VP9.3.4 A disk rotates around an axis through its center that is perpendicular to the plane of the disk. The disk has a line drawn on it that extends from the axis of the disk to the rim. At $t = 0$ this line lies along the x-axis and the disk is rotating with positive angular velocity ω_{0z}. The disk has constant positive angular acceleration α_z. At what time after $t = 0$ has the line on the disk rotated through an angle θ?

Be sure to review Examples 9.4 and 9.5 (Section 9.3) before attempting these problems.

VP9.5.1 Shortly after a vinyl record (radius 0.152 m) starts rotating on a turntable, its angular velocity is 1.60 rad/s and increasing at a rate of 8.00 rad/s^2. At this instant, for a point at the rim of the record, what are (a) the tangential component of acceleration, (b) the centripetal component of acceleration, and (c) the magnitude of acceleration?

VP9.5.2 A superhero swings a magic hammer over her head in a horizontal plane. The end of the hammer moves around a circular path of radius 1.50 m at an angular speed of 6.00 rad/s. As the superhero swings the hammer, she then ascends vertically at a constant 2.00 m/s. (a) What is the speed of the end of the hammer relative to the ground? (b) What is the acceleration (magnitude and direction) of the end of the hammer?

VP9.5.3 If the magnitude of the acceleration of a propeller blade's tip exceeds a certain value a_{max}, the blade tip will fracture. If the propeller has radius r, is initially at rest, and has angular acceleration of magnitude α, at what angular speed ω will the blade tip fracture?

VP9.5.4 At a certain instant, a rotating turbine wheel of radius R has angular speed ω (measured in rad/s). (a) What must be the magnitude α of its angular acceleration (measured in rad/s^2) at this instant if the acceleration vector $\vec{a}$ of a point on the rim of the wheel makes an angle of exactly 30° with the velocity vector $\vec{v}$ of that point? (b) At this same instant, what is the angle between $\vec{a}$ and $\vec{v}$ for a point on the wheel halfway between the axis of rotation and the rim?

Be sure to review Examples 9.7 and 9.8 (Section 9.4) before attempting these problems.

VP9.8.1 A solid cylinder of mass 12.0 kg and radius 0.250 m is free to rotate without friction around its central axis. If you do 75.0 J of work on the cylinder to increase its angular speed, what will be its final angular speed if the cylinder (a) starts from rest; (b) is initially rotating at 12.0 rad/s?

VP9.8.2 A square plate has mass 0.600 kg and sides of length 0.150 m. It is free to rotate without friction around an axis through its center and perpendicular to the plane of the plate. How much work must you do on the plate to change its angular speed (a) from 0 to 40.0 rad/s and (b) from 40.0 rad/s to 80.0 rad/s?

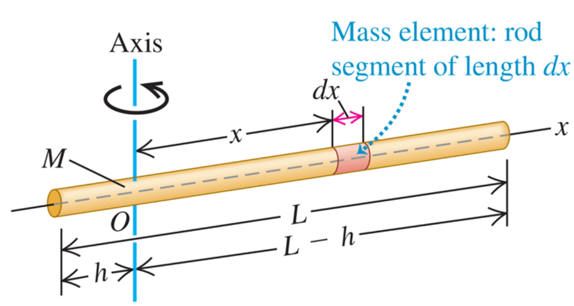
VP9.8.3 A hollow cylinder of mass 2.00 kg, inner radius 0.100 m, and outer radius 0.200 m is free to rotate without friction around a horizontal shaft of radius 0.100 m along the axis of the cylinder. You wrap a light, nonstretching cable around the cylinder and tie the free end to a 0.500 kg block of cheese. You release the cheese from rest a distance h above the floor. (a) If the cheese is moving downward at 4.00 m/s just before it hits the ground, what is the value of h? (b) What is the angular speed of the cylinder just before the cheese hits the ground?

VP9.8.4 A pulley in the shape of a solid cylinder of mass 1.50 kg and radius 0.240 m is free to rotate around a horizontal shaft along the axis of the pulley. There is friction between the pulley and this shaft. A light, nonstretching cable is wrapped around the pulley, and the free end is tied to a 2.00 kg textbook. You release the textbook from rest a distance 0.900 m above the floor. Just before the textbook hits the floor, the angular speed of the pulley is 10.0 rad/s. (a) What is the speed of the textbook just before it hits the floor? (b) How much work was done on the pulley by the force of friction while the textbook was falling to the floor?

Bridging Problem: A Rotating, Uniform Thin Rod

Figure 9.25 shows a slender uniform rod with mass M and length L. It might be a baton held by a twirler in a marching band (without the rubber end caps). (a) Use integration to compute its moment of inertia about an axis through O, at an arbitrary distance h from one end. (b) Initially the rod is at rest. It is given a constant angular acceleration of magnitude α around the axis through O. Find how much work is done on the rod in a time t. (c) At time t, what is the *linear* acceleration of the point on the rod farthest from the axis?

Figure 9.25



A thin rod with an axis through O.

Solution Guide

IDENTIFY and SET UP

1. Make a list of the target variables for this problem.
2. To calculate the moment of inertia of the rod, you'll have to divide the rod into infinitesimal elements of mass. If an element has length dx, what is the mass of the element? What are the limits of integration?
3. What is the angular speed of the rod at time t? How does the work required to accelerate the rod from rest to this angular speed compare to the rod's kinetic energy at time t?
4. At time t, does the point on the rod farthest from the axis have a centripetal acceleration? A tangential acceleration? Why or why not?

EXECUTE

5. Do the integration required to find the moment of inertia.
6. Use your result from step 5 to calculate the work done in time t to accelerate the rod from rest.
7. Find the linear acceleration components for the point in question at time t. Use these to find the magnitude of the acceleration.

EVALUATE

8. Check your results for the special cases $h = 0$ (the axis passes through one end of the rod) and $h = L/2$ (the axis passes through the middle of the rod). Are these limits consistent with Table 9.2? With the parallel-axis theorem?
9. Is the acceleration magnitude from step 7 constant? Would you expect it to be?

Video Tutor Solution: **Chapter 9 Bridging Problem**

Questions/Exercises/Problems: Rotation of Rigid Bodies

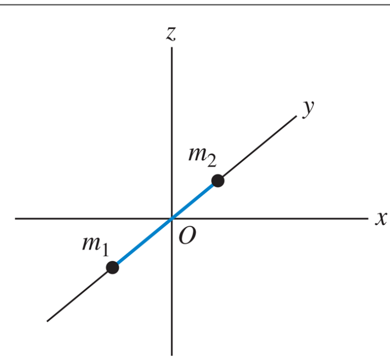
•, ••, ••• : Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

Discussion Questions

Q9.1 Which of the following formulas is valid if the angular acceleration of an object is *not* constant? Explain your reasoning in each case. (a) $v = r\omega$; (b) $a_{\tan} = r\alpha$; (c) $\omega = \omega_0 + \alpha t$; (d) $a_{\tan} = r\omega^2$; (e) $K = \frac{1}{2} I\omega^2$.

Q9.2 A diatomic molecule can be modeled as two point masses, m_1 and m_2, slightly separated (Fig. Q9.2). If the molecule is oriented along the y-axis, it has kinetic energy K when it spins about the x-axis. What will its kinetic energy (in terms of K) be if it spins at the same angular speed about (a) the z-axis and (b) the y-axis?

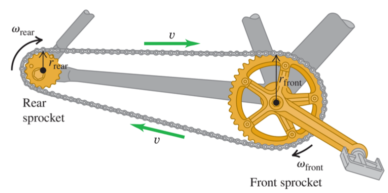
Figure Q9.2



Q9.3 What is the difference between tangential and radial acceleration for a point on a rotating body?

Q9.4 In Fig. Q9.4, all points on the chain have the same linear speed. Is the magnitude of the linear acceleration also the same for all points on the chain? How are the angular accelerations of the two sprockets related? Explain.

Figure Q9.4



Q9.5 In Fig. Q9.4, how are the radial accelerations of points at the teeth of the two sprockets related? Explain.

Q9.6 A flywheel rotates with constant angular velocity. Does a point on its rim have a tangential acceleration? A radial acceleration? Are these accelerations constant in magnitude? In direction? In each case give your reasoning.

Q9.7 What is the purpose of the spin cycle of a washing machine? Explain in terms of acceleration components.

Q9.8 You are designing a flywheel to store kinetic energy. If all of the following uniform objects have the same mass and same angular velocity, which one will store the greatest amount of kinetic energy? Which will store the least? Explain. (a) A solid sphere of diameter D rotating about a diameter; (b) a solid cylinder of diameter D rotating about an axis perpendicular to each face through its center; (c) a thin-walled hollow cylinder of diameter D rotating about an axis perpendicular to the plane of the circular face at its center; (d) a solid, thin bar of length D rotating about an axis perpendicular to it at its center.

Q9.9 Can you think of a body that has the same moment of inertia for all possible axes? If so, give an example, and if not, explain why this is not possible. Can you think of a body that has the same moment of inertia for all axes passing through a certain point? If so, give an example and indicate where the point is located.

Q9.10 To maximize the moment of inertia of a flywheel while minimizing its weight, what shape and distribution of mass should it have? Explain.

Q9.11 How might you determine experimentally the moment of inertia of an irregularly shaped body about a given axis?

Q9.12 A cylindrical body has mass M and radius R. Can the mass be distributed within the body in such a way that its moment of inertia about its axis of symmetry is greater than MR^2? Explain.

Q9.13 Describe how you could use part (b) of Table 9.2 to derive the result in part (d).

Q9.14 A hollow spherical shell of radius R that is rotating about an axis through its center has rotational kinetic energy K. If you want to modify this sphere so that it has three times as much kinetic energy at the same angular speed while keeping the same mass, what should be its radius in terms of R?

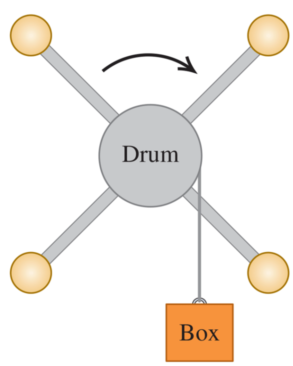
Q9.15 For the equations for I given in parts (a) and (b) of Table 9.2 to be valid, must the rod have a circular cross section? Is there any restriction on the size of the cross section for these equations to apply? Explain.

Q9.16 In part (d) of Table 9.2, the thickness of the plate must be much less than a for the expression given for I to apply. But in part (c), the expression given for I applies no matter how thick the plate is. Explain.

Q9.17 Two identical balls, A and B, are each attached to very light string, and each string is wrapped around the rim of a pulley of mass M on a frictionless axle. The only difference is that the pulley for ball A is a solid disk, while the one for ball B is a hollow disk, like part (e) in Table 9.2. If both balls are released from rest and fall the same distance, which one will have more kinetic energy, or will they have the same kinetic energy? Explain your reasoning.

Q9.18 An elaborate pulley consists of four identical balls at the ends of spokes extending out from a rotating drum (Fig. Q9.18). A box is connected to a light, thin rope wound around the rim of the drum. When it is released from rest, the box acquires a speed V after having fallen a distance d. Now the four balls are moved inward closer to the drum, and the box is again released from rest. After it has fallen a distance d, will its speed be equal to V, greater than V, or less than V? Show or explain why.

Figure Q9.18



Q9.19 You can use any angular measure—radians, degrees, or revolutions—in some of the equations in Chapter 9, but you can use only radian measure in others. Identify those for which using radians is necessary and those for which it is not, and in each case give your reasoning.

Q9.20 When calculating the moment of inertia of an object, can we treat all its mass as if it were concentrated at the center of mass of the object? Justify your answer.

Q9.21 A wheel is rotating about an axis perpendicular to the plane of the wheel and passing through the center of the wheel. The angular speed of the wheel is increasing at a constant rate. Point A is on the rim of the wheel and point B is midway between the rim and center of the wheel. For each of the following quantities, is its magnitude larger at point A or at point B, or is it the same at both points? (a) Angular speed; (b) tangential speed; (c) angular acceleration; (d) tangential acceleration; (e) radial acceleration. Justify each answer.

Q9.22 Estimate your own moment of inertia about a vertical axis through the center of the top of your head when you are standing up straight with your arms outstretched. Make reasonable approximations and measure or estimate necessary quantities.

Exercises

Section 9.1 Angular Velocity and Acceleration

9.1 • (a) What angle in radians is subtended by an arc 1.50 m long on the circumference of a circle of radius 2.50 m? What is this angle in degrees? (b) An arc 14.0 cm long on the circumference of a circle subtends an angle of 128°. What is the radius of the circle? (c) The angle between two radii of a circle with radius 1.50 m is 0.700 rad. What length of arc is intercepted on the circumference of the circle by the two radii?

9.2 • An airplane propeller is rotating at 1900 rpm (rev/min). (a) Compute the propeller's angular velocity in rad/s. (b) How many seconds does it take for the propeller to turn through 35°?

9.3 • **CP CALC** The angular velocity of a flywheel obeys the equation $\omega_z(t) = A + Bt^2$, where t is in seconds and A and B are constants having numerical values 2.75 (for A) and 1.50 (for B). (a) What are the units of A and B if ω_z is in rad/s? (b) What is the angular acceleration of the wheel at (i) $t = 0$ and (ii) $t = 5.00$ s? (c) Through what angle does the flywheel turn during the first 2.00 s? (**HINT:** See Section 2.6.)

9.4 •• **CALC** A fan blade rotates with angular velocity given by $\omega_z(t) = \gamma - \beta t^2$, where $\gamma = 5.00\ \mathrm{rad}/s$ and $\beta = 0.800\ \mathrm{rad/s^3}$. (a) Calculate the angular acceleration as a function of time. (b) Calculate the instantaneous angular acceleration α_z at $t = 3.00$ s and the average angular acceleration $\alpha_{\mathrm{av}-z}$ for the time interval $t = 0$ to $t = 3.00$ s. How do these two quantities compare? If they are different, why?

9.5 •• **CALC** A child is pushing a merry-go-round. The angle through which the merry-go-round has turned varies with time according to $\theta(t) = \gamma t + \beta t^3$, where $\gamma = 0.400\ \mathrm{rad}/s$ and

β $=0.0120\ \mathrm{rad/s^3}$. (a) Calculate the angular velocity of the merry-go-round as a function of time. (b) What is the initial value of the angular velocity? (c) Calculate the instantaneous value of the angular velocity ω_z at $t = 5.00$ s and the average angular velocity $\omega_{\text{av-}z}$ for the time interval $t = 0$ to $t = 5.00$ s. Show that $\omega_{\text{av-}z}$ is *not* equal to the average of the instantaneous angular velocities at $t = 0$ and $t = 5.00$ s, and explain.

9.6 • **CALC** At $t = 0$ the current to a dc electric motor is reversed, resulting in an angular displacement of the motor shaft given by $\theta(t) = (250\ \mathrm{rad/s})t - (20.0\ \mathrm{rad/s^2})t^2 - (1.50\ \mathrm{rad/s^3})t^3$. (a) At what time is the angular velocity of the motor shaft zero? (b) Calculate the angular acceleration at the instant that the motor shaft has zero angular velocity. (c) How many revolutions does the motor shaft turn through between the time when the current is reversed and the instant when the angular velocity is zero? (d) How fast was the motor shaft rotating at $t = 0$, when the current was reversed? (e) Calculate the average angular velocity for the time period from $t = 0$ to the time calculated in part (a).

9.7 • **CALC** The angle θ through which a disk drive turns is given by $\theta(t) = a + bt - ct^3$, where a, b, and c are constants, t is in seconds, and θ is in radians. When $t = 0$, $\theta = \pi/4$ rad and the angular velocity is $2.00\ \mathrm{rad}/s$. When $t = 1.50$ s, the angular acceleration is $1.25\ \mathrm{rad/s^2}$. (a) Find a, b, and c, including their units. (b) What is the angular acceleration when $\theta = \pi/4$ rad? (c) What are θ and the angular velocity when the angular acceleration is $3.50\ \mathrm{rad/s^2}$?

9.8 • A wheel is rotating about an axis that is in the z-direction. The angular velocity ω_z is $-6.00\ \mathrm{rad}/s$ at $t = 0$, increases linearly with time, and is $+4.00\ \mathrm{rad}/s$ at $t = 7.00$ s. We have taken counterclockwise rotation to be positive. (a) Is the angular acceleration during this time interval positive or negative? (b) During what time interval is the speed of the wheel increasing? Decreasing? (c) What is the angular displacement of the wheel at $t = 7.00$ s?

Section 9.2 Rotation with Constant Angular Acceleration

9.9 • A bicycle wheel has an initial angular velocity of 1.50 rad/s. (a) If its angular acceleration is constant and equal to 0.200 rad/s^2, what is its angular velocity at $t = 2.50$ s? (b) Through what angle has the wheel turned between $t = 0$ and $t = 2.50$ s?

9.10 •• An electric fan is turned off, and its angular velocity decreases uniformly from 500 rev/min to 200 rev/min in 4.00 s. (a) Find the angular acceleration in rev/s^2 and the number of revolutions made by the motor in the 4.00 s interval. (b) How many more seconds are required for the fan to come to rest if the angular acceleration remains constant at the value calculated in part (a)?

9.11 •• The rotating blade of a blender turns with constant angular acceleration 1.50 rad/s^2. (a) How much time does it take to reach an angular velocity of 36.0 rad/s, starting from rest? (b) Through how many revolutions does the blade turn in this time interval?

9.12 •• A wheel rotates from rest with constant angular acceleration. If it rotates through 8.00 revolutions in the first 2.50 s, how many more revolutions will it rotate through in the next 5.00 s?

9.13 •• A turntable rotates with a constant 2.25 rad/s^2 clockwise angular acceleration. After 4.00 s it has rotated through a clockwise angle of 30.0 rad. What was the angular velocity of the wheel at the beginning of the 4.00 s interval?

9.14 • A circular saw blade 0.200 m in diameter starts from rest. In 6.00 s it accelerates with constant angular acceleration to an angular velocity of 140 rad/s. Find the angular acceleration and the angle through which the blade has turned.

