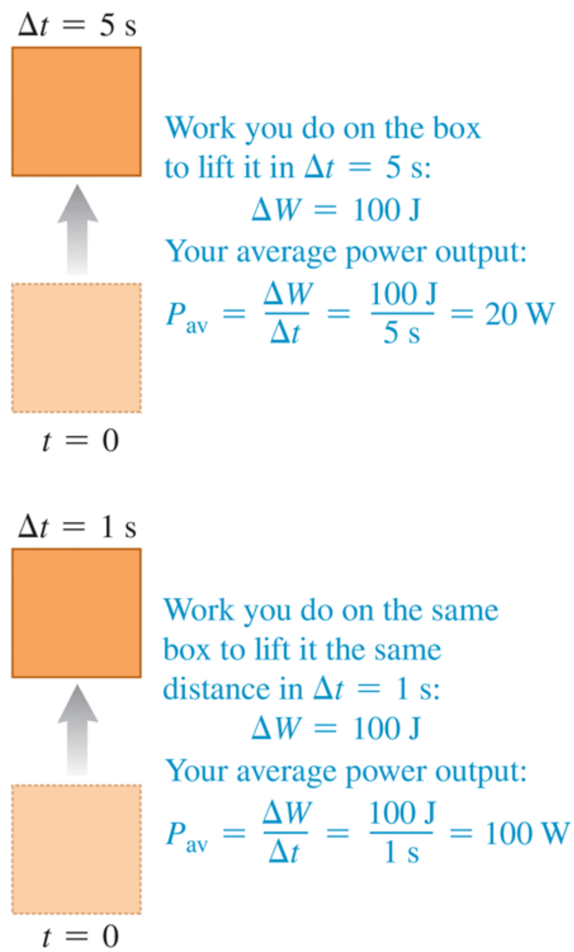
Muscle Power

Skeletal muscles provide the power that makes animals move. Muscle fibers that rely on anaerobic metabolism do not require oxygen; they produce large amounts of power but are useful for short sprints only. Muscle fibers that metabolize aerobically use oxygen and produce smaller amounts of power for long intervals. Both fiber types are visible in a fish fillet: The pale (anaerobic) muscle is used for brief bursts of speed, while the darker (aerobic) muscle is used for sustained swimming.



The SI unit of power is the **watt** (W), named for the English inventor James Watt. One watt equals 1 joule per second: $1\ \mathrm{W} = 1\ \mathrm{J}/s$ (Fig. 6.25). The kilowatt ($1\ \mathrm{kW} = 10^3\ \mathrm{W}$) and the megawatt ($1\ \mathrm{MW} = 10^6\ \mathrm{W}$) are also commonly used.

Figure 6.25



The same amount of work is done in both of these situations, but the power (the rate at which work is done) is different.

Another common unit of power is the *horsepower* (hp) (Fig. 6.26). The value of this unit derives from experiments by James Watt, who measured that in one minute a horse could do an amount of work equivalent to lifting 33,000 pounds (lb) a distance of 1 foot (ft), or 33,000 ft · lb. Thus $1 \text{ hp} = 33{,}000 \text{ ft} \cdot \text{lb/min}$. Using $1 \text{ ft} = 0.3048 \text{ m}$, $1 \text{ lb} = 4.448 \text{ N}$, and $1 \text{ min} = 60 \text{ s}$, we can show that

$$1 \text{ hp} = 746 \text{ W} = 0.746 \text{ kW}$$

Figure 6.26

A one-horsepower (746 W) propulsion system.

The watt is a familiar unit of *electrical* power; a 100 W light bulb converts 100 J of electrical energy into light and heat each second. But there's nothing inherently electrical about a watt. A light bulb could be rated in horsepower, and an engine can be rated in kilowatts.

The *kilowatt-hour* ($\mathrm{kW \cdot h}$) is the usual commercial unit of electrical energy. One kilowatt-hour is the total work done in 1 hour (3600 s) when the power is 1 kilowatt (10^3 J/s), so

$$1\ \mathrm{kW{\cdot}h} = (10^3\ \mathrm{J/s})(3600\ \mathrm{s}) = 3.6 \times 10^6\ \mathrm{J} = 3.6\ \mathrm{MJ}$$

The kilowatt-hour is a unit of *work* or *energy*, not power.

In mechanics we can also express power in terms of force and velocity. Suppose that a force $\vec{\boldsymbol{F}}$ acts on an object while it undergoes a vector displacement $\Delta\vec{\boldsymbol{s}}$. If $F_{\parallel}$ is the component of $\vec{\boldsymbol{F}}$ tangent to the path (parallel to $\Delta\vec{\boldsymbol{s}}$), then the work done by the force is $\Delta W = F_{\parallel}\Delta s$. The average power is

(6.17)

$$P_{\mathrm{av}} = \frac{F_{\|}\Delta s}{\Delta t} = F_{\|}\frac{\Delta s}{\Delta t} = F_{\|}\, v_{\mathrm{av}}$$

Instantaneous power P is the limit of this expression as $\Delta t \to 0$:

(6.18)

$$P = F_{\|}\, v$$

where v is the magnitude of the instantaneous velocity. We can also express Eq. (6.18) in terms of the scalar product:

(6.19)

Instantaneous power for a force doing work on a particle

Force that acts on particle

$$P = \vec{F} \cdot \vec{v} \qquad (6.19)$$

Velocity of particle

Example 6.9 Force and power

Each of the four jet engines on an Airbus A380 airliner develops a thrust (a forward force on the airliner) of 322,000 N (72,000 lb). When the airplane is flying at $250\,\mathrm{m/s}$ ($900\,\mathrm{km/h}$, or roughly $560\,\mathrm{mi/h}$), what horsepower does each engine develop?

IDENTIFY, SET UP, and EXECUTE Our target variable is the instantaneous power P, which is the rate at which the thrust does work. We use Eq. (6.18). The thrust is in the direction of motion, so $F_{\|}$ is just equal to the thrust. At $v = 250$ m/s, the power developed by each engine is

$$\begin{aligned} P &= F_{\|}v = (3.22 \times 10^5\ \mathrm{N})(250\ \mathrm{m/s}) = 8.05 \times 10^7\ \mathrm{W} \\ &= \left(8.05 \times 10^7\ \mathrm{W}\right)\frac{1\,\mathrm{hp}}{746\ \mathrm{W}} = 108,000\,\mathrm{hp} \end{aligned}$$

EVALUATE The speed of modern airliners is directly related to the power of their engines (Fig. 6.27). The largest propeller-driven airliners of the 1950s had engines that each developed about 3400 hp (2.5×10^6 W), giving them maximum speeds of about $600\,\mathrm{km/h}$ ($370\,\mathrm{mi/h}$). Each engine on an Airbus A380 develops more than 30 times more power, enabling it to fly at about $900\,\mathrm{km/h}$ ($560\,\mathrm{mi/h}$) and to carry a much heavier load.

Figure 6.27

(a)

(b)

(a) Propeller-driven and (b) jet airliners.

If the engines are at maximum thrust while the airliner is at rest on the ground so that $v = 0$, the engines develop *zero* power. Force and power are not the same thing!

KEY CONCEPT

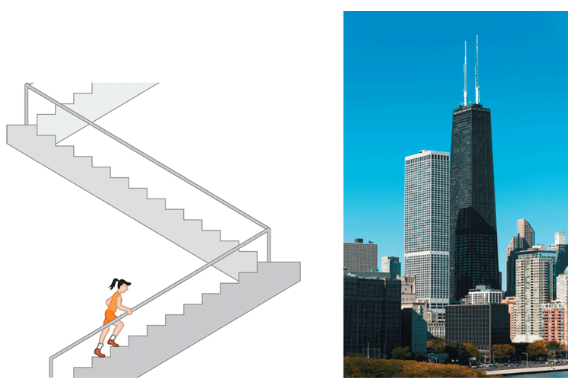
To find the power of a force acting on a moving object, multiply the component of force in the direction of motion by the object's speed.

Video Tutor Solution: **Example 6.9**

Example 6.10 A “power climb”

A 50.0 kg marathon runner runs up the stairs to the top of Chicago’s 443-m-tall Willis Tower, the second tallest building in the United States (Fig. 6.28). To lift herself to the top in 15.0 minutes, what must be her average power output? Express your answer in watts, in kilowatts, and in horsepower.

Figure 6.28



How much power is required to run up the stairs of Chicago’s Willis Tower in 15 minutes?

IDENTIFY and SET UP We'll treat the runner as a particle of mass m. Her average power output P_{av} must be enough to lift her at constant speed against gravity.

We can find P_{av} in two ways: (1) by determining how much work she must do and dividing that quantity by the elapsed time, as in Eq. (6.15), or (2) by calculating the average upward force she must exert (in the direction of the climb) and multiplying that quantity by her upward velocity, as in Eq. (6.17).

EXECUTE

a. As in Example 6.8, lifting a mass m against gravity requires an amount of work equal to the weight mg multiplied by the height h it is lifted. Hence the work the runner must do is

$$W = mgh = (50.0\ \mathrm{kg})(9.80\ \mathrm{m/s^2})(443\ \mathrm{m}) = 2.17 \times 10^5\ \mathrm{J}$$

She does this work in a time 15.0 min $= 900$ s, so from Eq. (6.15) the average power is

$$P_{\mathrm{av}} = \frac{2.17 \times 10^5\ \mathrm{J}}{900\ \mathrm{s}} = 241\ \mathrm{W} = 0.241\ \mathrm{kW} = 0.323\ \mathrm{hp}$$

b. The force exerted is vertical and the average vertical component of velocity is $(443\ \mathrm{m})/(900\ \mathrm{s}) = 0.492\ \mathrm{m/s}$, so from Eq. (6.17) the average power is

$$\begin{aligned} P_{\mathrm{av}} &= F_{\parallel} v_{\mathrm{av}} = (mg) v_{\mathrm{av}} \\ &= (50.0\,\mathrm{kg}) \left(9.80\,\mathrm{m/s^2}\right) (0.492\,\mathrm{m/s}) = 241\,\mathrm{W} \end{aligned}$$

which is the same result as before.

EVALUATE The runner's *total* power output will be several times greater than 241 W. The reason is that the runner isn't really a particle but a collection of parts that exert forces on each other and do work, such as the work done to inhale and exhale and to make her arms and legs swing. What we've calculated is only the part of her power output that lifts her to the top of the building.

KEY CONCEPT

To calculate average power (the average rate of doing work), divide the work done by the time required to do that work.

Video Tutor Solution: **Example 6.10**

Test Your Understanding of Section 6.4

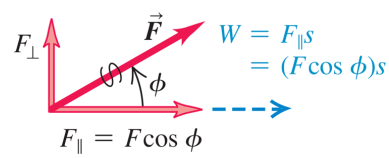
The air surrounding an airplane in flight exerts a drag force that acts opposite to the airplane's motion. When the Airbus A380 in Example 6.9 is flying in a straight line at a constant altitude at a constant $250 \mathrm{~m/s}$, what is the rate at which the drag force does work on it? (i) 432,000 hp; (ii) 108,000 hp; (iii) 0; (iv) $-108,000$ hp; (v) $-432,000$ hp.

Chapter 6 Summary

Work done by a force: When a constant force $\vec{F}$ acts on a particle that undergoes a straight-line displacement $\vec{s}$, the work done by the force on the particle is defined to be the scalar product of $\vec{F}$ and $\vec{s}$. The unit of work in SI units is $1 \text{ joule} = 1 \text{ newton-meter } (1 \text{ J} = 1 \text{ N} \cdot \text{m})$. Work is a scalar quantity; it can be positive or negative, but it has no direction in space. (See Examples 6.1 and 6.2.)

(6.2), (6.3)

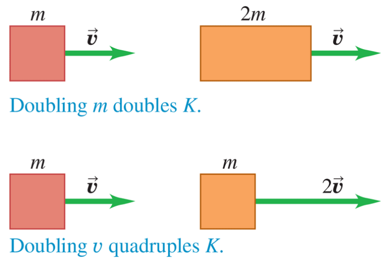
$$\begin{aligned} W &= \vec{F} \cdot \vec{s} = Fs \cos \phi \\ \phi &= \text{angle between } \vec{F} \text{ and } \vec{s} \end{aligned}$$



Kinetic energy: The kinetic energy K of a particle equals the amount of work required to accelerate the particle from rest to speed v. It is also equal to the amount of work the particle can do in the process of being brought to rest. Kinetic energy is a scalar that has no direction in space; it is always positive or zero. Its units are the same as the units of work: $1 \text{ J} = 1 \text{ N} \cdot \text{m} = 1 \text{ kg} \cdot \text{m}^2/\text{s}^2$.

(6.5)

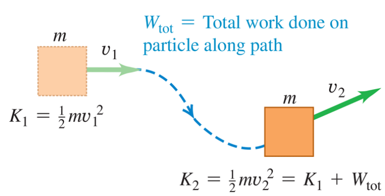
$$K = \frac{1}{2}mv^2$$



The work–energy theorem: When forces act on a particle while it undergoes a displacement, the particle's kinetic energy changes by an amount equal to the total work done on the particle by all the forces. This relationship, called the work–energy theorem, is valid whether the forces are constant or varying and whether the particle moves along a straight or curved path. It is applicable only to objects that can be treated as particles. (See Examples 6.3, 6.4 and 6.5.)

(6.6)

$$W_{\text{tot}} = K_2 - K_1 = \Delta K$$



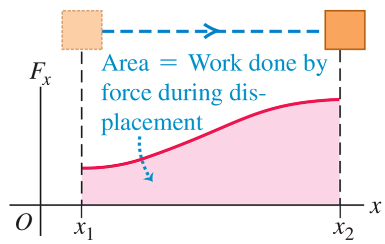
Work done by a varying force or on a curved path: When a force varies during a straight-line displacement, the work done by the force is given by an integral, Eq. (6.7). (See Examples 6.6 and 6.7.) When a particle follows a curved path, the work done on it by a force $\vec{F}$ is given by an integral that involves the angle ϕ between the force and the displacement. This expression is valid even if the force magnitude and the angle ϕ vary during the displacement. (See Example 6.8.)

(6.7)

$$W = \int_{x_1}^{x_2} F_x \, dx$$

(6.14)

$$\begin{aligned} W &= \int_{P_1}^{P_2} \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{l}} \\ &= \int_{P_1}^{P_2} F \, \cos\phi \, dl = \int_{P_1}^{P_2} F_{\|} \, dl \end{aligned}$$



Power: Power is the time rate of doing work. The average power P_{av} is the amount of work ΔW done in time Δt divided by that time. The instantaneous power is the limit of the average power as Δt goes to zero. When a force $\vec{\boldsymbol{F}}$ acts on a particle moving with velocity $\vec{\boldsymbol{v}}$, the instantaneous power (the rate at which the force does work) is the scalar product of $\vec{\boldsymbol{F}}$ and $\vec{\boldsymbol{v}}$. Like work and kinetic energy, power is a scalar quantity. The SI unit of power is 1 watt = 1 joule/second (1 W = 1 J/s). (See Examples 6.9 and 6.10.)

(6.15)

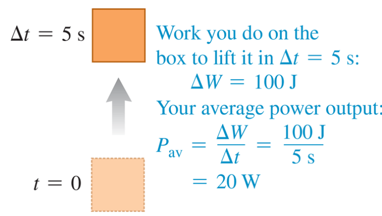
$$P_{av} = \frac{\Delta W}{\Delta t}$$

(6.16)

$$P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt}$$

(6.19)

$$P = \vec{F} \cdot \vec{v}$$



Guided Practice: Work and Kinetic Energy

For assigned homework and other learning materials, go to
Mastering Physics.

Key Example Variation Problems

Be sure to review Examples 6.1 and 6.2 (Section 6.1) before attempting these problems.

VP6.2.1 As a football player moves in a straight line [displacement $(3.00\ \text{m})\hat{\imath} - (6.50\ \text{m})\hat{\jmath}$], an opponent exerts a constant force $(126\ \text{N})\hat{\imath} + (168\ \text{N})\hat{\jmath}$ on him. (a) How much work does the opponent do on the football player? (b) How much work does the football player do on the opponent?

VP6.2.2 You push a stalled car with a constant force of 215 N as it moves a distance 8.40 m in a straight line. The amount of work that you do in this process is 1.47×10^3J. What is the angle between the direction of your push and the direction of the car's motion?

VP6.2.3 A block of mass 15.0 kg slides down a ramp inclined at 28.0° above the horizontal. As it slides, a kinetic friction force of 30.0 N parallel to the ramp acts on it. If the block slides for 3.00 m along the ramp, find (a) the work done on the block by friction, (b) the work done on the block by the force of gravity, (c) the work done on the block by the normal force, and (d) the total work done on the block.

VP6.2.4 Three students are fighting over a T-shirt. Student 1 exerts a constant force $\vec{\boldsymbol{F}}_1 = F_0\,\hat{\boldsymbol{\imath}}$ on the shirt, student 2 exerts a constant force $\vec{\boldsymbol{F}}_2 = -\boldsymbol{3}F_0\,\hat{\boldsymbol{\jmath}}$ and student 3 exerts a constant force $\vec{\boldsymbol{F}}_3 = -4F_0\,\hat{\boldsymbol{\imath}} + G\,\hat{\boldsymbol{\jmath}}$. (In these expressions F_0 and G are positive constants with units of force.) As the three students exert these forces, the T-shirt undergoes a straight-line displacemen $2d\hat{\boldsymbol{\imath}} + d\hat{\boldsymbol{\jmath}}$ where d is a positive constant with units of distance. (a) Find the work done on the T-shirt by each student. (b) What must be the value of G in order for the total work to be equal to zero?

Be sure to review Examples 6.3 and 6.4 (Section 6.2) before attempting these problems.

VP6.4.1 A nail is partially inserted into a block of wood, with a length of 0.0300 m protruding above the top of the block. To hammer the nail in the rest of the way, you drop a 20.0 kg metal cylinder onto it. The cylinder rides on vertical tracks that exert an upward friction force of 16.0 N on the cylinder as it falls. You release the cylinder from rest at a height of 1.50 m above the top of the nail. The cylinder comes to rest on top of the block of wood, with the nail fully inside the block. Use the work–energy theorem to find (a) the speed of the cylinder just as it hits the nail and (b) the average force the cylinder exerts on the nail while pushing it into the block. Ignore the effects of the air.

VP6.4.2 You are using a rope to lift a 14.5 kg crate of fruit. Initially you are lifting the crate at 0.500 m/s. You then increase the tension in the rope to 175 N and lift the crate an additional

1.25 m. During this 1.25 m motion, how much work is done on the crate by (a) the tension force, (b) the force of gravity, and (c) the net force? (d) What are the kinetic energy and speed of the crate after being lifted the additional 1.25 m?

VP6.4.3 A helicopter of mass 1.40×10^3 kg is descending vertically at 3.00 m/s. The pilot increases the upward thrust provided by the main rotor so that the vertical speed decreases to 0.450 m/s as the helicopter descends 2.00 m. (a) What are the initial and final kinetic energies of the helicopter? (b) As the helicopter descends this 2.00 m distance, how much total work is done on it? How much work is done on it by gravity? How much work is done on it by the upward thrust force? (c) What is the magnitude of the upward thrust force (assumed constant)?

VP6.4.4 A block of mass m is released from rest on a ramp that is inclined at an angle θ from the horizontal. The coefficient of kinetic friction between the block and the ramp is μ_k. The block slides a distance d along the ramp until it reaches the bottom. (a) How much work is done on the block by the force of gravity? (b) How much work is done on the block by the friction force? (c) Use the work–energy theorem to find the speed of the block just as it reaches the bottom of the ramp.

Be sure to review Examples 6.7 and 6.8 (Section 6.3) before attempting these problems.

VP6.8.1 An air-track glider of mass 0.150 kg is attached to the end of a horizontal air track by a spring with force constant 30.0 N/m (see Fig.6.22a). Initially the spring is unstretched and the glider is moving at 1.25 m/s to the right. Find the maximum distance d that the glider moves to the right (a) if the air track is turned on, so that there is no friction, and (b) if the air is turned off, so that there is kinetic friction with coefficient $\mu_{\mathrm{k}} = 0.320$.

VP6.8.2 An air-track glider of mass m is attached to the end of a horizontal air track by a spring with force constant k as in Fig.6.22a. The air track is turned off, so there is friction between the glider and the track. Initially the spring is unstretched, but unlike the situation in Fig. 6.22a, the glider is initially moving to the *left* at speed v_1 The glider moves a distance d to the left before coming momentarily to rest. Use the work–energy theorem to find the coefficient of kinetic friction between the glider and the track.

VP6.8.3 A pendulum is made up of a small sphere of mass 0.500 kg attached to a string of length 0.750 m. The sphere is swinging back and forth between point A, where the string is at the maximum angle of 35.0° to the left of vertical, and point C, where the string is at the maximum angle of 35.0° to the right of vertical. The string is vertical when the sphere is at point B. Calculate how much work the force of gravity does on the sphere (a) from A to B, (b) from B to C, and (c) from A to C.

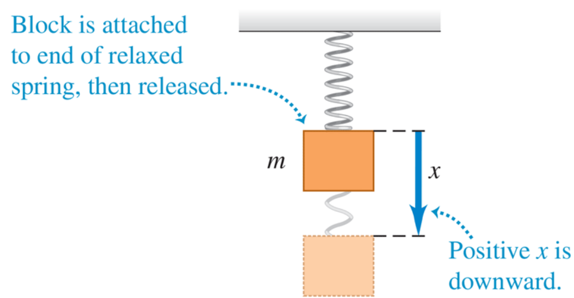
VP6.8.4 A spider of mass m is swinging back and forth at the end of a strand of silk of length L. During the spider's swing the strand makes a maximum angle of θ with the vertical. What is the speed of the spider at the low point of its motion, when the strand of silk is vertical?

Bridging Problem: A Spring That Disobeys Hooke's Law

Consider a hanging spring of negligible mass that does *not* obey Hooke's law. When the spring is pulled downward by a distance x, the spring exerts an upward force of magnitude αx^2, where α is a positive constant.

Initially the hanging spring is relaxed (not extended). We then attach a block of mass m to the spring and release the block. The block stretches the spring as it falls (Fig. 6.29). (a) How fast is the block moving when it has fallen a distance x_1? (b) At what rate does the spring do work on the block at this point? (c) Find the maximum distance x_2 that the spring stretches. (d) Will the block *remain* at the point found in part (c)?

Figure 6.29



The block is attached to a spring that does not obey Hooke's law.

Solution Guide

IDENTIFY and SET UP

1. The spring force in this problem isn't constant, so you have to use the work–energy theorem. You'll also need Eq. (6.7) to find the work done by the spring over a given displacement.
2. Draw a free-body diagram for the block, including your choice of coordinate axes. Note that x represents how far the spring is *stretched*, so choose the positive x-direction to be downward, as in Fig. 6.29. On your coordinate axis, label the points $x = x_1$ and $x = x_2$.
3. Make a list of the unknown quantities, and decide which of these are the target variables.

EXECUTE

4. Calculate the work done on the block by the spring as the block falls an arbitrary distance x. (The integral isn't a difficult one. Use Appendix B if you need a reminder.) Is the work done by the spring positive, negative, or zero?
5. Calculate the work done on the block by any other forces as the block falls an arbitrary distance x. Is this work positive, negative, or zero?
6. Use the work–energy theorem to find the target variables. (You'll also need an equation for power.) **HINT:** When the spring is at its maximum stretch, what is the speed of the block?
7. To answer part (d), consider the *net* force that acts on the block when it is at the point found in part (c).

EVALUATE

8. We learned in Section 2.5 that after an object dropped from rest has fallen freely a distance x_1, its speed is $\sqrt{2gx_1}$. Use this to decide whether your answer in part (a) makes sense. In addition, ask yourself whether the algebraic sign of your answer in part (b) makes sense.
9. Find the value of x where the net force on the block would be zero. How does this compare to your result for x_2? Is this consistent with your answer in part (d)?

Video Tutor Solution: **Chapter 6 Bridging Problem**

Questions/Exercises/Problems: Work and Kinetic Energy

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

Discussion Questions

Q6.1 The sign of many physical quantities depends on the choice of coordinates. For example, a_y for free-fall motion can be negative or positive, depending on whether we choose upward or downward as positive. Is the same true of work? In other words, can we make positive work negative by a different choice of coordinates? Explain.

Q6.2 An elevator is hoisted by its cables at constant speed. Is the total work done on the elevator positive, negative, or zero? Explain.

Q6.3 A rope tied to an object is pulled, causing the object to accelerate. But according to Newton's third law, the object pulls back on the rope with a force of equal magnitude and opposite direction. Is the total work done then zero? If so, how can the object's kinetic energy change? Explain.

Q6.4 If it takes total work W to give an object a speed v and kinetic energy K, starting from rest, what will be the object's speed (in terms of v) and kinetic energy (in terms of K) if we do twice as much work on it, again starting from rest?

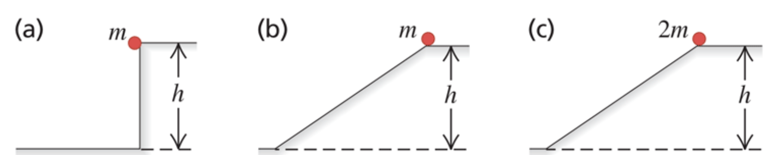
Q6.5 If there is a net nonzero force on a moving object, can the total work done on the object be zero? Explain, using an example.

Q6.6 In Example 5.5 (Section 5.1), how does the work done on the bucket by the tension in the cable compare with the work done on the cart by the tension in the cable?

Q6.7 In the conical pendulum of Example 5.20 (Section 5.4), which of the forces do work on the bob while it is swinging?

Q6.8 For the cases shown in Fig. Q6.8, the object is released from rest at the top and feels no friction or air resistance. In which (if any) cases will the mass have (i) the greatest speed at the bottom and (ii) the most work done on it by the time it reaches the bottom?

Figure Q6.8



Q6.9 A force $\vec{F}$ is in the x-direction and has a magnitude that depends on x. Sketch a possible graph of F versus x such that the force does zero work on an object that moves from x_1 to x_2, even though the force magnitude is not zero at all x in this range.

Q6.10 Does a car's kinetic energy change more when the car speeds up from 10 to 15 m/s or from 15 to 20 m/s? Explain.

Q6.11 A falling brick has a mass of 1.5 kg and is moving straight downward with a speed of 5.0 m/s. A 1.5 kg physics book is sliding across the floor with a speed of 5.0 m/s. A 1.5 kg melon is traveling with a horizontal velocity component 3.0 m/s to the right and a vertical component 4.0 m/s upward. Do all of these objects have the same velocity? Do all of them have the same kinetic energy? For both questions, give your reasoning.

Q6.12 Can the *total* work done on an object during a displacement be negative? Explain. If the total work is negative, can its magnitude be larger than the initial kinetic energy of the object? Explain.

Q6.13 A net force acts on an object and accelerates it from rest to a speed v_1. In doing so, the force does an amount of work W_1. By what factor must the work done on the object be increased to produce three times the final speed, with the object again starting from rest?

Q6.14 A truck speeding down the highway has a lot of kinetic energy relative to a stopped state trooper but no kinetic energy relative to the truck driver. In these two frames of reference, is the same amount of work required to stop the truck? Explain.

Q6.15 You are holding a briefcase by the handle, with your arm straight down by your side. Does the force your hand exerts do work on the briefcase when (a) you walk at a constant speed down a horizontal hallway and (b) you ride an escalator from the first to second floor of a building? In both cases justify your answer.

Q6.16 When a book slides along a tabletop, the force of friction does negative work on it. Can friction ever do *positive* work? Explain. (**HINT:** Think of a box in the back of an accelerating truck.)

Q6.17 Time yourself while running up a flight of steps, and compute the average rate at which you do work against the force of gravity. Express your answer in watts and in horsepower.

Q6.18 **Fractured Physics**. Many terms from physics are badly misused in everyday language. In both cases, explain the errors involved. (a) A *strong* person is called *powerful*. What is wrong with this use of *power*? (b) When a worker carries a bag of concrete along a level construction site, people say he did a lot of *work*. Did he?

Q6.19 An advertisement for a portable electrical generating unit claims that the unit's diesel engine produces 28,000 hp to drive an electrical generator that produces 30 MW of electrical power. Is this possible? Explain.

Q6.20 A car speeds up while the engine delivers constant power. Is the acceleration greater at the beginning of this process or at the end? Explain.

Q6.21 Consider a graph of instantaneous power versus time, with the vertical P-axis starting at $P = 0$. What is the physical significance of the area under the P-versus-T curve between vertical lines at t_1 and t_2 ? How could you find the average power from the graph? Draw a P-versus-T curve that consists of two straight-line sections and for which the peak power is equal to twice the average power.

Q6.22 A nonzero net force acts on an object. Is it possible for any of the following quantities to be constant: the object's (a) speed; (b) velocity; (c) kinetic energy?

Q6.23 When a certain force is applied to an ideal spring, the spring stretches a distance x from its unstretched length and does work W. If instead twice the force is applied, what distance (in terms of x) does the spring stretch from its unstretched length, and how much work (in terms of W) is required to stretch it this distance?

Q6.24 If work W is required to stretch an ideal spring a distance x from its unstretched length, what work (in terms of W) is required to stretch the spring an *additional* distance x?

Exercises

Section 6.1 Work

6.1 • You push your physics book 1.50 m along a horizontal tabletop with a horizontal push of 2.40 N while the opposing force of friction is 0.600 N. How much work does each of the following forces do on the book: (a) your 2.40 N push, (b) the friction force, (c) the normal force from the tabletop, and (d) gravity? (e) What is the net work done on the book?

6.2 • Using a cable with a tension of 1350 N, a tow truck pulls a car 5.00 km along a horizontal roadway. (a) How much work does the cable do on the car if it pulls horizontally? If it pulls at 35.0° above the horizontal? (b) How much work does the cable do on the tow truck in both cases of part (a)? (c) How much work does gravity do on the car in part (a)?

6.3 • A factory worker pushes a 30.0 kg crate a distance of 4.5 m along a level floor at constant velocity by pushing horizontally on it. The coefficient of kinetic friction between the crate and the floor is 0.25. (a) What magnitude of force must the worker apply? (b) How much work is done on the crate by this force? (c) How much work is done on the crate by friction? (d) How much work is done on the crate by the normal force? By gravity? (e) What is the total work done on the crate?

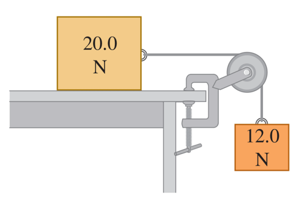
6.4 •• Suppose the worker in Exercise 6.3 pushes downward at an angle of 30° below the horizontal. (a) What magnitude of force must the worker apply to move the crate at constant velocity? (b) How much work is done on the crate by this force when the crate is pushed a distance of 4.5 m? (c) How much work is done on the crate by friction during this displacement? (d) How much work is done on the crate by the normal force? By gravity? (e) What is the total work done on the crate?

6.5 •• A 75.0 kg painter climbs a ladder that is 2.75 m long and leans against a vertical wall. The ladder makes a 30.0° angle with the wall. (a) How much work does gravity do on the painter? (b) Does the answer to part (a) depend on whether the painter climbs at constant speed or accelerates up the ladder?

6.6 •• Two tugboats pull a disabled supertanker. Each tug exerts a constant force of 1.80×10^6 N, one 14° west of north and the other 14° east of north, as they pull the tanker 0.75 km toward the north. What is the total work they do on the supertanker?

6.7 • Two blocks are connected by a very light string passing over a massless and frictionless pulley (Fig. E6.7). Traveling at constant speed, the 20.0 N block moves 75.0 cm to the right and the 12.0 N block moves 75.0 cm downward. (a) How much work is done on the 12.0 N block by (i) gravity and (ii) the tension in the string? (b) How much work is done on the 20.0 N block by (i) gravity, (ii) the tension in the string, (iii) friction, and (iv) the normal force? (c) Find the total work done on each block.

Figure E6.7



6.8 •• A loaded grocery cart is rolling across a parking lot in a strong wind. You apply a constant force

$\vec{F} = (30\,\mathrm{N})\hat{\imath} - (40\,\mathrm{N})\hat{\jmath}$ to the cart as it undergoes a displacement $\vec{s} = (-9.0\,\mathrm{m})\hat{\imath} - (3.0\,\mathrm{m})\hat{\jmath}$. How much work does the force you apply do on the grocery cart?

6.9 • Your physics book is resting in front of you on a horizontal table in the campus library. You push the book over to your friend, who is seated at the other side of the table, 0.400 m north and 0.300 m east of you. If you push the book in a straight line to your friend, friction does -4.8 J of work on the book. If instead you push the book 0.400 m due north and then 0.300 m due east, how much work is done by friction?

6.10 •• A 12.0 kg package in a mail-sorting room slides 2.00 m down a chute that is inclined at 53.0° below the horizontal. The coefficient of kinetic friction between the package and the chute's surface is 0.40. Calculate the work done on the package by (a) friction, (b) gravity, and (c) the normal force. (d) What is the net work done on the package?

6.11 • A force $\vec{F}$ that is at an angle 60° above the horizontal is applied to a box that moves on a horizontal frictionless surface, and the force does work W as the box moves a distance d. (a) At what angle above the horizontal would the force have to be directed in order for twice the work to be done for the same displacement of the box? (b) If the angle is kept at 60° and the box is initially at rest, by what factor would F have to be increased to double the final speed of the box after moving distance d?

6.12 •• A boxed 10.0 kg computer monitor is dragged by friction 5.50 m upward along a conveyor belt inclined at an angle of 36.9° above the horizontal. If the monitor's speed is a constant 2.10 cm/s, how much work is done on the monitor by (a) friction, (b) gravity, and (c) the normal force of the conveyor belt?

6.13 •• A large crate sits on the floor of a warehouse. Paul and Bob apply constant horizontal forces to the crate. The force applied by Paul has magnitude 48.0 N and direction 61.0° south of west. How much work does Paul's force do during a displacement of the crate that is 12.0 m in the direction 22.0° east of north?

6.14 •• You apply a constant force $\vec{\boldsymbol{F}} = (-68.0\,\mathrm{N})\hat{\imath} + (36.0\,\mathrm{N})\hat{\jmath}$ to a 380 kg car as the car travels 48.0 m in a direction that is 240.0° counterclockwise from the $+x$-axis. How much work does the force you apply do on the car?

6.15 •• On a farm, you are pushing on a stubborn pig with a constant horizontal force with magnitude 30.0 N and direction 37.0° counterclockwise from the $+x$-axis. How much work does this force do during a displacement of the pig that is (a) $\vec{\boldsymbol{s}} = (5.00\ \mathrm{m})\hat{\boldsymbol{\imath}}$; (b) $\vec{\boldsymbol{s}} = -(6.00\ \mathrm{m})\hat{\boldsymbol{\jmath}}$; (c) $\vec{\boldsymbol{s}} = -\,(2.00\,\mathrm{m})\hat{\boldsymbol{\imath}} + (4.00\,\mathrm{m})\hat{\boldsymbol{\jmath}}$?

Section 6.2 Kinetic Energy and the Work–Energy Theorem

6.16 •• A 1.50 kg book is sliding along a rough horizontal surface. At point A it is moving at $3.21\,\mathrm{m/s}$, and at point B it has slowed to $1.25\,\mathrm{m/s}$. (a) How much total work was done on the book between A and B? (b) If -0.750 J of total work is done on the book from B to C, how fast is it moving at point C? (c) How fast would it be moving at C if $+0.750$ J of total work was done on it from B to C?

6.17 •• BIO **Animal Energy**. Adult cheetahs, the fastest of the great cats, have a mass of about 70 kg and have been clocked to run at up to $72\ \mathrm{mi/h}\ \ (32\ \mathrm{m/s})$. (a) How many joules of kinetic energy does such a swift cheetah have? (b) By what factor would its kinetic energy change if its speed were doubled?

6.18 •• A baseball has a mass of 0.145 kg. (a) In batting practice a batter hits a ball that is sitting at rest on top of a post. The ball leaves the post with a horizontal speed of 30.0 $\mathrm{m/s}$. How much work did the force applied by the bat do on the ball? (b) During a game the same batter swings at a ball thrown by the pitcher and hits a line drive. Just before the ball is hit it is traveling at a speed of 20.0 $\mathrm{m/s}$, and just after it is hit it is traveling in the opposite direction at a speed of 30.0 $\mathrm{m/s}$. What is the total work done on the baseball by the force exerted by the bat? (c) How do the results of parts (a) and (b) compare? Explain.

6.19 • **Meteor Crater**. About 50,000 years ago, a meteor crashed into the earth near present-day Flagstaff, Arizona. Measurements from 2005 estimate that this meteor had a mass of about 1.4×10^8 kg (around 150,000 tons) and hit the ground at a speed of 12 $\mathrm{km/s}$. (a) How much kinetic energy did this meteor deliver to the ground? (b) How does this energy compare to the energy released by a 1.0 megaton nuclear bomb? (A megaton bomb releases the same amount of energy as a million tons of TNT, and 1.0 ton of TNT releases 4.184×10^9 J of energy.)

6.20 • A 4.80 kg watermelon is dropped from rest from the roof of an 18.0 m-tall building and feels no appreciable air resistance. (a) Calculate the work done by gravity on the watermelon during its displacement from the roof to the ground. (b) Just before it strikes the ground, what are the watermelon's (i) kinetic energy and (ii) speed? (c) Which of the answers in parts (a) and (b) would be *different* if there were appreciable air resistance?

6.21 • **CP** You are pushing a large box across a frictionless floor by applying a constant horizontal force. If the box starts at rest, you have to do work W_1 in order for the box to travel a distance d in time T. How much work would you have to do, in terms of W_1, to make the box go the same distance in half the time?

6.22 •• Use the work–energy theorem to solve each of these problems. You can use Newton's laws to check your answers. (a) A skier moving at 5.00 m/s encounters a long, rough horizontal patch of snow having a coefficient of kinetic friction of 0.220 with her skis. How far does she travel on this patch before stopping? (b) Suppose the rough patch in part (a) was only 2.90 m long. How fast would the skier be moving when she reached the end of the patch? (c) At the base of a frictionless icy hill that rises at 25.0° above the horizontal, a toboggan has a speed of 12.0 m/s toward the hill. How high vertically above the base will it go before stopping?

6.23 •• You are a member of an Alpine Rescue Team. You must project a box of supplies up an incline of constant slope angle α so that it reaches a stranded skier who is a vertical distance h above the bottom of the incline. The incline is slippery, but there is some friction present, with kinetic friction coefficient

μ_k. Use the work–energy theorem to calculate the minimum speed you must give the box at the bottom of the incline so that it will reach the skier. Express your answer in terms of g, h, μ_k, and α.

6.24 •• You throw a 3.00 N rock vertically into the air from ground level. You observe that when it is 15.0 m above the ground, it is traveling at 25.0 m/s upward. Use the work–energy theorem to find (a) the rock's speed just as it left the ground and (b) its maximum height.

6.25 • A sled with mass 12.00 kg moves in a straight line on a frictionless, horizontal surface. At one point in its path, its speed is 4.00 m/s; after it has traveled 2.50 m beyond this point, its speed is 6.00 m/s. Use the work–energy theorem to find the net force acting on the sled, assuming that this force is constant and that it acts in the direction of the sled's motion.

6.26 •• A soccer ball with mass 0.420 kg is initially moving with speed 2.00 m/s. A soccer player kicks the ball, exerting a constant force of magnitude 40.0 N in the same direction as the ball's motion. Over what distance must the player's foot be in contact with the ball to increase the ball's speed to 6.00 m/s?

6.27 • A 12-pack of Omni-Cola (mass 4.30 kg) is initially at rest on a horizontal floor. It is then pushed in a straight line for 1.20 m by a trained dog that exerts a horizontal force with magnitude 36.0 N. Use the work–energy theorem to find the final speed of the 12-pack if (a) there is no friction between the 12-pack and the floor, and (b) the coefficient of kinetic friction between the 12-pack and the floor is 0.30.

6.28 •• A block of ice with mass 2.00 kg slides 1.35 m down an inclined plane that slopes downward at an angle of 36.9° below the horizontal. If the block of ice starts from rest, what is its final speed? Ignore friction.

6.29 • Object A has 27 J of kinetic energy. Object B has one-quarter the mass of object A. (a) If object B also has 27 J of kinetic energy, is it moving faster or slower than object A? By what factor? (b) By what factor does the speed of each object change if total work -18 J is done on each?

6.30 •• A 30.0 kg crate is initially moving with a velocity that has magnitude 3.90 m/s in a direction 37.0° west of north. How much work must be done on the crate to change its velocity to 5.62 m/s in a direction 63.0° south of east?

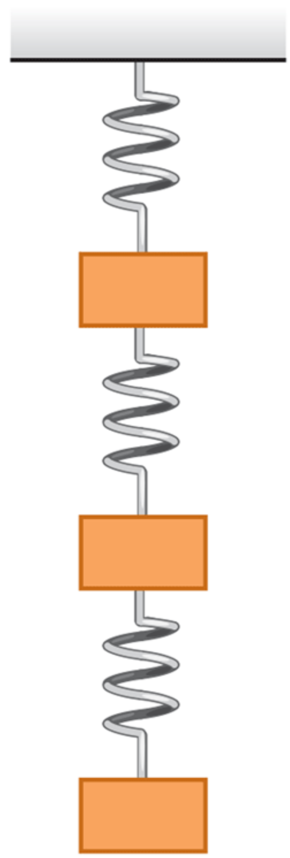
6.31 • **Stopping Distance**. A car is traveling on a level road with speed v_0 at the instant when the brakes lock, so that the tires slide rather than roll. (a) Use the work–energy theorem to calculate the minimum stopping distance of the car in terms of v_0, g, and the coefficient of kinetic friction μ_k between the tires and the road. (b) By what factor would the minimum stopping distance change if (i) the coefficient of kinetic friction were doubled, or (ii) the initial speed were doubled, or (iii) both the coefficient of kinetic friction and the initial speed were doubled?

Section 6.3 Work and Energy with Varying Forces

6.32 •• To stretch an ideal spring 3.00 cm from its unstretched length, 12.0 J of work must be done. (a) What is the force constant of this spring? (b) What magnitude force is needed to stretch the spring 3.00 cm from its unstretched length? (c) How much work must be done to compress this spring 4.00 cm from its unstretched length, and what force is needed to compress it this distance?

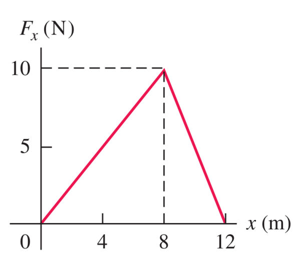
6.33 • Three identical 8.50 kg masses are hung by three identical springs (Fig. E6.33). Each spring has a force constant of 7.80 kN/m and was 12.0 cm long before any masses were attached to it. (a) Draw a free-body diagram of each mass. (b) How long is each spring when hanging as shown? (**HINT:** First isolate only the bottom mass. Then treat the bottom two masses as a system. Finally, treat all three masses as a system.)

Figure E6.33



6.34 • A child applies a force $\vec{F}$ parallel to the x-axis to a 10.0 kg sled moving on the frozen surface of a small pond. As the child controls the speed of the sled, the x-component of the force she applies varies with the x-coordinate of the sled as shown in Fig. E6.34. Calculate the work done by $\vec{F}$ when the sled moves (a) from $x = 0$ to $x = 8.0$ m; (b) from $x = 8.0$ m to $x = 12.0$ m; (c) from $x = 0$ to 12.0 m.

Figure E6.34



6.35 •• Suppose the sled in Exercise 6.34 is initially at rest at $x = 0$. Use the work–energy theorem to find the speed of the sled at (a) $x = 8.0$ m and (b) $x = 12.0$ m. Ignore friction between the sled and the surface of the pond.

6.36 •• A spring of force constant 300.0 N/m and unstretched length 0.240 m is stretched by two forces, pulling in opposite directions at opposite ends of the spring, that increase to 15.0 N. How long will the spring now be, and how much work was required to stretch it that distance?

6.37 •• A 6.0 kg box moving at 3.0 m/s on a horizontal, frictionless surface runs into one end of a light horizontal spring of force constant 75 N/cm that is fixed at the other end. Use the work–energy theorem to find the maximum compression of the spring.

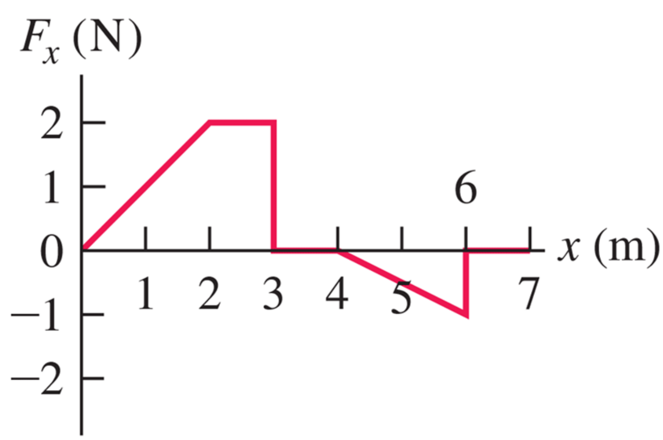
6.38 •• **Leg Presses**. As part of your daily workout, you lie on your back and push with your feet against a platform attached to two stiff ideal springs arranged side by side so that they are parallel to each other. When you push the platform, you compress the springs. You do 80.0 J of work when you compress the springs 0.200 m from their uncompressed length. (a) What magnitude of force must you apply to hold the platform in this position? (b) How much *additional* work must you do to move the platform 0.200 m *farther*, and what maximum force must you apply?

6.39 •• (a) In Example 6.7 (Section 6.3) it was calculated that with the air track turned off, the glider travels 8.6 cm before it stops instantaneously. How large would the coefficient of static friction μ_s have to be to keep the glider from springing back to the left? (b) If the coefficient of static friction between the glider and the track is $\mu_s = 0.60$, what is the maximum initial speed v_1 that the glider can be given and still remain at rest after it stops instantaneously? With the air track turned off, the coefficient of kinetic friction is $\mu_k = 0.47$.

6.40 • A 4.00 kg block of ice is placed against one end of a horizontal spring that is fixed at the other end, has force constant $k = 200$ N/m and is compressed 0.025 m. The spring is released and accelerates the block along a horizontal surface. Ignore friction and the mass of the spring. (a) Calculate the work done on the block by the spring during the motion of the block from its initial position to where the spring has returned to its uncompressed length. (b) What is the speed of the block after it leaves the spring?

6.41 • A force $\vec{F}$ is applied to a 2.0 kg, radio-controlled model car parallel to the x-axis as it moves along a straight track. The x-component of the force varies with the x-coordinate of the car (Fig. E6.41). Calculate the work done by the force $\vec{F}$ when the car moves from (a) $x = 0$ to $x = 3.0$ m; (b) $x = 3.0$ m to $x = 4.0$ m; (c) $x = 4.0$ m to $x = 7.0$ m; (d) $x = 0$ to $x = 7.0$ m; (e) $x = 7.0$ m to $x = 2.0$ m.

Figure E6.41



6.42 • Suppose the 2.0 kg model car in Exercise 6.41 is initially at rest at $x = 0$ and $\vec{F}$ is the net force acting on it. Use the work–energy theorem to find the speed of the car at (a) $x = 3.0\text{ m}$; (b) $x = 4.0\text{ m}$; (c) $x = 7.0\text{ m}$.

6.43 •• At a waterpark, sleds with riders are sent along a slippery, horizontal surface by the release of a large compressed spring. The spring, with force constant $k = 40.0\text{ N/cm}$ and negligible mass, rests on the frictionless horizontal surface. One end is in contact with a stationary wall. A sled and rider with total mass 70.0 kg are pushed against the other end, compressing the spring 0.375 m. The sled is then released with zero initial velocity. What is the sled's speed when the spring (a) returns to its uncompressed length and (b) is still compressed 0.200 m?

6.44 •• A small glider is placed against a compressed spring at the bottom of an air track that slopes upward at an angle of 40.0° above the horizontal. The glider has mass 0.0900 kg. The spring has $k = 640\text{ N/m}$ and negligible mass. When the spring is released, the glider travels a maximum distance of 1.80 m along the air track before sliding back down. Before reaching this maximum distance, the glider loses contact with the spring. (a) What distance was the spring originally compressed? (b) When the glider has traveled along the air track 0.80 m from its initial position against the compressed spring, is it still in contact with the spring? What is the kinetic energy of the glider at this point?

6.45 •• **CALC** A force in the $+x$-direction with magnitude $F(x) = 18.0\text{ N} - (0.530\text{ N/m})x$ is applied to a 6.00 kg box that is sitting on the horizontal, frictionless surface of a frozen lake. $F(x)$ is the only horizontal force on the box. If the box is initially at rest at $x = 0$, what is its speed after it has traveled 14.0 m?

Section 6.4 Power

6.46 •• A crate on a motorized cart starts from rest and moves with a constant eastward acceleration of $a = 2.80\ \mathrm{m/s^2}$. A worker assists the cart by pushing on the crate with a force that is eastward and has magnitude that depends on time according to $F(t) = (5.40\ \mathrm{N/s})t$. What is the instantaneous power supplied by this force at $t = 5.00\ \mathrm{s}$?

6.47 • How many joules of energy does a 100 watt light bulb use per hour? How fast would a 70 kg person have to run to have that amount of kinetic energy?

6.48 •• BIO **Should You Walk or Run?** It is 5.0 km from your home to the physics lab. As part of your physical fitness program, you could run that distance at 10 km/h (which uses up energy at the rate of 700 W), or you could walk it leisurely at 3.0 km/h (which uses energy at 290 W). Which choice would burn up more energy, and how much energy (in joules) would it burn? Why does the more intense exercise burn up less energy than the less intense exercise?

6.49 • Estimate how many 30 lb bags of mulch an average student in your physics class can load into the bed of a pickup truck in 5.0 min. The truck bed is 4.0 ft off the ground. If you assume that the magnitude of the work done on each bag by the student equals the magnitude of the work done on the bag by gravity when the bag is lifted into the truck, what is the average power output of the student? Express your result in watts and in horsepower.

6.50 •• A 20.0 kg rock is sliding on a rough, horizontal surface at 8.00 m/s and eventually stops due to friction. The coefficient of kinetic friction between the rock and the surface is 0.200. What average power is produced by friction as the rock stops?

6.51 • A student walks up three flights of stairs, a vertical height of about 50 ft. Estimate the student's weight to be the average for students in your physics class. If the magnitude of the average rate at which the gravity force does work on the student equals 500 W, how long would it take the student to travel up the three flights of stairs?

6.52 •• When its 75 kW (100 hp) engine is generating full power, a small single-engine airplane with mass 700 kg gains altitude at a rate of 2.5 m/s (150 m/min, or 500 ft/min). What fraction of the engine power is being used to make the airplane climb? (The remainder is used to overcome the effects of air resistance and of inefficiencies in the propeller and engine.)

6.53 •• **Working Like a Horse**. Your job is to lift 30 kg crates a vertical distance of 0.90 m from the ground onto the bed of a truck. How many crates would you have to load onto the truck in 1 minute (a) for the average power output you use to lift the crates to equal 0.50 hp; (b) for an average power output of 100 W?

6.54 •• An elevator has mass 600 kg, not including passengers. The elevator is designed to ascend, at constant speed, a vertical distance of 20.0 m (five floors) in 16.0 s, and it is driven by a motor that can provide up to 40 hp to the elevator. What is the maximum number of passengers that can ride in the elevator? Assume that an average passenger has mass 65.0 kg.

6.55 •• A ski tow operates on a 15.0° slope of length 300 m. The rope moves at 12.0 km/h and provides power for 50 riders at one time, with an average mass per rider of 70.0 kg. Estimate the power required to operate the tow.

6.56 • You are applying a constant horizontal force $\vec{F} = (-8.00\,\mathrm{N})\hat{i} + (3.00\,\mathrm{N})\hat{j}$ to a crate that is sliding on a factory floor. At the instant that the velocity of the crate is

$\vec{\boldsymbol{v}} = (3.20\ \mathrm{m/s})\hat{\boldsymbol{i}} + (2.20\,\mathrm{m/s})\hat{\boldsymbol{j}}$, what is the instantaneous power supplied by this force?

6.57 • **BIO** While hovering, a typical flying insect applies an average force equal to twice its weight during each downward stroke. Take the mass of the insect to be 10 g, and assume the wings move an average downward distance of 1.0 cm during each stroke. Assuming 100 downward strokes per second, estimate the average power output of the insect.

Problems

6.58 ••• **CALC** A balky cow is leaving the barn as you try harder and harder to push her back in. In coordinates with the origin at the barn door, the cow walks from $x = 0$ to $x = 6.9$ m as you apply a force with x-component $F_x = -[20.0\ \mathrm{N} + (3.0\ \mathrm{N/m})x]$. How much work does the force you apply do on the cow during this displacement?

6.59 • A luggage handler pulls a 20.0 kg suitcase up a ramp inclined at 32.0° above the horizontal by a force $\vec{\boldsymbol{F}}$ of magnitude 160 N that acts parallel to the ramp. The coefficient of kinetic friction between the ramp and the incline is $\mu_{\mathrm{k}} = 0.300$. If the suitcase travels 3.80 m along the ramp, calculate (a) the work done on the suitcase by $\vec{\boldsymbol{F}}$; (b) the work done on the suitcase by the gravitational force; (c) the work done on the suitcase by the normal force; (d) the work done on the suitcase by the friction force; (e) the total work done on the suitcase. (f) If the speed of the suitcase is zero at the bottom of the ramp, what is its speed after it has traveled 3.80 m along the ramp?

6.60 •• **CP** A can of beans that has mass M is launched by a spring-powered device from level ground. The can is launched at an angle of α_0 above the horizontal and is in the air for time T before it returns to the ground. Air resistance can be neglected. (a) How much work was done on the can by the launching device? (b) How much work is done on the can if it is launched at the same angle α_0 but stays in the air twice as long? How does your result compare to the answer to part (a)?

6.61 •• A 5.00 kg block is released from rest on a ramp that is inclined at an angle of 60.0° below the horizontal. The initial position of the block is a vertical distance of 2.00 m above the bottom of the ramp. (a) If the speed of the block is 5.00 m/s when it reaches the bottom of the ramp, what was the work done on it by the friction force? (b) If the angle of the ramp is changed but the block is released from a point that is still 2.00 m above the base of the ramp, both the magnitude of the friction force and the distance along the ramp that the block travels change. If the angle of the incline is changed to 50.0°, does the magnitude of the work done by the friction force increase or decrease compared to the value calculated in part (a)? (c) How much work is done by friction when the ramp angle is 50.0°?

6.62 •• A block of mass m is released from rest at the top of an incline that makes an angle α with the horizontal. The coefficient of kinetic friction between the block and incline is μ_k. The top of the incline is a vertical distance h above the bottom of the incline. Derive an expression for the work W_f done on the block by friction as it travels from the top of the incline to the bottom. When α is decreased, does the magnitude of W_f increase or decrease?

6.63 ••• Consider the blocks in Exercise 6.7 as they move 75.0 cm. Find the total work done on each one (a) if there is no friction between the table and the 20.0 N block, and (b) if

$\mu_s = 0.500$ and $\mu_k = 0.325$ between the table and the 20.0 N block.

6.64 •• A 5.00 kg package slides 2.80 m down a long ramp that is inclined at 24.0° below the horizontal. The coefficient of kinetic friction between the package and the ramp is $\mu_k = 0.310$. Calculate (a) the work done on the package by friction; (b) the work done on the package by gravity; (c) the work done on the package by the normal force; (d) the total work done on the package. (e) If the package has a speed of 2.20 m/s at the top of the ramp, what is its speed after it has slid 2.80 m down the ramp?

6.65 •• CP BIO **Whiplash Injuries**. When a car is hit from behind, its passengers undergo sudden forward acceleration, which can cause a severe neck injury known as *whiplash*. During normal acceleration, the neck muscles play a large role in accelerating the head so that the bones are not injured. But during a very sudden acceleration, the muscles do not react immediately because they are flexible; most of the accelerating force is provided by the neck bones. Experiments have shown that these bones will fracture if they absorb more than 8.0 J of energy. (a) If a car waiting at a stoplight is rear-ended in a collision that lasts for 10.0 ms, what is the greatest speed this car and its driver can reach without breaking neck bones if the driver's head has a mass of 5.0 kg (which is about right for a 70 kg person)? Express your answer in m/s and in mi/h. (b) What is the acceleration of the passengers during the collision in part (a), and how large a force is acting to accelerate their heads? Express the acceleration in m/s^2 and in *g*'s.

6.66 •• CALC A net force along the *x*-axis that has *x*-component $F_x = -12.0\ \mathrm{N} + (0.300\ \mathrm{N/m^2})x^2$ is applied to a 5.00 kg object that is initially at the origin and moving in the $-x$-

direction with a speed of 6.00 m/s. What is the speed of the object when it reaches the point $x = 5.00$ m?

6.67 • CALC **Varying Coefficient of Friction.** A box is sliding with a speed of 4.50 m/s on a horizontal surface when, at point P, it encounters a rough section. The coefficient of friction there is not constant; it starts at 0.100 at P and increases linearly with distance past P, reaching a value of 0.600 at 12.5 m past point P. (a) Use the work–energy theorem to find how far this box slides before stopping. (b) What is the coefficient of friction at the stopping point? (c) How far would the box have slid if the friction coefficient didn't increase but instead had the constant value of 0.100?

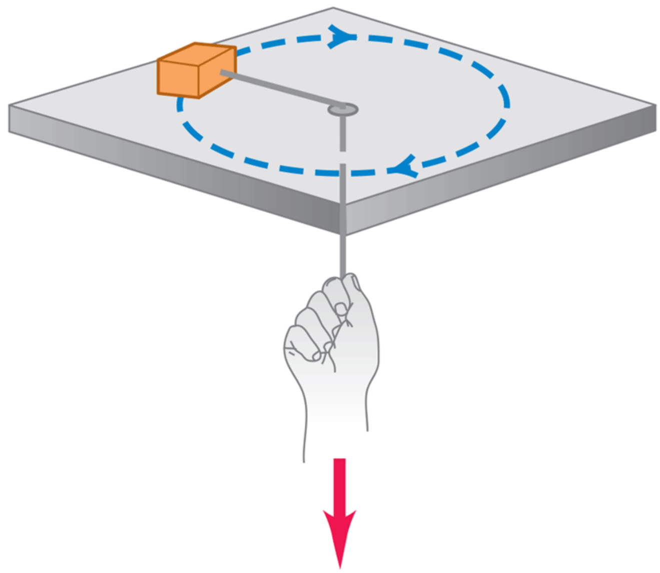
6.68 •• CALC Consider a spring that does not obey Hooke's law very faithfully. One end of the spring is fixed. To keep the spring stretched or compressed an amount x, a force along the x-axis with x-component $F_x = kx - bx^2 + cx^3$ must be applied to the free end. Here $k = 100\ \mathrm{N}/m$, $b = 700\ \mathrm{N/m^2}$, and $c = 12,000\ \mathrm{N/m^3}$. Note that $x > 0$ when the spring is stretched and $x < 0$ when it is compressed. How much work must be done (a) to stretch this spring by 0.050 m from its unstretched length? (b) To *compress* this spring by 0.050 m from its unstretched length? (c) Is it easier to stretch or compress this spring? Explain why in terms of the dependence of F_x on x. (Many real springs behave qualitatively in the same way.)

6.69 •• A net horizontal force $\vec{F}$ is applied to a box with mass M that is on a horizontal, frictionless surface. The box is initially at rest and then moves in the direction of the force. After the box has moved a distance D, the work that the constant force has done on it is W_D and the speed of the box is V. The equation $P = Fv$ tells us that the instantaneous rate at which $\vec{F}$ is doing work on the box depends on the speed of the box. (a) At the point in the motion of the box where the force has done half the total work, and so has done work $W_D/2$ on the box that started from rest, in terms of V what is the speed of the box? Is the speed at this point less than, equal to, or greater than half the final speed? (b) When the box has reached half its final speed, so its speed is $V/2$, how much work has been done on the box? Express your answer in terms of W_D. Is the amount of work done to produce this speed less than, equal to, or greater than half the work W_D done for the full displacement D?

6.70 •• You weigh 530 N. Your bathroom scale contains a light but very stiff ideal spring. When you stand at rest on the scale, the spring is compressed 1.80 cm. Your 180 N dog then gently jumps into your arms. How much work is done by the spring as the two of you are brought to rest by friction?

6.71 •• **CP** A small block with a mass of 0.0600 kg is attached to a cord passing through a hole in a frictionless, horizontal surface (Fig. P6.71). The block is originally revolving at a distance of 0.40 m from the hole with a speed of 0.70 m/s. The cord is then pulled from below, shortening the radius of the circle in which the block revolves to 0.10 m. At this new distance, the speed of the block is 2.80 m/s. (a) What is the tension in the cord in the original situation, when the block has speed $v = 0.70$ m/s? (b) What is the tension in the cord in the final situation, when the block has speed $v = 2.80$ m/s? (c) How much work was done by the person who pulled on the cord?

Figure P6.71

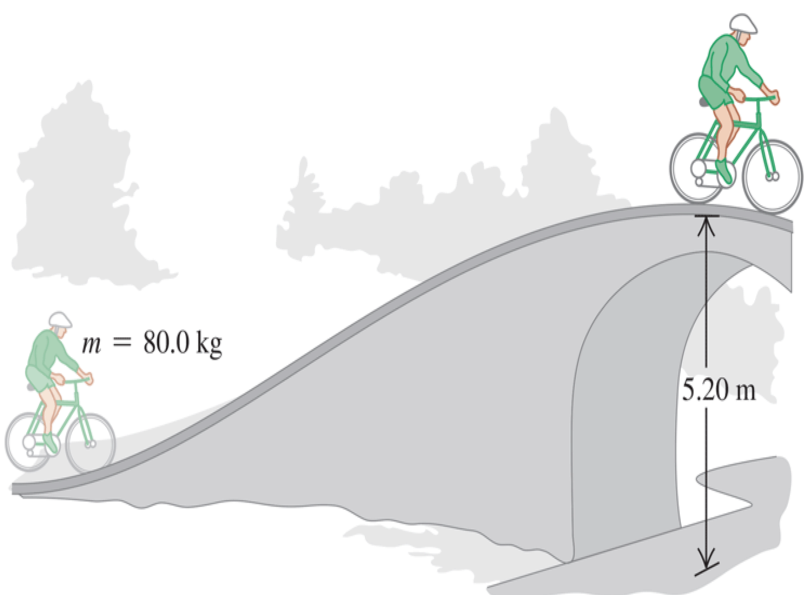


6.72 •• CALC **Proton Bombardment.** A proton with mass 1.67×10^{-27} kg is propelled at an initial speed of 3.00×10^{5} m/s directly toward a uranium nucleus 5.00 m away. The proton is repelled by the uranium nucleus with a force of magnitude $F = \alpha/x^2$, where x is the separation between the two objects and $\alpha = 2.12 \times 10^{-26}\ \mathrm{N{\cdot}m^2}$. Assume that the uranium nucleus remains at rest. (a) What is the speed of the proton when it is 8.00×10^{-10} m from the uranium nucleus? (b) As the proton approaches the uranium nucleus, the repulsive force slows down the proton until it comes momentarily to rest, after which the proton moves away from the uranium nucleus. How close to the uranium nucleus does the proton get? (c) What is the speed of the proton when it is again 5.00 m away from the uranium nucleus?

6.73 •• You are asked to design spring bumpers for the walls of a parking garage. A freely rolling 1200 kg car moving at 0.65 m/s is to compress the spring no more than 0.090 m before stopping. What should be the force constant of the spring? Assume that the spring has negligible mass.

6.74 •• You and your bicycle have combined mass 80.0 kg. When you reach the base of a bridge, you are traveling along the road at 5.00 m/s (Fig. P6.74). At the top of the bridge, you have climbed a vertical distance of 5.20 m and slowed to 1.50 m/s. Ignore work done by friction and any inefficiency in the bike or your legs. (a) What is the total work done on you and your bicycle when you go from the base to the top of the bridge? (b) How much work have you done with the force you apply to the pedals?

Figure P6.74



6.75 ••• A 2.50 kg textbook is forced against one end of a horizontal spring of negligible mass that is fixed at the other end and has force constant 250 N/m, compressing the spring a distance of 0.250 m. When released, the textbook slides on a horizontal tabletop with coefficient of kinetic friction $\mu_k = 0.30$. Use the work–energy theorem to find how far the textbook moves from its initial position before it comes to rest.

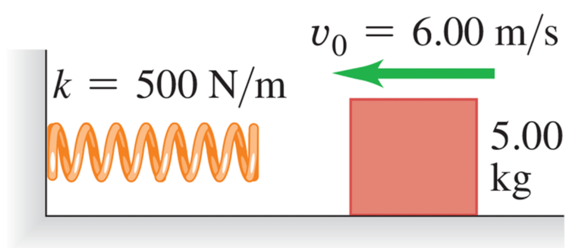
6.76 •• The spring of a spring gun has force constant $k = 400$ N/m and negligible mass. The spring is compressed 6.00 cm, and a ball with mass 0.0300 kg is placed in the horizontal barrel against the compressed spring. The spring is then released, and the ball is propelled out the barrel of the gun. The barrel is 6.00 cm long, so the ball leaves the barrel at the same point that it loses contact with the spring. The gun is held so that the barrel is horizontal. (a) Calculate the speed with which the ball leaves the barrel if you can ignore friction. (b) Calculate the speed of the ball as it leaves the barrel if a constant resisting force of 6.00 N acts on the ball as it moves along the barrel. (c) For the situation in part (b), at what position along the barrel does the ball have the greatest speed, and what is that speed? (In this case, the maximum speed does not occur at the end of the barrel.)

6.77 •• One end of a horizontal spring with force constant 130.0 N/m is attached to a vertical wall. A 4.00 kg block sitting on the floor is placed against the spring. The coefficient of kinetic friction between the block and the floor is $\mu_k = 0.400$. You apply a constant force $\vec{F}$ to the block. $\vec{F}$ has magnitude $F = 82.0$ N and is directed toward the wall. At the instant that the spring is compressed 80.0 cm, what are (a) the speed of the block, and (b) the magnitude and direction of the block's acceleration?

6.78 •• One end of a horizontal spring with force constant 76.0 N/m is attached to a vertical post. A 2.00 kg block of frictionless ice is attached to the other end and rests on the floor. The spring is initially neither stretched nor compressed. A constant horizontal force of 54.0 N is then applied to the block, in the direction away from the post. (a) What is the speed of the block when the spring is stretched 0.400 m? (b) At that instant, what are the magnitude and direction of the acceleration of the block?

6.79 • A 5.00 kg block is moving at $v_0 = 6.00\ \mathrm{m/s}$ along a frictionless, horizontal surface toward a spring with force constant $k = 500\ \mathrm{N/m}$ that is attached to a wall (Fig. P6.79). The spring has negligible mass. (a) Find the maximum distance the spring will be compressed. (b) If the spring is to compress by no more than 0.150 m, what should be the maximum value of v_0?

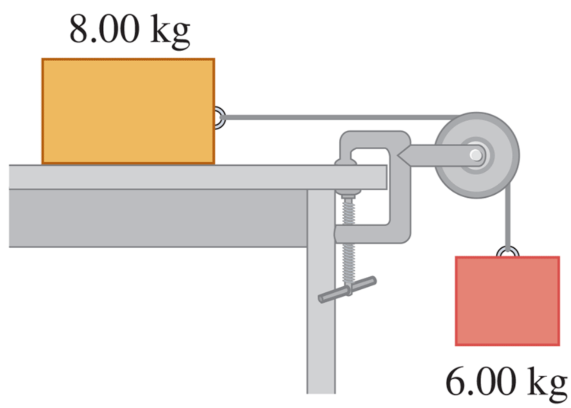
Figure P6.79



6.80 ••• A physics professor is pushed up a ramp inclined upward at 30.0° above the horizontal as she sits in her desk chair, which slides on frictionless rollers. The combined mass of the professor and chair is 85.0 kg. She is pushed 2.50 m along the incline by a group of students who together exert a constant horizontal force of 600 N. The professor's speed at the bottom of the ramp is 2.00 m/s. Use the work–energy theorem to find her speed at the top of the ramp.

6.81 •• Consider the system shown in Fig. P6.81. The rope and pulley have negligible mass, and the pulley is frictionless. Initially the 6.00 kg block is moving downward and the 8.00 kg block is moving to the right, both with a speed of 0.900 m/s. The blocks come to rest after moving 2.00 m. Use the work–energy theorem to calculate the coefficient of kinetic friction between the 8.00 kg block and the tabletop.

Figure P6.81



6.82 •• Consider the system shown in Fig. P6.81. The rope and pulley have negligible mass, and the pulley is frictionless. The coefficient of kinetic friction between the 8.00 kg block and the tabletop is $\mu_k = 0.250$. The blocks are released from rest. Use energy methods to calculate the speed of the 6.00 kg block after it has descended 1.50 m.

6.83 •• On an essentially frictionless, horizontal ice rink, a skater moving at 3.0 m/s encounters a rough patch that reduces her speed to 1.65 m/s due to a friction force that is 25% of her weight. Use the work–energy theorem to find the length of this rough patch.

6.84 •• **BIO** All birds, independent of their size, must maintain a power output of 10–25 watts per kilogram of object mass in order to fly by flapping their wings. (a) The Andean giant hummingbird (*Patagona gigas*) has mass 70 g and flaps its wings 10 times per second while hovering. Estimate the amount of work done by such a hummingbird in each wingbeat. (b) A 70 kg athlete can maintain a power output of 1.4 kW for no more than a few seconds; the *steady* power output of a typical athlete is only 500 W or so. Is it possible for a human-powered aircraft to fly for extended periods by flapping its wings? Explain.

6.85 •• A pump is required to lift 800 kg of water (about 210 gallons) per minute from a well 14.0 m deep and eject it with a speed of 18.0 m/s. (a) How much work is done per minute in lifting the water? (b) How much work is done in giving the water the kinetic energy it has when ejected? (c) What must be the power output of the pump?

6.86 •• The upper end of a light rope of length $L = 0.600$ m is attached to the ceiling, and a small steel ball with mass $m = 0.200$ kg is suspended from the lower end of the rope. Initially the ball is at rest and the rope is vertical. Then a force $\vec{F}$ with constant magnitude $F = 0.760$ N and a direction that is maintained tangential to the path of the ball is applied and the ball moves in an arc of a circle of radius L. What is the speed of the ball when the rope makes an angle $\alpha = 37.0^\circ$ with the vertical?

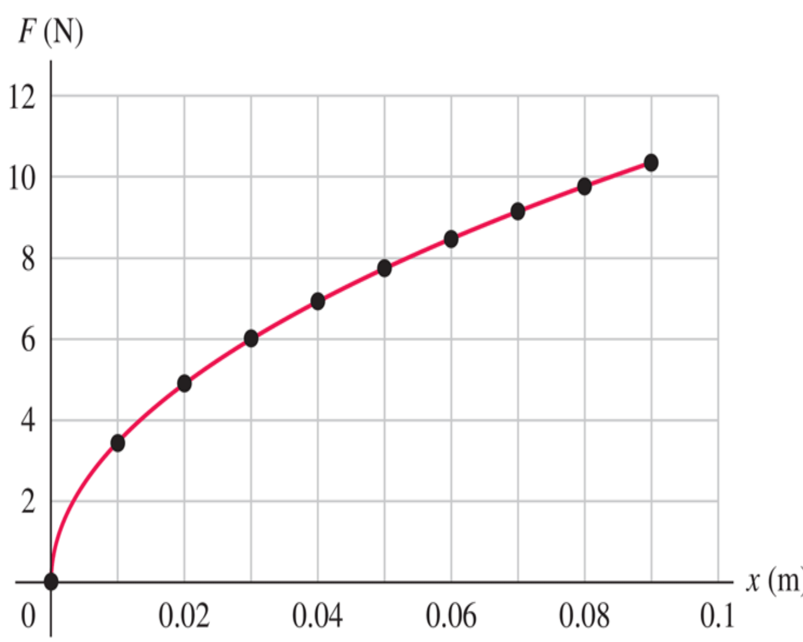
6.87 •• Consider the system of two blocks shown in Fig. P6.81, but with a different friction force on the 8.00 kg block. The blocks are released from rest. While the two blocks are moving, the tension in the light rope that connects them is 37.0 N. (a) During a 0.800 m downward displacement of the 6.00 kg block, how much work has been done on it by gravity? By the tension T in the rope? Use the work–energy theorem to find the speed of the 6.00 kg block after it has descended 0.800 m. (b) During the 0.800 m displacement of the 6.00 kg block, what is the total work done on the 8.00 kg block? During this motion how much work was done on the 8.00 kg block by the tension T in the cord? By the friction force exerted on the 8.00 kg block? (c) If the work–energy theorem is applied to the two blocks considered together as a composite system, use the theorem to find the net work done on the system during the 0.800 m downward displacement of the 6.00 kg block. How much work was done on the system of two blocks by gravity? By friction? By the tension in the rope?

6.88 • **CALC** An object has several forces acting on it. One of these forces is $\vec{F} = \alpha xy\hat{\imath}$, a force in the x-direction whose magnitude depends on the position of the object, with $\alpha = 2.50\ \mathrm{N/m^2}$. Calculate the work done on the object by this force for the following displacements of the object: (a) The object starts at the point ($x = 0$, $y = 3.00\ \mathrm{m}$) and moves parallel to the x-axis to the point ($x = 2.00\ \mathrm{m}$, $y = 3.00\ \mathrm{m}$). (b) The object starts at the point ($x = 2.00\ \mathrm{m}$, $y = 0$) and moves in the y-direction to the point ($x = 2.00\ \mathrm{m}$, $y = 3.00\ \mathrm{m}$). (c) The object starts at the origin and moves on the line $y = 1.5x$ to the point ($x = 2.00\ \mathrm{m}$, $y = 3.00\ \mathrm{m}$).

6.89 • **BIO** **Power of the Human Heart**. The human heart is a powerful and extremely reliable pump. Each day it takes in and discharges about 7500 L of blood. Assume that the work done by the heart is equal to the work required to lift this amount of blood a height equal to that of the average American woman (1.63 m). The density (mass per unit volume) of blood is $1.05 \times 10^3\ \mathrm{kg/m^3}$. (a) How much work does the heart do in a day? (b) What is the heart's power output in watts?

6.90 •• **DATA** Figure P6.90 shows the results of measuring the force F exerted on both ends of a rubber band to stretch it a distance x from its unstretched position. (Source: www.sciencebuddies.org) The data points are well fit by the equation $F = 33.55x^{0.4871}$, where F is in newtons and x is in meters. (a) Does this rubber band obey Hooke's law over the range of x shown in the graph? Explain. (b) The stiffness of a spring that obeys Hooke's law is measured by the value of its force constant k, where $k = F/x$. This can be written as $k = dF/dx$ to emphasize the quantities that are changing. Define $k_{\text{eff}} = dF/dx$ and calculate k_{eff} as a function of x for this rubber band. For a spring that obeys Hooke's law, k_{eff} is constant, independent of x. Does the stiffness of this band, as measured by k_{eff}, increase or decrease as x is increased, within the range of the data? (c) How much work must be done to stretch the rubber band from $x = 0$ to $x = 0.0400$ m? From $x = 0.0400$ m to $x = 0.0800$ m? (d) One end of the rubber band is attached to a stationary vertical rod, and the band is stretched horizontally 0.0800 m from its unstretched length. A 0.300 kg object on a horizontal, frictionless surface is attached to the free end of the rubber band and released from rest. What is the speed of the object after it has traveled 0.0400 m?

Figure P6.90

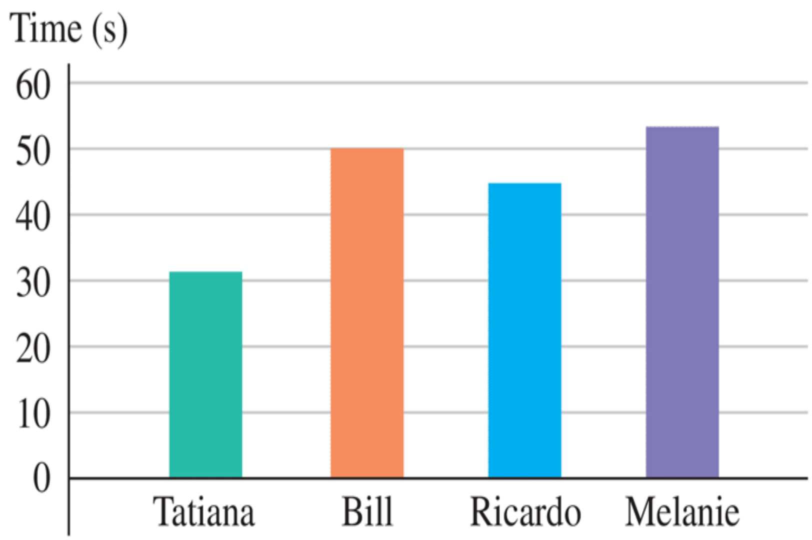


6.91 **••• DATA** In a physics lab experiment, one end of a horizontal spring that obeys Hooke's law is attached to a wall. The spring is compressed 0.400 m, and a block with mass 0.300 kg is attached to it. The spring is then released, and the block moves along a horizontal surface. Electronic sensors measure the speed v of the block after it has traveled a distance d from its initial position against the compressed spring. The measured values are listed in the table. (a) The data show that the speed v of the block increases and then decreases as the spring returns to its unstretched length. Explain why this happens, in terms of the work done on the block by the forces that act on it. (b) Use the work–energy theorem to derive an expression for v^2 in terms of d. (c) Use a computer graphing program (for example, Excel or Matlab) to graph the data as v^2 (vertical axis) versus d (horizontal axis). The equation that you derived in part (b) should show that v^2 is a quadratic function of d, so, in your graph, fit the data by a second-order polynomial (quadratic) and have the graphing program display the equation for this trendline. Use that equation to find the block's maximum speed v and the value of d at which this speed occurs. (d) By comparing the equation from the graphing program to the formula you derived in part (b), calculate the force constant k for the spring and the coefficient of kinetic friction for the friction force that the surface exerts on the block.

d (m)	v (m/s)
0	0
0.05	0.85
0.10	1.11
0.15	1.24
0.25	1.26
0.30	1.14
0.35	0.90
0.40	0.36

6.92 •• **DATA** For a physics lab experiment, four classmates run up the stairs from the basement to the top floor of their physics building—a vertical distance of 16.0 m. The classmates and their masses are: Tatiana, 50.2 kg; Bill, 68.2 kg; Ricardo, 81.8 kg; and Melanie, 59.1 kg. The time it takes each of them is shown in Fig. P6.92. (a) Considering only the work done against gravity, which person had the largest average power output? The smallest? (b) Chang is very fit and has mass 62.3 kg. If his average power output is 1.00 hp, how many seconds does it take him to run up the stairs?