9.15 •• A high-speed flywheel in a motor is spinning at 500 rpm when a power failure suddenly occurs. The flywheel has mass 40.0 kg and diameter 75.0 cm. The power is off for 30.0 s, and during this time the flywheel slows due to friction in its axle bearings. During the time the power is off, the flywheel makes 200 complete revolutions. (a) At what rate is the flywheel spinning when the power comes back on? (b) How long after the beginning of the power failure would it have taken the flywheel to stop if the power had not come back on, and how many revolutions would the wheel have made during this time?

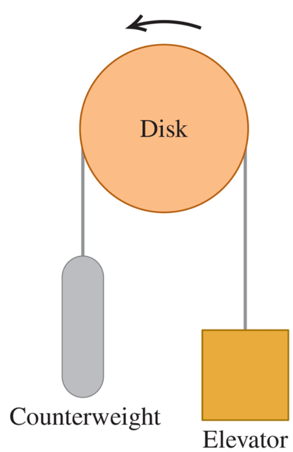
9.16 •• At $t = 0$ a grinding wheel has an angular velocity of 24.0 rad/s. It has a constant angular acceleration of 30.0 rad/s^2 until a circuit breaker trips at $t = 2.00$ s. From then on, it turns through 432 rad as it coasts to a stop at constant angular acceleration. (a) Through what total angle did the wheel turn between $t = 0$ and the time it stopped? (b) At what time did it stop? (c) What was its acceleration as it slowed down?

9.17 •• A safety device brings the blade of a power mower from an initial angular speed of ω_1 to rest in 1.00 revolution. At the same constant acceleration, how many revolutions would it take the blade to come to rest from an initial angular speed ω_3 that was three times as great, $\omega_3 = 3\omega_1$?

Section 9.3 Relating Linear and Angular Kinematics

9.18 • In a charming 19th-century hotel, an old-style elevator is connected to a counterweight by a cable that passes over a rotating disk 2.50 m in diameter (Fig. E9.18). The elevator is raised and lowered by turning the disk, and the cable does not slip on the rim of the disk but turns with it. (a) At how many rpm must the disk turn to raise the elevator at 25.0 cm/s? (b) To start the elevator moving, it must be accelerated at $\frac{1}{8}g$. What must be the angular acceleration of the disk, in $\mathrm{rad/s^2}$? (c) Through what angle (in radians and degrees) has the disk turned when it has raised the elevator 3.25 m between floors?

Figure E9.18



9.19 • Spin cycles of washing machines remove water from clothes by producing a large radial acceleration at the rim of the cylindrical tub that holds the water and clothes. Estimate the diameter of the tub in a typical home washing machine. (a) What is the rotation rate, in rev/min, of the tub during the spin cycle if the radial acceleration of points on the tub wall is /s of a point on the tub wall?

9.20 • **Compact Disc.** A compact disc (CD) stores music in a coded pattern of tiny pits 10^{-7} m deep. The pits are arranged in a track that spirals outward toward the rim of the disc; the inner and outer radii of this spiral are 25.0 mm and 58.0 mm, respectively. As the disc spins inside a CD player, the track is scanned at a constant *linear* speed of 1.25 m/s. (a) What is the angular speed of the CD when the innermost part of the track is scanned? The outermost part of the track? (b) The maximum playing time of a CD is 74.0 min. What would be the length of the track on such a maximum-duration CD if it were stretched out in a straight line? (c) What is the average angular acceleration of a maximum-duration CD during its 74.0 min playing time? Take the direction of rotation of the disc to be positive.

9.21 •• A wheel of diameter 40.0 cm starts from rest and rotates with a constant angular acceleration of 3.00 rad/s^2. Compute the radial acceleration of a point on the rim for the instant the wheel completes its second revolution from the relationship (a) $a_{\text{rad}} = \omega^2 r$ and (b) $a_{\text{rad}} = v^2/r$.

9.22 •• You are to design a rotating cylindrical axle to lift 800 N buckets of cement from the ground to a rooftop 78.0 m above the ground. The buckets will be attached to a hook on the free end of a cable that wraps around the rim of the axle; as the axle turns, the buckets will rise. (a) What should the diameter of the axle be in order to raise the buckets at a steady 2.00 cm/s when it is turning at 7.5 rpm? (b) If instead the axle must give the buckets an upward acceleration of 0.400 m/s^2, what should the angular acceleration of the axle be?

9.23 • The blade of an electric saw rotates at 2600 rev/min. Estimate the diameter of a typical saw that is used to saw boards in home construction and renovation. What is the linear speed in m/s of a point on the rim of the circular saw blade?

9.24 •• An electric turntable 0.750 m in diameter is rotating about a fixed axis with an initial angular velocity of 0.250 rev/s and a constant angular acceleration of 0.900 rev/s^2. (a) Compute the angular velocity of the turntable after 0.200 s. (b) Through how many revolutions has the turntable spun in this time interval? (c) What is the tangential speed of a point on the rim of the turntable at $t = 0.200$ s? (d) What is the magnitude of the *resultant* acceleration of a point on the rim at $t = 0.200$ s?

9.25 •• **Centrifuge.** An advertisement claims that a centrifuge takes up only 0.127 m of bench space but can produce a radial acceleration of $3000g$ at 5000 rev/min. Calculate the required radius of the centrifuge. Is the claim realistic?

9.26 • At $t = 3.00$ s a point on the rim of a 0.200-m-radius wheel has a tangential speed of 50.0 m/s as the wheel slows down with a tangential acceleration of constant magnitude 10.0 m/s^2. (a) Calculate the wheel's constant angular acceleration. (b) Calculate the angular velocities at $t = 3.00$ s and $t = 0$. (c) Through what angle did the wheel turn between $t = 0$ and $t = 3.00$ s? (d) At what time will the radial acceleration equal g?

9.27 • A rotating wheel with diameter 0.600 m is speeding up with constant angular acceleration. The speed of a point on the rim of the wheel increases from 3.00 m/s to 6.00 m/s while the wheel turns through 4.00 revolutions. What is the angular acceleration of the wheel?

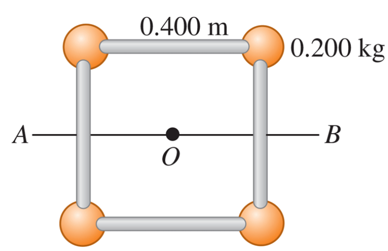
9.28 • The earth is approximately spherical, with a diameter of 1.27×10^7 m. It takes 24.0 hours for the earth to complete one revolution. What are the tangential speed and radial acceleration of a point on the surface of the earth, at the equator?

9.29 • A flywheel with radius 0.300 m starts from rest and accelerates with a constant angular acceleration of 0.600 rad/s^2. For a point on the rim of the flywheel, what are the magnitudes of the tangential, radial, and resultant accelerations after 2.00 s of acceleration?

Section 9.4 Energy in Rotational Motion

9.30 • Four small spheres, each of which you can regard as a point of mass 0.200 kg, are arranged in a square 0.400 m on a side and connected by extremely light rods (Fig. E9.30). Find the moment of inertia of the system about an axis (a) through the center of the square, perpendicular to its plane (an axis through point O in the figure); (b) bisecting two opposite sides of the square (an axis along the line AB in the figure); (c) that passes through the centers of the upper left and lower right spheres and through point O.

Figure E9.30



9.31 • Calculate the moment of inertia of each of the following uniform objects about the axes indicated. Consult Table 9.2 as needed. (a) A thin 2.50 kg rod of length 75.0 cm, about an axis perpendicular to it and passing through (i) one end and (ii) its center, and (iii) about an axis parallel to the rod and passing through it. (b) A 3.00 kg sphere 38.0 cm in diameter, about an axis through its center, if the sphere is (i) solid and (ii) a thin-walled hollow shell. (c) An 8.00 kg cylinder, of length 19.5 cm and diameter 12.0 cm, about the central axis of the cylinder, if the cylinder is (i) thin-walled and hollow, and (ii) solid.

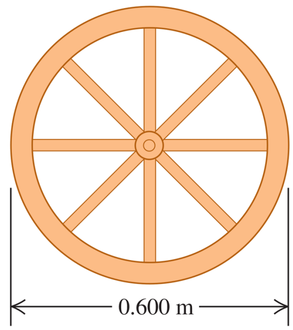
9.32 •• Three small blocks, each with mass m, are clamped at the ends and at the center of a rod of length L and negligible mass. Compute the moment of inertia of the system about an axis perpendicular to the rod and passing through (a) the center of the rod and (b) a point one-fourth of the length from one end.

9.33 • A uniform bar has two small balls glued to its ends. The bar is 2.00 m long and has mass 4.00 kg, while the balls each have mass 0.300 kg and can be treated as point masses. Find the moment of inertia of this combination about an axis (a) perpendicular to the bar through its center; (b) perpendicular to the bar through one of the balls; (c) parallel to the bar through both balls; and (d) parallel to the bar and 0.500 m from it.

9.34 •• You are a project manager for a manufacturing company. One of the machine parts on the assembly line is a thin, uniform rod that is 60.0 cm long and has mass 0.400 kg. (a) What is the moment of inertia of this rod for an axis at its center, perpendicular to the rod? (b) One of your engineers has proposed to reduce the moment of inertia by bending the rod at its center into a V-shape, with a 60.0° angle at its vertex. What would be the moment of inertia of this bent rod about an axis perpendicular to the plane of the V at its vertex?

9.35 •• A wagon wheel is constructed as shown in Fig. E9.35. The radius of the wheel is 0.300 m, and the rim has mass 1.40 kg. Each of the eight spokes that lie along a diameter and are 0.300 m long has mass 0.280 kg. What is the moment of inertia of the wheel about an axis through its center and perpendicular to the plane of the wheel? (Use Table 9.2.)

Figure E9.35



9.36 •• A uniform sphere made of modeling clay has radius R and moment of inertia I_1 for rotation about a diameter. It is flattened to a disk with the same radius R. In terms of I_1, what is the moment of inertia of the disk for rotation about an axis that is at the center of the disk and perpendicular to its flat surface?

9.37 •• A rotating flywheel has moment of inertia $12.0 \text{ kg} \cdot \text{m}^2$ for an axis along the axle about which the wheel is rotating. Initially the flywheel has 30.0 J of kinetic energy. It is slowing down with an angular acceleration of magnitude 0.500 rev/s^2. How long does it take for the rotational kinetic energy to become half its initial value, so it is 15.0 J?

9.38 •• An airplane propeller is 2.08 m in length (from tip to tip) with mass 117 kg and is rotating at 2400 rpm (rev/min) about an axis through its center. You can model the propeller as a slender rod. (a) What is its rotational kinetic energy? (b) Suppose that, due to weight constraints, you had to reduce the propeller's mass to 75.0% of its original mass, but you still needed to keep the same size and kinetic energy. What would its angular speed have to be, in rpm?

9.39 •• A uniform sphere with mass M and radius R is rotating with angular speed ω_1 about a frictionless axle along a diameter of the sphere. The sphere has rotational kinetic energyK_1. A thin-walled hollow sphere has the same mass and radius as the uniform sphere. It is also rotating about a fixed axis along its diameter. In terms of ω_1, what angular speed must the hollow sphere have if its kinetic energy is also K_1, the same as for the uniform sphere?

9.40 • A wheel is turning about an axis through its center with constant angular acceleration. Starting from rest, at $t = 0$, the wheel turns through 8.20 revolutions in 12.0 s. At $t = 12.0$ s the kinetic energy of the wheel is 36.0 J. For an axis through its center, what is the moment of inertia of the wheel?

9.41 • A uniform sphere with mass 28.0 kg and radius 0.380 m is rotating at constant angular velocity about a stationary axis that lies along a diameter of the sphere. If the kinetic energy of the sphere is 236 J, what is the tangential velocity of a point on the rim of the sphere?

9.42 •• A hollow spherical shell has mass 8.20 kg and radius 0.220 m. It is initially at rest and then rotates about a stationary axis that lies along a diameter with a constant acceleration of 0.890 rad/s^2. What is the kinetic energy of the shell after it has turned through 6.00 rev?

9.43 • Wheel A has three times the moment of inertia about its axis of rotation as wheel B. Wheel B's angular speed is four times that of wheel A. (a) Which wheel has the greater rotational kinetic energy? (b) If K_A and K_B are the rotational kinetic energies of the wheels, what is K_A/K_B?

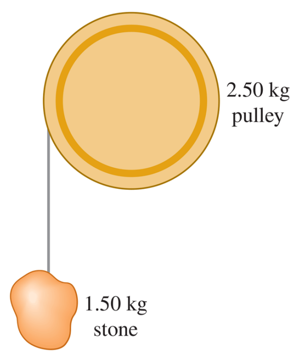
9.44 •• You need to design an industrial turntable that is 60.0 cm in diameter and has a kinetic energy of 0.250 J when turning at 45.0 rpm (rev/min). (a) What must be the moment of inertia of the turntable about the rotation axis? (b) If your workshop makes this turntable in the shape of a uniform solid disk, what must be its mass?

9.45 •• Energy is to be stored in a 70.0 kg flywheel in the shape of a uniform solid disk with radius $R = 1.20$ m. To prevent structural failure of the flywheel, the maximum allowed radial acceleration of a point on its rim is 3500 m/s^2. What is the maximum kinetic energy that can be stored in the flywheel?

9.46 •• You are designing a flywheel. It is to start from rest and then rotate with a constant angular acceleration of 0.200 rev/s^2. The design specifications call for it to have a rotational kinetic energy of 240 J after it has turned through 30.0 revolutions. What should be the moment of inertia of the flywheel about its rotation axis?

9.47 •• A pulley on a frictionless axle has the shape of a uniform solid disk of mass 2.50 kg and radius 20.0 cm. A 1.50 kg stone is attached to a very light wire that is wrapped around the rim of the pulley (Fig. E9.47), and the system is released from rest. (a) How far must the stone fall so that the pulley has 4.50 J of kinetic energy? (b) What percent of the total kinetic energy does the pulley have?

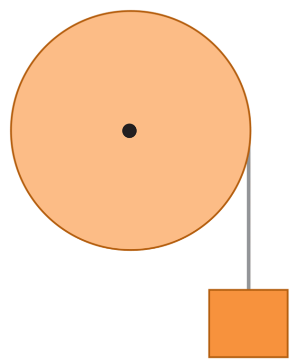
Figure E9.47



9.48 •• A bucket of mass m is tied to a massless cable that is wrapped around the outer rim of a uniform pulley of radius R, on a frictionless axle, similar to the system shown in Fig. E9.47. In terms of the stated variables, what must be the moment of inertia of the pulley so that it always has half as much kinetic energy as the bucket?

9.49 •• **CP** A thin, light wire is wrapped around the rim of a wheel (Fig. E9.49). The wheel rotates without friction about a stationary horizontal axis that passes through the center of the wheel. The wheel is a uniform disk with radius $R = 0.280$ m. An object of mass $m = 4.20$ kg is suspended from the free end of the wire. The system is released from rest and the suspended object descends with constant acceleration. If the suspended object moves downward a distance of 3.00 m in 2.00 s, what is the mass of the wheel?

Figure E9.49



Section 9.5 Parallel-Axis Theorem

9.50 •• Find the moment of inertia of a hoop (a thin-walled, hollow ring) with mass M and radius R about an axis perpendicular to the hoop's plane at an edge.

9.51 •• About what axis will a uniform, balsa-wood sphere have the same moment of inertia as does a thin-walled, hollow, lead sphere of the same mass and radius, with the axis along a diameter?

9.52 • (a) For the thin rectangular plate shown in part (d) of Table 9.2, find the moment of inertia about an axis that lies in the plane of the plate, passes through the center of the plate, and is parallel to the axis shown. (b) Find the moment of inertia of the plate for an axis that lies in the plane of the plate, passes through the center of the plate, and is perpendicular to the axis in part (a).

9.53 •• A thin, rectangular sheet of metal has mass M and sides of length a and b. Use the parallel-axis theorem to calculate the moment of inertia of the sheet for an axis that is perpendicular to the plane of the sheet and that passes through one corner of the sheet.

9.54 •• A thin uniform rod of mass M and length L is bent at its center so that the two segments are now perpendicular to each other. Find its moment of inertia about an axis perpendicular to its plane and passing through (a) the point where the two segments meet and (b) the midpoint of the line connecting its two ends.

Section 9.6 Moment-of-Inertia Calculations

9.55 •• CALC Use Eq. (9.20) to calculate the moment of inertia of a uniform, solid disk with mass M and radius R for an axis perpendicular to the plane of the disk and passing through its center.

9.56 • CALC Use Eq. (9.20) to calculate the moment of inertia of a slender, uniform rod with mass M and length L about an axis at one end, perpendicular to the rod.

9.57 •• CALC A slender rod with length L has a mass per unit length that varies with distance from the left end, where $x = 0$, according to $dm/dx = \gamma x$, where γ has units of $\mathrm{kg/m^2}$. (a) Calculate the total mass of the rod in terms of γ and L. (b) Use Eq. (9.20) to calculate the moment of inertia of the rod for an axis at the left end, perpendicular to the rod. Use the expression you derived in part (a) to express I in terms of M and L. How does your result compare to that for a uniform rod? Explain. (c) Repeat part (b) for an axis at the right end of the rod. How do the results for parts (b) and (c) compare? Explain.

Problems

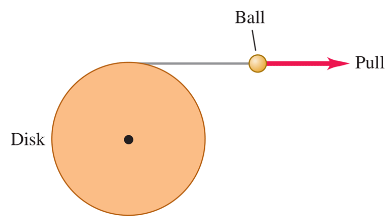
9.58 •• CALC A uniform disk with radius $R = 0.400$ m and mass 30.0 kg rotates in a horizontal plane on a frictionless vertical axle that passes through the center of the disk. The angle through which the disk has turned varies with time according to $\theta(t) = (1.10\ \mathrm{rad/s})t + (6.30\ \mathrm{rad/s^2})t^2$. What is the resultant linear acceleration of a point on the rim of the disk at the instant when the disk has turned through 0.100 rev?