Figure P6.92

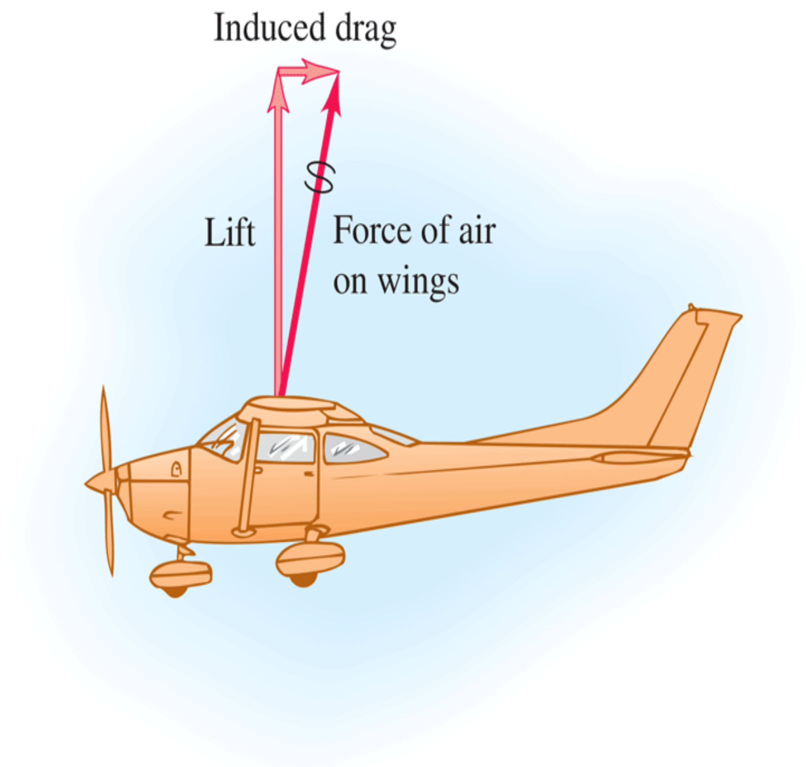


Challenge Problems

6.93 **••• CALC A Spring with Mass.** We usually ignore the kinetic energy of the moving coils of a spring, but let's try to get a reasonable approximation to this. Consider a spring of mass M, equilibrium length L_0, and force constant k. The work done to stretch or compress the spring by a distance L is $\frac{1}{2}kX^2$, where $X = L - L_0$. Consider a spring, as described above, that has one end fixed and the other end moving with speed v. Assume that the speed of points along the length of the spring varies linearly with distance l from the fixed end. Assume also that the mass M of the spring is distributed uniformly along the length of the spring. (a) Calculate the kinetic energy of the spring in terms of M and v (**HINT:** Divide the spring into pieces of length dl; find the speed of each piece in terms of l, v and L; find the mass of each piece in terms of dl, M, and L; and integrate from 0 to L. The result is *not* $\frac{1}{2}Mv^2$, since not all of the spring moves with the same speed.) In a spring gun, a spring of mass 0.243 kg and force constant 3200 N/m is compressed 2.50 cm from its unstretched length. When the trigger is pulled, the spring pushes horizontally on a 0.053 kg ball. The work done by friction is negligible. Calculate the ball's speed when the spring reaches its uncompressed length (b) ignoring the mass of the spring and (c) including, using the results of part (a), the mass of the spring. (d) In part (c), what is the final kinetic energy of the ball and of the spring?

6.94 ••• **CALC** An airplane in flight is subject to an air resistance force proportional to the square of its speed v But there is an additional resistive force because the airplane has wings. Air flowing over the wings is pushed down and slightly forward, so from Newton's third law the air exerts a force on the wings and airplane that is up and slightly backward (Fig. P6.94). The upward force is the lift force that keeps the airplane aloft, and the backward force is called *induced drag*. At flying speeds, induced drag is inversely proportional to v^2, so the total air resistance force can be expressed by $F_{\text{air}} = \alpha v^2 + \beta / v^2$, where α and β are positive constants that depend on the shape and size of the airplane and the density of the air. For a Cessna 150, a small single-engine airplane, $\alpha = 0.30\ \mathrm{N} \cdot s^2/m^2$ and $\beta = 3.5 \times 10^5\ \ \mathrm{N} \cdot \mathrm{m}^2/\mathrm{s}^2$. In steady flight, the engine must provide a forward force that exactly balances the air resistance force. (a) Calculate the speed (in km/h) at which this airplane will have the maximum *range* (that is, travel the greatest distance) for a given quantity of fuel. (b) Calculate the speed (in km/h) for which the airplane will have the maximum *endurance* (that is, remain in the air the longest time).

Figure P6.94



MCAT-Style Passage Problems

BIO Energy of locomotion. On flat ground, a 70 kg person requires about 300 W of metabolic power to walk at a steady pace of 5.0 km/h (1.4 m/s). Using the same metabolic power output, that person can bicycle over the same ground at 15 km/h.

6.95 Based on the given data, how does the energy used in biking 1 km compare with that used in walking 1 km? Biking takes (a) $\frac{1}{3}$ of the energy of walking the same distance; (b) the same energy as walking the same distance; (c) 3 times the energy of walking the same distance; (d) 9 times the energy of walking the same distance.

6.96 A 70 kg person walks at a steady pace of 5.0 km/h on a treadmill at a 5.0% grade. (That is, the vertical distance covered is 5.0% of the horizontal distance covered.) If we assume the metabolic power required is equal to that required for walking on a flat surface plus the rate of doing work for the vertical climb, how much power is required? (a) 300 W; (b) 315 W; (c) 350 W; (d) 370 W.

6.97 How many times greater is the kinetic energy of the person when biking than when walking? Ignore the mass of the bike. (a) 1.7; (b) 3; (c) 6; (d) 9.

Answers: Work and Kinetic Energy

Chapter Opening Question ?

(ii) The expression for kinetic energy is $K = \frac{1}{2}mv^2$. If we calculate K for the three balls, we find (i) $K = \frac{1}{2}(0.145\ \mathrm{kg}) \times (20.0\ \mathrm{m/s})^2 = 29.0\ \mathrm{kg \cdot m^2/s^2} = 29.0\ \mathrm{J}$, (ii) $K = \frac{1}{2}(0.0145\ \mathrm{kg}) \times (200\ \mathrm{m/s})^2 = 290\ \mathrm{J}$, and (iii) $K = \frac{1}{2}(1.45\ \mathrm{kg})(2.00\ \mathrm{m/s})^2 = 2.90\ \mathrm{J}$. The smaller ball has the least mass of all three, but it also has the greatest speed and so the most kinetic energy. Since kinetic energy is a scalar, it does not depend on the direction of motion.

Test Your Understanding

6.1 **(iii)** The electron has constant velocity, so its acceleration is zero and (by Newton's second law) the net force on the electron is also zero. Therefore the total work done by all the forces (equal to the work done by the net force) must be zero as well. The individual forces may do nonzero work, but that's not what the question asks.

6.2 **(iv), (i), (iii), (ii)** Object (i) has kinetic energy $K = \frac{1}{2}mv^2 = \frac{1}{2}(2.0\ \mathrm{kg})(5.0\ \mathrm{m/s})^2 = 25\ \mathrm{J}$. Object (ii) had zero kinetic energy initially and then had 30 J of work done on it, so its final kinetic energy is $K_2 = K_1 + W = 0 + 30\ \mathrm{J} = 30\ \mathrm{J}$. Object (iii) had initial kinetic energy $K_1 = \frac{1}{2}mv_1^2 = \frac{1}{2}(1.0\ \mathrm{kg})(4.0\ \mathrm{m/s})^2 = 8.0\ \mathrm{J}$ and then had 20 J of work done on it, so its final kinetic energy is $K_2 = K_1 + W = 8.0\ \mathrm{J} + 20\ \mathrm{J} = 28\ \mathrm{J}$. Object (iv) had initial kinetic energy $K_1 = \frac{1}{2}mv_1^2 = \frac{1}{2}(2.0\ \mathrm{kg})(10\ \mathrm{m/s})^2 = 100\ \mathrm{J}$; when it did 80 J of work on another object, the other object did -80 J of work on object (iv), so the final kinetic energy of object (iv) is $K_2 = K_1 + W = 100\ \mathrm{J} + (-80\ \mathrm{J}) = 20\ \mathrm{J}$.

6.3 (a) **(iii)** (b) **(iii)** At any point during the pendulum bob's motion, both the tension force and the weight act perpendicular to the motion—that is, perpendicular to an infinitesimal displacement $d\vec{l}$ of the bob. (In Fig. 5.32b, the displacement $d\vec{l}$ would be directed outward from the plane of the free-body diagram.) Hence for either force the scalar product inside the integral in Eq. (6.14) is $\vec{F} \cdot d\vec{l} = 0$, and the work done along any part of the circular path (including a complete circle) is $W = \int \vec{F} \cdot d\vec{l} = 0$.

6.4 **(v)** The airliner has a constant horizontal velocity, so the net horizontal force on it must be zero. Hence the backward drag force must have the same magnitude as the forward force due to the combined thrust of the four engines. This means that the drag force must do *negative* work on the airplane at the same rate that the combined thrust force does *positive* work. The combined thrust does work at a rate of $4(108,000 \text{ hp}) = 432,000 \text{ hp}$, so the drag force must do work at a rate of $-432,000 \text{ hp}$.

Key Example Variation Problems

VP6.2.1
- **a.** -714 J
- **b.** $+714 \text{ J}$

VP6.2.2 35.5°

VP6.2.3
- **a.** -90.0 J
- **b.** 207 J
- **c.** zero
- **d.** 117 J

VP6.2.4
- **a.** Student 1: $2F_0 d$; student 2: $-3F_0 d$; student 3: $-8F_0 d + Gd$
- **b.** $9F_0$

VP6.4.1
- **a.** 5.20 m/s
- **b.** $9.18 \times 10^3 \text{ N}$

VP6.4.2

a. 219 J

b. -178 J

c. 41 J

d. 43 J, 2.4 m/s

VP6.4.3

a. Initial: 6.30×10^3 J; final: 142 J

b. Total: -6.16×10^3 J; gravity: 2.74×10^4 J; thrust: -3.36×10^4 J

c. 1.68×10^4 N

VP6.4.4

a. $mgd \sin\theta$

b. $-\mu_\mathrm{k} mgd\cos\ \theta$

c. $\sqrt{2gd(\sin\theta - \mu_\mathrm{k} cos\ \ \theta)}$

VP6.8.1

a. 8.84 cm

b. 7.41 cm

VP6.8.2 $\left(mv^2 - kd^2\right)/2mgd$

VP6.8.3

a. $+0.665$ J

b. -0.665 J

c. zero

VP6.8.4 $\sqrt{2gL(1-\cos\theta)}$

Bridging Problem

(a)

$$v_1 = \sqrt{\frac{2}{m}\left(mgx_1 - \frac{1}{3}\alpha x_1^3\right)} = \sqrt{2gx_1 - \frac{2\alpha x_1^3}{3m}}$$

(b)

$$P = -F_{\text{spring-1}} v_1 = -\alpha x_1^2 \sqrt{2gx_1 - \frac{2\alpha x_1^3}{3m}}$$

(c)

$$x_2 = \sqrt{\frac{3mg}{\alpha}}$$

(d) no

Chapter 7
Potential Energy and Energy Conservation

? As this sandhill crane (*Grus canadensis*) glides in to a landing, it descends along a straight-line path at a constant speed. During the glide, what happens to the total mechanical energy (the sum of kinetic energy and gravitational potential energy)? (i) It stays the same; (ii) it increases due to the effect of gravity; (iii) it increases due to the effect of the air; (iv) it decreases due to the effect of gravity; (v) it decreases due to the effect of the air.

Learning Outcomes

In this chapter, you'll learn...

7.1 How to use the concept of gravitational potential energy in problems that involve vertical motion.

7.2 How to use the concept of elastic potential energy in problems that involve a moving object attached to a stretched or compressed spring.

7.3 The distinction between conservative and nonconservative forces, and how to solve problems in which both kinds of forces act on a moving object.

7.4 How to calculate the properties of a conservative force if you know the corresponding potential-energy function.

7.5 How to use energy diagrams to understand how an object moves in a straight line under the influence of a conservative force.

You'll need to review...

5.3 Kinetic friction and fluid resistance.

5.4 Dynamics of circular motion.

6.1, 6.2 Work and the work–energy theorem.

6.3 Work done by an ideal spring.

When a diver jumps off a high board into a swimming pool, she hits the water moving pretty fast, with a lot of kinetic energy—energy of *motion*. Where does that energy come from? The answer we learned in Chapter 6 was that the gravitational force does work on the diver as she falls, and her kinetic energy increases by an amount equal to the work done.

However, there's a useful alternative way to think about work and kinetic energy. This new approach uses the idea of *potential energy*, which is associated with the *position* of a system rather than with its motion. In this approach, there is *gravitational potential energy* even when the diver is at rest on the high board. As she falls, this potential energy is *transformed* into her kinetic energy.

If the diver bounces on the end of the board before she jumps, the bent board stores a second kind of potential energy called *elastic potential energy*. We'll discuss elastic potential energy of simple systems such as a stretched or compressed spring. (An important third kind of potential energy is associated with the forces between electrically charged objects. We'll return to this in Chapter 23.)

We'll prove that in some cases the sum of a system's kinetic and potential energies, called the *total mechanical energy* of the system, is constant during the motion of the system. This will lead us to the general statement of the *law of conservation of energy*, one of the most fundamental principles in all of science.

7.1 Gravitational Potential Energy

In many situations it seems as though energy has been stored in a system, to be recovered later. For example, you must do work to lift a heavy stone over your head. It seems reasonable that in hoisting the stone into the air you are storing energy in the system, energy that is later converted into kinetic energy when you let the stone fall.

This example points to the idea of an energy associated with the *position* of objects in a system. This kind of energy is a measure of the *potential* or *possibility* for work to be done; if you raise a stone into the air, there is a potential for the gravitational force to do work on it, but only if you allow the stone to fall to the ground. For this reason, energy associated with position is called **potential energy.** The potential energy associated with an object's weight and its height above the ground is called *gravitational potential energy* (Fig. 7.1).

Figure 7.1

The greater the height of a basketball, the greater the associated gravitational potential energy. As the basketball descends, gravitational potential energy is converted to kinetic energy and the basketball's speed increases.

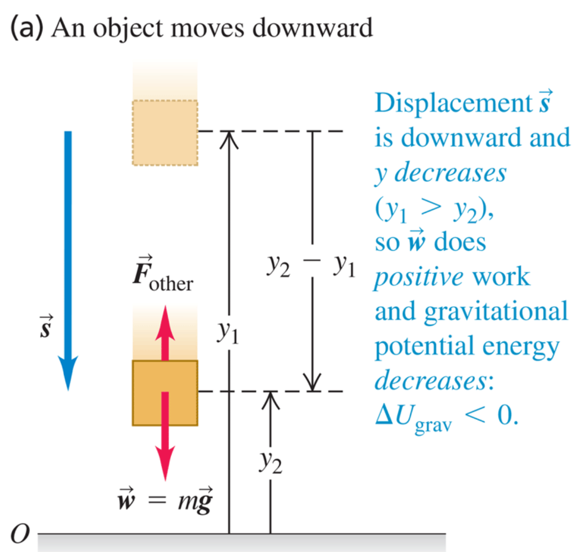
We now have *two* ways to describe what happens when an object falls without air resistance. One way, which we learned in Chapter 6, is to say that a falling object's kinetic energy increases because the force of the earth's gravity does work on the object. The other way is to say that the kinetic energy increases as the gravitational potential energy decreases. Later in this section we'll use the work–energy theorem to show that these two descriptions are equivalent.

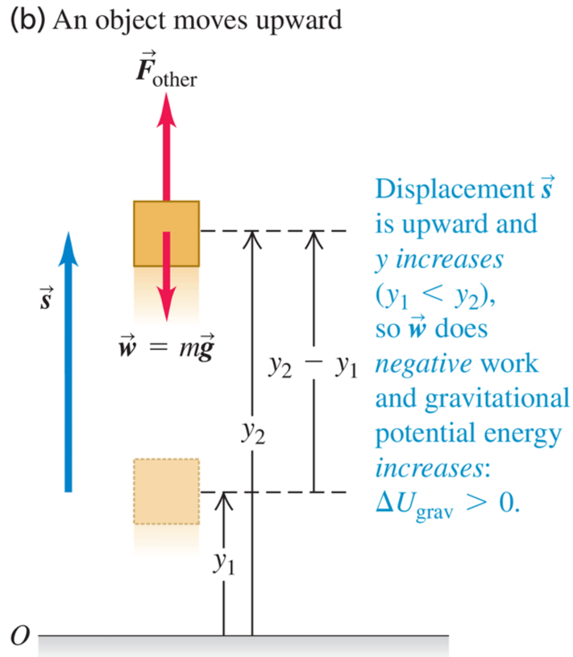
Let's derive the expression for gravitational potential energy. Suppose an object with mass m moves along the (vertical) y-axis, as in Fig. 7.2. The forces acting on it are its weight, with magnitude $w = mg$, and possibly some other forces; we call the vector sum (resultant) of all the other forces $\vec{\boldsymbol{F}}_{\text{other}}$. We'll assume that the object stays close enough to the earth's surface that the weight is constant. (We'll find in Chapter 13 that weight decreases with altitude.) We want to find the work done by the weight when the object moves downward from a height y_1 above the origin to a lower height y_2 (Fig. 7.2a). The weight and displacement are in the same direction, so the work W_{grav} done on the object by its weight is positive:

(7.1)

$$W_{\text{grav}} = Fs = w(y_1 - y_2) = mgy_1 - mgy_2$$

Figure 7.2





When an object moves vertically from an initial height y_1 to a final height y_2, the gravitational force $\vec{w}$ does work and the gravitational potential energy changes.

This expression also gives the correct work when the object moves *upward* and y_2 is greater than y_1 (Fig. 7.2b). In that case the quantity $(y_1 - y_2)$ is negative, and W_{grav} is negative because the weight and displacement are opposite in direction.

Equation (7.1) shows that we can express W_{grav} in terms of the values of the quantity mgy at the beginning and end of the displacement. This quantity is called the **gravitational potential energy,** U_{grav} :

(7.2)

Gravitational potential energy associated with a particle

$$U_{grav} = mgy \qquad (7.2)$$

Vertical coordinate of particle (y increases if particle moves upward)

Mass of particle

Acceleration due to gravity

Its initial value is $U_{\text{grav, 1}} = mgy_1$ and its final value is $U_{\text{grav, 2}} = mgy_2$. The change in U_{grav} is the final value minus the initial value, or $\Delta U_{\text{grav}} = U_{\text{grav, 2}} - U_{\text{grav, 1}}$. Using Eq. (7.2), we can rewrite Eq. (7.1) for the work done by the gravitational force during the displacement from y_1 to y_2 :

$$W_{\text{grav}} = U_{\text{grav, 1}} - \ U_{\text{grav, 2}} = -\left(U_{\text{grav, 2}} - U_{\text{grav, 1}}\right) = -\Delta U_{\text{grav}}$$

or

(7.3)

Work done by the gravitational force on a particle ...

... equals the **negative of the change in the gravitational potential energy.**

$$W_{\text{grav}} = mgy_1 - mgy_2 = U_{\text{grav},1} - U_{\text{grav},2} = -\Delta U_{\text{grav}} \qquad (7.3)$$

Mass of particle

Acceleration due to gravity

Initial and final vertical coordinates of particle

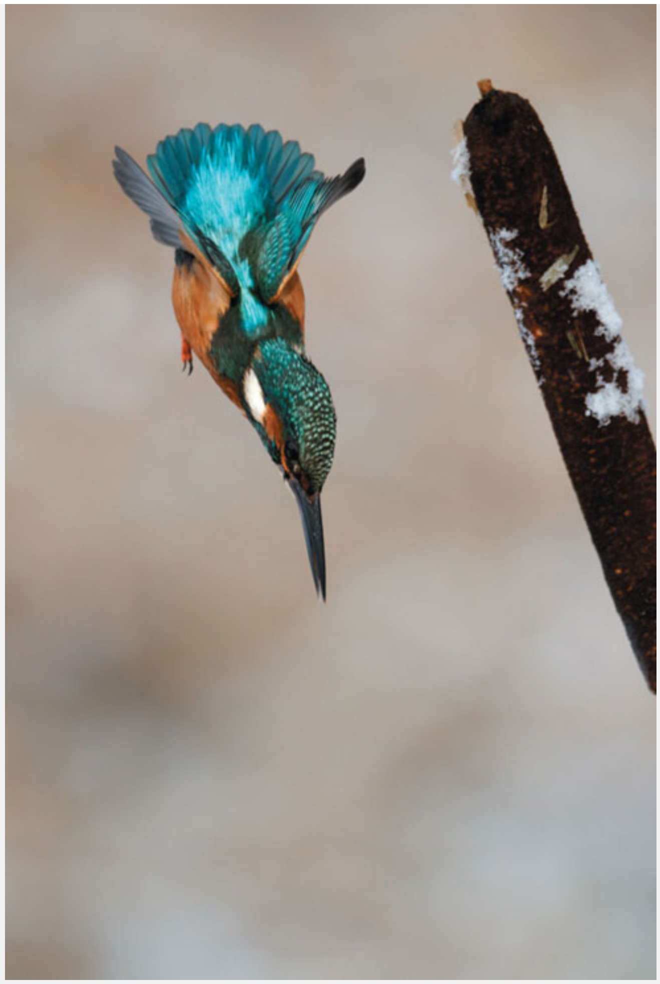
The negative sign in front of ΔU_{grav} is *essential.* When the object moves up, y increases, the work done by the gravitational force is negative, and the gravitational potential energy increases ($\Delta U_{\text{grav}} > 0$). When the object moves down, y decreases, the gravitational force does positive work, and the gravitational potential energy decreases ($\Delta U_{\text{grav}} < 0$). It's like drawing money out of the bank (decreasing U_{grav}) and spending it (doing positive work). The unit of potential energy is the joule (J), the same unit as is used for work.

CAUTION To what object does gravitational potential energy "belong"? It is *not* correct to call $U_{\text{grav}} = mgy$ the "gravitational potential energy of the object." The reason is that U_{grav} is a *shared* property of the object and the earth. The value of U_{grav} increases if the earth stays fixed and the object moves upward, away from the earth; it also increases if the object stays fixed and the earth is moved away from it. Notice that the formula $U_{\text{grav}} = mgy$ involves characteristics of both the object (its mass m) and the earth (the value of g).

BIO Application

Converting Gravitational Potential Energy to Kinetic Energy

When a kingfisher (*Alcedo atthis*) spots a tasty fish, the bird dives from its perch with its wings tucked in to minimize air resistance. Effectively the only force acting on the diving kingfisher is the force of gravity, so the total mechanical energy is conserved: The gravitational potential energy lost as the kingfisher descends is converted into the bird's kinetic energy.



Conservation of Total Mechanical Energy (Gravitational Forces Only)

Video Tutor Demo: **Chin Basher?**

To see what gravitational potential energy is good for, suppose an object's weight is the *only* force acting on it, so $\vec{\boldsymbol{F}}_{\text{other}} = \mathbf{0}$. The object is then falling freely with no air resistance and can be moving either up or down. Let its speed at point y_1 be v_1 and let its speed at y_2 be v_2. The work–energy theorem, Eq. (6.6), says that the total work done on the object equals the change in the object's kinetic energy: $W_{\text{tot}} = \Delta K = K_2 - K_1$. If gravity is the only force that acts, then from Eq. (7.3), $W_{\text{tot}} = W_{\text{grav}} = -\Delta U_{\text{grav}} = U_{\text{grav, 1}} - U_{\text{grav, 2}}$. Putting these together, we get

$$\Delta K = -\Delta U_{\text{grav}} \qquad \text{or} \qquad K_2 - K_1 = U_{\text{grav, 1}} - U_{\text{grav, 2}}$$

which we can rewrite as

(7.4)

If only the gravitational force does work, total mechanical energy is conserved:

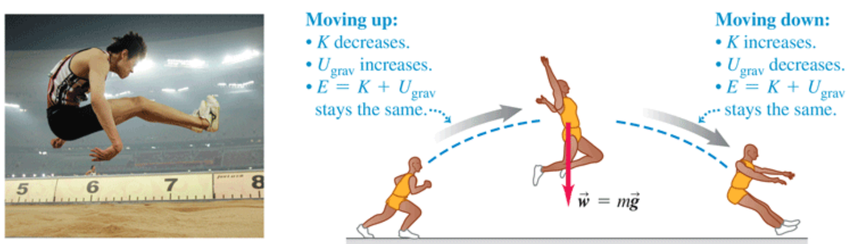
$$\overbrace{K_1}^{\substack{\text{Initial kinetic energy} \\ K_1 = \frac{1}{2}mv_1^2}} + \overbrace{U_{\text{grav},1}}^{\substack{\text{Initial gravitational potential energy} \\ U_{\text{grav},1} = mgy_1}} = \underbrace{K_2}_{\substack{\text{Final kinetic energy} \\ K_2 = \frac{1}{2}mv_2^2}} + \underbrace{U_{\text{grav},2}}_{\substack{\text{Final gravitational potential energy} \\ U_{\text{grav},2} = mgy_2}} \qquad (7.4)$$

The sum $K + U_{\text{grav}}$ of kinetic and potential energies is called E, the **total mechanical energy of the system.** By "system" we mean the object of mass m and the earth considered together, because gravitational potential energy U is a shared property of both objects. Then $E_1 = K_1 + U_{\text{grav},\,1}$ is the total mechanical energy at y_1 and $E_2 = K_2 + U_{\text{grav},\,2}$ is the total mechanical energy at y_2. Equation (7.4) says that when the object's weight is the only force doing work on it, $E_1 = E_2$. That is, E is constant; it has the same value at y_1 and y_2. But since positions y_1 and y_2 are arbitrary points in the motion of the object, the total mechanical energy E has the same value at *all* points during the motion:

$$E = K + U_{\text{grav}} = \quad \text{constant} \quad (\text{if only gravity does work})$$

A quantity that always has the same value is called a *conserved* quantity. *When only the force of gravity does work, the total mechanical energy is constant—that is, it is conserved* (Fig. 7.3). This is our first example of the **conservation of total mechanical energy.**

Figure 7.3



While this athlete is in midair, only gravity does work on him (if we neglect the minor effects of air resistance). Total mechanical energy E—the sum of kinetic and gravitational potential energies—is conserved.

When we throw a ball into the air, its speed decreases on the way up as kinetic energy is converted to potential energy: $\Delta K < 0$ and $\Delta U_{\text{grav}} > 0$. On the way back down, potential energy is converted back to kinetic energy and the ball's speed increases: $\Delta K > 0$ and $\Delta U_{\text{grav}} < 0$. But the *total* mechanical energy (kinetic plus potential) is the same at every point in the motion, provided that no force other than gravity does work on the ball (that is, air resistance must be negligible). It's still true that the gravitational force does work on the object as it moves up or down, but we no longer have to calculate work directly; keeping track of changes in the value of U_{grav} takes care of this completely.

Equation (7.4) is also valid if forces other than gravity are present but do *not* do work. We'll see a situation of this kind later, in Example 7.4.

CAUTION Choose "zero height" to be wherever you like When working with gravitational potential energy, we may choose any height to be $y = 0$. If we shift the origin for y, the values of y_1 and y_2 change, as do the values of $U_{\text{grav, 1}}$ and $U_{\text{grav, 2}}$. But this shift has no effect on the *difference* in height $y_2 - y_1$ or on the *difference* in gravitational potential energy $U_{\text{grav, 2}} - U_{\text{grav, 1}} = mg(y_2 - y_1)$. As Example 7.1 shows, the physically significant quantity is not the value of U_{grav} at a particular point but the *difference* in U_{grav} between two points. We can define U_{grav} to be zero at whatever point we choose.

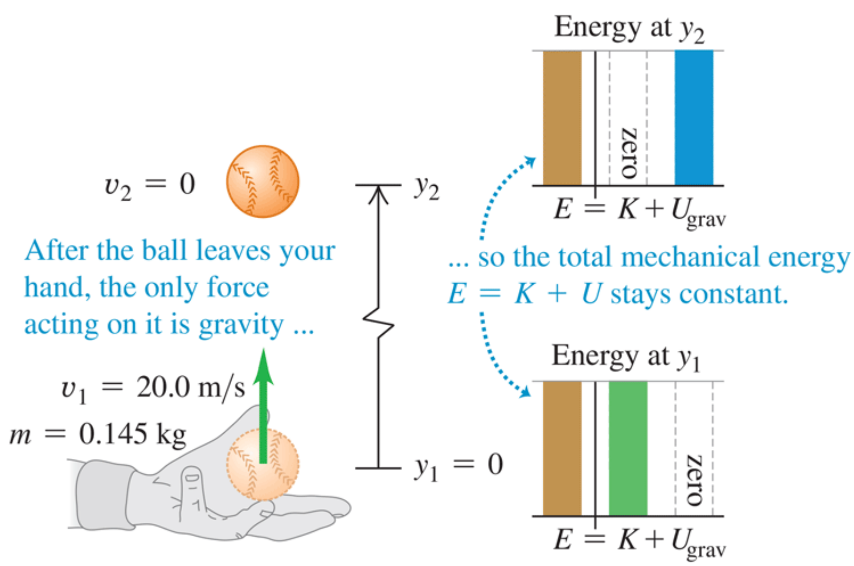
Example 7.1 Height of a baseball from energy conservation

WITH VARIATION PROBLEMS

You throw a 0.145 kg baseball straight up, giving it an initial velocity of magnitude 20.0 m/s. Find how high it goes, ignoring air resistance.

IDENTIFY and SET UP After the ball leaves your hand, only gravity does work on it. Hence total mechanical energy is conserved, and we can use Eq. (7.4). We take point 1 to be where the ball leaves your hand and point 2 to be where it reaches its maximum height. As in Fig. 7.2, we take the positive y-direction to be upward. The ball's speed at point 1 is $v_1 = 20.0\ \text{m/s}$; at its maximum height it is instantaneously at rest, so $v_2 = 0$. We take the origin at point 1, so $y_1 = 0$ (Fig. 7.4). Our target variable, the distance the ball moves vertically between the two points, is the displacement $y_2 - y_1 = y_2 - 0 = y_2$.

Figure 7.4



After a baseball leaves your hand, total mechanical energy $E = K + U$ is conserved.

EXECUTE We have $y_1 = 0$, $U_{grav,\,1} = mgy_1 = 0$, and $K_2 = \frac{1}{2}mv_2^2$. Then Eq. (7.4), $K_1 + U_{grav,\,1} = K_2 + U_{grav,\,2}$, becomes

$$K_1 = U_{grav,\,2}$$

As the energy bar graphs in Fig. 7.4 show, this equation says that the kinetic energy of the ball at point 1 is completely converted to gravitational potential energy at point 2. We substitute $K_1 = \frac{1}{2}mv_1{}^2$ and $U_{grav,\,2} = mgy_2$ and solve for y_2:

$$\begin{aligned}\frac{1}{2}mv_1^2 &= mgy_2\\ y_2 &= \frac{v_1^2}{2g} = \frac{(20.0\ \mathrm{m/s})^2}{2\left(9.80\ \mathrm{m/s^2}\right)} = 20.4\ \mathrm{m}\end{aligned}$$

EVALUATE As a check, use the given value of v_1 and our result for y_2 to calculate the kinetic energy at point 1 and the gravitational potential energy at point 2. You should find that these are equal: $K_1 = \frac{1}{2}mv_1^2$ and $U_{\text{grav, 2}} = mgy_2 = 29.0\ \text{J}$. Note that we could have found the result $y_2 = v_1^2/2g$ by using Eq. (2.13) in the form ${v_{2y}}^2 = {v_{1y}}^2 - 2g(y_2 - y_1)$.

What if we put the origin somewhere else—for example, 5.0 m below point 1, so that $y_1 = 5.0$ m? Then the total mechanical energy at point 1 is part kinetic and part potential; at point 2 it's still purely potential because $v_2 = 0$. You'll find that this choice of origin yields $y_2 = 25.4$ m, but again $y_2 - y_1 = 20.4$ m. In problems like this, you are free to choose the height at which $U_{\text{grav}} = 0$. The physics doesn't depend on your choice.

KEY CONCEPT

Total mechanical energy (the sum of kinetic energy and gravitational potential energy) is conserved when only the force of gravity does work.

Video Tutor Solution: **Example 7.1**

When Forces Other Than Gravity Do Work

If other forces act on the object in addition to its weight, then $\vec{F}_{\text{other}}$ in Fig. 7.2 is *not* zero. For the pile driver described in Example 6.4 (Section 6.2), the force applied by the hoisting cable and the friction with the vertical guide rails are examples of forces that might be included in $\vec{F}_{\text{other}}$. The gravitational work W_{grav} is still given by Eq. (7.3), but the total work W_{tot} is then the sum of W_{grav} and the work done by $\vec{F}_{\text{other}}$. We'll call this additional work W_{other}, so the total work done by all forces is $W_{\text{tot}} = W_{\text{grav}} + W_{\text{other}}$. Equating this to the change in kinetic energy, we have

(7.5)

$$W_{\text{other}} + W_{\text{grav}} = K_2 - K_1$$

Also, from Eq. (7.3), $W_{\text{grav}} = U_{\text{grav, 1}} - U_{\text{grav, 2}}$, so Eq. (7.5) becomes

$$W_{\text{other}} + U_{\text{grav, 1}} - U_{\text{grav, 2}} = K_2 - K_1$$

which we can rearrange in the form

(7.6)

$$K_1 + U_{\text{grav, 1}} + W_{\text{other}} = K_2 + U_{\text{grav, 2}} \quad \text{(if forces other than gravity do work)}$$

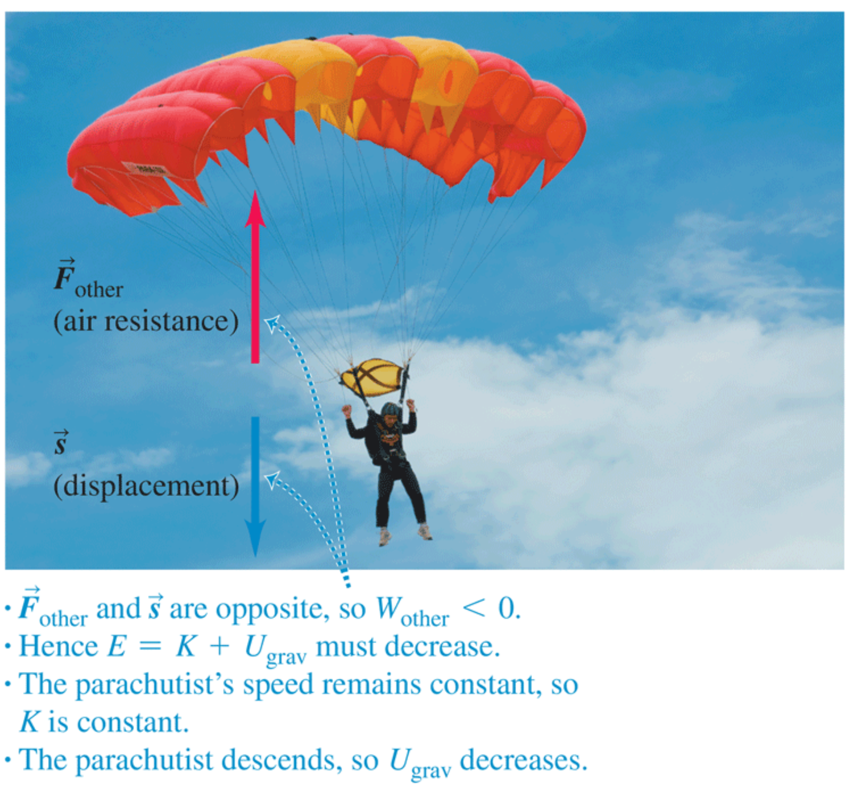
We can use the expressions for the various energy terms to rewrite Eq. (7.6):

(7.7)

$$\frac{1}{2}\,mv_1^2 + mgy_1 + W_{\text{other}} = \frac{1}{2}\,mv_2^2 + mgy_2 \quad \text{(if forces other than gravity do work)}$$

The meaning of Eqs. (7.6) and (7.7) is this: *The work done by all forces other than the gravitational force equals the change in the total mechanical energy $E = K + U_{\text{grav}}$ of the system, where U_{grav} is the gravitational potential energy*. When W_{other} is positive, E increases and $K_2 + U_{\text{grav},\ 2}$ is greater than $K_1 + U_{\text{grav},\ 1}$. When W_{other} is negative, E decreases (Fig. 7.5). In the special case in which no forces other than the object's weight do work, $W_{\text{other}} = 0$. The total mechanical energy is then constant, and we are back to Eq. (7.4).

Figure 7.5



As this parachutist moves downward at a constant speed, the upward force of air resistance does negative work W_{other} on him. Hence the total mechanical energy $E = K + U$ decreases.

Problem-Solving Strategy 7.1 Problems Using Total Mechanical Energy I

IDENTIFY *the relevant concepts:* Decide whether the problem should be solved by energy methods, by using $\sum \vec{F} = m\vec{a}$ directly, or by a combination of these. The energy approach is best when the problem involves varying forces or motion along a curved path (discussed later in this section). If the problem involves elapsed time, the energy approach is usually *not* the best choice because it doesn't involve time directly.

SET UP *the problem* using the following steps:

1. When using the energy approach, first identify the initial and final states (the positions and velocities) of the objects in question. Use the subscript 1 for the initial state and the subscript 2 for the final state. Draw sketches showing these states.
2. Define a coordinate system, and choose the level at which $y = 0$. Choose the positive y-direction to be upward. (The equations in this section require this.)
3. Identify any forces that do work on each object and that *cannot* be described in terms of potential energy. (So far, this means any forces other than gravity. In Section 7.2 we'll see that the work done by an ideal spring can also be expressed as a change in potential energy.) Sketch a free-body diagram for each object.
4. List the unknown and known quantities, including the coordinates and velocities at each point. Identify the target variables.

EXECUTE *the solution:* Write expressions for the initial and final kinetic and potential energies K_1, K_2, $U_{\text{grav, 1}}$, and $U_{\text{grav, 2}}$. If no other forces do work, use Eq. (7.4). If there are other forces that do work, use Eq. (7.6). Draw bar graphs showing the initial and final values of K, $U_{\text{grav, 1}}$, and $E = K + U_{\text{grav}}$. Then solve to find your target variables.

EVALUATE *your answer:* Check whether your answer makes physical sense. Remember that the gravitational work is included in ΔU_{grav}, so do not include it in W_{other}.

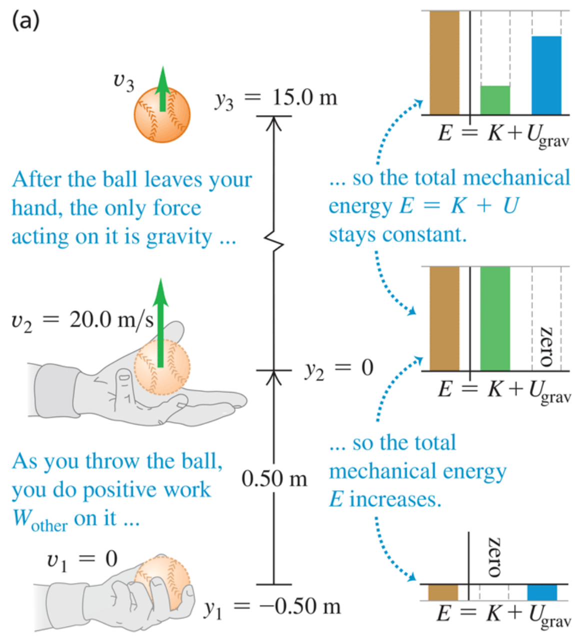
Example 7.2 Work and energy in throwing a baseball

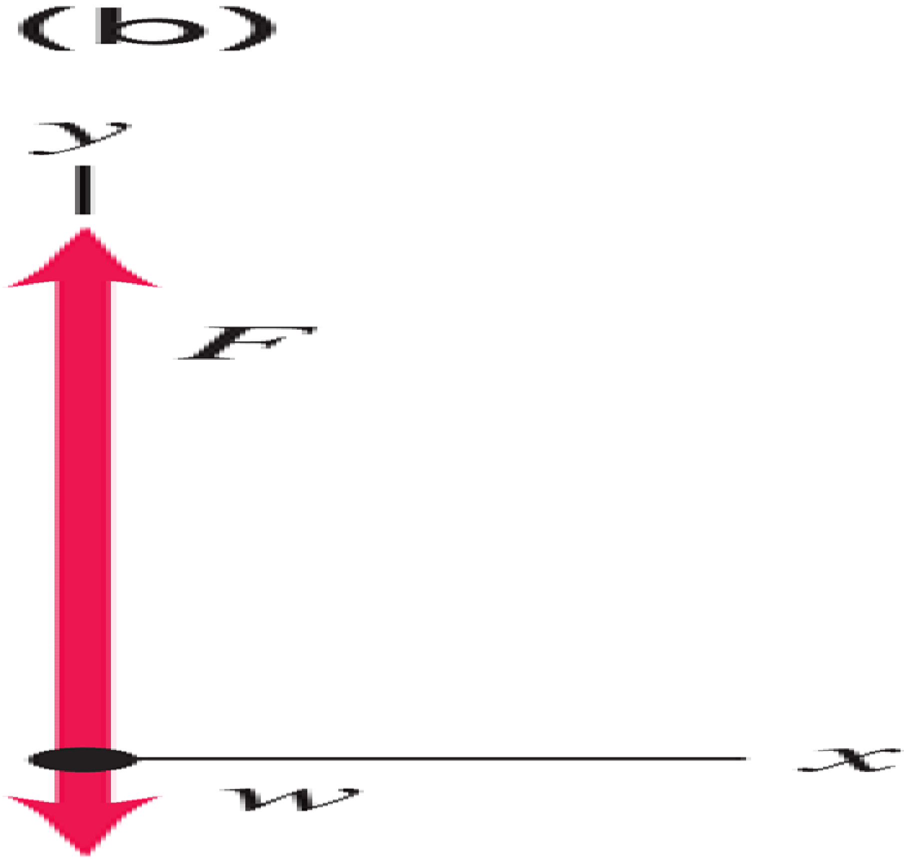
WITH VARIATION PROBLEMS

In Example 7.1 suppose your hand moves upward by 0.50 m while you are throwing the ball. The ball leaves your hand with an upward velocity of 20.0 m/s. (a) Find the magnitude of the force (assumed constant) that your hand exerts on the ball. (b) Find the speed of the ball at a point 15.0 m above the point where it leaves your hand. Ignore air resistance.

IDENTIFY and SET UP In Example 7.1 only gravity did work. Here we must include the nongravitational, “other” work done by your hand. Figure 7.6 shows a diagram of the situation, including a free-body diagram for the ball while it is being thrown. We let point 1 be where your hand begins to move, point 2 be where the ball leaves your hand, and point 3 be where the ball is 15.0 m above point 2. The nongravitational force $\vec{F}$ of your hand acts only between points 1 and 2. Using the same coordinate system as in Example 7.1, we have $y_1 = -0.50\text{ m}$, $y_2 = 0$, and $y_3 = 15.0\text{ m}$. The ball starts at rest at point 1, so $v_1 = 0$, and the ball's speed as it leaves your hand is $v_2 = 20.0 \text{ m/s}$. Our target variables are (a) the magnitude F of the force of your hand and (b) the magnitude of the ball's velocity v_{3y} at point 3.

Figure 7.6





(a) Applying energy ideas to a ball thrown vertically upward. (b) Free-body diagram for the ball as you throw it.

EXECUTE (a) To determine F, we'll first use Eq. (7.6) to calculate the work W_{other} done by this force. We have

$$\begin{aligned} K_1 &= 0 \\ U_{\text{grav},1} &= mgy_1 = (0.145\,\text{kg})\left(9.80\,\text{m/s}^2\right)(-0.50\,\text{m}) = -0.71\,\text{J} \\ K_2 &= \frac{1}{2}mv_2^2 = \frac{1}{2}(0.145\text{ kg})(20.0\text{ m/s})^2 = 29.0\text{ J} \\ U_{\text{grav},\,2} &= mgy_2 = (0.145\text{ kg})(9.80\text{ m/s}^2)(0) = 0 \end{aligned}$$

(Don't worry that $U_{\text{grav},\,1}$ is less than zero; all that matters is the *difference* in potential energy from one point to another.) From Eq. (7.6),

$$\begin{aligned} K_1 + U_{\text{grav, 1}} + W_{\text{other}} &= K_2 + U_{\text{grav, 2}} \\ W_{\text{other}} &= (K_2 - K_1) + (U_{\text{grav, 2}} - U_{\text{grav, 1}}) \\ &= (29.0\text{ J} - 0) + [0 - (-0.71\text{ J})] = 29.7\text{ J} \end{aligned}$$

But since $\vec{F}$ is constant and upward, the work done by $\vec{F}$ equals the force magnitude times the displacement: $W_{\text{other}} = F(y_2 - y_1)$. So

$$F = \frac{W_{\text{other}}}{y_2 - y_1} = \frac{29.7\text{ J}}{0.50\text{ m}} = 59\text{ N}$$

This is more than 40 times the weight of the ball (1.42 N).

(b) To find v_{3y}, note that between points 2 and 3 only gravity acts on the ball. So between these points the total mechanical energy is conserved and $W_{\text{other}} = 0$. From Eq. (7.4), we can solve for K_3 and from that solve for v_{3y}:

$$\begin{aligned} K_2 + U_{\text{grav, 2}} &= K_3 + U_{\text{grav, 3}} \\ U_{\text{grav, 3}} &= mgy_3 = (0.145\text{ kg})(9.80\text{ m/s}^2)(15.0\text{ m}) = 21.3\text{ J} \\ K_3 &= (K_2 + U_{\text{grav, 2}}) - U_{\text{grav, 3}} \\ &= (29.0\text{ J} + 0) - 21.3\text{ J} = 7.7\text{ J} \end{aligned}$$

Since $K_3 = \frac{1}{2} m v_{3y}{}^2$ we find

$$v_{3y} = \pm\sqrt{\frac{2K_3}{m}} = \pm\sqrt{\frac{2(7.7\text{ J})}{0.145\text{ kg}}} = \pm 10\text{ m}/s$$

The plus-or-minus sign reminds us that the ball passes point 3 on the way up and again on the way down. The ball's kinetic energy $K_3 = 7.7$ J at point 3, and hence its speed at that point, doesn't depend on the direction the ball is moving. The velocity v_{3y} is positive ($+10\text{ m}/s$) when the ball is moving up and negative ($-10\text{ m}/s$) when it is moving down; the speed v_3 is $10\text{ m}/s$ in either case.

EVALUATE In Example 7.1 we found that the ball reaches a maximum height $y = 20.4$ m. At that point all of the kinetic energy it had when it left your hand at $y = 0$ has been converted to gravitational potential energy. At $y = 15.0$ m, the ball is about three-fourths of the way to its maximum height, so about three-fourths of its total mechanical energy should be in the form of potential energy. Can you verify this from our results for K_3 and $U_{\text{grav, 3}}$?

KEY CONCEPT

When a force that cannot be described in terms of potential energy does work W_{other}, the final value of the total mechanical energy equals the initial value of the mechanical energy plus W_{other}.

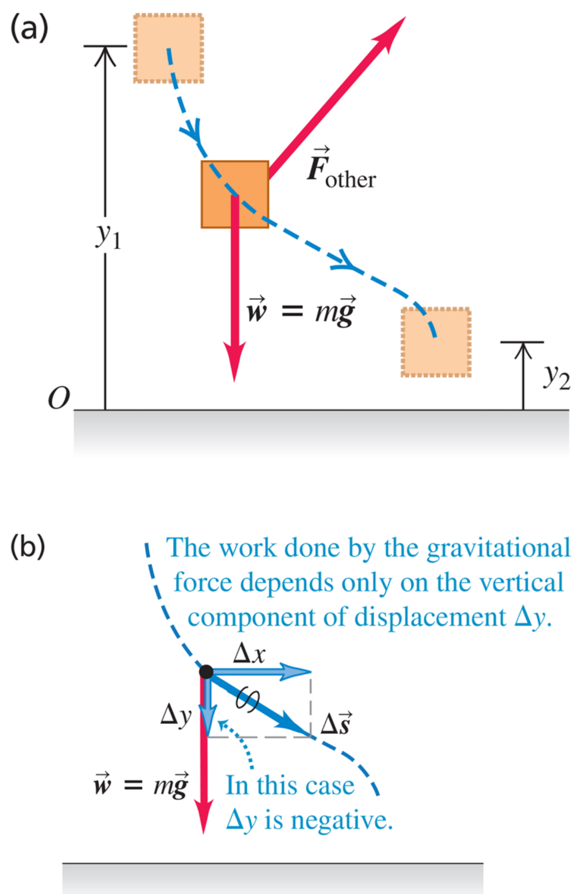
Video Tutor Solution: **Example 7.2**

Gravitational Potential Energy for Motion Along a Curved Path

In our first two examples the object moved along a straight vertical line. What happens when the path is slanted or curved (Fig. 7.7a)? The object is acted on by the gravitational force $\vec{w} = m\vec{g}$ and possibly by other forces whose resultant we call $\vec{F}_{\text{other}}$. To find the work W_{grav} done by the gravitational force during this displacement, we divide the path into small segments $\Delta\vec{s}$; Fig. 7.7b shows a typical segment. The work done by the gravitational force over this segment is the scalar product of the force and the displacement. In terms of unit vectors, the force is $\vec{w} = m\vec{g} = -mg\hat{j}$ and the displacement is $\Delta\vec{s} = \Delta x\hat{i} + \Delta y\hat{j}$, so

$$W_{\text{grav}} = \vec{w} \cdot \Delta\vec{s} = -mg\hat{j} \cdot \left(\Delta x\hat{i} + \Delta y\hat{j}\right) = -mg\Delta y$$

Figure 7.7



Calculating the change in gravitational potential energy for a displacement along a curved path.

The work done by gravity is the same as though the object had been displaced vertically a distance Δy, with no horizontal displacement. This is true for every segment, so the *total* work done by the gravitational force is $-mg$ multiplied by the *total* vertical displacement $(y_2 - y_1)$:

$$W_{\text{grav}} = -mg\,(y_2 - y_1) = mgy_1 - mgy_2 = U_{\text{grav, 1}} - U_{\text{grav, 2}}$$

This is the same as Eq. (7.1) or (7.3), in which we assumed a purely vertical path. So even if the path an object follows between two points is curved, the total work done by the gravitational force depends on only the difference in height between the two points of the path. This work is unaffected by any horizontal motion that may occur. So *we can use the same expression for gravitational potential energy whether the object's path is curved or straight.*

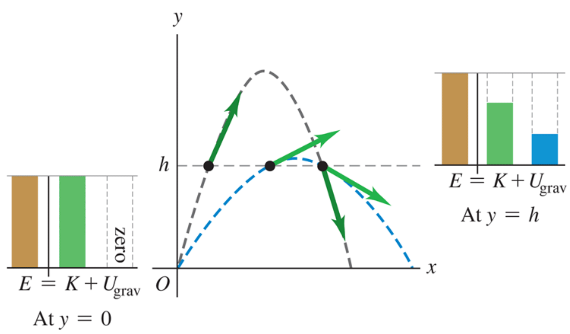
CAUTION **With gravitational potential energy, only the change in height matters** The change in gravitational potential energy along a curved path depends only on the difference between the final and initial heights, not on the shape of the path. If gravity is the only force that does work along a curved path, then the total mechanical energy is conserved.

Conceptual Example 7.3 Energy in projectile motion

A batter hits two identical baseballs with the same initial speed and from the same initial height but at different initial angles. Prove that both balls have the same speed at any height h if air resistance can be ignored.

SOLUTION The only force acting on each ball after it is hit is its weight. Hence the total mechanical energy for each ball is constant. Figure 7.8 shows the trajectories of two balls batted at the same height with the same initial speed, and thus the same total mechanical energy, but with different initial angles. At all points at the same height the potential energy is the same. Thus the kinetic energy at this height must be the same for both balls, and the speeds are the same.

Figure 7.8



For the same initial speed and initial height, the speed of a projectile at a given elevation h is always the same, if we ignore air resistance.

KEY CONCEPT

The gravitational potential energy of an object depends on its height, not on the path the object took to reach that height.

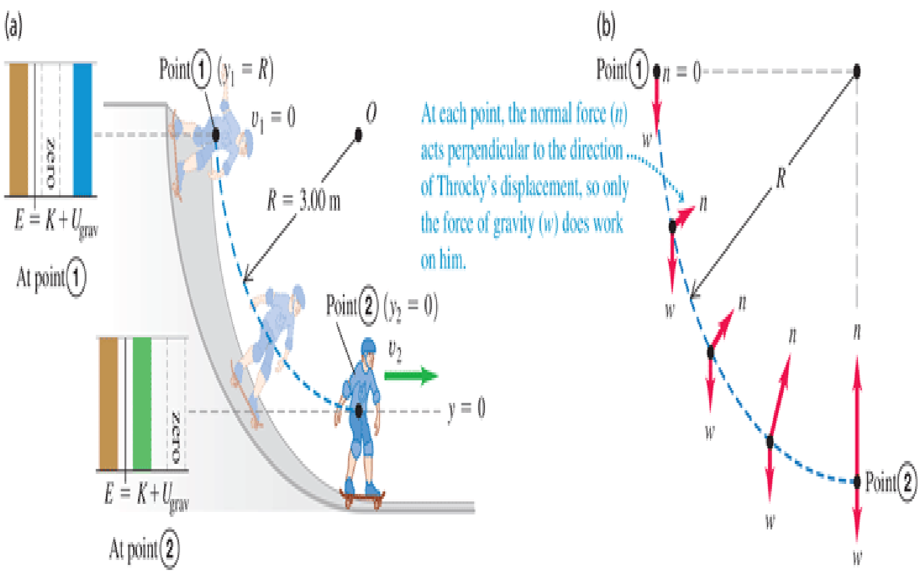
Video Tutor Solution: **Example 7.3**

Example 7.4 Speed at the bottom of a vertical circle

WITH VARIATION PROBLEMS

Your cousin Throckmorton skateboards from rest down a curved, frictionless ramp. If we treat Throcky and his skateboard as a particle, he moves through a quarter-circle with radius $R = 3.00$ m (Fig. 7.9, next page). Throcky and his skateboard have a total mass of 25.0 kg. (a) Find his speed at the bottom of the ramp. (b) Find the normal force that acts on him at the bottom of the curve.

Figure 7.9



(a) Throcky skateboarding down a frictionless circular ramp. The total mechanical energy is constant. (b) Free-body diagrams for Throcky and his skateboard at various points on the ramp.

IDENTIFY We can't use the constant-acceleration equations of Chapter 2 because Throcky's acceleration isn't constant; the slope decreases as he descends. Instead, we'll use the energy approach. Throcky moves along a circular arc, so we'll also use what we learned about circular motion in Section 5.4.

SET UP The only forces on Throcky are his weight and the normal force $\vec{n}$ exerted by the ramp (Fig. 7.9b). Although $\vec{n}$ acts all along the path, it does zero work because $\vec{n}$ is perpendicular to Throcky's displacement at every point. Hence $W_{\text{other}} = 0$ and the total mechanical energy is conserved. We treat Throcky as a particle located at the center of his body, take point 1 at the particle's starting point, and take point 2 (which we let be $y = 0$) at the particle's low point. We take the positive y-direction upward; then $y_1 = R$ and $y_2 = 0$. Throcky starts at rest at the top, so $v_1 = 0$. In part (a) our target variable is his speed v_2 at the bottom; in part (b) the target variable is the magnitude n of the normal force at point 2. To find n, we'll use Newton's second law and the relationship $a = v^2/R$.

EXECUTE (a) The various energy quantities are

$$\begin{aligned} K_1 &= 0 & U_{\text{grav, 1}} &= mgR \\ K_2 &= \frac{1}{2}mv_2^2 & U_{\text{grav, 2}} &= 0 \end{aligned}$$

From conservation of total mechanical energy, Eq. (7.4),

$$\begin{aligned} K_1 + U_{\text{grav, 1}} &= K_2 + U_{\text{grav, 2}} \\ 0 + mgR &= \frac{1}{2}mv_2^2 + 0 \\ v_2 &= \sqrt{2gR} = \sqrt{2(9.80 \text{ m/s}^2)(3.00 \text{ m})} = 7.67 \text{ m/}s \end{aligned}$$

This answer doesn't depend on the ramp being circular; Throcky would have the same speed $v_2 = \sqrt{2gR}$ at the bottom of any ramp of height R, no matter what its shape.

(b) To use Newton's second law to find n at point 2, we need the free-body diagram at that point (Fig. 7.9b). At point 2, Throcky is moving at speed $v_2 = \sqrt{2gR}$ in a circle of radius R; his acceleration is toward the center of the circle and has magnitude

$$a_{\mathrm{rad}} = \frac{v_1^2}{R} = \frac{2gR}{R} = 2g$$

The y-component of Newton's second law is

$$\begin{aligned} \sum F_y &= n + (-w) = ma_{\mathrm{rad}} = 2mg \\ n &= w + 2mg = 3mg \\ &= 3(25.0\ \mathrm{kg})(9.80\ \mathrm{m/s^2}) = 735\ \mathrm{N} \end{aligned}$$

At point 2 the normal force is three times Throcky's weight. This result doesn't depend on the radius R of the ramp. We saw in Examples 5.9 and 5.23 that the magnitude of n is the *apparent weight*, so at the bottom of the *curved part* of the ramp Throcky feels as though he weighs three times his true weight mg. But when he reaches the *horizontal* part of the ramp, immediately to the right of point 2, the normal force decreases to $w = mg$ and thereafter Throcky feels his true weight again. Can you see why?

EVALUATE This example shows a general rule about the role of forces in problems in which we use energy techniques: What matters is not simply whether a force *acts*, but whether that force *does work*. If the force does no work, like the normal force $\vec{n}$ here, then it does not appear in Eqs. (7.4) and (7.6).

KEY CONCEPT

If one of the forces that acts on a moving object is always perpendicular to the object's path, that force does no work on the object and plays no role in the equation for total mechanical energy.

Video Tutor Solution: **Example 7.4**

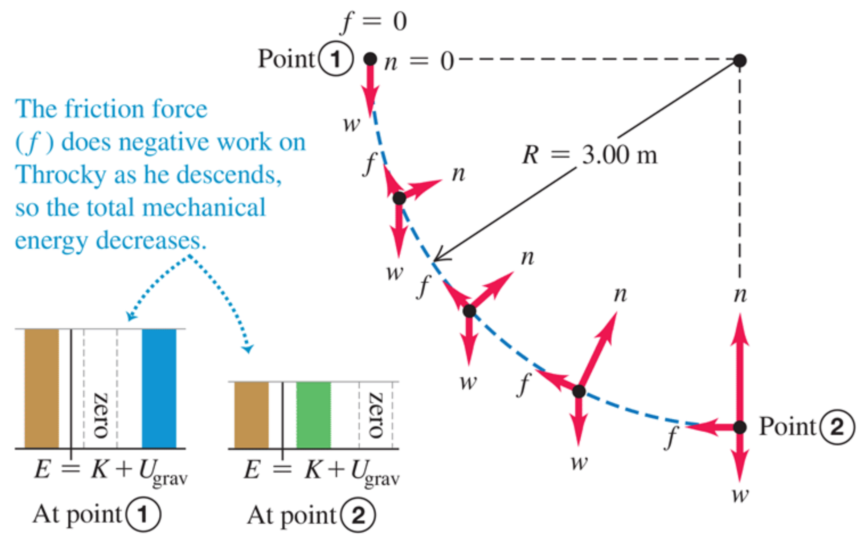
Example 7.5 A vertical circle with friction

WITH VARIATION PROBLEMS

Suppose that the ramp of Example 7.4 is not frictionless and that Throcky's speed at the bottom is only 6.00 m/s, not the 7.67 m/s we found there. What work was done on him by the friction force?

IDENTIFY and SET UP The setup is the same as in Example 7.4. Figure 7.10 shows that again the normal force does no work, but now there is a friction force $\vec{f}$ that *does* do work W_f . Hence the nongravitational work W_{other} done on Throcky between points 1 and 2 is equal to W_f and is not zero. Our target variable is $W_f = W_{\text{other}}$, which we'll find by using Eq. (7.6). Since $\vec{f}$ points opposite to Throcky's motion, W_f is negative.

Figure 7.10



Energy bar graphs and free-body diagrams for Throcky skateboarding down a ramp with friction.

EXECUTE The energy quantities are

$$\begin{aligned} K_1 &= 0 \\ U_{\text{grav},1} &= mgR = (25.0\,\text{kg})\left(9.80\,\text{m/s}^2\right)(3.00\,\text{m}) = 735\,\text{J} \\ K_2 &= \frac{1}{2}mv_2^2 = \frac{1}{2}(25.0\text{ kg})(6.00\text{ m/s})^2 = 450\text{ J} \\ U_{\text{grav},2} &= 0 \end{aligned}$$

From Eq. (7.6),

$$\begin{aligned} W_f &= W_{\text{other}} \\ &= K_2 + U_{\text{grav},2} - K_1 - U_{\text{grav},1} \\ &= 450\text{ J} + 0 - 0 - 735\text{ J} = -285\text{ J} \end{aligned}$$

The work done by the friction force is -285 J, and the total mechanical energy *decreases* by 285 J.

EVALUATE Our result for W_f is negative. Can you see from the free-body diagrams in Fig. 7.10 why this must be so?

It would be very difficult to apply Newton's second law, $\sum \vec{F} = m\vec{a}$, directly to this problem because the normal and friction forces and the acceleration are continuously changing in both magnitude and direction as Throcky descends. The energy approach, by contrast, relates the motions at the top and bottom of the ramp without involving the details of the motion in between.

KEY CONCEPT

Whether an object's path is straight or curved, the relationship is the same among the initial total mechanical energy, the final total mechanical energy, and the work done by forces other than gravity.

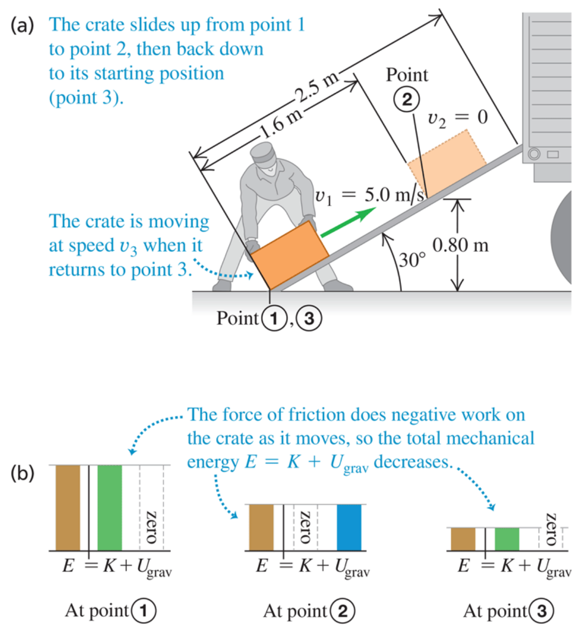
Video Tutor Solution: Example 7.5

Example 7.6 An inclined plane with friction

We want to slide a 12 kg crate up a 2.5-m-long ramp inclined at 30°. A worker, ignoring friction, calculates that he can do this by giving it an initial speed of 5.0 m/s at the bottom and letting it go. But friction is *not* negligible; the crate slides only 1.6 m up the ramp, stops, and slides back down (Fig. 7.11a). (a) Find the magnitude of the friction force acting on the crate, assuming that it is constant. (b)

How fast is the crate moving when it reaches the bottom of the ramp?

Figure 7.11



(a) A crate slides partway up the ramp, stops, and slides back down. (b) Energy bar graphs for points 1, 2, and 3.

IDENTIFY and SET UP The friction force does work on the crate as it slides from point 1, at the bottom of the ramp, to point 2, where the crate stops instantaneously ($v_2 = 0$). Friction also does work as the crate returns to the bottom of the ramp, which we'll call point 3 (Fig. 7.11a). We take the positive y-direction upward. We take $y = 0$ (and hence $U_{grav} = 0$) to be at ground level (point 1), so $y_1 = 0$, $y_2 = (1.6\text{ m})\sin 30^\circ = 0.80\text{ m}$, and $y_3 = 0$. We are given $v_1 = 5.0\text{ m/s}$. In part (a) our target variable is f, the magnitude of the friction force as the crate slides up; we'll find this by using the energy approach. In part (b) our target variable is v, the crate's speed at the bottom of the ramp. We'll calculate the work done by friction as the crate slides back down, then use the energy approach to find v_3.

EXECUTE (a) The energy quantities are

$$\begin{aligned} K_1 &= \frac{1}{2}(12\ \mathrm{kg})(5.0\ \mathrm{m/s})^2 = 150\ \mathrm{J} \\ U_{\mathrm{grav},\,1} &= 0 \\ K_2 &= 0 \\ U_{\mathrm{grav},\,2} &= (12\ \mathrm{kg})(9.8\ \mathrm{m/s^2})(0.80\ \mathrm{m}) = 94\ \mathrm{J} \\ W_{\mathrm{other}} &= -fs \end{aligned}$$

Here $s = 1.6$ m. Using Eq. (7.6), we find

$$\begin{aligned} K_1 + U_{\mathrm{grav},\,1} + W_{\mathrm{other}} &= K_2 + U_{\mathrm{grav},\,2} \\ W_{\mathrm{other}} &= -fs = (K_2 + U_{\mathrm{grav},\,2}) - (K_1 + U_{\mathrm{grav},\,1}) \\ &= (0 + 94\ \mathrm{J}) - (150\ \mathrm{J} + 0) = -56\ \mathrm{J} = -fs \\ f = \frac{W_{\mathrm{other}}}{s} &= \frac{56\ \mathrm{J}}{1.6\ \mathrm{m}} = 35\ \mathrm{N} \end{aligned}$$

The friction force of 35 N, acting over 1.6 m, causes the total mechanical energy of the crate to decrease from 150 J to 94 J (Fig. 7.11b).

(b) As the crate moves from point 2 to point 3, the work done by friction has the same negative value as from point 1 to point 2. (Both the friction force and the displacement reverse direction, but their magnitudes don't change.) The total work done by friction between points 1 and 3 is therefore

$$W_{\mathrm{other}} = W_{\mathrm{fric}} = -2fs = -2(56\ \mathrm{J}) = -112\ \mathrm{J}$$

From part (a), $K_1 = 150$ J and $U_{\mathrm{grav},\,1} = 0$; in addition, $U_{\mathrm{grav},\,3} = 0$ since $y_3 = 0$. Equation (7.6) then gives

$$\begin{aligned} K_1 + U_{\mathrm{grav},\,1} + W_{\mathrm{other}} &= K_3 + U_{\mathrm{grav},\,3} \\ K_3 &= K_1 + U_{\mathrm{grav},\,1} - U_{\mathrm{grav},\,3} + W_{\mathrm{other}} \\ &= 150\ \mathrm{J} + 0 - 0 + (-112\ \mathrm{J}) = 38\ \mathrm{J} \end{aligned}$$

The crate returns to the bottom of the ramp with only 38 J of the original 150 J of total mechanical energy (Fig. 7.11b). Since $K_3 = \frac{1}{2}mv_3^2$

$$v = \sqrt{\frac{2K_3}{m}} = \sqrt{\frac{2(38\ \mathrm{J})}{12\ \mathrm{kg}}} = 2.5\ \mathrm{m}/s$$

EVALUATE Energy is lost due to friction, so the crate's speed $v = 2.5\ \mathrm{m}/s$ when it returns to the bottom of the ramp is less than the speed $v_1 = 5.0\ \mathrm{m}/s$ at which it left that point. In part (b) we applied Eq. (7.6) to points 1 and 3, considering the round trip as a whole. Alternatively, we could have considered the second part of the motion by itself and applied Eq. (7.6) to points 2 and 3. Try it; do you get the same result for v_3?

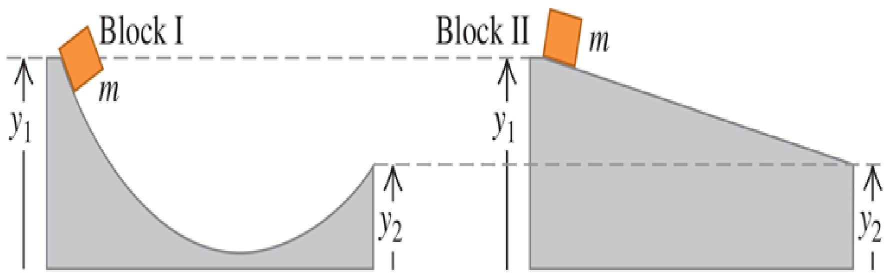
KEY CONCEPT

For straight-line motion problems in which the forces are constant in each stage of the motion, you can use the total mechanical energy to find the magnitude of an unknown force that does work.

Video Tutor Solution: **Example 7.6**

Test Your Understanding of Section 7.1

The figure shows two frictionless ramps. The heights y_1 and y_2 are the same for both ramps. If a block of mass m is released from rest at the left-hand end of each ramp, which block arrives at the right-hand end with the greater speed? (i) Block I; (ii) block II; (iii) the speed is the same for both blocks.

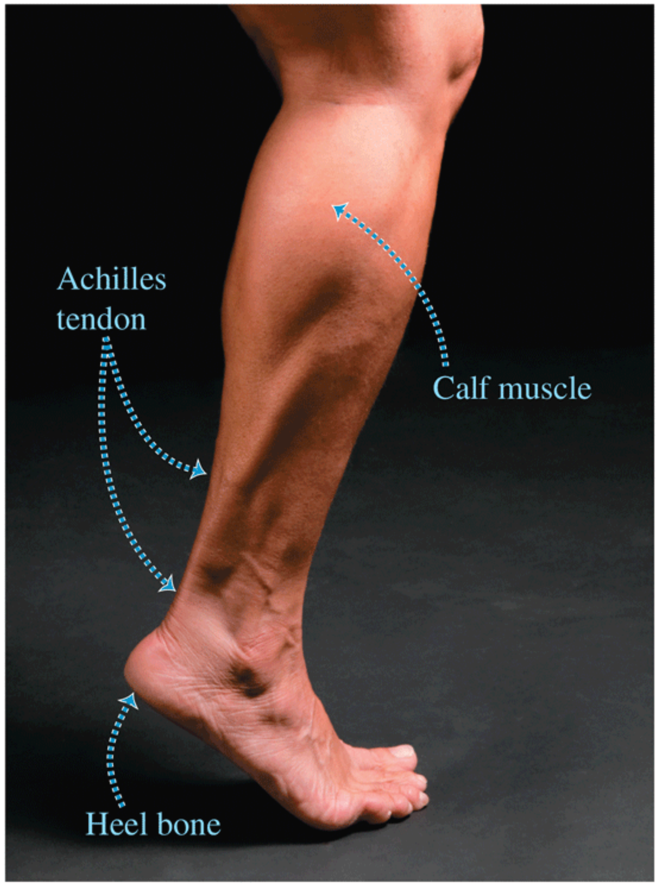


7.2 Elastic Potential Energy

In many situations we encounter potential energy that is not gravitational in nature. One example is a rubber-band slingshot. Work is done on the rubber band by the force that stretches it, and that work is stored in the rubber band until you let it go. Then the rubber band gives kinetic energy to the projectile.

This is the same pattern we saw with the baseball in Example 7.2: Do work on the system to store energy, which can later be converted to kinetic energy. We'll describe the process of storing energy in a deformable object such as a spring or rubber band in terms of *elastic potential energy* (Fig. 7.12). An object is called *elastic* if it returns to its original shape and size after being deformed.

Figure 7.12



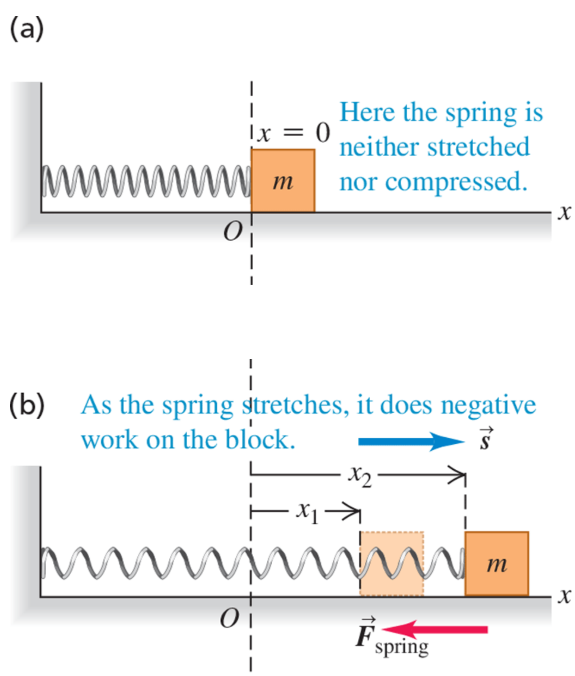
The Achilles tendon, which runs along the back of the ankle to the heel bone, acts like a natural spring. When it stretches and then relaxes, this tendon stores and then releases elastic potential energy. This spring action reduces the amount of work your leg muscles must do as you run.

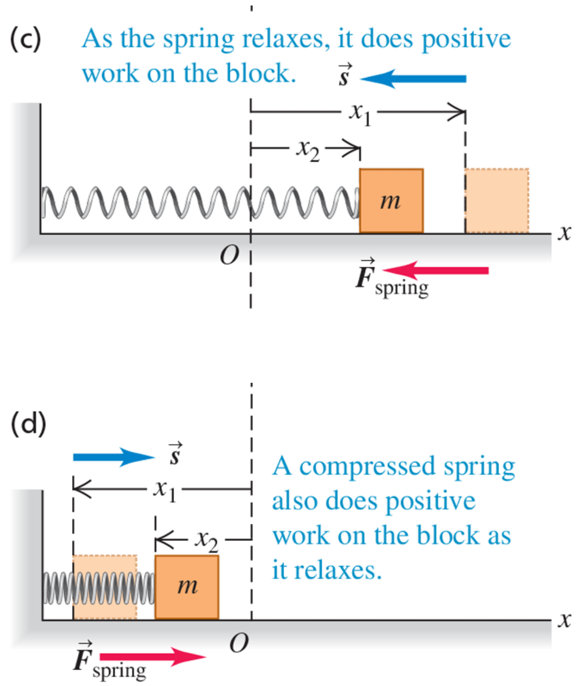
To be specific, we'll consider storing energy in an ideal spring, like the ones we discussed in Section 6.3. To keep such an ideal spring stretched by a distance x, we must exert a force $F = kx$, where k is the force constant of the spring. Many elastic objects show this same direct proportionality between force $\vec{F}$ and displacement x, provided that x is sufficiently small.

Let's proceed just as we did for gravitational potential energy. We begin with the work done by the elastic (spring) force and then combine this with the work–energy theorem. The difference is that gravitational potential energy is a shared property of an object and the earth, but elastic potential energy is stored in just the spring (or other deformable object).

Figure 7.13 shows the ideal spring from Fig. 6.18 but with its left end held stationary and its right end attached to a block with mass m that can move along the x-axis. In Fig. 7.13a the block is at $x = 0$ when the spring is neither stretched nor compressed. We move the block to one side, thereby stretching or compressing the spring, then let it go. As the block moves from a different position x_1 to a different position x_2, how much work does the elastic (spring) force do on the block?

Figure 7.13





Calculating the work done by a spring attached to a block on a horizontal surface. The quantity x is the extension or compression of the spring.

We found in Section 6.3 that the work we must do *on* the spring to move one end from an elongation x_1 to a different elongation x_2 is

(7.8)

$$W = \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2 \qquad \text{(work done } \textit{on} \text{ a spring)}$$

where k is the force constant of the spring. If we stretch the spring farther, we do positive work on the spring; if we let the spring relax while holding one end, we do negative work on it. This expression for work is also correct when the spring is compressed such that x_1, x_2, or both are negative. Now, from Newton's third law the work done *by* the spring is just the negative of the work done *on* the spring. So by changing the signs in Eq. (7.8), we find that in a displacement from x_1 to x_2 the spring does an amount of work W_{el} given by

(7.9)

$$W_{\text{el}} = \frac{1}{2}\,kx_1^2 - \frac{1}{2}\,kx_2^2 \qquad \text{(work done } \textit{by} \text{ a spring)}$$

The subscript "el" stands for *elastic*. When both x_1 and x_2 are positive and $x_2 > x_1$ (Fig. 7.13b), the spring does negative work on the block, which moves in the $+x-$ direction while the spring pulls on it in the $-x-$ direction. The spring stretches farther, and the block slows down. When both x_1 and x_2 are positive and $x_2 < x_1$ (Fig. 7.13c), the spring does positive work as it relaxes and the block speeds up. If the spring can be compressed as well as stretched, x_1, x_2, or both may be negative, but the expression for W_{el} is still valid. In Fig. 7.13d, both x_1 and x_2 are negative, but x_2 is less negative than x_1 ; the compressed spring does positive work as it relaxes, speeding the block up.

Just as for gravitational work, we can express Eq. (7.9) for the work done by the spring in terms of a quantity at the beginning and end of the displacement. This quantity is $\frac{1}{2}\,kx^2$, and we define it to be the **elastic potential energy:**

(7.10)

Force constant of spring

Elastic potential energy stored in a spring

$$U_{\text{el}} = \frac{1}{2}kx^2 \qquad (7.10)$$

Elongation of spring ($x > 0$ if stretched, $x < 0$ if compressed)

Figure 7.14 is a graph of Eq. (7.10). As for all other energy and work quantities, the unit of U_{el} is the joule (J); to see this from Eq. (7.10), recall that the units of k are N/m and that $1\ \text{N} \cdot m = 1\ J$. We can now use Eq. (7.10) to rewrite Eq. (7.9) for the work W_{el} done by the spring:

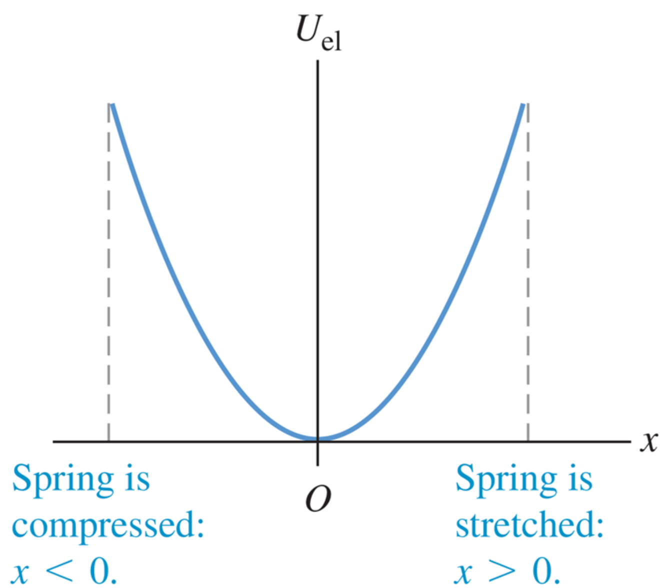
(7.11)

Work done by the elastic force ... ... equals the negative of the change in elastic potential energy.

$$W_{\mathrm{el}} = \tfrac{1}{2}kx_1^2 - \tfrac{1}{2}kx_2^2 = U_{\mathrm{el},1} - U_{\mathrm{el},2} = -\Delta U_{\mathrm{el}} \qquad (7.11)$$

Force constant of spring Initial and final elongations of spring

Figure 7.14



The graph of elastic potential energy for an ideal spring is a parabola: $U_{\mathrm{el}} = \frac{1}{2}kx^2$, where x is the extension or compression of the spring. Elastic potential energy U_{el} is never negative.

When a stretched spring is stretched farther, as in Fig. 7.13b, W_{el} is negative and U_{el} increases; more elastic potential energy is stored in the spring. When a stretched spring relaxes, as in Fig. 7.13c, x decreases, W_{el} is positive, and U_{el} decreases; the spring loses elastic potential energy. Figure 7.14 shows that U_{el} is positive for both positive and negative x values; Eqs. (7.10) and (7.11) are valid for both cases. The more a spring is compressed or stretched, the greater its elastic potential energy.

CAUTION Gravitational potential energy vs. elastic potential energy An important difference between gravitational potential energy $U_{\text{grav}} = mgy$ and elastic potential energy $U_{\text{el}} = \frac{1}{2}kx^2$ is that we *cannot* choose $x = 0$ to be wherever we wish. In Eq. (7.10), $x = 0$ *must* be the position at which the spring is neither stretched nor compressed. At that position, both its elastic potential energy and the force that it exerts are zero.

The work–energy theorem says that $W_{\text{tot}} = K_2 - K_1$, no matter what kind of forces are acting on an object. If the elastic force is the *only* force that does work on the object, then

$$W_{\text{tot}} = W_{\text{el}} = U_{\text{el, 1}} - U_{\text{el, 2}}$$

and so

(7.12)

If only the elastic force does work, total mechanical energy is conserved:

Initial kinetic energy $K_1 = \frac{1}{2}mv_1^2$

Initial elastic potential energy $U_{\text{el},1} = \frac{1}{2}kx_1^2$

$$K_1 + U_{\text{el},1} = K_2 + U_{\text{el},2} \qquad (7.12)$$

Final kinetic energy $K_2 = \frac{1}{2}mv_2^2$

Final elastic potential energy $U_{\text{el},2} = \frac{1}{2}kx_2^2$

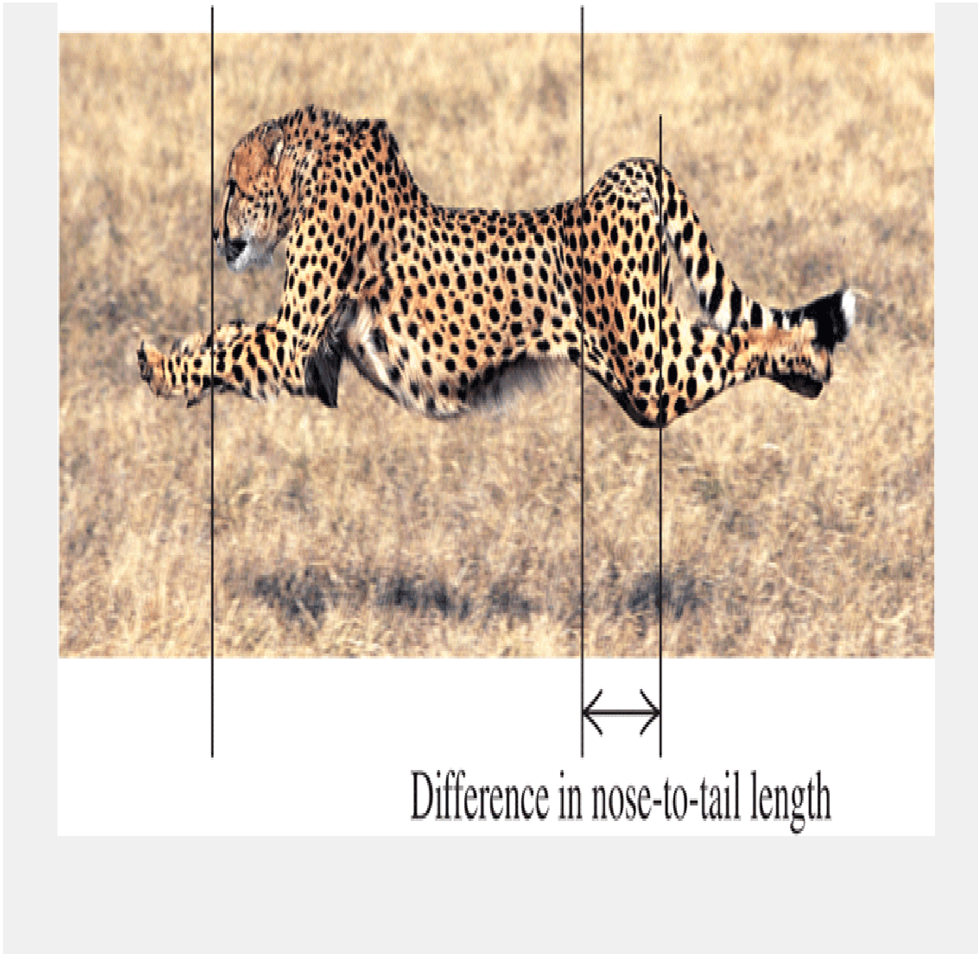
In this case the total mechanical energy $E = K + U_{\text{el}}$—the sum of kinetic and *elastic* potential energies—is *conserved*. An example of this is the motion of the block in Fig. 7.13, provided the horizontal surface is frictionless so no force does work other than that exerted by the spring.

For Eq. (7.12) to be strictly correct, the ideal spring that we've been discussing must also be *massless*. If the spring has mass, it also has kinetic energy as the coils of the spring move back and forth. We can ignore the kinetic energy of the spring if its mass is much less than the mass m of the object attached to the spring. For instance, a typical automobile has a mass of 1200 kg or more. The springs in its suspension have masses of only a few kilograms, so their mass can be ignored if we want to study how a car bounces on its suspension.

BIO Application

Elastic Potential Energy of a Cheetah

When a cheetah (*Acinonyx jubatus*) gallops, its back flexes and extends dramatically. Flexion of the back stretches tendons and muscles along the top of the spine and also compresses the spine, storing elastic potential energy. When the cheetah launches into its next bound, this energy is released, enabling the cheetah to run more efficiently.



Situations with Both Gravitational and Elastic Potential Energy

Equation (7.12) is valid when the only potential energy in the system is elastic potential energy. What happens when we have *both* gravitational and elastic forces, such as a block attached to the lower end of a vertically hanging spring? And what if work is also done by other forces that *cannot* be described in terms of potential energy, such as the force of air resistance on a moving block? Then the total work is the sum of the work done by the gravitational force (W_{grav}), the work done by the elastic force (W_{el}), and the work done by other forces

(W_{other}) : $W_{\text{tot}} = W_{\text{grav}} + W_{\text{el}} + W_{\text{other}}$. The work–energy theorem then gives

$$W_{\text{grav}} + W_{\text{el}} + W_{\text{other}} = K_2 - K_1$$

The work done by the gravitational force is $W_{\text{grav}} = U_{\text{grav, 1}} - U_{\text{grav, 2}}$ and the work done by the spring is $W_{\text{el}} = U_{\text{el, 1}} - U_{\text{el, 2}}$. Hence we can rewrite the work–energy theorem for this most general case as

(7.13)

$$K_1 + U_{\text{grav, 1}} + U_{\text{el, 1}} + W_{\text{other}} = K_2 + U_{\text{grav, 2}} + U_{\text{el, 2}} \quad \text{(valid in general)}$$

or, equivalently,

(7.14)

General relationship for kinetic energy and potential energy:

Initial kinetic energy

Final kinetic energy

$$K_1 + U_1 + W_{\text{other}} = K_2 + U_2 \qquad (7.14)$$

Initial potential energy of all kinds

Work done by other forces (not associated with potential energy)

Final potential energy of all kinds

where $U = U_{\text{grav}} + U_{\text{el}} = mgy + \frac{1}{2}kx^2$ is the *sum* of gravitational potential energy and elastic potential energy. We call U simply "the potential energy."

Equation (7.14) is *the most general statement* of the relationship among kinetic energy, potential energy, and work done by other forces. It says:

> The work done by all forces other than the gravitational force or elastic force equals the change in the total mechanical energy $E = K + U$ of the system.

The "system" is made up of the object of mass m, the earth with which it interacts through the gravitational force, and the spring of force constant k.

If W_{other} is positive, $E = K + U$ increases; if W_{other} is negative, E decreases. If the gravitational and elastic forces are the *only* forces that do work on the object, then $W_{\text{other}} = 0$ and the total mechanical energy $E = K + U$ is conserved. [Compare Eq. (7.14) to Eqs. (7.6) and (7.7), which include gravitational potential energy but not elastic potential energy.]

Trampoline jumping (Fig. 7.15) involves transformations among kinetic energy, elastic potential energy, and gravitational potential energy. As the jumper descends through the air from the high point of the bounce, gravitational potential energy U_{grav} decreases and kinetic energy K increases. Once the jumper touches the trampoline, some of the total mechanical energy goes into elastic potential energy U_{el} stored in the trampoline's springs. At the lowest point of the trajectory (U_{grav} is minimum), the jumper comes to a momentary halt ($K = 0$) and the springs are maximally stretched (U_{el} is maximum). The springs then convert their energy back into K and U_{grav}, propelling the jumper upward.

Figure 7.15

Trampoline jumping involves an interplay among kinetic energy, gravitational potential energy, and elastic potential energy. Due to air resistance and friction forces within the trampoline, total mechanical energy is not conserved. That's why the bouncing eventually stops unless the jumper does work with his or her legs to compensate for the lost energy.

Problem-Solving Strategy 7.2 Problems Using Total Mechanical Energy II

Problem-Solving Strategy 7.1 (Section 7.1) is useful in solving problems that involve elastic forces as well as gravitational forces. The only new wrinkle is that the potential energy U now includes the elastic potential energy $U_{\text{el}} = \frac{1}{2}kx^2$, where x is the displacement of the spring *from its unstretched length*. The work done by the gravitational and elastic forces is accounted for by their potential energies; the work done by other forces, W_{other}, must still be included separately.

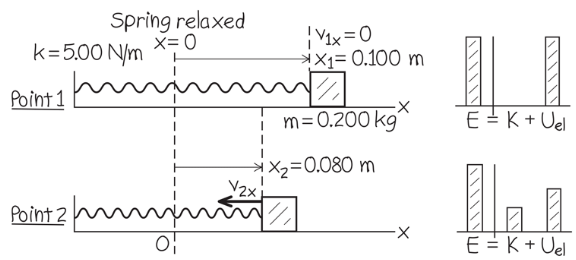
Example 7.7 Motion with elastic potential energy

WITH VARIATION PROBLEMS

A glider with mass $m = 0.200$ kg sits on a frictionless, horizontal air track, connected to a spring with force constant $k = 5.00$ N/m. You pull on the glider, stretching the spring 0.100 m, and release it from rest. The glider moves back toward its equilibrium position ($x = 0$). What is its x-velocity when $x = 0.080$ m?

IDENTIFY and SET UP There is no friction, so total mechanical energy is conserved. As the glider starts to move, elastic potential energy is converted to kinetic energy. The glider remains at the same height throughout the motion, so gravitational potential energy is not a factor and $U = U_{\text{el}} = \frac{1}{2}kx^2$. Figure 7.16 shows our sketches. Only the spring force does work on the glider, so $W_{\text{other}} = 0$ in Eq. (7.14). We designate the point where the glider is released as point 1 (that is, $_1 = 0.100$ m) and $x_2 = 0.080$ m as point 2. We are given $v_{1x} = 0$; our target variable is v_{2x}.

Figure 7.16



Our sketches and energy bar graphs for this problem.

EXECUTE The energy quantities are

$$\begin{aligned} K_1 &= \frac{1}{2}m{v_{1x}}^2 = \frac{1}{2}(0.200\text{ kg})(0)^2 = 0 \\ U_1 &= \frac{1}{2}\,kx_1^2 = \frac{1}{2}(5.00\,\text{N/m})(0.100\,\text{m})^2 = 0.0250\text{ J} \\ K_2 &= \frac{1}{2}\,m{_{2x}}^2 \\ U_2 &= \frac{1}{2}\,kx_2^2 = \frac{1}{2}(5.00\,\text{N/m})(0.080\,\text{m})^2 = 0.0160\text{ J} \end{aligned}$$

We use Eq. (7.14) with $W_{\text{other}} = 0$ to solve for K_2 and then find v_{2x}:

$$K_2 = K_1 + U_1 - U_2 = 0 + 0.0250\text{ J} - 0.0160\text{ J} = 0.0090\text{ J}$$

$$v_{2x} = \pm\sqrt{\frac{2K_2}{m}} = \pm\sqrt{\frac{2(0.0090\text{ J})}{0.200\text{ kg}}} = \pm 0.30\text{ m}/s$$

We choose the negative root because the glider is moving in the $-x-$ direction. Our answer is $v_{2x} = -0.30\text{ m/s}$.

EVALUATE Eventually the spring will reverse the glider's motion, pushing it back in the $+x$-direction (see Fig. 7.13d). The solution $v_{2x} = +0.30\text{ m}/s$ tells us that when the glider passes through $x = 0.080\text{ m}$ on this return trip, its speed will be $0.30\text{ m}/s$, just as when it passed through this point while moving to the left.

KEY CONCEPT

You can use elastic potential energy to describe the work done by an ideal spring that obeys Hooke's law.

Video Tutor Solution: **Example 7.7**

Example 7.8 Motion with elastic potential energy and work done by other forces

WITH VARIATION PROBLEMS

Suppose the glider in Example 7.7 is initially at rest at $x = 0$, with the spring unstretched. You then push on the glider with a constant force $\vec{F}$ (magnitude 0.610 N) in the $+x$-direction. What is the glider's velocity when it has moved to $x = 0.100 \text{ m}$?

IDENTIFY and SET UP Although the force $\vec{F}$ you apply is constant, the spring force isn't, so the acceleration of the glider won't be constant. Total mechanical energy is not conserved because of the work done by force $\vec{F}$, so W_{other} in Eq. (7.14) is not zero. As in Example 7.7, we ignore gravitational potential energy because the glider's height doesn't change. Hence we again have $U = U_{\text{el}} = \frac{1}{2}kx^2$. This time, we let point 1 be at $x_1 = 0$, where the velocity is $v_{1x} = 0$, and let point 2 be at $x = 0.100$ m. The glider's displacement is then $\Delta x = x_2 - x_1 = 0.100$ m. Our target variable is v_{2x}, the velocity at point 2.

EXECUTE Force $\vec{F}$ is constant and in the same direction as the displacement, so the work done by this force is $F\Delta x$. Then the energy quantities are

$$\begin{aligned} K_1 &= 0 \\ U_1 &= \frac{1}{2}kx_1^2 = 0 \\ K_2 &= \frac{1}{2}m{v_{2x}}^2 \\ U_2 &= \frac{1}{2}\,kx_2^2 = \frac{1}{2}\,(5.00\text{ N/m})(0.100\text{ m})^2 = 0.0250\text{ J} \\ W_{\text{other}} &= F\Delta x = (0.610\text{ N})(0.100\text{ m}) = 0.0610\text{ J} \end{aligned}$$

The initial total mechanical energy is zero; the work done by $\vec{F}$ increases the total mechanical energy to 0.0610 J, of which $U_2 = 0.0250$ J is elastic potential energy. The remainder is kinetic energy. From Eq. (7.14),

$$\begin{aligned} K_1 + U_1 + W_{\text{other}} &= K_2 + U_2 \\ K_2 &= K_1 + U_1 + W_{\text{other}} - U_2 \\ &= 0 + 0 + 0.0610\text{ J} - 0.0250\text{ J} = 0.0360\text{ J} \\ v_{2x} &= \sqrt{\frac{2K_2}{m}} = \sqrt{\frac{2(0.0360\text{ J})}{0.200\text{ kg}}} = 0.60\text{ m}/s \end{aligned}$$

We choose the positive square root because the glider is moving in the $+x$-direction.

EVALUATE What would be different if we disconnected the glider from the spring? Then only $\vec{F}$ would do work, there would be zero elastic potential energy at all times, and Eq. (7.14) would give us

$$\begin{aligned} K_2 &= K_1 + W_{\text{other}} = 0 + 0.0610\text{ J} \\ v_{2x} &= \sqrt{\frac{2K_2}{m}} = \sqrt{\frac{2(0.0610\text{ J})}{0.200\text{ kg}}} = 0.78\text{ m}/s \end{aligned}$$

Our answer $v_{2x} = 0.60 \text{ m}/s$ is less than $0.78 \text{ m}/s$ because the spring does negative work on the glider as it stretches (see Fig. 7.13b).

If you stop pushing on the glider when it reaches $x = 0.100 \text{ m}$, only the spring force does work on it thereafter. Hence for $x > 0.100 \text{ m}$, the total mechanical energy $E = K + U = 0.0610 \text{ J}$ is constant. As the spring continues to stretch, the glider slows down and the kinetic energy K decreases as the potential energy increases. The glider comes to rest at some point $x = x_3$, at which the kinetic energy is zero and the potential energy $U = U_{\text{el}} = \frac{1}{2} kx_3^2$ equals the total mechanical energy 0.0610 J. Can you show that $x_3 = 0.156 \text{ m}$? (It moves an additional 0.056 m after you stop pushing.) If there is no friction, will the glider remain at rest?

KEY CONCEPT

You can solve problems that involve elastic potential energy by using the same steps as for problems that involve gravitational potential energy, even when work is done by other forces.

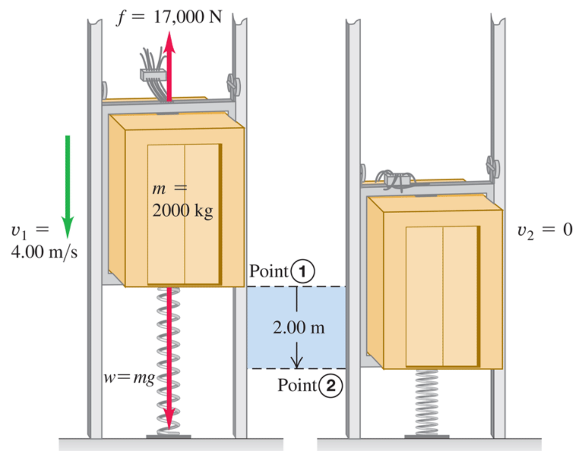
Video Tutor Solution: **Example 7.8**

Example 7.9 Motion with gravitational, elastic, and friction forces

WITH VARIATION PROBLEMS

A 2000 kg $(19,600\ N)$ elevator with broken cables in a test rig is falling at 4.00 m/s when it contacts a cushioning spring at the bottom of the shaft. The spring is intended to stop the elevator, compressing 2.00 m as it does so (Fig. 7.17). During the motion a safety clamp applies a constant 17,000 N friction force to the elevator. What is the necessary force constant k for the spring?

Figure 7.17



The fall of an elevator is stopped by a spring and by a constant friction force.

IDENTIFY and SET UP We'll use the energy approach and Eq. (7.14) to determine k, which appears in the expression for elastic potential energy. This problem involves *both* gravitational and elastic potential energies. Total mechanical energy is not conserved because the friction force does negative work W_{other} on the elevator. We take point 1 as the position of the bottom of the elevator when it contacts the spring, and point 2 as its position when it stops. We choose the origin to be at point 1, so $y_1 = 0$ and $y_2 = -2.00$ m. With this choice the coordinate of the upper end of the spring after contact is the same as the coordinate of the elevator, so the elastic potential energy at any point between points 1 and 2 is $U_{\text{el}} = \frac{1}{2}ky^2$. The gravitational potential energy is $U_{\text{grav}} = mgy$ as usual. We know the initial and final speeds of the elevator and the magnitude of the friction force, so the only unknown is the force constant k (our target variable).

EXECUTE The elevator's initial speed is $v_1 = 4.00\ \text{m/s}$, so its initial kinetic energy is

$$K_1 = \frac{1}{2}mv_1^2 = \frac{1}{2}(2000\ \text{kg})(4.00\ \text{m/s})^2 = 16{,}000\ \text{J}$$

The elevator stops at point 2, so $K_2 = 0$. At point 1 the potential energy $U_1 = U_{\text{grav}} + U_{\text{el}}$ is zero; U_{grav} is zero because $y_1 = 0$, and $U_{\text{el}} = 0$ because the spring is uncompressed. At point 2 there are both gravitational and elastic potential energies, so

$$U_2 = mgy_2 + \frac{1}{2}ky_2^2$$

The gravitational potential energy at point 2 is

$$mgy_2 = (2000\ \text{kg})(9.80\ \text{m/s}^2)(-2.00\ \text{m}) = -39{,}200\ \text{J}$$

The "other" force is the constant 17,000 N friction force. It acts opposite to the 2.00 m displacement, so

$$W_{\text{other}} = -(17{,}000\ \text{N})(2.00\ \text{m}) = -34{,}000\ \text{J}$$

We put these terms into Eq. (7.14), $K_1 + U_1 + W_{\text{other}} = K_2 + U_2$:

$$
\begin{aligned}
K_1 + 0 + W_{\text{other}} &= 0 + \left(mgy_2 + \frac{1}{2}\, ky_2^2 \right) \\
k &= \frac{2(K_1 + W_{\text{other}} - mgy_2)}{y_2^2} \\
&= \frac{2[16,000\ \text{J} + (-34,000\ \text{J}) - (-39,200\ \text{J})]}{(-2.00\ \text{m})^2} \\
&= 1.06 \times 10^4\ \text{N/m}
\end{aligned}
$$

This is about one-tenth the force constant of a spring in an automobile suspension.

EVALUATE There might seem to be a paradox here. The elastic potential energy at point 2 is

$$
\frac{1}{2}\, ky_2^2 = \frac{1}{2}\,(1.06 \times 10^4\ \text{N/m})(-2.00\ \text{m})^2 = 21,200\ \text{J}
$$

This is *more* than the total mechanical energy at point 1:

$$
E_1 = K_1 + U_1 = 16,000\ \text{J} + 0 = 16,000\ \text{J}
$$

But the friction force *decreased* the total mechanical energy of the system by 34,000 J between points 1 and 2. Did energy appear from nowhere? No. At point 2, which is below the origin, there is also *negative* gravitational potential energy $mgy_2 = -39,200$ J. The total mechanical energy at point 2 is therefore not 21,200 J but

$$
\begin{aligned}
E_2 &= K_2 + U_2 = 0 + \frac{1}{2}\, ky_2^2 + mgy_2 \\
&= 0 + 21,200\ \text{J} + (-39,200\ \text{J}) = -18,000\ \text{J}
\end{aligned}
$$

This is just the initial total mechanical energy of 16,000 J minus 34,000 J lost to friction.

Will the elevator stay at the bottom of the shaft? At point 2 the compressed spring exerts an upward force of magnitude $F_{\text{spring}} = (1.06 \times 10^4\ \text{N/m})(2.00\ \text{m}) = 21,200\ \text{N}$, while the downward force of gravity is only $w = mg = (2000\ \text{kg})(9.80\ \text{m/s}^2)\ = 19,600\ \text{N}$.

If there were no friction, there would be a net upward force of $21,200 \text{ N} - 19,600 \text{ N} = 1600 \text{ N}$, and the elevator would rebound. But the safety clamp can exert a kinetic friction force of 17,000 N, and it can presumably exert a maximum static friction force greater than that. Hence the clamp will keep the elevator from rebounding.

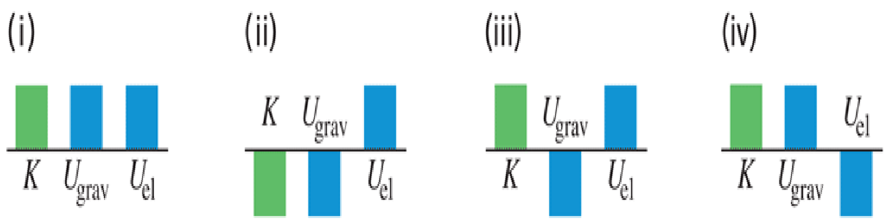
KEY CONCEPT

For problems in which you use an energy approach to analyze an object that both changes height and interacts with an ideal spring, you must include both gravitational potential energy and elastic potential energy.

Video Tutor Solution: **Example 7.9**

Test Your Understanding of Section 7.2

Consider the situation in Example 7.9 at the instant when the elevator is still moving downward and the spring is compressed by 1.00 m. Which of the energy bar graphs in the figure most accurately shows the kinetic energy K, gravitational potential energy U_{grav}, and elastic potential energy U_{el} at this instant?



7.3 Conservative and Nonconservative Forces

In our discussions of potential energy we have talked about "storing" kinetic energy by converting it to potential energy, with the idea that we can retrieve it again as kinetic energy. For example, when you throw a ball up in the air, it slows down as kinetic energy is converted to gravitational potential energy. But on the way down the ball speeds up as potential energy is converted back to kinetic energy. If there is no air resistance, the ball is moving just as fast when you catch it as when you threw it.

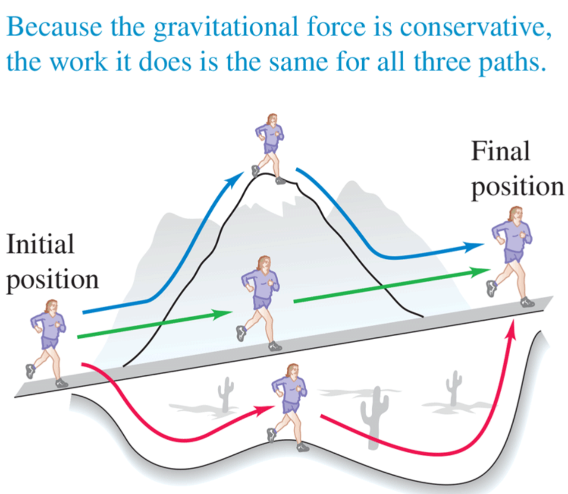
Another example is a glider moving on a frictionless horizontal air track that runs into a spring bumper. The glider compresses the spring and then bounces back. If there is no friction, the glider ends up with the same speed and kinetic energy it had before the collision. Again, there is a two-way conversion from kinetic to potential energy and back. In both cases the total mechanical energy, kinetic plus potential, is constant or *conserved* during the motion.

Conservative Forces

A force that offers this opportunity of two-way conversion between kinetic and potential energies is called a **conservative force.** We have seen two examples of conservative forces: the gravitational force and the spring force. (Later in this book we'll study another conservative force, the electric force between charged objects.) An essential feature of conservative forces is that their work is always *reversible.* Anything that we deposit in the energy "bank" can later be withdrawn without loss.

Another important aspect of conservative forces is that if an object follows different paths from point 1 to point 2, the work done by a conservative force is the same for all of these paths (Fig. 7.18). For example, if an object stays close to the surface of the earth, the gravitational force $m\vec{g}$ is independent of height, and the work done by this force depends on only the change in height. If the object moves around a closed path, ending at the same height where it started, the *total* work done by the gravitational force is always zero.

Figure 7.18



The work done by a conservative force such as gravity depends on only the endpoints of a path, not the specific path taken between those points.

In summary, the work done by a conservative force has four properties:

1. It can be expressed as the difference between the initial and final values of a *potential-energy* function.
2. It is reversible.
3. It is independent of the path of the object and depends on only the starting and ending points.

4. When the starting and ending points are the same, the total work is zero.

When the *only* forces that do work are conservative forces, the total mechanical energy $E = K + U$ is constant.

Nonconservative Forces

Not all forces are conservative. Consider the friction force acting on the crate sliding on a ramp in Example 7.6 (Section 7.1). When the crate slides up and then back down to the starting point, the total work done on it by the friction force is *not* zero. When the direction of motion reverses, so does the friction force, and friction does *negative* work in *both* directions. Friction also acts when a car with its brakes locked skids with decreasing speed (and decreasing kinetic energy). The lost kinetic energy can't be recovered by reversing the motion or in any other way, and total mechanical energy is *not* conserved. So there is *no* potential-energy function for the friction force.

In the same way, the force of fluid resistance (see Section 5.3) is not conservative. If you throw a ball up in the air, air resistance does negative work on the ball while it's rising *and* while it's descending. The ball returns to your hand with less speed and less kinetic energy than when it left, and there is no way to get back the lost mechanical energy.

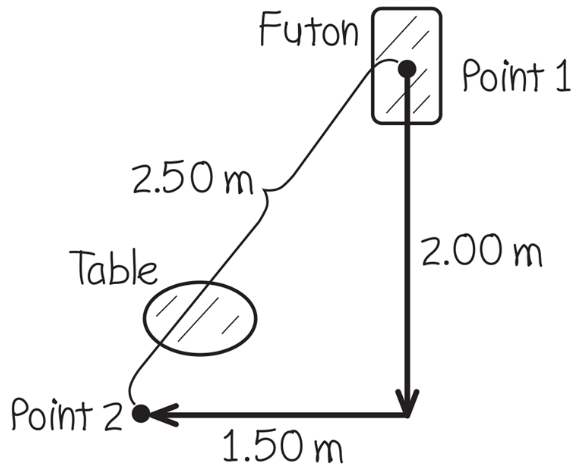
A force that is not conservative is called a **nonconservative force.** The work done by a nonconservative force *cannot* be represented by a potential-energy function. Some nonconservative forces, like kinetic friction or fluid resistance, cause mechanical energy to be lost or dissipated; a force of this kind is called a **dissipative force.** There are also nonconservative forces that *increase* mechanical energy. The fragments of an exploding firecracker fly off with very large kinetic energy, thanks to a chemical reaction of gunpowder with oxygen. The forces unleashed by this reaction are nonconservative because the process is not reversible. (The fragments never spontaneously reassemble themselves into a complete firecracker!)

Example 7.10 Frictional work depends on the path

You are rearranging your furniture and wish to move a 40.0 kg futon 2.50 m across the room. A heavy coffee table, which you don't want to move, blocks this straight-line path. Instead, you slide the futon along a dogleg path; the doglegs are 2.00 m and 1.50 m long. How much more work must you do to push the futon along the dogleg path than along the straight-line path? The coefficient of kinetic friction is $\mu_\mathrm{k} = 0.200$.

IDENTIFY and SET UP Here both you and friction do work on the futon, so we must use the energy relationship that includes "other" forces. We'll use this relationship to find a connection between the work that *you* do and the work that *friction* does. Figure 7.19 shows our sketch. The futon is at rest at both point 1 and point 2, so $K_1 = K_2 = 0$. There is no elastic potential energy (there are no springs), and the gravitational potential energy does not change because the futon moves only horizontally, so $U_1 = U_2$. From Eq. (7.14) it follows that $W_\mathrm{other} = 0$. That "other" work done on the futon is the sum of the positive work you do, W_you, and the negative work done by friction, W_fric. Since the sum of these is zero, we have

Figure 7.19



Our sketch for this problem.

$$W_{\text{you}} = -W_{\text{fric}}$$

So we can calculate the work done by friction to determine W_{you}.

EXECUTE The floor is horizontal, so the normal force on the futon equals its weight mg and the magnitude of the friction force is $f_{\text{k}} = \mu_k n = \mu_{\text{k}} mg$. The work you do over each path is then

$$\begin{aligned} W_{\text{you}} &= -W_{\text{fric}} = -(-f_{\text{k}} s) = +\mu_{\text{k}} mgs \\ &= (0.200)(40.0 \quad \text{kg})(9.80 \quad \text{m/s}^2)(2.50 \text{ m}) \\ &= 196 \text{ J} \qquad \text{(straight} - \text{line path)} \\ W_{\text{you}} &= -W_{\text{fric}} = +\mu_k mgs \\ &= (0.200)(40.0 \text{ kg})(9.80 \text{ m/s}^2)(2.00 \quad \text{m} + 1.50 \text{ m}) \\ &= 274 \text{ J} \qquad \text{(dogleg path)} \end{aligned}$$

The extra work you must do is $274 \text{ J} - 196 \text{ J} = 78 \text{ J}$.

EVALUATE Friction does different amounts of work on the futon, -196 J and -274 J, on these different paths between points 1 and 2. Hence friction is a *nonconservative* force.

KEY CONCEPT

The work done by a nonconservative force on an object that moves between two points depends on the path that the object follows.

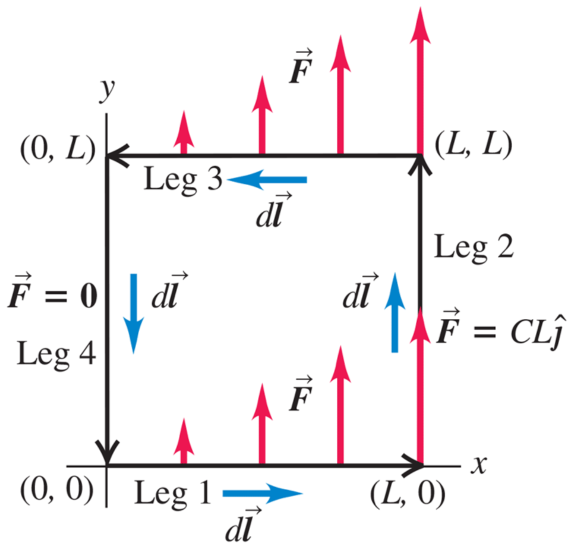
Unlike for conservative forces, you *cannot* express the work done by a nonconservative force in terms of a change in potential energy.

Video Tutor Solution: Example 7.10

Example 7.11 Conservative or nonconservative?

In a region of space the force on an electron is $\vec{F} = Cx\hat{j}$, where C is a positive constant. The electron moves around a square loop in the xy-plane (Fig. 7.20). Calculate the work done on the electron by force $\vec{F}$ during a counterclockwise trip around the square. Is this force conservative or nonconservative?

Figure 7.20



An electron moving around a square loop while being acted on by the force $\vec{F} = Cx\hat{j}$.

IDENTIFY and SET UP Force $\vec{F}$ is not constant and in general is not in the same direction as the displacement. To calculate the work done by $\vec{F}$, we'll use the general expression Eq. (6.14):

$$W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{l}$$

where $d\vec{l}$ is an infinitesimal displacement. We'll calculate the work done on each leg of the square separately, and add the results to find the work done on the round trip. If this round-trip work is zero, force $\vec{F}$ is conservative and can be represented by a potential-energy function.

EXECUTE On the first leg, from $(0,\ 0)$ to $(L,\ 0)$, the force is everywhere perpendicular to the displacement. So $\vec{F} \cdot d\vec{l} = 0$, and the work done on the first leg is $W_1 = 0$. The force has the same value $\vec{F} = CL\hat{j}$ everywhere on the second leg, from $(L,\ 0)$ to $(L,\ L)$. The displacement on this leg is in the $+y-$ direction, so $d\vec{l} = dy\hat{j}$ and

$$\vec{F} \cdot d\vec{l} = CL\hat{j} \cdot dy\hat{j} = CL\, dy$$

The work done on the second leg is then

$$W_2 = \int_{(L,\,0)}^{(L,\,L)} \vec{F} \cdot d\vec{l} = \int_{y=0}^{y=L} CL\, dy = CL \int_0^L dy = CL^2$$

On the third leg, from $(L,\ L)$ to $(0,\ L)$, $\vec{F}$ is again perpendicular to the displacement and so $W_3 = 0$. The force is zero on the final leg, from $(0,\ L)$ to $(0,\ 0)$, so $W_4 = 0$. The work done by $\vec{F}$ on the round trip is therefore

$$W = W_1 + W_2 + W_3 + W_4 = 0 + CL^2 + 0 + 0 = CL^2$$

The starting and ending points are the same, but the total work done by $\vec{F}$ is not zero. This is a *nonconservative* force; it *cannot* be represented by a potential-energy function.

EVALUATE Because $W > 0$, the total mechanical energy *increases* as the electron goes around the loop. This is actually what happens in an electric generating plant: A loop of wire is moved through a magnetic field, which gives rise to a nonconservative force similar to the one here. Electrons in the wire gain energy as they move around the loop, and this energy is carried via transmission lines to the consumer. (We'll discuss this in Chapter 29.)

If the electron went *clockwise* around the loop, $\vec{F}$ would be unaffected but the direction of each infinitesimal displacement $d\vec{l}$ would be reversed. Thus the sign of work would also reverse, and the work for a clockwise round trip would be $W = -CL^2$. This is a different behavior than the nonconservative friction force. The work done by friction on an object that slides in any direction over a stationary surface is always negative (see Example 7.6 in Section 7.1).

KEY CONCEPT

The work done on an object that makes a complete trip around a closed path is zero if the force is conservative, but nonzero if the force is nonconservative.

Video Tutor Solution: **Example 7.11**

The Law of Conservation of Energy

Nonconservative forces cannot be represented in terms of potential energy. But we can describe the effects of these forces in terms of kinds of energy other than kinetic or potential energy. When a car with locked brakes skids to a stop, both the tires and the road surface become hotter. The energy associated with this change in the state of the materials is called **internal energy.** Raising the temperature of an object increases its internal energy; lowering the object's temperature decreases its internal energy.

Application

Nonconservative Forces and Internal Energy in a Tire

An automobile tire deforms and flexes like a spring as it rolls, but it is not an ideal spring: Nonconservative internal friction forces act within the rubber. As a result, mechanical energy is lost and converted to internal energy of the tire. Thus the temperature of a tire increases as it rolls, which causes the pressure of the air inside the tire to increase as well. That's why tire pressures are best checked before the car is driven, when the tire is cold.

To see the significance of internal energy, let's consider a block sliding on a rough surface. Friction does *negative* work on the block as it slides, and the change in internal energy of the block and surface (both of which get hotter) is *positive*. Careful experiments show that the increase in the internal energy is *exactly* equal to the absolute value of the work done by friction. In other words,

$$\Delta U_{\text{int}} = -W_{\text{other}}$$

where ΔU_{int} is the change in internal energy. We substitute this into Eq. (7.14):

$$K_1 + U_1 - \Delta U_{\text{int}} = K_2 + U_2$$

Writing $\Delta K = K_2 - K_1$ and $\Delta U = U_2 - U_1$, we can finally express this as

(7.15)

Law of conservation of energy:

$$\underbrace{\Delta K}_{\text{Change in kinetic energy}} + \underbrace{\Delta U}_{\text{Change in potential energy}} + \underbrace{\Delta U_{\text{int}}}_{\text{Change in internal energy}} = 0 \qquad (7.15)$$

This remarkable statement is the general form of the **law of conservation of energy.** In a given process, the kinetic energy, potential energy, and internal energy of a system may all change. But the *sum* of those changes is always zero. If there is a decrease in one form of energy, it is made up for by an increase in the other forms (Fig. 7.21). When we expand our definition of energy to include internal energy, Eq. (7.15) says: *Energy is never created or destroyed; it only changes form.* No exception to this rule has ever been found.

Figure 7.21

The battery pack in this radio- controlled helicopter contains 2.4×10^4 J of electric energy. When this energy is used up, the internal energy of the battery pack decreases by this amount, so $\Delta U_{\text{int}} = -2.4 \times 10^4$ J. This energy can be converted to kinetic energy to make the rotor blades and helicopter go faster, or to gravitational potential energy to make the helicopter climb.

? The concept of work has been banished from Eq. (7.15); instead, it suggests that we think purely in terms of the conversion of energy from one form to another. For example, when you throw a baseball straight up, you convert a portion of the internal energy of your molecules to kinetic energy of the baseball. This is converted to gravitational potential energy as the ball climbs and back to kinetic energy as the ball falls. If there is air resistance, part of the energy is used to heat up the air and the ball and increase their internal energy. Energy is converted back to the kinetic form as the ball falls. If you catch the ball in your hand, whatever energy was not lost to the air once again becomes internal energy; the ball and your hand are now warmer than they were at the beginning.

In Chapters 19 and 20, we'll study the relationship of internal energy to temperature changes, heat, and work. This is the heart of the area of physics called *thermodynamics*.

Conceptual Example 7.12 Work done by friction

Let's return to Example 7.5 (Section 7.1), in which Throcky skateboards down a curved ramp. He starts with zero kinetic energy and 735 J of potential energy, and at the bottom he has 450 J of kinetic energy and zero potential energy; hence $\Delta K = +450$ J and $\Delta U = -735$ J. The work $W_{\text{other}} = W_{\text{fric}}$ done by the friction forces is -285 J, so the change in internal energy is $\Delta U_{\text{int}} = -W_{\text{other}} = +285$ J. The skateboard wheels and bearings and the ramp all get a little warmer. In accordance with Eq. (7.15), the sum of the energy changes equals zero:

$$\Delta K + \Delta U + \Delta U_{\text{int}} = +450 \text{ J} + (-735 \text{ J}) + 285 \text{ J} = 0$$

The total energy of the system (including internal, nonmechanical forms of energy) is conserved.

KEY CONCEPT

In any physical process, energy is never created or destroyed; it is merely converted among the forms of kinetic energy, potential energy, and internal energy.

Video Tutor Solution: **Example 7.12**

Test Your Understanding of Section 7.3

In a hydroelectric generating station, falling water is used to drive turbines ("water wheels"), which in turn run electric generators. Compared to the amount of gravitational potential energy released by the falling water, how much electrical energy is produced? (i) The same; (ii) more; (iii) less.

7.4 Force and Potential Energy

For the two kinds of conservative forces (gravitational and elastic) we have studied, we started with a description of the behavior of the *force* and derived from that an expression for the *potential energy*. For example, for an object with mass m in a uniform gravitational field, the gravitational force is $F_y = -mg$. We found that the corresponding potential energy is $U(y) = mgy$. The force that an ideal spring exerts on an object is $F_x = -kx$, and the corresponding potential-energy function is $U(x) = \frac{1}{2} kx^2$.

In studying physics, however, you'll encounter situations in which you are given an expression for the *potential energy* as a function of position and have to find the corresponding *force*. We'll see several examples of this kind when we study electric forces later in this book: It's often far easier to calculate the electric potential energy first and then determine the corresponding electric force afterward.

Here's how we find the force that corresponds to a given potential-energy expression. First let's consider motion along a straight line, with coordinate x. We denote the x-component of force, a function of x, by $F_x(x)$ and the potential energy as $U(x)$. This notation reminds us that both F_x and U are *functions* of x. Now we recall that in any displacement, the work W done by a conservative force equals the negative of the change ΔU in potential energy:

$$W = -\Delta U$$

Let's apply this to a small displacement Δx. The work done by the force $F_x(x)$ during this displacement is approximately equal to $F_x(x)\,\Delta x$. We have to say “approximately” because $F_x(x)$ may vary a little over the interval Δx. So

$$F_x(x)\,\Delta x = -\Delta U \qquad \text{and} \qquad F_x(x) = -\frac{\Delta U}{\Delta x}$$

You can probably see what’s coming. We take the limit as $\Delta x \to 0$; in this limit, the variation of F_x becomes negligible, and we have the exact relationship

(7.16)

Force from potential energy: In **one-dimensional motion,** the value of a conservative force at point x ...

$$F_x(x) = -\frac{dU(x)}{dx} \tag{7.16}$$

... is the negative of the derivative at x of the associated potential-energy function.

This result makes sense; in regions where $U(x)$ changes most rapidly with x (that is, where $dU(x)/dx$ is large), the greatest amount of work is done during a given displacement, and this corresponds to a large force magnitude. Also, when $F_x(x)$ is in the positive x-direction, $U(x)$ *decreases* with increasing x. So $F_x(x)$ and $dU(x)/dx$ should indeed have opposite signs. The physical meaning of Eq. (7.16) is that *a conservative force always acts to push the system toward lower potential energy.*

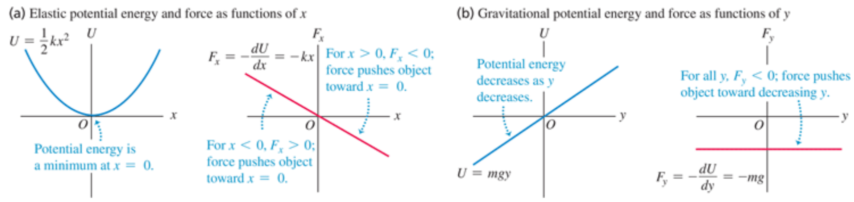
As a check, let’s consider the function for elastic potential energy, $U(x) = \frac{1}{2}kx^2$. Substituting this into Eq. (7.16) yields

$$F_x(x) = -\frac{d}{dx}\left(\frac{1}{2}kx^2\right) = -kx$$

which is the correct expression for the force exerted by an ideal spring (Fig. 7.22a). Similarly, for gravitational potential energy we have

$U(y) = mgy$; taking care to change x to y for the choice of axis, we get $F_y = -dU/dy = -d(mgy)/dy = -mg$, which is the correct expression for gravitational force (Fig. 7.22b).

Figure 7.22



A conservative force is the negative derivative of the corresponding potential energy.

Example 7.13 An electric force and its potential energy

An electrically charged particle is held at rest at the point $x = 0$; a second particle with equal charge is free to move along the positive x-axis. The potential energy of the system is $U(x) = C/x$, where C is a positive constant that depends on the magnitude of the charges. Derive an expression for the x-component of force acting on the movable particle as a function of its position.

IDENTIFY and SET UP We are given the potential-energy function $U(x)$. We'll find the corresponding force function by using Eq. (7.16), $F_x(x) = -dU(x)/dx$.

EXECUTE The derivative of $1/x$ with respect to x is $-1/x^2$. So for $x > 0$ the force on the movable charged particle is

$$F_x(x) = -\frac{dU(x)}{dx} = -C\left(-\frac{1}{x^2}\right) = \frac{C}{x^2}$$

EVALUATE The x-component of force is positive, corresponding to a repulsion between like electric charges. Both the potential energy and the force are very large when the particles are close together (small x), and both get smaller as the particles move farther apart (large x). The force pushes the movable particle toward large positive values of x, where the potential energy is lower. (We'll study electric forces in detail in Chapter 21.)

KEY CONCEPT

For motion in one dimension, the force associated with a potential-energy function equals the negative derivative of that function with respect to position.

Video Tutor Solution: Example 7.13

Force and Potential Energy in Three Dimensions

We can extend this analysis to three dimensions for a particle that may move in the x-, y-, or z-direction, or all at once, under the action of a conservative force that has components F_x, F_y, and F_z. Each component of force may be a function of the coordinates x, y, and z. The potential-energy function U is also a function of all three space coordinates. The potential-energy change ΔU when the particle moves a small distance Δx in the x-direction is again given by $-F_x \, \Delta x$; it doesn't depend on F_y and F_z, which represent force components that are perpendicular to the displacement and do no work. So we again have the approximate relationship

$$F_x = -\frac{\Delta U}{\Delta x}$$

We determine the y- and z-components in exactly the same way:

$$F_y = -\frac{\Delta U}{\Delta y} \qquad F_z = -\frac{\Delta U}{\Delta z}$$

To make these relationships exact, we take the limits $\Delta x \to 0$, $\Delta y \to 0$, and $\Delta z \to 0$ so that these ratios become derivatives. Because U may be a function of all three coordinates, we need to remember that when we calculate each of these derivatives, only one coordinate changes at a time. We compute the derivative of U with respect to x by assuming that y and z are constant and only x varies, and so on. Such a derivative is called a *partial derivative.* The usual notation for a partial derivative is $\partial U / \partial x$ and so on; the symbol ∂ is a modified d. So we write

(7.17)

Force from potential energy: In three-dimensional motion, the value at a given point of each component of a conservative force ...

$$F_x = -\frac{\partial U}{\partial x} \qquad F_y = -\frac{\partial U}{\partial y} \qquad F_z = -\frac{\partial U}{\partial z} \tag{7.17}$$

... is the negative of the partial derivative at that point of the associated potential-energy function.

We can use unit vectors to write a single compact vector expression for the force $\vec{F}$:

(7.18)

Force from potential energy: The vector value of a conservative force at a given point ...

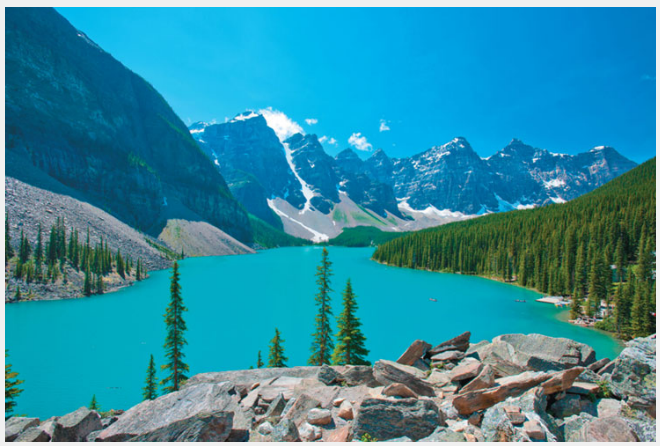
$$\vec{F} = -\left(\frac{\partial U}{\partial x}\hat{\imath} + \frac{\partial U}{\partial y}\hat{\jmath} + \frac{\partial U}{\partial z}\hat{k}\right) = -\vec{\nabla} U \tag{7.18}$$

... is the negative of the gradient at that point of the associated potential-energy function.

Application

Topography and Potential Energy Gradient

The greater the elevation of a hiker in Canada's Banff National Park, the greater the gravitational potential energy U_{grav}. Think of an x-axis that runs horizontally from west to east and a y-axis that runs horizontally from south to north. Then the function $U_{\text{grav}}\,(x, y)$ tells us the elevation as a function of position in the park. Where the mountains have steep slopes, $\vec{F} = -\vec{\nabla}U_{\text{grav}}$ has a large magnitude and there's a strong force pushing you along the mountain's surface toward a region of lower elevation (and hence lower U_{grav}). There's zero force along the surface of the lake, which is all at the same elevation. Hence U_{grav} is constant at all points on the lake surface, and $\vec{F} = -\vec{\nabla}\,U_{\text{grav}} = 0$.



In Eq. (7.18) we take the partial derivative of U with respect to each coordinate, multiply by the corresponding unit vector, and then take the vector sum. This operation is called the **gradient** of U and is often abbreviated as $\vec{\nabla}\,U$.

As a check, let's substitute into Eq. (7.18) the function $U = mgy$ for gravitational potential energy:

$$\vec{\boldsymbol{F}} = -\vec{\nabla}(mgy) = -(\frac{\partial(mgy)}{\partial x}\,\hat{\boldsymbol{i}} + \frac{\partial(mgy)}{\partial y}\,\hat{\boldsymbol{j}} + \frac{\partial(mgy)}{\partial z}\,\widehat{\boldsymbol{k}}\Big) = (-mg)\hat{\boldsymbol{j}}$$

This is just the familiar expression for the gravitational force.

Example 7.14 Force and potential energy in two dimensions

A puck with coordinates x and y slides on a level, frictionless air-hockey table. It is acted on by a conservative force described by the potential-energy function

$$U(x,\ y) = \frac{1}{2}\,k\left(x^2 + y^2\right)$$

Note that $r = \sqrt{x^2 + y^2}$ is the distance on the table surface from the puck to the origin. Find a vector expression for the force acting on the puck, and find an expression for the magnitude of the force.

IDENTIFY and SET UP Starting with the function $U(x,\ y)$, we need to find the vector components and magnitude of the corresponding force $\vec{F}$. We'll use Eq. (7.18) to find the components. The function U doesn't depend on z, so the partial derivative of U with respect to z is $\partial U/\partial z = 0$ and the force has no z-component. We'll determine the magnitude F of the force by using $F = \sqrt{F_x^2 + F_y^2}$.

EXECUTE The x- and y-components of $\vec{F}$ are

$$F_x = -\frac{\partial U}{\partial x} = -kx \qquad F_y = -\frac{\partial U}{\partial y} = -ky$$

From Eq. (7.18), the vector expression for the force is

$$\vec{F} = (-kx)\hat{\imath} + (-ky)\hat{\jmath} = -k\left(x\hat{\imath} + y\hat{\jmath}\right)$$

The magnitude of the force is

$$F = \sqrt{(-kx)^2 + (-ky)^2} = k\sqrt{x^2 + y^2} = kr$$

EVALUATE Because $x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$ is just the position vector $\vec{\boldsymbol{r}}$ of the particle, we can rewrite our result as $\vec{\boldsymbol{F}} = -k\vec{\boldsymbol{r}}$. This represents a force that is opposite in direction to the particle's position vector—that is, a force directed toward the origin, $r = 0$. This is the force that would be exerted on the puck if it were attached to one end of a spring that obeys Hooke's law and has a negligibly small unstretched length compared to the other distances in the problem. (The other end is attached to the air-hockey table at $r = 0$.)

To check our result, note that $U = \frac{1}{2}kr^2$. We can find the force from this expression using Eq. (7.16) with x replaced by r:

$$F_r = -\frac{dU}{dr} = -\frac{d}{dr}\left(\frac{1}{2}kr^2\right) = -kr$$

As we found above, the force has magnitude kr; the minus sign indicates that the force is toward the origin (at $\mathrm{r} = 0$).

KEY CONCEPT

For motion in two or three dimensions, the force associated with a potential-energy function equals the negative gradient of that function.

Video Tutor Solution: **Example 7.14**

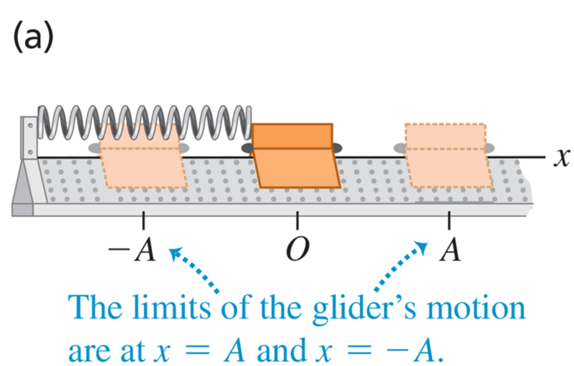
Test Your Understanding of Section 7.4

A particle moving along the x-axis is acted on by a conservative force F_x . At a certain point, the force is zero. (a) Which of the following statements about the value of the potential-energy function $U(x)$ at that point is correct? (i) $U(x) = 0$; (ii) $U(x) > 0$; (iii) $U(x) < 0$; (iv) not enough information is given to decide. (b) Which of the following statements about the value of the derivative of $U(x)$ at that point is correct? (i) $dU(x)/dx = 0$; (ii) $dU(x)/dx > 0$; (iii) $dU(x)/dx < 0$; (iv) not enough information is given to decide.

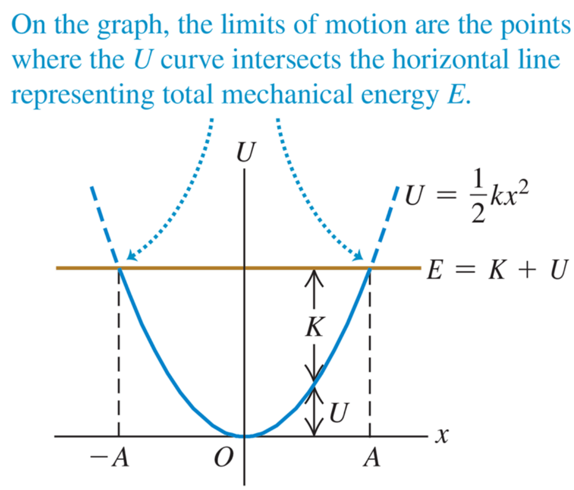
7.5 Energy Diagrams

When a particle moves along a straight line under the action of a conservative force, we can get a lot of insight into its possible motions by looking at the graph of the potential-energy function $U(x)$. Figure 7.23a shows a glider with mass m that moves along the x-axis on an air track. The spring exerts on the glider a force with x-component $F_x = -kx$. Figure 7.23b is a graph of the corresponding potential-energy function $U(x) = \frac{1}{2}kx^2$. If the elastic force of the spring is the *only* horizontal force acting on the glider, the total mechanical energy $E = K + U$ is constant, independent of x. A graph of E as a function of x is thus a straight horizontal line. We use the term **energy diagram** for a graph like this, which shows both the potential-energy function $U(x)$ and the energy of the particle subjected to the force that corresponds to $U(x)$.

Figure 7.23



(b)



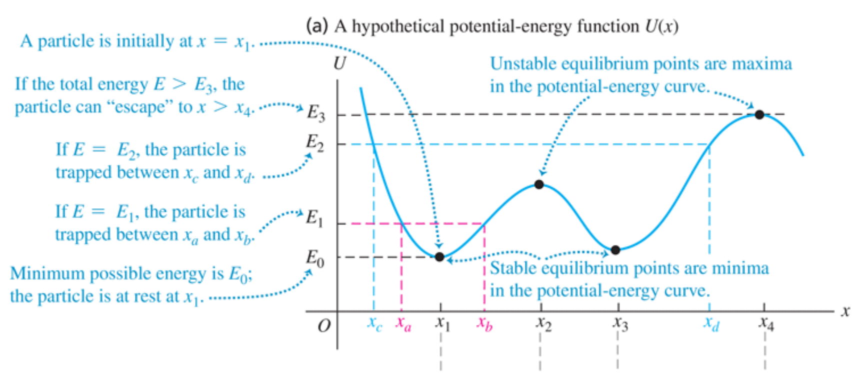
(a) A glider on an air track. The spring exerts a force $F_x = -kx$. (b) The potential-energy function.

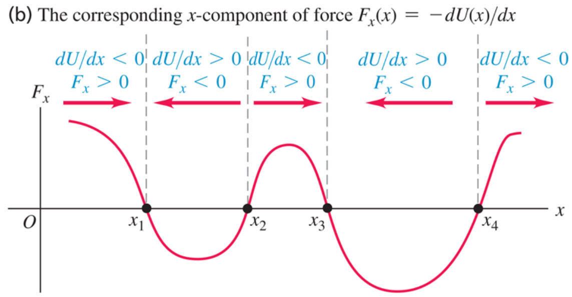
The vertical distance between the U and E graphs at each point represents the difference $E - U$, equal to the kinetic energy K at that point. We see that K is greatest at $x = 0$. It is zero at the values of x where the two graphs cross, labeled A and $-A$ in Fig. 7.23b. Thus the speed v is greatest at $x = 0$, and it is zero at $x = \pm A$, the points of *maximum* possible displacement from $x = 0$ for a given value of the total energy E. The potential energy U can never be greater than the total energy E; if it were, K would be negative, and that's impossible. The motion is a back-and-forth oscillation between the points $x = A$ and $x = -A$.

From Eq. (7.16), at each point the force F_x on the glider is equal to the negative of the slope of the $U(x)$ curve: $F_x = -dU/dx$ (see Fig. 7.22a). When the particle is at $x = 0$, the slope and the force are zero, so this is an *equilibrium* position. When x is positive, the slope of the $U(x)$ curve is positive and the force F_x is negative, directed toward the origin. When x is negative, the slope is negative and F_x is positive, again directed toward the origin. Such a force is called a *restoring force;* when the glider is displaced to either side of $x = 0$, the force tends to “restore” it back to $x = 0$. An analogous situation is a marble rolling around in a round-bottomed bowl. We say that $x = 0$ is a point of **stable equilibrium.** More generally, *any minimum in a potential-energy curve is a stable equilibrium position.*

Figure 7.24a shows a hypothetical but more general potential-energy function $U(x)$. Figure 7.24b shows the corresponding force $F_x = -dU/dx$. Points x_1 and x_3 are stable equilibrium points. At both points, F_x is zero because the slope of the $U(x)$ curve is zero. When the particle is displaced to either side, the force pushes back toward the equilibrium point. The slope of the $U(x)$ curve is also zero at points x_2 and x_4, and these are also equilibrium points. But when the particle is displaced a little to the right of either point, the slope of the $U(x)$ curve becomes negative, corresponding to a positive F_x that tends to push the particle still farther from the point. When the particle is displaced a little to the left, F_x is negative, again pushing away from equilibrium. This is analogous to a marble rolling on the top of a bowling ball. Points x_2 and x_4 are called **unstable equilibrium** points; *any maximum in a potential-energy curve is an unstable equilibrium position.*

Figure 7.24





The maxima and minima of a potential-energy function $U(x)$ correspond to points where $F_x = 0$.

CAUTION Potential energy and the direction of a conservative force The direction of the force on an object is *not* determined by the sign of the potential energy U. Rather, it's the sign of $F_x = -dU/dx$ that matters. The physically significant quantity is the *difference* in the values of U between two points (Section 7.1), which is what the derivative $F_x = -dU/dx$ measures. You can always add a constant to the potential-energy function without changing the physics.

Application

Acrobats in Equilibrium

Each of these acrobats is in *unstable* equilibrium. The gravitational potential energy is lower no matter which way an acrobat tips, so if she begins to fall she will keep on falling. Staying balanced requires the acrobats' constant attention.

If the total energy is E_1 and the particle is initially near x_1, it can move only in the region between x_a and x_b determined by the intersection of the E_1 and U graphs (Fig. 7.24a). Again, U cannot be greater than E_1 because K can't be negative. We speak of the particle as moving in a *potential well*, and x_a and x_b are the *turning points* of the particle's motion (since at these points, the particle stops and reverses direction). If we increase the total energy to the level E_2, the particle can move over a wider range, from x_c to x_d. If the total energy is greater than E_3, the particle can "escape" and move to indefinitely large values of x. At the other extreme, E_0 represents the minimum total energy the system can have.

Test Your Understanding of Section 7.5

The curve in Fig. 7.24b has a maximum at a point between x_2 and x_3 . Which statement correctly describes the particle's acceleration (with magnitude a) at this point? (i) $a_x = 0$. (ii) The particle accelerates in the $+x$-direction, so $a_x > 0$; a is less than at any other point between x_2 and x_3. (iii) The particle accelerates in the $+x$-direction, so $a_x > 0$; a is greater than at any other point between x_2 and x_3. (iv) The particle accelerates in the $-x$-direction, so $a_x < 0$; a is less than at any other point between x_2 and x_3. (v) The particle accelerates in the $-x$-direction, so $a_x < 0$; a is greater than at any other point between x_2 and x_3.

Chapter 7 Summary

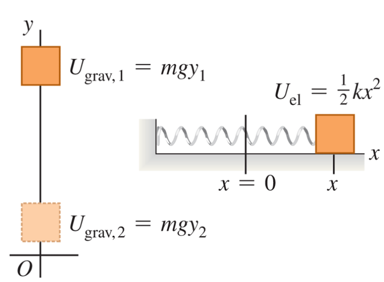
Gravitational potential energy and elastic potential energy: The work done on a particle by a constant gravitational force can be represented as a change in the gravitational potential energy, $U_{\text{grav}} = mgy$. This energy is a shared property of the particle and the earth. A potential energy is also associated with the elastic force $F_x = -kx$ exerted by an ideal spring, where x is the amount of stretch or compression. The work done by this force can be represented as a change in the elastic potential energy of the spring, $U_{\text{el}} = \frac{1}{2}kx^2$.

(7.2), (7.3)

$$\begin{aligned} W_{\text{grav}} &= mgy_1 - mgy_2 \\ &= U_{\text{grav, 1}} - U_{\text{grav, 2}} \\ &= -\Delta U_{\text{grav}} \end{aligned}$$

(7.10), (7.11)

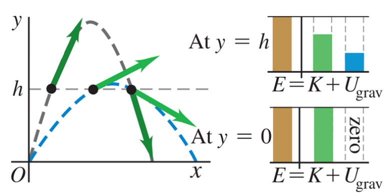
$$\begin{aligned} W_{\text{el}} &= \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2 \\ &= U_{\text{el, 1}} - U_{\text{el, 2}} = -\Delta U_{\text{el}} \end{aligned}$$



When total mechanical energy is conserved: The total potential energy U is the sum of the gravitational and elastic potential energies: $U = U_{\text{grav}} + U_{\text{el}}$. If no forces other than the gravitational and elastic forces do work on a particle, the sum of kinetic and potential energies is conserved. This sum $E = K + U$ is called the total mechanical energy. (See Examples 7.1, 7.3, 7.4, and 7.7.)

(7.4), (7.12)

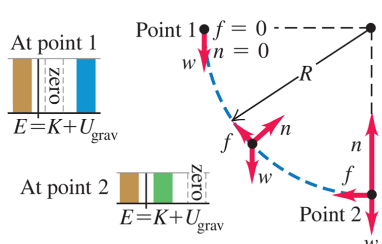
$$K_1 + U_1 = K_2 + U_2$$



When total mechanical energy is not conserved: When forces other than the gravitational and elastic forces do work on a particle, the work W_{other} done by these other forces equals the change in total mechanical energy (kinetic energy plus total potential energy). (See Examples 7.2, 7.5, 7.6, 7.8, and 7.9.)

(7.14)

$$K_1 + U_1 + W_{\text{other}} = K_2 + U_2$$

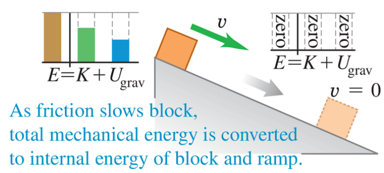


Conservative forces, nonconservative forces, and the law of conservation of energy: All forces are either conservative or nonconservative. A conservative force is one for which the work–

kinetic energy relationship is completely reversible. The work of a conservative force can always be represented by a potential-energy function, but the work of a nonconservative force cannot. The work done by nonconservative forces manifests itself as changes in the internal energy of objects. The sum of kinetic, potential, and internal energies is always conserved. (See Examples 7.10, 7.11 and 7.12.)

(7.15)

$$\Delta K + \Delta U + \Delta U_{\text{int}} = 0$$



Determining force from potential energy: For motion along a straight line, a conservative force $F_x\ (x)$ is the negative derivative of its associated potential-energy function U. In three dimensions, the components of a conservative force are negative partial derivatives of U. (See Examples 7.13 and 7.14.)

(7.16)

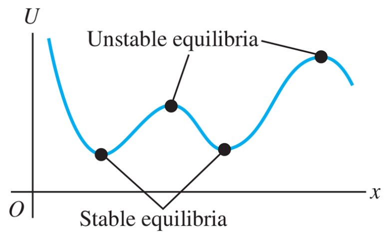
$$F_x\ (x) = -\frac{dU\ (x)}{dx}$$

(7.17)

$$F_x = -\frac{\partial U}{\partial x} \quad F_y = -\frac{\partial U}{\partial y} \quad F_z = -\frac{\partial U}{\partial z}$$

(7.18)

$$\begin{aligned}\vec{\boldsymbol{F}} &= -\left(\frac{\partial U}{\partial x}\hat{\boldsymbol{i}} + \frac{\partial U}{\partial y}\hat{\boldsymbol{j}} + \frac{\partial U}{\partial z}\widehat{\boldsymbol{k}}\right) \\ &= -\vec{\nabla} U\end{aligned}$$



Guided Practice: Potential Energy and Energy Conservation

For assigned homework and other learning materials, go to
Mastering Physics.

Key Example Variation Problems

Be sure to review Examples 7.1 and 7.2 (Section 7.1) before attempting these problems.

VP7.2.1 You throw a baseball (mass 0.145 kg) vertically upward. It leaves your hand moving at 12.0 m/s. Air resistance can be neglected. At what height above your hand does the ball have (a) half as much upward velocity, (b) half as much kinetic energy as when it left your hand?

VP7.2.2 You toss a rock of mass m vertically upward. Air resistance can be neglected. The rock reaches a maximum height h above your hand. What is the speed of the rock when it is at height (a) $h/4$ and (b) $3h/4$?

VP7.2.3 You throw a tennis ball (mass 0.0570 kg) vertically upward. It leaves your hand moving at 15.0 m/s. Air resistance cannot be neglected, and the ball reaches a maximum height of 8.00 m. (a) By how much does the total mechanical energy decrease from when the ball leaves your hand to when it reaches its maximum height? (b) What is the magnitude of the average force of air resistance?

VP7.2.4 You catch a volleyball (mass 0.270 kg) that is moving downward at 7.50 m/s. In stopping the ball, your hands and the volleyball descend together a distance of 0.150 m. (a) How much work do your hands do on the volleyball in the process of stopping it? (b) What is the magnitude of the force (assumed constant) that your hands exert on the volleyball?

Be sure to review Examples 7.4 and 7.5 (Section 7.1) before attempting these problems.

VP7.5.1 A well-greased, essentially frictionless, metal bowl has the shape of a hemisphere of radius 0.150 m. You place a pat of butter of mass 5.00×10^{-3}kg at the rim of the bowl and let it slide to the bottom of the bowl. (a) What is the speed of the pat of butter when it reaches the bottom of the bowl? (b) At the bottom of the bowl, what is the force that the bowl exerts on the pat of butter? How does this compare to the weight of the pat?

VP7.5.2 A snowboarder and her board (combined mass 40.0 kg) are moving at 9.30 m/s at the bottom of a curved ditch. (a) If friction can be ignored, what is the maximum vertical distance that she can travel up the sides of the ditch? Does this answer depend on the shape of the ditch? (b) The snowboarder finds that, due to friction, the maximum vertical distance she can travel up the sides of the ramp is 3.50 m. How much work did the force of friction do on her?

VP7.5.3 A pendulum is made of a small sphere of mass 0.250 kg attached to a lightweight string 1.20 m in length. As the pendulum swings back and forth, the maximum angle that the string makes with the vertical is 34.0°. Friction can be ignored. At the low point of the sphere's trajectory, what are (a) the kinetic energy of the sphere and (b) the tension in the string?

VP7.5.4 You are testing a new roller coaster ride in which a car of mass m moves around a vertical circle of radius R. In one test, the car starts at the bottom of the circle (point A) with initial kinetic energy K_{i}. When the car reaches the top of the circle (point B), its kinetic energy is $\frac{1}{4}K_{\mathrm{i}}$, and its gravitational potential energy has increased by $\frac{1}{2}K_{\mathrm{i}}$. (a) What was the speed of the car at point A, in terms of g and R? (b) How much work was done on the car by the force of friction as it moved from point A to point B, in terms of m, g, and R? (c) What was the magnitude of the friction force (assumed to be constant throughout the motion), in terms of m and g?

Be sure to review Examples 7.7, 7.8, and 7.9 (Section 7.2) before attempting these problems.

VP7.9.1 A glider of mass 0.240 kg is on a frictionless, horizontal track, attached to a horizontal spring of force constant 6.00 N/m. Initially the spring (whose other end is fixed) is stretched by 0.100 m and the attached glider is moving at 0.400 m/s in the direction that causes the spring to stretch farther. (a) What is the total mechanical energy (kinetic energy plus elastic potential energy) of the system? (b) When the glider comes momentarily to rest, by what distance is the spring stretched?

VP7.9.2 A glider of mass 0.240 kg is on a horizontal track, attached to a horizontal spring of force constant 6.00 N/m. There is friction between the track and the glider. Initially the spring (whose other end is fixed) is stretched by 0.100 m and the attached glider is moving at 0.400 m/s in the direction that causes the spring to stretch farther. The glider comes momentarily to rest when the spring is stretched by 0.112 m. (a) How much work does the force of friction do on the glider as the stretch of the spring increases from 0.100 m to 0.112 m? (b) What is the coefficient of kinetic friction between the glider and the track?

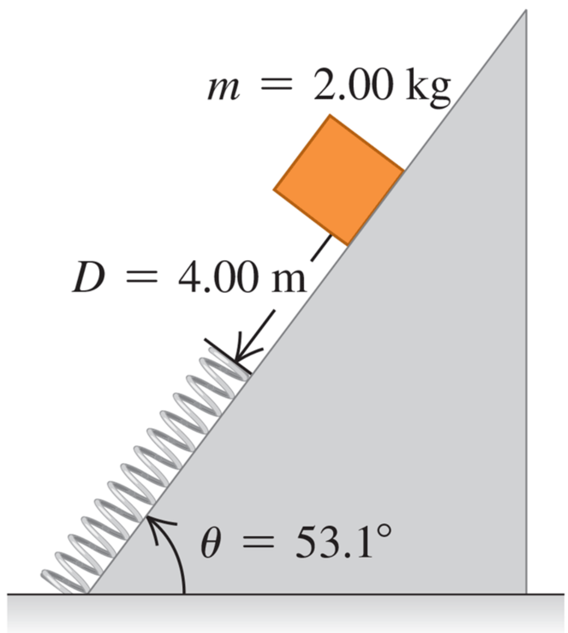
VP7.9.3 A lightweight vertical spring of force constant k has its lower end mounted on a table. You compress the spring by a distance d, place a block of mass m on top of the compressed spring, and then release the block. The spring launches the block upward, and the block rises to a maximum height some distance above the now-relaxed spring. (a) Find the speed of the block just as it loses contact with the spring. (b) Find the total vertical distance that the block travels from when it is first released to when it reaches its maximum height.

VP7.9.4 A cylinder of mass m is free to slide in a vertical tube. The kinetic friction force between the cylinder and the walls of the tube has magnitude f. You attach the upper end of a lightweight vertical spring of force constant k to the cap at the top of the tube, and attach the lower end of the spring to the top of the cylinder. Initially the cylinder is at rest and the spring is relaxed. You then release the cylinder. What vertical distance will the cylinder descend before it comes momentarily to rest?

Bridging Problem: A Spring and Friction on an Incline

A 2.00 kg package is released on a 53.1° incline, 4.00 m from a long spring with force constant $1.20 \times 10^2 \mathrm{N}/m$ that is attached at the bottom of the incline (Fig. 7.25). The coefficients of friction between the package and incline are $\mu_\mathrm{s} = 0.400$ and $\mu_\mathrm{k} = 0.200$. The mass of the spring is negligible. (a) What is the maximum compression of the spring? (b) The package rebounds up the incline. When it stops again, how close does it get to its original position? (c) What is the change in the internal energy of the package and incline from the point at which the package is released until it rebounds to its maximum height?

Figure 7.25



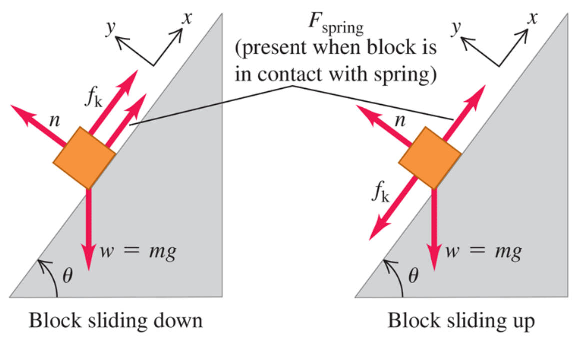
The initial situation.

Solution Guide

IDENTIFY and SET UP

1. This problem involves the gravitational force, a spring force, and the friction force, as well as the normal force that acts on the package. Since the spring force isn't constant, you'll have to use energy methods. Is total mechanical energy conserved during any part of the motion? Why or why not?

2. Draw free-body diagrams for the package as it is sliding down the incline and sliding back up the incline. Include your choice of coordinate axes (see below). (*Hint:* If you choose $x = 0$ to be at the end of the uncompressed spring, you'll be able to use $U_{\text{el}} = \frac{1}{2} kx^2$ for the elastic potential energy of the spring.)



3. Label the three critical points in the package's motion: its starting position, its position when it comes to rest with the spring maximally compressed, and its position when it has rebounded as far as possible up the incline. (*Hint:* You can assume that the package is no longer in contact with the spring at the last of these positions. If this turns out to be incorrect, you'll calculate a value of x that tells you the spring is still partially compressed at this point.)

4. List the unknown quantities and decide which of these are the target variables.

EXECUTE

5. Find the magnitude of the friction force that acts on the package. Does the magnitude of this force depend on whether the package is moving up or down the incline, or on whether the package is in contact with the spring? Does the *direction* of the friction force depend on any of these?

6. Write the general energy equation for the motion of the package between the first two points you labeled in step 3. Use this equation to solve for the distance that the spring is compressed when the package is at its lowest point. (*Hint:* You'll have to solve a quadratic equation. To decide which of the two solutions of this equation is the correct one, remember that the distance the spring is compressed is positive.)
7. Write the general energy equation for the motion of the package between the second and third points you labeled in step 3. Use this equation to solve for how far the package rebounds.
8. Calculate the change in internal energy for the package's trip down and back up the incline. Remember that the amount the internal energy *increases* is equal to the amount the total mechanical energy *decreases.*

EVALUATE

9. Was it correct to assume in part (b) that the package is no longer in contact with the spring when it reaches its maximum rebound height?
10. Check your result for part (c) by finding the total work done by the force of friction over the entire trip. Is this in accordance with your result from step 8?

Video Tutor Solution: Chapter 7 Bridging Problem

Questions/Exercises/Problems: Potential Energy and Energy Conservation

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

Discussion Questions

Q7.1 A baseball is thrown straight up with initial speed v_0. If air resistance cannot be ignored, when the ball returns to its initial height its speed is less than v_0. Explain why, using energy concepts.

Q7.2 A projectile has the same initial kinetic energy no matter what the angle of projection. Why doesn’t it rise to the same maximum height in each case?

Q7.3 An object is released from rest at the top of a ramp. If the ramp is frictionless, does the object’s speed at the bottom of the ramp depend on the shape of the ramp or just on its height? Explain. What if the ramp is *not* frictionless?

Q7.4 An egg is released from rest from the roof of a building and falls to the ground. Its fall is observed by a student on the roof of the building, who uses coordinates with origin at the roof, and by a student on the ground, who uses coordinates with origin at the ground. Do the values the two students assign to the following quantities match each other: initial gravitational potential energy, final gravitational potential energy, change in gravitational potential energy, and kinetic energy of the egg just before it strikes the ground? Explain.

Q7.5 A physics teacher had a bowling ball suspended from a very long rope attached to the high ceiling of a large lecture hall. To illustrate his faith in conservation of energy, he would back up to one side of the stage, pull the ball far to one side until the taut rope brought it just to the end of his nose, and then release it. The massive ball would swing in a mighty arc across the stage and then return to stop momentarily just in front of the nose of the stationary, unflinching teacher. However, one day after the demonstration he looked up just in time to see a student at the other side of the stage *push* the ball away from his nose as he tried to duplicate the demonstration. Tell the rest of the story, and explain the reason for the potentially tragic outcome.

Q7.6 Is it possible for a friction force to *increase* the total mechanical energy of a system? If so, give examples.

Q7.7 A woman bounces on a trampoline, going a little higher with each bounce. Explain how she increases the total mechanical energy.

Q7.8 **Fractured Physics.** People often call their electric bill a *power* bill, yet the quantity on which the bill is based is expressed in *kilowatt-hours.* What are people really being billed for?

Q7.9 (a) A book is lifted upward a vertical distance of 0.800 m. During this displacement, does the gravitational force acting on the book do positive work or negative work? Does the gravitational potential energy of the book increase or decrease? (b) A can of beans is released from rest and falls downward a vertical distance of 2.00 m. During this displacement, does the gravitational force acting on the can do positive work or negative work? Does the gravitational potential energy of the can increase or decrease?

Q7.10 (a) A block of wood is pushed against a spring, which is compressed 0.080 m. Does the force on the block exerted by the spring do positive or negative work? Does the potential energy stored in the spring increase or decrease? (b) A block of wood is placed against a vertical spring that is compressed 6.00 cm. The spring is released and pushes the block upward. From the point where the spring is compressed 6.00 cm to where it is compressed 2.00 cm from its equilibrium length and the block has moved 4.00 cm upward, does the spring force do positive or negative work on the block? During this motion, does the potential energy stored in the spring increase or decrease?

Q7.11 A 1.0 kg stone and a 10.0 kg stone are released from rest at the same height above the ground. Ignore air resistance. Which of these statements about the stones are true? Justify each answer. (a) Both have the same initial gravitational potential energy. (b) Both will have the same acceleration as they fall. (c) Both will have the same speed when they reach the ground. (d) Both will have the same kinetic energy when they reach the ground.

Q7.12 Two objects with different masses are launched vertically into the air by placing them on identical compressed springs and then releasing the springs. The two springs are compressed by the same amount before launching. Ignore air resistance and the masses of the springs. Which of these statements about the masses are true? Justify each answer. (a) Both reach the same maximum height. (b) At their maximum height, both have the same gravitational potential energy, if the initial gravitational potential of each mass is taken to be zero.

Q7.13 When people are cold, they often rub their hands together to warm up. How does doing this produce heat? Where does the heat come from?

Q7.14 A box slides down a ramp and work is done on the box by the forces of gravity and friction. Can the work of each of these forces be expressed in terms of the change in a potential-energy function? For each force explain why or why not.

Q7.15 In physical terms, explain why friction is a nonconservative force. Does it store energy for future use?

Q7.16 Since only changes in potential energy are important in any problem, a student decides to let the elastic potential energy of a spring be zero when the spring is stretched a distance x_1. The student decides, therefore, to let $U = \frac{1}{2}k(x - x_1)^2$. Is this correct? Explain.

Q7.17 Figure 7.22a shows the potential-energy function for the force $F_x = -kx$. Sketch the potential-energy function for the force $F_x = +kx$. For this force, is $x = 0$ a point of equilibrium? Is this equilibrium stable or unstable? Explain.

Q7.18 Figure 7.22b shows the potential-energy function associated with the gravitational force between an object and the earth. Use this graph to explain why objects always fall toward the earth when they are released.

Q7.19 For a system of two particles we often let the potential energy for the force between the particles approach zero as the separation of the particles approaches infinity. If this choice is made, explain why the potential energy at noninfinite separation is positive if the particles repel one another and negative if they attract.

Q7.20 Explain why the points $x = A$ and $x = -A$ in Fig. 7.23b are called *turning points*. How are the values of E and U related at a turning point?

Q7.21 A particle is in *neutral equilibrium* if the net force on it is zero and remains zero if the particle is displaced slightly in any direction. Sketch the potential-energy function near a point of neutral equilibrium for the case of one-dimensional motion. Give an example of an object in neutral equilibrium.

Q7.22 The net force on a particle of mass m has the potential-energy function graphed in Fig. 7.24a. If the total energy is E_1, graph the speed v of the particle versus its position x. At what value of x is the speed greatest? Sketch v versus x if the total energy is E_2.

Q7.23 The potential-energy function for a force $\vec{F}$ is $U = \alpha x^3$, where α is a positive constant. What is the direction of $\vec{F}$?

Exercises

Section 7.1 Gravitational Potential Energy

7.1 • In one day, a 75 kg mountain climber ascends from the 1500 m level on a vertical cliff to the top at 2400 m. The next day, she descends from the top to the base of the cliff, which is at an elevation of 1350 m. What is her change in gravitational potential energy (a) on the first day and (b) on the second day?

7.2 • BIO **How High Can We Jump?** The maximum height a typical human can jump from a crouched start is about 60 cm. By how much does the gravitational potential energy increase for a 72 kg person in such a jump? Where does this energy come from?

7.3 •• CP A 90.0 kg mail bag hangs by a vertical rope 3.5 m long. A postal worker then displaces the bag to a position 2.0 m sideways from its original position, always keeping the rope taut. (a) What horizontal force is necessary to hold the bag in the new position? (b) As the bag is moved to this position, how much work is done (i) by the rope and (ii) by the worker?

7.4 •• BIO **Food Calories.** The *food calorie*, equal to 4186 J, is a measure of how much energy is released when the body metabolizes food. A certain fruit-and-cereal bar contains 140 food calories. (a) If a 65 kg hiker eats one bar, how high a mountain must he climb to “work off” the calories, assuming that all the food energy goes into increasing gravitational potential energy? (b) If, as is typical, only 20% of the food calories go into mechanical energy, what would be the answer to part (a)? (*Note:* In this and all other problems, we are assuming that 100% of the food calories that are eaten are absorbed and used by the body. This is not true. A person’s “metabolic efficiency” is the percentage of calories eaten that are actually used; the body eliminates the rest. Metabolic efficiency varies considerably from person to person.)