9.59 •• CP A circular saw blade with radius 0.120 m starts from rest and turns in a vertical plane with a constant angular acceleration of 2.00 $\mathrm{rev/s^2}$. After the blade has turned through 155 rev, a small piece of the blade breaks loose from the top of the blade. After the piece breaks loose, it travels with a velocity that is initially horizontal and equal to the tangential velocity of the rim of the blade. The piece travels a vertical distance of 0.820 m to the floor. How far does the piece travel horizontally, from where it broke off the blade until it strikes the floor?

9.60 • **CALC** A roller in a printing press turns through an angle $\theta(t)$ given by $\theta(t) = \gamma t^2 - \beta t^3$, where $\gamma = 3.20\ \mathrm{rad/s^2}$ and $\beta = 0.500\ \mathrm{rad/s^3}$. (a) Calculate the angular velocity of the roller as a function of time. (b) Calculate the angular acceleration of the roller as a function of time. (c) What is the maximum positive angular velocity, and at what value of t does it occur?

9.61 •• **CP CALC** A disk of radius 25.0 cm is free to turn about an axle perpendicular to it through its center. It has very thin but strong string wrapped around its rim, and the string is attached to a ball that is pulled tangentially away from the rim of the disk (Fig. P9.61). The pull increases in magnitude and produces an acceleration of the ball that obeys the equation $a(t) = At$, where t is in seconds and A is a constant. The cylinder starts from rest, and at the end of the third second, the ball's acceleration is 1.80 $\mathrm{m/s^2}$. (a) Find A. (b) Express the angular acceleration of the disk as a function of time. (c) How much time after the disk has begun to turn does it reach an angular speed of 15.0 rad/s? (d) Through what angle has the disk turned just as it reaches 15.0 rad/s? (**HINT:** See Section 2.6.)

Figure P9.61

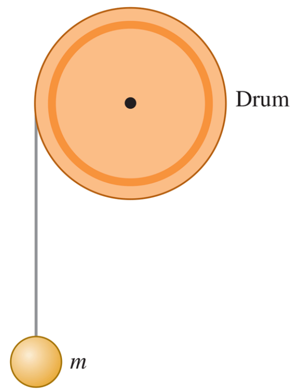


9.62 •• You are designing a rotating metal flywheel that will be used to store energy. The flywheel is to be a uniform disk with radius 25.0 cm. Starting from rest at $t = 0$, the flywheel rotates with constant angular acceleration 3.00 rad/s^2 about an axis perpendicular to the flywheel at its center. If the flywheel has a density (mass per unit volume) of 8600 kg/m^3, what thickness must it have to store 800 J of kinetic energy at $t = 8.00$ s?

9.63 •• A uniform wheel in the shape of a solid disk is mounted on a frictionless axle at its center. The wheel has mass 5.00 kg and radius 0.800 m. A thin rope is wrapped around the wheel, and a block is suspended from the free end of the rope. The system is released from rest and the block moves downward. What is the mass of the block if the wheel turns through 8.00 revolutions in the first 5.00 s after the block is released?

9.64 •• Engineers are designing a system by which a falling mass m imparts kinetic energy to a rotating uniform drum to which it is attached by thin, very light wire wrapped around the rim of the drum (Fig. P9.64). There is no appreciable friction in the axle of the drum, and everything starts from rest. This system is being tested on earth, but it is to be used on Mars, where the acceleration due to gravity is 3.71 m/s^2. In the earth tests, when m is set to 15.0 kg and allowed to fall through 5.00 m, it gives 250.0 J of kinetic energy to the drum. (a) If the system is operated on Mars, through what distance would the 15.0 kg mass have to fall to give the same amount of kinetic energy to the drum? (b) How fast would the 15.0 kg mass be moving on Mars just as the drum gained 250.0 J of kinetic energy?

Figure P9.64



9.65 •• Consider the system of two blocks shown in Fig. P9.77. There is no friction between block A and the tabletop. The mass of block B is 5.00 kg. The pulley rotates about a frictionless axle, and the light rope doesn't slip on the pulley surface. The pulley has radius 0.200 m and moment of inertia /s after the block has descended 1.20 m, what is the mass of block A?

9.66 •• The motor of a table saw is rotating at 3450 rev/min. A pulley attached to the motor shaft drives a second pulley of half the diameter by means of a V-belt. A circular saw blade of diameter 0.208 m is mounted on the same rotating shaft as the second pulley. (a) The operator is careless and the blade catches and throws back a small piece of wood. This piece of wood moves with linear speed equal to the tangential speed of the rim of the blade. What is this speed? (b) Calculate the radial acceleration of points on the outer edge of the blade to see why sawdust doesn't stick to its teeth.

9.67 ••• While riding a multispeed bicycle, the rider can select the radius of the rear sprocket that is fixed to the rear axle. The front sprocket of a bicycle has radius 12.0 cm. If the angular speed of the front sprocket is 0.600 rev/s, what is the radius of the rear sprocket for which the tangential speed of a point on the rim of the rear wheel will be 5.00 m/s? The rear wheel has radius 0.330 m.

9.68 ••• A computer disk drive is turned on starting from rest and has constant angular acceleration. If it took 0.0865 s for the drive to make its *second* complete revolution, (a) how long did it take to make the first complete revolution, and (b) what is its angular acceleration, in rad/s^2?

9.69 •• CP Consider the system shown in Fig. E9.49. The suspended block has mass 1.50 kg. The system is released from rest and the block descends as the wheel rotates on a frictionless axle. As the wheel is rotating, the tension in the light wire is 9.00 N. What is the kinetic energy of the wheel 2.00 s after the system is released?

9.70 •• A uniform disk has radius R_0 and mass M_0. Its moment of inertia for an axis perpendicular to the plane of the disk at the disk's center is $\frac{1}{2}M_0R^2{}_0$. You have been asked to halve the disk's moment of inertia by cutting out a circular piece at the center of the disk. In terms of R_0, what should be the radius of the circular piece that you remove?

9.71 •• **Measuring *I*.** As an intern at an engineering firm, you are asked to measure the moment of inertia of a large wheel for rotation about an axis perpendicular to the wheel at its center. You measure the diameter of the wheel to be 0.640 m. Then you mount the wheel on frictionless bearings on a horizontal frictionless axle at the center of the wheel. You wrap a light rope around the wheel and hang an 8.20 kg block of wood from the free end of the rope, as in Fig.

E9.49. You release the system from rest and find that the block descends 12.0 m in 4.00 s. What is the moment of inertia of the wheel for this axis?

9.72 ••• A uniform, solid disk with mass m and radius R is pivoted about a horizontal axis through its center. A small object of the same mass m is glued to the rim of the disk. If the disk is released from rest with the small object at the end of a horizontal radius, find the angular speed when the small object is directly below the axis.

9.73 •• **CP** A meter stick with a mass of 0.180 kg is pivoted about one end so it can rotate without friction about a horizontal axis. The meter stick is held in a horizontal position and released. As it swings through the vertical, calculate (a) the change in gravitational potential energy that has occurred; (b) the angular speed of the stick; (c) the linear speed of the end of the stick opposite the axis. (d) Compare the answer in part (c) to the speed of a particle that has fallen 1.00 m, starting from rest.

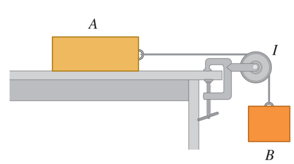
9.74 •• A physics student of mass 43.0 kg is standing at the edge of the flat roof of a building, 12.0 m above the sidewalk. An unfriendly dog is running across the roof toward her. Next to her is a large wheel mounted on a horizontal axle at its center. The wheel, used to lift objects from the ground to the roof, has a light crank attached to it and a light rope wrapped around it; the free end of the rope hangs over the edge of the roof. The student grabs the end of the rope and steps off the roof. If the wheel has radius 0.300 m and a moment of inertia of 9.60 $kg{\cdot}m^2$ for rotation about the axle, how long does it take her to reach the sidewalk, and how fast will she be moving just before she lands? Ignore friction in the axle.

9.75 ••• A slender rod is 80.0 cm long and has mass 0.120 kg. A small 0.0200 kg sphere is welded to one end of the rod, and a small 0.0500 kg sphere is welded to the other end. The rod, pivoting about a stationary, frictionless axis at its center, is held horizontal and released from rest. What is the linear speed of the 0.0500 kg sphere as it passes through its lowest point?

9.76 •• Exactly one turn of a flexible rope with mass m is wrapped around a uniform cylinder with mass M and radius R. The cylinder rotates without friction about a horizontal axle along the cylinder axis. One end of the rope is attached to the cylinder. The cylinder starts with angular speed ω_0. After one revolution of the cylinder the rope has unwrapped and, at this instant, hangs vertically down, tangent to the cylinder. Find the angular speed of the cylinder and the linear speed of the lower end of the rope at this time. Ignore the thickness of the rope. [**HINT:** Use Eq. (9.18).]

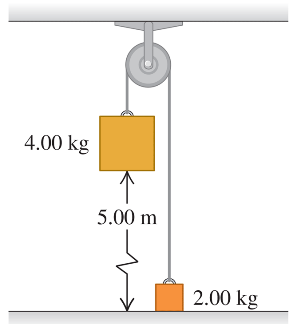
9.77 • The pulley in Fig. P9.77 has radius R and a moment of inertia I. The rope does not slip over the pulley, and the pulley spins on a frictionless axle. The coefficient of kinetic friction between block A and the tabletop is μ_k. The system is released from rest, and block B descends. Block A has mass m_A and block B has mass m_B. Use energy methods to calculate the speed of block B as a function of the distance d that it has descended.

Figure P9.77



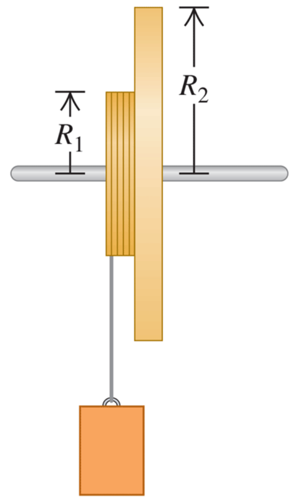
9.78 •• The pulley in Fig. P9.78 has radius 0.160 m and moment of inertia 0.380 kg·m^2. The rope does not slip on the pulley rim. Use energy methods to calculate the speed of the 4.00 kg block just before it strikes the floor.

Figure P9.78



9.79 •• Two metal disks, one with radius $R_1 = 2.50$ cm and mass $M_1 = 0.80$ kg and the other with radius $R_2 = 5.00$ cm and mass $M_2 = 1.60$ kg, are welded together and mounted on a frictionless axis through their common center (Fig. P9.79). (a) What is the total moment of inertia of the two disks? (b) A light string is wrapped around the edge of the smaller disk, and a 1.50 kg block is suspended from the free end of the string. If the block is released from rest at a distance of 2.00 m above the floor, what is its speed just before it strikes the floor? (c) Repeat part (b), this time with the string wrapped around the edge of the larger disk. In which case is the final speed of the block greater? Explain.

Figure P9.79

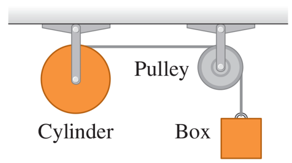


9.80 •• A thin, light wire is wrapped around the rim of a wheel as shown in Fig. E9.49. The wheel rotates about a stationary horizontal axle that passes through the center of the wheel. The wheel has radius 0.180 m and moment of inertia for rotation about the axle of $I = 0.480\ \text{kg}\cdot\text{m}^2$. A small block with mass 0.340 kg is suspended from the free end of the wire. When the system is released from rest, the block descends with constant acceleration. The bearings in the wheel at the axle are rusty, so friction there does -9.00 J of work as the block descends 3.00 m. What is the magnitude of the angular velocity of the wheel after the block has descended 3.00 m?

9.81 ••• In the system shown in Fig. 9.17, a 12.0 kg mass is released from rest and falls, causing the uniform 10.0 kg cylinder of diameter 30.0 cm to turn about a frictionless axle through its center. How far will the mass have to descend to give the cylinder 480 J of kinetic energy?

9.82 • In Fig. P9.82, the cylinder and pulley turn without friction about stationary horizontal axles that pass through their centers. A light rope is wrapped around the cylinder, passes over the pulley, and has a 3.00 kg box suspended from its free end. There is no slipping between the rope and the pulley surface. The uniform cylinder has mass 5.00 kg and radius 40.0 cm. The pulley is a uniform disk with mass 2.00 kg and radius 20.0 cm. The box is released from rest and descends as the rope unwraps from the cylinder. Find the speed of the box when it has fallen 2.50 m.

Figure P9.82



9.83 •• BIO **Human Rotational Energy.** A dancer is spinning at 72 rpm about an axis through her center with her arms outstretched (Fig. P9.83). From biomedical measurements, the typical distribution of mass in a human body is as follows:

Figure P9.83

Head: 7.0%

Arms: 13% (for both)

Trunk and legs: 80.0%

Suppose you are this dancer. Using this information plus length measurements on your own body, calculate (a) your moment of inertia about your spin axis and (b) your rotational kinetic energy. Use Table 9.2 to model reasonable approximations for the pertinent parts of your body.

9.84 ••• A thin, uniform rod is bent into a square of side length a. If the total mass is M, find the moment of inertia about an axis through the center and perpendicular to the plane of the square. (**HINT:** Use the parallel-axis theorem.)

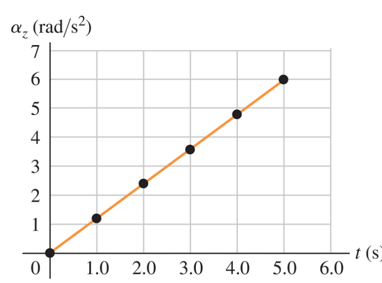
9.85 •• CALC A sphere with radius $R = 0.200$ m has density ρ that decreases with distance r from the center of the sphere according to $\rho = 3.00 \times 10^3\ \text{kg/m}^3 - (9.00 \times 10^3\ \text{kg/m}^4)r$. (a) Calculate the total mass of the sphere. (b) Calculate the moment of inertia of the sphere for an axis along a diameter.

9.86 •• CALC **Neutron Stars and Supernova Remnants.** The Crab Nebula is a cloud of glowing gas about 10 light-years across, located about 6500 light-years from the earth (Fig. P9.86). It is the remnant of a star that underwent a *supernova explosion,* seen on earth in 1054 A.D. Energy is released by the Crab Nebula at a rate of about 5×10^{31} W, about 10^5 times the rate at which the sun radiates energy. The Crab Nebula obtains its energy from the rotational kinetic energy of a rapidly spinning *neutron star* at its center. This object rotates once every 0.0331 s, and this period is increasing by 4.22×10^{-13} s for each second of time that elapses. (a) If the rate at which energy is lost by the neutron star is equal to the rate at which energy is released by the nebula, find the moment of inertia of the neutron star. (b) Theories of supernovae predict that the neutron star in the Crab Nebula has a mass about 1.4 times that of the sun. Modeling the neutron star as a solid uniform sphere, calculate its radius in kilometers. (c) What is the linear speed of a point on the equator of the neutron star? Compare to the speed of light. (d) Assume that the neutron star is uniform and calculate its density. Compare to the density of ordinary rock ($3000\ \mathrm{kg/m^3}$) and to the density of an atomic nucleus (about $10^{17}\ \mathrm{kg/m^3}$). Justify the statement that a neutron star is essentially a large atomic nucleus.

Figure P9.86

9.87 •• **DATA** A technician is testing a computer-controlled, variable-speed motor. She attaches a thin disk to the motor shaft, with the shaft at the center of the disk. The disk starts from rest, and sensors attached to the motor shaft measure the angular acceleration /s, of the disk (b) at $t = 5.0$ s; (c) when it has turned through 2.00 rev?

Figure P9.87



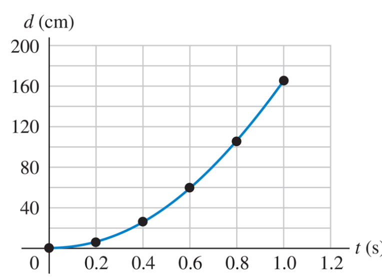
9.88 •• **DATA** You are analyzing the motion of a large flywheel that has radius 0.800 m. In one test run, the wheel starts from rest and turns in a horizontal plane with constant angular acceleration. An accelerometer on the rim of the flywheel measures the magnitude of the resultant acceleration a of a point on the rim of the flywheel as a function of the angle $\theta - \theta_0$ through which the wheel has turned. You collect these results:

$\theta - \theta_0$ (rad)	0.50	1.00	1.50	2.00	2.50	3.00	3.50	4.00
a (m/s^2)	0.678	1.07	1.52	1.98	2.45	2.92	3.39	3.87

Construct a graph of a^2 (in m^2/s^4) versus $(\theta - \theta_0)^2$ (in rad^2). (a) What are the slope and y-intercept of the straight line that gives the best fit to the data? (b) Use the slope from part (a) to find the angular acceleration of the flywheel. (c) What is the linear speed of a point on the rim of the flywheel when the wheel has turned through an angle of 135°? (d) When the flywheel has turned through an angle of 90.0°, what is the angle between the linear velocity of a point on its rim and the resultant acceleration of that point?

9.89 •• **DATA** You are rebuilding a 1965 Chevrolet. To decide whether to replace the flywheel with a newer, lighter-weight one, you want to determine the moment of inertia of the original, 35.6-cm-diameter flywheel. It is not a uniform disk, so you can't use $I = \frac{1}{2}MR^2$ to calculate the moment of inertia. You remove the flywheel from the car and use low-friction bearings to mount it on a horizontal, stationary rod that passes through the center of the flywheel, which can then rotate freely (about 2 m above the ground). After gluing one end of a long piece of flexible fishing line to the rim of the flywheel, you wrap the line a number of turns around the rim and suspend a 5.60 kg metal block from the free end of the line. When you release the block from rest, it descends as the flywheel rotates. With high-speed photography you measure the distance d the block has moved downward as a function of the time since it was released. The equation for the graph shown in Fig. P9.89 that gives a good fit to the data points is $d = (165\ \mathrm{cm/s^2})t^2$. (a) Based on the graph, does the block fall with constant acceleration? Explain. (b) Use the graph to calculate the speed of the block when it has descended 1.50 m. (c) Apply conservation of mechanical energy to the system of flywheel and block to calculate the moment of inertia of the flywheel. (d) You are relieved that the fishing line doesn't break. Apply Newton's second law to the block to find the tension in the line as the block descended.

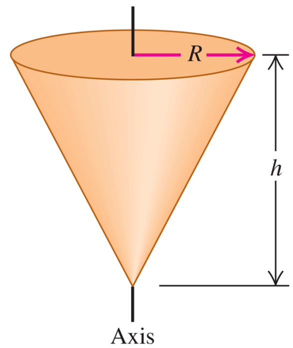
Figure P9.89



Challenge Problems

9.90 ••• **CALC** Calculate the moment of inertia of a uniform solid cone about an axis through its center (Fig. P9.90). The cone has mass M and altitude h. The radius of its circular base is R.

Figure P9.90



9.91 ••• **CALC** On a compact disc (CD), music is coded in a pattern of tiny pits arranged in a track that spirals outward toward the rim of the disc. As the disc spins inside a CD player, the track is scanned at a constant *linear* speed of $v = 1.25$ m/s. Because the radius of the track varies as it spirals outward, the *angular* speed of the disc must change as the CD is played. (See Exercise 9.20.) Let's see what angular acceleration is required to keep v constant. The equation of a spiral is $r(\theta) = r_0 + \beta\theta$, where r_0 is the radius of the spiral at $\theta = 0$ and β is a constant. On a CD, r_0 is the inner radius of the spiral track. If we take the rotation direction of the CD to be positive, β must be positive so that r increases as the disc turns and θ increases. (a) When the disc rotates through a small angle $d\theta$, the distance scanned along the track is $ds = r\,d\theta$. Using the above expression for $r(\theta)$, integrate ds to find the total distance s scanned along the track as a function of the total angle θ through which the disc has rotated. (b) Since the track is scanned at a constant linear speed v, the distance s found in part (a) is equal to vt. Use this to find θ as a function of time. There will be two solutions for θ; choose the positive one, and explain why this is the solution to choose. (c) Use your expression for $\theta(t)$ to find the angular velocity ω_z and the angular acceleration α_z as functions of time. Is α_z constant? (d) On a CD, the inner radius of the track is 25.0 mm, the track radius increases by 1.55 μm per revolution, and the playing time is 74.0 min. Find r_0, β, and the total number of revolutions made during the playing time. (e) Using your results from parts (c) and (d), make graphs of ω_z (in rad/s) versus t and α_z (in rad/s^2) versus t between $t = 0$ and $t = 74.0$ min.

MCAT-Style Passage Problems

BIO The Spinning Eel. American eels (*Anguilla rostrata*) are freshwater fish with long, slender bodies that we can treat as uniform cylinders 1.0 m long and 10 cm in diameter. An eel compensates for its small jaw and teeth by holding onto prey with its mouth and then rapidly spinning its body around its long axis to tear off a piece of flesh. Eels have been recorded to spin at up to 14 revolutions per second when feeding in this way. Although this feeding method is costly in terms of energy, it allows the eel to feed on larger prey than it otherwise could.

9.92 A field researcher uses the slow-motion feature on her phone's camera to shoot a video of an eel spinning at its maximum rate. The camera records at 120 frames per second. Through what angle does the eel rotate from one frame to the next? (a) 1°; (b) 10°; (c) 22°; (d) 42°.

9.93 The eel is observed to spin at 14 spins per second clockwise, and 10 seconds later it is observed to spin at 8 spins per second counterclockwise. What is the magnitude of the eel's average angular acceleration during this time? (a) $6/10 \text{ rad/s}^2$; (b) $6\pi/10 \text{ rad/s}^2$; (c) $12\pi/10 \text{ rad/s}^2$; (d) $44\pi/10 \text{ rad/s}^2$.

9.94 The eel has a certain amount of rotational kinetic energy when spinning at 14 spins per second. If it swam in a straight line instead, about how fast would the eel have to swim to have the same amount of kinetic energy as when it is spinning? (a) 0.5 m/s; (b) 0.7 m/s; (c) 3 m/s; (d) 5 m/s.

9.95 A new species of eel is found to have the same mass but one-quarter the length and twice the diameter of the American eel. How does its moment of inertia for spinning around its long axis compare to that of the American eel? The new species has (a) half the moment of inertia as the American eel; (b) the same moment of inertia as the American eel; (c) twice the moment of inertia as the American eel; (d) four times the moment of inertia as the American eel.

Answers: Rotation of Rigid Bodies

Chapter Opening Question ?

(ii) The rotational kinetic energy of a rigid body rotating around an axis is $K = \frac{1}{2}I\omega^2$, where I is the body's moment of inertia for that axis and ω is the rotational speed. Table 9.2 shows that the moment of inertia for a slender rod of mass M and length L with an axis through one end (like a wind turbine blade) is $I = \frac{1}{3}ML^2$. If we double L while M and ω stay the same, both the moment of inertia I and the kinetic energy K increase by a factor of $2^2 = 4$.

Test Your Understanding

9.1 **(a) (i) and (iii) (b) (ii)** The rotation is speeding up when the angular velocity and angular acceleration have the same sign, and slowing down when they have opposite signs. Hence it is speeding up for $0 < t < 2$ s (both ω_z and α_z are positive) and for $4\text{ s} < t < 6$ s (both ω_z and α_z are negative) but is slowing down for $2\text{ s} < t < 4$ s (ω_z is positive and α_z is negative). Note that the body is rotating in one direction for $t < 4$ s (ω_z is positive) and in the opposite direction for $t > 4$ s (ω_z is negative).

9.2 **(a) (i) (ii)** When the disc comes to rest, $\omega_z = 0$. From Eq. (9.7), the *time* when this occurs is $t = (\omega_z - \omega_{0z})/\alpha_z = -\omega_{0z}/\alpha_z$ (this is a positive time because α_z is negative). If we double the initial angular velocity ω_{0z} and also double the angular acceleration α_z, their ratio is unchanged and the rotation stops in the same amount of time. The *angle* through which the disc rotates is given by Eq. (9.10): $\theta - \theta_0 = \frac{1}{2}(\omega_{0z} + \omega_z)t = \frac{1}{2}\omega_{0z}t$ (since the final angular velocity is $\omega_z = 0$). The initial angular velocity ω_{0z} has been doubled but the time t is the same, so the angular displacement $\theta - \theta_0$ (and hence the number of revolutions) has doubled. You can also come to the same conclusion by using Eq. (9.12).

9.3 **(ii)** From Eq. (9.13), $v = r\omega$. To maintain a constant linear speed v, the angular speed ω must decrease as the scanning head moves outward (greater r).

9.4 **(i)** The kinetic energy in the falling block is $\frac{1}{2}mv^2$, and the kinetic energy in the rotating cylinder is $\frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{2}\,mR^2\right)(v/R)^2 = \frac{1}{4}mv^2$. Hence the total kinetic energy of the system is $\frac{3}{4}mv^2$, of which two-thirds is in the block and one-third is in the cylinder.

9.5 **(ii)** More of the mass of the pool cue is concentrated at the thicker end, so the center of mass is closer to that end. The moment of inertia through a point P at either end is $I_P = I_{\text{cm}} + Md^2$; the thinner end is farther from the center of mass, so the distance d and the moment of inertia I_P are greater for the thinner end.

9.6 **(iii)** Our result from Example 9.10 does *not* depend on the cylinder length L. The moment of inertia depends on only the *radial* distribution of mass, not on its distribution along the axis.

Key Example Variation Problems

VP9.3.1 **a.** 2.06 rad

b. 118°

c. 0.328 rev

VP9.3.2 **a.** 2.75 rad/s^2

b. 27.0 rad

VP9.3.3 0.693 s

VP9.3.4 $t = \left(-\omega_{0z} + \sqrt{\omega^2{}_{0z} + 2\alpha_z\theta}\right)/\alpha_z$

VP9.5.1 **A.** $1.22\,\text{m/s}^2$

B. $0.389\,\text{m/s}^2$

C. $1.28\,\text{m/s}^2$

VP9.5.2 **a.** 9.22 m/s

b. $54.0\,\text{m/s}^2$, toward the axis of rotation

VP9.5.3 $\left(\frac{a^2{}_{\text{max}}}{r^2} - \alpha^2\right)^{1/4}$

VP9.5.4 **a.** $\sqrt{3}\omega^2$

b. 30°

VP9.8.1 **a.** 20.0 rad/s

b. 23.3 rad/s

VP9.8.2 **a.** 1.80 J

b. 5.4 J

VP9.8.3 **a.** 2.86 m

b. 20.0 rad/s

VP9.8.4 **a.** 2.40 m/s

b. −9.72 J

Bridging Problem

(a) $$I = \left[\frac{M}{L}\left(\frac{x^3}{3}\right)\right]_{-h}^{L-h} = \frac{1}{3}M\left(L^2 - 3Lh + 3h^2\right)$$

(b) $W = \frac{1}{6}M\left(L^2 - 3Lh + 3h^2\right)\alpha^2 t^2$

(c) $a = (L - h)\alpha\sqrt{1 + \alpha^2 t^4}$

Chapter 10
Dynamics of Rotational Motion

? These jugglers toss the pins so that they rotate in midair. Each pin is of uniform composition, so its weight is concentrated toward its thick end. If we ignore air resistance but not the effects of gravity, will the angular speed of a pin in flight (i) increase continuously; (ii) decrease continuously; (iii) alternately increase and decrease; or (iv) remain the same?

Learning Outcomes

In this chapter, you'll learn...

10.1 What is meant by the torque produced by a force.

10.2 How the net torque on a rigid body affects the body's rotational motion.

10.3 How to analyze the motion of a rigid body that both rotates and moves as a whole through space.

10.4 How to solve problems that involve work and power for rotating rigid bodies.

10.5 What is meant by the angular momentum of a particle or rigid body.

10.6 How the angular momentum of an object can remain constant even if the object changes shape.

10.7 Why a spinning gyroscope undergoes precession.

You'll need to review...

1.10 Vector product of two vectors.

5.2 Newton's second law.

6.1, 6.4 Work, the work–energy theorem, and power.

8.2, 8.3, 8.5 External versus internal forces, inelastic collisions, and center-of-mass motion.

9.1, 9.2, 9.3, 9.4, 9.5 Rotational motion and the parallel-axis theorem.

We learned in Chapters 4 and 5 that a net force applied to an object gives that object an acceleration. But what does it take to give an object an *angular* acceleration? That is, what does it take to start a stationary object rotating or to bring a spinning object to a halt? A force is required, but it must be applied in a way that gives a twisting or turning action.

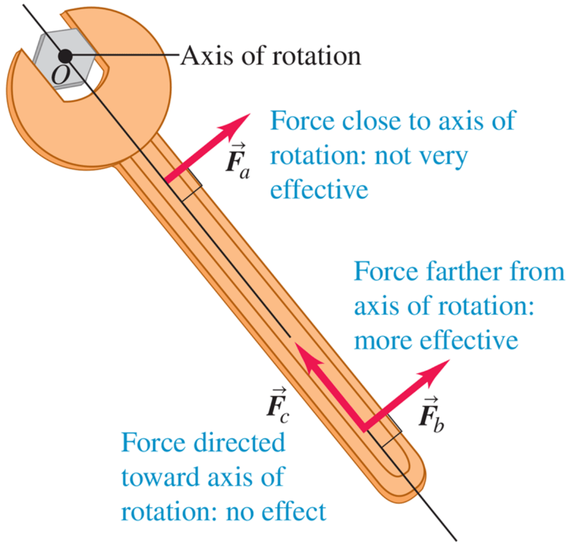
In this chapter we'll define a new physical quantity, *torque,* that describes the twisting or turning effort of a force. We'll find that the net torque acting on a rigid body determines its angular acceleration, in the same way that the net force on an object determines its linear acceleration. We'll also look at work and power in rotational motion so as to understand, for example, how energy is transferred by an electric motor. Next we'll develop a new conservation principle, *conservation of angular momentum,* that is tremendously useful for understanding the rotational motion of both rigid and nonrigid bodies. We'll finish this chapter by studying *gyroscopes,* rotating devices that don't fall over when you might think they should—but that actually behave in accordance with the dynamics of rotational motion.

10.1 Torque

Video Tutor Demo: Walking the Plank

We know that forces acting on an object can affect its **translational motion**—that is, the motion of the object as a whole through space. Now we want to learn which aspects of a force determine how effective it is in causing or changing *rotational* motion. The magnitude and direction of the force are important, but so is the point on the object where the force is applied. In Fig. 10.1 a wrench is being used to loosen a tight bolt. Force $\vec{F}_b$, applied near the end of the handle, is more effective than an equal force $\vec{F}_a$ applied near the bolt. Force $\vec{F}_c$ does no good; it's applied at the same point and has the same magnitude as $\vec{F}_b$, but it's directed along the length of the handle. The quantitative measure of the tendency of a force to cause or change an object's rotational motion is called *torque;* we say that $\vec{F}_a$ applies a torque about point O to the wrench in Fig. 10.1, $\vec{F}_b$ applies a greater torque about O, and $\vec{F}_c$ applies zero torque about O.

Figure 10.1



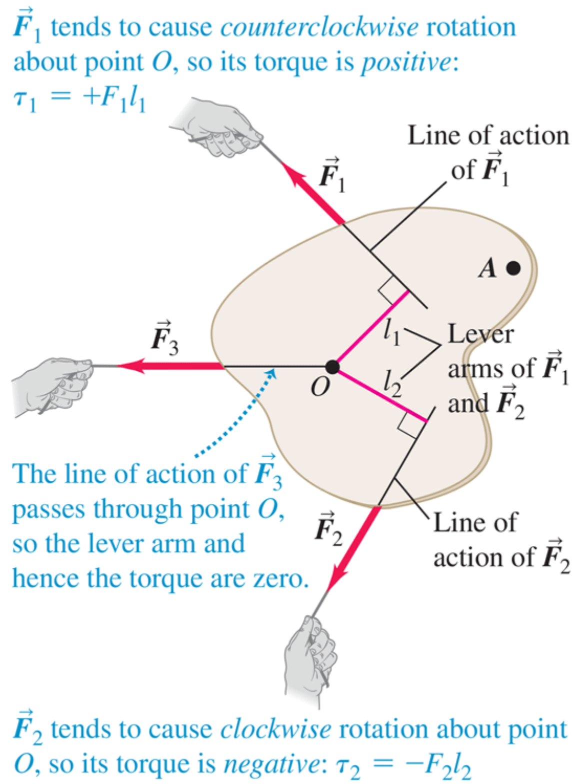
Which of these three equal- magnitude forces is most likely to loosen the tight bolt?

Figure 10.2 shows three examples of how to calculate torque. The object can rotate about an axis that is perpendicular to the plane of the figure and passes through point O. Three forces act on the object in the plane of the figure. The tendency of the first of these forces, $\vec{\boldsymbol{F}}_1$, to cause a rotation about O depends on its magnitude F_1. It also depends on the *perpendicular* distance l_1 between point O and the **line of action** of the force (that is, the line along which the force vector lies). We call the distance l_1 the **lever arm** (or **moment arm**) of force $\vec{\boldsymbol{F}}_1$ about O. The twisting effort is directly proportional to both F_1 and l_1, so we define the **torque** (or *moment*) of the force $\vec{\boldsymbol{F}}_1$ with respect to O as the product $F_1 l_1$. We use the Greek letter τ (tau) for torque. If a force of magnitude F has a line of action that is a perpendicular distance l from O, the torque is

(10.1)

$$\tau = Fl$$

Figure 10.2



The torque of a force about a point is the product of the force magnitude and the lever arm of the force.

Physicists usually use the term "torque," while engineers usually use "moment" (unless they are talking about a rotating shaft).

The lever arm of $\vec{F}_1$ in Fig. 10.2 is the perpendicular distance l_1, and the lever arm of $\vec{F}_2$ is the perpendicular distance l_2. The line of action of $\vec{F}_3$ passes through point O, so the lever arm for $\vec{F}_3$ is zero and its torque with respect to O is zero. In the same way, force $\vec{F}_c$ in Fig. 10.1 has zero torque with respect to point O; $\vec{F}_b$ has a greater torque than $\vec{F}_a$ because its lever arm is greater.

CAUTION Torque is always measured about a point Torque is *always* defined with reference to a specific point. If we shift the position of this point, the torque of each force may change. For example, the torque of force $\vec{F}_3$ in Fig. 10.2 is zero with respect to point O but *not* with respect to point A. It's not enough to refer to "the torque of $\vec{\boldsymbol{F}}$"; you must say "the torque of $\vec{\boldsymbol{F}}$ with respect to point X" or "the torque of $\vec{\boldsymbol{F}}$ about point X."

Force $\vec{\boldsymbol{F}}_1$ in Fig. 10.2 tends to cause *counterclockwise* rotation about O, while $\vec{\boldsymbol{F}}_2$ tends to cause *clockwise* rotation. To distinguish between these two possibilities, we need to choose a positive sense of rotation. With the choice that *counterclockwise torques are positive and clockwise torques are negative*, the torques of $\vec{\boldsymbol{F}}_1$ and $\vec{\boldsymbol{F}}_2$ about O are

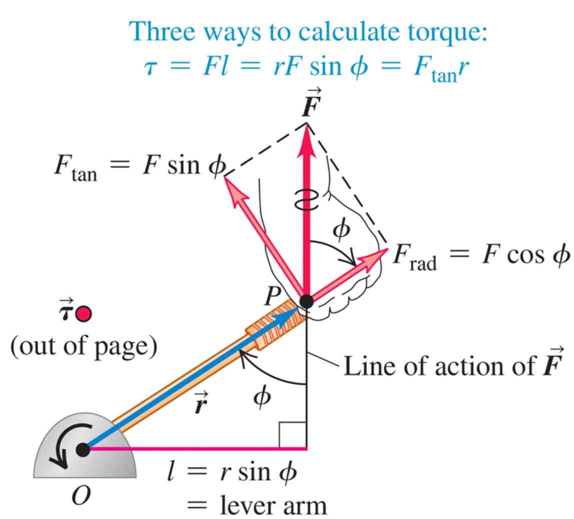
$$\tau_1 = +F_1 l_1 \qquad \tau_2 = -F_2 l_2$$

Figure 10.2 shows this choice for the sign of torque. We'll often use the symbol to indicate our choice of the positive sense of rotation.