7.5 • A baseball is thrown from the roof of a 22.0-m-tall building with an initial velocity of magnitude 12.0 m/s and directed at an angle of 53.1° above the horizontal. (a) What is the speed of the ball just before it strikes the ground? Use energy methods and ignore air resistance. (b) What is the answer for part (a) if the initial velocity is at an angle of 53.1° *below* the horizontal? (c) If the effects of air resistance are included, will part (a) or (b) give the higher speed?

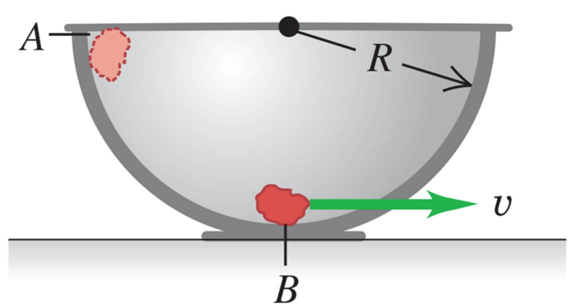
7.6 •• A crate of mass M starts from rest at the top of a frictionless ramp inclined at an angle α above the horizontal. Find its speed at the bottom of the ramp, a distance d from where it started. Do this in two ways: Take the level at which the potential energy is zero to be (a) at the bottom of the ramp with y positive upward, and (b) at the top of the ramp with y positive upward. (c) Why didn't the normal force enter into your solution?

7.7 •• BIO **Human Energy vs. Insect Energy.** For its size, the common flea is one of the most accomplished jumpers in the animal world. A 2.0-mm-long, 0.50 mg flea can reach a height of 20 cm in a single leap. (a) Ignoring air drag, what is the takeoff speed of such a flea? (b) Calculate the kinetic energy of this flea at takeoff and its kinetic energy per kilogram of mass. (c) If a 65 kg, 2.0-m-tall human could jump to the same height compared with his length as the flea jumps compared with its length, how high could the human jump, and what takeoff speed would the man need? (d) Most humans can jump no more than 60 cm from a crouched start. What is the kinetic energy per kilogram of mass at takeoff for such a 65 kg person? (e) Where does the flea store the energy that allows it to make sudden leaps?

7.8 • Estimate the maximum speed you can achieve while running a 100 m dash. Treat yourself as a point particle. (a) At this speed, what is your kinetic energy? (b) To what height above the ground would you have to climb in a tree to increase your gravitational potential energy by an amount equal to the kinetic energy you calculated in part (a)?

7.9 •• CP A small rock with mass 0.20 kg is released from rest at point A, which is at the top edge of a large, hemispherical bowl with radius $R = 0.50$ m (Fig. E7.9). Assume that the size of the rock is small compared to R, so that the rock can be treated as a particle, and assume that the rock slides rather than rolls. The work done by friction on the rock when it moves from point A to point B at the bottom of the bowl has magnitude 0.22 J. (a) Between points A and B, how much work is done on the rock by (i) the normal force and (ii) gravity? (b) What is the speed of the rock as it reaches point B? (c) Of the three forces acting on the rock as it slides down the bowl, which (if any) are constant and which are not? Explain. (d) Just as the rock reaches point B, what is the normal force on it due to the bottom of the bowl?

Figure E7.9



7.10 •• A 25.0 kg child plays on a swing having support ropes that are 2.20 m long. Her brother pulls her back until the ropes are 42.0° from the vertical and releases her from rest. (a) What is her potential energy just as she is released, compared with the potential energy at the bottom of the swing's motion? (b) How fast will she be moving at the bottom? (c) How much work does the tension in the ropes do as she swings from the initial position to the bottom of the motion?

7.11 •• You are testing a new amusement park roller coaster with an empty car of mass 120 kg. One part of the track is a vertical loop with radius 12.0 m. At the bottom of the loop (point A) the car has speed 25.0 m$/s$, and at the top of the loop (point B) it has speed 8.0 m/s. As the car rolls from point A to point B, how much work is done by friction?

7.12 • **Tarzan and Jane.** Tarzan, in one tree, sights Jane in another tree. He grabs the end of a vine with length 20 m that makes an angle of 45° with the vertical, steps off his tree limb, and swings down and then up to Jane's open arms. When he arrives, his vine makes an angle of 30° with the vertical. Determine whether he gives her a tender embrace or knocks her off her limb by calculating Tarzan's speed just before he reaches Jane. Ignore air resistance and the mass of the vine.

7.13 •• Two blocks are attached to either end of a light rope that passes over a light, frictionless pulley suspended from the ceiling. One block has mass 8.00 kg, and the other has mass 6.00 kg. The blocks are released from rest. (a) For a 0.200 m downward displacement of the 8.00 kg block, what is the change in the gravitational potential energy associated with each block? (b) If the tension in the rope is T, how much work is done on each block by the rope? (c) Apply conservation of energy to the system that includes both blocks. During the 0.200 m downward displacement, what is the total work done on the system by the tension in the rope? What is the change in gravitational potential energy associated with the system? Use energy conservation to find the speed of the 8.00 kg block after it has descended 0.200 m.

Section 7.2 Elastic Potential Energy

7.14 •• An ideal spring of negligible mass is 12.00 cm long when nothing is attached to it. When you hang a 3.15 kg weight from it, you measure its length to be 13.40 cm. If you wanted to store 10.0 J of potential energy in this spring, what would be its *total* length? Assume that it continues to obey Hooke's law.

7.15 •• A force of 520 N keeps a certain ideal spring stretched a distance of 0.200 m. (a) What is the potential energy of the spring when it is stretched 0.200 m? (b) What is its potential energy when it is compressed 5.00 cm?

7.16 • **BIO** **Tendons.** Tendons are strong elastic fibers that attach muscles to bones. To a reasonable approximation, they obey Hooke's law. In laboratory tests on a particular tendon, it was found that, when a 250 g object was hung from it, the tendon stretched 1.23 cm. (a) Find the force constant of this tendon in N/m. (b) Because of its thickness, the maximum tension this tendon can support without rupturing is 138 N. By how much can the tendon stretch without rupturing, and how much energy is stored in it at that point?

7.17 • An ideal spring stores potential energy U_0 when it is compressed a distance x_0 from its uncompressed length. (a) In terms of U_0, how much energy does the spring store when it is compressed (i) twice as much and (ii) half as much? (b) In terms of x_0, how much must the spring be compressed from its uncompressed length to store (i) twice as much energy and (ii) half as much energy?

7.18 • A small block of mass m on a horizontal frictionless surface is attached to a horizontal spring that has force constant k. The block is pushed against the spring, compressing the spring a distance d. The block is released, and it moves back and forth on the end of the spring. During its motion, what is the maximum speed of the block?

7.19 •• A spring of negligible mass has force constant $k = 800\ \mathrm{N/m}$. (a) How far must the spring be compressed for 1.20 J of potential energy to be stored in it? (b) You place the spring vertically with one end on the floor. You then lay a 1.60 kg book on top of the spring and release the book from rest. Find the maximum distance the spring will be compressed.

7.20 • A 1.20 kg piece of cheese is placed on a vertical spring of negligible mass and force constant $k = 1800\ \mathrm{N}/m$ that is compressed 15.0 cm. When the spring is released, how high does the cheese rise from this initial position? (The cheese and the spring are *not* attached.)

7.21 •• A spring of negligible mass has force constant $k = 1600\ \mathrm{N/m}$. (a) How far must the spring be compressed for 3.20 J of potential energy to be stored in it? (b) You place the spring vertically with one end on the floor. You then drop a 1.20 kg book onto it from a height of 0.800 m above the top of the spring. Find the maximum distance the spring will be compressed.

7.22 •• (a) For the elevator of Example 7.9 (Section 7.2), what is the speed of the elevator after it has moved downward 1.00 m from point 1 in Fig. 7.17? (b) When the elevator is 1.00 m below point 1 in Fig. 7.17, what is its acceleration?

7.23 •• A 2.50 kg mass is pushed against a horizontal spring of force constant 25.0 N/cm on a frictionless air table. The spring is attached to the tabletop, and the mass is not attached to the spring in any way. When the spring has been compressed enough to store 11.5 J of potential energy in it, the mass is suddenly released from rest. (a) Find the greatest speed the mass reaches. When does this occur? (b) What is the greatest acceleration of the mass, and when does it occur?

7.24 •• A 2.50 kg block on a horizontal floor is attached to a horizontal spring that is initially compressed 0.0300 m. The spring has force constant 840 N/m. The coefficient of kinetic friction between the floor and the block is $\mu_{\mathrm{k}} = 0.40$. The block and spring are released from rest, and the block slides along the floor. What is the speed of the block when it has moved a distance of 0.0200 m from its initial position? (At this point the spring is compressed 0.0100 m.)

7.25 •• You are asked to design a spring that will give a 1160 kg satellite a speed of 2.50 m/s relative to an orbiting space shuttle. Your spring is to give the satellite a maximum acceleration of $5.00g$. The spring's mass, the recoil kinetic energy of the shuttle, and changes in gravitational potential energy will all be negligible. (a) What must the force constant of the spring be? (b) What distance must the spring be compressed?

7.26 • It takes a force of 5.00 N to stretch an ideal spring 2.00 cm. (a) What force does it take to stretch the spring an additional 4.00 cm? (b) By what factor does the stored elastic potential energy increase when the spring, originally stretched 2.00 cm, is stretched 4.00 cm more?

Section 7.3 Conservative and Nonconservative Forces

7.27 • A 0.60 kg book slides on a horizontal table. The kinetic friction force on the book has magnitude 1.8 N. (a) How much work is done on the book by friction during a displacement of 3.0 m to the left? (b) The book now slides 3.0 m to the right, returning to its starting point. During this second 3.0 m displacement, how much work is done on the book by friction? (c) What is the total work done on the book by friction during the complete round trip? (d) On the basis of your answer to part (c), would you say that the friction force is conservative or nonconservative? Explain.

7.28 •• CALC In an experiment, one of the forces exerted on a proton is $\vec{F} = -\alpha x^2 \hat{i}$, where $\alpha = 12\ \mathrm{N/m^2}$. (a) How much work does $\vec{F}$ do when the proton moves along the straight-line path from the point (0.10 m, 0) to the point (0.10 m, 0.40 m)? (b) Along the straight-line path from the point (0.10 m, 0) to the point (0.30 m, 0)? (c) Along the straight-line path from the point (0.30 m, 0) to the point (0.10 m, 0)? (d) Is the force $\vec{F}$ conservative? Explain. If $\vec{F}$ is conservative, what is the potential-energy function for it? Let $U = 0$ when $x = 0$.

7.29 •• A 62.0 kg skier is moving at 6.50 m/s on a frictionless, horizontal, snow-covered plateau when she encounters a rough patch 4.20 m long. The coefficient of kinetic friction between this patch and her skis is 0.300. After crossing the rough patch and returning to friction-free snow, she skis down an icy, frictionless hill 2.50 m high. (a) How fast is the skier moving when she gets to the bottom of the hill? (b) How much internal energy was generated in crossing the rough patch?

7.30 • While a roofer is working on a roof that slants at 36° above the horizontal, he accidentally nudges his 85.0 N toolbox, causing it to start sliding downward from rest. If it starts 4.25 m from the lower edge of the roof, how fast will the toolbox be moving just as it reaches the edge of the roof if the kinetic friction force on it is 22.0 N?

Section 7.4 Force and Potential Energy

7.31 •• **CALC** A force parallel to the x-axis acts on a particle moving along the x-axis. This force produces potential energy $U(x)$ given by $U(x) = \alpha x^4$, where $\alpha = 0.630\ \mathrm{J/m^4}$. What is the force (magnitude and direction) when the particle is at $x = -0.800\ \mathrm{m}$?

7.32 •• **CALC** The potential energy of a pair of hydrogen atoms separated by a large distance x is given by $U(x) = -C_6/x^6$, where C_6 is a positive constant. What is the force that one atom exerts on the other? Is this force attractive or repulsive?

7.33 •• **CALC** A small block with mass 0.0400 kg is moving in the xy-plane. The net force on the block is described by the potential-energy function $U(x,\ y) = (5.80\ \mathrm{J/m^2})x^2 - (3.60\ \mathrm{J/m^3})y^3$. What are the magnitude and direction of the acceleration of the block when it is at the point ($x = 0.300\ \mathrm{m}$, $y = 0.600\ \mathrm{m}$)?

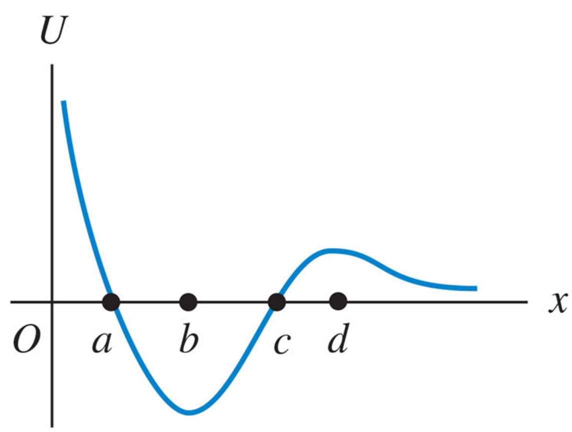
7.34 •• **CALC** An object moving in the xy-plane is acted on by a conservative force described by the potential-energy function $U(x,\ y) = \alpha\left[\left(1/x^2\right) + \left(1/y^2\right)\right]$, where α is a positive constant. Derive an expression for the force expressed in terms of the unit vectors $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$.

Section 7.5 Energy Diagrams

7.35 • **CALC** The potential energy of two atoms in a diatomic molecule is approximated by $U(r) = \left(a/r^{12}\right) - \left(b/r^6\right)$, where r is the spacing between atoms and a and b are positive constants. (a) Find the force $F(r)$ on one atom as a function of r. Draw two graphs: one of $U(r)$ versus r and one of $F(r)$ versus r. (b) Find the equilibrium distance between the two atoms. Is this equilibrium stable? (c) Suppose the distance between the two atoms is equal to the equilibrium distance found in part (b). What minimum energy must be added to the molecule to *dissociate* it—that is, to separate the two atoms to an infinite distance apart? This is called the *dissociation energy* of the molecule. (d) For the molecule CO, the equilibrium distance between the carbon and oxygen atoms is 1.13×10^{-10} m and the dissociation energy is 1.54×10^{-18} J per molecule. Find the values of the constants a and b.

7.36 • A marble moves along the x-axis. The potential-energy function is shown in Fig. E7.36. (a) At which of the labeled x-coordinates is the force on the marble zero? (b) Which of the labeled x-coordinates is a position of stable equilibrium? (c) Which of the labeled x-coordinates is a position of unstable equilibrium?

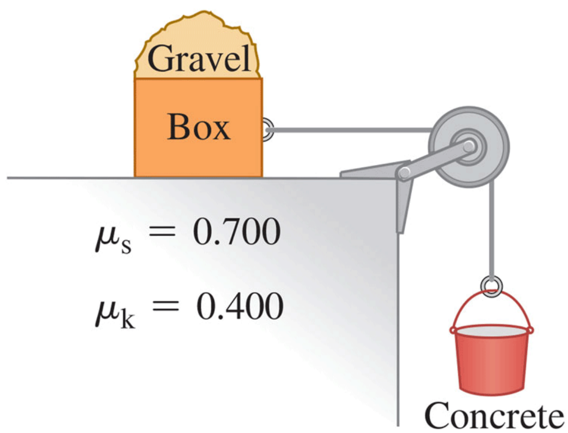
Figure E7.36



Problems

7.37 ••• At a construction site, a 65.0 kg bucket of concrete hangs from a light (but strong) cable that passes over a light, friction-free pulley and is connected to an 80.0 kg box on a horizontal roof (Fig. P7.37). The cable pulls horizontally on the box, and a 50.0 kg bag of gravel rests on top of the box. The coefficients of friction between the box and roof are shown. (a) Find the friction force on the bag of gravel and on the box. (b) Suddenly a worker picks up the bag of gravel. Use energy conservation to find the speed of the bucket after it has descended 2.00 m from rest. (Use Newton's laws to check your answer.)

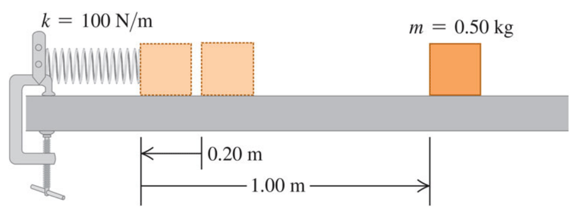
Figure P7.37



7.38 •• **CP** Estimate the maximum horizontal distance you can throw a baseball ($m = 0.145$ kg) if you throw it at an angle of $\alpha_0 = 45^\circ$ above the horizontal in order to achieve the maximum range. (a) What is the kinetic energy of the baseball just after it leaves your hand? Ignore air resistance and the small distance the ball is above the ground when it leaves your hand. Take the zero of potential energy to be at the ground. (b) At the ball's maximum height, what fraction of its total mechanical energy is kinetic energy and what fraction is gravitational potential energy? (c) If you throw the baseball at an initial angle of 60° above the horizontal, at its maximum height what fraction of its total energy is kinetic energy and what fraction is gravitational potential energy? (d) What fraction of the total mechanical energy is kinetic energy at the maximum height in the limiting cases of $\alpha_0 = 0^\circ$ and $\alpha_0 = 90^\circ$?

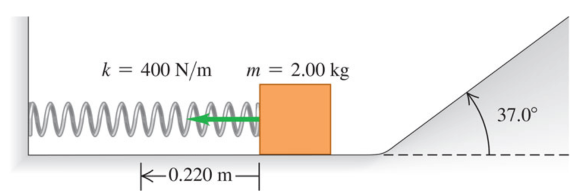
7.39 • A block with mass 0.50 kg is forced against a horizontal spring of negligible mass, compressing the spring a distance of 0.20 m (Fig. P7.39). When released, the block moves on a horizontal tabletop for 1.00 m before coming to rest. The force constant k is 100 N/m. What is the coefficient of kinetic friction μ_k between the block and the tabletop?

Figure P7.39



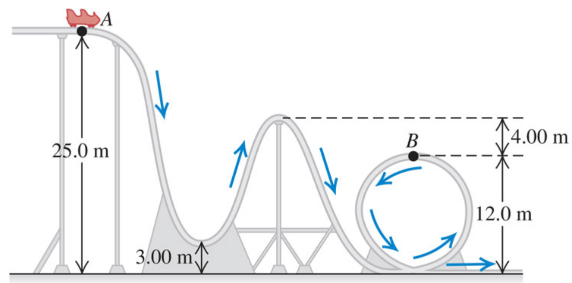
7.40 • A 2.00 kg block is pushed against a spring with negligible mass and force constant $k = 400\ \mathrm{N}/m$, compressing it 0.220 m. When the block is released, it moves along a frictionless, horizontal surface and then up a frictionless incline with slope 37.0° (Fig. P7.40). (a) What is the speed of the block as it slides along the horizontal surface after having left the spring? (b) How far does the block travel up the incline before starting to slide back down?

Figure P7.40



7.41 •• A 350 kg roller coaster car starts from rest at point A and slides down a frictionless loop-the-loop (Fig. P7.41). The car's wheels are designed to stay on the track. (a) How fast is this roller coaster car moving at point B? (b) How hard does it press against the track at point B?

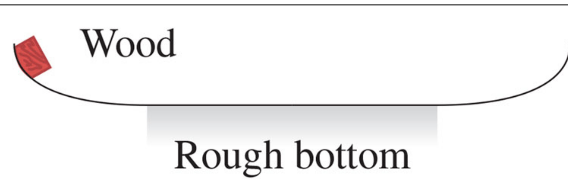
Figure P7.41



7.42 •• **CP** A small rock with mass m is released from rest at the inside rim of a large, hemispherical bowl (point A) that has radius R, as shown in Fig. E7.9. If the normal force exerted on the rock as it slides through its lowest point (point B) is twice the weight of the rock, how much work did friction do on the rock as it moved from A to B? Express your answer in terms of m, R, and g.

7.43 •• A 2.0 kg piece of wood slides on a curved surface (Fig. P7.43). The sides of the surface are perfectly smooth, but the rough horizontal bottom is 30 m long and has a kinetic friction coefficient of 0.20 with the wood. The piece of wood starts from rest 4.0 m above the rough bottom. (a) Where will this wood eventually come to rest? (b) For the motion from the initial release until the piece of wood comes to rest, what is the total amount of work done by friction?

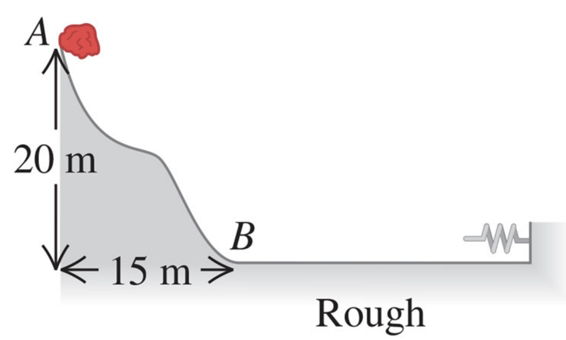
Figure P7.43



7.44 •• CP A small block with mass m slides without friction on the inside of a vertical circular track that has radius R. What minimum speed must the block have at the bottom of its path if it is not to fall off the track at the top of its path?

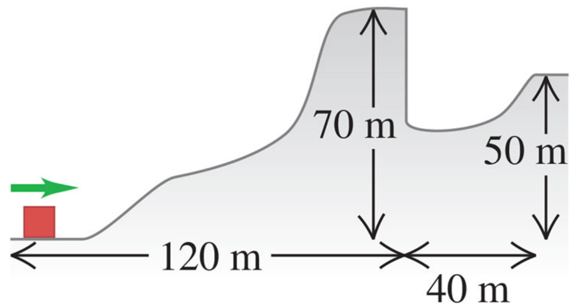
7.45 •• A 15.0 kg stone slides down a snow-covered hill (Fig. P7.45), leaving point A at a speed of 10.0 m/s. There is no friction on the hill between points A and B, but there is friction on the level ground at the bottom of the hill, between B and the wall. After entering the rough horizontal region, the stone travels 100 m and then runs into a very long, light spring with force constant 2.00 N/m. The coefficients of kinetic and static friction between the stone and the horizontal ground are 0.20 and 0.80, respectively. (a) What is the speed of the stone when it reaches point B? (b) How far will the stone compress the spring? (c) Will the stone move again after it has been stopped by the spring?

Figure P7.45



7.46 •• CP A 2.8 kg block slides over the smooth, icy hill shown in Fig. P7.46. The top of the hill is horizontal and 70 m higher than its base. What minimum speed must the block have at the base of the 70 m hill to pass over the pit at the far (right-hand) side of that hill?

Figure P7.46



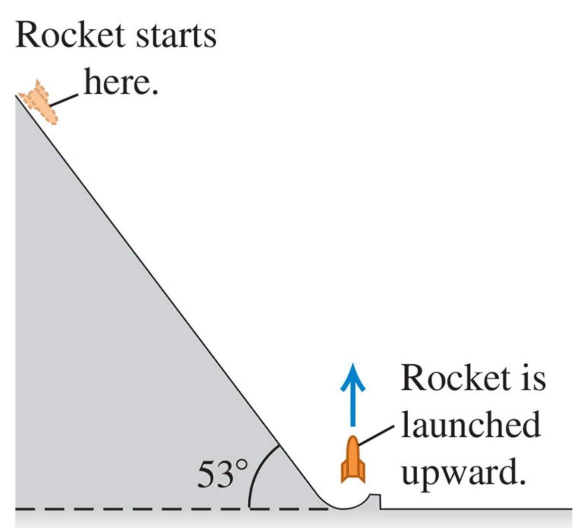
7.47 •• A small box with mass 0.600 kg is placed against a compressed spring at the bottom of an incline that slopes upward at 37.0° above the horizontal. The other end of the spring is attached to a wall. The coefficient of kinetic friction between the box and the surface of the incline is $\mu_k = 0.400$. The spring is released and the box travels up the incline, leaving the spring behind. What minimum elastic potential energy must be stored initially in the spring if the box is to travel 2.00 m from its initial position to the top of the incline?

7.48 ••• You are designing a delivery ramp for crates containing exercise equipment. The 1470 N crates will move at 1.8 m/s at the top of a ramp that slopes downward at 22.0°. The ramp exerts a 515 N kinetic friction force on each crate, and the maximum static friction force also has this value. Each crate will compress a spring at the bottom of the ramp and will come to rest after traveling a total distance of 5.0 m along the ramp. Once stopped, a crate must not rebound back up the ramp. Calculate the largest force constant of the spring that will be needed to meet the design criteria.

7.49 ••• The Great Sandini is a 60 kg circus performer who is shot from a cannon (actually a spring gun). You don't find many men of his caliber, so you help him design a new gun. This new gun has a very large spring with a very small mass and a force constant of 1100 N/m that he will compress with a force of 4400 N. The inside of the gun barrel is coated with Teflon, so the average friction force will be only 40 N during the 4.0 m he moves in the barrel. At what speed will he emerge from the end of the barrel, 2.5 m above his initial rest position?

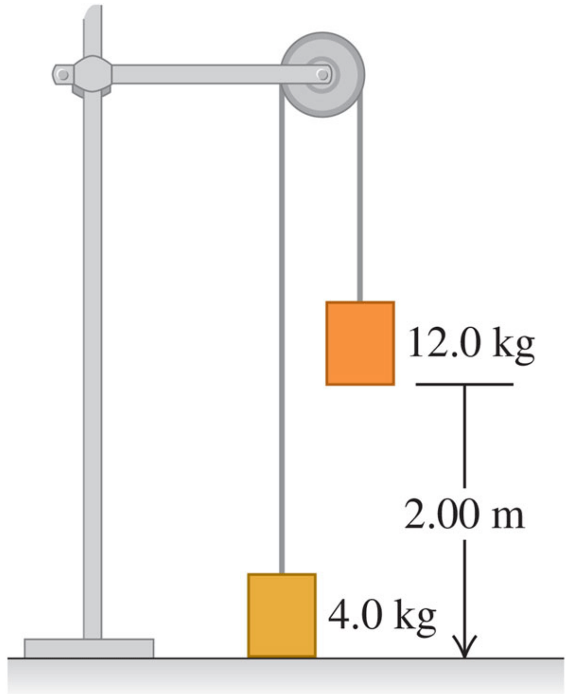
7.50 •• A 1500 kg rocket is to be launched with an initial upward speed of 50.0 m/s. In order to assist its engines, the engineers will start it from rest on a ramp that rises 53° above the horizontal (Fig. P7.50). At the bottom, the ramp turns upward and launches the rocket vertically. The engines provide a constant forward thrust of 2000 N, and friction with the ramp surface is a constant 500 N. How far from the base of the ramp should the rocket start, as measured along the surface of the ramp?

Figure P7.50



7.51 •• A system of two paint buckets connected by a lightweight rope is released from rest with the 12.0 kg bucket 2.00 m above the floor (Fig. P7.51). Use the principle of conservation of energy to find the speed with which this bucket strikes the floor. Ignore friction and the mass of the pulley.

Figure P7.51



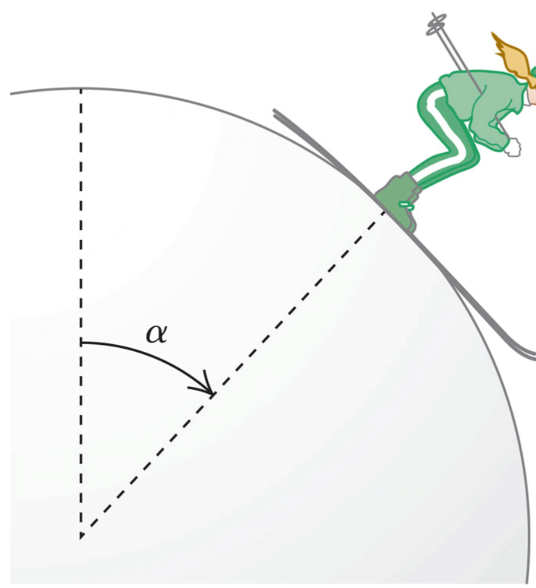
7.52 •• A block with mass $m = 0.200$ kg is placed against a compressed spring at the bottom of a ramp that is at an angle of 53.0° above the horizontal. The spring has 8.00 J of elastic potential energy stored in it. The spring is released, and the block moves up the incline. After the block has traveled a distance of 3.00 m, its speed is 4.00 m/s. What is the magnitude of the friction force that the ramp exerts on the block while the block is moving?

7.53 •• **CP** A 0.300 kg potato is tied to a string with length 2.50 m, and the other end of the string is tied to a rigid support. The potato is held straight out horizontally from the point of support, with the string pulled taut, and is then released. (a) What is the speed of the potato at the lowest point of its motion? (b) What is the tension in the string at this point?

7.54 •• A 60.0 kg skier starts from rest at the top of a ski slope 65.0 m high. (a) If friction forces do -10.5 kJ of work on her as she descends, how fast is she going at the bottom of the slope? (b) Now moving horizontally, the skier crosses a patch of soft snow where $\mu_k = 0.20$. If the patch is 82.0 m wide and the average force of air resistance on the skier is 160 N, how fast is she going after crossing the patch? (c) The skier hits a snowdrift and penetrates 2.5 m into it before coming to a stop. What is the average force exerted on her by the snowdrift as it stops her?

7.55 • **CP** A skier starts at the top of a very large, frictionless snowball, with a very small initial speed, and skis straight down the side (Fig. P7.55). At what point does she lose contact with the snowball and fly off at a tangent? That is, at the instant she loses contact with the snowball, what angle α does a radial line from the center of the snowball to the skier make with the vertical?

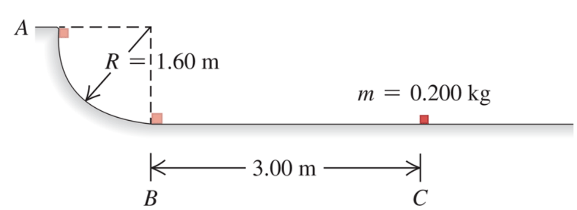
Figure P7.55



7.56 •• A block with mass 0.400 kg is on a horizontal frictionless surface and is attached to a horizontal compressed spring that has force constant $k = 200\ \mathrm{N}/m$. The other end of the spring is attached to a wall. The block is released, and it moves back and forth on the end of the spring. During this motion the block has speed 3.00 m/s when the spring is stretched 0.160 m. (a) During the motion of the block, what is its maximum speed? (b) During the block's motion, what is the maximum distance the spring is compressed from its equilibrium position? (c) When the spring has its maximum compression, what is the speed of the block and what is the magnitude of the acceleration of the block?

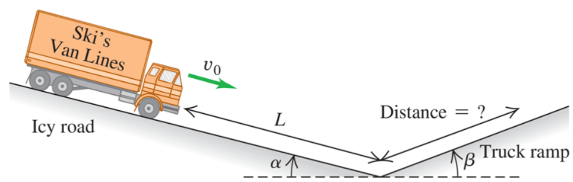
7.57 •• In a truck-loading station at a post office, a small 0.200 kg package is released from rest at point A on a track that is one-quarter of a circle with radius 1.60 m (Fig. P7.57). The size of the package is much less than 1.60 m, so the package can be treated as a particle. It slides down the track and reaches point B with a speed of 4.80 m/s. From point B, it slides on a level surface a distance of 3.00 m to point C, where it comes to rest. (a) What is the coefficient of kinetic friction on the horizontal surface? (b) How much work is done on the package by friction as it slides down the circular arc from A to B?

Figure P7.57



7.58 ••• A truck with mass m has a brake failure while going down an icy mountain road of constant downward slope angle α (Fig. P7.58). Initially the truck is moving downhill at speed v_0. After careening downhill a distance L with negligible friction, the truck driver steers the runaway vehicle onto a runaway truck ramp of constant upward slope angle β. The truck ramp has a soft sand surface for which the coefficient of rolling friction is μ_r. What is the distance that the truck moves up the ramp before coming to a halt? Solve by energy methods.

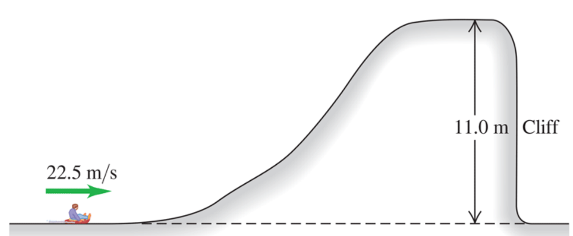
Figure P7.58



7.59 •• **CALC** A certain spring found *not* to obey Hooke's law exerts a restoring force $F_x(x) = -\alpha x - \beta x^2$ if it is stretched or compressed, where $\alpha = 60.0\ \mathrm{N}/m$ and $\beta = 18.0\ \mathrm{N/m}^2$. The mass of the spring is negligible. (a) Calculate the potential-energy function $U(x)$ for this spring. Let $U = 0$ when $x = 0$. (b) An object with mass 0.900 kg on a frictionless, horizontal surface is attached to this spring, pulled a distance 1.00 m to the right (the $+x-$ direction) to stretch the spring, and released. What is the speed of the object when it is 0.50 m to the right of the $x = 0$ equilibrium position?

7.60 •• **CP** A sled with rider having a combined mass of 125 kg travels over a perfectly smooth icy hill (Fig. 7.60). How far does the sled land from the foot of the cliff?

Figure P7.60



7.61 •• **CALC** A conservative force $\vec{F}$ is in the $+x$-direction and has magnitude $F(x) = \alpha/(x + x_0)^2$, where $\alpha = 0.800\ \mathrm{N{\cdot}m^2}$ and $x_0 = 0.200\ \mathrm{m}$. (a) What is the potential-energy function $U(x)$ for this force? Let $U(x) \to 0$ as $x \to \infty$. (b) An object with mass $m = 0.500\ \mathrm{kg}$ is released from rest at $x = 0$ and moves in the $+x$-direction. If $\vec{F}$ is the only force acting on the object, what is the object's speed when it reaches $x = 0.400\ \mathrm{m}$?

7.62 •• **CP** A light rope of length 1.40 m is attached to the ceiling. A small steel ball with mass 0.200 kg swings on the lower end of the rope as a pendulum. As the ball swings back and forth, the angle θ between the rope and the vertical direction has a maximum value of 37.0°. (a) What is the tension in the rope when $\theta = 37.0^\circ$? (b) What is the tension when $\theta = 25.0^\circ$?

7.63 •• A 0.150 kg block of ice is placed against a horizontal, compressed spring mounted on a horizontal tabletop that is 1.20 m above the floor. The spring has force constant $1900\ \mathrm{N}/m$ and is initially compressed 0.045 m. The mass of the spring is negligible. The spring is released, and the block slides along the table, goes off the edge, and travels to the floor. If there is negligible friction between the block of ice and the tabletop, what is the speed of the block of ice when it reaches the floor?

7.64 •• If a fish is attached to a vertical spring and slowly lowered to its equilibrium position, it is found to stretch the spring by an amount d. If the same fish is attached to the end of the unstretched spring and then allowed to fall from rest, through what maximum distance does it stretch the spring? (*Hint:* Calculate the force constant of the spring in terms of the distance d and the mass m of the fish.)

7.65 ••• **CALC** You are an industrial engineer with a shipping company. As part of the package-handling system, a small box with mass 1.60 kg is placed against a light spring that is compressed 0.280 m. The spring, whose other end is attached to a wall, has force constant $k = 45.0 \mathrm{~N}/m$. The spring and box are released from rest, and the box travels along a horizontal surface for which the coefficient of kinetic friction with the box is $\mu_{\mathrm{k}} = 0.300$. When the box has traveled 0.280 m and the spring has reached its equilibrium length, the box loses contact with the spring. (a) What is the speed of the box at the instant when it leaves the spring? (b) What is the maximum speed of the box during its motion?

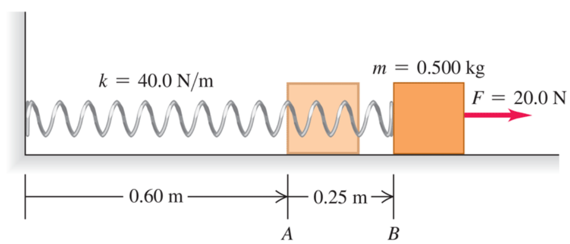
7.66 •• A basket of negligible weight hangs from a vertical spring scale of force constant $1500 \mathrm{~N}/m$. (a) If you suddenly put a 3.0 kg adobe brick in the basket, find the maximum distance that the spring will stretch. (b) If, instead, you release the brick from 1.0 m above the basket, by how much will the spring stretch at its maximum elongation?

7.67 ••• **CALC** A 3.00 kg fish is attached to the lower end of a vertical spring that has negligible mass and force constant $900 \mathrm{~N}/m$. The spring initially is neither stretched nor compressed. The fish is released from rest. (a) What is its speed after it has descended 0.0500 m from its initial position? (b) What is the maximum speed of the fish as it descends?

7.68 •• **CP** To test a slide at an amusement park, a block of wood with mass 3.00 kg is released at the top of the slide and slides down to the horizontal section at the end, a vertical distance of 23.0 m below the starting point. The block flies off the ramp in a horizontal direction and then lands on the ground after traveling through the air 30.0 m horizontally and 40.0 m downward. Neglect air resistance. How much work does friction do on the block as it slides down the ramp?

7.69 • A 0.500 kg block, attached to a spring with length 0.60 m and force constant 40.0 N/m, is at rest with the back of the block at point A on a frictionless, horizontal air table (Fig. P7.69). The mass of the spring is negligible. You move the block to the right along the surface by pulling with a constant 20.0 N horizontal force. (a) What is the block's speed when the back of the block reaches point B, which is 0.25 m to the right of point A? (b) When the back of the block reaches point B, you let go of the block. In the subsequent motion, how close does the block get to the wall where the left end of the spring is attached?

Figure P7.69



7.70 ••• **CP** A small block with mass 0.0400 kg slides in a vertical circle of radius $R = 0.500$ m on the inside of a circular track. During one of the revolutions of the block, when the block is at the bottom of its path, point A, the normal force exerted on the block by the track has magnitude 3.95 N. In this same revolution, when the block reaches the top of its path, point B, the normal force exerted on the block has magnitude 0.680 N. How much work is done on the block by friction during the motion of the block from point A to point B?

7.71 ••• **CP** A small block with mass 0.0500 kg slides in a vertical circle of radius $R = 0.800$ m on the inside of a circular track. There is no friction between the track and the block. At the bottom of the block's path, the normal force the track exerts on the block has magnitude 3.40 N. What is the magnitude of the normal force that the track exerts on the block when it is at the top of its path?

7.72 •• **CP** **Pendulum.** A small rock with mass 0.12 kg is fastened to a massless string with length 0.80 m to form a pendulum. The pendulum is swinging so as to make a maximum angle of 45° with the vertical. Air resistance is negligible. (a) What is the speed of the rock when the string passes through the vertical position? What is the tension in the string (b) when it makes an angle of 45° with the vertical, (c) as it passes through the vertical?

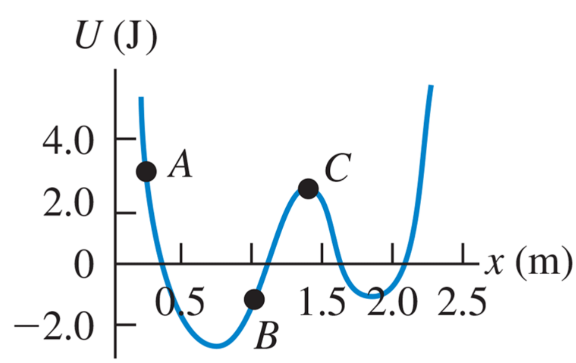
7.73 ••• A wooden block with mass 1.50 kg is placed against a compressed spring at the bottom of an incline of slope 30.0° (point A). When the spring is released, it projects the block up the incline. At point B, a distance of 6.00 m up the incline from A, the block is moving up the incline at 7.00 m/s and is no longer in contact with the spring. The coefficient of kinetic friction between the block and the incline is $\mu_\mathrm{k} = 0.50$. The mass of the spring is negligible. Calculate the amount of potential energy that was initially stored in the spring.

7.74 •• **CALC** A small object with mass $m = 0.0900$ kg moves along the $+x$-axis. The only force on the object is a conservative force that has the potential-energy function $U(x) = -\alpha x^2 + \beta x^3$, where $\alpha = 2.00\ \mathrm{J/m^2}$ and $\beta = 0.300\ \mathrm{J/m^3}$. The object is released from rest at small x. When the object is at $x = 4.00$ m, what are its (a) speed and (b) acceleration (magnitude and direction)? (c) What is the maximum value of x reached by the object during its motion?

7.75 ••• **CALC** A cutting tool under microprocessor control has several forces acting on it. One force is $\vec{\boldsymbol{F}} = -\alpha x y^2 \hat{\boldsymbol{j}}$, a force in the negative y-direction whose magnitude depends on the position of the tool. For $\alpha = 2.50\ \mathrm{N/m^3}$, consider the displacement of the tool from the origin to the point ($x = 3.00$ m, $y = 3.00$ m). (a) Calculate the work done on the tool by $\vec{F}$ if this displacement is along the straight line $y = x$ that connects these two points. (b) Calculate the work done on the tool by $\vec{F}$ if the tool is first moved out along the x-axis to the point ($x = 3.00$ m, y = 0) and then moved parallel to the y-axis to the point ($x = 3.00$ m, $y = 3.00$ m). (c) Compare the work done by $\vec{F}$ along these two paths. Is $\vec{F}$ conservative or nonconservative? Explain.

7.76 • A particle moves along the x-axis while acted on by a single conservative force parallel to the x-axis. The force corresponds to the potential-energy function graphed in Fig. P7.76. The particle is released from rest at point A. (a) What is the direction of the force on the particle when it is at point A? (b) At point B? (c) At what value of x is the kinetic energy of the particle a maximum? (d) What is the force on the particle when it is at point C? (e) What is the largest value of x reached by the particle during its motion? (f) What value or values of x correspond to points of stable equilibrium? (g) Of unstable equilibrium?

Figure P7.76



7.77 •• DATA You are designing a pendulum for a science museum. The pendulum is made by attaching a brass sphere with mass m to the lower end of a long, light metal wire of (unknown) length L. A device near the top of the wire measures the tension in the wire and transmits that information to your laptop computer. When the wire is vertical and the sphere is at rest, the sphere's center is 0.800 m above the floor and the tension in the wire is 265 N. Keeping the wire taut, you then pull the sphere to one side (using a ladder if necessary) and gently release it. You record the height h of the center of the sphere above the floor at the point where the sphere is released and the tension T in the wire as the sphere swings through its lowest point. You collect your results:

h (m)	0.800	2.00	4.00	6.00	8.00	10.0	12.0
T (N)	265	274	298	313	330	348	371

Assume that the sphere can be treated as a point mass, ignore the mass of the wire, and assume that total mechanical energy is conserved through each measurement. (a) Plot T versus h, and use this graph to calculate L. (b) If the breaking strength of the wire is 822 N, from what maximum height h can the sphere be released if the tension in the wire is not to exceed half the breaking strength? (c) The pendulum is swinging when you leave at the end of the day. You lock the museum doors, and no one enters the building until you return the next morning. You find that the sphere is hanging at rest. Using energy considerations, how can you explain this behavior?

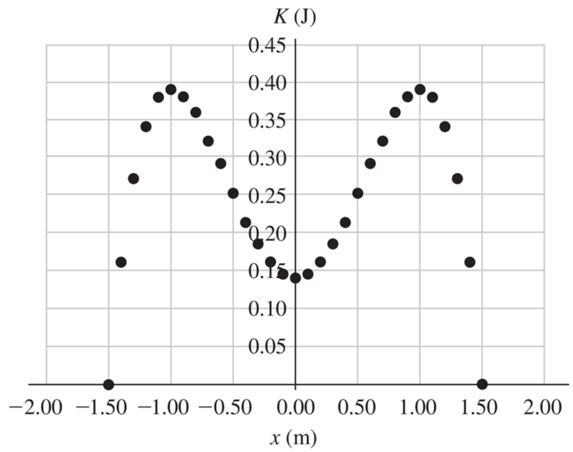
7.78 **•• DATA** A long ramp made of cast iron is sloped at a constant angle $\theta = 52.0^\circ$ above the horizontal. Small blocks, each with mass 0.42 kg but made of different materials, are released from rest at a vertical height h above the bottom of the ramp. In each case the coefficient of static friction is small enough that the blocks start to slide down the ramp as soon as they are released. You are asked to find h so that each block will have a speed of 4.00 m/s when it reaches the bottom of the ramp. You are given these coefficients of sliding (kinetic) friction for different pairs of materials:

Material 1	Material 2	Coefficient of Sliding Friction
Cast iron	Cast iron	0.15
Cast iron	Copper	0.29
Cast iron	Lead	0.43
Cast iron	Zinc	0.85

(a) Use work and energy considerations to find the required value of h if the block is made from (i) cast iron; (ii) copper; (iii) zinc. (b) What is the required value of h for the copper block if its mass is doubled to 0.84 kg? (c) For a given block, if θ is increased while h is kept the same, does the speed v of the block at the bottom of the ramp increase, decrease, or stay the same?

7.79 •• DATA A single conservative force $F(x)$ acts on a small sphere of mass m while the sphere moves along the x-axis. You release the sphere from rest at $x = -1.50$ m. As the sphere moves, you measure its velocity as a function of position. You use the velocity data to calculate the kinetic energy K; Fig. P7.79 shows your data. (a) Let $U(x)$ be the potential-energy function for $F(x)$. Is $U(x)$ symmetric about $x = 0$? [If so, then $U(x) = U(-x)$.] (b) If you set $U = 0$ at $x = 0$, what is the value of U at $x = -1.50$ m? (c) Sketch $U(x)$. (d) At what values of x (if any) is $F = 0$? (e) For what range of values of x between $x = -1.50$ m and $x = +1.50$ m is F positive? Negative? (f) If you release the sphere from rest at $x = -1.30$ m, what is the largest value of x that it reaches during its motion? The largest value of kinetic energy that it has during its motion?

Figure P7.79



Challenge Problem

7.80 ••• CALC A proton with mass m moves in one dimension. The potential-energy function is $U(x) = (\alpha/x^2) - (\beta/x)$, where α and β are positive constants. The proton is released from rest at $x_0 = \alpha/\beta$. (a) Show that $U(x)$ can be written as

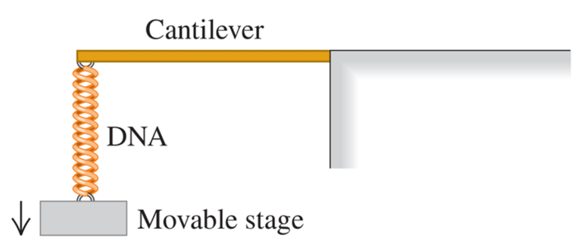
$$U(x) = \frac{\alpha}{{x_0}^2}\left[\left(\frac{x_0}{x}\right)^2 - \frac{x_0}{x}\right]$$

Graph $U(x)$. Calculate $U(x_0)$ and thereby locate the point x_0 on the graph. (b) Calculate $v(x)$, the speed of the proton as a function of position. Graph $v(x)$ and give a qualitative description of the motion. (c) For what value of x is the speed of the proton a maximum? What is the value of that maximum speed? (d) What is the force on the proton at the point in part (c)? (e) Let the proton be released instead at $x_1 = 3\alpha/\beta$. Locate the point x_1 on the graph of $U(x)$. Calculate $v(x)$ and give a qualitative description of the motion. (f) For each release point

($x = x_0$ and $x = x_1$), what are the maximum and minimum values of x reached during the motion?

MCAT-Style Passage Problems

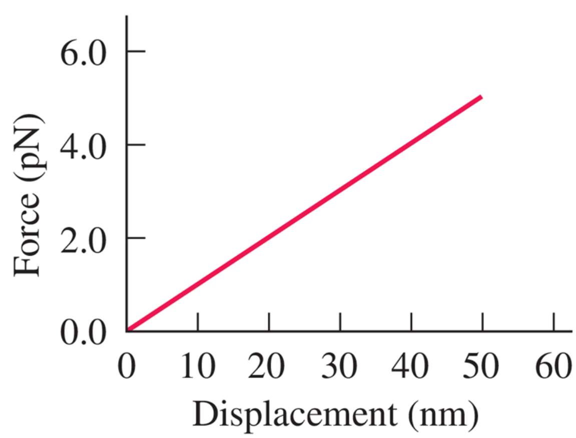
BIO **The DNA Spring.** A DNA molecule, with its double-helix structure, can in some situations behave like a spring. Measuring the force required to stretch single DNA molecules under various conditions can provide information about the biophysical properties of DNA. A technique for measuring the stretching force makes use of a very small cantilever, which consists of a beam that is supported at one end and is free to move at the other end, like a tiny diving board. The cantilever is constructed so that it obeys Hooke's law—that is, the displacement of its free end is proportional to the force applied to it. Because different cantilevers have different force constants, the cantilever's response must first be calibrated by applying a known force and determining the resulting deflection of the cantilever. Then one end of a DNA molecule is attached to the free end of the cantilever, and the other end of the DNA molecule is attached to a small stage that can be moved away from the cantilever, stretching the DNA. The stretched DNA pulls on the cantilever, deflecting the end of the cantilever very slightly. The measured deflection is then used to determine the force on the DNA molecule.



7.81 During the calibration process, the cantilever is observed to deflect by 0.10 nm when a force of 3.0 pN is applied to it. What deflection of the cantilever would correspond to a force of 6.0 pN? (a) 0.07 nm; (b) 0.14 nm; (c) 0.20 nm; (d) 0.40 nm.

7.82 A segment of DNA is put in place and stretched. Figure P7.82 shows a graph of the force exerted on the DNA as a function of the displacement of the stage. Based on this graph, which statement is the best interpretation of the DNA's behavior over this range of displacements? The DNA (a) does not follow Hooke's law, because its force constant increases as the force on it increases; (b) follows Hooke's law and has a force constant of about 0.1 pN/nm; (c) follows Hooke's law and has a force constant of about 10 pN/nm; (d) does not follow Hooke's law, because its force constant decreases as the force on it increases.

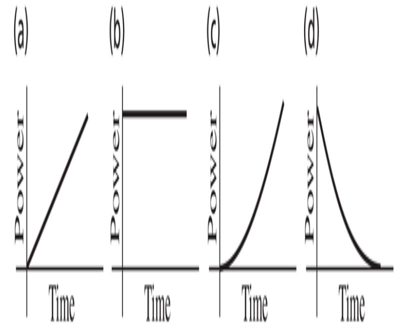
Figure P7.82



7.83 Based on Fig. P7.82, how much elastic potential energy is stored in the DNA when it is stretched 50 nm? (a) 2.5×10^{-19} J; (b) 1.2×10^{-19} J; (c) 5.0×10^{-12} J; (d) 2.5×10^{-12} J.

7.84 The stage moves at a constant speed while stretching the DNA. Which of the graphs in Fig. P7.84 best represents the power supplied to the stage versus time?

Figure P7.84



Answers: Potential Energy and Energy Conservation

Chapter Opening Question ?

(v) As the crane descends, air resistance directed opposite to the bird's motion prevents its speed from increasing. Because the crane's speed stays the same, its kinetic energy K remains constant, but the gravitational potential energy U_{grav} decreases as the crane descends. Hence the total mechanical energy $E = K + U_{\mathrm{grav}}$ decreases. The lost mechanical energy goes into warming the crane's skin (that is, an increase in the crane's internal energy) and stirring up the air through which the crane passes (an increase in the internal energy of the air). See Section 7.3.

Test Your Understanding

7.1 **(iii)** The initial kinetic energy $K_1 = 0$, the initial potential energy $U_1 = mgy_1$, and the final potential energy $U_2 = mgy_2$, are the same for both blocks. Total mechanical energy is conserved in both cases, so the final kinetic energy $K_2 = \frac{1}{2}m{v_2}^2$ is also the same for both blocks. Hence the speed at the right-hand end is the *same* in both cases!

7.2 **(iii)** The elevator is still moving downward, so the kinetic energy K is positive (remember that K can never be negative); the elevator is below point 1, so $y < 0$ and $U_{\mathrm{grav}} < 0$; and the spring is compressed, so $U_{\mathrm{el}} > 0$.

7.3 **(iii)** Because of friction in the turbines and between the water and turbines, some of the potential energy goes into raising the temperatures of the water and the mechanism.

7.4 **(a) (iv)**, **(b) (i)** If $F_x = 0$ at a point, then the derivative of $U(x)$ must be zero at that point because $F_x = -dU(x)/dx$. However, this tells us absolutely nothing about the *value* of $U(x)$ at that point.

7.5 **(iii)** Figure 7.24b shows the x-component of force, F_x. Where this is maximum (most positive), the x-component of force and the x-acceleration have more positive values than at adjacent values of x.

Key Example Variation Problems

VP7.2.1 **a.** 5.51 m

b. 3.67 m

VP7.2.2 **a.** $\sqrt{3gh/2}$

b. $\sqrt{gh/2}$

VP7.2.3 **a.** 1.94 J

b. 0.243 N

VP7.2.4 **a.** -7.99 J

b. 53.3 N

VP7.5.1 **a.** 1.71 m/s

b. 0.147 N, or 3.00 times the weight

VP7.5.2 **a.** 4.41 m; no

b. -358 J

VP7.5.3 **a.** 0.503 J

b. 3.29 N

VP7.5.4 **a.** $\sqrt{8gR}$

b. $-mgR$

c. mg/π

VP7.9.1 **a.** 0.0492 J

b. 0.128 m

VP7.9.2 **a.** -0.0116 J

b. 0.410

VP7.9.3 **a.** $\sqrt{(kd^2/m) - 2gx}$

b. $kd^2/2mg$

VP7.9.4 $2\,(mg-f)/k$

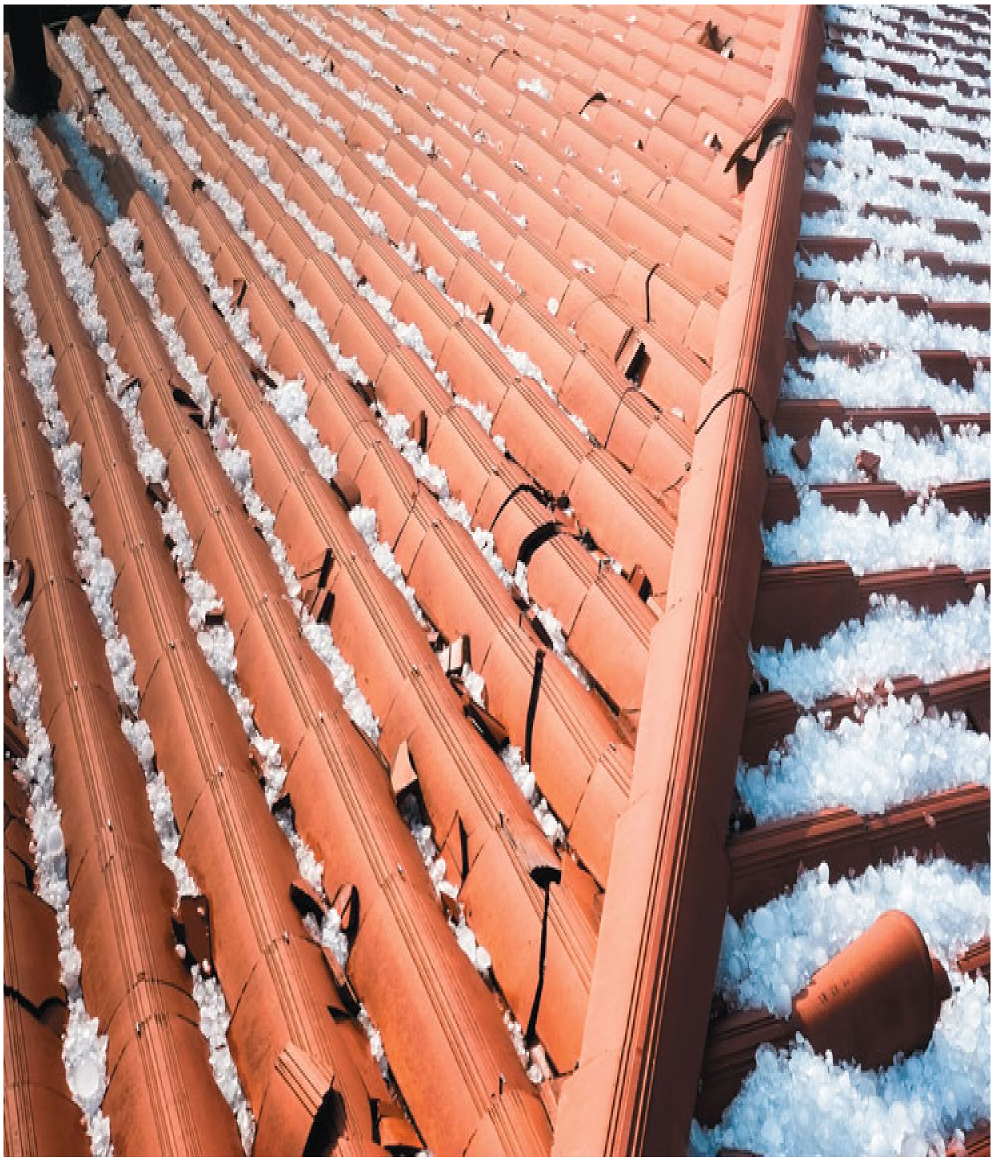
Bridging Problem

(a) 1.06 m

(b) 1.32 m

(c) 20.7 J

Chapter 8
Momentum, Impulse, and Collisions



?🗗 Hailstorms such as the one that broke the tiles on this roof cause billions of dollars of damage every year in North America alone. Which would do the greater amount of damage? (i) A 0.2 kg hailstone that hits at speed 15 m/s; (ii) a 0.1 kg hailstone that hits at speed 30 m/s; (iii) both would do the same amount of damage.

Learning Outcomes

In this chapter, you'll learn...

8.1 The meaning of the momentum of a particle, and how the impulse of the net external force acting on a particle causes its momentum to change.

8.2 The circumstances under which the total momentum of a system of particles is constant (conserved).

8.3 How to use momentum conservation to solve problems in which two objects collide with each other, and what the differences are among elastic, inelastic, and completely inelastic collisions.

8.4 How to analyze what happens in the important special case of an elastic collision.

8.5 What's meant by the center of mass of a system, and what determines how the center of mass moves.

8.6 How to analyze situations such as rocket propulsion in which the mass of an object changes as it moves.

You'll need to review...

3.5 Relative velocity.

4.2 Inertial frames of reference.

6.1, 6.2 Work, kinetic energy, and the work–energy theorem.

6.3 Work done by an ideal spring.

Many questions involving forces can't be answered by directly applying Newton's second law, $\sum \overrightarrow{\boldsymbol{F}} = m\overrightarrow{a}$. For example, when a truck collides head-on with a compact car, what determines which way the wreckage moves after the collision? In playing pool, how do you decide how to aim the cue ball in order to knock the eight ball into the pocket? And when a meteorite collides with the earth, how much of the meteorite's kinetic energy is released in the impact?

All of these questions involve forces about which we know very little: the forces between the car and the truck, between the two pool balls, or between the meteorite and the earth. Remarkably, we'll find in this chapter that we don't have to know *anything* about these forces to answer questions of this kind!

Our approach uses two new concepts, *momentum* and *impulse*, and a new conservation law, *conservation of momentum*. This conservation law is every bit as important as the law of conservation of energy. The law of conservation of momentum is valid even in situations in which Newton's laws are inadequate, such as objects moving at very high speeds (near the speed of light) or objects on a very small scale (such as the constituents of atoms). Within the domain of Newtonian mechanics, conservation of momentum enables us to analyze many situations that would be very difficult if we tried to use Newton's laws directly. Among these are *collision* problems, in which two objects collide and can exert very large forces on each other for a short time. We'll also use momentum ideas to solve problems in which an object's mass changes as it moves, including the important special case of a rocket (which loses mass as it expends fuel).

8.1 Momentum and Impulse

In Section 6.2 we re-expressed Newton's second law for a particle, $\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$, in terms of the work–energy theorem. This theorem helped us tackle a great number of problems and led us to the law of conservation of energy. Let's return to $\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$, and see yet another useful way to restate this fundamental law.

Newton's Second Law in Terms of Momentum

Consider a particle of constant mass m. Because $\vec{a} = d\vec{\boldsymbol{v}}/dt$, we can write Newton's second law for this particle as

(8.1)

$$\sum \vec{\boldsymbol{F}} = m\frac{d\vec{\boldsymbol{v}}}{dt} = \frac{d}{dt}(m\vec{\boldsymbol{v}})$$

We can move the mass m inside the derivative because it is constant. Thus Newton's second law says that the net external force $\sum \vec{\boldsymbol{F}}$ acting on a particle equals the time rate of change of the product of the particle's mass and velocity. We'll call this product the **momentum,** or **linear momentum,** of the particle:

(8.2)

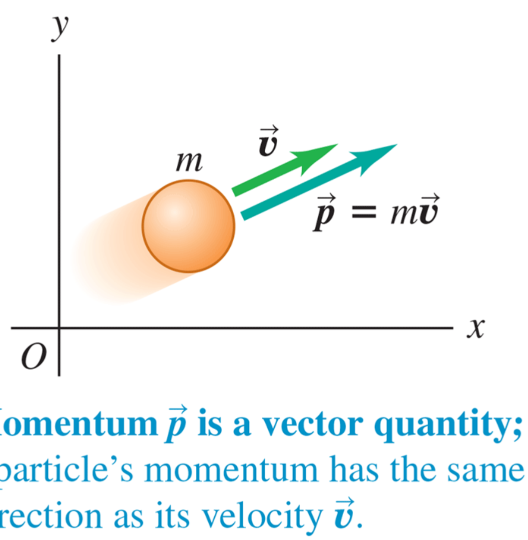
Particle mass

Momentum of a particle (a vector quantity) $\quad \vec{p} = m\vec{v} \qquad (8.2)$

Particle velocity

The greater the mass m and speed v of a particle, the greater is its magnitude of momentum mv. Keep in mind that momentum is a *vector* quantity with the same direction as the particle's velocity (Fig. 8.1). A car driving north at 20 m/s and an identical car driving east at 20 m/s have the same *magnitude* of momentum (mv) but different momentum *vectors* ($m\vec{v}$) because their directions are different.

Figure 8.1



The velocity and momentum vectors of a particle.

We often express the momentum of a particle in terms of its components. If the particle has velocity components v_x, v_y, and v_z, then its momentum components p_x, p_y, and p_z (which we also call the *x-momentum, y-momentum,* and *z-momentum*) are

(8.3)

$$p_x = mv_x \quad p_y = mv_y \quad p_z = mv_z$$

These three component equations are equivalent to Eq. (8.2).

The units of the magnitude of momentum are units of mass times speed; the SI units of momentum are kg·m/s. The plural of momentum is "momenta."

Let's now substitute the definition of momentum, Eq. (8.2), into Eq. (8.1):

(8.4)

Newton's second law in terms of momentum: The net external force acting on a particle ...

$$\sum \vec{F} = \frac{d\vec{p}}{dt} \tag{8.4}$$

... equals the rate of change of the particle's momentum.

The net external force (vector sum of all forces) acting on a particle equals the time rate of change of momentum of the particle. This, not $\sum \vec{F} = m\vec{a}$, is the form in which Newton originally stated his second law (although he called momentum the "quantity of motion"). This law is valid only in inertial frames of reference (see Section 4.2). As Eq. (8.4) shows, a rapid change in momentum requires a large net external force, while a gradual change in momentum requires a smaller net external force (Fig. 8.2).

Figure 8.2

When you land after jumping upward, your momentum changes from a downward value to zero. It's best to land with your knees bent so that your legs can flex: You then take a relatively long time to stop, and the force that the ground exerts on your legs is small. If you land with your legs extended, you stop in a short time, the force on your legs is larger, and the possibility of injury is greater.

The Impulse–Momentum Theorem

Both a particle's momentum $\vec{\boldsymbol{p}} = m\vec{\boldsymbol{v}}$ and its kinetic energy $K = \frac{1}{2}mv^2$ depend on the mass and velocity of the particle. What is the fundamental difference between these two quantities? A purely mathematical answer is that momentum is a vector whose magnitude is proportional to speed, while kinetic energy is a scalar proportional to the speed squared. But to see the *physical* difference between momentum and kinetic energy, we must first define a quantity closely related to momentum called *impulse*.

Let's first consider a particle acted on by a *constant* net external force $\sum \vec{\boldsymbol{F}}$ during a time interval Δt from t_1 to t_2. The **impulse** of the net external force, denoted by $\vec{J}$, is defined to be the product of the net external force and the time interval:

(8.5)

$$\underbrace{\vec{J}}_{\text{Impulse of a constant net external force}} = \overbrace{\sum \vec{F}}^{\text{Constant net external force}}\underbrace{(t_2 - t_1)}_{\text{Time interval over which net external force acts}} = \sum \vec{F}\,\Delta t \qquad (8.5)$$

Impulse is a vector quantity; its direction is the same as the net external force $\sum \vec{F}$. The SI unit of impulse is the newton-second $(\mathrm{N} \cdot \mathrm{s})$.> Because $1\ \mathrm{N} = 1\ \mathrm{kg{\cdot}m/s^2}$, an alternative set of units for impulse is kg·m/s, the same as for momentum.

BIO Application

Woodpecker Impulse

The pileated woodpecker *(Dryocopus pileatus)* has been known to strike its beak against a tree up to 20 times a second and up to 12,000 times a day. The impact force can be as much as 1200 times the weight of the bird’s head. Because the impact lasts such a short time, the impulse—the net external force during the impact multiplied by the duration of the impact—is relatively small. (The woodpecker has a thick skull of spongy bone as well as shock-absorbing cartilage at the base of the lower jaw, and so avoids injury.)

To see what impulse is good for, let’s go back to Newton’s second law as restated in terms of momentum, Eq. (8.4). If the net external force $\sum \vec{\boldsymbol{F}}$ is constant, then $d\vec{\boldsymbol{p}}/dt$ is also constant. In that case, $d\vec{\boldsymbol{p}}/dt$ is equal to the *total* change in momentum $\vec{\boldsymbol{p}}_2 - \vec{\boldsymbol{p}}_1$ during the time interval $t_2 - t_1$, divided by the interval:

$$\sum \vec{\boldsymbol{F}} = \frac{\vec{\boldsymbol{p}}_2 - \vec{\boldsymbol{p}}_1}{t_2 - t_1}$$

Multiplying this equation by $(t_2 - t_1)$, we have

$$\sum \vec{\boldsymbol{F}}(t_2 - t_1) = \vec{\boldsymbol{p}}_2 - \vec{\boldsymbol{p}}_1$$

Comparing with Eq. (8.5), we end up with

(8.6)

Impulse–momentum theorem: The impulse of the net external force on a particle during a time interval equals the change in momentum of that particle during that interval:

$$\underbrace{\vec{J}}_{\text{Impulse of net external force over a time interval}} = \underbrace{\vec{p}_2}_{\text{Final momentum}} - \underbrace{\vec{p}_1}_{\text{Initial momentum}} = \underbrace{\Delta\vec{p}}_{\text{Change in momentum}} \qquad (8.6)$$

The impulse–momentum theorem also holds when forces are not constant. To see this, we integrate both sides of Newton's second law $\sum \vec{F} = d\vec{p}/dt$ over time between the limits t_1 and t_2 :

$$\int_{t_1}^{t_2} \sum \vec{F}\, dt = \int_{t_1}^{t_2} \frac{d\vec{p}}{dt}\, dt = \int_{\vec{p}_1}^{\vec{p}_2} d\vec{p} = \vec{p}_2 - \vec{p}_1$$

We see from Eq. (8.6) that the integral on the left is the impulse of the net external force:

(8.7)

Upper limit = final time

$$\underbrace{\vec{J}}_{\text{Impulse of a general net external force (either constant or varying)}} = \int_{t_1}^{t_2} \underbrace{\sum \vec{F}\, dt}_{\text{Time integral of net external force}} \qquad (8.7)$$

Lower limit = initial time

If the net external force $\sum \vec{F}$ is constant, the integral in Eq. (8.7) reduces to Eq. (8.5). We can define an *average* net external force $\vec{F}_{\text{av}}$ such that even when $\sum \vec{F}$ is not constant, the impulse $\vec{J}$ is given by

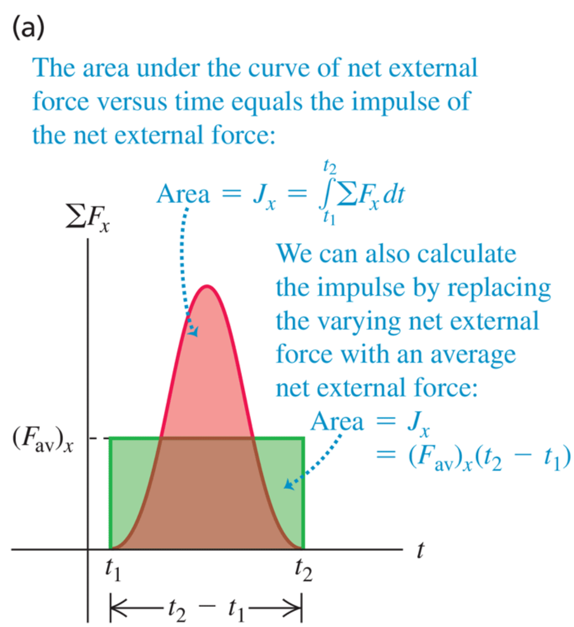
(8.8)

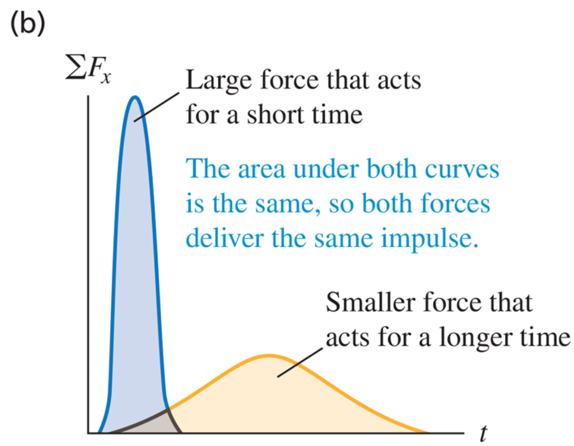
$$\vec{J} = \vec{F}_{\text{av}}\,(t_2 - t_1)$$

When $\sum \overrightarrow{\boldsymbol{F}}$ is constant, $\sum \vec{F} = \vec{F}_{\text{av}}$ and Eq. (8.8) reduces to Eq. (8.5).

Figure 8.3a shows the x-component of net external force $\sum F_x$ as a function of time during a collision. This might represent the force on a soccer ball that is in contact with a player's foot from time t_1 to t_2. The x-component of impulse during this interval is represented by the red area under the curve between t_1 and t_2. This area is equal to the green rectangular area bounded by t_1, t_2, and $(F_{\text{av}})_x$, so $(F_{\text{av}})_x\ (t_2 - t_1)$ is equal to the impulse of the actual time-varying force during the same interval. Note that a large force acting for a short time can have the same impulse as a smaller force acting for a longer time if the areas under the force–time curves are the same (Fig. 8.3b). We used this idea in Fig. 8.2: A small force acting for a relatively long time (as when you land with your legs bent) has the same effect as a larger force acting for a short time (as when you land stiff-legged). Automotive air bags use the same principle (Fig. 8.4).

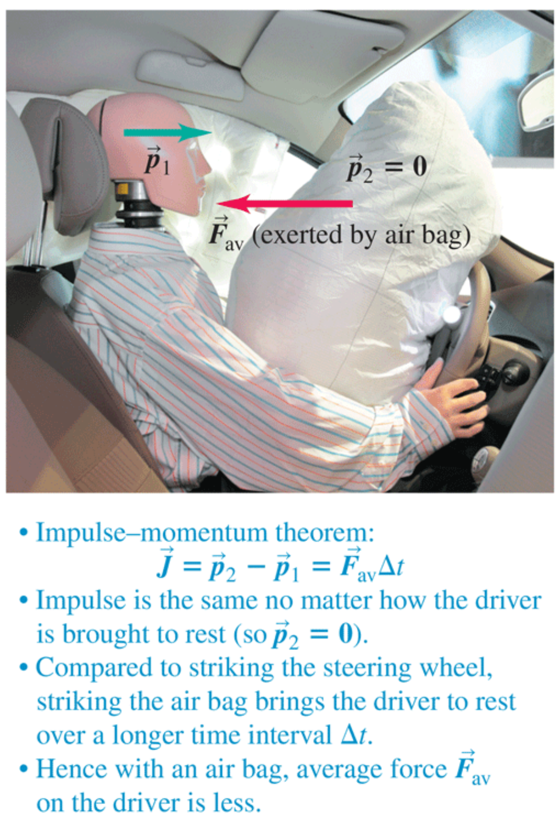
Figure 8.3





The meaning of the area under a graph of $\sum F_x$ versus t.

Figure 8.4



The impulse–momentum theorem explains how air bags reduce the chance of injury by minimizing the force on an occupant of a car.

Both impulse and momentum are vector quantities, and Eqs. (8.5), (8.6), (8.7) and (8.8) are vector equations. It's often easiest to use them in component form:

(8.9)

$$J_x = \int_{t_1}^{t_2} \sum F_x dt = (F_{\text{av}})_x \ (t_2 - t_1) = p_{2x} - p_{1x} = mv_{2x} - mv_{1x}$$

$$J_y = \int_{t_1}^{t_2} \sum F_y dt = (F_{\text{av}})_y \ (t_2 - t_1) = p_{2y} - p_{1y} = mv_{2y} - mv_{1y}$$

and similarly for the z-component.

Momentum and Kinetic Energy Compared

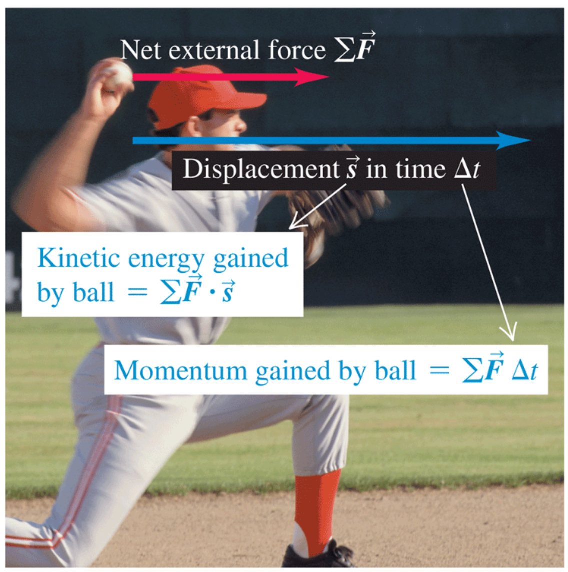
We can now see the fundamental difference between momentum and kinetic energy. The impulse–momentum theorem, $\vec{\boldsymbol{J}} = \vec{\boldsymbol{p}}_2 - \vec{\boldsymbol{p}}_1$, says that changes in a particle's momentum are due to impulse, which depends on the *time* over which the net external force acts. By contrast, the work–energy theorem, $W_{\text{tot}} = K_2 - K_1$, tells us that kinetic energy changes when work is done on a particle; the total work depends on the *distance* over which the net external force acts.

Let's consider a particle that starts from rest at t_1 so that $\vec{\boldsymbol{v}}_1 = \mathbf{0}$. Its initial momentum is $\vec{\boldsymbol{p}}_1 = m\vec{\boldsymbol{v}}_1 = \mathbf{0}$, and its initial kinetic energy is $K_1 = \frac{1}{2}mv_1^2 = 0$. Now let a constant net external force equal to $\vec{\boldsymbol{F}}$ act on that particle from time t_1 until time t_2. During this interval, the particle moves a distance s in the direction of the force. From Eq. (8.6), the particle's momentum at time t_2 is

$$\vec{p}_2 = \vec{p}_1 + \vec{J} = \vec{J}$$

where $\vec{J} = \vec{F}(t_2 - t_1)$ is the impulse that acts on the particle. So *the momentum of a particle equals the impulse that accelerated it from rest to its present speed;* impulse is the product of the net external force that accelerated the particle and the *time* required for the acceleration. By comparison, the kinetic energy of the particle at t_2 is $K_2 = W_{\text{tot}} = Fs$, the total *work* done on the particle to accelerate it from rest. The total work is the product of the net external force and the *distance* required to accelerate the particle (Fig. 8.5).

Figure 8.5



The *kinetic energy* of a pitched baseball is equal to the work the pitcher does on it (force multiplied by the distance the ball moves during the throw). The *momentum* of the ball is equal to the impulse the pitcher imparts to it (force multiplied by the time it took to bring the ball up to speed).

? Here's an application of the distinction between momentum and kinetic energy. Which is easier to catch: a 0.50 kg ball moving at 4.0 m/s or a 0.10 kg ball moving at 20 m/s? Both balls have the same magnitude of momentum,

$p = mv = (0.50 \text{ kg}) \times (4.0 \text{ m/s}) = (0.10 \text{ kg})(20 \text{ m/s}) = 2.0 \text{ kg} \cdot \text{m/s}$.

However, the two balls have different values of kinetic energy $K = \frac{1}{2}mv^2$: The large, slow-moving ball has $K = 4.0 \text{ J}$, while the small, fast-moving ball has $K = 20 \text{ J}$. Since the momentum is the same for both balls, both require the same *impulse* to be brought to rest. But stopping the 0.10 kg ball with your hand requires five times more *work* than stopping the 0.50 kg ball because the smaller ball has five times more kinetic energy. For a given force that you exert with your hand, it takes the same amount of time (the duration of the catch) to stop either ball, but your hand and arm will be pushed back five times farther if you choose to catch the small, fast-moving ball. To minimize arm strain, you should choose to catch the 0.50 kg ball with its lower kinetic energy.

Both the impulse–momentum and work–energy theorems rest on the foundation of Newton's laws. They are *integral* principles, relating the motion at two different times separated by a finite interval. By contrast, Newton's second law itself (in either of the forms $\sum \vec{\boldsymbol{F}} = m\vec{a}$ or $\sum \vec{\boldsymbol{F}} = d\vec{\boldsymbol{p}}/dt$) is a *differential* principle that concerns the rate of change of velocity or momentum at each instant.

Conceptual Example 8.1 Momentum versus kinetic energy

Consider again the race described in Conceptual Example 6.5 (Section 6.2) between two iceboats on a frictionless frozen lake. The boats have masses m and $2m$, and the wind exerts the same constant horizontal force $\vec{\boldsymbol{F}}$ on each boat (see Fig. 6.14). The boats start from rest and cross the finish line a distance s away. Which boat crosses the finish line with greater momentum?

SOLUTION In Conceptual Example 6.5 we asked how the *kinetic energies* of the boats compare when they cross the finish line. We answered this by remembering that *an object's kinetic energy equals the total work done to accelerate it from rest*. Both boats started from rest, and the total work done was the same for both boats (because the net external force and the displacement were the same for both). Hence both boats had the same kinetic energy at the finish line.

Similarly, to compare the *momenta* of the boats we use the idea that *the momentum of each boat equals the impulse that accelerated it from rest*. As in Conceptual Example 6.5, the net external force on each boat equals the constant horizontal wind force $\vec{F}$. Let Δt be the time a boat takes to reach the finish line, so that the impulse on the boat during that time is $\vec{J} = \vec{F}\Delta t$. Since the boat starts from rest, this equals the boat's momentum $\vec{p}$ at the finish line:

$$\vec{p} = \vec{F}\Delta t$$

Both boats are subjected to the same force $\vec{F}$, but they take different times Δt to reach the finish line. The boat of mass $2m$ accelerates more slowly and takes a longer time to travel the distance s; thus there is a greater impulse on this boat between the starting and finish lines. So the boat of mass $2m$ crosses the finish line with a greater magnitude of momentum than the boat of mass m (but with the same kinetic energy). Can you show that the boat of mass $2m$ has $\sqrt{2}$ times as much momentum at the finish line as the boat of mass m?

KEY CONCEPT

The momentum of an object equals the product of its mass and its velocity $\vec{v}$, and also equals the impulse (net external force multiplied by time) needed to accelerate it from rest to $\vec{v}$.