The SI unit of torque is the newton-meter. In our discussion of work and energy we called this combination the joule. But torque is *not* work or energy, and torque should be expressed in newton-meters, *not* joules.

Figure 10.3 shows a force $\vec{\boldsymbol{F}}$ applied at point P, located at position $\vec{\boldsymbol{r}}$ with respect to point O. There are three ways to calculate the torque of $\vec{\boldsymbol{F}}$:

Figure 10.3



Three ways to calculate the torque of force $\vec{\boldsymbol{F}}$ about point O. In this figure, $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{F}}$ are in the plane of the page and the torque vector $\vec{\boldsymbol{\tau}}$ points out of the page toward you.

---

1. Find the lever arm l and use $\tau = Fl$.
2. Determine the angle ϕ between the vectors $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{F}}$; the lever arm is $r \ \sin\phi$, so $\tau = rF \sin\phi$.
3. Represent $\vec{\boldsymbol{F}}$ in terms of a radial component F_{rad} along the direction of $\vec{\boldsymbol{r}}$ and a tangential component $F_{\tan}$ at right angles, perpendicular to $\vec{\boldsymbol{r}}$. (We call this component *tangential* because if the object rotates, the point where the force acts moves in a circle, and this component is tangent to that circle.) Then $F_{\tan} = F \sin\phi$ and $\tau = r(F \sin\phi) = F_{\tan} r$. The component F_{rad} produces *no* torque with respect to O because its lever arm with respect to that point is zero (compare to forces $\vec{\boldsymbol{F}}_c$ in Fig. 10.1 and $\vec{\boldsymbol{F}}_3$ in Fig. 10.2).

Summarizing these three expressions for torque, we have

(10.2)

Magnitude of $\vec{r}$ (vector from O to where $\vec{F}$ acts)

Lever arm of $\vec{F}$

Magnitude of torque due to force $\vec{F}$ relative to point O

$$\tau = Fl = rF\sin\phi = F_{\tan}r \qquad (10.2)$$

Magnitude of $\vec{F}$

Angle between $\vec{r}$ and $\vec{F}$

Tangential component of $\vec{F}$

Torque as a Vector

We saw in Section 9.1 that angular velocity and angular acceleration can be represented as vectors; the same is true for torque. To see how to do this, note that the quantity $rF \sin \phi$ in Eq. (10.2) is the magnitude of the *vector product* $\vec{r} \times \vec{F}$ that we defined in Section 1.10. (Go back and review that definition.) We generalize the definition of torque as follows: When a force $\vec{F}$ acts at a point having a position vector $\vec{r}$ with respect to an origin O, as in Fig. 10.3, the torque $\vec{\tau}$ of the force with respect to O is the *vector* quantity

(10.3)

Torque vector due to force $\vec{F}$ relative to point O

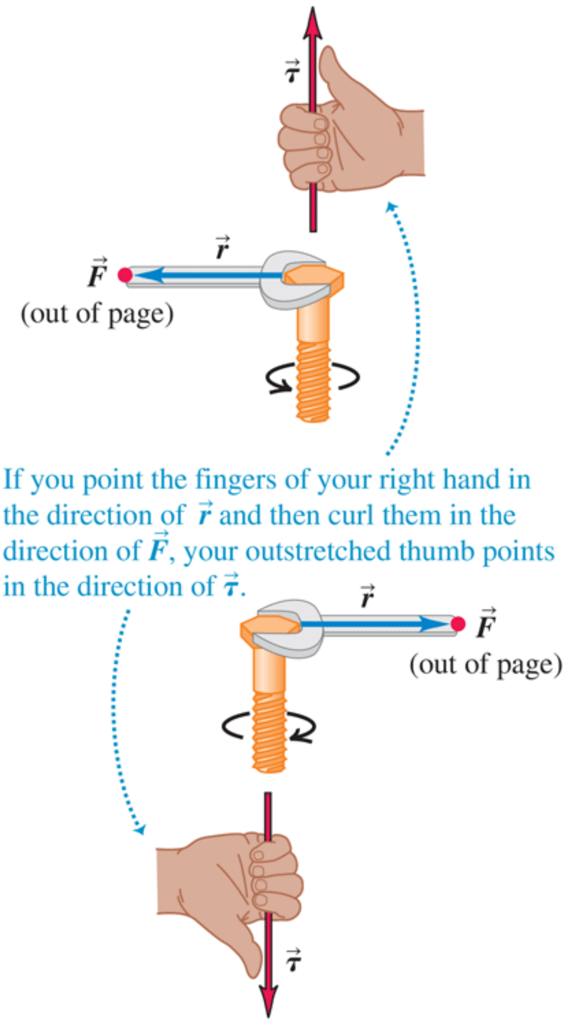
Vector from O to where $\vec{F}$ acts

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{10.3}$$

Force $\vec{F}$

The torque as defined in Eq. (10.2) is the magnitude of the torque vector $\vec{r} \times \vec{F}$. The direction of $\vec{\tau}$ is perpendicular to both $\vec{r}$ and $\vec{F}$. In particular, if both $\vec{r}$ and $\vec{F}$ lie in a plane perpendicular to the axis of rotation, as in Fig. 10.3, then the torque vector $\vec{\tau} = \vec{r} \times \vec{F}$ is directed along the axis of rotation, with a sense given by the right-hand rule (see Fig. 1.30 and Fig. 10.4).

Figure 10.4



The torque vector $\vec{\tau} = \vec{r} \times \vec{F}$ is directed along the axis of the bolt, perpendicular to both $\vec{r}$ and $\vec{F}$. The fingers of the right hand curl in the direction of the rotation that the torque tends to cause.

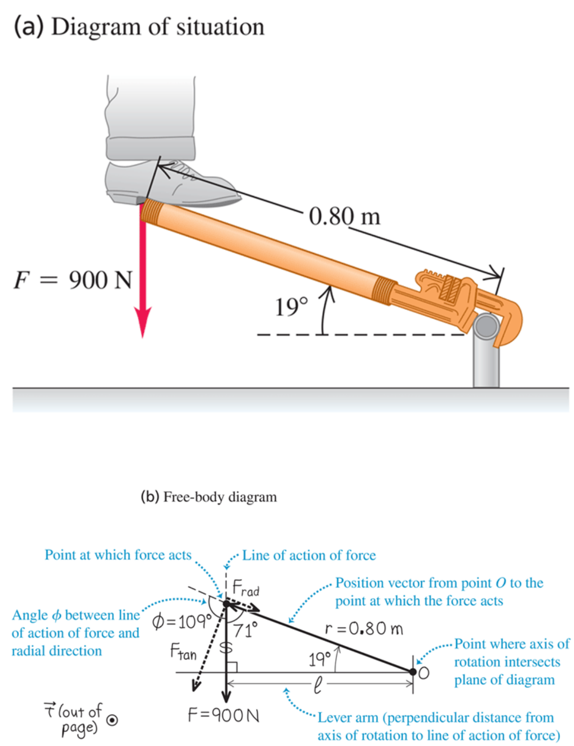
Because $\vec{\tau} = \vec{r} \times \vec{F}$ is perpendicular to the plane of the vectors $\vec{r}$ and $\vec{F}$, it's common to have diagrams like Fig. 10.4, in which one of the vectors is perpendicular to the page. We use a dot $(\bullet)$ to represent a vector that points out of the page and a cross $(\times)$ to represent a vector that points into the page (see Figs. 10.3 and 10.4).

In the following sections we'll usually be concerned with rotation of an object about an axis oriented in a specified constant direction. In that case, only the component of torque along that axis will matter. We often call that component the torque with respect to the specified *axis*.

Example 10.1 Applying a torque

To loosen a pipe fitting, a plumber slips a piece of scrap pipe (a "cheater") over his wrench handle. He stands on the end of the cheater, applying his 900 N weight at a point 0.80 m from the center of the fitting (Fig. 10.5a). The wrench handle and cheater make an angle of 19° with the horizontal. Find the magnitude and direction of the torque he applies about the center of the fitting.

Figure 10.5



(a) Loosening a pipe fitting by standing on a "cheater." (b) Our vector diagram to find the torque about *O*.

IDENTIFY and SET UP Figure 10.5b shows the vectors $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{F}}$ and the angle between them ($\phi = 109^\circ$). Equation (10.1) or (10.2) will tell us the magnitude of the torque. The right-hand rule with Eq. (10.3), $\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}}$, will tell us the direction of the torque.

EXECUTE To use Eq. (10.1), we first calculate the lever arm l. As Fig. 10.5b shows,

$$l = r \sin \phi = (0.80 \text{ m}) \sin 109^\circ = 0.76 \text{ m}$$

Then Eq. (10.1) tells us that the magnitude of the torque is

$$\tau = Fl = (900 \text{ N})(0.76 \text{ m}) = 680 \text{ N} \cdot \text{m}$$

We get the same result from Eq. (10.2):

$$\tau = rF \sin \phi = (0.80 \text{ m})(900 \text{ N})(\sin 109^\circ) = 680 \text{ N} \cdot \text{m}$$

Alternatively, we can find $F_{\tan}$, the tangential component of $\vec{\boldsymbol{F}}$ that acts perpendicular to $\vec{\boldsymbol{r}}$. Figure 10.5b shows that this component is at an angle of $109^\circ - 90^\circ = 19^\circ$ from $\vec{\boldsymbol{F}}$, so $F_{\tan} = F(\cos 19^\circ) = (900 \text{ N})(\cos 19^\circ) = 851 \text{ N}$. Then, from Eq. (10.2),

$$\tau = F_{\tan} r = (851 \text{ N})(0.80 \text{ m}) = 680 \text{ N} \cdot \text{m}$$

Curl the fingers of your right hand from the direction of $\vec{\boldsymbol{r}}$ (in the plane of Fig. 10.5b, to the left and up) into the direction of $\vec{\boldsymbol{F}}$ (straight down). Then your right thumb points out of the plane of the figure: This is the direction of $\vec{\boldsymbol{\tau}}$.

EVALUATE To check the direction of $\vec{\boldsymbol{\tau}}$, note that the force in Fig. 10.5 tends to produce a counterclockwise rotation about O. If you curl the fingers of your right hand in a counterclockwise direction, the thumb points out of the plane of Fig. 10.5, which is indeed the direction of the torque.

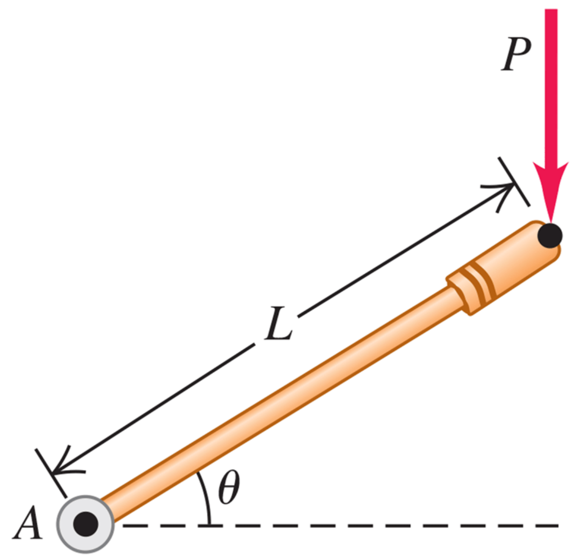
KEY CONCEPT

You can determine the magnitude of the torque due to a force $\vec{F}$ in any of three ways: (i) from the magnitude of $\vec{F}$ and the lever arm; (ii) from the magnitude of $\vec{F}$, the magnitude of the vector $\vec{r}$ from the origin to where $\vec{F}$ acts, and the angle between $\vec{r}$ and $\vec{F}$; or (iii) from the magnitude of $\vec{r}$ and the tangential component of $\vec{F}$. Find the direction of the torque using the right-hand rule.

Video Tutor Solution: **Example 10.1**

Test Your Understanding of Section 10.1

The accompanying figure shows a force of magnitude P being applied to one end of a lever of length L. What is the magnitude of the torque of this force about point A? (i) $PL \sin\theta$; (ii) $PL \cos\theta$; (iii) $PL \tan\theta$; (iv) $PL/\sin\theta$; (v) PL.



10.2 Torque and Angular Acceleration for a Rigid Body

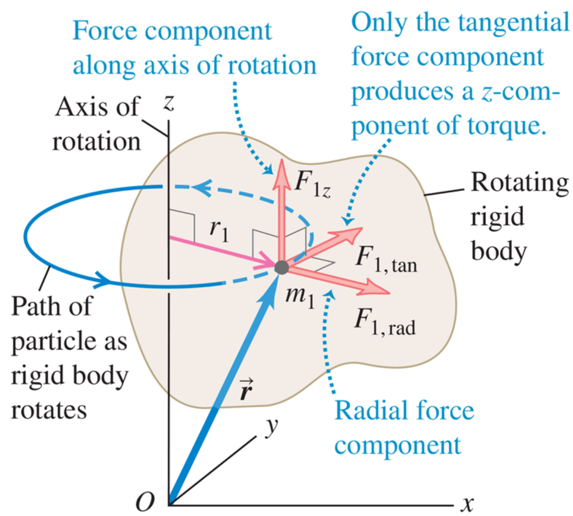
We're now ready to develop the fundamental relationship for the rotational dynamics of a rigid body (an object with a definite and unchanging shape and size). We'll show that the angular acceleration of a rotating rigid body is directly proportional to the sum of the torque components along the axis of rotation. The proportionality factor is the moment of inertia.

To develop this relationship, let's begin as we did in Section 9.4 by envisioning the rigid body as being made up of a large number of particles. We choose the axis of rotation to be the z-axis; the first particle has mass m_1 and distance r_1 from this axis (Fig. 10.6). The *net force* $\vec{\boldsymbol{F}}_1$ acting on this particle has a component $F_{1,\,\text{rad}}$ along the radial direction, a component $F_{1,\,\text{tan}}$ that is tangent to the circle of radius r_1 in which the particle moves as the body rotates, and a component F_{1z} along the axis of rotation. Newton's second law for the tangential component is

(10.4)

$$F_{1,\,\text{tan}} = m_1\, a_{1,\,\text{tan}}$$

Figure 10.6



As a rigid body rotates around the z-axis, a net force $\vec{\boldsymbol{F}}_1$ acts on one particle of the body. Only the force component $F_{1,\tan}$ can affect the rotation, because only $F_{1,\tan}$ exerts a torque about O with a z-component (along the rotation axis).

We can express the tangential acceleration of the first particle in terms of the angular acceleration α_z of the body by using Eq. (9.14): $a_{1,\tan} = r_1 \alpha_z$. Using this relationship and multiplying both sides of Eq. (10.4) by r_1, we obtain

(10.5)

$$F_{1,\tan} r_1 = m_1 r_1^2 \alpha_z$$

From Eq. (10.2), $F_{1,\tan} r_1$ is the *torque* of the net force with respect to the rotation axis, equal to the component τ_{1z} of the torque vector along the rotation axis. The subscript z is a reminder that the torque affects rotation around the z-axis, in the same way that the subscript on F_{1z} is a reminder that this force affects the motion of particle 1 along the z-axis.

Neither of the components $F_{1,\text{rad}}$ or F_{1z} contributes to the torque about the z-axis, since neither tends to change the particle's rotation about that axis. So $\tau_{1z} = F_{1,\tan} r_1$ is the total torque acting on the particle with respect to the rotation axis. Also, $m_1 r_1^2$ is I_1, the moment of inertia of the particle about the rotation axis. Hence we can rewrite Eq. (10.5) as

$$\tau_{1z} = I_1 \alpha_z = m_1 r_1^2 \alpha_z$$

We write such an equation for every particle in the body, then add all these equations:

$$\tau_{1z} + \tau_{2z} + \cdots = I_1 \alpha_z + I_2 \alpha_z + \cdots = m_1 r_1^2 \alpha_z + m_2 r_2^2 \alpha_z + \cdots$$

or

(10.6)

$$\sum \tau_{iz} = \left(\sum m_i r_i^2\right) \alpha_z$$

The left side of Eq. (10.6) is the sum of all the torques about the rotation axis that act on all the particles. The right side is $I = \sum m_i r_i^2$, the total moment of inertia about the rotation axis, multiplied by the angular acceleration α_z. Note that α_z is the same for every particle because this is a *rigid* body. Thus Eq. (10.6) says that for the rigid body as a whole,

(10.7)

Rotational analog of Newton's second law for a rigid body:

Net torque on a rigid body about z-axis

$$\sum \tau_z = I\alpha_z \qquad (10.7)$$

Moment of inertia of rigid body about z-axis

Angular acceleration of rigid body about z-axis

Just as Newton's second law says that a net *force* on a particle causes an *acceleration* in the direction of the net force, Eq. (10.7) says that a net *torque* on a rigid body about an axis causes an *angular acceleration* about that axis (Fig. 10.7).