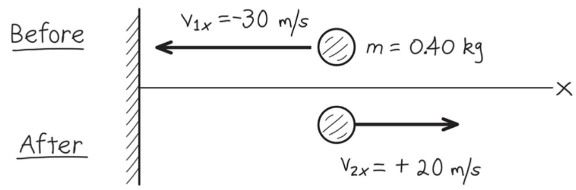
Video Tutor Solution: Example 8.1

Example 8.2 A ball hits a wall

You throw a ball with a mass of 0.40 kg against a brick wall. It is moving horizontally to the left at 30 m/s when it hits the wall; it rebounds horizontally to the right at 20 m/s. (a) Find the impulse of the net external force on the ball during its collision with the wall. (b) If the ball is in contact with the wall for 0.010 s, find the average horizontal force that the wall exerts on the ball during the impact.

IDENTIFY and SET UP We're given enough information to determine the initial and final values of the ball's momentum, so we can use the impulse–momentum theorem to find the impulse. We'll then use the definition of impulse to determine the average force. Figure 8.6 shows our sketch. We need only a single axis because the motion is purely horizontal. We'll take the positive x-direction to be to the right. In part (a) our target variable is the x-component of impulse, J_x, which we'll find by using Eqs. (8.9). In part (b), our target variable is the average x-component of force $(F_{\mathrm{av}})_x$; once we know J_x, we can also find this force by using Eqs. (8.9).

Figure 8.6



Our sketch for this problem.

EXECUTE (a) With our choice of x-axis, the initial and final x-components of momentum of the ball are

$$p_{1x} = mv_{1x} = (0.40 \text{ kg})(-30 \text{ m/s}) = -12 \text{ kg} \cdot \text{m/s}$$
$$p_{2x} = mv_{2x} = (0.40 \text{ kg})(+20 \text{ m/s}) = +8.0 \text{ kg} \cdot \text{m/s}$$

From the x-equation in Eqs. (8.9), the x-component of impulse equals the *change* in the x-momentum:

$$\begin{aligned} J_x &= p_{2x} - p_{1x} \\ &= 8.0 \text{ kg} \cdot \text{m/s} - (-12 \text{ kg} \cdot \text{m/s}) = 20 \text{ kg} \cdot \text{m/s} = 20 \text{ N} \cdot \text{s} \end{aligned}$$

(b) The collision time is $t_2 - t_1 = \Delta t = 0.010$ s. From the x-equation in Eqs. (8.9), $J_x = (F_{\text{av}})_x (t_2 - t_1) = (F_{\text{av}})_x \Delta t$, so

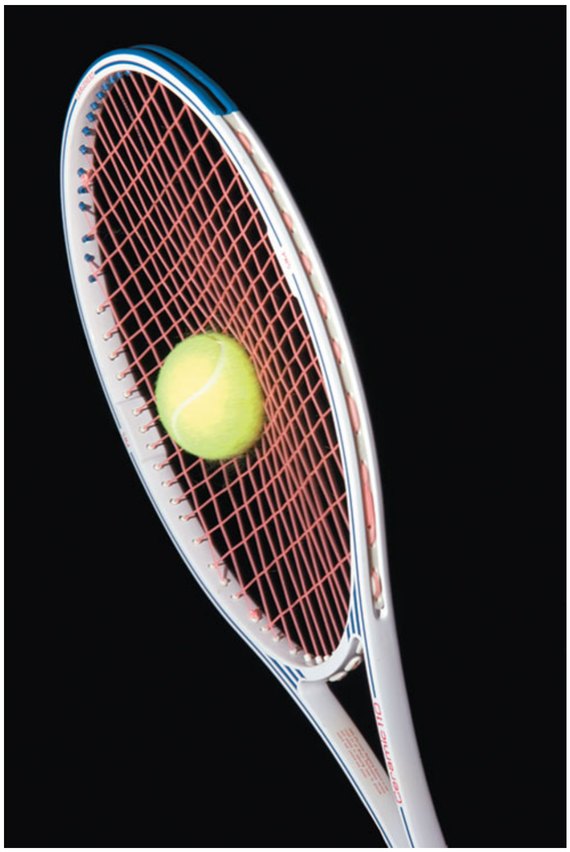
$$(F_{\text{av}})_x = \frac{J_x}{\Delta t} = \frac{20 \text{ N} \cdot \text{s}}{0.010 \text{ s}} = 2000 \text{ N}$$

EVALUATE The x-component of impulse J_x is positive—that is, to the right in Fig. 8.6. The impulse represents the "kick" that the wall imparts to the ball, and this "kick" is certainly to the right.

CAUTION **Momentum is a vector** Because momentum is a vector, we had to include the negative sign in writing $p_{1x} = -12 \text{ kg} \cdot \text{m/s}$. Had we omitted it, we would have calculated the impulse to be $8.0 \text{ kg} \cdot \text{m/s} - (12 \text{ kg} \cdot \text{m/s}) = -4 \text{ kg} \cdot \text{m/s}$. This would say that the wall had somehow given the ball a kick to the *left*! Remember the *direction* of momentum in your calculations.

The force that the wall exerts on the ball must have such a large magnitude (2000 N, equal to the weight of a 200 kg object) to change the ball's momentum in such a short time. Other forces that act on the ball during the collision are comparatively weak; for instance, the gravitational force is only 3.9 N. Thus, during the short time that the collision lasts, we can ignore all other forces on the ball. Figure 8.7 shows the impact of a tennis ball and racket.

Figure 8.7



Typically, a tennis ball is in contact with the racket for approximately 0.01 s. The ball flattens noticeably due to the tremendous force exerted by the racket.

Note that the 2000 N value we calculated is the *average* horizontal force that the wall exerts on the ball during the impact. It corresponds to the horizontal line $(F_{\mathrm{av}})_x$ in Fig. 8.3a. The horizontal force is zero before impact, rises to a maximum, and then decreases to zero when the ball loses contact with the wall. If the ball is relatively rigid, like a baseball or golf ball, the collision lasts a short time and the maximum force is large, as in the blue curve in Fig. 8.3b. If the ball is softer, like a tennis ball, the collision time is longer and the maximum force is less, as in the orange curve in Fig. 8.3b.

KEY CONCEPT

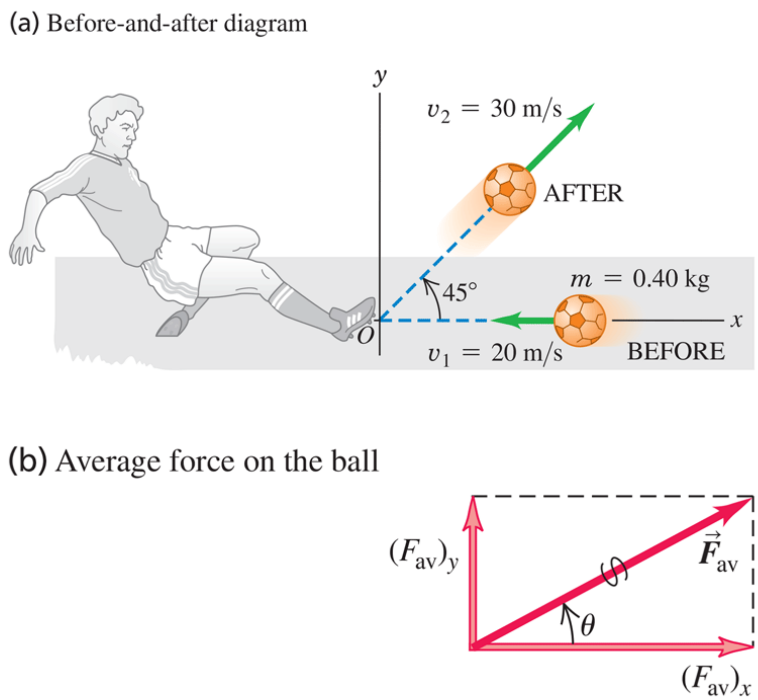
The impulse-momentum theorem states that when a net external force acts on an object, the object's momentum changes by an amount equal to the impulse of the net external force. Both momentum and impulse are vector quantities.

Video Tutor Solution: **Example 8.2**

Example 8.3 Kicking a soccer ball

A soccer ball has a mass of 0.40 kg. Initially it is moving to the left at 20 m/s, but then it is kicked. After the kick it is moving at 45° upward and to the right with speed 30 m/s (Fig. 8.8a). Find the impulse of the net external force and the average net external force, assuming a collision time $\Delta t = 0.010$ s.

Figure 8.8



(a) Kicking a soccer ball. (b) Finding the average force on the ball from its components.

IDENTIFY and SET UP The ball moves in two dimensions, so we must treat momentum and impulse as vector quantities. We take the x-axis to be horizontally to the right and the y-axis to be vertically upward. Our target variables are the components of the net impulse on the ball, J_x and J_y, and the components of the average net external force on the ball, $(F_{av})_x$ and $(F_{av})_y$. We'll find them by using the impulse–momentum theorem in its component form, Eqs. (8.9).

EXECUTE Using $\cos 45^\circ = \sin 45^\circ = 0.707$, we find the ball's velocity components before and after the kick:

$$v_{1x} = -20\ \mathrm{m/s} \quad v_{1y} = 0$$
$$v_{2x} = v_{2x} = (30\,\mathrm{m/s})(0.707) = 21.2\,\mathrm{m/s}$$

From Eqs. (8.9), the impulse components are

$$\begin{aligned} J_x &= p_{2x} - p_{1x} = m\,(v_{2x} - v_{1x}) \\ &= (0.40\,\mathrm{kg})[21.2\,\mathrm{m/s} - (-20\,\mathrm{m/s})] = 16.5\,\mathrm{kg}\cdot\mathrm{m/s} \\ J_y &= p_{2y} - p_{1y} = m(v_{2y} - v_{1y}) \\ &= (0.40\,\mathrm{kg})(21.2\,\mathrm{m/s} - 0) = 8.5\,\mathrm{kg}\cdot\mathrm{m/s} \end{aligned}$$

From Eq. (8.8), the average net external force components are

$$(F_{\mathrm{av}})_x = \frac{J_x}{\Delta t} = 1650\ \mathrm{N} \quad (F_{\mathrm{av}})_y = \frac{J_y}{\Delta t} = 850\ \mathrm{N}$$

The magnitude and direction of the $\vec{F}_{\mathrm{av}}$ vector (Fig. 8.8b) are

$$\begin{aligned} F_{\mathrm{av}} &= \sqrt{(1650\ \mathrm{N})^2 + (850\ \mathrm{N})^2} = 1.9 \times 10^3\ \mathrm{N} \\ \theta &= \arctan \frac{850\,\mathrm{N}}{1650\,\mathrm{N}} = 27^\circ \end{aligned}$$

The ball was not initially at rest, so its final velocity does *not* have the same direction as the average force that acted on it.

EVALUATE $\vec{F}_{\mathrm{av}}$ includes the force of gravity, which is very small; the weight of the ball is only 3.9 N. As in Example 8.2, the average force acting during the collision is exerted almost entirely by the object that the ball hit (in this case, the soccer player’s foot).

KEY CONCEPT

In two-dimensional problems involving impulse and momentum, you must apply the impulse–momentum theorem separately to the x-components and the y-components.

Video Tutor Solution: **Example 8.3**

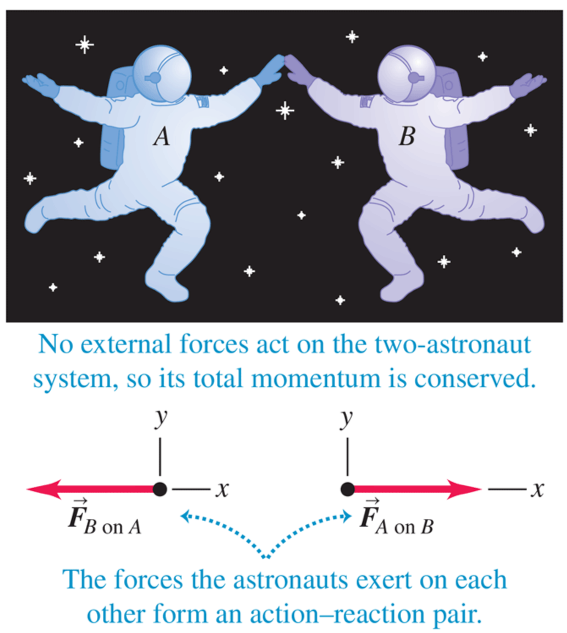
Test Your Understanding of Section 8.1

Rank the following situations according to the magnitude of the impulse of the net external force, from largest value to smallest value. In each situation a 1000 kg automobile is moving along a straight east–west road. The automobile is initially (i) moving east at 25 m/s and comes to a stop in 10 s; (ii) moving east at 25 m/s and comes to a stop in 5 s; (iii) at rest, and a 2000 N net external force toward the east is applied to it for 10 s; (iv) moving east at 25 m/s, and a 2000 N net external force toward the west is applied to it for 10 s; (v) moving east at 25 m/s; over a 30 s period, the automobile reverses direction and ends up moving west at 25 m/s.

8.2 Conservation of Momentum

The concept of momentum is particularly important in situations in which we have two or more objects that *interact*. To see why, let's consider first an idealized system of two objects that interact with each other but not with anything else—for example, two astronauts who touch each other as they float freely in the zero-gravity environment of outer space (Fig. 8.9). Think of the astronauts as particles. Each particle exerts a force on the other; according to Newton's third law, the two forces are always equal in magnitude and opposite in direction. Hence, the *impulses* that act on the two particles are equal in magnitude and opposite in direction, as are the changes in momentum of the two particles.

Figure 8.9



Two astronauts push each other as they float freely in the zero-gravity environment of space.

Let's go over that again with some new terminology. For any system, the forces that the particles of the system exert on each other are called **internal forces.** Forces exerted on any part of the system by some object outside it are called **external forces.** For the system shown in Fig. 8.9, the internal forces are $\vec{F}_{B \text{on A}}$, exerted by particle B on particle A, and $\vec{F}_{A \text{on B}}$, exerted by particle A on particle B. There are *no* external forces; when this is the case, we have an **isolated system.**

The net external force on particle A is $\vec{F}_{B \text{on A}}$ and the net external force on particle B is $\vec{F}_{A \text{on B}}$, so from Eq. (8.4) the rates of change of the momenta of the two particles are

(8.10)

$$\vec{F}_{B \text{on A}} = \frac{d\vec{p}_A}{dt} \quad \vec{F}_{A \text{on B}} = \frac{d\vec{p}_B}{dt}$$

The momentum of each particle changes, but these changes are related to each other by Newton's third law: Forces $\vec{F}_{B \text{on A}}$ and $\vec{F}_{A \text{on B}}$ are always equal in magnitude and opposite in direction. That is, $\vec{F}_{B \text{on A}} = -\vec{F}_{A \text{on B}}$, so $\vec{F}_{B \text{on A}} + \vec{F}_{A \text{on B}} = \mathbf{0}$. Adding together the two equations in Eq. (8.10), we have

(8.11)

$$\vec{F}_{B \text{on A}} + \vec{F}_{A \text{on B}} = \frac{d\vec{p}_A}{dt} + \frac{d\vec{p}_B}{dt} = \frac{d\left(\vec{p}_A + \vec{p}_B\right)}{dt} = \mathbf{0}$$

The rates of change of the two momenta are equal and opposite, so the rate of change of the vector sum $\vec{p}_A + \vec{p}_B$ is zero. We define the **total momentum** $\vec{P}$ of the system of two particles as the vector sum of the momenta of the individual particles; that is,

(8.12)

$$\vec{\boldsymbol{P}} = \vec{\boldsymbol{p}}_A + \vec{\boldsymbol{p}}_B$$

Then Eq. (8.11) becomes

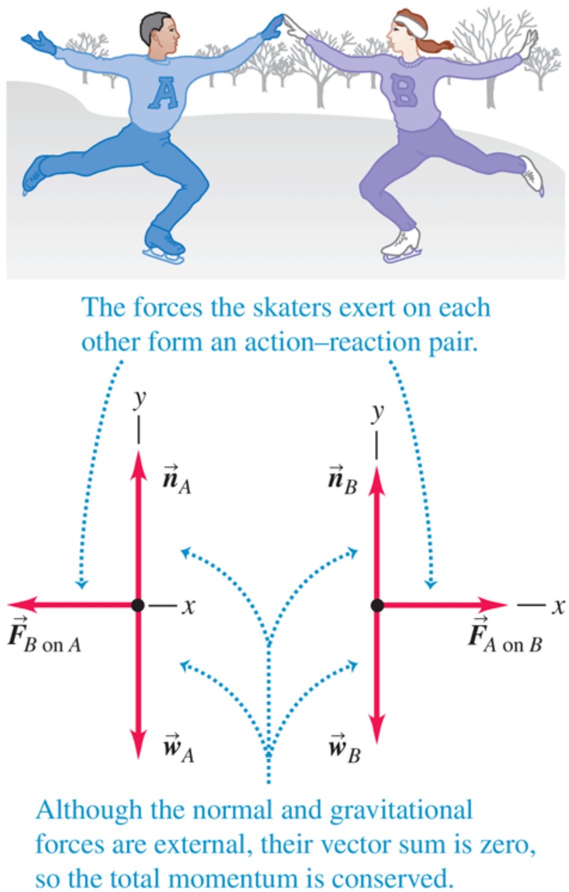
(8.13)

$$\vec{\boldsymbol{F}}_{B \text{on A}} + \vec{\boldsymbol{F}}_{A \text{on B}} = \frac{d\vec{\boldsymbol{P}}}{dt} = \mathbf{0}$$

The time rate of change of the *total* momentum $\vec{\boldsymbol{P}}$ is zero. Hence the total momentum of the system is constant, even though the individual momenta of the particles that make up the system can change.

If external forces are also present, they must be included on the left side of Eq. (8.13) along with the internal forces. Then the total momentum is, in general, not constant. But if the vector sum of the external forces is zero, as in Fig. 8.10, these forces have no effect on the left side of Eq. (8.13), and $d\vec{\boldsymbol{P}}/dt$ is again zero. Thus we have the following general result:

Figure 8.10



Two ice skaters push each other as they skate on a frictionless, horizontal surface. (Compare to Fig. 8.9.)

Conservation of momentum

If the vector sum of external forces on a system is zero, the total momentum of the system is constant.

This is the simplest form of the **principle of conservation of momentum.** This principle is a direct consequence of Newton's third law. What makes this principle useful is that it doesn't depend on the detailed nature of the internal forces that act between members of the system. This means that we can apply conservation of momentum even if (as is often the case) we know very little about the internal forces. We have used Newton's second law to derive this principle, so we have to be careful to use it only in inertial frames of reference.

We can generalize this principle for a system that contains any number of particles A, B, C, … interacting only with one another, with total momentum

(8.14)

Total momentum of a system of particles $A, B, C, \ldots$

$$\vec{P} = \vec{p}_A + \vec{p}_B + \cdots = m_A\vec{v}_A + m_B\vec{v}_B + \cdots \tag{8.14}$$

… equals vector sum of momenta of all particles in the system.

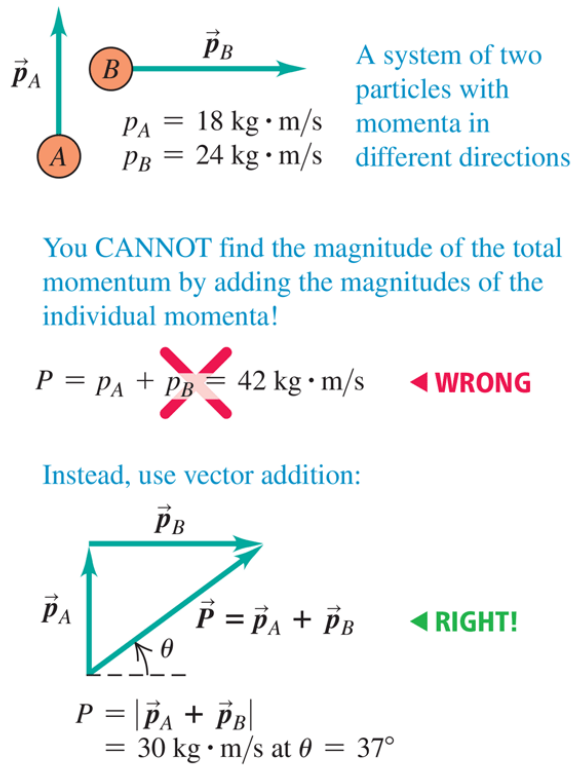
We make the same argument as before: The total rate of change of momentum of the system due to each action–reaction pair of internal forces is zero. Thus the total rate of change of momentum of the entire system is zero whenever the vector sum of the external forces acting on it is zero. The internal forces can change the momenta of individual particles but not the *total* momentum of the system.

CAUTION Conservation of momentum means conservation of its components When you apply the conservation of momentum to a system, remember that momentum is a *vector* quantity. Hence you must use vector addition to compute the total momentum of a system (Fig. 8.11). Using components is usually the simplest method. If p_{Ax} , p_{Ay} , and p_{Az} are the components of momentum of particle A, and similarly for the other particles, then Eq. (8.14) is equivalent to the component equations

(8.15)

$$P_x = p_{Ax} + p_{Bx} + \cdots, \quad P_y = p_{Ay} + p_{By} + \cdots, \quad P_z = p_{Az} + p_{Bz} + \cdots$$

Figure 8.11



When applying conservation of momentum, remember that momentum is a vector quantity!

If the vector sum of the external forces on the system is zero, then P_x, P_y, and P_z are all constant.

In some ways the principle of conservation of momentum is more general than the principle of conservation of total mechanical energy. For example, total mechanical energy is conserved only when the internal forces are *conservative*—that is, when the forces allow two-way conversion between kinetic and potential energies. But conservation of momentum is valid even when the internal forces are *not* conservative. In this chapter we'll analyze situations in which both momentum and total mechanical energy are conserved, and others in which only momentum is conserved. These two principles play a fundamental role in all areas of physics, and we'll encounter them throughout our study of physics.

Problem-Solving Strategy 8.1 Conservation of Momentum

IDENTIFY *the relevant concepts:* Confirm that the vector sum of the external forces acting on the system of particles is zero. If it isn't zero, you can't use conservation of momentum.

SET UP *the problem* using the following steps:

1. Treat each object as a particle. Draw "before" and "after" sketches, including velocity vectors. Assign algebraic symbols to each magnitude, angle, and component. Use letters to label each particle and subscripts 1 and 2 for "before" and "after" quantities. Include any given values.
2. Define a coordinate system and show it in your sketches; define the positive direction for each axis.
3. Identify the target variables.

EXECUTE *the solution:*

4. Write an equation in symbols equating the total initial and final x-components of momentum, using $p_x = mv_x$ for each particle. Write a corresponding equation for the y-components. Components can be positive or negative, so be careful with signs!
5. In some problems, energy considerations (discussed in Section 8.4) give additional equations relating the velocities.
6. Solve your equations to find the target variables.

EVALUATE *your answer:* Does your answer make physical sense? If your target variable is a certain object's momentum, check that the direction of the momentum is reasonable.

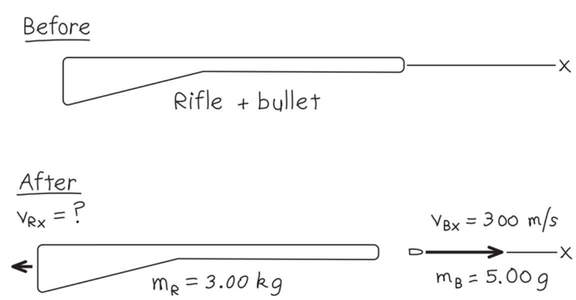
Example 8.4 Recoil of a rifle

WITH VARIATION PROBLEMS

A marksman holds a rifle of mass $m_{\mathrm{R}} = 3.00\mathrm{kg}$ loosely, so it can recoil freely. He fires a bullet of mass $m_{\mathrm{B}} = 5.00\mathrm{g}$ horizontally with a velocity relative to the ground of $v_{\mathrm{B}x} = 300\mathrm{m/s}$. What is the recoil velocity $v_{\mathrm{R}x}$ of the rifle? What are the final momentum and kinetic energy of the bullet and rifle?

IDENTIFY and SET UP If the marksman exerts negligible horizontal forces on the rifle, then there is no net horizontal force on the system (the bullet and rifle) during the firing, and the total horizontal momentum of the system is conserved. Figure 8.12 shows our sketch. We take the positive x-axis in the direction of aim. The rifle and the bullet are initially at rest, so the initial x-component of total momentum is zero. After the shot is fired, the bullet's x-momentum is $p_{\mathrm{B}x} = m_{\mathrm{B}} v_{Bx}$ and the rifle's x-momentum is $p_{\mathrm{R}x} = m_{\mathrm{R}} v_{Rx}$. Our target variables are $v_{\mathrm{R}x}$, $p_{\mathrm{B}x}$, $p_{\mathrm{R}x}$, and the final kinetic energies $K_{\mathrm{B}} = \frac{1}{2} m_{\mathrm{B}} v_{Bx}{}^2$ and $K_{\mathrm{R}} = \frac{1}{2} m_{\mathrm{R}} v_{Rx}{}^2$.

Figure 8.12



Our sketch for this problem.

EXECUTE Conservation of the x-component of total momentum gives

$$\begin{aligned} P_x &= 0 = m_{\mathrm{B}} v_{Bx} + m_{\mathrm{R}} v_{Rx} \\ v_{Rx} &= -\frac{m_B}{m_R} v_{Bx} = -\left(\frac{0.00500\,\mathrm{kg}}{3.00\,\mathrm{kg}}\right)(300\ \mathrm{m/s}) = -0.500\,\mathrm{m/s} \end{aligned}$$

The negative sign means that the recoil is in the direction opposite to that of the bullet.

The final momenta and kinetic energies are

$$\begin{aligned} p_{\mathrm{B}x} &= m_{\mathrm{B}} v_{Bx} = (0.00500\mathrm{kg})(300\mathrm{m/s}) = 1.50\mathrm{kg}\cdot\mathrm{m/s} \\ K_{\mathrm{B}} &= \frac{1}{2} m_{\mathrm{B}} {v_{Bx}}^2 = \frac{1}{2}(0.00500\mathrm{kg})(300\mathrm{m/s})^2 = 225\mathrm{J} \\ p_{\mathrm{R}x} &= m_{\mathrm{R}} v_{Rx} = (3.00\mathrm{kg})(-0.500\mathrm{m/s}) = -1.50\mathrm{kg}\cdot\mathrm{m/s} \\ K_{\mathrm{R}} &= \frac{1}{2} m_{\mathrm{R}} {v_{Rx}}^2 = \frac{1}{2}(3.00\mathrm{kg})(-0.500\mathrm{m/s})^2 = 0.375\mathrm{J} \end{aligned}$$

EVALUATE The bullet and rifle have equal and opposite final *momenta* thanks to Newton's third law: They experience equal and opposite interaction forces that act for the same *time,* so the impulses are equal and opposite. But the bullet travels a much greater *distance* than the rifle during the interaction. Hence the force on the bullet does more work than the force on the rifle, giving the bullet much greater *kinetic energy* than the rifle. The 600:1 ratio of the two kinetic energies is the inverse of the ratio of the masses; in fact, you can show that this always happens in recoil situations. (See Exercise 8.26 for an application of these ideas to the recoil of atomic nuclei from a fission reaction.)

KEY CONCEPT

The momentum of a system is conserved if no net external force acts on the system. If the momentum of one part of the system changes by $\Delta\vec{p}$, the momentum of the other parts of the system changes by $-\Delta\vec{p}$ so that the total momentum remains the same.

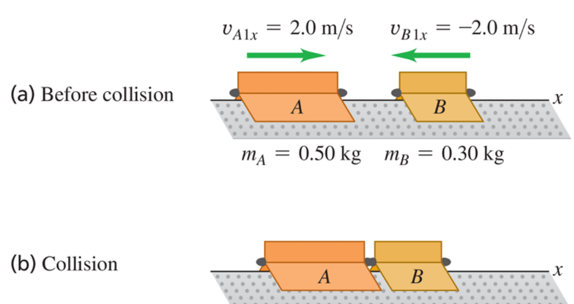
Video Tutor Solution: **Example 8.4**

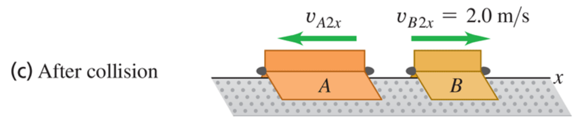
Example 8.5 Collision along a straight line

WITH VARIATION PROBLEMS

Two gliders with different masses move toward each other on a frictionless air track (Fig. 8.13a). After they collide (Fig. 8.13b), glider B has a final velocity of $+2.0\text{m/s}$ (Fig. 8.13c). What is the final velocity of glider A? How do the changes in momentum and in velocity compare?

Figure 8.13





Two gliders colliding on an air track.

IDENTIFY and SET UP As for the skaters in Fig. 8.10, the total vertical force on each glider is zero, and the net external force on each individual glider is the horizontal force exerted on it by the other glider. The net external force on the *system* of two gliders is zero, so their total momentum is conserved. We take the positive x-axis to be to the right. We are given the masses and initial velocities of both gliders and the final velocity of glider B. Our target variables are v_{A2x} (the final x-component of velocity of glider A) and the changes in momentum and in velocity of the two gliders (the value *after* the collision minus the value *before* the collision).

EXECUTE The x-component of total momentum before the collision is

$$\begin{aligned} P_x &= m_A v_{A1x} + m_B v_{B1x} \\ &= (0.50\,\mathrm{kg})(2.0\,\mathrm{m/s}) + (0.30\,\mathrm{kg})(-2.0\,\mathrm{m/s}) = 0.40\,\mathrm{kg\cdot m/s} \end{aligned}$$

This is positive (to the right in Fig. 8.13) because A has a greater magnitude of momentum than B. The x-component of total momentum has the same value after the collision, so

$$P_x = m_A v_{A2x} + m_B v_{B2x}$$

We solve for v_{A2x}:

$$\begin{aligned} v_{A2x} &= \frac{P_x - m_B v_{B2x}}{m_A} = \frac{0.40\mathrm{kg\cdot m/s} - (0.30\mathrm{kg})(2.0\mathrm{m/s})}{0.50\mathrm{kg}} \\ &= -0.40\,\mathrm{m/s} \end{aligned}$$

The changes in the x-momenta are

$$
\begin{aligned}
m_A u_{A2x} - m_A u_{A1x} &= (0.50\text{kg})(-0.40\text{m/s}) - (0.50\text{kg})(2.0\,\text{m/s}) \\
&= -1.2\,\text{kg}\cdot\text{m/s} \\
m_B u_{B2x} - m_B u_{B1x} &= (0.30\,\text{kg})(2.0\,\text{m/s}) - (0.30\,\text{kg})(-2.0\,\text{m/s}) \\
&= +1.2\,\text{kg}\cdot\text{m/s}
\end{aligned}
$$

The changes in x-velocities are

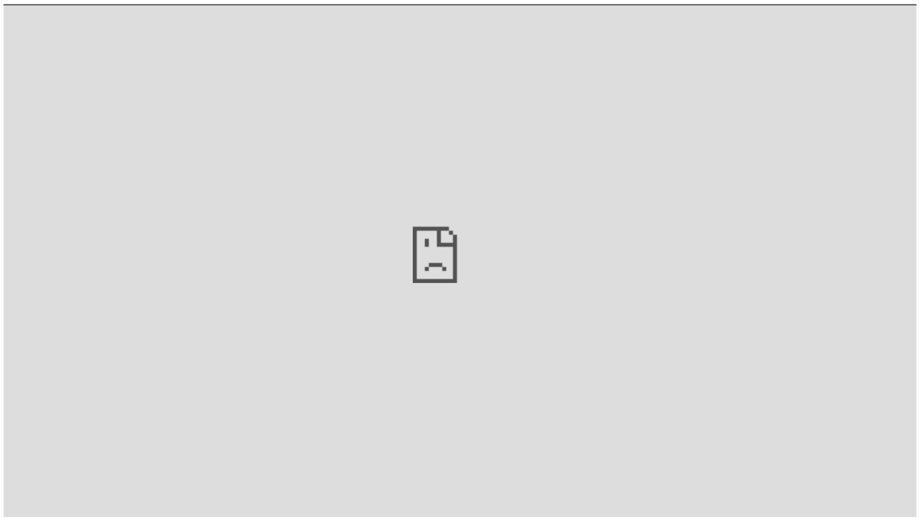
$$
\begin{aligned}
v_{A2x} - v_{A1x} &= (-0.40\text{m/s}) - 2.0\text{m/s} = -2.4\text{m/s} \\
v_{B2x} - v_{B1x} &= 2.0\,\text{m/s} - (-2.0\text{m/s} = +4.0\text{m/s}
\end{aligned}
$$

EVALUATE The gliders were subjected to equal and opposite interaction forces for the same time during their collision. By the impulse–momentum theorem, they experienced equal and opposite impulses and therefore equal and opposite changes in momentum. But by Newton's second law, the less massive glider (B) had a greater magnitude of acceleration and hence a greater velocity change.

KEY CONCEPT

In any collision, momentum is conserved: The total momentum of the colliding objects has the same value just after the collision as just before the collision.

Video Tutor Solution: **Example 8.5**

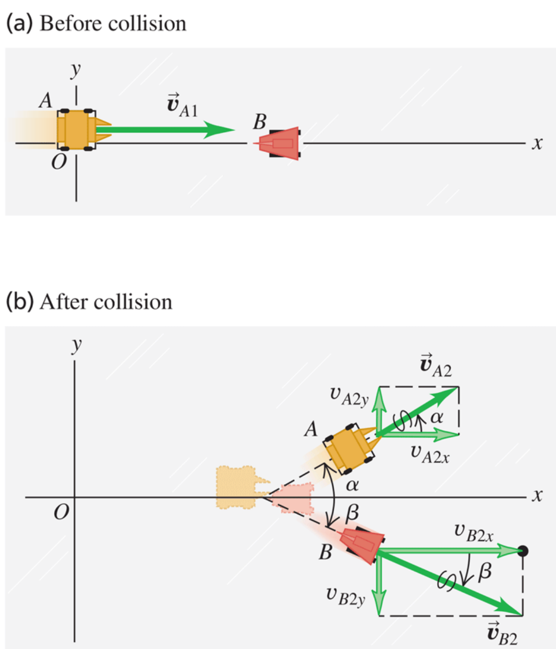


Example 8.6 Collision in a horizontal plane

WITH VARIATION PROBLEMS

Figure 8.14a shows two battling robots on a frictionless surface. Robot A, with mass 20 kg, initially moves at $2.0 \mathrm{m/s}$ parallel to the x-axis. It collides with robot B, which has mass 12 kg and is initially at rest. After the collision, robot A moves at $1.0 \mathrm{m/s}$ in a direction that makes an angle $\alpha = 30^\circ$ with its initial direction (Figure 8.14b). What is the final velocity of robot B?

Figure 8.14



Views from above of the robot velocities.

IDENTIFY and SET UP There are no horizontal external forces, so the x- and y-components of the total momentum of the system are conserved. Hence the sum of the x-components of momentum *before* the collision (subscript 1) must equal the sum *after* the collision (subscript 2), and similarly for the sums of the y-components. Our target variable is $\vec{v}_{B2}$, the final velocity of robot B.

EXECUTE The momentum-conservation equations and their solutions for v_{B2x} and v_{B2y} are

$$\begin{aligned}
m_A v_{A1x} + m_B v_{B1x} &= m_A v_{A2x} + m_B v_{B2x} \\
v_{B2x} &= \frac{m_A v_{A1x} + m_B v_{B1x} - m_A v_{A2x}}{m_B} \\
&= \frac{\begin{bmatrix}(20\,\mathrm{kg})(2.0\,\mathrm{m/s}) + (12\,\mathrm{kg})(0) \\ -(20\,\mathrm{kg})(1.0\,\mathrm{m/s})(\cos 30^\circ)\end{bmatrix}}{12\,\mathrm{kg}} = 1.89\,\mathrm{m/s} \\
m_A v_{A1y} + m_B v_{B1y} &= m_A v_{A2y} + m_B v_{B2y} \\
v_{B2y} &= \frac{m_A v_{A1y} + m_B v_{B1y} - m_A v_{A2y}}{m_B} \\
&= \frac{\begin{bmatrix}(20\,\mathrm{kg})(0) + (12\,\mathrm{kg})(0) \\ -(12\,\mathrm{kg})(1.0\,\mathrm{m/s})(\sin 30^\circ)\end{bmatrix}}{12\,\mathrm{kg}} = -0.83\,\mathrm{m/s}
\end{aligned}$$

Figure 8.14b shows the motion of robot B after the collision. The magnitude of $\vec{\boldsymbol{v}}_{B2}$ is

$$v_{B2} = \sqrt{(1.89\mathrm{m/s})^2 + (-0.83\mathrm{m/s})^2} = 2.1\mathrm{m/s}$$

and the angle of its direction from the positive x-axis is

$$\beta = \arctan\frac{-0.83\mathrm{m/s}}{1.89\mathrm{m/s}} = -24^\circ$$

EVALUATE Let's confirm that the components of total momentum before and after the collision are equal. Initially robot A has x-momentum $m_A v_{A1x} = (20\mathrm{kg})(2.0\mathrm{m/s}) = 40\mathrm{kg}\cdot\mathrm{m/s}$ and zero y-momentum; robot B has zero momentum. Afterward, the momentum components are $m_A v_{A2x} = (20\mathrm{kg})(1.0\mathrm{m/s})(\cos 30^\circ) = 17\mathrm{kg}\cdot\mathrm{m/s}$ and $m_B v_{B2x} = (12\mathrm{kg})(1.89\mathrm{m/s}) = 23\mathrm{kg{\cdot}m/s}$; the total x-momentum is 40kg·m/s, the same as before the collision. The final y-components are

$m_A v_{A2y} = (20\text{kg})(1.0\text{m/s})(\sin 30^\circ) = 10\text{kg} \cdot \text{m/s}$ and $m_B v_{B2y} = (12\text{kg})(-0.83\text{m/s}) = -10\text{kg·m/s}$; the total y-component of momentum is zero, as before the collision.

KEY CONCEPT

In problems that involve a two-dimensional collision, write separate conservation equations for the x-component and y-component of total momentum.

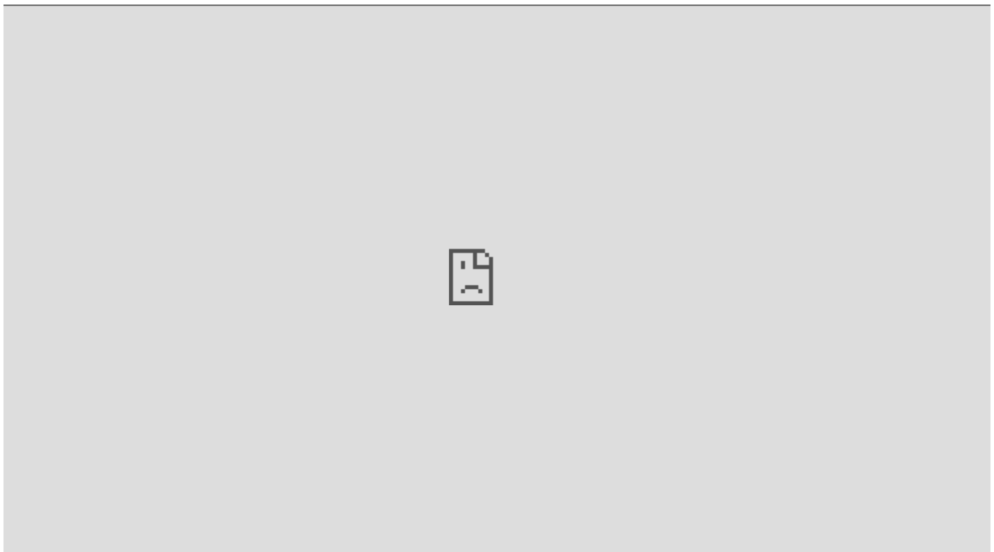
Video Tutor Solution: **Example 8.6**

Test Your Understanding of Section 8.2

A spring-loaded toy sits at rest on a horizontal, frictionless surface. When the spring releases, the toy breaks into equal-mass pieces A, B, and C, which slide along the surface. Piece A moves off in the negative x-direction, while piece B moves off in the negative y-direction. (a) What are the signs of the velocity components of piece C? (b) Which of the three pieces is moving the fastest?

8.3 Momentum Conservation and Collisions

Video Tutor Demo: **Happy/Sad Pendulums**



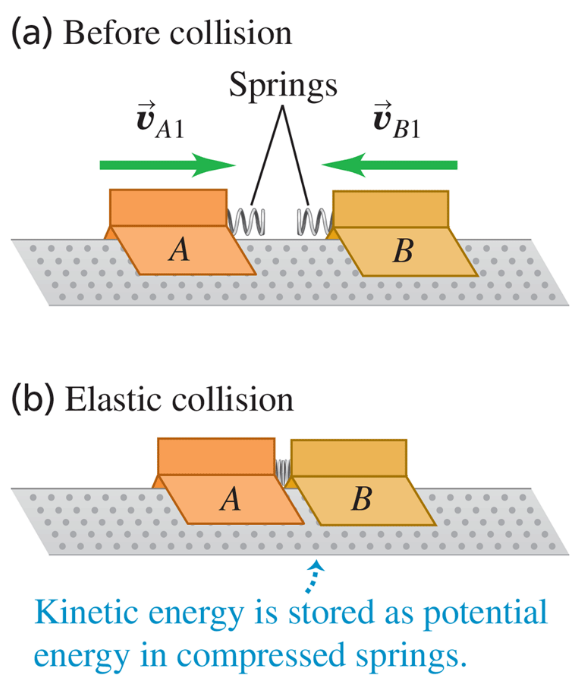
To most people the term "collision" is likely to mean some sort of automotive disaster. We'll broaden the meaning to include any strong interaction between objects that lasts a relatively short time. So we include not only car accidents but also balls colliding on a billiard table, neutrons hitting atomic nuclei in a nuclear reactor, and a close encounter of a spacecraft with the planet Saturn.

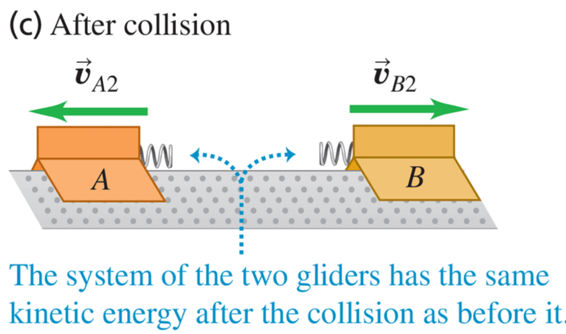
If the forces between the colliding objects are much larger than any external forces, as is the case in most collisions, we can ignore the external forces and treat the objects as an *isolated* system. Then momentum is conserved and the total momentum of the system has the same value before and after the collision. Two cars colliding at an icy intersection provide a good example. Even two cars colliding on dry pavement can be treated as an isolated system during the collision if the forces between the cars are much larger than the friction forces of pavement against tires.

Elastic and Inelastic Collisions

If the forces between the objects are also *conservative*, so no mechanical energy is lost or gained in the collision, the total *kinetic* energy of the system is the same after the collision as before. Such a collision is called an **elastic collision.** A collision between two marbles or two billiard balls is almost completely elastic. Figure 8.15 shows a model for an elastic collision. When the gliders collide, their springs are momentarily compressed and some of the original kinetic energy is momentarily converted to elastic potential energy. Then the gliders bounce apart, the springs expand, and this potential energy is converted back to kinetic energy.

Figure 8.15

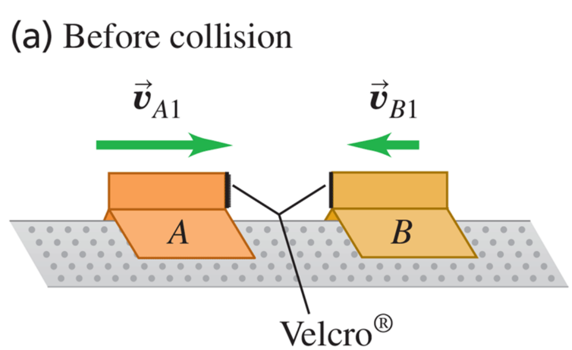


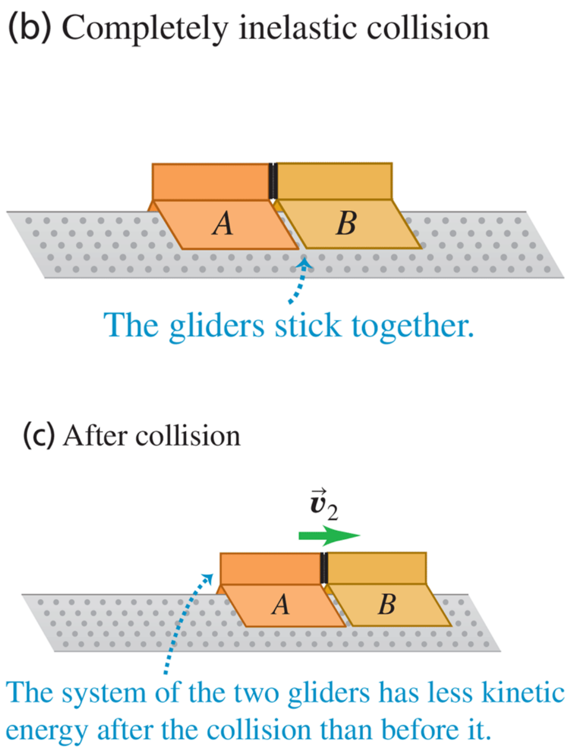


Two gliders undergoing an elastic collision on a frictionless surface. Each glider has a steel spring bumper that exerts a conservative force on the other glider.

A collision in which the total kinetic energy after the collision is *less* than before the collision is called an **inelastic collision.** A meatball landing on a plate of spaghetti and a bullet embedding itself in a block of wood are examples of inelastic collisions. An inelastic collision in which the colliding objects stick together and move as one object after the collision is called a **completely inelastic collision.** Figure 8.16 shows an example; we have replaced the spring bumpers in Fig. 8.15 with Velcro®, which sticks the two objects together.

Figure 8.16





Two gliders undergoing a completely inelastic collision. The spring bumpers on the gliders are replaced by Velcro, so the gliders stick together after collision.

CAUTION An inelastic collision doesn't have to be completely inelastic Inelastic collisions include many situations in which the objects do *not* stick. If two cars bounce off each other in a "fender bender," the work done to deform the fenders cannot be recovered as kinetic energy of the cars, so the collision is inelastic (Fig. 8.17).

Figure 8.17

Cars are designed so that collisions are inelastic—the structure of the car absorbs as much of the energy of the collision as possible. This absorbed energy cannot be recovered, since it goes into a permanent deformation of the car.

Remember this rule: **In any collision in which external forces can be ignored, momentum is conserved and the total momentum before equals the total momentum after; in elastic collisions *only*, the total kinetic energy before equals the total kinetic energy after.**

Completely Inelastic Collisions

Let's look at what happens to momentum and kinetic energy in a *completely* inelastic collision of two objects (A and B), as in Fig. 8.16. Because the two objects stick together after the collision, they have the same final velocity $\vec{\boldsymbol{v}}_2$:

$$\vec{\boldsymbol{v}}_{A2} = \vec{\boldsymbol{v}}_{B2} = \vec{\boldsymbol{v}}_2$$

Conservation of momentum gives the relationship

(8.16)

$$m_A \vec{\boldsymbol{v}}_{A1} + m_B \vec{\boldsymbol{v}}_{B1} = (m_A + m_B)\vec{\boldsymbol{v}}_2 \quad \text{(completely inelastic collision)}$$

If we know the masses and initial velocities, we can compute the common final velocity $\vec{v}_2$.

Suppose, for example, that an object with mass m_A and initial x-component of velocity v_{A1x} collides inelastically with an object with mass m_B that is initially at rest ($v_{B1x} = 0$). From Eq. (8.16) the common x-component of velocity v_{2x} of both objects after the collision is

(8.17)

$$v_{2x} = \frac{m_A}{m_A + m_B} v_{A1x} \quad \begin{array}{l}\text{(completely inelastic collision,}\\ B\,\text{initially at rest)}\end{array}$$

Let's verify that the total kinetic energy after this completely inelastic collision is less than before the collision. The motion is purely along the x-axis, so the kinetic energies K_1 and K_2 before and after the collision, respectively, are

$$\begin{aligned} K_1 &= \frac{1}{2} m_A {v_{A1x}}^2 \\ K_2 &= \frac{1}{2}(m_A + m_B){v_{2x}}^2 = \frac{1}{2}(m_A + m_B)\left(\frac{m_A}{m_A + m_B}\right)^2 {v_{A1x}}^2 \end{aligned}$$

The ratio of final to initial kinetic energy is

(8.18)

$$\frac{K_2}{K_1} = \frac{m_A}{m_A + m_B} \quad \begin{array}{l}\text{(completely inelastic collision,}\\ B\,\text{initially at rest)}\end{array}$$

The right side is always less than unity because the denominator is always greater than the numerator. Even when the initial velocity of m_B is not zero, the kinetic energy after a completely inelastic collision is always less than before.

PLEASE NOTE: Don't memorize Eq. (8.17) or (8.18)! We derived them only to prove that kinetic energy is always lost in a completely inelastic collision.

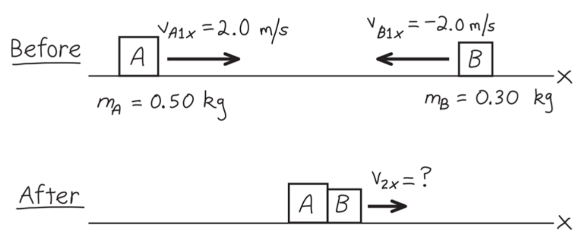
Example 8.7 A completely inelastic collision

WITH VARIATION PROBLEMS

We repeat the collision described in Example 8.5 (Section 8.2), but this time equip the gliders so that they stick together when they collide. Find the common final x-velocity, and compare the initial and final kinetic energies of the system.

IDENTIFY and SET UP There are no external forces in the x-direction, so the x-component of momentum is conserved. Figure 8.18 shows our sketch. Our target variables are the final x-velocity, v_{2x}, and the initial and final kinetic energies, K_1 and K_2.

Figure 8.18



Our sketch for this problem.

EXECUTE From conservation of momentum,

$$\begin{aligned} m_A v_{A1x} + m_B v_{B1x} &= (m_A + m_B) v_{2x} \\ v_{2x} &= \frac{m_A v_{A1x} + m_B v_{B1x}}{m_A + m_B} \\ &= \frac{(0.50\,\text{kg})(2.0\,\text{m/s}) + (0.30\,\text{kg})(-2.0\,\text{m/s})}{0.50\,\text{kg} + 0.3\,\text{kg}} \\ &= 0.50\,\text{m/s} \end{aligned}$$

Because v_{2x} is positive, the gliders move together to the right after the collision. Before the collision, the kinetic energies are

$$\begin{aligned} K_A &= \frac{1}{2}m_A {v_{A1x}}^2 = \frac{1}{2}(0.50 \text{ kg}) + (2.0 \text{ m/s})^2 = 1.0 \text{ J} \\ K_B &= \frac{1}{2}m_B {v_{B1x}}^2 = \frac{1}{2}(0.30 \text{ kg}) + (-2.0 \text{ m/s})^2 = 0.60 \text{ J} \end{aligned}$$

The total kinetic energy before the collision is $K_1 = K_A + K_B = 1.6$ J. The kinetic energy after the collision is

$$\begin{aligned} K_2 &= \frac{1}{2}(m_A + m_B){v_{2x}}^2 = \frac{1}{2}(0.50 \text{ kg} + 0.30 \text{ kg})(0.50 \text{ m/s})^2 \\ &= 0.10 \text{ J} \end{aligned}$$

EVALUATE The final kinetic energy is only $\frac{1}{16}$ of the original; $\frac{15}{16}$ of the total mechanical energy is converted to other forms. If there is a wad of chewing gum between the gliders, it squashes and becomes warmer. If there is a spring between the gliders that is compressed as they lock together, the energy is stored as potential energy of the spring. In both cases the *total* energy of the system is conserved, although *kinetic* energy is not. In an isolated system, however, momentum is *always* conserved whether the collision is elastic or not.

KEY CONCEPT

In a completely inelastic collision, the colliding objects come together and stick. Momentum is conserved, but kinetic energy is not.

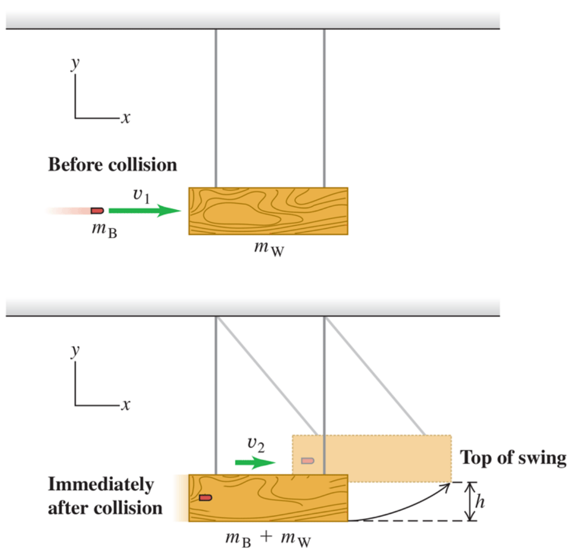
Video Tutor Solution: **Example 8.7**

Example 8.8 The ballistic pendulum

WITH VARIATION PROBLEMS

Figure 8.19 shows a ballistic pendulum, a simple system for measuring the speed of a bullet. A bullet of mass m_B makes a completely inelastic collision with a block of wood of mass m_W, which is suspended like a pendulum. After the impact, the block swings up to a maximum height h. In terms of h, m_B, and m_W, what is the initial speed v_1 of the bullet?

Figure 8.19



A ballistic pendulum.

IDENTIFY We'll analyze this event in two stages: (1) the bullet embeds itself in the block, and (2) the block swings upward. The first stage happens so quickly that the block does not move appreciably. The supporting strings remain nearly vertical, so negligible external horizontal force acts on the bullet–block system, and the horizontal component of momentum is conserved. Total mechanical energy is *not* conserved during this stage, however, because a nonconservative force does work (the force of friction between bullet and block).

In the second stage, the block and bullet move together. The only forces acting on this system are gravity (a conservative force) and the string tensions (which do no work). Thus, as the block swings, *total mechanical energy* is conserved. Momentum is *not* conserved during this stage, however, because there is a net external force (the forces of gravity and string tension don't cancel when the strings are inclined).

SET UP We take the positive x-axis to the right and the positive y-axis upward. Our target variable is v_1. Another unknown quantity is the speed v_2 of the system just after the collision. We'll use momentum conservation in the first stage to relate v_1 to v_2, and we'll use energy conservation in the second stage to relate v_2 to h.

EXECUTE In the first stage, all velocities are in the $+x$-direction. Momentum conservation gives

$$\begin{aligned} m_{\mathrm{B}} v_1 &= (m_{\mathrm{B}} + m_{\mathrm{W}2}) v_2 \\ v_1 &= \frac{m_B + m_W}{m_B} v_2 \end{aligned}$$

At the beginning of the second stage, the system has kinetic energy $K = \frac{1}{2}(m_{\mathrm{B}} + m_{\mathrm{W}})v_2^2$. The system swings up and comes to rest for an instant at a height h, where its kinetic energy is zero and the potential energy is $(m_{\mathrm{B}}+m_{\mathrm{W}})gh$; it then swings back down. Energy conservation gives

$$\begin{aligned}\tfrac{1}{2}(m_{\mathrm{B}}+m_{\mathrm{W}})v_2^2 &= (m_{\mathrm{B}}+m_{\mathrm{W}})gh \\ v_2 &= \sqrt{2gh}\end{aligned}$$

We substitute this expression for v_2 into the momentum equation:

$$v_1=\frac{m_{\mathrm{B}}+m_{\mathrm{W}}}{m_{\mathrm{B}}}\sqrt{2gh}$$

EVALUATE Let's plug in realistic numbers: $m_{\mathrm{B}}=5.00\ \mathrm{g}=0.00500\ \mathrm{kg}$, $m_{\mathrm{W}}=2.00\ \mathrm{kg}$, and $h=3.00\ \mathrm{cm}=0.0300\ \mathrm{m}$:

$$v_1=\frac{0.00500\ \mathrm{kg}+2.00\ \mathrm{kg}}{0.00500\ \mathrm{kg}}\sqrt{2(9.80\ \mathrm{m/s^2})(0.0300\ \mathrm{m})}=307\ \mathrm{m/s}$$

$$v_2=\sqrt{2gh}=\sqrt{2(9.80\,\mathrm{m/s^2})(0.0300\,\mathrm{m})}=0.767\,\mathrm{m/s}$$

The speed v_2 of the block after impact is *much* lower than the initial speed v_1 of the bullet. The kinetic energy of the bullet before impact is $\dfrac{1}{2}(0.00500\ \mathrm{kg})(307\ \mathrm{m/s})^2=236\ \mathrm{J}$. Just after impact the kinetic energy of the system is $\dfrac{1}{2}(2.005\ \mathrm{kg})(0.767\ \mathrm{m/s})^2=0.589\ \mathrm{J}$. Nearly all the kinetic energy disappears as the wood splinters and the bullet and block become warmer.

KEY CONCEPT

Conservation of momentum holds true only when the net external force is zero. In some situations momentum is conserved during part of the motion (such as during a collision) but not during other parts.

Video Tutor Solution: **Example 8.8**

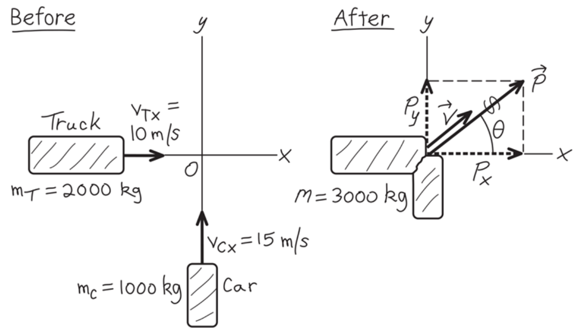
Example 8.9 An automobile collision

WITH VARIATION PROBLEMS

A 1000 kg car traveling north at 15 m/s collides with a 2000 kg truck traveling east at 10 m/s. The occupants, wearing seat belts, are uninjured, but the two vehicles move away from the impact point as one. The insurance adjustor asks you to find the velocity of the wreckage just after impact. What is your answer?

IDENTIFY and SET UP Any horizontal external forces (such as friction) on the vehicles during the collision are very small compared with the forces that the colliding vehicles exert on each other. (We'll verify this below.) So we can treat the cars as an isolated system, and the momentum of the system is conserved. Figure 8.20 shows our sketch and the x- and y-axes. We can use Eqs. (8.15) to find the total momentum $\vec{P}$ before the collision. The momentum has the same value just after the collision; hence we can find the velocity $\vec{V}$ just after the collision (our target variable) by using $\vec{P} = M\vec{V}$, where $M = m_{\mathrm{C}} + m_{\mathrm{T}} = 3000$ kg is the mass of the wreckage.

Figure 8.20



Our sketch for this problem.

EXECUTE From Eqs. (8.15), the components of $\vec{\boldsymbol{P}}$ are

$$\begin{aligned} P_x &= p_{\mathrm{C}x} + p_{\mathrm{T}x} = m_{\mathrm{C}} v_{Cx} + m_{\mathrm{T}} v_{Tx} \\ &= (1000\,\mathrm{kg})(0) + (2000\,\mathrm{kg})(10\,\mathrm{m/s}) = 2.0 \times 10^4 \mathrm{kg \cdot m/s} \\ P_y &= p_{\mathrm{C}y} + p_{\mathrm{T}y} = m_{\mathrm{C}} v_{\mathrm{C}y} + m_{\mathrm{T}} v_{\mathrm{T}y} \\ &= (1000\,\mathrm{kg})(15\,\mathrm{m/s}) + (2000\,\mathrm{kg})(0) = 1.5 \times 10^4 \mathrm{kg \cdot m/s} \end{aligned}$$

The magnitude of $\vec{\boldsymbol{P}}$ is

$$\begin{aligned} P &= \sqrt{\left(2.0 \times 10^4 \ \mathrm{kg \cdot m/s}\right)^2 + \left(1.5 \times 10^4 \ \mathrm{kg \cdot m/s}\right)^2} \\ &= 2.5 \times 10^4 \mathrm{kg \cdot m/s} \end{aligned}$$

and its direction is given by the angle θ shown in Fig. 8.20:

$$\tan \theta = \frac{P_y}{P_x} = \frac{1.5 \times 10^4 \ \mathrm{kg \cdot m/s}}{2.0 \times 10^4 \ \mathrm{kg \cdot m/s}} = 0.75 \quad \theta = 37^\circ$$

From $\vec{\boldsymbol{P}} = M\vec{\boldsymbol{V}}$, the direction of the velocity $\vec{\boldsymbol{V}}$ just after the collision is also $\theta = 37^\circ$. The velocity magnitude is

$$V = \frac{P}{M} = \frac{2.5 \times 10^4 \ \mathrm{kg{\cdot}m/s}}{3000 \ \mathrm{kg}} = 8.3 \ \mathrm{m/s}$$

EVALUATE As you can show, the initial kinetic energy is 2.1×10^5 J and the final value is 1.0×10^5 J. In this inelastic collision, the total kinetic energy is less after the collision than before.

We can now justify our neglect of the external forces on the vehicles during the collision. The car's weight is about 10,000 N; if the coefficient of kinetic friction is 0.5, the friction force on the car during the impact is about 5000 N. The car's initial kinetic energy is $\frac{1}{2}(1000\ \mathrm{kg})(15\ \mathrm{m/s})^2 = 1.1 \times 10^5\ \mathrm{J}$, so $-1.1 \times 10^5\ \mathrm{J}$ of work must be done to stop it. If the car crumples by 0.20 m in stopping, a force of magnitude $(1.1 \times 10^5\ \mathrm{J})/(0.20\ \mathrm{m}) = 5.5 \times 10^5 \mathrm{N}$ would be needed; that's 110 times the friction force. So it's reasonable to treat the external force of friction as negligible compared with the internal forces the vehicles exert on each other.

KEY CONCEPT

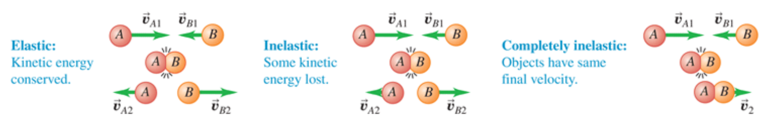
You can use conservation of momentum in collision problems even though external forces act on the system. That's because the external forces are typically small compared to the internal forces that the colliding objects exert on each other.

Video Tutor Solution: Example 8.9

Classifying Collisions

It’s important to remember that we can classify collisions according to energy considerations (Fig. 8.21). A collision in which kinetic energy is conserved is called *elastic.* (We’ll explore this type in more depth in the next section.) A collision in which the total kinetic energy decreases is called *inelastic.* When the two objects have a common final velocity, we say that the collision is *completely inelastic.* There are also cases in which the final kinetic energy is *greater* than the initial value. Rifle recoil, discussed in Example 8.4 (Section 8.2), is an example.

Figure 8.21



Collisions are classified according to energy considerations.

Finally, we emphasize again that we can typically use momentum conservation for collisions even when external forces are acting on the system. That’s because the net external force acting on the colliding objects is typically small in comparison with the internal forces during the collision (as in Example 8.9).

Test Your Understanding of Section 8.3

For each situation, state whether the collision is elastic or inelastic. If it is inelastic, state whether it is completely inelastic. (a) You drop a ball from your hand. It collides with the floor and bounces back up so that it just reaches your hand. (b) You drop a different ball from your hand and let it collide with the ground. This ball bounces back up to half the height from which it was dropped. (c) You drop a ball of clay from your hand. When it collides with the ground, it stops.

8.4 Elastic Collisions

We saw in Section 8.3 that an *elastic collision* in an isolated system is one in which kinetic energy (as well as momentum) is conserved. Elastic collisions occur when the forces between the colliding objects are *conservative*. When two billiard balls collide, they squash a little near the surface of contact, but then they spring back. Some of the kinetic energy is stored temporarily as elastic potential energy, but at the end it is reconverted to kinetic energy (Fig. 8.22).

Figure 8.22

Billiard balls deform very little when they collide, and they quickly spring back from any deformation they do undergo. Hence the force of interaction between the balls is almost perfectly conservative, and the collision is almost perfectly elastic.

Let's look at a *one-dimensional* elastic collision between two objects A and B, in which all the velocities lie along the same line. We call this line the x-axis, so each momentum and velocity has only an x-component. We call the x-velocities before the collision v_{A1x} and v_{B1x} , and those after the collision v_{A2x} and v_{B2x} . From conservation of kinetic energy we have

$$\frac{1}{2}m_A{v_{A1x}}^2 + \frac{1}{2}m_B{v_{B1x}}^2 = \frac{1}{2}m_A{v_{A2x}}^2 + \frac{1}{2}m_B{v_{B2x}}^2$$

and conservation of momentum gives

$$m_Av_{A1x} + m_Bv_{B1x} = m_Av_{A2x} + m_Bv_{B2x}$$

If the masses m_A and m_B and the initial velocities v_{A1x} and v_{B1x} are known, we can solve these two equations to find the two final velocities v_{A2x} and v_{B2x} .

Elastic Collisions, One Object Initially at Rest

The general solution to the above equations is a little complicated, so we'll concentrate on the particular case in which object B is at rest before the collision (so $v_{B1x} = 0$). Think of object B as a target for object A to hit. Then the kinetic energy and momentum conservation equations are, respectively,

(8.19)

$$\frac{1}{2}m_A{v_{A1x}}^2 = \frac{1}{2}m_A{v_{A2x}}^2 + \frac{1}{2}m_B{v_{B2x}}^2$$

(8.20)

$$m_A\,v_{A1x} = m_A\,v_{A2x} + m_B\,v_{B2x}$$

We can solve for v_{A2x} and v_{B2x} in terms of the masses and the initial velocity v_{A1x} . This involves some fairly strenuous algebra, but it's worth it. No pain, no gain! The simplest approach is somewhat indirect, but along the way it uncovers an additional interesting feature of elastic collisions.

First we rearrange Eqs. (8.19) and (8.20) as follows:

(8.21)

$$m_B {v_{B2x}}^2 = m_A \left({v_{A1x}}^2 - {v_{A2x}}^2\right) = m_A \left(v_{A1x} - v_{A2x}\right)\left(v_{A1x} + v_{A2x}\right)$$

(8.22)

$$m_B v_{B2x} = m_A \left(v_{A1x} - v_{A2x}\right)$$

Now we divide Eq. (8.21) by Eq. (8.22) to obtain

(8.23)

$$v_{B2x} = v_{A1x} + v_{A2x}$$

We substitute this expression back into Eq. (8.22) to eliminate v_{B2x} and then solve for v_{A2x} :

(8.24)

$$\begin{aligned} m_B \left(v_{A1x} + v_{A2x}\right) &= m_A \left(v_{A1x} - v_{A2x}\right) \\ v_{A2x} &= \frac{m_A - m_B}{m_A + m_B} v_{A1x} \end{aligned}$$

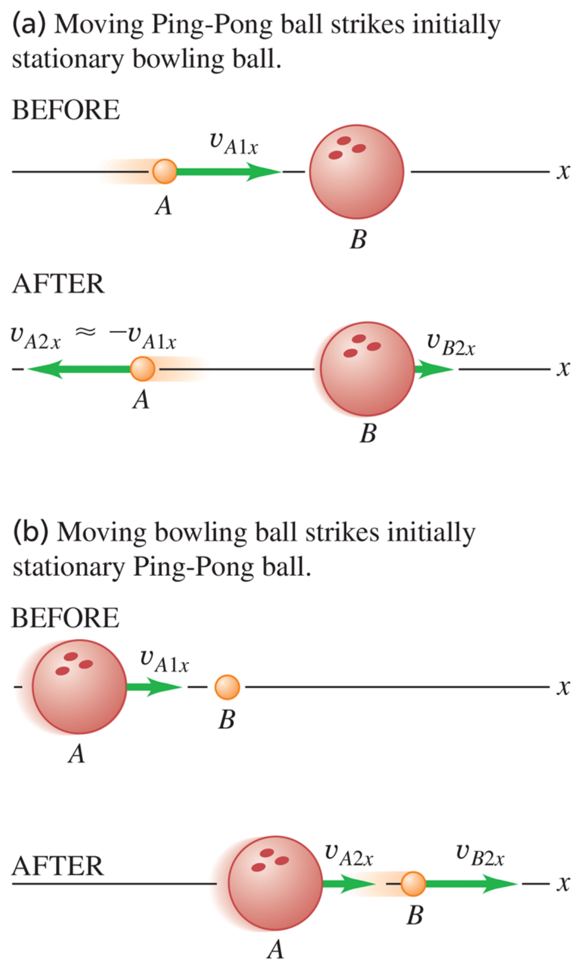
Finally, we substitute this result back into Eq. (8.23) to obtain

(8.25)

$$v_{B2x} = \frac{2m_A}{m_A + m_B} v_{A1x}$$

Now we can interpret the results. Suppose A is a Ping-Pong ball and B is a bowling ball. Then we expect A to bounce off after the collision with a velocity nearly equal to its original value but in the opposite direction (Fig. 8.23a), and we expect B's velocity to be much less. That's just what the equations predict. When m_A is much smaller than m_B, the fraction in Eq. (8.24) is approximately equal to (-1), so v_{A2x} is approximately equal to $-v_{A1x}$. The fraction in Eq. (8.25) is much smaller than unity, so v_{B2x} is much less than v_{A1x}. Figure 8.23b shows the opposite case, in which A is the bowling ball and B the Ping-Pong ball and m_A is much larger than m_B. What do you expect to happen then? Check your predictions against Eqs. (8.24) and (8.25).

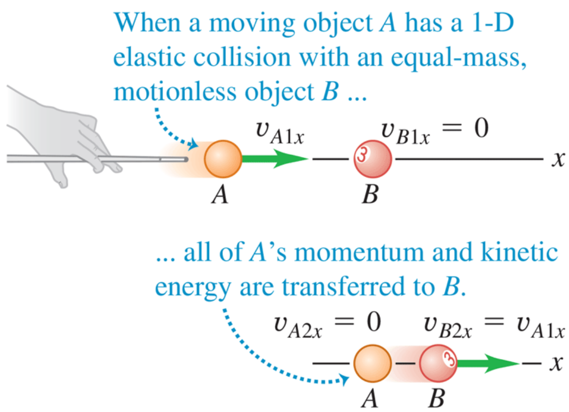
Figure 8.23



One-dimensional elastic collisions between objects with different masses.

Another interesting case occurs when the masses are equal (Fig. 8.24). If $m_A = m_B$, then Eqs. (8.24) and (8.25) give $v_{A2x} = 0$ and $v_{B2x} = v_{A1x}$. That is, the object that was moving stops dead; it gives all its momentum and kinetic energy to the object that was at rest. This behavior is familiar to all pool players.

Figure 8.24



A one-dimensional elastic collision between objects of equal mass.

Elastic Collisions and Relative Velocity

Let's return to the more general case in which A and B have different masses. Equation (8.23) can be rewritten as

(8.26)

$$v_{A1x} = v_{B2x} - v_{A2x}$$

Here $v_{B2x} - v_{A2x}$ is the velocity of B relative to A *after* the collision; from Eq. (8.26), this equals v_{A1x}, which is the *negative* of the velocity of B relative to A *before* the collision. (We discussed relative velocity in Section 3.5.) The relative velocity has the same magnitude, but opposite sign, before and after the collision. The sign changes because A and B are approaching each other before the collision but moving apart after the collision. If we view this collision from a second coordinate system moving with constant velocity relative to the first, the velocities of the objects are different but the *relative* velocities are the same. Hence our statement about relative velocities holds for *any* straight-line elastic collision, even when neither object is at rest initially. *In a straight-line elastic collision of two objects, the relative velocities before and after the collision have the same magnitude but opposite sign.* This means that if B is moving before the collision, Eq. (8.26) becomes

(8.27)

$$v_{B2x} - v_{A2x} = -\left(v_{B1x} - v_{A1x}\right)$$

It turns out that a *vector* relationship similar to Eq. (8.27) is a general property of *all* elastic collisions, even when both objects are moving initially and the velocities do not all lie along the same line. This result provides an alternative and equivalent definition of an elastic collision: *In an elastic collision, the relative velocity of the two objects has the same magnitude before and after the collision.* Whenever this condition is satisfied, the total kinetic energy is also conserved.

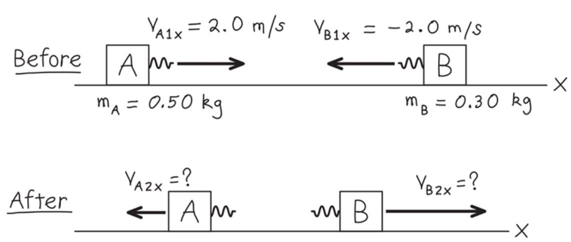
When an elastic two-object collision isn't head-on, the velocities don't all lie along a single line. If they all lie in a plane, then each final velocity has two unknown components, and there are four unknowns in all. Conservation of energy and conservation of the x- and y-components of momentum give only three equations. To determine the final velocities uniquely, we need additional information, such as the direction or magnitude of one of the final velocities.

Example 8.10 An elastic straight-line collision

We repeat the air-track collision of Example 8.5 (Section 8.2), but now we add ideal spring bumpers to the gliders so that the collision is elastic. What are the final velocities of the gliders?

IDENTIFY and SET UP The net external force on the system is zero, so the momentum of the system is conserved. Figure 8.25 shows our sketch. We'll find our target variables, v_{A2x} and v_{B2x}, by using Eq. (8.27), the relative-velocity relationship for an elastic collision, and the momentum-conservation equation.

Figure 8.25



Our sketch for this problem.

EXECUTE From Eq. (8.27),

$$\begin{aligned} v_{B2x} - v_{A2x} &= -(v_{B1x} - v_{A1x}) \\ &= -(-2.0\,\mathrm{m/s} - 2.0\,\mathrm{m/s}) = 4.0\,\mathrm{m/s} \end{aligned}$$

From conservation of momentum,

$$\begin{aligned} m_A v_{A1x} + m_B v_{B1x} &= m_A v_{A2x} + m_B v_{B2x} \\ (0.50\,\mathrm{kg})(2.0\,\mathrm{m/s}) + (0.30\,\mathrm{kg})(-2.0\,\mathrm{m/s}) & \\ &= (0.50\,\mathrm{kg})v_{A2x} + (0.30\,\mathrm{kg})v_{B2x} \\ 0.50v_{A2x} + 0.30v_{B2x} &= 0.40\,\mathrm{m/s} \end{aligned}$$

(To get the last equation we divided both sides of the equation just above it by 1 kg. This makes the units the same as in the first equation.) Solving these equations simultaneously, we find

$$v_{A2x} = -1.0\,\mathrm{m/s} \quad v_{B2x} = 3.0\,\mathrm{m/s}$$

EVALUATE Both objects reverse their direction of motion; A moves to the left at 1.0 m/s and B moves to the right at 3.0 m/s. This is unlike the result of Example 8.5 because that collision was *not* elastic. The more massive glider A slows down in the collision and so loses kinetic energy. The less massive glider B speeds up and gains kinetic energy. The total kinetic energy before the collision (which we calculated in Example 8.7) is 1.6 J. The total kinetic energy after the collision is

$$\frac{1}{2}(0.50\ \mathrm{kg})(-1.0\ \mathrm{m/s})^2 + \frac{1}{2}(0.30\ \mathrm{kg})(3.0\ \mathrm{m/s})^2 = 1.6\ \mathrm{J}$$

The kinetic energies before and after this elastic collision are equal. Kinetic energy is transferred from A to B, but none of it is lost.

CAUTION Be careful with the elastic collision equations You could *not* have solved this problem by using Eqs. (8.24) and (8.25), which apply only if object B is initially *at rest*. Always be sure that you use equations that are applicable!

KEY CONCEPT

In an elastic collision both total momentum and total kinetic energy are conserved. The relative velocity of the two colliding objects has the same magnitude after the collision as before, but in the opposite direction.

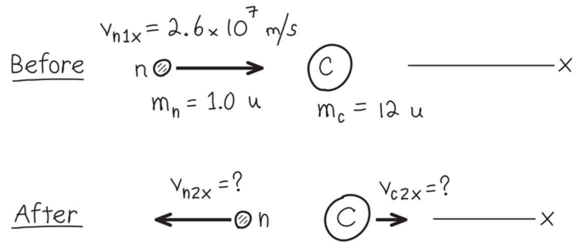
Video Tutor Solution: **Example 8.10**

Example 8.11 Moderating fission neutrons in a nuclear reactor

The fission of uranium nuclei in a nuclear reactor produces high-speed neutrons. Before such neutrons can efficiently cause additional fissions, they must be slowed down by collisions with nuclei in the *moderator* of the reactor. The first nuclear reactor (built in 1942 at the University of Chicago) used carbon (graphite) as the moderator. Suppose a neutron (mass 1.0 u) traveling at $2.6 \times 10^7 \mathrm{m/s}$ undergoes a head-on elastic collision with a carbon nucleus (mass 12 u) initially at rest. Neglecting external forces during the collision, find the velocities after the collision. (1 u is the *atomic mass unit,* equal to 1.66×10^{-27} kg.)

IDENTIFY and SET UP We ignore external forces, so momentum is conserved in the collision. The collision is elastic, so kinetic energy is also conserved. Figure 8.26 shows our sketch. We take the x-axis to be in the direction in which the neutron is moving initially. The collision is head-on, so both particles move along this same axis after the collision. The carbon nucleus is initially at rest, so we can use Eqs. (8.24) and (8.25); we replace A by n (for the neutron) and B by C (for the carbon nucleus). We have $m_{\mathrm{n}} = 1.0$ u, $m_{\mathrm{C}} = 12$ u, and $v_{\mathrm{n}1x} = 2.6 \times 10^7 \mathrm{m/s}$. The target variables are the final velocities $v_{\mathrm{n}2x}$ and $v_{\mathrm{C}2x}$.

Figure 8.26



Our sketch for this problem.

EXECUTE You can do the arithmetic. (*Hint:* There's no reason to convert atomic mass units to kilograms.) The results are

$$v_{n2x} = -2.2 \times 10^7 \mathrm{m/s} \quad v_{C2x} = 0.4 \times 10^7 \mathrm{m/s}$$

EVALUATE The neutron ends up with $|(m_n - m_C)/(m_n + m_C)| = \dfrac{11}{13}$ of its initial speed, and the speed of the recoiling carbon nucleus is $|2m_n/(m_n + m_C)| = \dfrac{2}{13}$ of the neutron's initial speed. Kinetic energy is proportional to speed squared, so the neutron's final kinetic energy is $\left(\dfrac{11}{13}\right)^2 \approx 0.72$ of its original value. After a second head-on collision, its kinetic energy is $(0.72)^2$, or about half its original value, and so on. After a few dozen collisions (few of which are head-on), the neutron speed will be low enough that it can efficiently cause a fission reaction in a uranium nucleus.

KEY CONCEPT

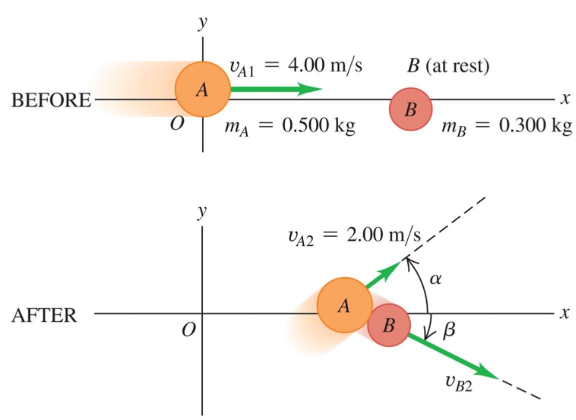
When a particle undergoes a head-on elastic collision with a more massive, initially stationary target, the particle bounces back in the opposite direction. The recoiling target carries away some of the particle's initial kinetic energy.

Video Tutor Solution: **Example 8.11**

Example 8.12 A two-dimensional elastic collision

Figure 8.27 shows an elastic collision of two pucks (masses $m_A = 0.500$ kg and $m_B = 0.300$ kg) on a frictionless air-hockey table. Puck A has an initial velocity of 4.00 m/s in the positive x-direction and a final velocity of 2.00 m/s in an unknown direction α. Puck B is initially at rest. Find the final speed v_{B2} of puck B and the angles α and β.

Figure 8.27



An elastic collision that isn't head-on.

IDENTIFY and SET UP We'll use the equations for conservation of energy and conservation of x- and y-momentum. These three equations should be enough to solve for the three target variables.

EXECUTE The collision is elastic, so the initial and final kinetic energies of the system are equal:

$$\begin{aligned}
\frac{1}{2}m_A{v_{A1}}^2 &= \frac{1}{2}m_A{v_{A2}}^2 + \frac{1}{2}m_B{v_{B2}}^2 \\
{v_{B2}}^2 &= \frac{m_A{v_{A1}}^2 - m_A{v_{A2}}^2}{m_B} \\
&= \frac{(0.500\,\mathrm{kg})(4.0\,\mathrm{m/s})^2 - (0.500\,\mathrm{kg})(2.00\,\mathrm{m/s})^2}{0.300\,\mathrm{kg}} \\
v_{B2} &= 4.47\,\mathrm{m/s}
\end{aligned}$$

Conservation of the x- and y-components of total momentum gives

$$\begin{aligned}
m_A v_{A1x} &= m_A v_{A2x} + m_B v_{B2x} \\
(0.500\,\mathrm{kg})(4.00\,\mathrm{m/s}) &= (0.500\,\mathrm{kg})(2.00\mathrm{m/s})(\cos\alpha) \\
&\quad + (0.300\,\mathrm{kg})(4.47\,\mathrm{m/s})(\cos\beta) \\
0 &= m_A v_{A2y} + m_B v_{B2y} \\
0 &= (0.500\,\mathrm{kg})(2.00\,\mathrm{m/s})(\sin\alpha) \\
&\quad - (0.300\,\mathrm{kg})(4.47\,\mathrm{m/s})(\sin\beta)
\end{aligned}$$

These are two simultaneous equations for α and β. You can supply the details of the solution. (*Hint:* Solve the first equation for cos β and the second for sin β; square each equation and add. Since $\sin^2\beta + \cos^2\beta = 1$, this eliminates β and leaves an equation that you can solve for cos α and hence for α. Substitute this value into either of the two equations and solve for β.) The results are

$$\alpha = 36.9^\circ \quad \beta = 26.6^\circ$$

EVALUATE To check the answers we confirm that the y-momentum, which was zero before the collision, is in fact zero after the collision. The y-momenta are

$$\begin{aligned}
p_{A2y} &= (0.500\ \mathrm{kg})(2.00\ \mathrm{m/s})(\sin 36.9^\circ) = +0.600\,\mathrm{kg}\cdot\mathrm{m/s} \\
p_{B2y} &= -(0.300\,\mathrm{kg})(4.47\mathrm{m/s})(\sin 26.6^\circ) = -0.600\,\mathrm{kg}\cdot\mathrm{m/s}
\end{aligned}$$

and their sum is indeed zero.

KEY CONCEPT

There are three conservation equations for a problem that involves a two-dimensional elastic collision: one for kinetic energy, one for the x-component of momentum, and one for the y-component of momentum.

Video Tutor Solution: **Example 8.12**

Test Your Understanding of Section 8.4

Most present-day nuclear reactors use water as a moderator (see Example 8.11). Are water molecules (mass $m_{\mathrm{w}} = 18.0\ \mathrm{u}$) a better or worse moderator than carbon atoms? (One advantage of water is that it also acts as a coolant for the reactor's radioactive core.)

8.5 Center of Mass

We can restate the principle of conservation of momentum in a useful way by using the concept of **center of mass**. Suppose we have several particles with masses m_1, m_2, and so on. Let the coordinates of m_1 be (x_1, y_1), those of m_2 be (x_2, y_2), and so on. We define the center of mass of the system as the point that has coordinates $(x_{\mathrm{cm}}, y_{\mathrm{cm}})$ given by

(8.28)

$$
\begin{aligned}
x_{\mathrm{cm}} &= \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i x_i}{\sum_i m_i} \\
y_{\mathrm{cm}} &= \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i y_i}{\sum_i m_i}
\end{aligned}
\quad \text{(center of mass)}
$$

We can express the position of the center of mass as a vector $\vec{r}_{\mathrm{cm}}$:

(8.29)

Position vectors of individual particles

Position vector of center of mass of a system of particles

$$
\vec{r}_{\mathrm{cm}} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i\vec{r}_i}{\sum_i m_i} \qquad (8.29)
$$

Masses of individual particles

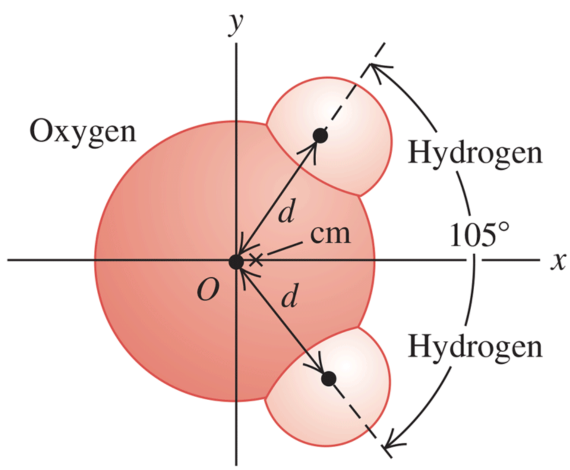
We say that the center of mass is a *mass-weighted average* position of the particles.

Example 8.13 Center of mass of a water molecule

WITH VARIATION PROBLEMS

Figure 8.28 shows a simple model of a water molecule. The oxygen–hydrogen separation is $d = 9.57 \times 10^{-11}$ m. Each hydrogen atom has mass 1.0 u, and the oxygen atom has mass 16.0 u. Find the position of the center of mass.

Figure 8.28



Where is the center of mass of a water molecule?

IDENTIFY and SET UP Nearly all the mass of each atom is concentrated in its nucleus, whose radius is only about 10^{-5} times the overall radius of the atom. Hence we can safely represent each atom as a point particle. Figure 8.28 shows our coordinate system, with the x-axis chosen to lie along the molecule's symmetry axis. We'll use Eqs. (8.28) to find x_{cm} and y_{cm}.

EXECUTE The oxygen atom is at $x = 0$, $y = 0$. The x-coordinate of each hydrogen atom is $d \cos\ (105^\circ/2)$; the y-coordinates are $\pm d \sin (105^\circ/2)$. From Eqs. (8.28),

$$x_{\mathrm{cm}} = \frac{\begin{bmatrix}(1.0\,\mathrm{u})(d\cos\,52.5^\circ)(1.0\,\mathrm{u})(d\cos 52.5^\circ)\\ +(16.0\,\mathrm{u})(0)\end{bmatrix}}{1.0\,\mathrm{u}+1.0\,\mathrm{u}+16.0\,\mathrm{u}} = 0.068d$$

$$y_{\mathrm{cm}} = \frac{\begin{bmatrix}(1.0\,\mathrm{u})(d\sin 52.5^\circ)+(1.0\,\mathrm{u})(-d\,\sin\,52.5^\circ)\\ +(16.0\,\mathrm{u})(0)\end{bmatrix}}{1.0\,\mathrm{u}+1.0\,\mathrm{u}+16.0\,\mathrm{u}} = 0$$

Substituting $d = 9.57 \times 10^{-11}$ m, we find

$$x_{\mathrm{cm}} = (0.068)(9.57 \times 10^{-11}\ \mathrm{m}) = 6.5 \times 10^{-12}\ \mathrm{m}$$

EVALUATE The center of mass is much closer to the oxygen atom (located at the origin) than to either hydrogen atom because the oxygen atom is much more massive. The center of mass lies along the molecule's *axis of symmetry*. If the molecule is rotated 180° around this axis, it looks exactly the same as before. The position of the center of mass can't be affected by this rotation, so it *must* lie on the axis of symmetry.

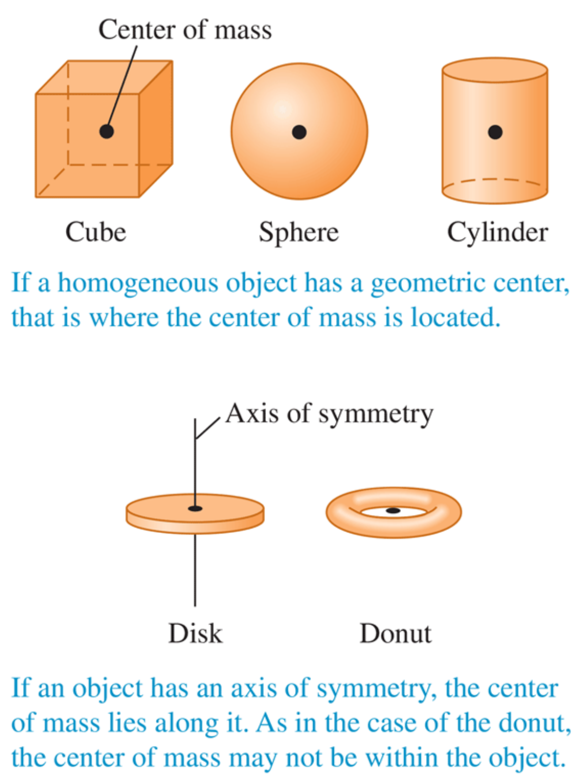
KEY CONCEPT

The x-coordinate of the center of mass of a collection of particles is a weighted sum of the x-coordinates of the individual particles, and similarly for the y-coordinate.

Video Tutor Solution: **Example 8.13**

For solid objects, in which we have (at least on a macroscopic level) a continuous distribution of matter, the sums in Eqs. (8.28) have to be replaced by integrals. The calculations can get quite involved, but we can say three general things about such problems (Fig. 8.29). First, whenever a homogeneous object has a geometric center, such as a billiard ball, a sugar cube, or a can of frozen orange juice, the center of mass is at the geometric center. Second, whenever an object has an axis of symmetry, such as a wheel or a pulley, the center of mass always lies on that axis. Third, there is no law that says the center of mass has to be within the object. For example, the center of mass of a donut is in the middle of the hole.

Figure 8.29



Locating the center of mass of a symmetric object.

Motion of the Center of Mass

To see the significance of the center of mass of a collection of particles, we must ask what happens to the center of mass when the particles move. The x- and y-components of velocity of the center of mass, $v_{\mathrm{cm}-x}$ and $v_{\mathrm{cm}-y}$, are the time derivatives of x_{cm} and y_{cm} . Also, dx_1/dt is the x-component of velocity of particle 1, so $dx_1/dt = v_{1x}$, and so on. Taking time derivatives of Eqs. (8.28), we get

(8.30)

$$v_{\mathrm{cm}-x} = \frac{m_1 v_{1x} + m_2 v_{2x} + m_3 v_{3x} + \cdots}{m_1 + m_2 + m_3 + \cdots}$$
$$v_{\mathrm{cm}-y} = \frac{m_1 v_{1x} + m_2 v_{2y} + m_3 v_{3y} + \cdots}{m_1 + m_2 + m_3 + \cdots}$$

These equations are equivalent to the single vector equation obtained by taking the time derivative of Eq. (8.29):

(8.31)

$$\vec{\boldsymbol{v}}_{\mathrm{cm}} = \frac{m_1 \vec{\boldsymbol{v}}_1 + m_2 \vec{\boldsymbol{v}}_2 + m_3 \vec{\boldsymbol{v}}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots}$$

We denote the *total* mass $m_1 + m_2 + \cdots$ by M. We can then rewrite Eq. (8.31) as

(8.32)

Total mass of a system of particles

Momenta of individual particles

$$M\vec{v}_{\mathrm{cm}} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots = \vec{P} \qquad (8.32)$$

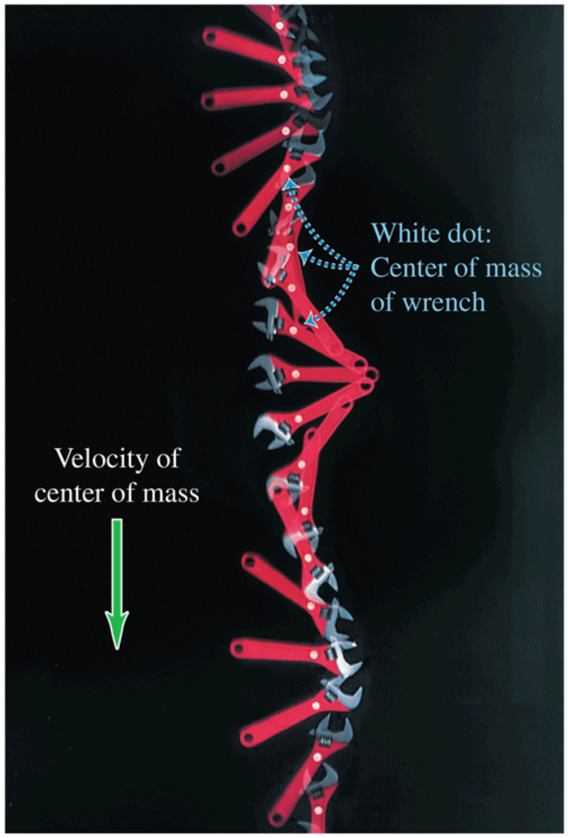
Velocity of center of mass

Total momentum of system

So *the total momentum* $\vec{\boldsymbol{P}}$ *of a system equals the total mass times the velocity of the center of mass*. When you catch a baseball, you are really catching a collection of a very large number of molecules of masses m_1 , m_2 , m_3 , ... The impulse you feel is due to the total momentum of this entire collection. But this impulse is the same as if you were catching a single particle of mass $M = m_1 + m_2 + m_3 + \cdots$ moving with $\vec{\boldsymbol{v}}_{\text{cm}}$, the velocity of the collection's center of mass. So Eq. (8.32) helps us justify representing an extended object as a particle.

For a system of particles on which the net external force is zero, so that the total momentum $\vec{\boldsymbol{P}}$ is constant, the velocity of the center of mass $\vec{\boldsymbol{v}}_{\text{cm}} = \vec{\boldsymbol{P}}/\boldsymbol{M}$ is also constant. Figure 8.30 shows an example. The overall motion of the wrench appears complicated, but the center of mass follows a straight line, as though all the mass were concentrated at that point.

Figure 8.30



The net external force on this wrench is almost zero as it spins on a smooth, horizontal surface (seen from above). Hence the center of mass moves in a straight line with nearly constant velocity.

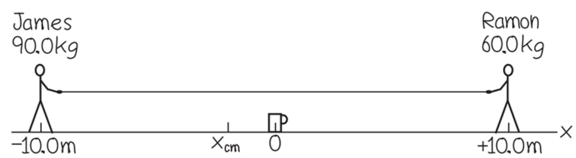
Example 8.14 A tug-of-war on the ice

WITH VARIATION PROBLEMS

James (mass 90.0 kg) and Ramon (mass 60.0 kg) are 20.0 m apart on a frozen pond. Midway between them is a mug of their favorite beverage. They pull on the ends of a light rope stretched between them. When James has moved 6.0 m toward the mug, how far and in what direction has Ramon moved?

IDENTIFY and SET UP The surface is horizontal and (we assume) frictionless, so the net external force on the system of James, Ramon, and the rope is zero; their total momentum is conserved. Initially there is no motion, so the total momentum is zero. The velocity of the center of mass is therefore zero, and it remains at rest. Let's take the origin at the position of the mug and let the $+x$-axis extend from the mug toward Ramon. Figure 8.31 shows our sketch. We use the first of Eqs. (8.28) to calculate the position of the center of mass; we ignore the mass of the light rope.

Figure 8.31



Our sketch for this problem.

EXECUTE The initial x-coordinates of James and Ramon are -10.0 m and $+10.0$ m, respectively, so the x-coordinate of the center of mass is

$$x_{\text{cm}} = \frac{(90.0 \text{ kg})(-10.0 \text{ m}) + (60.0 \text{ kg})(10.0 \text{ m})}{90.0 \text{ kg} + 60.0 \text{ kg}} = -2.0 \text{ m}$$

When James moves 6.0 m toward the mug, his new x-coordinate is -4.0 m; we'll call Ramon's new x-coordinate x_2. The center of mass doesn't move, so

$$\begin{aligned} x_{\text{cm}} &= \frac{(90.0 \text{ kg})(-4.0 \text{ m}) + (60.0 \text{ kg})x_2}{90.0 \text{ kg} + 60.0 \text{ kg}} = -2.0 \text{ m} \\ x_2 &= 1.0 \text{ m} \end{aligned}$$

James has moved 6.0 m and is still 4.0 m from the mug, but Ramon has moved 9.0 m and is only 1.0 m from it.

EVALUATE The ratio of the distances moved, $(6.0 \text{ m})/(9.0 \text{ m}) = \frac{2}{3}$, is the *inverse* ratio of the masses. Can you see why? Because the surface is frictionless, the two men will keep moving and collide at the center of mass; Ramon will reach the mug first. This is independent of how hard either person pulls; pulling harder just makes them move faster.

KEY CONCEPT

If there is no net external force on a system of particles, the center of mass of the system maintains the same velocity. As a special case, if the center of mass is at rest, it remains at rest.

Video Tutor Solution: **Example 8.14**

External Forces and Center-of-Mass Motion

If the net external force on a system of particles is not zero, then total momentum is not conserved and the velocity of the center of mass changes. Let's look at this situation in more detail.

Equations (8.31) and (8.32) give the *velocity* of the center of mass in terms of the velocities of the individual particles. We take the time derivatives of these equations to show that the *accelerations* are related in the same way. Let $\vec{\boldsymbol{a}}_{\text{cm}} = d\vec{\boldsymbol{v}}_{\text{cm}}/dt$ be the acceleration of the center of mass; then

(8.33)

$$M\vec{\boldsymbol{a}}_{\text{cm}} = m_1\vec{\boldsymbol{a}}_1 + m_2\vec{\boldsymbol{a}}_2 + m_3\vec{\boldsymbol{a}}_3 + \cdots$$

Now $m_1\vec{\boldsymbol{a}}_1$ is equal to the vector sum of forces on the first particle, and so on, so the right side of Eq. (8.33) is equal to the vector sum $\sum \vec{\boldsymbol{F}}$ of *all* the forces on *all* the particles. Just as we did in Section 8.2, we can classify each force as *external* or *internal*. The sum of all forces on all the particles is then

$$\sum \vec{\boldsymbol{F}} = \sum \vec{\boldsymbol{F}}_{\text{ext}} + \sum \vec{\boldsymbol{F}}_{\text{int}} = M\vec{\boldsymbol{a}}_{\text{cm}}$$

Because of Newton's third law, all of the internal forces cancel in pairs, and $\sum \vec{\boldsymbol{F}}_{\text{int}} = \mathbf{0}$. What survives on the left side is the sum of only the *external* forces:

(8.34)

Net external force on an object or a collection of particles

$$\sum \vec{F}_{\text{ext}} = M\vec{a}_{\text{cm}} \tag{8.34}$$

Total mass of object or collection of particles

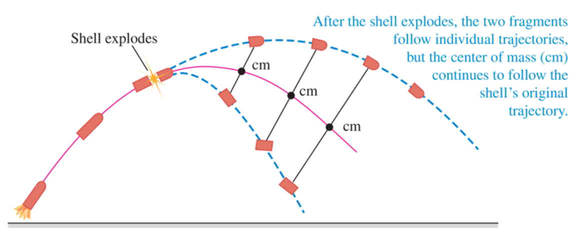
Acceleration of center of mass

When an object or a collection of particles is acted on by external forces, the center of mass moves as though all the mass were concentrated at that point and it were acted on by a net external force equal to the sum of the external forces on the system.

This result is central to the whole subject of mechanics. In fact, we've been using this result all along; without it, we would not be able to represent an extended object as a point particle when we apply Newton's laws. It explains why only *external* forces can affect the motion of an extended object. If you pull upward on your belt, your belt exerts an equal downward force on your hands; these are *internal* forces that cancel and have no effect on the overall motion of your body.

As an example, suppose that a cannon shell traveling in a parabolic trajectory (ignoring air resistance) explodes in flight, splitting into two fragments with equal mass (Fig. 8.32). The fragments follow new parabolic paths, but the center of mass continues on the original parabolic trajectory, as though all the mass were still concentrated at that point.

Figure 8.32



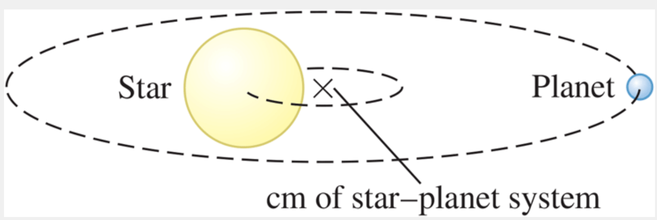
A shell explodes into two fragments in flight. If air resistance is ignored, the center of mass continues on the same trajectory as the shell's path before the explosion.

This property of the center of mass is important when we analyze the motion of rigid objects. In Chapter 10 we'll describe the motion of an extended object as a combination of translational motion of the center of mass and rotational motion about an axis through the center of mass. This property also plays an important role in the motion of astronomical objects. It's not correct to say that the moon orbits the earth; rather, both the earth and the moon move in orbits around their common center of mass.

Application

Finding Planets Beyond Our Solar System

Planets orbiting distant stars are so faint that they cannot be seen with even the best telescopes. But they can be detected by using the idea that a planet and its parent star orbit around their common center of mass (cm). If we observe a star "wobbling" around a point, we can infer that there is an unseen companion planet and can actually determine the planet's mass. Hundreds of planets around distant stars have been discovered in this way.



There's one more useful way to describe the motion of a system of particles. Using $\vec{a}_{cm} = d\vec{v}_{cm}/dt$, we can rewrite Eq. (8.33) as

(8.35)

$$M\vec{a}_{cm} = M\frac{d\vec{v}_{cm}}{dt} = \frac{d(M\vec{v}_{cm})}{dt} = \frac{d\vec{P}}{dt}$$

The total system mass M is constant, so we're allowed to move it inside the derivative. Substituting Eq. (8.35) into Eq. (8.34), we find

(8.36)

$$\sum \vec{\boldsymbol{F}}_{\text{ext}} = \frac{d\vec{\boldsymbol{P}}}{dt} \quad \text{(extended object or system of particles)}$$

This equation looks like Eq. (8.4). The difference is that Eq. (8.36) describes a *system* of particles, such as an extended object, while Eq. (8.4) describes a single particle. The interactions between the particles that make up the system can change the individual momenta of the particles, but the *total* momentum $\vec{\boldsymbol{P}}$ of the system can be changed only by external forces acting from outside the system.

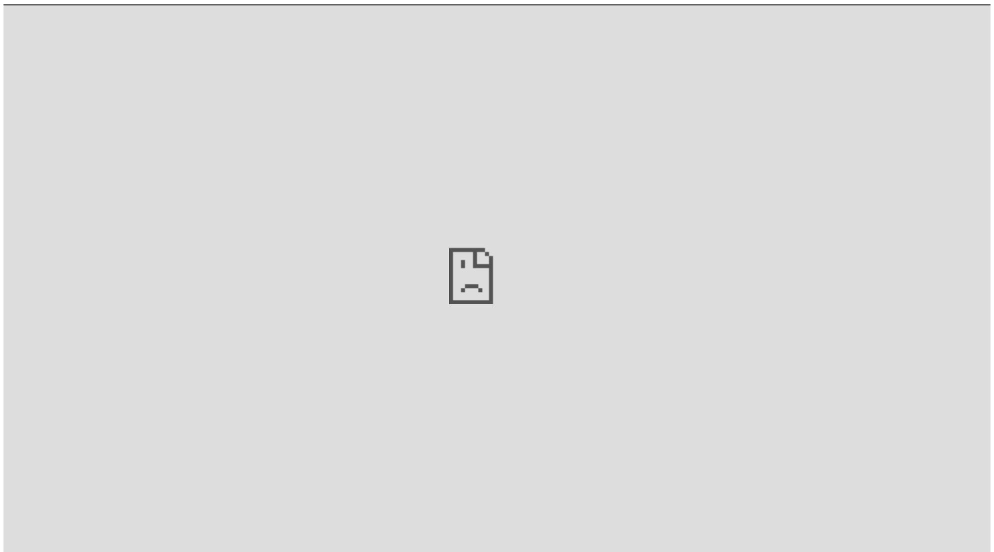
If the net external force is zero, Eqs. (8.34) and (8.36) show that the center-of-mass acceleration $\vec{\boldsymbol{a}}_{\text{cm}}$ is zero (so the center-of-mass velocity $\vec{\boldsymbol{v}}_{\text{cm}}$ is constant) and the total momentum $\vec{\boldsymbol{P}}$ is constant. This is just our statement from Section 8.3: If the net external force on a system is zero, momentum is conserved.

Test Your Understanding of Section 8.5

Will the center of mass in Fig. 8.32 continue on the same parabolic trajectory even after one of the fragments hits the ground? Why or why not?

8.6 Rocket Propulsion

Video Tutor Demo: Water Rocket

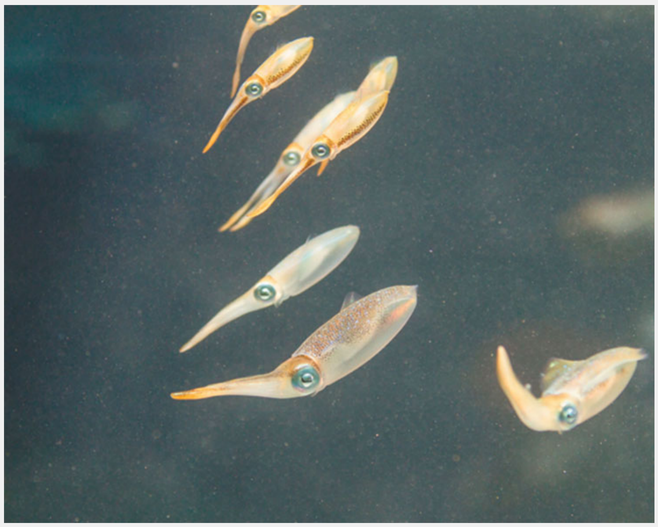


Momentum considerations are particularly useful for analyzing a system in which the masses of parts of the system change with time. In such cases we can't use Newton's second law $\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ directly because m changes. Rocket propulsion is an important example of this situation. A rocket is propelled forward by rearward ejection of burned fuel that initially was in the rocket (which is why rocket fuel is also called *propellant*). The forward force on the rocket is the reaction to the backward force on the ejected material. The total mass of the system is constant, but the mass of the rocket itself decreases as material is ejected.

BIO Application

Jet Propulsion in Squids

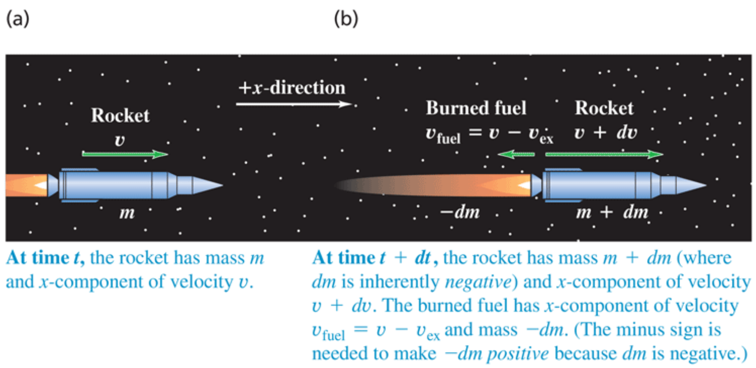
Both a jet engine and a squid use variations in their mass to provide propulsion: They increase their mass by taking in fluid (air for a jet engine, water for a squid) at low speed, then decrease their mass by ejecting that fluid at high speed. The Caribbean reef squid (*Sepioteuthis sepioidea*), shown here, can use jet propulsion to vault to a height of 2 m above the water and fly a total distance of 10 m—about 50 times its body length!



For simplicity, let's consider a rocket in outer space, where there is no gravitational force and no air resistance. Let m denote the mass of the rocket, which will change as it expends fuel. We choose our x-axis to be along the rocket's direction of motion. Figure 8.33a shows the rocket at a time t, when its mass is m and its x-velocity relative to our coordinate system is v (To simplify, we'll drop the subscript x in this discussion.) The x-component of total momentum at this instant is $P_1 = mv$. In a short time interval dt, the mass of the rocket changes by an amount dm. This is an inherently negative quantity because the rocket's mass m *decreases* with time. During dt, a *positive* mass $-dm$ of burned fuel is ejected from the rocket. Let v_{ex} be the exhaust *speed* of this material *relative to the* *rocket;* the burned fuel is ejected opposite the direction of motion, so its x-component of *velocity* relative to the rocket is $-v_{\text{ex}}$. The x-velocity v_{fuel} of the burned fuel relative to our coordinate system is then

$$v_{\text{fuel}} = v + (-v_{ex}) = v - v_{ex}$$

Figure 8.33



A rocket moving in gravity-free outer space at (a) time t and (b) time $t + dt$.

and the x-component of momentum of the ejected mass $(-dm)$ is

$$(-dm)v_{\text{fuel}} = (-dm)(v - v_{\text{ex}})$$

Figure 8.33b shows that at the end of the time interval dt, the x-velocity of the rocket and unburned fuel has increased to $v + dv$, and its mass has decreased to $m + dm$ (remember that dm is negative). The rocket's momentum at this time is

$$(m + dm)\,(v + dv)$$

Thus the *total* x-component of momentum P_2 of the rocket plus ejected fuel at time $t + dt$ is

$$P_2 = (m + dm)\,(v + dv) + (-dm)(v - v_{\text{ex}})$$

According to our initial assumption, the rocket and fuel are an isolated system. Thus momentum is conserved, and the total x-component of momentum of the system must be the same at time t and at time $t + dt$: $P_1 = P_2$. Hence

$$mv = (m + dm)\,(v + dv) + (-dm)(v - v_{\text{ex}})$$

This can be simplified to

$$m\,dv = -dm\,v_{\text{ex}} - dm\,dv$$

We can ignore the term $(-dm\,dv)$ because it is a product of two small quantities and thus is much smaller than the other terms. Dropping this term, dividing by dt, and rearranging, we find

(8.37)

$$m\frac{dv}{dt} = -v_{\text{ex}}\frac{dm}{dt}$$

Now dv/dt is the acceleration of the rocket, so the left side of Eq. (8.37) (mass times acceleration) equals the net external force F, or *thrust*, on the rocket:

(8.38)

$$F = -v_{\text{ex}}\frac{dm}{dt}$$

The thrust is proportional both to the relative speed v_{ex} of the ejected fuel and to the mass of fuel ejected per unit time, $-dm/dt$. (Remember that dm/dt is negative because it is the rate of change of the rocket's mass, so F is positive.)

The x-component of acceleration of the rocket is

(8.39)

$$a = \frac{dv}{dt} = -\frac{v_{\text{ex}}}{m}\frac{dm}{dt}$$

This is positive because v_{ex} is positive (remember, it's the exhaust *speed*) and dm/dt is negative. The rocket's mass m decreases continuously while the fuel is being consumed. If v_{ex} and dm/dt are constant, the acceleration increases until all the fuel is gone.

Equation (8.38) tells us that an effective rocket burns fuel at a rapid rate (large $-dm/dt$) and ejects the burned fuel at a high relative speed (large v_{ex}), as in Fig. 8.34. In the early days of rocket propulsion, people who didn't understand conservation of momentum thought that a rocket couldn't function in outer space because "it doesn't have anything to push against." In fact, rockets work *best* in outer space, where there is no air resistance! The launch vehicle in Fig. 8.34 is *not* "pushing against the ground" to ascend.

Figure 8.34

To provide enough thrust to lift its payload into space, this *Atlas V* launch vehicle ejects more than 1000 kg of burned fuel per second at speeds of nearly 4000 m/s.

If the exhaust speed v_{ex} is constant, we can integrate Eq. (8.39) to relate the velocity v at any time to the remaining mass m. At time $t = 0$, let the mass be m_0 and the velocity be v_0. Then we rewrite Eq. (8.39) as

$$dv = -v_{\text{ex}} \frac{dm}{m}$$

We change the integration variables to v' and m', so we can use v and m as the upper limits (the final speed and mass). Then we integrate both sides, using limits v_0 to v and m_0 to m, and take the constant v_{ex} outside the integral:

(8.40)

$$\begin{aligned}\int_{v_0}^{\vec{v}} dv\prime &= -\int_{m_0}^{m} v_{\text{ex}}\frac{dm\prime}{m\prime} = -v_{\text{ex}}\int_{m_0}^{m}\frac{dm\prime}{m\prime}\\ v - v_0 &= -v_{\text{ex}}\text{in}\frac{m}{m_0} = v_{\text{ex}}\ \ln\frac{m_0}{m}\end{aligned}$$

The ratio m_0/m is the original mass divided by the mass after the fuel has been exhausted. In practical spacecraft this ratio is made as large as possible to maximize the speed gain, which means that the initial mass of the rocket is almost all fuel. The final velocity of the rocket will be greater in magnitude (and is often *much* greater) than the relative speed v_{ex} if $\ln(m_0/\mathrm{m}) > 1$—that is, if $m_0/\mathrm{m} > e = 2.71828\ldots.$

We've assumed throughout this analysis that the rocket is in gravity-free outer space. However, gravity must be taken into account when a rocket is launched from the surface of a planet, as in Fig. 8.34.

Example 8.15 Acceleration of a rocket

The engine of a rocket in outer space, far from any planet, is turned on. The rocket ejects burned fuel at a constant rate; in the first second of firing, it ejects $\frac{1}{120}$ of its initial mass m_0 at a relative speed of 2400 m/s. What is the rocket's initial acceleration?

IDENTIFY and SET UP We are given the rocket's exhaust speed v_{ex} and the fraction of the initial mass lost during the first second of firing, from which we can find dm/dt. We'll use Eq. (8.39) to find the acceleration of the rocket.

EXECUTE The initial rate of change of mass is

$$\frac{dm}{dt} = -\frac{m_0/120}{1\ \text{s}} = -\frac{m_0}{120\ \text{s}}$$

From Eq. (8.39),

$$a = -\frac{v_{\text{ex}}}{m_0}\frac{dm}{dt} = -\frac{2400\ \text{m/s}}{m_0}\left(-\frac{m_0}{120\ \text{s}}\right) = 20\ \text{m/s}^2$$

EVALUATE The answer doesn't depend on m_0 . If v_{ex} is the same, the initial acceleration is the same for a 120,000 kg spacecraft that ejects 1000 kg/s as for a 60 kg astronaut equipped with a small rocket that ejects 0.5 kg/s.

KEY CONCEPT

The thrust provided by a rocket equals the exhaust speed multiplied by the mass of fuel ejected per unit time. The resulting acceleration equals the thrust divided by the rocket's mass.

Video Tutor Solution: **Example 8.15**

Example 8.16 Speed of a rocket

Suppose that $\frac{3}{4}$ of the initial mass of the rocket in Example 8.15 is fuel, so the fuel is completely consumed at a constant rate in 90 s.

The final mass of the rocket is $m = m_0/4$. If the rocket starts from rest in our coordinate system, find its speed at the end of this time.

IDENTIFY, SET UP, and EXECUTE We are given the initial velocity $v_0 = 0$, the exhaust speed $v_{\text{ex}} = 2400\ \text{m/s}$, and the final mass m as a fraction of the initial mass m_0. We'll use Eq. (8.40) to find the final speed v:

$$v = v_0 + v_{\text{ex}} \ln \frac{m_0}{m} = 0 + (2400\ \text{m/s})(\ln 4) = 3327\ \text{m/s}$$

EVALUATE Let's examine what happens as the rocket gains speed. (To illustrate our point, we use more figures than are significant.) At the start of the flight, when the velocity of the rocket is zero, the ejected fuel is moving backward at 2400 m/s relative to our frame of reference. As the rocket moves forward and speeds up, the fuel's speed relative to our system decreases; when the rocket speed reaches 2400 m/s, this relative speed is *zero*. [Knowing the rate of fuel consumption, you can solve Eq. (8.40) to show that this occurs at about t=75.6 s.] After this time the ejected burned fuel moves *forward,* not backward, in our system. Relative to our frame of reference, the last bit of ejected fuel has a forward velocity of $3327\ \text{m/s} - 2400\ \text{m/s} = 927\ \text{m/s}$.

KEY CONCEPT

Because the mass of a rocket decreases as it ejects fuel, its acceleration is not constant even if the thrust is constant.

Video Tutor Solution: **Example 8.16**

Test Your Understanding of Section 8.6

(a) If a rocket in gravity-free outer space has the same thrust at all times, is its acceleration constant, increasing, or decreasing? (b) If the rocket has the same acceleration at all times, is the thrust constant, increasing, or decreasing?

Chapter 8 Summary

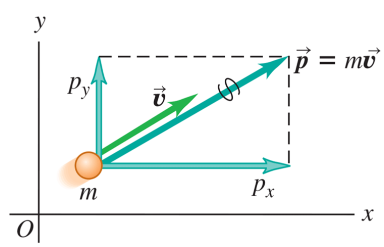
Momentum of a particle: The momentum $\vec{p}$ of a particle is a vector quantity equal to the product of the particle's mass m and velocity $\vec{v}$. Newton's second law says that the net external force on a particle is equal to the rate of change of the particle's momentum.

(8.2)

$$\vec{p} = m\vec{v}$$

(8.4)

$$\sum \vec{F} = \frac{d\vec{p}}{dt}$$



Impulse and momentum: If a constant net external force $\sum \vec{F}$ acts on a particle for a time interval Δt from t_1 to t_2, the impulse $\vec{J}$ of the net external force is the product of the net external force and the time interval. If $\sum \vec{F}$ varies with time, $\vec{J}$ is the integral of the net external force over the time interval. In any case, the change in a particle's momentum during a time interval equals the impulse of the net external force that acted on the particle during that interval. The momentum of a particle equals the impulse that accelerated it from rest to its present speed. (See Examples 8.1, 8.2 and 8.3.)

(8.5)

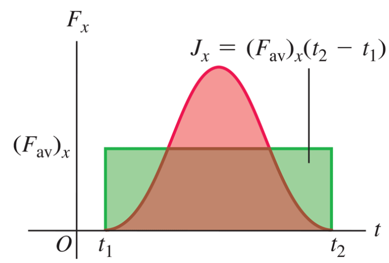
$$\vec{J} = \sum \vec{F}\,(t_2 - t_1) = \sum \vec{F}\,\Delta t$$

(8.7)

$$\vec{J} = \int_{t_1}^{t_2} \sum \vec{F}\,dt$$

(8.6)

$$\vec{J} = \vec{p}_2 - \vec{p}_1$$

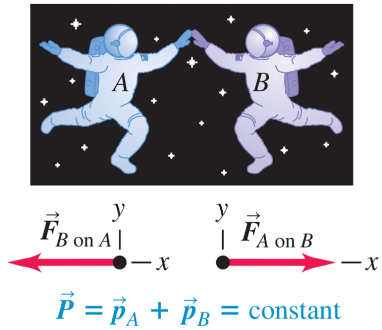


Conservation of momentum: An internal force is a force exerted by one part of a system on another. An external force is a force exerted on any part of a system by something outside the system. If the net external force on a system is zero, the total momentum of the system $\vec{P}$ (the vector sum of the momenta of the individual particles that make up the system) is constant, or conserved. Each component of total momentum is separately conserved. (See Examples 8.4, 8.5 and 8.6.)

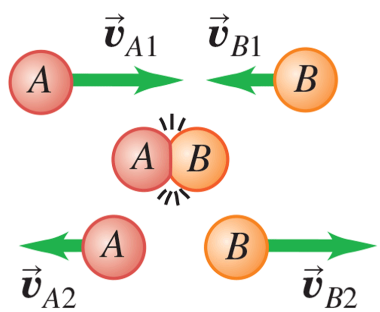
(8.14)

$$\begin{aligned}\overrightarrow{P} &= \overrightarrow{p}_A + \overrightarrow{p}_B + \cdots \\ &= m_A\overrightarrow{v}_A + m_B\overrightarrow{v}_B + \cdots\end{aligned}$$

If $\sum \vec{F} = \mathbf{0}$, then $\vec{P} = \text{constant}$.



Collisions: In typical collisions, the initial and final total momenta are equal. In an elastic collision between two objects, the initial and final total kinetic energies are also equal, and the initial and final relative velocities have the same magnitude. In an inelastic two-object collision, the total kinetic energy is less after the collision than before. If the two objects have the same final velocity, the collision is completely inelastic. (See Examples 8.7, 8.8, 8.9, 8.10, 8.11 and 8.12.)



Center of mass: The position vector of the center of mass of a system of particles, $\vec{r}_{\text{cm}}$, is a weighted average of the positions $\vec{r}_1, \vec{r}_2, \ldots$ of the individual particles. The total momentum $\vec{P}$ of a system equals the system's total mass M multiplied by the velocity of its center of mass, $\vec{v}_{\text{cm}}$. The center of mass moves as though all the mass M were concentrated at that point. If the net external force on the system is zero, the center-of-mass velocity $\vec{v}_{\text{cm}}$ is constant. If the net external force is not zero, the center of mass accelerates as though it were a particle of mass M being acted on by the same net external force. (See Examples 8.13 and 8.14.)

(8.29)

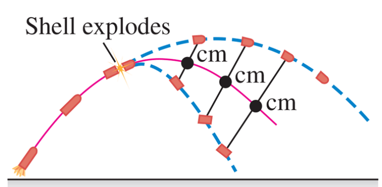
$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i\vec{r}_i}{\sum_i m_i}$$

(8.32)

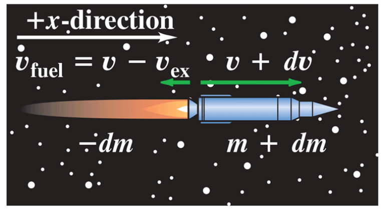
$$\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots = M\vec{v}_{cm}$$

(8.34)

$$\sum \vec{F}_{ext} = M\vec{a}_{cm}$$



Rocket propulsion: In rocket propulsion, the mass of a rocket changes as the fuel is used up and ejected from the rocket. Analysis of the motion of the rocket must include the momentum carried away by the spent fuel as well as the momentum of the rocket itself. (See Examples 8.15 and 8.16.)


+x-direction
$v_{\text{fuel}} = v - v_{\text{ex}}$
$v + dv$
$-dm$
$m + dm$

Guided Practice: Momentum, Impulse, and Collisions

For assigned homework and other learning materials, go to
Mastering Physics.

Key Example Variation Problems

Be sure to review Examples 8.4, 8.5, and 8.6 (Section 8.2) before attempting these problems.

VP8.6.1 You hold glider A of mass 0.125 kg and glider B of mass 0.375 kg at rest on an air track with a compressed spring of negligible mass between them. When you release the gliders, the spring pushes them apart. (a) Once the gliders are no longer in contact with the spring, glider A is moving to the right at 0.600 m/s. What is the velocity (magnitude and direction) of glider B at this time? (b) Glider A moving to the right at 0.600 m/s then collides head-on with a third glider C of mass 0.750 kg that is moving to the left at 0.400 m/s. After this collision, glider C is moving to the left at 0.150 m/s. What is the velocity (magnitude and direction) of glider A after this collision?

VP8.6.2 Two hockey players skating on essentially frictionless ice collide head-on. Madeleine, of mass 65.0 kg, is moving at

6.00 m/s to the east just before the collision and at 3.00 m/s to the west just after the collision. Buffy, of mass 55.0 kg, is moving at 3.50 m/s to the east just after the collision. (a) Find Buffy's velocity (magnitude and direction) just before the collision. (b) What are the changes in the velocities of the two hockey players during the collision? Take east to be the positive direction. Who has the greater magnitude of velocity change: more massive Madeleine or less massive Buffy?

VP8.6.3 A 2.40 kg stone is sliding in the $+x$-direction on a horizontal, frictionless surface. It collides with a 4.00 kg stone at rest. After the collision the 2.40 kg stone is moving at 3.60 m/s at an angle of 30.0° measured from the $+x$-direction toward the $+y$-direction, and the 4.00 kg stone is moving at an angle of 45.0° measured from the $+x$-direction toward the $-y$-direction. (a) What is the y-component of momentum of the 2.40 kg stone after the collision? What must be the y-component of momentum of the 4.00 kg stone after the collision? (b) What is the speed of the 4.00 kg stone after the collision? (c) What is the x-component of the total momentum of the two stones after the collision? (d) What is the speed of the 2.40 kg stone before the collision?

VP8.6.4 A hockey puck of mass m is moving in the $+x$-direction at speed v_{P1} on a frictionless, horizontal surface. It collides with a stone of mass $2m$ that is initially at rest. After the collision the hockey puck is moving at an angle θ measured from the $+x$-direction toward the $+y$-direction, and the stone is moving at the same angle θ but measured from the $+x$-direction toward the $-y$-direction. (a) In order for the y-component of momentum to be conserved, what must be the ratio of the final speed v_{S2} of the stone to the final speed v_{P2} of the hockey puck? (b) Use conservation of the x-component of momentum to find v_{S2} and v_{P2} in terms of v_{P1} and θ.

Be sure to review Examples 8.7, 8.8, and 8.9 (Section 8.3) before attempting these problems.

VP8.9.1 Two blocks of clay, one of mass 1.00 kg and one of mass 4.00 kg, undergo a completely inelastic collision. Before the collision one of the blocks is at rest and the other block is moving with kinetic energy 32.0 J. (a) If the 4.00 kg block is initially at rest and the 1.00 kg block is moving, what is the initial speed of the 1.00 kg block? What is the common final speed of the two blocks? How much kinetic energy is lost in the collision? (b) If the 1.00 kg block is initially at rest and the 4.00 kg block is moving, what is the initial speed of the 4.00 kg block? What is the common final speed of the two blocks? How much kinetic energy is lost in the collision? (c) In which case is more of the initial kinetic energy lost in a completely inelastic collision: a moving object collides (i) with a heavier object at rest or (ii) with a lighter object at rest?

VP8.9.2 A 0.500 kg block of cheese sliding on a frictionless tabletop collides with and sticks to a 0.200 kg apple. Before the collision the cheese was moving at 1.40 m/s and the apple was at rest. The cheese and apple then slide together off the edge of the table and fall to the floor 0.600 m below. (a) Find the speed of the cheese and apple just after the collision. In this collision, what is conserved: momentum, total mechanical energy, both, or neither? (b) What is the speed of the cheese and apple just before they hit the floor? During the fall from the tabletop to the floor, what is conserved: momentum, total mechanical energy, both, or neither?

VP8.9.3 A 2.40 kg can of coffee moving at 1.50 m/s in the $+x$-direction on a kitchen counter collides head-on with a 1.20 kg box of macaroni that is initially at rest. After the collision the can of coffee is moving at 0.825 m/s in the $+x$-direction. (a)

What is the velocity (magnitude and direction) of the box of macaroni after the collision? (b) What are the kinetic energies of the can before and after the collision, and of the box after the collision? (c) Is this collision elastic, inelastic, or completely inelastic? How can you tell?

VP8.9.4 A block of mass m moving due east at speed v collides with and sticks to a block of mass $2m$ that is moving at the same speed v but in a direction 45.0° north of east. Find the direction in which the two blocks move after the collision.

Be sure to review Examples 8.13 and 8.14 (Section 8.5) before attempting these problems.

VP8.14.1 Find the x- and y-coordinates of the center of mass of a system composed of the following particles: a 0.500 kg particle at the origin; a 1.25 kg particle at $x = 0.150$ m, $y = 0.200$ m; and a 0.750 kg particle at $x = 0.200$ m, $y = -0.800$ m.

VP8.14.2 Three objects lie along the x-axis. A 3.00 kg object is at the origin, a 2.00 kg object is at $x = 1.50$ m, and a 1.20 kg object is at an unknown position. The center of mass of the system of three objects is at $x = -0.200$ m. What is the position of the 1.20 kg object?

VP8.14.3 You hold a 0.125 kg glider A and a 0.500 kg glider B at rest on an air track with a compressed spring of negligible mass between them. When you release the gliders, the spring pushes them apart so that they move in opposite directions. When glider A has moved 0.960 m to the left from its starting position, how far to the right from its starting position has glider B moved?

VP8.14.4 Three objects each have mass m. Each object feels a force from the other two, but not from any other object. Initially the first object is at $x = -L$, $y = 0$; the second object is at $x = +L$, $y = 0$; and the third object is at $x = 0$, $y = L$. At a later time the first object is at $x = -L/3$, $y = +L/4$; and the second object is at $x = +L/2$, $y = -L$. At this later time, where is the third object?

Bridging Problem: One Collision After Another

Sphere A, of mass 0.600 kg, is initially moving to the right at 4.00 m/s. Sphere B, of mass 1.80 kg, is initially to the right of sphere A and moving to the right at 2.00 m/s. After the two spheres collide, sphere B is moving at 3.00 m/s in the same direction as before. (a) What is the velocity (magnitude and direction) of sphere A after this collision? (b) Is this collision elastic or inelastic? (c) Sphere B then has an off-center collision with sphere C, which has mass 1.20 kg and is initially at rest. After this collision, sphere B is moving at 19.0° to its initial direction at 2.00 m/s. What is the velocity (magnitude and direction) of sphere C after this collision? (d) What is the impulse (magnitude and direction) imparted to sphere B by sphere C when they collide? (e) Is this second collision elastic or inelastic? (f) What is the velocity (magnitude and direction) of the center of mass of the system of three spheres (A, B, and C) after the second collision? No external forces act on any of the spheres in this problem.

Solution Guide

IDENTIFY and SET UP

1. Momentum is conserved in these collisions. Can you explain why?

2. Choose the x- and y-axes, and use your choice of axes to draw three figures that show the spheres (i) before the first collision, (ii) after the first collision but before the second collision, and (iii) after the second collision. Assign subscripts to values in each of situations (i), (ii), and (iii).
3. Make a list of the target variables, and choose the equations that you'll use to solve for these.

EXECUTE

4. Solve for the velocity of sphere A after the first collision. Does A slow down or speed up in the collision? Does this make sense?
5. Now that you know the velocities of both A and B after the first collision, decide whether or not this collision is elastic. (How will you do this?)
6. The second collision is two-dimensional, so you'll have to demand that *both* components of momentum are conserved. Use this to find the speed and direction of sphere C after the second collision. (**HINT:** After the first collision, sphere B maintains the same velocity until it hits sphere C.)
7. Use the definition of impulse to find the impulse imparted to sphere B by sphere C. Remember that impulse is a vector.
8. Use the same technique that you employed in step 5 to decide whether the second collision is elastic.
9. Find the velocity of the center of mass after the second collision.

EVALUATE

10. Compare the directions of the vectors you found in steps 6 and 7. Is this a coincidence? Why or why not?
11. Find the velocity of the center of mass before and after the first collision. Compare to your result from step 9. Again, is this a

coincidence? Why or why not?

Video Tutor Solution: Chapter 8 Bridging Problem

Questions/Exercises/Problems: Momentum, Impulse, and Collisions

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

Discussion Questions

Q8.1 In splitting logs with a hammer and wedge, is a heavy hammer more effective than a lighter hammer? Why?

Q8.2 Suppose you catch a baseball and then someone invites you to catch a bowling ball with either the same momentum or the same kinetic energy as the baseball. Which would you choose? Explain.

Q8.3 When rain falls from the sky, what happens to its momentum as it hits the ground? Is your answer also valid for Newton's famous apple?

Q8.4 A car has the same kinetic energy when it is traveling south at 30 m/s as when it is traveling northwest at 30 m/s. Is the momentum of the car the same in both cases? Explain.

Q8.5 A truck is accelerating as it speeds down the highway. One inertial frame of reference is attached to the ground with its origin at a fence post. A second frame of reference is attached to a police car that is traveling down the highway at constant velocity. Is the momentum of the truck the same in these two reference frames? Explain. Is the rate of change of the truck's momentum the same in these two frames? Explain.

Q8.6 (a) If the momentum of a *single* point object is equal to zero, must the object's kinetic energy also be zero? (b) If the momentum of a *pair* of point objects is equal to zero, must the kinetic energy of those objects also be zero? (c) If the kinetic energy of a pair of point objects is equal to zero, must the momentum of those objects also be zero? Explain your reasoning in each case.

Q8.7 A woman holding a large rock stands on a frictionless, horizontal sheet of ice. She throws the rock with speed v_0 at an angle α above the horizontal. Consider the system consisting of the woman plus the rock. Is the momentum of the system conserved? Why or why not? Is any component of the momentum of the system conserved? Again, why or why not?

Q8.8 In Example 8.7 (Section 8.3), where the two gliders of Fig. 8.18 stick together after the collision, the collision is inelastic because $K_2 < K_1$. In Example 8.5 (Section 8.2), is the collision inelastic? Explain.

Q8.9 In a completely inelastic collision between two objects, where the objects stick together after the collision, is it possible for the final kinetic energy of the system to be zero? If so, give an example in which this would occur. If the final kinetic energy is zero, what must the initial momentum of the system be? Is the initial kinetic energy of the system zero? Explain.

Q8.10 Since for a particle the kinetic energy is given by $K = \frac{1}{2}mv^2$ and the momentum by $\vec{\boldsymbol{P}} = m\vec{\boldsymbol{v}}$, it is easy to show that $K = p^2/2m$. How, then, is it possible to have an event during which the total momentum of the system is constant but the total kinetic energy changes?

Q8.11 In each of Examples 8.10, 8.11, and 8.12 (Section 8.4), verify that the relative velocity vector of the two objects has the same magnitude before and after the collision. In each case, what happens to the *direction* of the relative velocity vector?

Q8.12 A glass dropped on the floor is more likely to break if the floor is concrete than if it is wood. Why? (Refer to Fig. 8.3b.)

Q8.13 In Fig. 8.23b, the kinetic energy of the Ping-Pong ball is larger after its interaction with the bowling ball than before. From where does the extra energy come? Describe the event in terms of conservation of energy.

Q8.14 A machine gun is fired at a steel plate. Is the average force on the plate from the bullet impact greater if the bullets bounce off or if they are squashed and stick to the plate? Explain.

Q8.15 A net external force of 4 N acts on an object initially at rest for 0.25 s and gives it a final speed of 5 m/s. How could a net external force of 2 N produce the same final speed?

Q8.16 A net external force with x-component $\sum F_x$ acts on an object from time t_1 to time t_2. The x-component of the momentum of the object is the same at t_1 as it is at t_2, but $\sum F_x$ is not zero at all times between t_1 and t_2. What can you say about the graph of $\sum F_x$ versus t?

Q8.17 A tennis player hits a tennis ball with a racket. Consider the system made up of the ball and the racket. Is the total momentum of the system the same just before and just after the hit? Is the total momentum just after the hit the same as 2 s later, when the ball is in midair at the high point of its trajectory? Explain any differences between the two cases.

Q8.18 In Example 8.4 (Section 8.2), consider the system consisting of the rifle plus the bullet. What is the speed of the system's center of mass after the rifle is fired? Explain.

Q8.19 An egg is released from rest from the roof of a building and falls to the ground. As the egg falls, what happens to the momentum of the system of the egg plus the earth?

Q8.20 A woman stands in the middle of a perfectly smooth, frictionless, frozen lake. She can set herself in motion by throwing things, but suppose she has nothing to throw. Can she propel herself to shore *without* throwing anything?

Q8.21 At the highest point in its parabolic trajectory, a shell explodes into two fragments. Is it possible for *both* fragments to fall straight down after the explosion? Why or why not?

Q8.22 When an object breaks into two pieces (explosion, radioactive decay, recoil, etc.), the lighter fragment gets more kinetic energy than the heavier one. This is a consequence of momentum conservation, but can you also explain it by using Newton's laws of motion?

Q8.23 An apple falls from a tree and feels no air resistance. As it is falling, which of these statements about it are true? (a) Only its momentum is conserved; (b) only its total mechanical energy is conserved; (c) both its momentum and its total mechanical energy are conserved; (d) its kinetic energy is conserved.

Q8.24 Two pieces of clay collide and stick together. During the collision, which of these statements are true? (a) Only the momentum of the clay is conserved; (b) only the total mechanical energy of the clay is conserved; (c) both the momentum and the total mechanical energy of the clay are conserved; (d) the kinetic energy of the clay is conserved.

Q8.25 Two objects of mass M and $5m$ are at rest on a horizontal, frictionless table with a compressed spring of negligible mass between them. When the spring is released, which of the following statements are true? (a) The two objects receive equal magnitudes of momentum; (b) the two objects receive equal amounts of kinetic energy from the spring; (c) the heavier object gains more kinetic energy than the lighter object; (d) the lighter object gains more kinetic energy than the heavier object. Explain your reasoning in each case.

Q8.26 A very heavy SUV collides head-on with a very light compact car. Which of these statements about the collision are correct? (a) The amount of kinetic energy lost by the SUV is equal to the amount of kinetic energy gained by the compact; (b) the amount of momentum lost by the SUV is equal to the amount of momentum gained by the compact; (c) the compact feels a considerably greater force during the collision than the SUV does; (d) both cars lose the same amount of kinetic energy.

Exercises

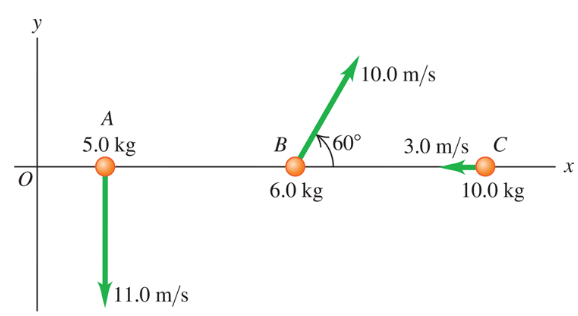
Section 8.1 Momentum and Impulse

8.1 • (a) What is the magnitude of the momentum of a 10,000 kg truck whose speed is 12.0 m/s? (b) What speed would a 2000 kg SUV have to attain in order to have (i) the same momentum? (ii) the same kinetic energy?

8.2 • In a certain track and field event, the shotput has a mass of 7.30 kg and is released with a speed of 15.0 m/s at 40.0° above the horizontal over a competitor's straight left leg. What are the initial horizontal and vertical components of the momentum of this shotput?

8.3 • Objects A, B, and C are moving as shown in Fig. E8.3. Find the x- and y-components of the net momentum of the particles if we define the system to consist of (a) A and C, (b) B and C, (c) all three objects.

Figure E8.3



8.4 • Two vehicles are approaching an intersection. One is a 2500 kg pickup traveling at 14.0 m/s from east to west (the $-x$-direction), and the other is a 1500 kg sedan going from south to north (the $+y-$ direction) at 23.0 m/s. (a) Find the x- and y-components of the net momentum of this system. (b) What are the magnitude and direction of the net momentum?

8.5 • One 110 kg football lineman is running to the right at 2.75 m/s while another 125 kg lineman is running directly toward him at 2.60 m/s. What are (a) the magnitude and direction of the net momentum of these two athletes, and (b) their total kinetic energy?

8.6 •• BIO **Biomechanics.** The mass of a regulation tennis ball is 57 g (although it can vary slightly), and tests have shown that the ball is in contact with the tennis racket for 30 ms. (This number can also vary, depending on the racket and swing.) We shall assume a 30.0 ms contact time. One of the fastest-known served tennis balls was served by "Big Bill" Tilden in 1931, and its speed was measured to be 73 m/s. (a) What impulse and what total force did Big Bill exert on the tennis ball in his record serve? (b) If Big Bill's opponent returned his serve with a speed of 55 m/s, what total force and what impulse did he exert on the ball, assuming only horizontal motion?

8.7 • **Force of a Golf Swing.** A 0.0450 kg golf ball initially at rest is given a speed of 25.0 m/s when a club strikes it. If the club and ball are in contact for 2.00 ms, what average force acts on the ball? Is the effect of the ball's weight during the time of contact significant? Why or why not?

8.8 • **Force of a Baseball Swing.** A baseball has mass 0.145 kg. (a) If the velocity of a pitched ball has a magnitude of 45.0 m/s and the batted ball's velocity is 55.0 m/s in the opposite direction, find the magnitude of the change in momentum of the ball and of the impulse applied to it by the bat. (b) If the ball remains in contact with the bat for 2.00 ms, find the magnitude of the average force applied by the bat.

8.9 • A 0.160 kg hockey puck is moving on an icy, frictionless, horizontal surface. At $t = 0$, the puck is moving to the right at 3.00 m/s. (a) Calculate the velocity of the puck (magnitude and direction) after a force of 25.0 N directed to the right has been applied for 0.050 s. (b) If, instead, a force of 12.0 N directed to the left is applied from $t = 0$ to $t = 0.050$ s, what is the final velocity of the puck?

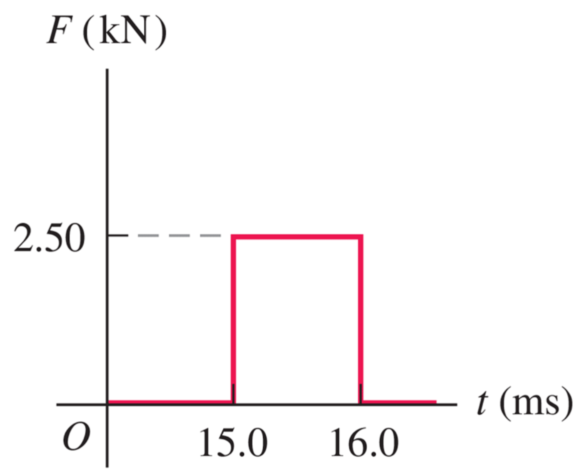
8.10 •• A bat strikes a 0.145 kg baseball. Just before impact, the ball is traveling horizontally to the right at 40.0 m/s; when it leaves the bat, the ball is traveling to the left at an angle of 30° above horizontal with a speed of 52.0 m/s. If the ball and bat are in contact for 1.75 ms, find the horizontal and vertical components of the average force on the ball.

8.11 • **CALC** At time $t = 0$ a 2150 kg rocket in outer space fires an engine that exerts an increasing force on it in the $+x-$direction. This force obeys the equation $F_x = At^2$, where t is time, and has a magnitude of 781.25 N when $t = 1.25$ s. (a) Find the SI value of the constant A, including its units. (b) What impulse does the engine exert on the rocket during the 1.50 s interval starting 2.00 s after the engine is fired? (c) By how much does the rocket's velocity change during this interval? Assume constant mass.

8.12 •• A packing crate with mass 80.0 kg is at rest on a horizontal, frictionless surface. At $t = 0$ a net horizontal force in the $+x$-direction is applied to the crate. The force has a constant value of 80.0 N for 12.0 s and then decreases linearly with time so it becomes zero after an additional 6.00 s. What is the final speed of the crate, 18.0 s after the force was first applied?

8.13 • A 2.00 kg stone is sliding to the right on a frictionless, horizontal surface at 5.00 m/s when it is suddenly struck by an object that exerts a large horizontal force on it for a short period of time. The graph in Fig. E8.13 shows the magnitude of this force as a function of time. (a) What impulse does this force exert on the stone? (b) Just after the force stops acting, find the magnitude and direction of the stone's velocity if the force acts (i) to the right or (ii) to the left.

Figure E8.13



8.14 •• **CALC** Starting at $t = 0$, a horizontal net external force $\vec{F} = (0.280\ \mathrm{N/s})t\hat{\imath} + (-0.450\ \mathrm{N/s^2})t^2\hat{\jmath}$ is applied to a box that has an initial momentum $\vec{p} = (-3.00\ \mathrm{kg \cdot m/s})\hat{\imath} + (4.00\ \mathrm{kg \cdot m/s})\hat{\jmath}$. What is the momentum of the box at $t = 2.00$ s?

8.15 • A young ice skater with mass 40.0 kg has fallen and is sliding on the frictionless ice of a skating rink with a speed of 20.0 m/s. (a) What is the magnitude of her linear momentum when she has this speed? (b) What is her kinetic energy? (c) What constant net horizontal force must be applied to the skater to bring her to rest in 5.00 s?

Section 8.2 Conservation of Momentum

8.16 • A 68.5 kg astronaut is doing a repair in space on the orbiting space station. She throws a 2.25 kg tool away from her at 3.20 m/s relative to the space station. What will be the change in her speed as a result of this throw?

8.17 •• The expanding gases that leave the muzzle of a rifle also contribute to the recoil. A .30 caliber bullet has mass 0.00720 kg and a speed of 601 m/s relative to the muzzle when fired from a rifle that has mass 2.80 kg. The loosely held rifle recoils at a speed of 1.85 m/s relative to the earth. Find the momentum of the propellant gases in a coordinate system attached to the earth as they leave the muzzle of the rifle.

8.18 • Two figure skaters, one weighing 625 N and the other 725 N, push off against each other on frictionless ice. (a) If the heavier skater travels at 1.50 m/s, how fast will the lighter one travel? (b) How much kinetic energy is "created" during the skaters' maneuver, and where does this energy come from?

8.19 • BIO **Animal Propulsion.** Squids and octopuses propel themselves by expelling water. They do this by keeping water in a cavity and then suddenly contracting the cavity to force out the water through an opening. A 6.50 kg squid (including the water in the cavity) at rest suddenly sees a dangerous predator. (a) If the squid has 1.75 kg of water in its cavity, at what speed must it expel this water suddenly to achieve a speed of 2.50 m/s to escape the predator? Ignore any drag effects of the surrounding water. (b) How much kinetic energy does the squid create by this maneuver?

8.20 •• You are standing on a sheet of ice that covers the football stadium parking lot in Buffalo; there is negligible friction between your feet and the ice. A friend throws you a 0.600 kg ball that is traveling horizontally at 10.0 m/s. Your mass is 70.0 kg. (a) If you catch the ball, with what speed do you and the ball move afterward? (b) If the ball hits you and bounces off your chest, so afterward it is moving horizontally at 8.0 m/s in the opposite direction, what is your speed after the collision?

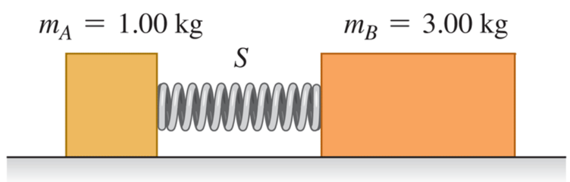
8.21 •• On a frictionless, horizontal air table, puck A (with mass 0.250 kg) is moving toward puck B (with mass 0.350 kg), which is initially at rest. After the collision, puck A has a velocity of 0.120 m/s to the left, and puck B has a velocity of 0.650 m/s to the right. (a) What was the speed of puck A before the collision? (b) Calculate the change in the total kinetic energy of the system that occurs during the collision.

8.22 •• When cars are equipped with flexible bumpers, they will bounce off each other during low-speed collisions, thus causing less damage. In one such accident, a 1750 kg car traveling to the right at 1.50 m/s collides with a 1450 kg car going to the left at 1.10 m/s. Measurements show that the heavier car's speed just after the collision was 0.250 m/s in its original direction. Ignore any road friction during the collision. (a) What was the speed of the lighter car just after the collision? (b) Calculate the change in the combined kinetic energy of the two-car system during this collision.

8.23 •• Two identical 0.900 kg masses are pressed against opposite ends of a light spring of force constant 1.75 N/cm, compressing the spring by 20.0 cm from its normal length. Find the speed of each mass when it has moved free of the spring on a frictionless, horizontal table.

8.24 • Block A in Fig. E8.24 has mass 1.00 kg, and block B has mass 3.00 kg. The blocks are forced together, compressing a spring S between them; then the system is released from rest on a level, frictionless surface. The spring, which has negligible mass, is not fastened to either block and drops to the surface after it has expanded. Block B acquires a speed of 1.20 m/s. (a) What is the final speed of block A? (b) How much potential energy was stored in the compressed spring?

Figure E8.24



8.25 •• A hunter on a frozen, essentially frictionless pond uses a rifle that shoots 4.20 g bullets at 965 m/s. The mass of the hunter (including his gun) is 72.5 kg, and the hunter holds tight to the gun after firing it. Find the recoil velocity of the hunter if he fires the rifle (a) horizontally and (b) at 56.0° above the horizontal.

8.26 • An atomic nucleus suddenly bursts apart (fissions) into two pieces. Piece A, of mass m_A, travels off to the left with speed v_A. Piece B, of mass m_B, travels off to the right with speed v_B. (a) Use conservation of momentum to solve for v_B in terms of m_A, m_B, and v_A. (b) Use the results of part (a) to show that $K_A/K_B = m_B/m_A$,where K_A and K_B are the kinetic energies of the two pieces.

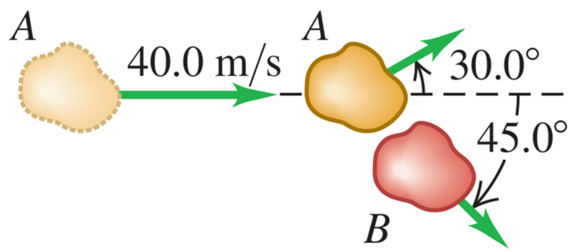
8.27 •• Two ice skaters, Daniel (mass 65.0 kg) and Rebecca (mass 45.0 kg), are practicing. Daniel stops to tie his shoelace and, while at rest, is struck by Rebecca, who is moving at 13.0 m/s before she collides with him. After the collision, Rebecca has a velocity of magnitude 8.00 m/s at an angle of 53.1° from her initial direction. Both skaters move on the frictionless, horizontal surface of the rink. (a) What are the magnitude and direction of Daniel's velocity after the collision? (b) What is the change in total kinetic energy of the two skaters as a result of the collision?

8.28 •• You are standing on a large sheet of frictionless ice and holding a large rock. In order to get off the ice, you throw the rock so it has velocity 12.0 m/s relative to the earth at an angle of 35.0° above the horizontal. If your mass is 70.0 kg and the rock's mass is 3.00 kg, what is your speed after you throw the rock? (See Discussion Question Q8.7.)

8.29 •• You (mass 55 kg) are riding a frictionless skateboard (mass 5.0 kg) in a straight line at a speed of 4.5 m/s. A friend standing on a balcony above you drops a 2.5 kg sack of flour straight down into your arms. (a) What is your new speed while you hold the sack? (b) Since the sack was dropped vertically, how can it affect your *horizontal* motion? Explain. (c) Now you try to rid yourself of the extra weight by throwing the sack straight up. What will be your speed while the sack is in the air? Explain.

8.30 •• **Asteroid Collision.** Two asteroids of equal mass in the asteroid belt between Mars and Jupiter collide with a glancing blow. Asteroid A, which was initially traveling at 40.0 m/s, is deflected 30.0° from its original direction, while asteroid B, which was initially at rest, travels at 45.0° to the original direction of A (Fig. E8.30). (a) Find the speed of each asteroid after the collision. (b) What fraction of the original kinetic energy of asteroid A dissipates during this collision?

Figure E8.30



Section 8.3 Momentum Conservation and Collisions

8.31 • An ice hockey forward with mass 70.0 kg is skating due north with a speed of 5.5 m/s. As the forward approaches the net for a slap shot, a defensive player (mass 110 kg) skates toward him in order to apply a body-check. The defensive player is traveling south at 4.0 m/s just before they collide. If the two players become intertwined and move together after they collide, in what direction and at what speed do they move after the collision? Friction between the two players and the ice can be neglected.

8.32 • Two skaters collide and grab on to each other on frictionless ice. One of them, of mass 70.0 kg, is moving to the right at 4.00 m/s, while the other, of mass 65.0 kg, is moving to the left at 2.50 m/s. What are the magnitude and direction of the velocity of these skaters just after they collide?

8.33 •• A 15.0 kg fish swimming at 1.10 m/s suddenly gobbles up a 4.50 kg fish that is initially stationary. Ignore any drag effects of the water. (a) Find the speed of the large fish just after it eats the small one. (b) How much total mechanical energy was dissipated during this meal?

8.34 •• **CP** An apple with mass M is hanging at rest from the lower end of a light vertical rope. A dart of mass $M/4$ is shot vertically upward, strikes the bottom of the apple, and remains embedded in it. If the speed of the dart is v_0 just before it strikes the apple, how high does the apple move upward because of its collision with the dart?

8.35 •• Two large blocks of wood are sliding toward each other on the frictionless surface of a frozen pond. Block A has mass 4.00 kg and is initially sliding east at 2.00 m/s. Block B has mass 6.00 kg and is initially sliding west at 2.50 m/s. The blocks collide head-on. After the collision block B is sliding east at 0.50 m/s. What is the decrease in the total kinetic energy of the two blocks as a result of the collision?

8.36 • A 1050 kg sports car is moving westbound at 15.0 m/s on a level road when it collides with a 6320 kg truck driving east on the same road at 10.0 m/s. The two vehicles remain locked together after the collision. (a) What is the velocity (magnitude and direction) of the two vehicles just after the collision? (b) At what speed should the truck have been moving so that both it and the car are stopped in the collision? (c) Find the change in kinetic energy of the system of two vehicles for the situations of parts (a) and (b). For which situation is the change in kinetic energy greater in magnitude?

8.37 •• On a very muddy football field, a 110 kg linebacker tackles an 85 kg halfback. Immediately before the collision, the linebacker is slipping with a velocity of 8.8 m/s north and the halfback is sliding with a velocity of 7.2 m/s east. What is the velocity (magnitude and direction) at which the two players move together immediately after the collision?

8.38 •• **Accident Analysis.** Two cars collide at an intersection. Car A, with a mass of 2000 kg, is going from west to east, while car B, of mass 1500 kg, is going from north to south at 15 m/s. As a result, the two cars become enmeshed and move as one. As an expert witness, you inspect the scene and determine that, after the collision, the enmeshed cars moved at an angle of 65° south of east from the point of impact. (a) How fast were the enmeshed cars moving just after the collision? (b) How fast was car A going just before the collision?

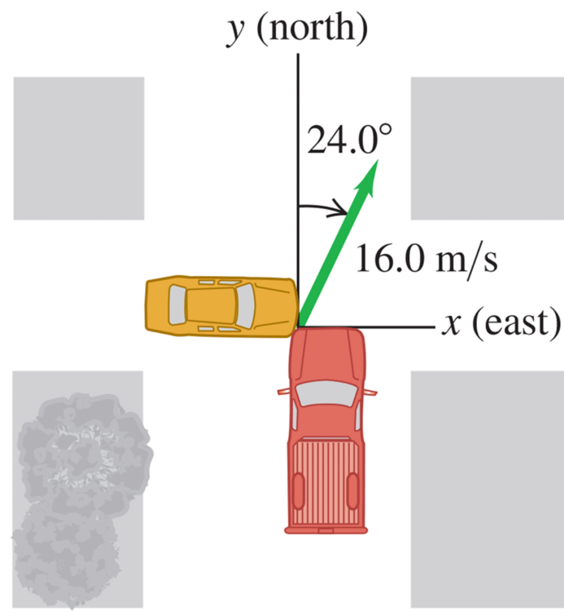
8.39 •• Jack (mass 55.0 kg) is sliding due east with speed 8.00 m/s on the surface of a frozen pond. He collides with Jill (mass

48.0 kg), who is initially at rest. After the collision, Jack is traveling at 5.00 m/s in a direction 34.0° north of east. What is Jill's velocity (magnitude and direction) after the collision? Ignore friction.

8.40 •• BIO **Bird Defense.** To protect their young in the nest, peregrine falcons will fly into birds of prey (such as ravens) at high speed. In one such episode, a 600 g falcon flying at 20.0 m/s hit a 1.50 kg raven flying at 9.0 m/s. The falcon hit the raven at right angles to the raven's original path and bounced back at 5.0 m/s. (These figures were estimated by the author as he watched this attack occur in northern New Mexico.) (a) By what angle did the falcon change the raven's direction of motion? (b) What was the raven's speed right after the collision?

8.41 • At the intersection of Texas Avenue and University Drive, a yellow subcompact car with mass 950 kg traveling east on University collides with a red pickup truck with mass 1900 kg that is traveling north on Texas and has run a red light (Fig. E8.41). The two vehicles stick together as a result of the collision, and the wreckage slides at 16.0 m/s in the direction 24.0° east of north. Calculate the speed of each vehicle before the collision. The collision occurs during a heavy rainstorm; ignore friction forces between the vehicles and the wet road.

Figure E8.41

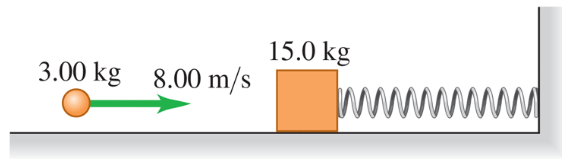


8.42 •• A 5.00 g bullet is fired horizontally into a 1.20 kg wooden block resting on a horizontal surface. The coefficient of kinetic friction between block and surface is 0.20. The bullet remains embedded in the block, which is observed to slide 0.310 m along the surface before stopping. What was the initial speed of the bullet?

8.43 •• **A Ballistic Pendulum.** A 12.0 g rifle bullet is fired with a speed of 380 m/s into a ballistic pendulum with mass 6.00 kg, suspended from a cord 70.0 cm long (see Example 8.8 in Section 8.3). Compute (a) the vertical height through which the pendulum rises, (b) the initial kinetic energy of the bullet, and (c) the kinetic energy of the bullet and pendulum immediately after the bullet becomes embedded in the wood.

8.44 •• **Combining Conservation Laws.** A 15.0 kg block is attached to a very light horizontal spring of force constant 500.0 N/m and is resting on a frictionless horizontal table (Fig. E8.44). Suddenly it is struck by a 3.00 kg stone traveling horizontally at 8.00 m/s to the right, whereupon the stone rebounds at 2.00 m/s horizontally to the left. Find the maximum distance that the block will compress the spring after the collision.

Figure E8.44



8.45 • • **CP** A 0.800 kg ornament is hanging by a 1.50 m wire when the ornament is suddenly hit by a 0.200 kg missile traveling horizontally at 12.0 m/s. The missile embeds itself in the ornament during the collision. What is the tension in the wire immediately after the collision?

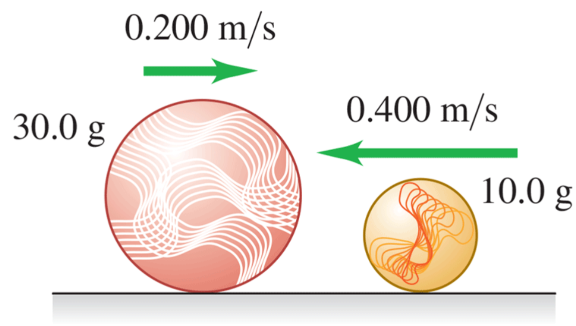
Section 8.4 Elastic Collisions

8.46 •• A 0.150 kg glider is moving to the right with a speed of 0.80 m/s on a frictionless, horizontal air track. The glider has a head-on collision with a 0.300 kg glider that is moving to the left with a speed of 2.20 m/s. Find the final velocity (magnitude and direction) of each glider if the collision is elastic.

8.47 •• Blocks A (mass 2.00 kg) and B (mass 6.00 kg) move on a frictionless, horizontal surface. Initially, block B is at rest and block A is moving toward it at 2.00 m/s. The blocks are equipped with ideal spring bumpers, as in Example 8.10 (Section 8.4). The collision is head-on, so all motion before and after the collision is along a straight line. (a) Find the maximum energy stored in the spring bumpers and the velocity of each block at that time. (b) Find the velocity of each block after they have moved apart.

8.48 • A 10.0 g marble slides to the left at a speed of 0.400 m/s on the frictionless, horizontal surface of an icy New York sidewalk and has a head-on, elastic collision with a larger 30.0 g marble sliding to the right at a speed of 0.200 m/s (Fig. E8.48). (a) Find the velocity of each marble (magnitude and direction) after the collision. (Since the collision is head-on, all motion is along a line.) (b) Calculate the *change in momentum* (the momentum after the collision minus the momentum before the collision) for each marble. Compare your values for each marble. (c) Calculate the *change in kinetic energy* (the kinetic energy after the collision minus the kinetic energy before the collision) for each marble. Compare your values for each marble.

Figure E8.48



8.49 •• **Moderators.** Canadian nuclear reactors use *heavy water* moderators in which elastic collisions occur between the neutrons and deuterons of mass 2.0 u (see Example 8.11 in Section 8.4). (a) What is the speed of a neutron, expressed as a fraction of its original speed, after a head-on, elastic collision with a deuteron that is initially at rest? (b) What is its kinetic energy, expressed as a fraction of its original kinetic energy? (c) How many such successive collisions will reduce the speed of a neutron to $1/59,000$ of its original value?

8.50 •• You are at the controls of a particle accelerator, sending a beam of $1.50 \times 10^7 \mathrm{m/s}$ protons (mass m) at a gas target of an unknown element. Your detector tells you that some protons bounce straight back after a collision with one of the nuclei of the unknown element. All such protons rebound with a speed of $1.20 \times 10^7 \mathrm{m/s}$. Assume that the initial speed of the target nucleus is negligible and the collision is elastic. (a) Find the mass of one nucleus of the unknown element. Express your answer in terms of the proton mass m. (b) What is the speed of the unknown nucleus immediately after such a collision?