Figure 10.7

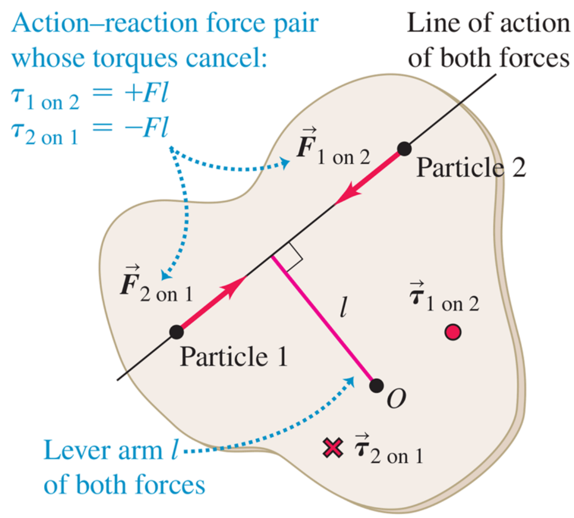
Loosening or tightening a screw requires giving it an angular acceleration and hence applying a torque. To make this easier, use a screwdriver with a large-radius handle. This provides a large lever arm for the force your hand applies.

Our derivation assumed that the angular acceleration α_z is the same for all particles in the body. So Eq. (10.7) is valid *only* for *rigid* bodies. Hence this equation doesn't apply to a rotating tank of water or a swirling tornado of air, different parts of which have different angular accelerations. Note that since our derivation used Eq. (9.14), $a_{\tan} = r\alpha_z$, α_z must be measured in $\mathrm{rad/s^2}$.

The torque on each particle is due to the net force on that particle, which is the vector sum of external and internal forces (see Section 8.2). According to Newton's third law, the *internal* forces that any pair of particles in the rigid body exert on each other are equal in magnitude and opposite in direction (Fig. 10.8). If these forces act along the line joining the two particles, their lever arms with respect to any axis are also equal. So the torques for each such pair are equal and opposite, and add to zero.

Hence *all* the internal torques add to zero, so the sum $\sum \tau_z$ in Eq. (10.7) includes only the torques of the *external* forces.

Figure 10.8



Why only *external* torques affect a rigid body's rotation: Any two particles in the body exert equal and opposite forces on each other. If the forces act along the line joining the particles, the lever arms of the forces with respect to an axis through O are the same and the torques due to the two forces are equal and opposite.

Often, an important external force acting on a rigid body is its *weight*. This force is not concentrated at a single point; it acts on every particle in the entire body. Nevertheless, if $\vec{g}$ has the same value at all points, we always get the correct torque (about any specified axis) if we assume that all the weight is concentrated at the *center of mass* of the body. We'll prove this statement in Chapter 11, but meanwhile we'll use it for some of the problems in this chapter.

Problem-Solving Strategy 10.1 Rotational Dynamics for Rigid Bodies

Our strategy for solving problems in rotational dynamics is very similar to Problem-Solving Strategy 5.2 for solving problems involving Newton's second law.

IDENTIFY *the relevant concepts:* Equation (10.7), $\sum \tau_z = I\alpha_z$, is useful whenever torques act on a rigid body. Sometimes you can use an energy approach instead, as we did in Section 9.4. However, if the target variable is a force, a torque, an acceleration, an angular acceleration, or an elapsed time, using $\sum \tau_z = I\alpha_z$ is almost always best.

SET UP *the problem* using the following steps:

1. Sketch the situation and identify the body or bodies to be analyzed. Indicate the rotation axis.
2. For each body, draw a free-body diagram that shows the body's *shape,* including all dimensions and angles. Label pertinent quantities with algebraic symbols.
3. Choose coordinate axes for each body and indicate a positive sense of rotation (clockwise or counterclockwise) for each rotating body. If you know the sense of α_z, pick that as the positive sense of rotation.

EXECUTE *the solution:*

1. For each body, decide whether it undergoes translational motion, rotational motion, or both. Then apply $\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ (as in Section 5.2), $\sum \tau_z = I\alpha_z$, or both to the body.
2. Express in algebraic form any *geometrical* relationships between the motions of two or more bodies. An example is a string that unwinds, without slipping, from a pulley or a wheel that rolls without slipping (discussed in Section 10.3). These relationships usually appear as relationships between linear and/or angular accelerations.

3. Ensure that you have as many independent equations as there are unknowns. Solve the equations to find the target variables.

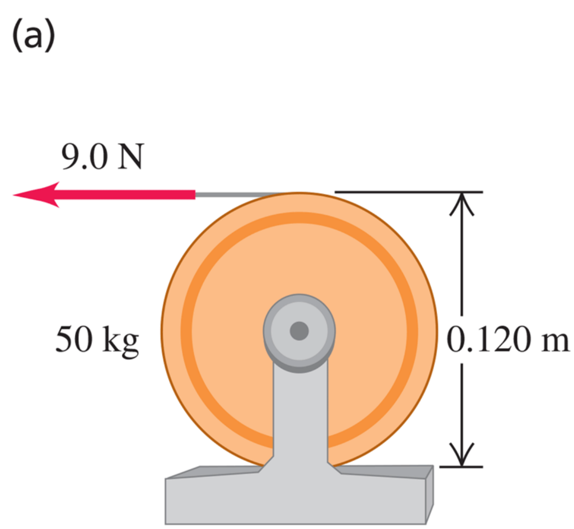
EVALUATE *your answer:* Check that the algebraic signs of your results make sense. As an example, if you are unrolling thread from a spool, your answers should not tell you that the spool is turning in the direction that rolls the thread back onto the spool! Check that any algebraic results are correct for special cases or for extreme values of quantities.

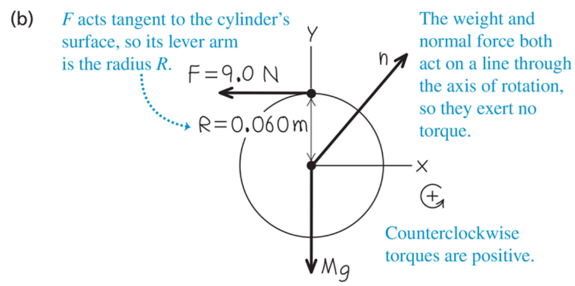
Example 10.2 An unwinding cable I

WITH VARIATION PROBLEMS

Figure 10.9a shows the situation that we analyzed in Example 9.7 using energy methods. What is the cable's acceleration?

Figure 10.9





(a) Cylinder and cable. (b) Our free-body diagram for the cylinder.

IDENTIFY and SET UP We can't use the energy method of Section 9.4, which doesn't involve acceleration. Instead we'll apply rotational dynamics to find the angular acceleration of the cylinder (Fig. 10.9b). We'll then find a relationship between the motion of the cable and the motion of the cylinder rim, and use this to find the acceleration of the cable. The cylinder rotates counterclockwise when the cable is pulled, so we take counterclockwise rotation to be positive. The net force on the cylinder must be zero because its center of mass remains at rest. The force F exerted by the cable produces a torque about the rotation axis. The weight (magnitude Mg) and the normal force (magnitude n) exerted by the cylinder's bearings produce *no* torque about the rotation axis because both act along lines through that axis.

EXECUTE The lever arm of F is equal to the radius $R = 0.060\text{ m}$ of the cylinder, so the torque is $\tau_z = FR$. (This torque is positive, as it tends to cause a counterclockwise rotation.) From Table 9.2, case (f), the moment of inertia of the cylinder about the rotation axis is $I = \frac{1}{2}MR^2$. Then Eq. (10.7) tells us that

$$\alpha_z = \frac{\tau_z}{I} = \frac{FR}{MR^2/2} = \frac{2F}{MR} = \frac{2(9.0\text{ N})}{(50\text{ kg})(0.060\text{ m})} = 6.0\text{ rad/s}^2$$

(We can add "rad" to our result because radians are dimensionless.)

To get the linear acceleration of the cable, recall from Section 9.3 that the acceleration of a cable unwinding from a cylinder is the same as the tangential acceleration of a point on the surface of the cylinder where the cable is tangent to it. This tangential acceleration is given by Eq. (9.14):

$$a_{\tan} = R\alpha_z = (0.060\ \text{m})(6.0\ \text{rad/s}^2) = 0.36\ \text{m}/s^2$$

EVALUATE Can you use this result, together with an equation from Chapter 2, to determine the speed of the cable after it has been pulled 2.0 m? Does your result agree with that of Example 9.7?

KEY CONCEPT

For any problem involving torques on a rigid body, first draw a free-body diagram to identify where on the rigid body each external force acts with respect to the axis of rotation. Then apply the rotational analog of Newton's second law, $\sum \tau_z = I\alpha_z$.

Video Tutor Solution: **Example 10.2**

Example 10.3 An unwinding cable II

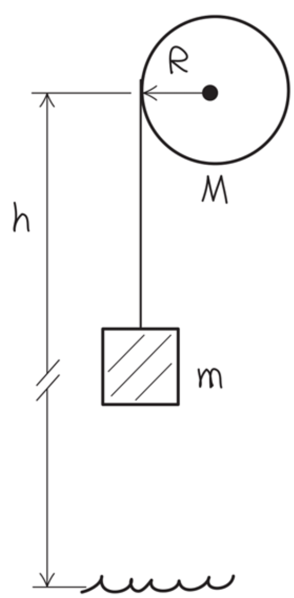
WITH VARIATION PROBLEMS

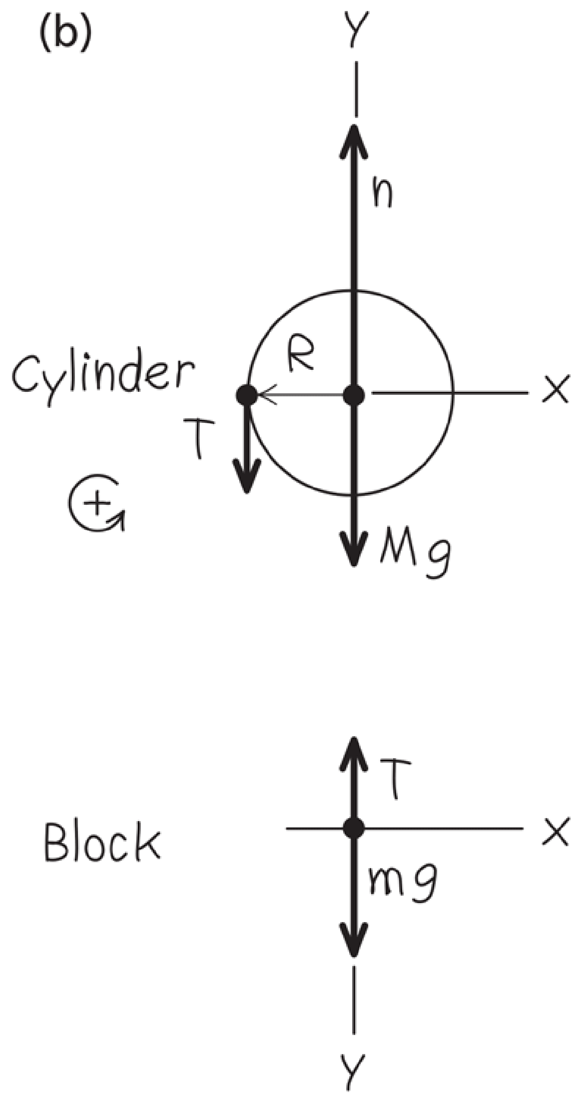
In Example 9.8 (Section 9.4), what are the acceleration of the falling block and the tension in the cable?

IDENTIFY and SET UP We'll apply translational dynamics to the block and rotational dynamics to the cylinder. As in Example 10.2, we'll relate the linear acceleration of the block (our target variable) to the angular acceleration of the cylinder. Figure 10.10 (next page) shows our sketch of the situation and a free-body diagram for each object. We take the positive sense of rotation for the cylinder to be counterclockwise and the positive direction of the y-coordinate for the block to be downward.

Figure 10.10

(a)





(a) Our diagram of the situation. (b) Our free-body diagrams for the cylinder and the block. We assume the cable has negligible mass.

EXECUTE For the block, Newton's second law gives

$$\sum F_y = mg + (-T) = ma_y$$

For the cylinder, the only torque about its axis is that due to the cable tension T. Hence Eq. (10.7) gives

$$\sum \tau_z = RT = I\alpha_z = \frac{1}{2}\, MR^2\alpha_z$$

As in Example 10.2, the acceleration of the cable is the same as the tangential acceleration of a point on the cylinder rim. From Eq. (9.14), this acceleration is $a_y = a_{\tan} = R\alpha_z$. We use this to replace $R\alpha_z$ with a_y in the cylinder equation above, and divide by R. The result is $T = \frac{1}{2} Ma_y$. Now we substitute this expression for T into Newton's second law for the block and solve for the acceleration a_y :

$$\begin{aligned} mg - \tfrac{1}{2} Ma_y &= ma_y \\ a_y &= \frac{g}{1 + M/2m} \end{aligned}$$

To find the cable tension T, we substitute our expression for a_y into the block equation:

$$T = mg - ma_y = mg - m\left(\frac{g}{1 + M/2m}\right) = \frac{mg}{1 + 2m/M}$$

EVALUATE The acceleration is positive (in the downward direction) and less than g, as it should be, since the cable is holding back the block. The cable tension is *not* equal to the block's weight mg; if it were, the block could not accelerate.

Let's check some particular cases. When M is much larger than m, the tension is nearly equal to mg and the acceleration is correspondingly much less than g. When M is zero, $T = 0$ and $a_y = g$; the object falls freely. If the object starts from rest ($v_{0y} = 0$) a height h above the floor, its y-velocity when it strikes the floor is given by $v_y^2 = v_{0y}^2 + 2a_y h = 2a_y h$, so

$$v_y = \sqrt{2a_y h} = \sqrt{\frac{2gh}{1 + M/2m}}$$

We found this result from energy considerations in Example 9.8.

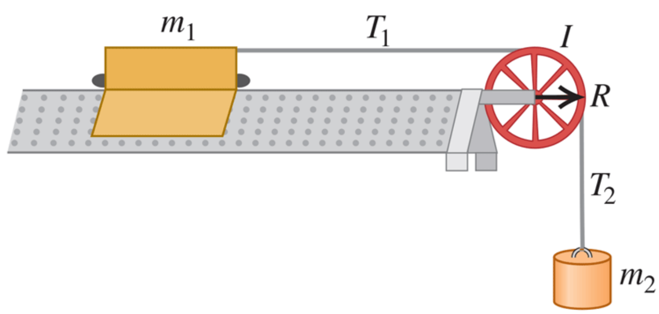
KEY CONCEPT

When an object is connected to a string that wraps around a rotating pulley of radius R, the linear acceleration a_y of the object is related to the angular acceleration α_z of the pulley by $a_y = R\alpha_z$.

Video Tutor Solution: **Example 10.3**

Test Your Understanding of Section 10.2

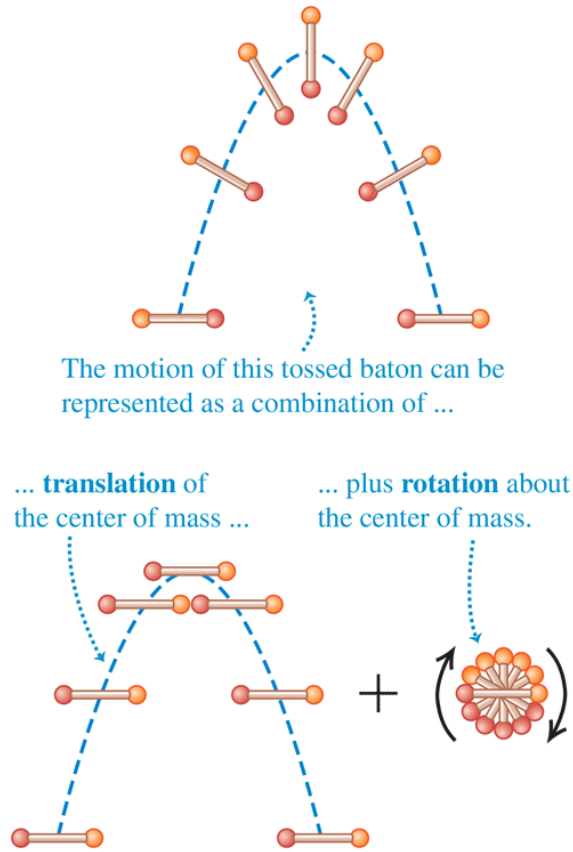
The figure shows a glider of mass m_1 that can slide without friction on a horizontal air track. It is attached to an object of mass m_2 by a massless string. The pulley has radius R and moment of inertia I about its axis of rotation. When released, the hanging object accelerates downward, the glider accelerates to the right, and the string turns the pulley without slipping or stretching. Rank the magnitudes of the following forces that act during the motion, in order from largest to smallest magnitude. (i) The tension force (magnitude T_1) in the horizontal part of the string; (ii) the tension force (magnitude T_2) in the vertical part of the string; (iii) the weight m_2g of the hanging object.



10.3 Rigid-Body Rotation About a Moving Axis

We can extend our analysis of rigid-body rotational dynamics to some cases in which the axis of rotation moves. When that happens, the motion of the rigid body is **combined translation and rotation.** The key to understanding such situations is this: Every possible motion of a rigid body can be represented as a combination of *translational motion of the center of mass* and *rotation about an axis through the center of mass.* This is true even when the center of mass accelerates, so it is not at rest in any inertial frame. Figure 10.11 illustrates this for the motion of a tossed baton: The center of mass of the baton follows a parabolic curve, as though the baton were a particle located at the center of mass. A rolling ball is another example of combined translational and rotational motions.

Figure 10.11



The motion of a rigid body is a combination of translational motion of the center of mass and rotation around the center of mass.