8.51 •• Object B is at rest when object A collides with it. The collision is one-dimensional and elastic. After the collision object B has half the velocity that object A had before the collision. (a) Which object has the greater mass? (b) How much greater? (c) If the velocity of object A before the collision was 6.0 m/s to the right, what is its velocity after the collision?

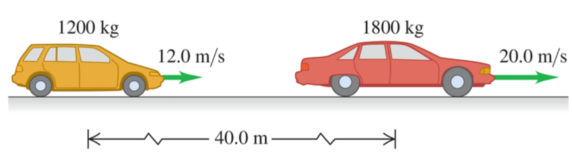
Section 8.5 Center of Mass

8.52 • Find the position of the center of mass of the system of the sun and Jupiter. (Since Jupiter is more massive than the rest of the solar planets combined, this is essentially the position of the center of mass of the solar system.) Does the center of mass lie inside or outside the sun? Use the data in Appendix F.

8.53 •• **Pluto and Charon.** Pluto's diameter is approximately 2370 km, and the diameter of its satellite Charon is 1250 km. Although the distance varies, they are often about 19,700 km apart, center to center. Assuming that both Pluto and Charon have the same composition and hence the same average density, find the location of the center of mass of this system relative to the center of Pluto.

8.54 • A 1200 kg SUV is moving along a straight highway at 12.0 m/s. Another car, with mass 1800 kg and speed 20.0 m/s, has its center of mass 40.0 m ahead of the center of mass of the SUV (Fig. E8.54). Find (a) the position of the center of mass of the system consisting of the two cars; (b) the magnitude of the system's total momentum, by using the given data; (c) the speed of the system's center of mass; (d) the system's total momentum, by using the speed of the center of mass. Compare your result with that of part (b).

Figure E8.54



8.55 • A uniform cube with mass 0.500 kg and volume 0.0270 m^3 is sitting on the floor. A uniform sphere with radius 0.400 m and mass 0.800 kg sits on top of the cube. How far is the center of mass of the two-object system above the floor?

8.56 • At one instant, the center of mass of a system of two particles is located on the x-axis at $x = 2.0$ m and has a velocity of $(5.0\ \mathrm{m/s})\hat{\imath}$. One of the particles is at the origin. The other particle has a mass of 0.10 kg and is at rest on the x-axis at $x = 8.0$ m. (a) What is the mass of the particle at the origin? (b) Calculate the total momentum of this system. (c) What is the velocity of the particle at the origin?

8.57 •• In Example 8.14 (Section 8.5), Ramon pulls on the rope to give himself a speed of 1.10 m/s. What is James's speed?

8.58 • **CALC** A system consists of two particles. At $t = 0$ one particle is at the origin; the other, which has a mass of 0.50 kg, is on the y-axis at $y = 6.0$ m. At $t = 0$ the center of mass of the system is on the y-axis at $y = 2.4$ m. The velocity of the center of mass is given by $(0.75\mathrm{m/s^3})t^2\hat{\imath}$. (a) Find the total mass of the system. (b) Find the acceleration of the center of mass at any time t. (c) Find the net external force acting on the system at $t = 3.0$ s.

8.59 • **CALC** A radio-controlled model airplane has a momentum given by

$$\left[(-0.75\ \mathrm{kg\cdot m/s^3})t^2 + (3.0\ \mathrm{kg\cdot m/s})\right]\hat{\imath} + (0.25\ \mathrm{kg\cdot m/s^2})t\hat{\jmath}.$$

What are the x-, y-, and z-components of the net external force on the airplane?

8.60 •• You have three identical, uniform, square pieces of wood, each with side length L. You stack the three pieces of wood at the edge of the horizontal top of a table. The first block extends a distance $L/4$ past the edge of the table. The next block extends a distance $L/4$ past the edge of the first block, so a distance $L/2$ past the edge of the table. The third block extends a distance $L/4$ past the edge of the block beneath it, so $3L/4$ past the edge of the table. The stack is unstable if the center of mass of the stack extends beyond the edge of the table. Calculate the horizontal location of the center of mass of the three-block stack.

Section 8.6 Rocket Propulsion

8.61 •• A 70 kg astronaut floating in space in a 110 kg MMU (manned maneuvering unit) experiences an acceleration of $0.029\ \mathrm{m/s^2}$ when he fires one of the MMU's thrusters. (a) If the speed of the escaping N_2 gas relative to the astronaut is 490 m/s, how much gas is used by the thruster in 5.0 s? (b) What is the thrust of the thruster?

8.62 • A small rocket burns 0.0500 kg of fuel per second, ejecting it as a gas with a velocity relative to the rocket of magnitude 1600 m/s. (a) What is the thrust of the rocket? (b) Would the rocket operate in outer space where there is no atmosphere? If so, how would you steer it? Could you brake it?

Problems

8.63 •• **CP** A large shipping crate is at rest on the horizontal floor of a warehouse. The coefficient of static friction between the crate and the floor is $\mu_s = 0.500$; the coefficient of kinetic friction is $\mu_k = 0.300$. (a) Estimate the weight, in pounds, of the heaviest crate you could start sliding by pushing on it horizontally. Based on this estimate, what is the magnitude of the maximum force you can apply? (b) If you continue to push on the crate with a constant force equal to the force calculated in part (a), use the impulse–momentum theorem to calculate how long you must push on the crate to give it a speed of 8.0 m/s. Don't forget to take into account the kinetic friction force. (c) With the same force, how far would the crate travel before reaching a speed of 8.0 m/s? Use the work–energy theorem.

8.64 •• A steel ball with mass 40.0 g is dropped from a height of 2.00 m onto a horizontal steel slab. The ball rebounds to a height of 1.60 m. (a) Calculate the impulse delivered to the ball during impact. (b) If the ball is in contact with the slab for 2.00 ms, find the average force on the ball during impact.

8.65 •• Just before it is struck by a racket, a tennis ball weighing 0.560 N has a velocity of $(20.0\,\mathrm{m/s}\hat{\imath} - (4.0\ \mathrm{m/s})\hat{\jmath}$. During the 3.00 ms that the racket and ball are in contact, the net external force on the ball is constant and equal to $-(380\,\mathrm{N})\hat{\imath} + (110\ \mathrm{N})\hat{\jmath}$. What are the x- and y-components (a) of the impulse of the net external force applied to the ball; (b) of the final velocity of the ball?

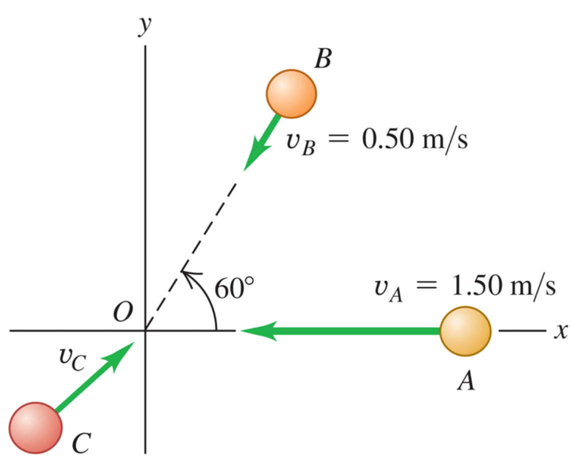
8.66 •• Starting at $t = 0$ a net external force in the $+x$-direction is applied to an object that has mass 2.00 kg. A graph of the force as a function of time is a straight line that passes through the origin and has slope 3.00 N/s. If the object is at rest at $t = 0$, what is the magnitude of the force when the object has reached a speed of 9.00 m/s?

8.67 •• Blocks A (mass 2.00 kg) and B (mass 10.00 kg, to the right of A) move on a frictionless, horizontal surface. Initially, block B is moving to the left at 0.500 m/s and block A is moving to the right at 2.00 m/s. The blocks are equipped with ideal spring bumpers, as in Example 8.10 (Section 8.4). The collision is head-on, so all motion before and after it is along a straight line. Find (a) the maximum energy stored in the spring bumpers and the velocity of each block at that time; (b) the velocity of each block after they have moved apart.

8.68 •• A railroad handcar is moving along straight, frictionless tracks with negligible air resistance. In the following cases, the car initially has a total mass (car and contents) of 200 kg and is traveling east with a velocity of magnitude 5.00 m/s. Find the *final velocity* of the car in each case, assuming that the handcar does not leave the tracks. (a) A 25.0 kg mass is thrown sideways out of the car with a velocity of magnitude 2.00 m/s relative to the car's initial velocity. (b) A 25.0 kg mass is thrown backward out of the car with a velocity of 5.00 m/s relative to the initial motion of the car. (c) A 25.0 kg mass is thrown into the car with a velocity of 6.00 m/s relative to the ground and opposite in direction to the initial velocity of the car.

8.69 • Spheres A (mass 0.020 kg), B (mass 0.030 kg), and C (mass 0.050 kg) are approaching the origin as they slide on a frictionless air table. The initial velocities of A and B are given in Fig. P8.69. All three spheres arrive at the origin at the same time and stick together. (a) What must the x- and y-components of the initial velocity of C be if all three objects are to end up moving at 0.50 m/s in the $+x-$ direction after the collision? (b) If C has the velocity found in part (a), what is the change in the kinetic energy of the system of three spheres as a result of the collision?

Figure P8.69



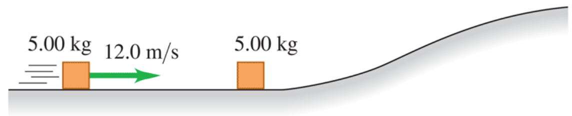
8.70 •• Starting at $t = 0$ a net external force $F(t)$ in the $+x$-direction is applied to an object that has mass 3.00 kg and is initially at rest. The force is zero at $t = 0$ and increases linearly to 5.00 N at $t = 4.00$ s. The force then decreases linearly until it becomes zero at $t = 10.0$ s. (a) Draw a graph of F versus t from $t = 0$ to $t = 10.0$ s. (b) What is the speed of the object at $t = 10.0$ s?

8.71 •• **CP** An 8.00 kg block of wood sits at the edge of a frictionless table, 2.20 m above the floor. A 0.500 kg blob of clay slides along the length of the table with a speed of 24.0 m/s, strikes the block of wood, and sticks to it. The combined object leaves the edge of the table and travels to the floor. What horizontal distance has the combined object traveled when it reaches the floor?

8.72 ••• **CP** A small wooden block with mass 0.800 kg is suspended from the lower end of a light cord that is 1.60 m long. The block is initially at rest. A bullet with mass 12.0 g is fired at the block with a horizontal velocity v_0. The bullet strikes the block and becomes embedded in it. After the collision the combined object swings on the end of the cord. When the block has risen a vertical height of 0.800 m, the tension in the cord is 4.80 N. What was the initial speed v_0 of the bullet?

8.73 •• **Combining Conservation Laws.** A 5.00 kg chunk of ice is sliding at 12.0 m/s on the floor of an ice-covered valley when it collides with and sticks to another 5.00 kg chunk of ice that is initially at rest (Fig. P8.73). Since the valley is icy, there is no friction. After the collision, how high above the valley floor will the combined chunks go?

Figure P8.73



8.74 •• CP Block B (mass 4.00 kg) is at rest at the edge of a smooth platform, 2.60 m above the floor. Block A (mass 2.00 kg) is sliding with a speed of 8.00 m/s along the platform toward block B. A strikes B and rebounds with a speed of 2.00 m/s. The collision projects B horizontally off the platform. What is the speed of B just before it strikes the floor?

8.75 •• Two carts of equal mass are on a horizontal, frictionless air track. Initially cart A is moving toward stationary cart B with a speed of v_A. The carts undergo an inelastic collision, and after the collision the total kinetic energy of the two carts is one-half their initial total kinetic energy before the collision. What is the speed of each cart after the collision?

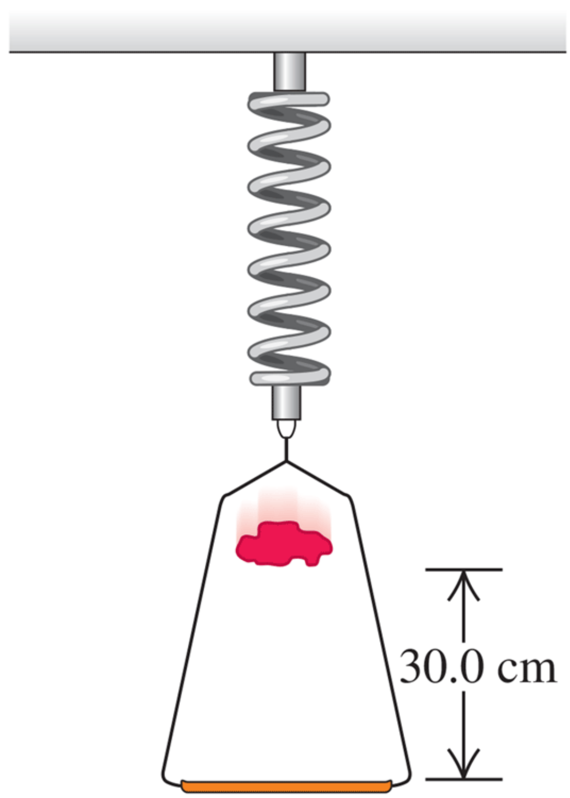
8.76 •• **Automobile Accident Analysis.** You are called as an expert witness to analyze the following auto accident: Car B, of mass 1900 kg, was stopped at a red light when it was hit from behind by car A, of mass 1500 kg. The cars locked bumpers during the collision and slid to a stop with brakes locked on all wheels. Measurements of the skid marks left by the tires showed them to be 7.15 m long. The coefficient of kinetic friction between the tires and the road was 0.65. (a) What was the speed of car A just before the collision? (b) If the speed limit was 35 mph, was car A speeding, and if so, by how many miles per hour was it *exceeding* the speed limit?

8.77 •• A 1500 kg sedan goes through a wide intersection traveling from north to south when it is hit by a 2200 kg

SUV traveling from east to west. The two cars become enmeshed due to the impact and slide as one thereafter. On-the-scene measurements show that the coefficient of kinetic friction between the tires of these cars and the pavement is 0.75, and the cars slide to a halt at a point 5.39 m west and 6.43 m south of the impact point. How fast was each car traveling just before the collision?

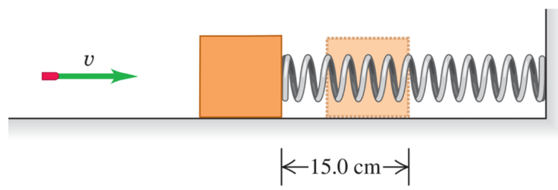
8.78 ••• **CP** A 0.150 kg frame, when suspended from a coil spring, stretches the spring 0.0400 m. A 0.200 kg lump of putty is dropped from rest onto the frame from a height of 30.0 cm (Fig. P8.78). Find the maximum distance the frame moves downward from its initial equilibrium position.

Figure P8.78



8.79 • A rifle bullet with mass 8.00 g strikes and embeds itself in a block with mass 0.992 kg that rests on a frictionless, horizontal surface and is attached to an ideal spring (Fig. P8.79). The impact compresses the spring 15.0 cm. Calibration of the spring shows that a force of 0.750 N is required to compress the spring 0.250 cm. (a) Find the magnitude of the block's velocity just after impact. (b) What was the initial speed of the bullet?

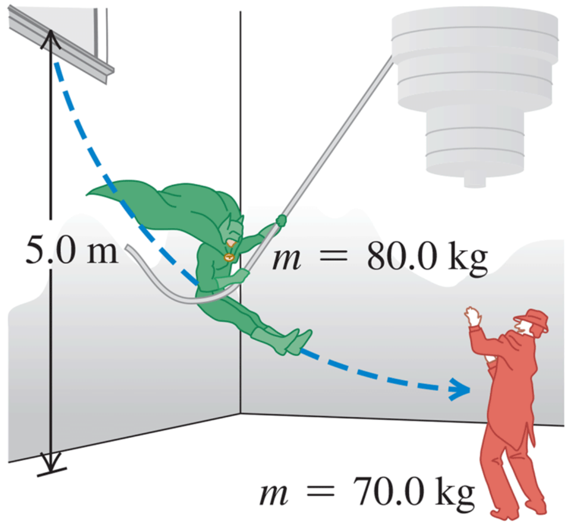
Figure P8.79



8.80 •• **A Ricocheting Bullet.** A 0.100 kg stone rests on a frictionless, horizontal surface. A bullet of mass 6.00 g, traveling horizontally at 350 m/s, strikes the stone and rebounds horizontally at right angles to its original direction with a speed of 250 m/s. (a) Compute the magnitude and direction of the velocity of the stone after it is struck. (b) Is the collision perfectly elastic?

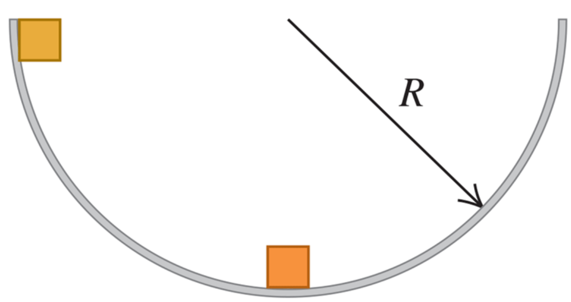
8.81 •• A movie stuntman (mass 80.0 kg) stands on a window ledge 5.0 m above the floor (Fig. P8.81). Grabbing a rope attached to a chandelier, he swings down to grapple with the movie's villain (mass 70.0 kg), who is standing directly under the chandelier. (Assume that the stuntman's center of mass moves downward 5.0 m. He releases the rope just as he reaches the villain.) (a) With what speed do the entwined foes start to slide across the floor? (b) If the coefficient of kinetic friction of their bodies with the floor is $\mu_k = 0.250$, how far do they slide?

Figure P8.81



8.82 •• CP Two identical masses are released from rest in a smooth hemispherical bowl of radius R from the positions shown in Fig. P8.82. Ignore friction between the masses and the surface of the bowl. If the masses stick together when they collide, how high above the bottom of the bowl will they go after colliding?

Figure P8.82



8.83 •• Objects A and B undergo a one-dimensional elastic collision. The initial speed of A is $v_{A\mathrm{i}}$ and the initial speed of B is zero. Equations (8.24) and **(8.25)** give the final velocity components of the objects after the collision. Let $m_A = \alpha m_B$, where α is a constant. (a) What is the value of α if the final kinetic energy of B equals the initial kinetic energy of A? (b) What is α if the final kinetic energies of A and B are equal?

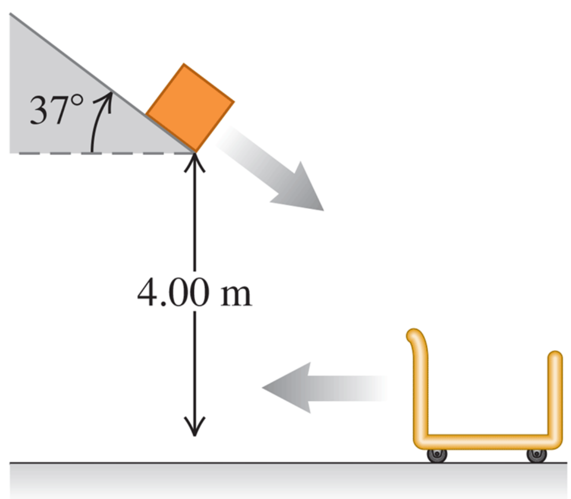
8.84 ••• **CP** A 20.00 kg lead sphere is hanging from a hook by a thin wire 2.80 m long and is free to swing in a complete circle. Suddenly it is struck horizontally by a 5.00 kg steel dart that embeds itself in the lead sphere. What must be the minimum initial speed of the dart so that the combination makes a complete circular loop after the collision?

8.85 •• A 4.00 g bullet, traveling horizontally with a velocity of magnitude 400 m/s, is fired into a wooden block with mass 0.800 kg, initially at rest on a level surface. The bullet passes through the block and emerges with its speed reduced to 190 m/s. The block slides a distance of 72.0 cm along the surface from its initial position. (a) What is the coefficient of kinetic friction between block and surface? (b) What is the decrease in kinetic energy of the bullet? (c) What is the kinetic energy of the block at the instant after the bullet passes through it?

8.86 •• A 5.00 g bullet is shot *through* a 1.00 kg wood block suspended on a string 2.00 m long. The center of mass of the block rises a distance of 0.38 cm. Find the speed of the bullet as it emerges from the block, along a horizontal, straight line, if its initial speed is 450 m/s.

8.87 •• **CP** In a shipping company distribution center, an open cart of mass 50.0 kg is rolling to the left at a speed of 5.00 m/s (Fig. P8.87). Ignore friction between the cart and the floor. A 15.0 kg package slides down a chute that is inclined at 37° from the horizontal and leaves the end of the chute with a speed of 3.00 m/s. The package lands in the cart and they roll together. If the lower end of the chute is a vertical distance of 4.00 m above the bottom of the cart, what are (a) the speed of the package just before it lands in the cart and (b) the final speed of the cart?

Figure P8.87



8.88 ••• **Neutron Decay**. A neutron at rest decays (breaks up) to a proton and an electron. Energy is released in the decay and appears as kinetic energy of the proton and electron. The mass of a proton is 1836 times the mass of an electron. What fraction of the total energy released goes into the kinetic energy of the proton?

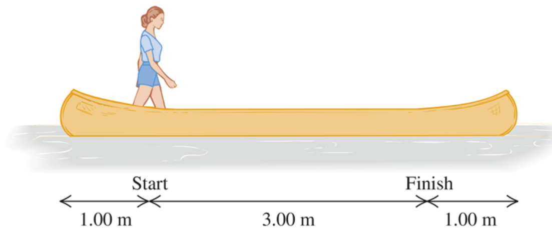
8.89 •• CP In a game of physics and skill, a rigid block (A) with mass m sits at rest at the edge of a frictionless air table, 1.20 m above the floor. You slide an identical block (B) with initial speed v_B toward A. The blocks have a head-on elastic collision, and block A leaves the table with a horizontal velocity. The goal of the game is to have block A land on a target on the floor. The target is a horizontal distance of 2.00 m from the edge of the table. What is the initial speed v_B that accomplishes this? Neglect air resistance.

8.90 •• Jonathan and Jane are sitting in a sleigh that is at rest on frictionless ice. Jonathan's weight is 800 N, Jane's weight is 600 N, and that of the sleigh is 1000 N. They see a poisonous spider on the floor of the sleigh and immediately jump off. Jonathan jumps to the left with a velocity of 5.00 m/s at 30.0° above the horizontal (relative to the ice), and Jane jumps to the right at 7.00 m/s at 36.9° above the horizontal (relative to the ice). Calculate the sleigh's horizontal velocity (magnitude and direction) after they jump out.

8.91 •• Friends Burt and Ernie stand at opposite ends of a uniform log that is floating in a lake. The log is 3.0 m long and has mass 20.0 kg. Burt has mass 30.0 kg; Ernie has mass 40.0 kg. Initially, the log and the two friends are at rest relative to the shore. Burt then offers Ernie a cookie, and Ernie walks to Burt's end of the log to get it. Relative to the shore, what distance has the log moved by the time Ernie reaches Burt? Ignore any horizontal force that the water exerts on the log, and assume that neither friend falls off the log.

8.92 •• A 45.0 kg woman stands up in a 60.0 kg canoe 5.00 m long. She walks from a point 1.00 m from one end to a point 1.00 m from the other end (Fig. P8.92). If you ignore resistance to motion of the canoe in the water, how far does the canoe move during this process?

Figure P8.92



8.93 •• CP Two sticky spheres are suspended from light ropes of length L that are attached to the ceiling at a common point. Sphere A has mass $2m$ and is hanging at rest with its rope vertical. Sphere B has mass m and is held so that its rope makes an angle with the vertical that puts B a vertical height H above A. Sphere B is released from rest and swings down, collides with sphere A, and sticks to it. In terms of H, what is the maximum height above the original position of A reached by the combined spheres after their collision?

8.94 •• CP In a fireworks display, a rocket is launched from the ground with a speed of 18.0 m/s and a direction of 51.0° above the horizontal. During the flight, the rocket explodes into two pieces of equal mass (see Fig. 8.32). (a) What horizontal distance from the launch point will the center of mass of the two pieces be after both have landed on the ground? (b) If one piece lands a horizontal distance of 26.0 m from the launch point, where does the other piece land?

8.95 •• A block with mass 0.500 kg sits at rest on a light but not long vertical spring that has spring constant 80.0 N/m and one end on the floor. (a) How much elastic potential energy is stored in the spring when the block is sitting at rest on it? (b) A second identical block is dropped onto the first from a height of 4.00 m above the first block and sticks to it. What is the maximum elastic potential energy stored in the spring during the motion of the blocks after the collision? (c) What is the maximum distance the first block moves down after the second block has landed on it?

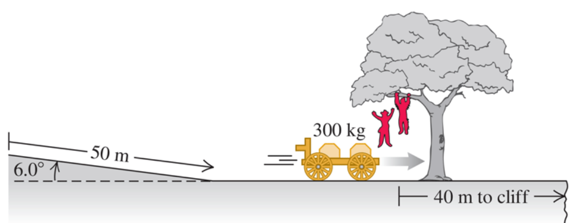
8.96 •• **CP** A 20.0 kg projectile is fired at an angle of 60.0° above the horizontal with a speed of 80.0 m/s. At the highest point of its trajectory, the projectile explodes into two fragments with equal mass, one of which falls vertically with zero initial speed. Ignore air resistance. (a) How far from the point of firing does the other fragment strike if the terrain is level? (b) How much energy is released during the explosion?

8.97 ••• **CP** A fireworks rocket is fired vertically upward. At its maximum height of 80.0 m, it explodes and breaks into two pieces: one with mass 1.40 kg and the other with mass 0.28 kg. In the explosion, 860 J of chemical energy is converted to kinetic energy of the two fragments. (a) What is the speed of each fragment just after the explosion? (b) It is observed that the two fragments hit the ground at the same time. What is the distance between the points on the ground where they land? Assume that the ground is level and air resistance can be ignored.

8.98 ••• A 12.0 kg shell is launched at an angle of 55.0° above the horizontal with an initial speed of 150 m/s. At its highest point, the shell explodes into two fragments, one three times heavier than the other. The two fragments reach the ground at the same time. Ignore air resistance. If the heavier fragment lands back at the point from which the shell was launched, where will the lighter fragment land, and how much energy was released in the explosion?

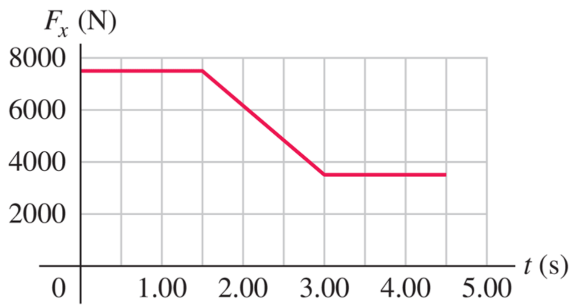
8.99 • **CP** An outlaw cuts loose a wagon with two boxes of gold, of total mass 300 kg, when the wagon is at rest 50 m up a 6.0° slope. The outlaw plans to have the wagon roll down the slope and across the level ground, and then fall into a canyon where his accomplices wait. But in a tree 40 m from the canyon's cliff wait the Lone Ranger (mass 75.0 kg) and Tonto (mass 60.0 kg). They drop vertically into the wagon as it passes beneath them (Fig. P8.99). (a) If they require 5.0 s to grab the gold and jump out, will they make it before the wagon goes over the cliff? The wagon rolls with negligible friction. (b) When the two heroes drop into the wagon, is the kinetic energy of the system of heroes plus wagon conserved? If not, does it increase or decrease, and by how much?

Figure P8.99



8.100 •• **DATA** A 2004 Prius with a 150 lb driver and no passengers weighs 3071 lb. The car is initially at rest. Starting at $t = 0$, a net horizontal force $F_x(t)$ in the $+x$-direction is applied to the car. The force as a function of time is given in Fig. P8.100. (a) For the time interval $t = 0$ to $t = 4.50$ s, what is the impulse applied to the car? (b) What is the speed of the car at $t = 4.50$ s? (c) At $t = 4.50$ s, the 3500 N net external force is replaced by a constant net braking force $B_x = -5200$ N. Once the braking force is first applied, how long does it take the car to stop? (d) How much work must be done on the car by the braking force to stop the car? (e) What distance does the car travel from the time the braking force is first applied until the car stops?

Figure P8.100



8.101 •• **DATA** In your job in a police lab, you must design an apparatus to measure the muzzle velocities of bullets fired from handguns. Your solution is to attach a 2.00 kg wood block that rests on a horizontal surface to a light horizontal spring. The other end of the spring is attached to a wall. Initially the spring is at its equilibrium length. A bullet is fired horizontally into the block and remains embedded in it. After the bullet strikes the block, the block compresses the spring a maximum distance d. You have measured that the coefficient of kinetic friction between the block and the horizontal surface is 0.38. The table lists some firearms that you'll test:

Bullet ID	Type	Bullet Mass (grains)	Muzzle Velocity (ft/s)
A	.38Spec Glaser Blue	80	1667
B	.38Spec Federal	125	945
C	.44Spec Remington	240	851
D	.44Spec Winchester	200	819
E	.45ACP Glaser Blue	140	1355

Source: www.chuckhawks.com

Source: www.chuckhawks.com

A grain is a unit of mass equal to 64.80 mg. (a) Of bullets A through E, which will produce the maximum compression of the spring? The minimum? (b) You want the maximum compression of the spring to be 0.25 m. What must be the force constant of the spring? (c) For the bullet that produces the minimum spring compression, what is the compression d if the spring has the force constant calculated in part (b)?

8.102 •• **DATA** For the Texas Department of Public Safety, you are investigating an accident that occurred early on a foggy morning in a remote section of the Texas Panhandle. A 2012 Prius traveling due north collided in a highway intersection with a 2013 Dodge Durango that was traveling due east. After the collision, the wreckage of the two vehicles was locked together and skidded across the level ground until it struck a tree. You measure that the tree is 35 ft from the point of impact. The line from the point of impact to the tree is in a direction 39° north of east. From experience, you estimate that the coefficient of kinetic friction between the ground and the wreckage is 0.45. Shortly before the collision, a highway patrolman with a radar gun measured the speed of the Prius to be 50 mph and, according to a witness, the Prius driver made no attempt to slow down. Four people with a total weight of 460 lb were in the Durango. The only person in the Prius was the 150 lb driver. The Durango with its passengers had a weight of 6500 lb, and the Prius with its driver had a weight of 3042 lb. (a) What was the Durango's speed just before the collision? (b) How fast was the wreckage traveling just before it struck the tree?

Challenge Problems

8.103 •• CALC **A Variable-Mass Raindrop.** In a rocket-propulsion problem the mass is variable. Another such problem is a raindrop falling through a cloud of small water droplets. Some of these small droplets adhere to the raindrop, thereby *increasing* its mass as it falls. The force on the raindrop is

$$F_{\text{ext}} = \frac{dp}{dt} = m\frac{dv}{dt} + v\frac{dm}{dt}$$

Suppose the mass of the raindrop depends on the distance x that it has fallen. Then $m = kx$, where k is a constant, and $dm/dt = kv$. This gives, since $F_{\text{ext}} = mg$,

$$mg = m\frac{dv}{dt} + v\,(kv)$$

Or, dividing by k,

$$xg = x\frac{dv}{dt} + v^2$$

This is a differential equation that has a solution of the form $v = at$, where a is the acceleration and is constant. Take the initial velocity of the raindrop to be zero. (a) Using the proposed solution for v, find the acceleration a. (b) Find the distance the raindrop has fallen in $t = 3.00$ s. (c) Given that $k = 2.00$ g/m, find the mass of the raindrop at $t = 3.00$ s. (For many more intriguing aspects of this problem, see K. S. Krane, *American Journal of Physics*, Vol. 49 (1981), pp. 113–117.)

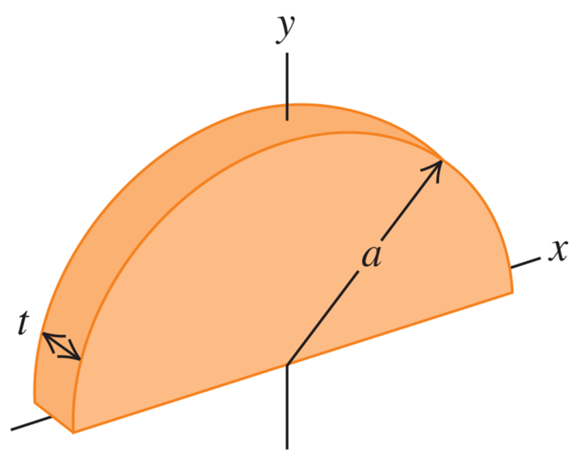
8.104 •• CALC In Section 8.5 we calculated the center of mass by considering objects composed of a *finite* number of point masses or objects that, by symmetry, could be represented by a finite number of point masses. For a solid object whose mass distribution does not allow for a simple determination of the center of mass by symmetry, the sums of Eqs. (8.28) must be generalized to integrals

$$x_{\mathrm{cm}} = \frac{1}{M}\int x\,dm \quad y_{\mathrm{cm}} = \frac{1}{M}\int y\,dm$$

where x and y are the coordinates of the small piece of the object that has mass dm. The integration is over the whole of the object. Consider a thin rod of length L, mass M, and cross-sectional area A. Let the origin of the coordinates be at the left end of the rod and the positive x-axis lie along the rod. (a) If the density $\rho = M/V$ of the object is uniform, perform the integration described above to show that the x-coordinate of the center of mass of the rod is at its geometrical center. (b) If the density of the object varies linearly with x-that is, $\rho = \alpha x$, where α is a positive constant —calculate the x-coordinate of the rod's center of mass.

8.105 •• **CALC** Use the methods of Challenge Problem 8.104 to calculate the x- and y-coordinates of the center of mass of a semicircular metal plate with uniform density ρ and thickness t. Let the radius of the plate be a. The mass of the plate is thus $M = \dfrac{1}{2}\rho\pi a^2 t$. Use the coordinate system indicated in Fig. P8.105.

Figure P8.105



MCAT-Style Passage Problems

BIO Momentum and the Archerfish. Archerfish are tropical fish that hunt by shooting drops of water from their mouths at insects above the water's surface to knock them into the water, where the fish can eat them. A 65 g fish at rest just at the surface of the water can expel a 0.30 g drop of water in a short burst of 5.0 ms. High-speed measurements show that the water has a speed of 2.5 m/s just after the archerfish expels it.

8.106 What is the momentum of one drop of water immediately after it leaves the fish's mouth? (a) $7.5 \times 10^{-4}\ \mathrm{kg \cdot m/s}$; (b) $1.5 \times 10^{-4}\ \mathrm{kg \cdot m/s}$; (c) $7.5 \times 10^{-3}\ \mathrm{kg \cdot m/s}$; (d) $1.5 \times 10^{-3}\ \mathrm{kg \cdot m/s}$.

8.107 What is the speed of the archerfish immediately after it expels the drop of water? (a) $0.0025\ \mathrm{m/s}$; (b) $0.012\ \mathrm{m/s}$; (c) $0.75\ \mathrm{m/s}$; (d) $2.5\ \mathrm{m/s}$.

8.108 What is the average force the fish exerts on the drop of water? (a) 0.00015 N; (b) 0.00075 N; (c) 0.075 N; (d) 0.15 N.

8.109 The fish shoots the drop of water at an insect that hovers on the water's surface, so just before colliding with the insect, the drop is still moving at the speed it had when it left the fish's mouth.

In the collision, the drop sticks to the insect, and the speed of the insect and water just after the collision is measured to be 2.0 m/s. What is the insect's mass? (a) 0.038 g; (b) 0.075 g; (c) 0.24 g; (d) 0.38 g.

Answers: Momentum, Impulse, and Collisions

Chapter Opening Question ?

(ii) Both hailstones have the same magnitude of momentum $p = mv$ (the product of mass and speed), but the faster, lighter hailstone has twice the kinetic energy $K = \frac{1}{2}mv^2$ of the slower, heavier one. Hence, the lightweight hailstone can do the most work on whatever it hits (and do the most damage) in the process of coming to a halt (see Section 8.1).

Test Your Understanding

8.1 (v), (i) and (ii) (tied for second place), (iii) and (iv) (tied for third place) We use two interpretations of the impulse of the net external force: (1) the net external force multiplied by the time that the net external force acts, and (2) the change in momentum of the particle on which the net external force acts. Which interpretation we use depends on what information we are given. We take the positive x-direction to be to the east. (i) The force is not given, so we use interpretation 2:

$J_x = mv_{2x} - mv_{1x} = (1000\text{ kg})(0) - (1000\text{ kg})(25\text{ m/s}) = -25{,}000\text{ kg·m/s}$, so the magnitude of the impulse is $25{,}000\text{ kg} \cdot \text{m/s} = 25{,}000\text{ N} \cdot \text{s}$. (ii) For the same reason and values as in (i), we use interpretation 2, and the magnitude of the impulse is again $25{,}000\text{ N} \cdot \text{s}$. (iii) The final velocity is not given, so we use interpretation 1:

$J_x = (\sum F_x)_{\text{av}}(t_2 - t_1) = (2000\text{ N})(10\text{ s}) = 20{,}000\text{ N·s}$, so the magnitude of the impulse is $20{,}000\text{ N} \cdot \text{s}$. (iv) For the same reason as in (iii), we use interpretation 1:

$J_x = (\sum F_x)_{\text{av}}(t_2 - t_1) = (-2000\text{ N})(10\text{ s}) = -20{,}000\text{ N} \cdot \text{s}$, so the magnitude of the impulse is $20{,}000\text{ N} \cdot \text{s}$. (v) The force is not given, so we use interpretation 2:

$J_x = mv_{2x} - mv_{1x} = (1000 \text{ kg})(-25 \text{ m/s}) -$ $(1000 \text{ kg})(25 \text{ m/s}) = -50,000 \text{ kg·m/s}$, so the magnitude of the impulse is $50,000 \text{ kg} \cdot \text{m/s} = 50,000 \text{ N} \cdot \text{s}$.

8.2 (a) $\boldsymbol{v_{C2x} > 0, v_{C2y} > 0,}$ (b) **piece *C*** There are no external horizontal forces, so the x- and y-components of the total momentum of the system are conserved. Both components of the total momentum are zero before the spring releases, so they must be zero after the spring releases. Hence,

$$P_x = 0 = m_A v_{A2x} + m_B v_{B2x} + m_C v_{C2x}$$
$$P_y = 0 = m_A v_{A2y} + m_B v_{B2y} + m_C v_{C2y}$$

We are given that $m_A = m_B = m_C$, $v_{A2x} < 0$, $v_{A2y} = 0$, $v_{B2x} = 0$, and $v_{B2y} < 0$. You can solve the above equations to show that $v_{C2x} = -v_{A2x} > 0$ and $v_{C2y} = -v_{B2y} > 0$, so both velocity components of piece C are positive. Piece C has speed $\sqrt{v_{C2x}{}^2 + v_{C2y}{}^2} = \sqrt{v_{A2x}{}^2 + v_{B2y}{}^2}$, which is greater than the speed of either piece A or piece B.

8.3 (a) **elastic,** (b) **inelastic,** (c) **completely inelastic** In each case gravitational potential energy is converted to kinetic energy as the ball falls, and the collision is between the ball and the ground. In (a) all of the initial energy is converted back to gravitational potential energy, so no kinetic energy is lost in the bounce and the collision is elastic. In (b) there is less gravitational potential energy at the end than at the beginning, so some kinetic energy is lost in the bounce. Hence the collision is inelastic. In (c) the ball loses all of its kinetic energy, the ball and the ground stick together, and the collision is completely inelastic.

8.4 **worse** After colliding with a water molecule initially at rest, the neutron has speed $|(m_\text{n} - m_\text{w})/(m_\text{n} + m_\text{w})| = |(1.0 \text{ u} - 18 \text{ u})/(1.0 \text{ u} + 18 \text{ u})| = \dfrac{17}{19}$ of its initial speed, and its kinetic energy is $\left(\dfrac{17}{19}\right)^2 = 0.80$ of the initial value. Hence a water molecule is a worse moderator than a carbon atom, for which the corresponding numbers are $\frac{11}{13}$ and $\left(\frac{11}{13}\right)^2 = 0.72.$

8.5 **no** If gravity is the only force acting on the system of two fragments, the center of mass will follow the parabolic trajectory of a freely falling object. Once a fragment lands, however, the ground exerts a normal force on that fragment. Hence the net external force on the system has changed, and the trajectory of the center of mass changes in response.

8.6 **(a) increasing, (b) decreasing** From Eqs. (8.37) and (8.38), the thrust F is equal to $m\left(dv/dt\right)$, where m is the rocket's mass and dv/dt is its acceleration. Because m decreases with time, if the thrust F is constant, then the acceleration must increase with time (the same force acts on a smaller mass); if the acceleration dv/dt is constant, then the thrust must decrease with time (a smaller force is all that's needed to accelerate a smaller mass).

Key Example Variation Problems

VP8.6.1 **a.** $0.200\ \mathrm{m/s}$, to the left

b. $0.900\ \mathrm{m/s}$, to the left

VP8.6.2 **a.** $7.14\ \mathrm{m/s}$, to the west

b. $-9.00\ \mathrm{m/s}$ for Madeleine, $+10.6\ \ +10.6\,\mathrm{m/s}$ for Buffy; Buffy

VP8.6.3 **a.** $+4.32$ kg·m/s; $-4.32\ \mathrm{kg}\cdot\mathrm{m/s}$

b. $1.53\ \mathrm{m/s}$

c. $+11.8\ \mathrm{kg}\cdot\mathrm{m/s}$

d. $4.92\ \mathrm{m/s}$

VP8.6.4 **a.** $1/2$

b. $\upsilon_{\mathrm{P2}} = \upsilon_{\mathrm{P1}}/2\cos\theta,\ \upsilon_{\mathrm{S2}} = \upsilon_{\mathrm{P1}}/4\cos\theta$

VP8.9.1 **a.** $8.00\ \mathrm{m/s}$ before, $1.60\ \mathrm{m/s}$ after; 25.6 J lost

b. 4.00 m/s before, 3.20 m/s after; 6.4 J lost (c) case (i)

VP8.9.2 **a.** 1.00 m/s; momentum

b. 3.57 m/s; total mechanical energy

VP8.9.3 **a.** 1.35 m/s, in the $+x$-direction

b. can before, 2.70 J; can after, 0.817 J; box after, 1.09 J

c. inelastic; kinetic energy is lost, but the objects do not stick together after the collision

VP8.9.4 30.4°, north of east

VP8.14.1 $x_{\mathrm{cm}} = +0.135 \mathrm{~m}$, $y_{\mathrm{cm}} = -0.140 \mathrm{~m}$

VP8.14.2 $x = -3.53 \mathrm{~m}$

VP8.14.3 $0.240 \mathrm{~m}$

VP8.14.4 $x = -L/6$, $y = +7L/4$

Bridging Problem

(a) $1.00 \mathrm{~m/s}$, to the right

(b) elastic

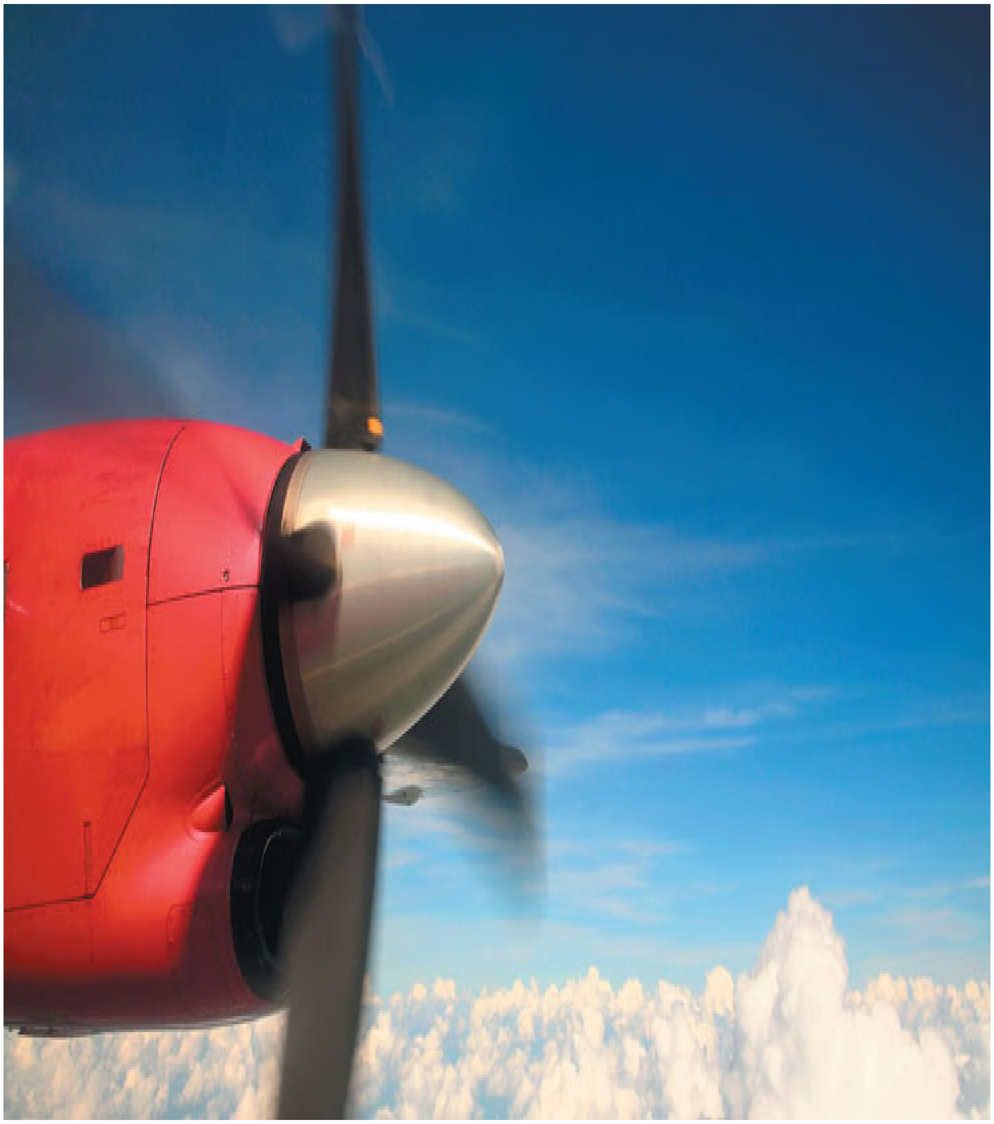
(c) $1.93 \mathrm{~m/s}$, at -30.4°

(d) $2.31 \mathrm{~kg} \cdot \mathrm{m/s}$, at 149.6°

(e) inelastic

(f) $1.67 \mathrm{~m/s}$, in the $+x$-direction

Chapter 9
Rotation of Rigid Bodies



?☐ Each blade on a rotating airplane propeller is like a long, thin rod. If each blade were stretched to double its length (while the mass of each blade and the propeller's angular speed stay the same), by what factor would the kinetic energy of the rotating blades increase? (i) 2; (ii) 4; (iii) 8; (iv) the kinetic energy would not change; (v) the kinetic energy would decrease, not increase.

Learning Outcomes

In this chapter, you'll learn...

- 9.1 How to describe the rotation of a rigid body in terms of angular coordinate, angular velocity, and angular acceleration.
- 9.2 How to analyze rigid-body rotation when the angular acceleration is constant.
- 9.3 How to relate the rotation of a rigid body to the linear velocity and linear acceleration of a point on the body.
- 9.4 The meaning of a body's moment of inertia about a rotation axis, and how it relates to rotational kinetic energy.
- 9.5 How to relate the values of a body's moment of inertia for two different but parallel rotation axes.
- 9.6 How to calculate the moment of inertia of bodies with various shapes.

You'll need to review...

- 1.10 Vector product of two vectors.
- 2.2, 2.3, 2.4 Linear velocity, linear acceleration, and motion with constant acceleration.
- 3.4 Motion in a circle.
- 7.1 Using mechanical energy to solve problems.

What do the motions of an airplane propeller, a Blu-ray disc, a Ferris wheel, and a circular saw blade have in common? None of these can be represented adequately as a moving *point;* each involves an object that *rotates* about an axis that is stationary in some inertial frame of reference.

Rotation occurs at all scales, from the motions of electrons in atoms to the motions of entire galaxies. We need to develop some general methods for analyzing the motion of a rotating object. In this chapter and the next we consider objects that have definite size and definite shape, and that in general can have rotational as well as translational motion.

Real-world objects can be very complicated; the forces that act on them can deform them—stretching, twisting, and squeezing them. We'll ignore these deformations for now and assume that the object has a perfectly definite and unchanging shape and size. We call this idealized model a **rigid body.** This chapter and the next are mostly about rotational motion of a rigid body.

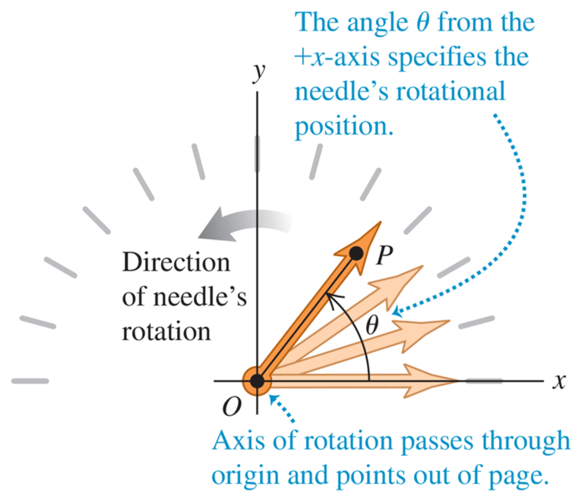
We begin with kinematic language for *describing* rotational motion. Next we look at the kinetic energy of rotation, the key to using energy methods for rotational motion. Then in Chapter 10 we'll develop dynamic principles that relate the forces on a body to its rotational motion.

9.1 Angular Velocity and Acceleration

In analyzing rotational motion, let's think first about a rigid body that rotates about a *fixed axis*—an axis that is at rest in some inertial frame of reference and does not change direction relative to that frame. The rotating rigid body might be a motor shaft, a chunk of beef on a barbecue skewer, or a merry-go-round.

Figure 9.1 shows a rigid body rotating about a fixed axis. The axis passes through point O and is perpendicular to the plane of the diagram, which we'll call the xy-plane. One way to describe the rotation of this body would be to choose a particular point P on the body and to keep track of the x- and y-coordinates of P. This isn't very convenient, since it takes two numbers (the two coordinates x and y) to specify the rotational position of the body. Instead, we notice that the line OP is fixed in the body and rotates with it. The angle θ that OP makes with the $+x$-axis is a single **angular cooordinate** that completely describes the body's rotational position.

Figure 9.1



A speedometer needle (an example of a rigid body) rotating counterclockwise about a fixed axis.

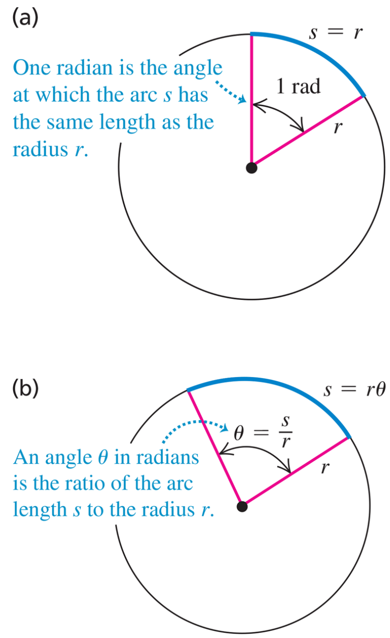
The angular coordinate θ of a rigid body rotating around a fixed axis can be positive or negative. If we choose positive angles to be measured counterclockwise from the positive x-axis, then the angle θ in Fig. 9.1 is positive. If we instead choose the positive rotation direction to be clockwise, then θ in Fig. 9.1 is negative. When we considered the motion of a particle along a straight line, it was essential to specify the direction of positive displacement along that line; when we discuss rotation around a fixed axis, it's just as essential to specify the direction of positive rotation.

The most natural way to measure the angle θ is not in degrees but in **radians.** As Fig. 9.2a shows, one radian (1 rad) is the angle subtended at the center of a circle by an arc with a length equal to the radius of the circle. In Fig. 9.2b an angle θ is subtended by an arc of length s on a circle of radius r. The value of θ (in radians) is equal to s divided by r:

(9.1)

$$\theta = \frac{s}{r} \qquad \text{or} \qquad s = r\theta \qquad (\theta \text{ in radians})$$

Figure 9.2



Measuring angles in radians.

An angle in radians is the ratio of two lengths, so it is a pure number, without dimensions. If $s = 3.0$ m and $r = 2.0$ m, then $\theta = 1.5$, but we'll often write this as 1.5 rad to distinguish it from an angle measured in degrees or revolutions.

The circumference of a circle (that is, the arc length all the way around the circle) is 2π times the radius, so there are 2π (about 6.283) radians in one complete revolution (360°). Therefore

$$1 \text{ rad} = \frac{360^\circ}{2\pi} = 57.3^\circ$$

Similarly, $180^\circ = \pi$ rad, $90^\circ = \pi/2$ rad, and so on. If we had measured angle θ in degrees, we would have needed an extra factor of $(2\pi/360)$ on the right-hand side of $s = r\,\theta$ in Eq. (9.1). By measuring angles in radians, we keep the relationship between angle and distance along an arc as simple as possible.

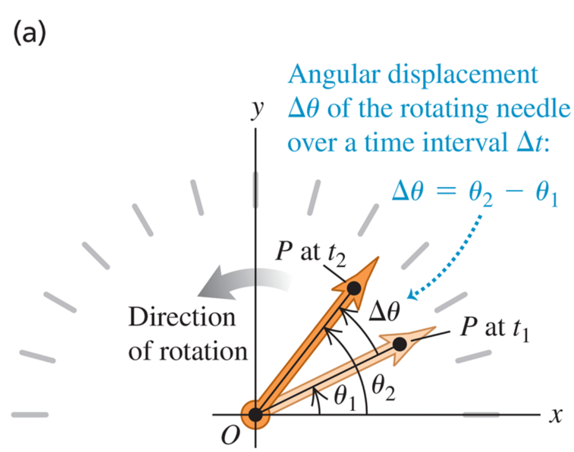
Angular Velocity

The coordinate θ shown in Fig. 9.1 specifies the rotational position of a rigid body at a given instant. We can describe the rotational *motion* of such a rigid body in terms of the rate of change of θ. We'll do this in an analogous way to our description of straight-line motion in Chapter 2. In Fig. 9.3a, a reference line OP in a rotating body makes an angle θ_1 with the $+x$-axis at time t_1. At a later time t_2 the angle has changed to θ_2. We define the **average angular velocity** $\omega_{\text{av-}z}$ (the Greek letter omega) of the body in the time interval $\Delta t = t_2 - t_1$ as the ratio of the **angular displacement** $\Delta\theta = \theta_2 - \theta_1$ to Δt:

(9.2)

$$\omega_{\text{av-}z} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

Figure 9.3



(b)

(a) Angular displacement $\Delta\theta$ of a rotating body. (b) Every part of a rotating rigid body has the same average angular velocity $\Delta\theta/\Delta t$.

The subscript z indicates that the body in Fig. 9.3a is rotating about the z-axis, which is perpendicular to the plane of the diagram. The **instantaneous angular velocity** ω_z is the limit of $\omega_{\text{av-}z}$ as Δt approaches zero:

(9.3)

The **instantaneous angular velocity** of a rigid body rotating around the z-axis ...

$$\omega_z = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \qquad (9.3)$$

... equals the limit of the body's average angular velocity as the time interval approaches zero ...

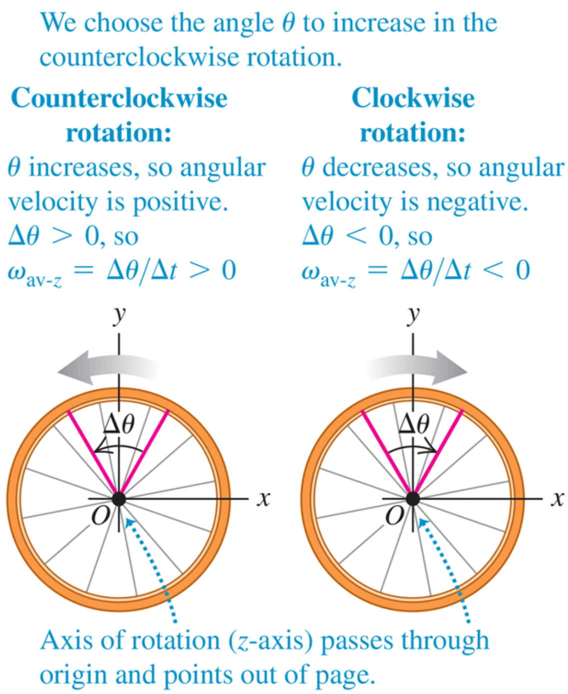
... and equals the instantaneous rate of change of the body's angular coordinate.

When we refer simply to "angular velocity," we mean the instantaneous angular velocity, not the average angular velocity.

The angular velocity ω_z can be positive or negative, depending on the direction in which the rigid body is rotating (Fig. 9.4). The angular *speed*

ω, which we'll use in Sections 9.3 and 9.4, is the magnitude of angular velocity. Like linear speed v, the angular speed is never negative.

Figure 9.4



A rigid body's average angular velocity (shown here) and instantaneous angular velocity can be positive or negative.

CAUTION Angular velocity vs. linear velocity Keep in mind the distinction between *angular* velocity ω_z and *linear* velocity v_x (see Section 2.2). If an object has a linear velocity v_x, the object as a whole is *moving* along the x-axis. By contrast, if an object has an angular velocity ω_z, then it is *rotating* around the z-axis. We do *not* mean that the object is moving along the z-axis.

Different points on a rotating rigid body move different distances in a given time interval, depending on how far each point lies from the rotation axis. But because the body is rigid, *all* points rotate through the same angle in the same time (Fig. 9.3b). Hence *at any instant, every part of a rotating rigid body has the same angular velocity.*

If angle θ is in radians, the unit of angular velocity is the radian per second (rad/s). Other units, such as the revolution per minute (rev/min or rpm), are often used. Since 1 rev $= 2\pi$ rad, two useful conversions are

$$1 \text{ rev/s} = 2\pi \text{ rad/s} \qquad \text{and} \qquad 1 \text{ rev/min} = 1 \text{ rpm} = \frac{2\pi}{60} \text{ rad/s}$$

That is, 1 rad/s is about 10 rpm.

Example 9.1 Calculating angular velocity

The angular position θ of a 0.36-m-diameter flywheel is given by

$$\theta = (2.0 \text{ rad/s}^3)t^3$$

(a) Find θ, in radians and in degrees, at $t_1 = 2.0$ s and $t_2 = 5.0$ s. (b) Find the distance that a particle on the flywheel rim moves from $t_1 = 2.0$ s to $t_2 = 5.0$ s. (c) Find the average angular velocity, in rad / s and in rev / min, over that interval. (d) Find the instantaneous angular velocities at $t_1 = 2.0$ s and $t_2 = 5.0$ s.

IDENTIFY and SET UP Our target variables are θ_1 and θ_2 (the angular positions at times t_1 and t_2) and the angular displacement $\Delta\theta = \theta_2 - \theta_1$. We'll find these from the given expression for θ as a function of time. Knowing $\Delta\theta$, we'll find the distance traveled and the average angular velocity between t_1 and t_2 by using Eqs. (9.1) and (9.2), respectively. To find the instantaneous angular velocities ω_{1z} (at time t_1) and ω_{2z} (at time t_2), we'll take the derivative of the given equation for θ with respect to time, as in Eq. (9.3).

EXECUTE (a) We substitute the values of t into the equation for θ:

$$\begin{aligned}\theta_1 &= (2.0\ \text{rad/s}^3)(2.0\ \text{s})^3 = 16\ \text{rad}\\ &= (16\ \text{rad})\frac{360^\circ}{2\pi\ \text{rad}} = 920^\circ\\ \theta_2 &= (2.0\ \text{rad/s}^3)(5.0\ \text{s})^3 = 250\ \text{rad}\\ &= (250\ \text{rad})\frac{360^\circ}{2\pi\ \text{rad}} = 14{,}000^\circ\end{aligned}$$

(b) During the interval from t_1 to t_2 the flywheel's angular displacement is $\Delta\theta = \theta_2 - \theta_1 = 250\ \text{rad} - 16\ \text{rad} = 234\ \text{rad}$. The radius r is half the diameter, or 0.18 m. To use Eq. (9.1), the angles *must* be expressed in radians:

$$s = r\,\theta_2 - r\,\theta_1 = r\Delta\theta = (0.18\ \text{m})(234\ \text{rad}) = 42\ \text{m}$$

We can drop "radians" from the unit for s because θ is a dimensionless number; like r, s is measured in meters.

(c) From Eq. (9.2),

$$\begin{aligned}\omega_{\text{av}-z} &= \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{250\ \text{rad} - 16\ \text{rad}}{5.0\ \text{s} - 2.0\ \text{s}} = 78\ \text{rad/s}\\ &= \left(78\ \frac{\text{rad}}{\text{s}}\right)\left(\frac{1\ \text{rev}}{2\pi\ \text{rad}}\right)\left(\frac{60\ \text{s}}{1\ \text{min}}\right) = 740\ \text{rev/min}\end{aligned}$$

(d) From Eq. (9.3),

$$\omega_z = \frac{d\theta}{dt} = \frac{d}{dt}\left[(2.0\ \text{rad/s}^3)t^3\right] = (2.0\text{rad/s}^3)\left(3t^2\right) = (6.0\text{rad/s}^3)t^2$$

At times $t_1 = 2.0$ s and $t_2 = 5.0$ s we have

$$\begin{aligned}\omega_{1z} &= (6.0\ \text{rad/s}^3)(2.0\ \text{s})^2 = 24\ \text{rad/s}\\ \omega_{2z} &= (6.0\ \text{rad/s}^3)(5.0\ \text{s})^2 = 150\ \text{rad/s}\end{aligned}$$

EVALUATE The angular velocity $\omega_z = (6.0\ \text{rad/s}^3)t^2$ increases with time. Our results are consistent with this; the instantaneous angular velocity at the end of the interval ($\omega_{2z} = 150$ rad/s) is greater than at the beginning ($\omega_{1z} = 24\ \text{rad/s}$), and the average angular velocity $\omega_{\text{av}-z} = 78\,\text{rad/s}$ over the interval is intermediate between these two values.

KEY CONCEPT

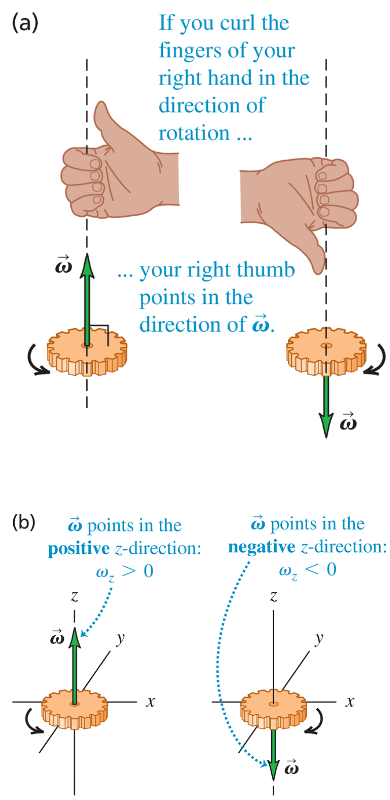
To find the *average* angular velocity of a rotating rigid body, first find the body's angular displacement (final angular position minus initial angular position) during a time interval. Then divide the result by that time interval. To find the rigid body's *instantaneous* angular velocity, take the derivative of its angular position with respect to time.

Video Tutor Solution: **Example 9.1**

Angular Velocity as a Vector

As we have seen, our notation for the angular velocity ω_z about the z-axis is reminiscent of the notation v_x for the ordinary velocity along the x-axis (see Section 2.2). Just as v_x is the x-component of the velocity vector $\vec{\boldsymbol{v}}$, ω_z is the z-component of an angular velocity *vector* $\vec{\boldsymbol{\omega}}$ directed along the axis of rotation. As Fig. 9.5a shows, the direction of $\vec{\boldsymbol{\omega}}$ is given by the right-hand rule that we used to define the vector product in Section 1.10. If the rotation is about the z-axis, then $\vec{\omega}$ has only a z-component. This component is positive if $\vec{\boldsymbol{\omega}}$ is along the positive z-axis and negative if $\vec{\boldsymbol{\omega}}$ is along the negative z-axis (Fig. 9.5b).

Figure 9.5



(a) The right-hand rule for the direction of the angular velocity vector $\vec{\omega}$. Reversing the direction of rotation reverses the direction of $\vec{\omega}$. (b) The sign of ω_z for rotation along the z-axis.

The vector formulation is especially useful when the direction of the rotation axis *changes*. We'll examine such situations briefly at the end of Chapter 10. In this chapter, however, we'll consider only situations in which the rotation axis is fixed. Hence throughout this chapter we'll use "angular velocity" to refer to ω_z, the component of $\vec{\boldsymbol{\omega}}$ along the axis.

CAUTION The angular velocity vector is perpendicular to the plane of rotation, not in it It's a common error to think that an object's angular velocity vector $\vec{\boldsymbol{\omega}}$ points in the direction in which some particular part of the object is moving. Another error is to think that $\vec{\boldsymbol{\omega}}$ is a "curved vector" that points around the rotation axis in the direction of rotation (like the curved arrows in Figs. 9.1, 9.3, and 9.4). Neither of these is true! Angular velocity is an attribute of the *entire* rotating rigid body, not any one part, and there's no such thing as a curved vector. We choose the direction of $\vec{\boldsymbol{\omega}}$ to be along the rotation axis—*perpendicular* to the plane of rotation—because that axis is common to every part of a rotating rigid body.

Angular Acceleration

A rigid body whose angular velocity changes has an *angular acceleration.* When you pedal your bicycle harder to make the wheels turn faster or apply the brakes to bring the wheels to a stop, you're giving the wheels an angular acceleration.

If ω_{1z} and ω_{2z} are the instantaneous angular velocities at times t_1 and t_2 , we define the **average angular acceleration** $\alpha_{\mathrm{av}-z}$ over the interval $\Delta t = t_2 - t_1$ as the change in angular velocity divided by Δt (Fig. 9.6):

(9.4)

$$\alpha_{\mathrm{av}-z} = \frac{\omega_{2z} - \omega_{1z}}{t_2 - t_1} = \frac{\Delta\omega_z}{\Delta t}$$

(9.5)

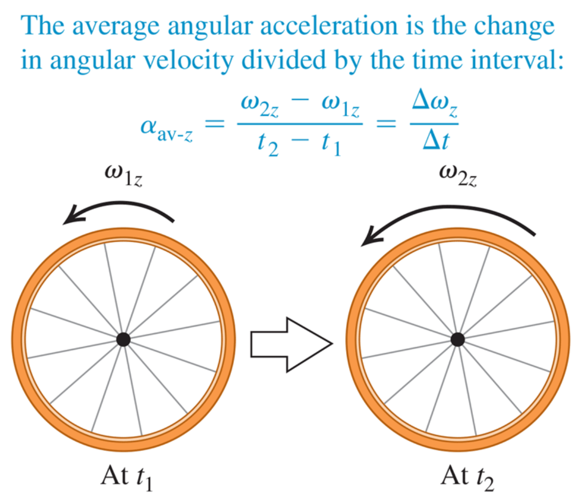
The **instantaneous angular acceleration** of a rigid body rotating around the z-axis ...

$$\alpha_z = \lim_{\Delta t \to 0} \frac{\Delta \omega_z}{\Delta t} = \frac{d\omega_z}{dt} \tag{9.5}$$

... equals the limit of the body's average angular acceleration as the time interval approaches zero ...

... and equals the instantaneous rate of change of the body's angular velocity.

Figure 9.6



Calculating the average angular acceleration of a rotating rigid body.

The **instantaneous angular acceleration** α_z is the limit of $\alpha_{\text{av}-z}$ as $\Delta t \to 0$:

The usual unit of angular acceleration is the radian per second per second, or rad/s^2. From now on we'll use the term "angular acceleration" to mean the instantaneous angular acceleration rather than the average angular acceleration.

Because $\omega_z = d\theta/dt$, we can also express angular acceleration as the second derivative of the angular coordinate:

(9.6)

$$\alpha_z = \frac{d}{dt}\frac{d\theta}{dt} = \frac{d^2\theta}{dt^2}$$

You've probably noticed that we use Greek letters for angular kinematic quantities: θ for angular position, ω_z for angular velocity, and α_z for angular acceleration. These are analogous to x for position, v_x for velocity, and a_x for acceleration in straight-line motion. In each case, velocity is the rate of change of position with respect to time and acceleration is the rate of change of velocity with respect to time. We sometimes use the terms "*linear* velocity" for v_x and "*linear* acceleration" for a_x to distinguish clearly between these and the *angular* quantities introduced in this chapter.

If the angular acceleration α_z is positive, then the angular velocity ω_z is increasing; if α_z is negative, then ω_z is decreasing. The rotation is speeding up if α_z and ω_z have the same sign and slowing down if α_z and ω_z have opposite signs. (These are exactly the same relationships as those between *linear* acceleration a_x and *linear* velocity v_x for straight-line motion; see Section 2.3.)

Example 9.2 Calculating angular acceleration

For the flywheel of Example 9.1, (a) find the average angular acceleration between $t_1 = 2.0$ s and $t_2 = 5.0$ s. (b) Find the instantaneous angular accelerations at $t_1 = 2.0$ s and $t_2 = 5.0$ s.

IDENTIFY and SET UP We use Eqs. (9.4) and (9.5) for the average and instantaneous angular accelerations.

EXECUTE (a) From Example 9.1, the values of ω_z at the two times are

$$\omega_{1z} = 24 \text{ rad/s} \quad \omega_{2z} = 150 \text{ rad/s}$$

From Eq. (9.4), the average angular acceleration is

$$\alpha_{\text{av}-z} = \frac{150\ \text{rad/s} - 24\ \text{rad/s}}{5.0\ \text{s} - 2.0\ \text{s}} = 42\ \text{rad/s}^2$$

(b) We found in Example 9.1 that $\omega_z = (6.0\ \text{rad/s}^3)t^2$ for the flywheel. From Eq. (9.5), the value of α_z at any time t is

$$\begin{aligned} \alpha_z &= \frac{d\omega_z}{dt} = \frac{d}{dt}[(6.0\ \text{rad/s}^3)(t^2)] = (6.0\ \text{rad/s}^3)(2t) \\ &= (12\ \text{rad/s}^3)t \end{aligned}$$

Hence

$$\begin{aligned} \alpha_{1z} &= (12\ \text{rad/s}^3)(2.0\ \text{s}) = 24\ \text{rad/s}^2 \\ \alpha_{2z} &= (12\ \text{rad/s}^3)(5.0\ \text{s}) = 60\ \text{rad/s}^2 \end{aligned}$$

EVALUATE The angular acceleration is *not* constant in this situation. The angular velocity ω_z is always increasing because α_z is always positive. Furthermore, the rate at which angular velocity increases is itself increasing, since α_z increases with time.

KEY CONCEPT

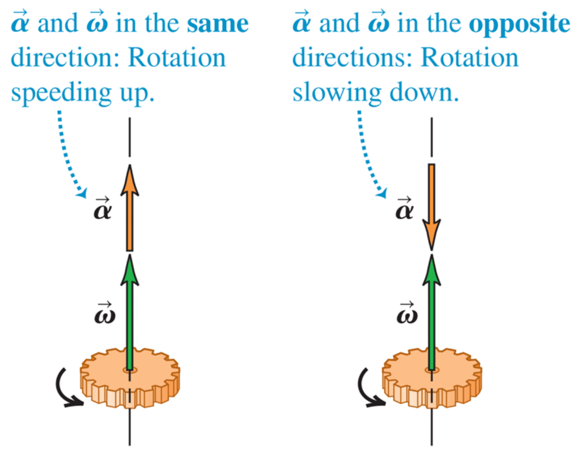
To find the average angular acceleration of a rotating rigid body, first find the change in its angular velocity (final angular velocity minus initial angular velocity) during a time interval. Then divide the result by the time interval.

Video Tutor Solution: **Example 9.2**

Angular Acceleration as a Vector

Just as we did for angular velocity, it's useful to define an angular acceleration *vector* $\vec{\boldsymbol{\alpha}}$. Mathematically, $\vec{\boldsymbol{\alpha}}$ is the time derivative of the angular velocity vector $\vec{\boldsymbol{\omega}}$. If the object rotates around the fixed z-axis, then $\vec{\boldsymbol{\alpha}}$ has only a z-component α_z. In this case, $\vec{\boldsymbol{\alpha}}$ is in the same direction as $\vec{\boldsymbol{\omega}}$ if the rotation is speeding up and opposite to $\vec{\boldsymbol{\omega}}$ if the rotation is slowing down (Fig. 9.7).

Figure 9.7

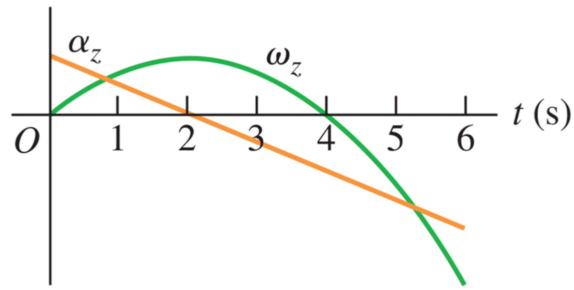


When the rotation axis is fixed, both the angular acceleration and angular velocity vectors lie along that axis.

The vector $\vec{\boldsymbol{\alpha}}$ will be particularly useful in Chapter 10 when we discuss what happens when the rotation axis changes direction. In this chapter, however, the rotation axis will always be fixed and we need only the z-component α_z.

Test Your Understanding of Section 9.1

The figure shows a graph of ω_z and α_z versus time for a particular rotating body. (a) During which time intervals is the rotation speeding up? (i) $0 < t < 2$ s; (ii) 2 s $< t <$ 4 s; (iii) 4 s $< t <$ 6 s. (b) During which time intervals is the rotation slowing down? (i) $0 < t < 2$ s; (ii) 2 s $< t <$ 4 s; (iii) 4 s<t $<$ 6 s.



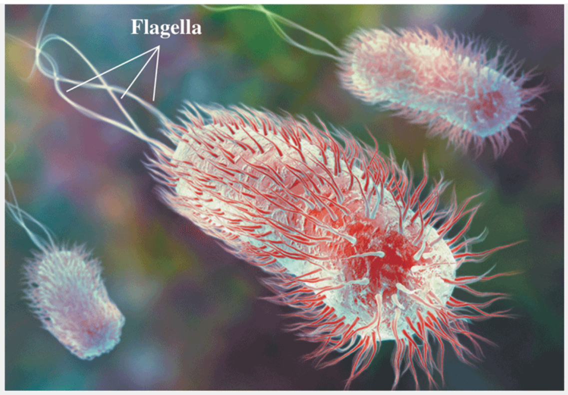
9.2 Rotation with Constant Angular Acceleration

In Chapter 2 we found that straight-line motion is particularly simple when the acceleration is constant. This is also true of rotational motion about a fixed axis. When the angular acceleration is constant, we can derive equations for angular velocity and angular position by using the same procedure that we used for straight-line motion in Section 2.4. In fact, the equations we are about to derive are identical to Eqs. (2.8), (2.12), (2.13), and (2.14) if we replace x with θ, v_x with ω_z , and a_x with α_z. We suggest that you review Section 2.4 before continuing.

BIO Application

Rotational Motion in Bacteria

Escherichia coli bacteria (about 2 μm by 0.5 μm) are found in the lower intestines of humans and other warm-blooded animals. The bacteria swim by rotating their long, corkscrew-shaped flagella, which act like the blades of a propeller. Each flagellum is powered by a remarkable motor (made of protein) located at the base of the bacterial cell. The motor can rotate the flagellum at angular speeds from 200 to 1000 rev/min (about 20 to 100 rad/s) and can vary its speed to give the flagellum an angular acceleration.



Let ω_{0z} be the angular velocity of a rigid body at time $t = 0$ and ω_z be its angular velocity at a later time t. The angular acceleration α_z is constant and equal to the average value for any interval. From Eq. (9.4) with the interval from 0 to t,

$$\alpha_z = \frac{\omega_z - \omega_{0z}}{t - 0} \qquad \text{or}$$

(9.7)

Angular velocity at time t of a rigid body with **constant angular acceleration**

Angular velocity of body at time 0

$$\omega_z = \omega_{0z} + \alpha_z t \tag{9.7}$$

Time

Constant angular acceleration of body

The product $\alpha_z t$ is the total change in ω_z between $t = 0$ and the later time t; angular velocity ω_z at time t is the sum of the initial value ω_{0z} and this total change.

With constant angular acceleration, the angular velocity changes at a uniform rate, so its average value between 0 and t is the average of the initial and final values:

(9.8)

$$\omega_{\text{av-}z} = \frac{\omega_{0z} + \omega_z}{2}$$

We also know that $\omega_{\text{av-}z}$ is the total angular displacement $(\theta - \theta_0)$ divided by the time interval $(t - 0)$:

(9.9)

$$\omega_{\text{av-}z} = \frac{\theta - \theta_0}{t - 0}$$

When we equate Eqs. (9.8) and (9.9) and multiply the result by t, we get

(9.10)

Angular position at time t of a rigid body with **constant angular acceleration**

Angular position of body at time 0

$$\theta - \theta_0 = \frac{1}{2}(\omega_{0z} + \omega_z)t \tag{9.10}$$

Time

Angular velocity of body at time 0

Angular velocity of body at time t

To obtain a relationship between θ and t that doesn't contain ω_z , we substitute Eq. (9.7) into Eq. (9.10):

$$\theta - \theta_0 = \frac{1}{2}\left[\omega_{0z} + (\omega_{0z} + \alpha_z t)\right]t \quad \text{or}$$

(9.11)

Angular position at time t of a rigid body with **constant angular acceleration**

Angular position of body at time 0

Time

$$\theta = \theta_0 + \omega_{0z}t + \frac{1}{2}\alpha_z t^2 \tag{9.11}$$

Angular velocity of body at time 0

Constant angular acceleration of body

That is, if at the initial time $t = 0$ the body is at angular position θ_0 and has angular velocity ω_{0z} , then its angular position θ at any later time t is θ_0 , plus the rotation $\omega_{0z}\,t$ it would have if the angular velocity were constant, plus an additional rotation $\frac{1}{2}\,\alpha_z\,t^2$ caused by the changing angular velocity.

Following the same procedure as for straight-line motion in Section 2.4, we can combine Eqs. (9.7) and (9.11) to obtain a relationship between θ and ω_z that does not contain t. We invite you to work out the details, following the same procedure we used to get Eq. (2.13). We get

(9.12)

Angular velocity at time t of a rigid body with **constant angular acceleration**

Angular velocity of body at time 0

$$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0) \tag{9.12}$$

Constant angular acceleration of body

Angular position of body at time t

Angular position of body at time 0

CAUTION Constant angular acceleration Keep in mind that all of these results are valid *only* when the angular acceleration α_z is *constant;* do not try to apply them to problems in which α_z is *not* constant. Table 9.1 shows the analogy between Eqs. (9.7), (9.10), (9.11), and (9.12) for fixed-axis rotation with constant angular acceleration and the corresponding equations for straight-line motion with constant linear acceleration.

Table 9.1 **Comparison of Linear and Angular Motions with Constant Acceleration**

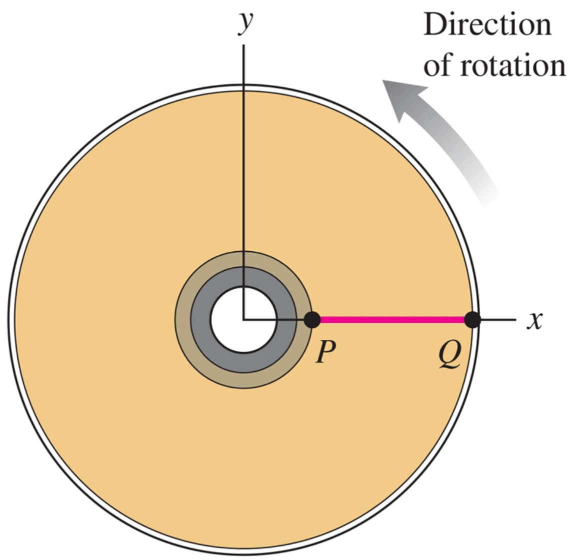
Straight-Line Motion with Constant Linear Acceleration		Fixed-Axis Rotation with Constant Angular Acceleration	
$a_x = \text{constant}$		$\alpha_z = \text{constant}$	
$v_x = v_{0x} + a_x t$	(2.8)	$\omega_z = \omega_{0z} + \alpha_z t$	(9.7)
$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$	(2.12)	$\theta = \theta_0 + \omega_{0z}t + \frac{1}{2}\alpha_z t^2$	(9.11)
$v_x^2 = v_{0x}{}^2 + 2a_x(x - x_0)$	(2.13)	$\omega_z^2 = \omega_{0z}{}^2 + 2\alpha_z(\theta - \theta_0)$	(9.12)
$x - x_0 = \frac{1}{2}(v_{0x} + v_x)t$	(2.14)	$\theta - \theta_0 = \frac{1}{2}(\omega_{0z} + \omega_z)t$	(9.10)

Example 9.3 Rotation with constant angular acceleration

WITH VARIATION PROBLEMS

You have finished watching a movie on Blu-ray and the disc is slowing to a stop. The disc's angular velocity at $t = 0$ is 27.5 rad/s, and its angular acceleration is a constant $-10.0\ \text{rad/s}^2$. A line PQ on the disc's surface lies along the $+x$-axis at $t = 0$ (Fig. 9.8). (a) What is the disc's angular velocity at $t = 0.300$ s? (b) What angle does the line PQ make with the $+x$-axis at this time?

Figure 9.8



A line PQ on a rotating Blu-ray disc at $t = 0$.

IDENTIFY and SET UP The angular acceleration of the disc is constant, so we can use any of the equations derived in this section (Table 9.1). Our target variables are the angular velocity ω_z and the angular displacement θ at $t = 0.300$ s. Given $\omega_{0z} = 27.5$ rad/s, $\theta_0 = 0$, and $\alpha_z = -10.0\ \text{rad/s}^2$, it's easiest to use Eqs. (9.7) and (9.11) to find the target variables.

EXECUTE (a) From Eq. (9.7), at $t = 0.300$s we have

$$\begin{aligned}\omega_z &= \omega_{0z} + \alpha_z t = 27.5\ \text{rad/s} + (-10.0\ \text{rad/s}^2)(0.300\ \text{s}) \\ &= 24.5\ \text{rad/s}\end{aligned}$$

(b) From Eq. (9.11),

$$\begin{aligned}\theta &= \theta_0 + \omega_{0z} t + \frac{1}{2}\alpha_z t^2 \\ &= 0 + (27.5\ \text{rad/s})(0.300\ \text{s}) + \frac{1}{2}\left(-10.0\ \text{rad/s}^2\right)(0.300\ \text{s})^2 \\ &= 7.80\ \text{rad} = 7.80\ \text{rad}\left(\frac{1\ \text{rev}}{2\pi\ \text{rad}}\right) = 1.24\ \text{rev}\end{aligned}$$

The disc has made one complete revolution plus an additional 0.24 revolution—that is, 360° plus $(0.24\ \text{rev})(360^\circ/\text{rev}) = 87^\circ$. Hence the line PQ makes an angle of 87° with the $+x$-axis.

EVALUATE Our answer to part (a) tells us that the disc's angular velocity has decreased, as it should since $\alpha_z < 0$. We can use our result for ω_z from part (a) with Eq. (9.12) to check our result for θ from part (b). To do so, we solve Eq. (9.12) for θ:

$$\begin{aligned}
\omega_z^2 &= {\omega_{0z}}^2 + 2\alpha_z(\theta - \theta_0) \\
\theta &= \theta_0 + \left(\frac{\omega_z^2 - {\omega_{0z}}^2}{2\alpha_z}\right) \\
&= 0 + \frac{(24.5\ \mathrm{rad/s})^2 - (27.5\ \mathrm{rad/s})^2}{2(-10.0\ \mathrm{rad/s^2})} = 7.80\ \mathrm{rad}
\end{aligned}$$

This agrees with our previous result from part (b).

KEY CONCEPT

The relationships among angular position θ, angular velocity ω_z, and angular acceleration α_z for a rigid body rotating with constant angular acceleration are the same as the relationships among position x, velocity v_x, and acceleration a_x for an object moving in a straight line with constant linear acceleration.

Video Tutor Solution: **Example 9.3**

Test Your Understanding of Section 9.2

Suppose the disc in Example 9.3 was initially spinning at twice the rate (55.0 rad/s rather than 27.5 rad/s) and slowed down at twice the rate ($-20.0\ \text{rad/s}^2$ rather than $-10.0\ \text{rad/s}^2$). (a) Compared to the situation in Example 9.3, how long would it take the disc to come to a stop? (i) The same amount of time; (ii) twice as much time; (iii) 4 times as much time; (iv) $\frac{1}{2}$ as much time; (v) $\frac{1}{4}$ as much time. (b) Compared to the situation in Example 9.3, through how many revolutions would the disc rotate before coming to a stop? (i) The same number of revolutions; (ii) twice as many revolutions; (iii) 4 times as many revolutions; (iv) $\frac{1}{2}$ as many revolutions; (v) $\frac{1}{4}$ as many revolutions.

9.3 Relating Linear and Angular Kinematics

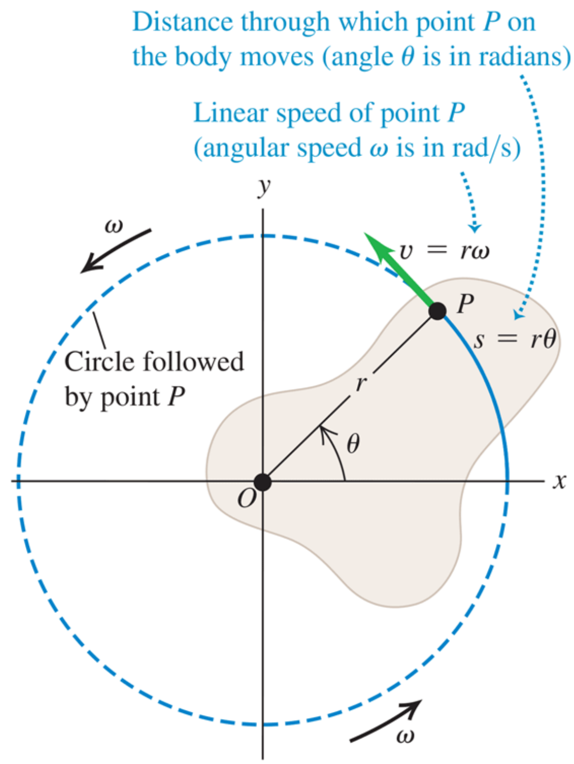
How do we find the linear speed and acceleration of a particular point in a rotating rigid body? We need to answer this question to proceed with our study of rotation. For example, to find the kinetic energy of a rotating body, we have to start from $K = \frac{1}{2}mv^2$ for a particle, and this requires that we know the speed v for each particle in the body. So it's worthwhile to develop general relationships between the *angular* speed and acceleration of a rigid body rotating about a fixed axis and the *linear* speed and acceleration of a specific point or particle in the body.

Linear Speed in Rigid-Body Rotation

When a rigid body rotates about a fixed axis, every particle in the body moves in a circular path that lies in a plane perpendicular to the axis and is centered on the axis. A particle's speed is directly proportional to the body's angular velocity; the faster the rotation, the greater the speed of each particle. In Fig. 9.9, point P is a constant distance r from the axis, so it moves in a circle of radius r. At any time, Eq. (9.1) relates the angle θ (in radians) and the arc length s:

$$s = r\,\theta$$

Figure 9.9



A rigid body rotating about a fixed axis through point O.

We take the time derivative of this, noting that r is constant for any specific particle, and take the absolute value of both sides:

$$\left|\frac{ds}{dt}\right| = r\left|\frac{d\theta}{dt}\right|$$

Now $|ds/dt|$ is the absolute value of the rate of change of arc length, which is equal to the instantaneous *linear* speed v of the particle. The absolute value of the rate of change of the angle, $|d\theta/dt|$, is the instantaneous **angular speed** ω—that is, the magnitude of the instantaneous angular velocity in rad/s. Thus

(9.13)

Linear speed of a point on a rotating rigid body

$$v = r\omega \tag{9.13}$$

Angular speed of the rotating rigid body

Distance of that point from rotation axis

The farther a point is from the axis, the greater its linear speed. The *direction* of the linear velocity *vector* is tangent to its circular path at each point (Fig. 9.9).

CAUTION **Speed vs. velocity** Keep in mind the distinction between the linear and angular *speeds* v and ω, which appear in Eq. (9.13), and the linear and angular *velocities* v_x and ω_z. The quantities without subscripts, v and ω, are never negative; they are the magnitudes of the vectors $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{\omega}}$, respectively, and their values tell you only how fast a particle is moving v or how fast a body is rotating (ω). The quantities with subscripts, v_x and ω_z, can be either positive or negative; their signs tell you the direction of the motion.

Linear Acceleration in Rigid-Body Rotation

We can represent the acceleration $\vec{\boldsymbol{a}}$ of a particle moving in a circle in terms of its centripetal and tangential components, a_{rad} and a_{tan} (Fig. 9.10), as we did in Section 3.4. (You should review that section now.) We found that the **tangential component of acceleration** a_{tan}, the component parallel to the instantaneous velocity, acts to change the *magnitude* of the particle's velocity (i.e., the speed) and is equal to the rate of change of speed. Taking the derivative of Eq. (9.13), we find

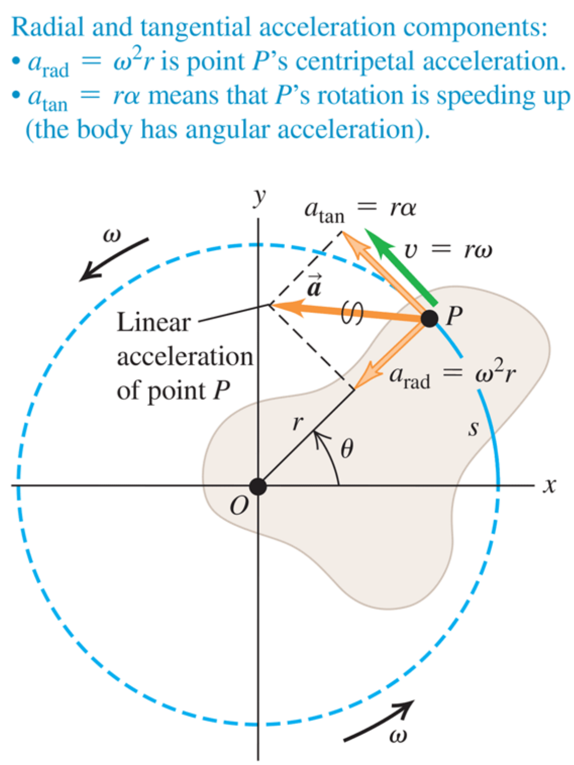
(9.14)

Tangential acceleration of a point on a rotating rigid body

Distance of that point from rotation axis

$$a_{\text{tan}} = \underbrace{\frac{dv}{dt}}_{\text{Rate of change of linear speed of that point}} = r\underbrace{\frac{d\omega}{dt}}_{\text{Rate of change of angular speed of body}} = r\alpha \qquad (9.14)$$

Figure 9.10



A rigid body whose rotation is speeding up. The acceleration of point P has a component a_{rad} toward the rotation axis (perpendicular to $\vec{\mathbf{v}}$ and a component a_{tan} along the circle that point P follows (parallel to $\vec{\mathbf{v}}$)

This component of $\vec{\mathbf{a}}$ is always tangent to the circular path of point P (Fig. 9.10).

The quantity $\alpha = d\omega/dt$ in Eq. (9.14) is the rate of change of the angular *speed*. It is not quite the same as $\alpha_z = d\omega_z/dt$, which is the rate of change of the angular *velocity*. For example, consider a object rotating so that its angular velocity vector points in the $-z$-direction (see Fig. 9.5b). If the body is gaining angular speed at a rate of 10 rad/s per second, then $\alpha = 10\ \text{rad/s}^2$. But ω_z is negative and becoming more negative as the rotation gains speed, so $\alpha_z = -10\ \text{rad/s}^2$. The rule for rotation about a fixed axis is that α is equal to α_z if ω_z is positive but equal to $-\alpha_z$ if ω_z is negative.

The component of $\vec{\mathbf{a}}$ in Fig. 9.10 directed toward the rotation axis, the **centripetal component of acceleration** a_{rad}, is associated with the change of *direction* of the velocity of point P. In Section 3.4 we worked out the relationship $a_{\text{rad}} = v^2/r$. We can express this in terms of ω by using Eq. (9.13):

(9.15)

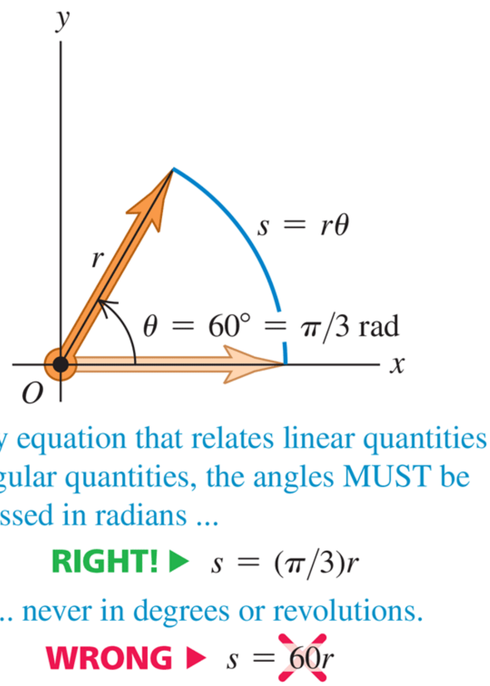
$$\underbrace{a_{\text{rad}}}_{\text{Centripetal acceleration of a point on a rotating rigid body}} = \frac{\overbrace{v^2}^{\text{Linear speed of that point}}}{\underbrace{r}_{\text{Distance of that point from rotation axis}}} = \overbrace{\omega}^{\text{Angular speed of body}}{}^2\, r \qquad (9.15)$$

This is true at each instant, *even when ω and v are not constant.* The centripetal component always points toward the axis of rotation.

CAUTION Use angles in radians Remember that Eq. (9.1), $s = r\,\theta$, is valid *only* when θ is measured in radians. The same is true of any equation derived from this, including Eqs. (9.13), (9.14), and (9.15).

When you use these equations, you *must* express the angular quantities in radians, not revolutions or degrees (Fig. 9.11).

Figure 9.11



Always use radians when relating linear and angular quantities.

Equations (9.1), (9.13), and (9.14) also apply to any particle that has the same tangential velocity as a point in a rotating rigid body. For example, when a rope wound around a circular cylinder unwraps without stretching or slipping, its speed and acceleration at any instant are equal to the speed and tangential acceleration of the point at which it is tangent to the cylinder. The same principle holds for situations such as bicycle chains and sprockets, belts and pulleys that turn without slipping, and so on. We'll have several opportunities to use these relationships later in this chapter and in Chapter 10. Note that Eq. (9.15) for the centripetal component a_{rad} is applicable to the rope or chain *only* at points that are in contact with the cylinder or sprocket. Other points do not have the same acceleration toward the center of the circle that points on the cylinder or sprocket have.

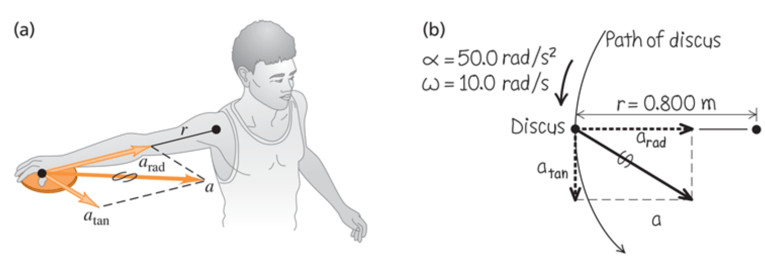
Example 9.4 Throwing a discus

WITH VARIATION PROBLEMS

An athlete whirls a discus in a circle of radius 80.0 cm. At a certain instant, the athlete is rotating at 10.0 rad/s and the angular speed is increasing at 50.0 $\mathrm{rad/s^2}$. For this instant, find the tangential and centripetal components of the acceleration of the discus and the magnitude of the acceleration.

IDENTIFY and SET UP We treat the discus as a particle traveling in a circular path (Fig. 9.12a), so we can use the ideas developed in this section. We are given $r = 0.800\ \mathrm{m}$, $\omega = 10.0\ \mathrm{rad/s}$, and $\alpha = 50.0\ \mathrm{rad/s^2}$ (Fig. 9.12b). We'll use Eqs. (9.14) and (9.15) to find the acceleration components a_{tan} and a_{rad}, respectively; we'll then find the magnitude a by using the Pythagorean theorem.

Figure 9.12



(a) Whirling a discus in a circle. (b) Our sketch showing the acceleration components for the discus.

EXECUTE From Eqs. (9.14) and (9.15),

$$\begin{aligned} a_{\mathrm{tan}} &= r\alpha = (0.800\ \mathrm{m})(50.0\ \mathrm{rad/s^2}) = 40.0\ \mathrm{m/s^2} \\ a_{\mathrm{rad}} &= \omega^2 r = (10.0\ \mathrm{rad/s})^2(0.800\ \mathrm{m}) = 80.0\ \mathrm{m/s^2} \end{aligned}$$

Then

$$a = \sqrt{{a_{\tan}}^2 + {a_{\text{rad}}}^2} = 89.4 \text{ m/s}^2$$

EVALUATE Note that we dropped the unit "radian" from our results for $a_{\tan}$, a_{rad}, and a. We can do this because "radian" is a dimensionless quantity. Can you show that if the angular speed doubles to 20.0 rad/s while α remains the same, the acceleration magnitude a increases to 322 m/s^2?

KEY CONCEPT

Points on a rigid body have a *centripetal* (radial) acceleration component $a_{\text{rad}} = \omega^2 r$ whenever the rigid body is rotating; they have a *tangential* acceleration component $a_{\tan} = r\alpha$ *only* if the angular speed ω is changing. These two acceleration components are perpendicular, so you can use the Pythagorean theorem to relate them to the magnitude of the acceleration.

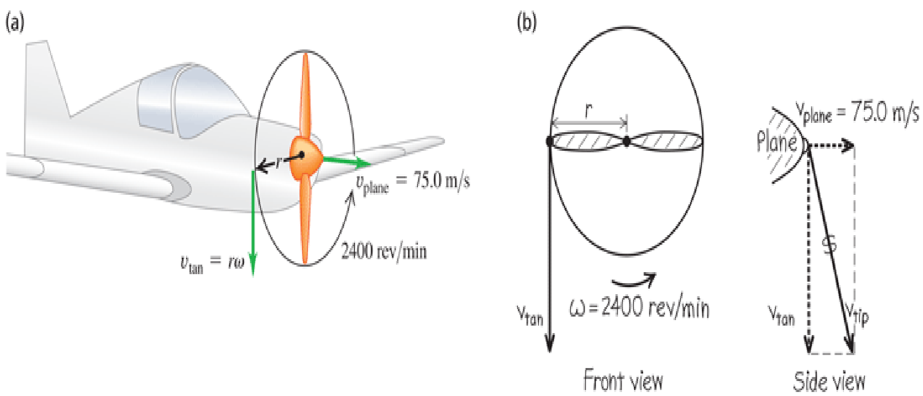
Video Tutor Solution: **Example 9.4**

Example 9.5 Designing a propeller

WITH VARIATION PROBLEMS

You are designing an airplane propeller that is to turn at 2400 rpm (Fig. 9.13a). The forward airspeed of the plane is to be 75.0 m/s, and the speed of the propeller tips through the air must not exceed 270 m/s. (This is about 80% of the speed of sound in air. If the propeller tips moved faster, they would produce a lot of noise.) (a) What is the maximum possible propeller radius? (b) With this radius, what is the acceleration of the propeller tip?

Figure 9.13



(a) A propeller-driven airplane in flight. (b) Our sketch showing the velocity components for the propeller tip.

IDENTIFY and SET UP We consider a particle at the tip of the propeller; our target variables are the particle's distance from the axis and its acceleration. The speed of this particle through the air, which cannot exceed 270 m/s, is due to both the propeller's rotation and the forward motion of the airplane. Figure 9.13b shows that the particle's velocity $\vec{v}_{tip}$ is the vector sum of its tangential velocity due to the propeller's rotation of magnitude $v_{tan} = \omega r$, given by Eq. (9.13), and the forward velocity of the airplane of magnitude $v_{plane} = 75.0\ m/s$. The propeller rotates in a plane perpendicular to the direction of flight, so $\vec{v}_{tan}$ and $\vec{v}_{plane}$ are perpendicular to each other, and we can use the Pythagorean theorem to obtain an expression for v_{tip} from v_{tan} and v_{plane}. We'll then set $v_{\mathrm{tip}} = 270\ \mathrm{m}/s$ and solve for the radius r. The angular speed of the propeller is constant, so the acceleration of the propeller tip has only a radial component; we'll find it by using Eq. (9.15).

EXECUTE We first convert ω to rad/s (see Fig. 9.11):

$$\omega = 2400\ \mathrm{rpm} = \left(2400\ \frac{\mathrm{rev}}{\mathrm{min}}\right)\left(\frac{2\pi\ \mathrm{rad}}{1\ \mathrm{rev}}\right)\left(\frac{1\ \mathrm{min}}{60\ \mathrm{s}}\right) = 251\ \mathrm{rad/s}$$

(a) From Fig. 9.13b and Eq. (9.13),

$$\begin{aligned} {v_{\mathrm{tip}}}^2 &= {v_{\mathrm{plane}}}^2 + {v_{\mathrm{tan}}}^2 = {v_{\mathrm{plane}}}^2 + r^2\omega^2 \quad \text{so} \\ r^2 &= \frac{{v_{\mathrm{tip}}}^2 - {v_{\mathrm{plane}}}^2}{\omega^2} \qquad \text{and} \qquad r = \frac{\sqrt{{v_{\mathrm{tip}}}^2 - {v_{\mathrm{plane}}}^2}}{\omega} \end{aligned}$$

If $v_{\mathrm{tip}} = 270\ \mathrm{m/s}$, the maximum propeller radius is

$$r = \frac{\sqrt{(270\ \mathrm{m/s})^2 - (75.0\ \mathrm{m/s})^2}}{251\ \mathrm{rad/s}} = 1.03\ \mathrm{m}$$

(b) The centripetal acceleration of the particle is, from Eq. (9.15),

$$\begin{aligned} a_{\mathrm{rad}} &= \omega^2 r = (251\ \mathrm{rad/s})^2(1.03\ \mathrm{m}) \\ &= 6.5 \times 10^4\ \mathrm{m/s}^2 \end{aligned}$$

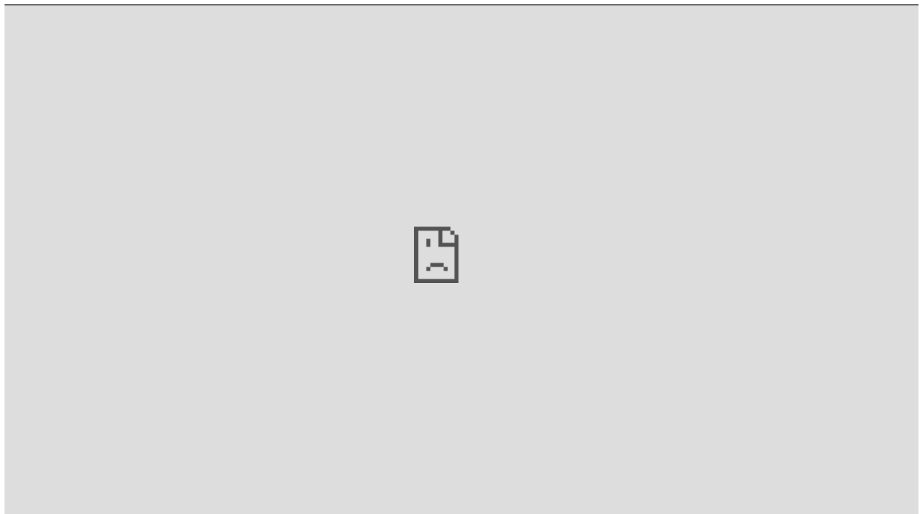
The tangential acceleration a_{tan} is zero because ω is constant.

EVALUATE From $\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$, the propeller must exert a force of 6.5×10^4 N on each kilogram of material at its tip! This is why propellers are made out of tough material such as aluminum alloy.

KEY CONCEPT

If a rotating rigid body is also moving as a whole through space, use vector addition to find the velocity of a point on the rigid body.

Video Tutor Solution: **Example 9.5**

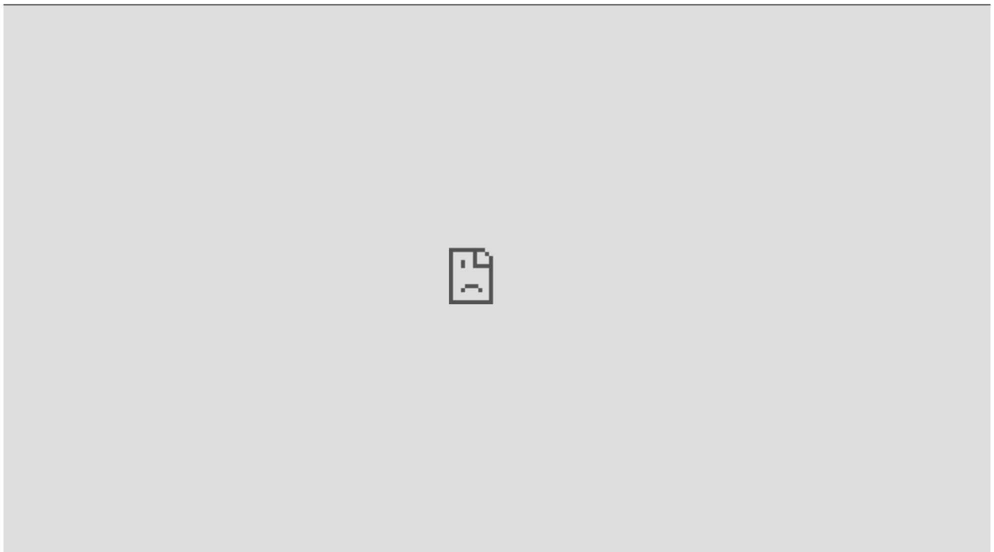


Test Your Understanding of Section 9.3

Information is stored on a Blu-ray disc (see Fig. 9.8) in a coded pattern of tiny pits. The pits are arranged in a track that spirals outward toward the rim of the disc. As the disc spins inside a player, the track is scanned at a constant *linear* speed. How must the rotation speed ω of the disc change as the player's scanning head moves outward over the track? (i) ω must increase; (ii) ω must decrease; (iii) ω must stay the same.

9.4 Energy in Rotational Motion

Video Tutor Demo: **Canned Food Race**



A rotating rigid body consists of mass in motion, so it has kinetic energy. As we'll see, we can express this kinetic energy in terms of the body's angular speed and a new quantity, called *moment of inertia*, that depends on the body's mass and how the mass is distributed.

To begin, we think of a body as being made up of a large number of particles, with masses m_1, m_2, ... at distances r_1, r_2, ... from the axis of rotation. We label the particles with the index i: The mass of the ith particle is m_i and r_i is the *perpendicular* distance from the axis to the ith particle. (The particles need not all lie in the same plane.)

When a rigid body rotates about a fixed axis, the speed v_i of the ith particle is given by Eq. (9.13), $v_i = r_i \omega$, where ω is the body's angular speed. Different particles have different values of r_i, but ω is the same for all (otherwise, the body wouldn't be rigid). The kinetic energy of the ith particle can be expressed as

$$\frac{1}{2}\, m_i\, v_i^2 = \frac{1}{2}\, m_i\, r_i^2 \omega^2$$

The body's *total* kinetic energy is the sum of the kinetic energies of all its particles:

$$K = \frac{1}{2}\, m_1\, r_1^2 \omega^2 + \frac{1}{2}\, m_2\, r_2^2 \omega^2 + \cdots = \sum_i \frac{1}{2}\, m_i\, r_i^2 \omega^2$$

Taking the common factor $\frac{1}{2}\omega^2$ out of this expression, we get

$$K = \frac{1}{2}\, \left(m_1\, r_1^2 + m_2\, r_2^2 + \cdots\right)\omega^2 = \frac{1}{2}\left(\sum_i m_i\, r_i^2\right)\omega^2$$

The quantity in parentheses, obtained by multiplying the mass of each particle by the square of its distance from the axis of rotation and adding these products, is called the **moment of inertia *I*** of the body for this rotation axis:

(9.16)

Masses of the particles that make up the body

Moment of inertia of a body for a given rotation axis

$$I = m_1 r_1^2 + m_2 r_2^2 + \cdots = \sum_i m_i r_i^2 \qquad (9.16)$$

Perpendicular distances of the particles from rotation axis

"Moment" means that I depends on how the body's mass is distributed in space; it has nothing to do with a "moment" of time. For a body with a given rotation axis and a given total mass, the greater the distances from the axis to the particles that make up the body, the greater the moment of inertia I. In a rigid body, all distances r_i are constant and I is independent of how the body rotates around the given axis. The SI unit of I is the kilogram$-$meter^2 kilogram-meter^2($\mathrm{kg} \cdot \mathrm{m}^2$).

Using Eq. (9.16), we see that the **rotational kinetic energy** K of a rigid body is

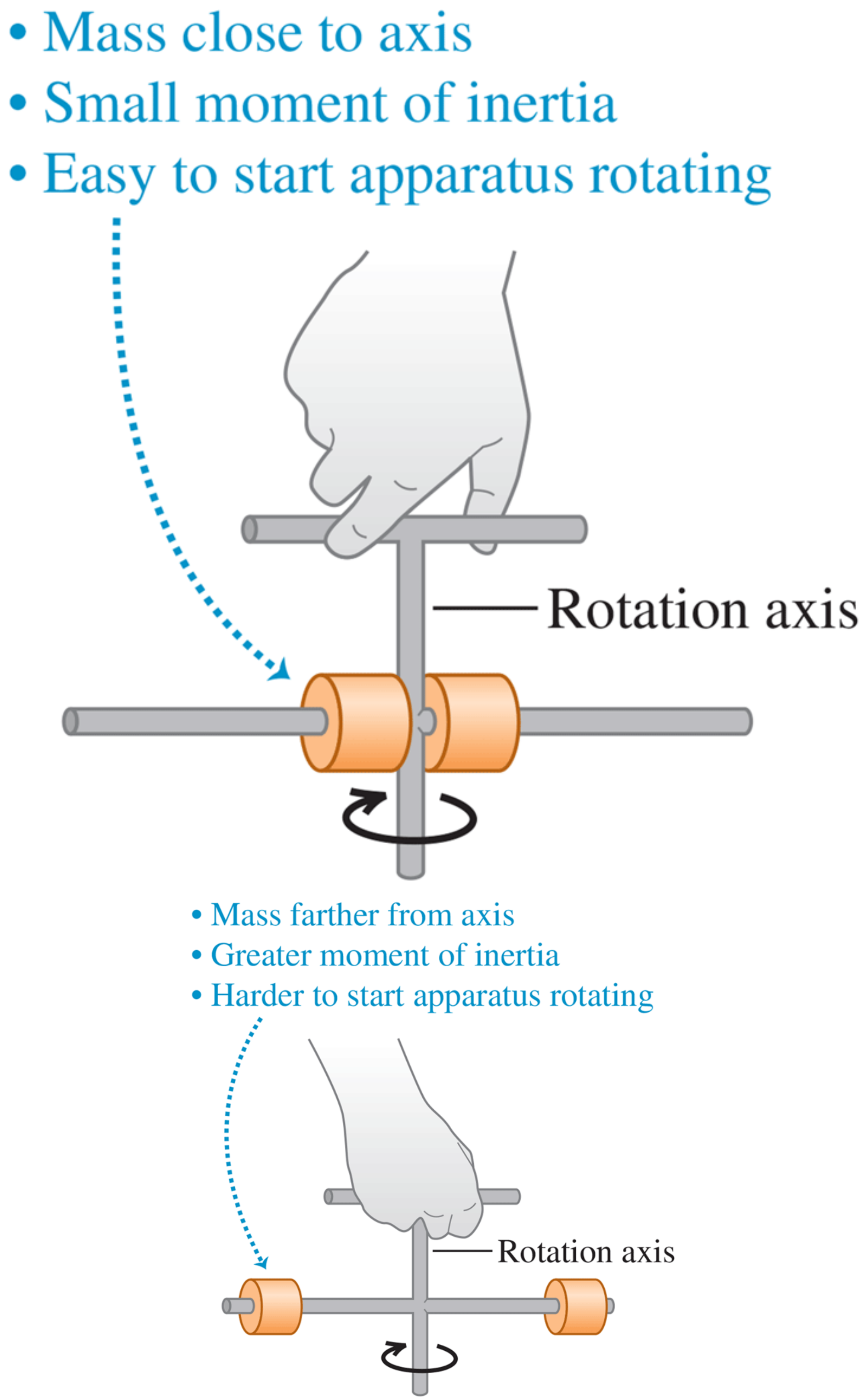
(9.17)

$$\underbrace{K}_{\text{Rotational kinetic energy of a rigid body rotating around an axis}} = \frac{1}{2}\overbrace{I}^{\text{Moment of inertia of body for given rotation axis}}\underbrace{\omega}_{\text{Angular speed of body}}^{2} \qquad (9.17)$$

The kinetic energy given by Eq. (9.17) is *not* a new form of energy; it's simply the sum of the kinetic energies of the individual particles that make up the rotating rigid body. To use Eq. (9.17), ω *must* be measured in radians per second, not revolutions or degrees per second, to give K in joules. That's because we used $v_i = r_i \omega$ in our derivation.

Equation (9.17) gives a simple physical interpretation of moment of inertia: *The greater the moment of inertia, the greater the kinetic energy of a rigid body rotating with a given angular speed* ω. We learned in Chapter 6 that the kinetic energy of an object equals the amount of work done to accelerate that object from rest. So the greater a body's moment of inertia, the harder it is to start the body rotating if it's at rest and the harder it is to stop its rotation if it's already rotating (Fig. 9.14). For this reason, I is also called the *rotational inertia*.

Figure 9.14

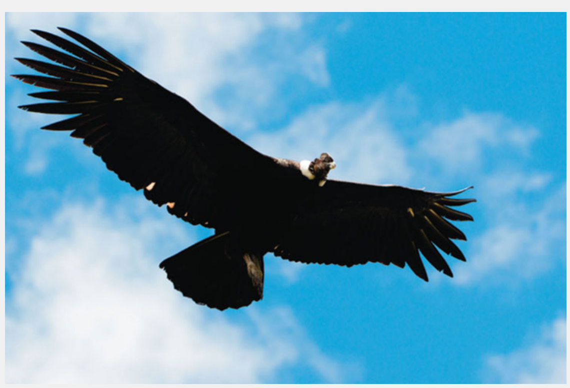


An apparatus free to rotate around a vertical axis. To vary the moment of inertia, the two equal-mass cylinders can be locked into different positions on the horizontal shaft.

BIO Application

Moment of Inertia of a Bird's Wing

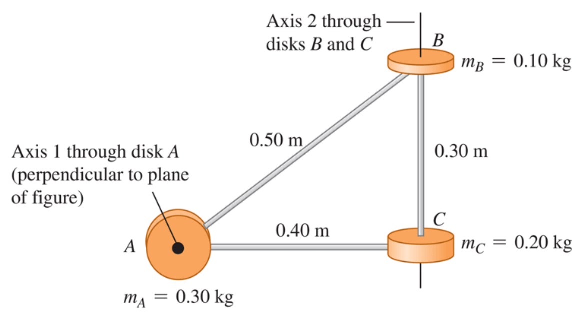
When a bird flaps its wings, it rotates the wings up and down around the shoulder. A hummingbird has small wings with a small moment of inertia, so the bird can move its wings rapidly (up to 70 beats per second). By contrast, the Andean condor (*Vultur gryphus*) has immense wings that are hard to move due to their large moment of inertia. Condors flap their wings at about one beat per second on takeoff, but at most times prefer to soar while holding their wings steady.



Example 9.6 Moments of inertia for different rotation axes

A machine part (Fig. 9.15) consists of three small disks linked by lightweight struts. (a) What is this body's moment of inertia about axis 1 through the center of disk A, perpendicular to the plane of the diagram? (b) What is its moment of inertia about axis 2 through the centers of disks B and C? (c) What is the body's kinetic energy if it rotates about axis 1 with angular speed $\omega = 4.0\ \text{rad/s}$?

Figure 9.15



An oddly shaped machine part.

IDENTIFY and SET UP We'll consider the disks as massive particles located at the centers of the disks, and consider the struts as massless. In parts (a) and (b), we'll use Eq. (9.16) to find the moments of inertia. Given the moment of inertia about axis 1, we'll use Eq. (9.17) in part (c) to find the rotational kinetic energy.

EXECUTE (a) The particle at point A lies *on* axis 1 through A, so its distance r from the axis is zero and it contributes nothing to the moment of inertia. Hence only B and C contribute in Eq. (9.16):

$$
\begin{aligned}
I_1 = \sum m_i r_i^2 &= (0.10\ \text{kg})(0.50\ \text{m})^2 + (0.20\ \text{kg})(0.40\ \text{m})^2 \\
&= 0.057\ \text{kg} \cdot \text{m}^2
\end{aligned}
$$

(b) The particles at B and C both lie on axis 2, so neither particle contributes to the moment of inertia. Hence only A contributes:

$$
I_2 = \sum m_i r_i^2 = (0.30\ \text{kg})(0.40\ \text{m})^2 = 0.048\ \text{kg} \cdot \text{m}^2
$$

(c) From Eq. (9.17),

$$
K_1 = \frac{1}{2} I_1 \omega^2 = \frac{1}{2}(0.057\ \text{kg}\cdot\text{m}^2)(4.0\ \text{rad/s})^2 = 0.46\ \text{J}
$$

EVALUATE The moment of inertia about axis 2 is smaller than that about axis 1. Hence, of the two axes, it's easier to make the machine part rotate about axis 2.

KEY CONCEPT

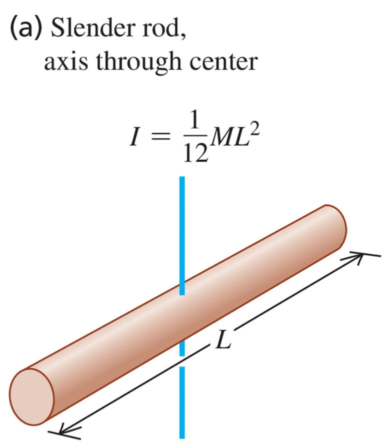
The moment of inertia I of a rigid body about an axis depends on the position and orientation of the axis. The value of I for a given rigid body can be very different for different axes.

Video Tutor Solution: **Example 9.6**

CAUTION Moment of inertia depends on the choice of axis Example 9.6 shows that the moment of inertia of a body depends on the location and orientation of the axis. It's not enough to say, "The moment of inertia is 0.048 $kg \cdot m^2$." We have to be specific and say, "The moment of inertia *about the axis through B and C* is 0.048 $kg{\cdot}m^2$."

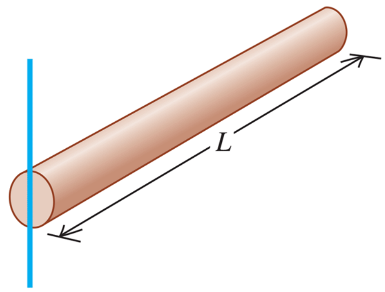
? In Example 9.6 we represented the body as several point masses, and we evaluated the sum in Eq. (9.16) directly. When the body is a *continuous* distribution of matter, such as a solid cylinder or plate, the sum becomes an integral, and we need to use calculus to calculate the moment of inertia. We'll give several examples of such calculations in Section 9.6; meanwhile, Table 9.2 gives moments of inertia for several familiar shapes in terms of their masses and dimensions. Each body shown in Table 9.2 is *uniform;* that is, the density has the same value at all points within the solid parts of the body.

Table 9.2 Moments of Inertia of Various Bodies



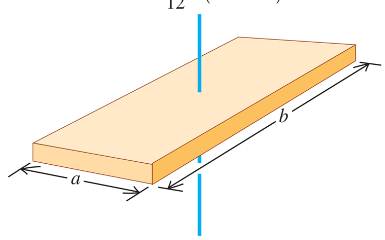
(b) Slender rod,
axis through one end

$$I = \frac{1}{3}ML^2$$



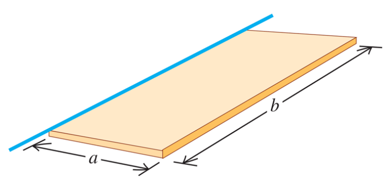
(c) Rectangular plate,
axis through center

$$I = \frac{1}{12}M\left(a^2 + b^2\right)$$



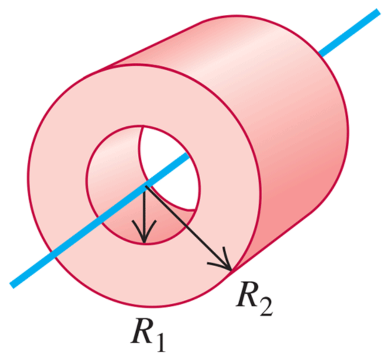
(d) Thin rectangular plate,
axis along edge

$$I = \frac{1}{3}Ma^2$$



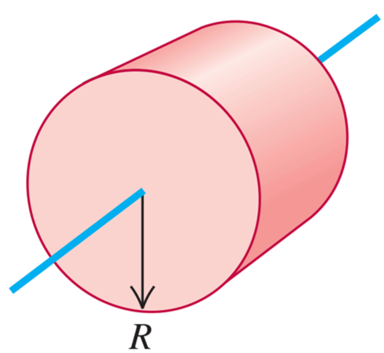
(e) Hollow cylinder

$$I = \frac{1}{2}M\left(R_1^2 + R_2^2\right)$$



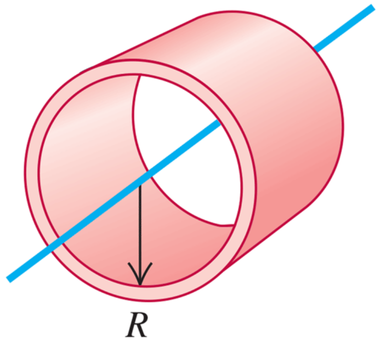
(f) Solid cylinder

$$I = \frac{1}{2}MR^2$$



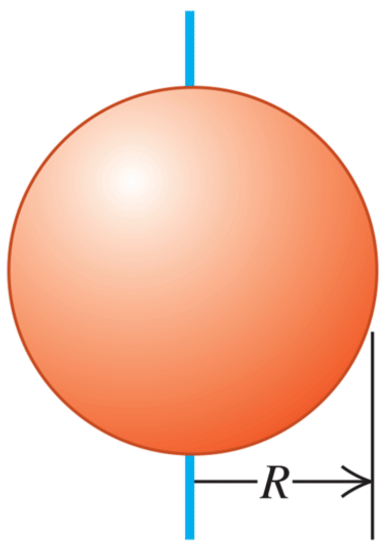
(g) Thin-walled hollow cylinder

$$I = MR^2$$



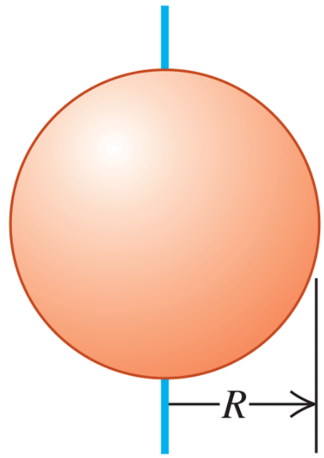
(h) Solid sphere

$$I = \frac{2}{5}MR^2$$



(i) Thin-walled hollow sphere

$I = \frac{2}{3}MR^2$



CAUTION Computing moments of inertia You may be tempted to try to compute the moment of inertia of a body by assuming that all the mass is concentrated at the center of mass and multiplying the total mass by the square of the distance from the center of mass to the axis. That doesn't work! For example, when a uniform thin rod of length L and mass M is pivoted about an axis through one end, perpendicular to the rod, the moment of inertia is $I = ML^2/3$ [case (b) in Table 9.2]. If we took the mass as concentrated at the center, a distance $L/2$ from the axis, we would obtain the *incorrect* result $I = M(L/2)^2 = ML^2/4$.

Now that we know how to calculate the kinetic energy of a rotating rigid body, we can apply the energy principles of Chapter 7 to rotational motion. The Problem-Solving Strategy on the next page, along with the examples that follow, shows how this is done.

Problem-Solving Strategy 9.1 Rotational Energy

IDENTIFY *the relevant concepts:* You can use work–energy relationships and conservation of energy to find relationships involving the position and motion of a rigid body rotating around a fixed axis. The energy method is usually not helpful for problems that involve elapsed time. In Chapter 10 we'll see how to approach rotational problems of this kind.

SET UP *the problem* using Problem-Solving Strategy 7.1 (Section 7.1) with the following additional steps:

5. You can use Eqs. (9.13) and (9.14) in problems involving a rope (or the like) wrapped around a rotating rigid body, if the rope doesn't slip. These equations relate the linear speed and tangential acceleration of a point on the body to the body's angular velocity and angular acceleration. (See Examples 9.7 and 9.8.)
6. Use Table 9.2 to find moments of inertia. Use the parallel-axis theorem, Eq. (9.19) (to be derived in Section 9.5), to find moments of inertia for rotation about axes parallel to those shown in the table.

EXECUTE *the solution:* Write expressions for the initial and final kinetic and potential energies K_1, K_2, U_1, and U_2 and for the nonconservative work W_{other} (if any), where K_1 and must now include any rotational kinetic energy $K = \frac{1}{2}I\omega^2$. Substitute these expressions into Eq. (7.14), $K_1 + U_1 + W_{\text{other}} = K_2 + U_2$ (if nonconservative work is done), or Eq. (7.12), $K_1 + U_1 = K_2 + U_2$ (if only conservative work is done), and solve for the target variables. It's helpful to draw bar graphs showing the initial and final values of K, U, and $E = K + U$.

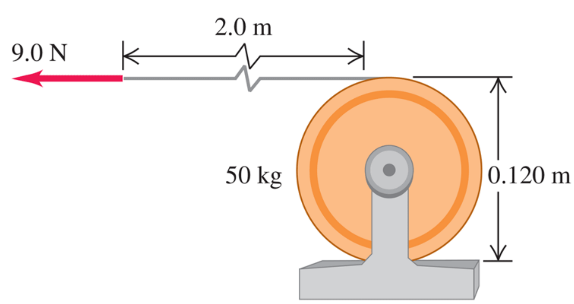
EVALUATE *your answer:* Check whether your answer makes physical sense.

Example 9.7 An unwinding cable I

WITH VARIATION PROBLEMS

We wrap a light, nonstretching cable around a solid cylinder, of mass 50 kg and diameter 0.120 m, that rotates in frictionless bearings about a stationary horizontal axis (Fig. 9.16). We pull the free end of the cable with a constant 9.0 N force for a distance of 2.0 m; it turns the cylinder as it unwinds without slipping. The cylinder is initially at rest. Find its final angular speed and the final speed of the cable.

Figure 9.16



A cable unwinds from a cylinder (side view).

IDENTIFY We'll solve this problem by using energy methods. We'll assume that the cable is massless, so only the cylinder has kinetic energy. There are no changes in gravitational potential energy. There is friction between the cable and the cylinder, but because the cable doesn't slip, there is no motion of the cable relative to the cylinder and no mechanical energy is lost in frictional work. Because the cable is massless, the force that the cable exerts on the cylinder rim is equal to the applied force F.

SET UP Point 1 is when the cable begins to move. The cylinder starts at rest, so $K_1 = 0$. Point 2 is when the cable has moved a distance $s = 2.0$ m and the cylinder has kinetic energy $K_2 = \frac{1}{2}I\omega^2$. One of our target variables is ω; the other is the speed of the cable at point 2, which is equal to the tangential speed v of the cylinder at that point. We'll use Eq. (9.13) to find v from ω.

EXECUTE The work done on the cylinder is $W_{\text{other}} = Fs = (9.0 \text{ N})(2.0 \text{ m}) = 18 \text{ J}$. From Table 9.2 the moment of inertia is

$$I = \frac{1}{2}mR^2 = \frac{1}{2}(50 \text{ kg})(0.060 \text{ m})^2 = 0.090 \text{ kg·m}^2$$

(The radius R is half the diameter.) From Eq. (7.14), $K_1 + U_1 + W_{\text{other}} = K_2 + U_2$, so

$$\begin{aligned} 0 + 0 + W_{\text{other}} &= \frac{1}{2}I\omega^2 + 0 \\ \omega = \sqrt{\frac{2W_{\text{other}}}{I}} &= \sqrt{\frac{2(18 \text{ J})}{0.090 \text{ kg} \cdot \text{m}^2}} = 20 \text{ rad/s} \end{aligned}$$

From Eq. (9.13), the final tangential speed of the cylinder, and hence the final speed of the cable, is

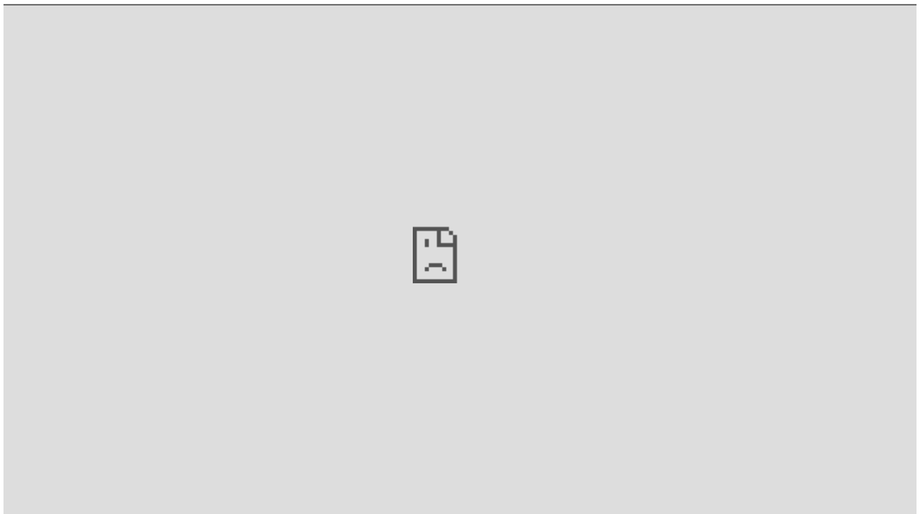
$$v = R\omega = (0.060 \text{ m})(20 \text{ rad/s}) = 1.2 \text{ m/s}$$

EVALUATE If the cable mass is not negligible, some of the 18 J of work would go into the kinetic energy of the cable. Then the cylinder would have less kinetic energy and a lower angular speed than we calculated here.

KEY CONCEPT

When using energy methods to solve problems about rotating rigid bodies, follow the same general steps as in Chapter 7, but include any rotational kinetic energy, $K = \frac{1}{2}I\omega^2$.

Video Tutor Solution: **Example 9.7**



Example 9.8 An unwinding cable II

WITH VARIATION PROBLEMS

We wrap a light, nonstretching cable around a solid cylinder with mass M and radius R. The cylinder rotates with negligible friction about a stationary horizontal axis. We tie the free end of the cable to a block of mass m and release the block from rest at a distance h above the floor. As the block falls, the cable unwinds without stretching or slipping. Find the speed of the falling block and the angular speed of the cylinder as the block strikes the floor.

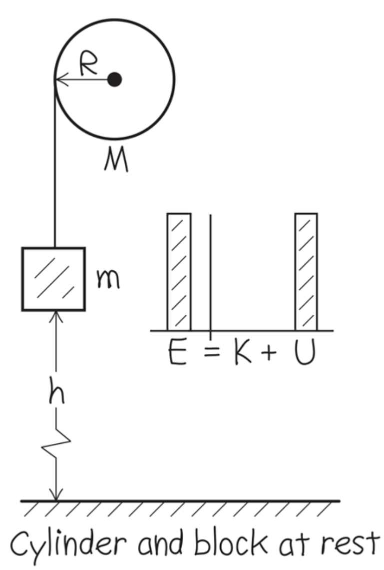
IDENTIFY As in Example 9.7, the cable doesn't slip and so friction does no work. We assume that the cable is massless, so that the forces it exerts on the cylinder and the block have equal magnitudes. At its upper end the force and displacement are in the same direction, and at its lower end they are in opposite directions, so the cable does no *net* work and $W_{\text{other}} = 0$. Only gravity does work, and mechanical energy is conserved.

SET UP Figure 9.17a shows the situation before the block begins to fall (point 1). The initial kinetic energy is $K_1 = 0$. We take the gravitational potential energy to be zero when the block is at floor level (point 2), so $U_1 = mgh$ and $U_2 = 0$. (We ignore the gravitational potential energy for the rotating cylinder, since its height doesn't change.) Just before the block hits the floor (Fig. 9.17b), the block has kinetic energy due to its translational motion and the cylinder has kinetic energy due to its rotation. The total kinetic energy is the sum of these:

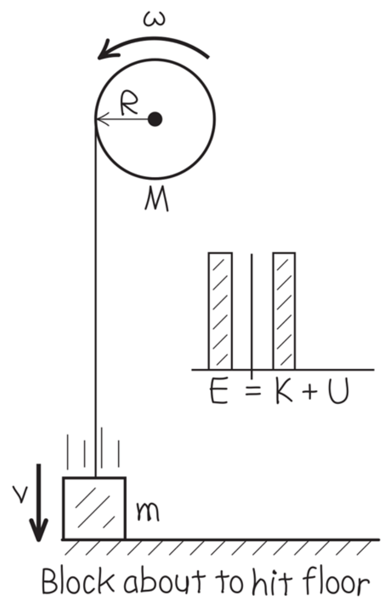
$$K_2 = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

Figure 9.17

(a) Initial (block at point 1)



(b) Final (block at point 2)



Our sketches for this problem.

The moment of inertia of the cylinder is $I = \dfrac{1}{2} MR^2$. Also, $v = R\omega$ since the speed of the falling block must be equal to the tangential speed at the outer surface of the cylinder.

EXECUTE We use our expressions for K_1, U_1, K_2, and U_2 and the relationship $\omega = v/R$ in Eq. (7.4), $K_1 + U_1 = K_2 + U_2$, and solve for v :

$$\begin{aligned} 0 + mgh &= \tfrac{1}{2} m^2 + \tfrac{1}{2} \left(\tfrac{1}{2} MR^2\right) \left(\frac{v}{R}\right)^2 + 0 = \tfrac{1}{2} \left(m + \tfrac{1}{2}\ M\right) v^2 \\ v &= \sqrt{\frac{2gh}{1 + M/2m}} \end{aligned}$$

The final angular speed of the cylinder is $\omega = v/R$.

EVALUATE When M is much larger than m, v is very small; when M is much smaller than m, v is nearly equal to $\sqrt{2gh}$, the speed of a body that falls freely from height h. Both of these results are as we would expect.

KEY CONCEPT

When a block is attached to a string that wraps around a cylinder or pulley of radius R, the speed v of the block is related to the angular speed ω of the cylinder or pulley by $v = R\omega$. You can use this to find the combined kinetic energy of the two objects.

Video Tutor Solution: Example 9.8

Gravitational Potential Energy for an Extended Body

In Example 9.8 the cable was of negligible mass, so we could ignore its kinetic energy as well as the gravitational potential energy associated with it. If the mass is *not* negligible, we need to know how to calculate the *gravitational potential energy* associated with such an extended body. If the acceleration of gravity g is the same at all points on the body, the gravitational potential energy is the same as though all the mass were concentrated at the center of mass of the body. Suppose we take the y-

axis vertically upward. Then for a body with total mass M, the gravitational potential energy U is simply

(9.18)

$$U = Mgy_{\text{cm}} \quad \text{(gravitational potential energy for an extended body)}$$

where y_{cm} is the y-coordinate of the center of mass. This expression applies to any extended body, whether it is rigid or not (Fig. 9.18).

Figure 9.18

In a technique called the "Fosbury flop" after its innovator, this athlete arches her body as she passes over the bar in the high jump. As a result, her center of mass actually passes *under* the bar. This technique requires a smaller increase in gravitational potential energy [Eq. (9.18)] than the older method of straddling the bar.

To prove Eq. (9.18), we again represent the body as a collection of mass elements m_i. The potential energy for element m_i is $m_i g y_i$, so the total potential energy is

$$U = m_1 gy_1 + m_2 gy_2 + \cdots = (m_1 y_1 + m_2 y_2 + \cdots)g$$

But from Eq. (8.28), which defines the coordinates of the center of mass,

$$m_1 y_1 + m_2 y_2 + \cdots = (m_1 + m_2 + \cdots)y_{\text{cm}} = My_{\text{cm}}$$

where $M = m_1 + m_2 + \cdots$ is the total mass. Combining this with the above expression for U, we find $U = Mgy_{\text{cm}}$ in agreement with Eq. (9.18).

We leave the application of Eq. (9.18) to the problems. In Chapter 10 we'll use this equation to help us analyze rigid-body problems in which the axis of rotation moves.

Test Your Understanding of Section 9.4

Suppose the cylinder and block in Example 9.8 have the same mass, so $m = M$. Just before the block strikes the floor, which statement is correct about the relationship between the kinetic energy K_{block} of the falling block and the rotational kinetic energy K_{cylinder} of the cylinder? (i) $K_{\text{block}} > K_{\text{cylinder}}$; (ii) $K_{\text{block}} < K_{\text{cylinder}}$; (iii) $K_{\text{block}} = K_{\text{cylinder}}$.

9.5 Parallel-Axis Theorem

We pointed out in Section 9.4 that a body doesn't have just one moment of inertia. In fact, it has infinitely many, because there are infinitely many axes about which it might rotate. But there is a simple relationship, called the **parallel-axis theorem,** between the moment of inertia of a body about an axis through its center of mass and the moment of inertia about any other axis parallel to the original axis (Fig. 9.19):

(9.19)

Parallel-axis theorem:

$$I_P = I_{\mathrm{cm}} + Md^2 \qquad (9.19)$$

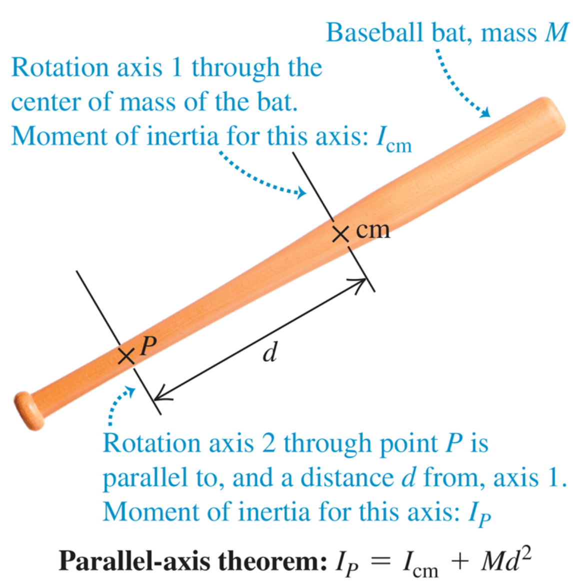
I_P: Moment of inertia of a body for a rotation axis through point P

I_{cm}: Moment of inertia of body for a parallel axis through center of mass

M: Mass of body

d: Distance between two parallel axes

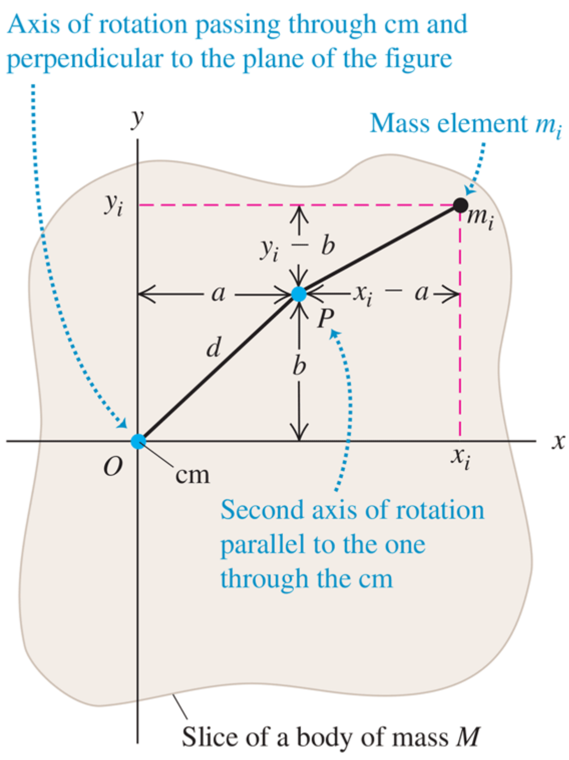
Figure 9.19



The parallel-axis theorem.

To prove this theorem, we consider two axes, both parallel to the z-axis: one through the center of mass and the other through a point P (Fig. 9.20). First we take a very thin slice of the body, parallel to the xy-plane and perpendicular to the z-axis. We take the origin of our coordinate system to be at the center of mass of the body; the coordinates of the center of mass are then $x_{\text{cm}} = y_{\text{cm}} = z_{\text{cm}} = 0$. The axis through the center of mass passes through this thin slice at point O, and the parallel axis passes through point P, whose x- and y-coordinates are $(a,\ b)$. The distance of this axis from the axis through the center of mass is d, where $d^2 = a^2 + b^2$.

Figure 9.20



The mass element m_i has coordinates $(x_i,\ y_i)$ with respect to an axis of rotation through the center of mass (cm) and coordinates $(x_i - a, y_i - b)$ with respect to the parallel axis through point P.

We can write an expression for the moment of inertia I_P about the axis through point P. Let m_i be a mass element in our slice, with coordinates $(x_i\,, y_i\,, z_i)$. Then the moment of inertia I_{cm} of the slice about the axis through the center of mass (at O) is

$$I_{\text{cm}} = \sum_i m_i \left(x_i^2 + y_i^2\right)$$

The moment of inertia of the slice about the axis through P is

$$I_P = \sum_i m_i[(x_i - a)^2 + (y_i - b)^2]$$

These expressions don't involve the coordinates z_i measured perpendicular to the slices, so we can extend the sums to include *all* particles in *all* slices. Then I_P becomes the moment of inertia of the *entire* body for an axis through P. We then expand the squared terms and regroup, and obtain

$$I_P = \sum_i m_i \left(x_i^2 + y_i^2\right) - 2a \sum_i m_i\, x_i - 2b \sum_i m_i\, y_i + \left(a^2 + b^2\right) \sum_i m_i$$

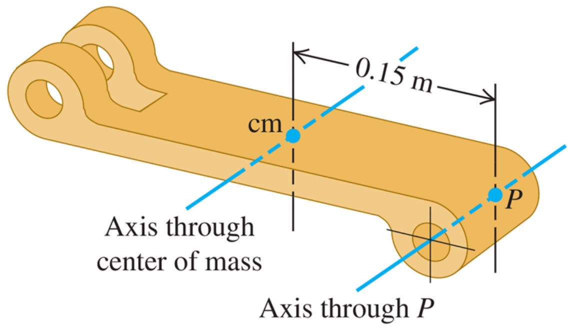
The first sum is I_{cm}. From Eq. (8.28), the definition of the center of mass, the second and third sums are proportional to x_{cm} and y_{cm}; these are zero because we have taken our origin to be the center of mass. The final term is d^2 multiplied by the total mass, or Md^2. This completes our proof that $I_P = I_{\text{cm}} + Md^2$.

As Eq. (9.19) shows, a rigid body has a lower moment of inertia about an axis through its center of mass than about any other parallel axis. Thus it's easier to start a body rotating if the rotation axis passes through the center of mass. This suggests that it's somehow most natural for a rotating body to rotate about an axis through its center of mass; we'll make this idea more quantitative in Chapter 10.

Example 9.9 Using the parallel-axis theorem

A part of a mechanical linkage (Fig. 9.21) has a mass of 3.6 kg. Its moment of inertia I_P about an axis 0.15 m from its center of mass is $I_P = 0.132\ \text{kg·m}^2$. What is the moment of inertia I_{cm} about a parallel axis through the center of mass?

Figure 9.21



Calculating I_{cm} from a measurement of I_P .

IDENTIFY, SET UP, and EXECUTE We'll determine the target variable I_{cm} by using the parallel-axis theorem, Eq. (9.19). Rearranging the equation, we obtain

$$\begin{aligned} I_{\text{cm}} &= I_P - Md^2 = 0.132\ \text{kg} \cdot \text{m}^2 - (3.6\ \text{kg})(0.15\ \text{m})^2 \\ &= 0.051\ \text{kg} \cdot \text{m}^2 \end{aligned}$$

EVALUATE As we expect, I_{cm} is less than I_P; the moment of inertia for an axis through the center of mass has a lower value than for any other parallel axis.

KEY CONCEPT

You can use the parallel-axis theorem to relate the moment of inertia of a rigid body about any axis to the moment of inertia of the same rigid body through a parallel axis through its center of mass.

Video Tutor Solution: **Example 9.9**

Test Your Understanding of Section 9.5

A pool cue is a wooden rod of uniform composition and is tapered with a larger diameter at one end than at the other end. Use the parallel-axis theorem to decide whether a pool cue has a larger moment of inertia for an axis perpendicular to the length of the rod that is (i) through the thicker end or (ii) through the thinner end.

9.6 Moment-of-Inertia Calculations

If a rigid body is a continuous distribution of mass—like a solid cylinder or a solid sphere—it cannot be represented by a few point masses. In this case the *sum* of masses and distances that defines the moment of inertia [Eq. (9.16)] becomes an *integral*. Imagine dividing the body into elements of mass dm that are very small, so that all points in a particular element are at essentially the same perpendicular distance from the axis of rotation. We call this distance r, as before. Then the moment of inertia is

(9.20)

$$I = \int r^2 \, dm$$

To evaluate the integral, we have to represent r and dm in terms of the same integration variable. When the body is effectively one-dimensional, such as the slender rods (a) and (b) in Table 9.2, we can use a coordinate x along the length and relate dm to an increment dx. For a three-dimensional body it is usually easiest to express dm in terms of an element of volume dV and the *density* ρ of the body. Density is mass per unit volume, $\rho = dm/dV$, so we may write Eq. (9.20) as

$$I = \int r^2 \rho \, dV$$

This expression tells us that a body's moment of inertia depends on how its density varies within its volume (Fig. 9.22). If the body is uniform in density, then we may take ρ outside the integral:

(9.21)

$$I = \rho \int r^2 \, dV$$

Figure 9.22

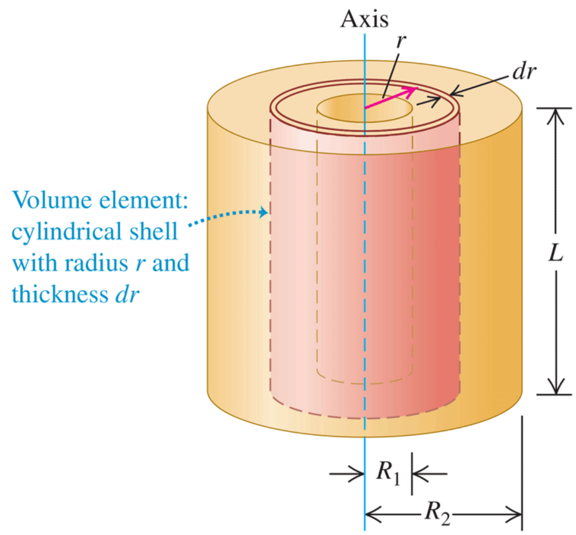
By measuring small variations in the orbits of satellites, geophysicists can measure the earth's moment of inertia. This tells us how our planet's mass is distributed within its interior. The data show that the earth is far denser at the core than in its outer layers.

To use this equation, we have to express the volume element dV in terms of the differentials of the integration variables, such as $dV = dx\, dy\, dz$. The element dV must always be chosen so that all points within it are at very nearly the same distance from the axis of rotation. The limits on the integral are determined by the shape and dimensions of the body. For regularly shaped bodies, this integration is often easy to do.

Example 9.10 Hollow or solid cylinder, rotating about axis of symmetry

[Figure 9.23] shows a hollow cylinder of uniform mass density ρ with length L, inner radius R_1, and outer radius R_2. (It might be a steel cylinder in a printing press.) Using integration, find its moment of inertia about its axis of symmetry.

Figure 9.23



Finding the moment of inertia of a hollow cylinder about its symmetry axis.

IDENTIFY and SET UP We choose as a volume element a thin cylindrical shell of radius r, thickness dr, and length L. All parts of this shell are at very nearly the same distance r from the axis. The volume of the shell is very nearly that of a flat sheet with thickness dr, length L, and width $2\pi r$ (the circumference of the shell). The mass of the shell is

$$dm = \rho\, dV = \rho\,(2\pi r L\, dr)$$

We'll use this expression in Eq. (9.20), integrating from $r = R_1$ to $r = R_2$.

EXECUTE From Eq. (9.20), the moment of inertia is

$$
\begin{aligned}
I &= \int r^2\, dm = \int_{R_1}^{R_2} r^2 \rho \,(2\pi r L\, dr) \\
&= 2\pi\rho L \int_{R_1}^{R_2} r^3\, dr = \frac{2\pi\rho L}{4} \quad R_2^4 - R_1^4 \\
&= \frac{\pi\rho L}{2} \quad R_2^2 - R_1^2 \quad R_2^2 + R_1^2
\end{aligned}
$$

[In the last step we used the identity $a^2 - b^2 = (a - b)\,(a + b)$.] Let's express this result in terms of the total mass M of the body, which is its density ρ multiplied by the total volume V. The cylinder's volume is

$$V = \pi L \quad R_2^2 - R_1^2$$

so its total mass M is

$$M = \rho V = \pi L \rho \quad R_2^2 - R_1^2$$

Comparing with the above expression for I, we see that

$$I = \frac{1}{2} M \quad R_1^2 + R_2^2$$

EVALUATE Our result agrees with Table 9.2, case (e). If the cylinder is solid, with outer radius $R_2 = R$ and inner radius $R_1 = 0$, its moment of inertia is

$$I = \frac{1}{2} M R^2$$

in agreement with case (f). If the cylinder wall is very thin, we have $R_1 \approx R_2 = R$ and the moment of inertia is

$$I = M R^2$$

in agreement with case (g). We could have predicted this last result without calculation; in a thin-walled cylinder, all the mass is at the same distance $r = R$ from the axis, so

$$I = \quad r^2\,dm = R^2 \quad dm = MR^2.$$

KEY CONCEPT

Use integration to calculate the moment of inertia of a rigid body that is a continuous distribution of mass. If the body is symmetrical, divide it into volume elements that make use of its symmetry.

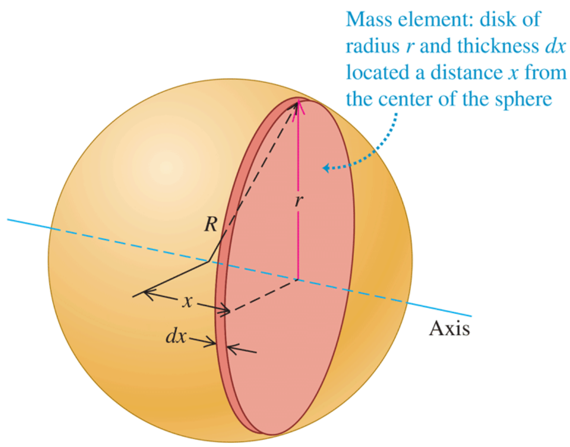
Video Tutor Solution: **Example 9.10**

Example 9.11 Uniform sphere with radius *R*, axis through center

Find the moment of inertia of a solid sphere of uniform mass density ρ (like a billiard ball) about an axis through its center.

IDENTIFY and SET UP We divide the sphere into thin, solid disks of thickness dx (Fig. 9.24), whose moment of inertia we know from Table 9.2, case (f). We'll integrate over these to find the total moment of inertia.

Figure 9.24



Finding the moment of inertia of a sphere about an axis through its center.

EXECUTE The radius and hence the volume and mass of a disk depend on its distance x from the center of the sphere. The radius r of the disk shown in Fig. 9.24 is

$$r = \sqrt{R^2 - x^2}$$

Its volume is

$$dV = \pi r^2\, dx = \pi\left(R^2 - x^2\right) dx$$

and so its mass is

$$dm = \rho\, dV = \pi\rho\left(R^2 - x^2\right) dx$$

From Table 9.2, case (f), the moment of inertia of a disk of radius r and mass dm is

$$\begin{aligned} dI &= \frac{1}{2} r^2\, dm = \frac{1}{2}(R^2 - x^2)[\pi\rho(R^2 - x^2)dx] \\ &= \frac{\pi\rho}{2}\left(R^2 - x^2\right)^2 dx \end{aligned}$$

Integrating this expression from $x = 0$ to $x = R$ gives the moment of inertia of the right hemisphere. The total I for the entire sphere, including both hemispheres, is just twice this:

$$I = (2)\frac{\pi\rho}{2}\int_{0}^{R} \; R^2 - x^2 \;\; ^2 \, dx$$

Carrying out the integration, we find

$$I = \frac{8\pi\rho R^5}{15}$$

The volume of the sphere is $V = 4\pi R^3/3$, so in terms of its mass M its density is

$$\rho = \frac{M}{V} = \frac{3M}{4\pi R^3}$$

Hence our expression for I becomes

$$I = \quad \frac{8\pi R^5}{15} \quad\quad \frac{3M}{4\pi R^3} \quad = \tfrac{2}{5}\, MR^2$$

EVALUATE This is just as in Table 9.2, case (h). Note that the moment of inertia $I = \frac{2}{5}MR^2$ of a solid sphere of mass M and radius R is less than the moment of inertia $I = \frac{1}{2}MR^2$ of a solid *cylinder* of the same mass and radius, because more of the sphere's mass is located close to the axis.

KEY CONCEPT

When calculating a moment of inertia by integration, in general use geometry to determine the size and moment of inertia of each volume element.

Video Tutor Solution: **Example 9.11**

Test Your Understanding of Section 9.6

Two hollow cylinders have the same inner and outer radii and the same mass, but they have different lengths. One is made of low-density wood and the other of high-density lead. Which cylinder has the greater moment of inertia around its axis of symmetry? (i) The wood cylinder; (ii) the lead cylinder; (iii) the two moments of inertia are equal.

Chapter 9 Summary

Rotational kinematics: When a rigid body rotates about a stationary axis (usually called the z-axis), the body's position is described by an angular coordinate θ. The angular velocity ω_z is the time derivative of θ, and the angular acceleration α_z is the time derivative of ω_z or the second derivative of θ. (See Examples 9.1 and 9.2.) If the angular acceleration is constant, then θ, ω_z, and α_z are related by simple kinematic equations analogous to those for straight-line motion with constant linear acceleration. (See Example 9.3.)

(9.3)

$$\omega_z = \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t} = \frac{d\theta}{dt}$$

(9.5)

$$\alpha_z = \lim_{\Delta t \to 0} \frac{\Delta \omega_z}{\Delta t} = \frac{d\omega_z}{dt}$$

Constant α_z only:

(9.11)

$$\theta = \theta_0 + \omega_{0z} t + \frac{1}{2} \alpha_z t^2$$

(9.10)

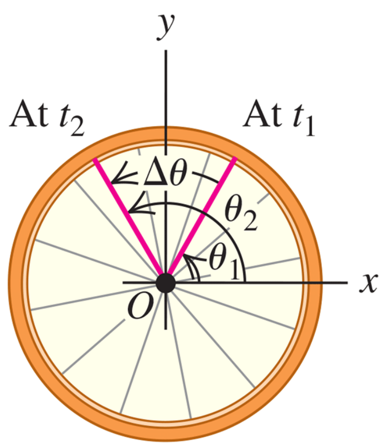
$$\theta - \theta_0 = \frac{1}{2}(\omega_{0z} + \omega_z) t$$

(9.7)

$$\omega_z = \omega_{0z} + \alpha_z t$$

(9.12)

$$\omega_z^2 = {\omega_{0z}}^2 + 2\alpha_z(\theta - \theta_0)$$



Relating linear and angular kinematics: The angular speed ω of a rigid body is the magnitude of the body's angular velocity. The rate of change of ω is $\alpha = d\omega/dt$. For a particle in the body a distance r from the rotation axis, the speed v and the components of the acceleration $\vec{a}$ are related to ω and α. (See Examples 9.4 and 9.5.)

(9.13)

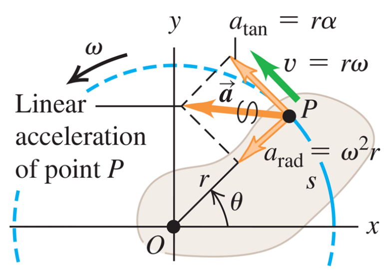
$$v = r\omega$$

(9.14)

$$a_{\tan} = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha$$

(9.15)

$$a_{\text{rad}} = \frac{v^2}{r} = \omega^2 r$$



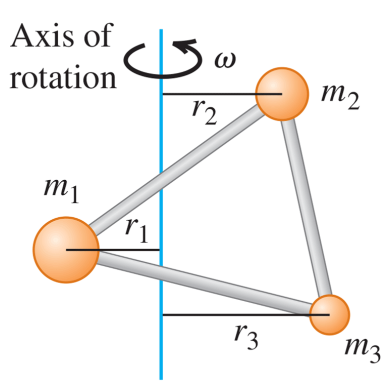
Moment of inertia and rotational kinetic energy: The moment of inertia I of a body about a given axis is a measure of its rotational inertia: The greater the value of I, the more difficult it is to change the state of the body rotation. The moment of inertia can be expressed as a sum over the particles m_i that make up the body, each of which is at its own perpendicular distance r_i from the axis. The rotational kinetic energy of a rigid body rotating about a fixed axis depends on the angular speed ω and the moment of inertia I for that rotation axis. (See Examples 9.6, 9.7 and 9.8.)

(9.16)

$$\begin{aligned} I &= m_1 {r_1}^2 + m_2 {r_2}^2 + \cdots \\ &= \sum_i m_i {r_i}^2 \end{aligned}$$

(9.17)

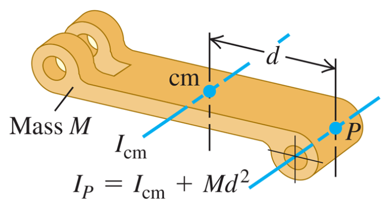
$$K = \frac{1}{2} I \omega^2$$



Calculating the moment of inertia: The parallel-axis theorem relates the moments of inertia of a rigid body of mass M about two parallel axes: an axis through the center of mass (moment of inertia I_{cm}) and a parallel axis a distance d from the first axis (moment of inertia I_P). (See Example 9.9.) If the body has a continuous mass distribution, the moment of inertia can be calculated by integration. (See Examples 9.10 and 9.11.)

(9.19)

$$I_P = I_{cm} + Md^2$$



Guided Practice: Rotation of Rigid Bodies

For assigned homework and other learning materials, go to
Mastering Physics.

Key Example Variation Problems

Be sure to review Example 9.3 (Section 9.2) before attempting these problems.

VP9.3.1 A machine part is initially rotating at 0.500 rad/s. Its rotation speeds up with constant angular acceleration 2.50 rad/s^2. Through what angle has the machine part rotated when its angular speed equals 3.25 rad/s? Give your answer in (a) radians, (b) degrees, and (c) revolutions.

VP9.3.2 The rotor of a helicopter is gaining angular speed with constant angular acceleration. At $t = 0$ it is rotating at 1.25 rad/s. From $t = 0$ to $t = 2.00$ s, the rotor rotates through 8.00 rad. (a) What is the angular acceleration of the rotor? (b) Through what angle (in radians) does the rotor rotate from $t = 0$ to $t = 4.00$ s?

VP9.3.3 A jeweler's grinding wheel slows down at a constant rate from 185 rad/s to 105 rad/s while it rotates through 16.0 revolutions. How much time does this take?