Combined Translation and Rotation: Energy Relationships

It's beyond our scope to prove that rigid-body motion can always be divided into translation of the center of mass and rotation about the center of mass. But we *can* prove this for the kinetic energy K of a rigid body that has both translational and rotational motions. For such a rigid body, K is the sum of two parts:

(10.8)

$$\text{Kinetic energy of a rigid body with both translation and rotation} \rightarrow K = \overbrace{\tfrac{1}{2}Mv_{\mathrm{cm}}^{2}}^{\substack{\text{Kinetic energy of } \textit{translation} \\ \text{of center of mass (cm)}}} + \overbrace{\tfrac{1}{2}I_{\mathrm{cm}}\omega^{2}}^{\substack{\text{Kinetic energy of } \textit{rotation} \\ \text{around axis through cm}}} \tag{10.8}$$

Mass of rigid body (M)

Speed of cm (v_{cm})

Moment of inertia of rigid body about axis through cm (I_{cm})

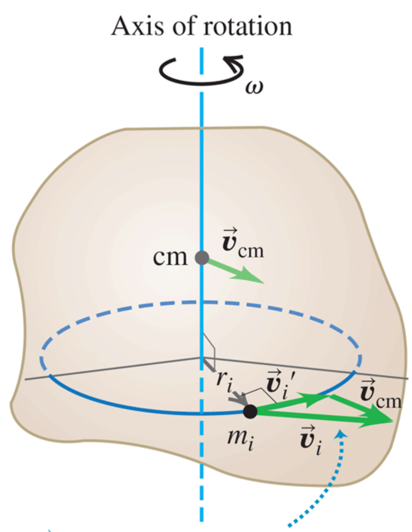
Angular speed of rigid body (ω)

To prove this relationship, we again imagine the rigid body to be made up of particles. For a typical particle with mass m_i (Fig. 10.12), the velocity $\vec{v}_i$ of this particle relative to an inertial frame is the vector sum of the velocity $\vec{v}_{\mathrm{cm}}$ of the center of mass and the velocity $\vec{v}'_i$ of the particle *relative to* the center of mass:

(10.9)

$$\vec{v}_i = \vec{v}_{\mathrm{cm}} + \vec{v}'_i$$

Figure 10.12



Velocity $\vec{v}_i$ of particle in rotating, translating rigid body $=$ (velocity $\vec{v}_{\mathrm{cm}}$ of center of mass) $+$ (particle's velocity $\vec{v}_i{}'$ relative to center of mass)

A rigid body with both translational and rotational motions.

The kinetic energy K_i of this particle in the inertial frame is $\frac{1}{2}\,m_i\,v_i^2$, which we can also express as $\frac{1}{2}\,m_i\left(\vec{\boldsymbol{v}}_i \cdot \vec{\boldsymbol{v}}_i\right)$. Substituting Eq. (10.9) into this, we get

$$\begin{aligned} K_i &= \tfrac{1}{2}\,m_i\,(\overrightarrow{\boldsymbol{v}}_{\mathrm{cm}} + \overrightarrow{\boldsymbol{v}}_i') \cdot (\overrightarrow{\boldsymbol{v}}_{\mathrm{cm}} + \overrightarrow{\boldsymbol{v}}_i') \\ &= \tfrac{1}{2}\,m_i\,(\overrightarrow{\boldsymbol{v}}_{\mathrm{cm}}\overrightarrow{\boldsymbol{v}}_{\mathrm{cm}} + 2\overrightarrow{\boldsymbol{v}}_{\mathrm{cm}} \cdot \overrightarrow{\boldsymbol{v}}_i' + \overrightarrow{\boldsymbol{v}}_i' \cdot \overrightarrow{\boldsymbol{v}}_i') \\ &= \tfrac{1}{2}\,m_i\,(v_{\mathrm{cm}}^2 + 2\overrightarrow{\boldsymbol{v}}_{\mathrm{cm}}\overrightarrow{\boldsymbol{v}}_i' + v_i'^2) \end{aligned}$$

The total kinetic energy is the sum $\sum K_i$ for all the particles making up the rigid body. Expressing the three terms in this equation as separate sums, we get

$$K = \sum K_i = \sum\left(\tfrac{1}{2}\,m_i\,v_{\mathrm{cm}}^2\right) + \sum\left(m_i\,\vec{\boldsymbol{v}}_{\mathrm{cm}} \cdot \vec{\boldsymbol{v}}_i'\right) + \sum\left(\tfrac{1}{2}\,m_i\,v_i'^2\right)$$

The first and second terms have common factors that we take outside the sum:

(10.10)

$$K = \tfrac{1}{2}\,\left(\sum m_i\right)v_{\mathrm{cm}}^2 + \vec{\boldsymbol{v}}_{\mathrm{cm}} \cdot \left(\sum m_i\,\vec{\boldsymbol{v}}_i'\right) + \sum\left(\tfrac{1}{2}\,m_i\,v_i'^2\right)$$

Now comes the reward for our effort. In the first term, $\sum m_i$ is the total mass M. The second term is zero because $\sum m_i\,\vec{\boldsymbol{v}}_i'$ is M times the velocity of the center of mass *relative to the center of mass,* and this is zero by definition. The last term is the sum of the kinetic energies of the particles computed by using their speeds with respect to the center of mass; this is just the kinetic energy of rotation around the center of mass. Using the same steps that led to Eq. (9.17) for the rotational kinetic energy of a rigid body, we can write this last term as $\frac{1}{2}\,I_{\mathrm{cm}}\omega^2$, where I_{cm} is the moment of inertia with respect to the axis through the center of mass and ω is the angular speed. So Eq. (10.10) becomes Eq. (10.8):

$$K = \frac{1}{2}\, M v_{\mathrm{cm}}^2 + \frac{1}{2}\, I_{\mathrm{cm}} \omega^2$$

Rolling Without Slipping

An important case of combined translation and rotation is **rolling without slipping.** The rolling wheel in Fig. 10.13 (next page) is symmetrical, so its center of mass is at its geometric center. We view the motion in an inertial frame of reference in which the surface on which the wheel rolls is at rest. In this frame, the point on the wheel that contacts the surface must be instantaneously *at rest* so that it does not slip. Hence the velocity $\vec{v}_1'$ of the point of contact relative to the center of mass must have the same magnitude but opposite direction as the center-of-mass velocity $\vec{v}_{\mathrm{cm}}$. If the wheel's radius is R and its angular speed about the center of mass is ω, then the magnitude of $\vec{v}_1'$ is $R\omega$; hence

(10.11)

Condition for rolling without slipping:

Speed of center of mass of rolling wheel

$$v_{\mathrm{cm}} = R\omega \qquad (10.11)$$

Radius of wheel

Angular speed of wheel

BIO Application

Combined Translation and Rotation

A maple seed consists of a pod attached to a much lighter, flattened wing. Airflow around the wing slows the falling seed to about 1 m/s and causes the seed to rotate about its center of mass. The seed's slow fall means that a breeze can carry the seed some distance from the parent tree. In the absence of wind, the seed's center of mass falls straight down.

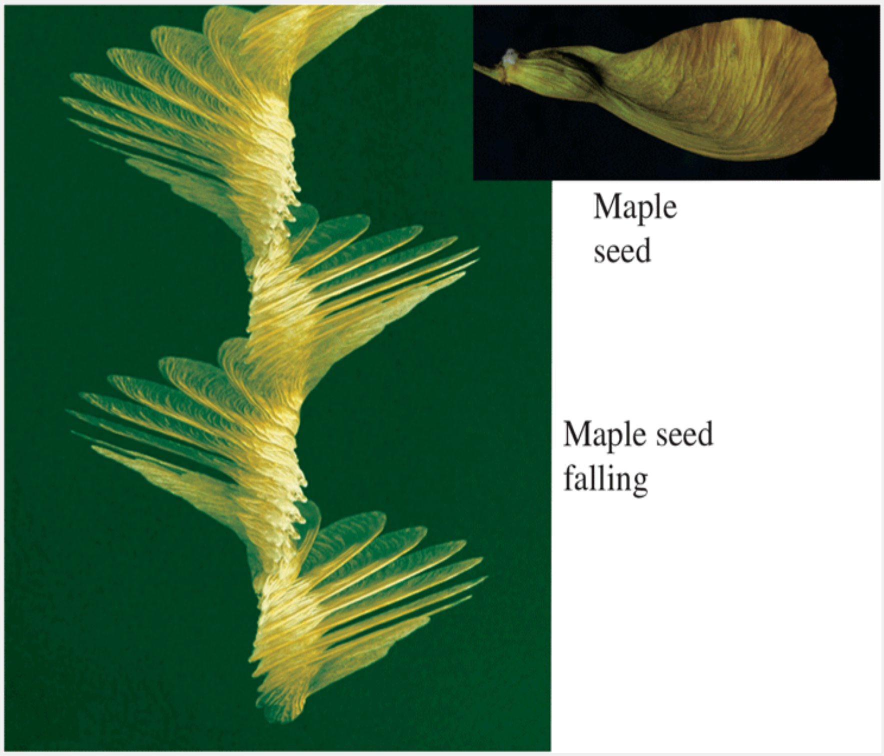
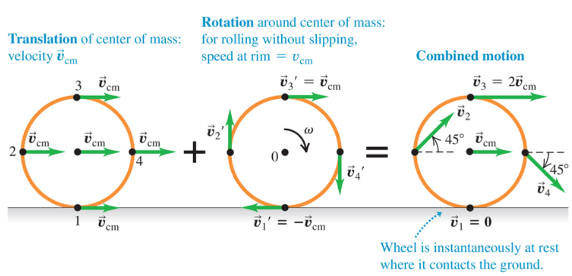


Figure 10.13



The motion of a rolling wheel is the sum of the translational motion of the center of mass and the rotational motion of the wheel around the center of mass.

As Fig. 10.13 shows, the velocity of a point on the wheel is the vector sum of the velocity of the center of mass and the velocity of the point relative to the center of mass. Thus while point 1, the point of contact, is instantaneously at rest, point 3 at the top of the wheel is moving forward *twice as fast* as the center of mass, and points 2 and 4 at the sides have velocities at 45° to the horizontal.

At any instant we can think of the wheel as rotating about an "instantaneous axis" of rotation that passes through the point of contact with the ground. The angular velocity ω is the same for this axis as for an axis through the center of mass; an observer at the center of mass sees the rim make the same number of revolutions per second as does an observer at the rim watching the center of mass spin around him. If we think of the motion of the rolling wheel in Fig. 10.13 in this way, the kinetic energy of the wheel is $K = \frac{1}{2} I_1 \omega^2$, where I_1 is the moment of inertia of the wheel about an axis through point 1. But by the parallel-axis theorem, Eq. (9.19), $I_1 = I_{\text{cm}} + MR^2$, where M is the total mass of the wheel and I_{cm} is the moment of inertia with respect to an axis through the center of mass. Using Eq. (10.11), we find that the wheel's kinetic energy is as given by Eq. (10.8):

$$K = \frac{1}{2} I_1 \omega^2 = \frac{1}{2} I_{\text{cm}}\omega^2 + \frac{1}{2} MR^2\omega^2 = \frac{1}{2} I_{\text{cm}}\omega^2 + \frac{1}{2} Mv_{\text{cm}}^2$$

CAUTION Rolling without slipping The relationship $v_{\text{cm}} = R\omega$ holds *only* if there is rolling without slipping. When a drag racer first starts to move, the rear tires are spinning very fast even though the racer is hardly moving, so $R\omega$ is greater than v_{cm} (Fig. 10.14). If a driver applies the brakes too heavily so that the car skids, the tires will spin hardly at all and $R\omega$ is less than v_{cm}.

Figure 10.14

The smoke rising from this drag racer's rear tires shows that the tires are slipping on the road, so v_{cm} is *not* equal to $R\omega$.

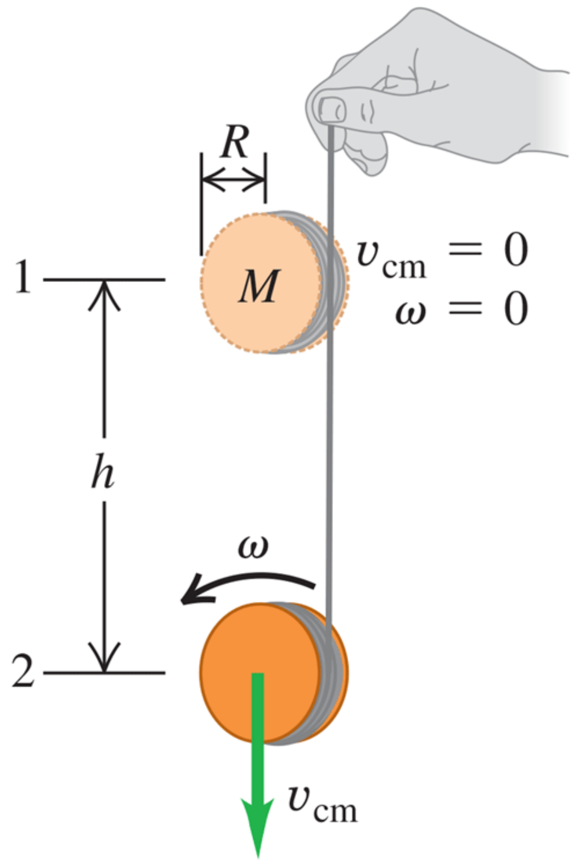
If a rigid body changes height as it moves, we must also consider gravitational potential energy. We saw in Section 9.4 that for any extended object of mass M, rigid or not, the gravitational potential energy U is the same as if we replaced the object by a particle of mass M located at the object's center of mass, so

$$U = Mgy_{\mathrm{cm}}$$

Example 10.4 Speed of a primitive yo-yo

A primitive yo-yo has a massless string wrapped around a solid cylinder with mass M and radius R (Fig. 10.15). You hold the free end of the string stationary and release the cylinder from rest. The string unwinds but does not slip or stretch as the cylinder descends and rotates. Using energy considerations, find the speed v_{cm} of the cylinder's center of mass after it has descended a distance h.

Figure 10.15



Calculating the speed of a primitive yo-yo.

IDENTIFY and SET UP Since you hold the upper end of the string fixed, your hand does no work on the string–cylinder system. There is friction between the string and the cylinder, but the string doesn't slip so no mechanical energy is lost. Hence we can use conservation of mechanical energy. The initial kinetic energy of the cylinder is $K_1 = 0$, and its final kinetic energy K_2 is given by Eq. (10.8); the massless string has no kinetic energy. The moment of inertia is $I_{cm} = \frac{1}{2} MR^2$, and by Eq. (9.13) $\omega = v_{cm}/R$ because the string doesn't slip. The potential energies are $U_1 = Mgh$ and $U_2 = 0$.

EXECUTE From Eq. (10.8), the kinetic energy at point 2 is

$$K_2 = \tfrac{1}{2} Mv_{cm}^2 + \frac{1}{2}\left(\frac{1}{2} MR^2\right)\left(\frac{v_{cm}}{R}\right)^2 = \frac{3}{4} Mv_{cm}^2$$

The kinetic energy is $1\,\frac{1}{2}$ times what it would be if the yo-yo were falling at speed v_{cm} without rotating. Two-thirds of the total kinetic energy $\left(\frac{1}{2}\ Mv_{\mathrm{cm}}^2\right)$ is translational and one-third $\left(\frac{1}{4}\ Mv_{\mathrm{cm}}^2\right)$ is rotational. Using conservation of energy,

$$\begin{aligned} K_1 + U_1 &= K_2 + U_2 \\ 0 + Mgh &= \tfrac{3}{4} Mv_{\mathrm{cm}}^2 + 0 \\ v_{\mathrm{cm}} &= \sqrt{\tfrac{4}{3} gh} \end{aligned}$$

EVALUATE No mechanical energy was lost or gained, so from the energy standpoint the string is merely a way to convert some of the gravitational potential energy (which is released as the cylinder falls) into rotational kinetic energy rather than translational kinetic energy. Because not all of the released energy goes into translation, v_{cm} is less than the speed $\sqrt{2gh}$ of an object dropped from height h with no strings attached.

KEY CONCEPT

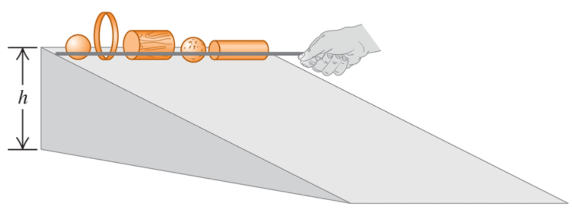
If a rigid body is both translating (moving as a whole through space) and rotating, its total kinetic energy is the sum of the kinetic energy of translation of the center of mass and the kinetic energy of rotation around an axis through the center of mass.

Video Tutor Solution: **Example 10.4**

Example 10.5 Race of the rolling bodies

In a physics demonstration, an instructor "races" various rigid bodies that roll without slipping from rest down an inclined plane (Fig. 10.16). What shape should a body have to reach the bottom of the incline first?

Figure 10.16



Which body rolls down the incline fastest, and why?

IDENTIFY and SET UP Kinetic friction does no work if the bodies roll without slipping. We can also ignore the effects of *rolling friction*, introduced in Section 5.3, if the bodies and the surface of the incline are rigid. (Later in this section we'll explain why this is so.) We can therefore use conservation of energy. Each body starts from rest at the top of an incline with height h, so $K_1 = 0$, $U_1 = Mgh$, and $U_2 = 0$. Equation (10.8) gives the kinetic energy at the bottom of the incline; since the bodies roll without slipping, $\omega = v_{\mathrm{cm}}/R$. We can express the moments of inertia of the four round bodies in Table 9.2, cases (f)–(i), as $I_{\mathrm{cm}} = cMR^2$, where c is a number less than or equal to 1 that depends on the shape of the body, Our goal is to find the value of c that gives the body the greatest speed v_{cm} after its center of mass has descended a vertical distance h.

EXECUTE From conservation of energy,

$$
\begin{aligned}
K_1 + U_1 &= K_2 + U_2 \\
0 + Mgh &= \tfrac{1}{2}\, Mv_{\text{cm}}^2 + \tfrac{1}{2}\, cMR^2 \left(\tfrac{v_{\text{cm}}}{R}\right)^2 + 0 \\
Mgh &= \tfrac{1}{2}\, (1 + c)Mv_{\text{cm}}^2 \\
v_{\text{cm}} &= \sqrt{\frac{2gh}{1 + c}}
\end{aligned}
$$

EVALUATE For a given value of c, the speed v_{cm} after descending a distance h is *independent* of the body's mass M and radius R. Hence *all* uniform solid cylinders $\left(c = \frac{1}{2}\right)$ have the same speed at the bottom, regardless of their mass and radii. The values of c tell us that the order of finish for uniform bodies will be as follows: (1) any solid sphere $\left(c = \frac{2}{5}\right)$, (2) any solid cylinder $\left(c = \frac{1}{2}\right)$, (3) any thin-walled, hollow sphere $\left(c = \frac{2}{3}\right)$, and (4) any thin-walled, hollow cylinder $(c = 1)$. Small-c bodies always beat large-c bodies because less of their kinetic energy is tied up in rotation, so more is available for translation.

KEY CONCEPT

For a rigid body that rolls without slipping, has a given mass and radius, and moves with a given center-of-mass speed, the kinetic energy of rotation depends on the shape of the rigid body.

Video Tutor Solution: **Example 10.5**

Combined Translation and Rotation: Dynamics

We can also analyze the combined translational and rotational motions of a rigid body from the standpoint of dynamics. We showed in Section 8.5 that for an extended object, the acceleration of the center of mass is the same as that of a particle of the same mass acted on by all the external forces on the actual object:

(10.12)

$$\underbrace{\sum \vec{F}_{\text{ext}}}_{\text{Net external force on an object}} = \overbrace{M}^{\text{Total mass of object}}\underbrace{\vec{a}_{\text{cm}}}_{\text{Acceleration of center of mass}} \qquad (10.12)$$

The rotational motion about the center of mass is described by the rotational analog of Newton's second law, Eq. (10.7):

(10.13)

$$\underbrace{\sum \tau_z}_{\text{Net torque on a rigid body about } z\text{-axis through center of mass}} = \overbrace{I_{\text{cm}}}^{\text{Moment of inertia of rigid body about } z\text{-axis}}\underbrace{\alpha_z}_{\text{Angular acceleration of rigid body about } z\text{-axis}} \qquad (10.13)$$

It's not immediately obvious that Eq. (10.13) should apply to the motion of a translating rigid body; after all, our derivation of $\sum \tau_z = I\alpha_z$ in Section 10.2 assumed that the axis of rotation was stationary. But Eq.

(10.13) is valid *even when the axis of rotation moves,* provided the following two conditions are met:

1. The axis through the center of mass must be an axis of symmetry.
2. The axis must not change direction.

These conditions are satisfied for many types of rotation (Fig. 10.17). Note that in general this moving axis of rotation is *not* at rest in an inertial frame of reference.

Figure 10.17

The axle of a bicycle wheel passes through the wheel's center of mass and is an axis of symmetry. Hence the rotation of the wheel is described by Eq. (10.13), provided the bicycle doesn't turn or tilt to one side (which would change the orientation of the axle).

We can now solve dynamics problems involving a rigid body that undergoes translational and rotational motions at the same time, provided that the rotation axis satisfies the two conditions just mentioned. Problem-Solving Strategy 10.1 (Section 10.2) is equally useful here, and you should review it now. Keep in mind that when a rigid body undergoes translational and rotational motions at the same time, we need two separate equations of motion *for the same/body:* Eq.

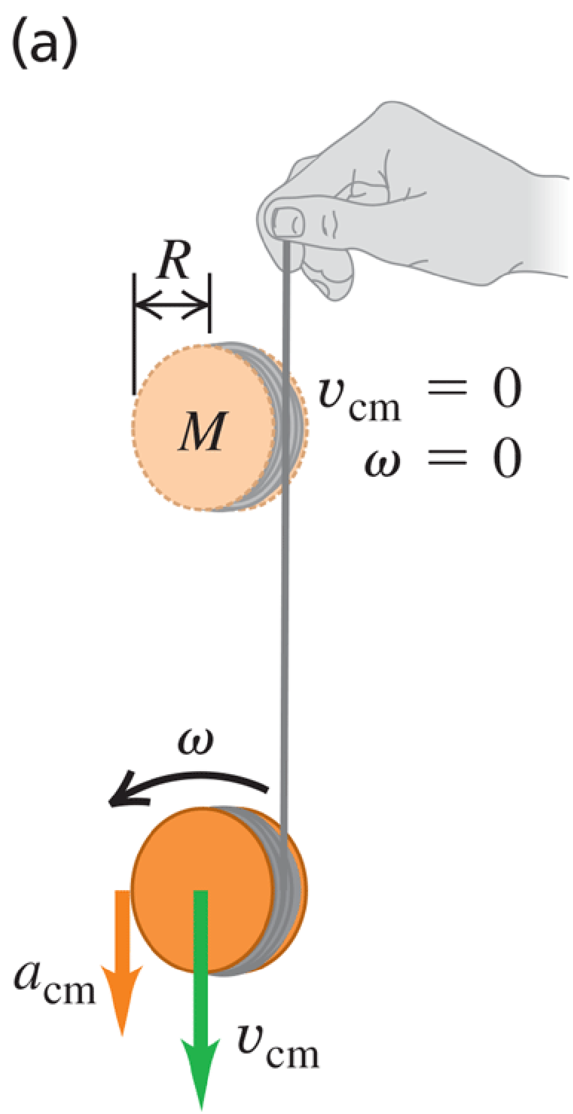
(10.12) for the translation of the center of mass and Eq. (10.13) for rotation about an axis through the center of mass.

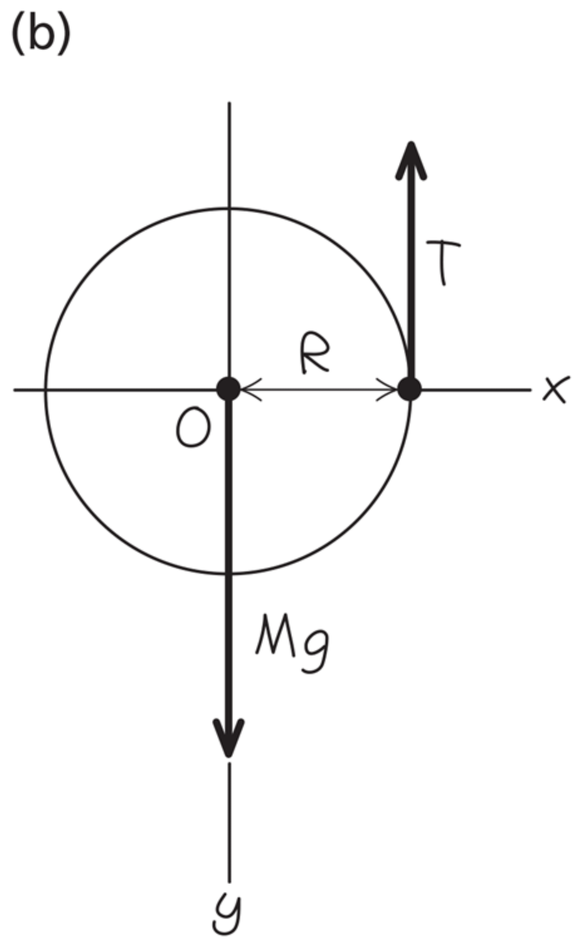
Example 10.6 Acceleration of a primitive yo-yo

WITH VARIATION PROBLEMS

For the primitive yo-yo in Example 10.4 (Fig. 10.18a), find the downward acceleration of the cylinder and the tension in the string.

Figure 10.18





Dynamics of a primitive yo-yo (see Fig. 10.15).

IDENTIFY and SET UP Figure 10.18b shows our free-body diagram for the yo-yo, including our choice of positive coordinate directions. Our target variables are $a_{\mathrm{cm}-y}$ and T. We'll use Eq. (10.12) for the translational motion of the center of mass and Eq. (10.13) for the rotational motion around the center of mass. We'll also use Eq. (10.11), which says that the string unwinds without slipping. As in Example 10.4, the moment of inertia of the yo-yo for an axis through its center of mass is $I_{\mathrm{cm}} = \frac{1}{2}\,MR^2$.

EXECUTE From Eq. (10.12),

(10.14)

$$\sum F_y = Mg + (-T) = Ma_{\mathrm{cm}-y}$$

From Eq. (10.13),

(10.15)

$$\sum \tau_z = TR = I_{\text{cm}}\alpha_z = \tfrac{1}{2}\,MR^2\alpha_z$$

From Eq. (10.11), $v_{\text{cm}-z} = R\omega_z$; the derivative of this expression with respect to time gives us

(10.16)

$$a_{\text{cm}-y} = R\alpha_z$$

We now use Eq. (10.16) to eliminate α_z from Eq. (10.15) and then solve Eqs. (10.14) and (10.15) simultaneously for T and $a_{\text{cm}-y}$:

$$a_{\text{cm}-y} = \frac{2}{3}\,g \qquad T = \frac{1}{3}\,Mg$$

EVALUATE The string slows the fall of the yo-yo, but not enough to stop it completely. Hence $a_{\text{cm}-y}$ is less than the free-fall value g and T is less than the yo-yo weight Mg.

KEY CONCEPT

To analyze the motion of a rigid body that is both translating and rotating, use Newton's second law for the translational motion of the center of mass and the rotational analog of Newton's second law for the rotation around the center of mass.

Video Tutor Solution: **Example 10.6**

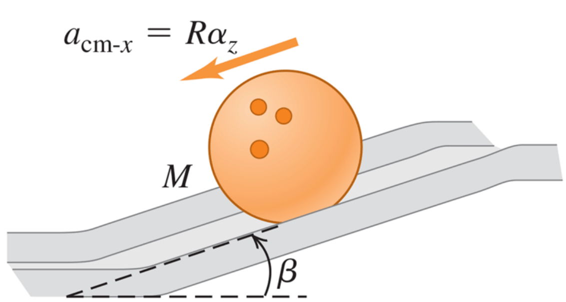
Example 10.7 Acceleration of a rolling sphere

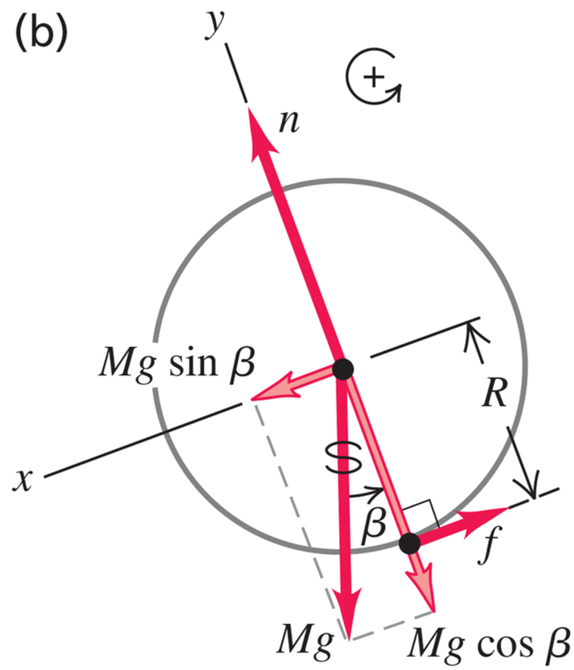
WITH VARIATION PROBLEMS

A bowling ball of mass M rolls without slipping down a ramp that is inclined at an angle β to the horizontal (Fig. 10.19a). What are the ball's acceleration and the magnitude of the friction force on the ball? Treat the ball as a uniform solid sphere, ignoring the finger holes.

Figure 10.19

(a)





A bowling ball rolling down a ramp.

IDENTIFY and SET UP The free-body diagram (Fig. 10.19b) shows that only the friction force exerts a torque about the center of mass. Our target variables are the acceleration $a_{\mathrm{cm}-x}$ of the ball's center of mass and the magnitude f of the friction force. (Because the ball does not slip at the instantaneous point of contact with the ramp, this is a *static* friction force; it prevents slipping and gives the ball its angular acceleration.) We use Eqs. (10.12) and (10.13) as in Example 10.6.

EXECUTE The ball's moment of inertia is $I_{\mathrm{cm}} = \frac{2}{5}MR^2$. The equations of motion are

(10.17)

$$\sum F_x = Mg\sin\beta + (-f) = Ma_{\mathrm{cm}-x}$$

(10.18)

$$\sum \tau_z = fR = I_{\mathrm{cm}}\alpha_z = \left(\frac{2}{5}MR^2\right)\alpha_z$$

The ball rolls without slipping, so as in Example 10.6 we use $a_{\mathrm{cm}-x} = R\alpha_z$ to eliminate α_z from Eq. (10.18):

$$fR = \frac{2}{5} MRa_{\mathrm{cm}-x}$$

This equation and Eq. (10.17) are two equations for the unknowns $a_{\mathrm{cm}-x}$ and f. We solve Eq. (10.17) for f, substitute that expression into the above equation to eliminate f, and solve for $a_{\mathrm{cm}-x}$:

$$a_{\mathrm{cm}-x} = \frac{5}{7} g \sin \beta$$

Finally, we substitute this acceleration into Eq. (10.17) and solve for f :

$$f = \frac{2}{7} Mg \sin \beta$$

EVALUATE The ball's acceleration is just $\frac{5}{7}$ as large as that of an object *sliding* down the slope without friction. If the ball descends a vertical distance h as it rolls down the ramp, its displacement along the ramp is $h/\sin \beta$. You can show that the speed of the ball at the bottom of the ramp is $v_{\mathrm{cm}} = \sqrt{\frac{10}{7} gh}$, the same as our result from Example 10.5 with $c = \frac{2}{5}$.

If the ball were rolling *uphill* without slipping, the force of friction would still be directed uphill as in Fig. 10.19b. Can you see why?

KEY CONCEPT

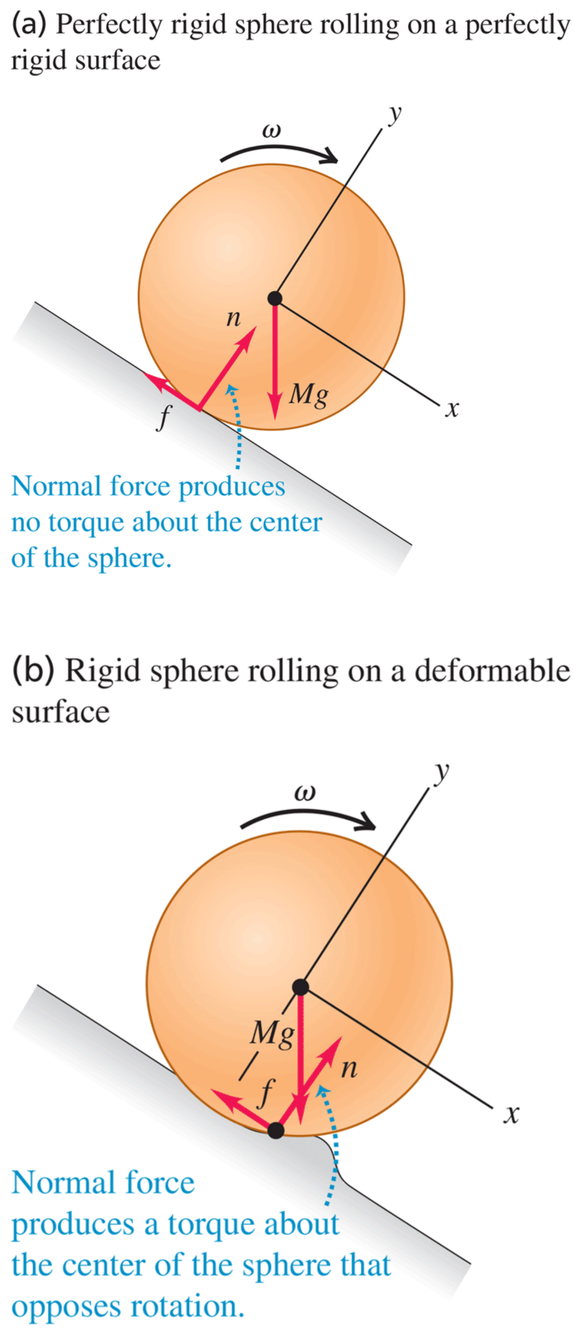
If an object is rolling without slipping on an incline, a friction force must act on it. The direction of this friction force is always such as to prevent slipping.

Video Tutor Solution: **Example 10.7**

Rolling Friction

In Example 10.5 we said that we can ignore rolling friction if both the rolling body and the surface over which it rolls are perfectly rigid. In Fig. 10.20a a perfectly rigid sphere is rolling down a perfectly rigid incline. The line of action of the normal force passes through the center of the sphere, so its torque is zero; there is no sliding at the point of contact, so the friction force does no work. Figure 10.20b shows a more realistic situation, in which the surface "piles up" in front of the sphere and the sphere rides in a shallow trench. Because of these deformations, the contact forces on the sphere no longer act along a single point but over an area; the forces are concentrated on the front of the sphere as shown. As a result, the normal force now exerts a torque that opposes the rotation. In addition, there is some sliding of the sphere over the surface due to the deformation, causing mechanical energy to be lost. The combination of these two effects is the phenomenon of *rolling friction.* Rolling friction also occurs if the rolling body is deformable, such as an automobile tire. Often the rolling body and the surface are rigid enough that rolling friction can be ignored, as we have assumed in all the examples in this section.

Figure 10.20



Rolling down (a) a perfectly rigid surface and (b) a deformable surface. In (b) the deformation is greatly exaggerated, and the force n is the component of the contact force that points normal to the plane of the surface before it is deformed.

BIO Application

Rolling for Reproduction

One of the few organisms that uses rolling as a means of locomotion is the weed called Russian thistle (*Kali tragus*). The plant breaks off at its base, forming a rounded tumbleweed that disperses its seeds as it rolls. Because a tumbleweed deforms easily, it is subject to substantial rolling friction. Hence it quickly slows to a stop unless propelled by the wind.

Test Your Understanding of Section 10.3

Suppose the solid cylinder used as a yo-yo in Example 10.6 is replaced by a hollow cylinder of the same mass and radius. (a) Will the acceleration of the yo-yo (i) increase, (ii) decrease, or (iii) remain the same? (b) Will the string tension (i) increase, (ii) decrease, or (iii) remain the same?

10.4 Work and Power in Rotational Motion

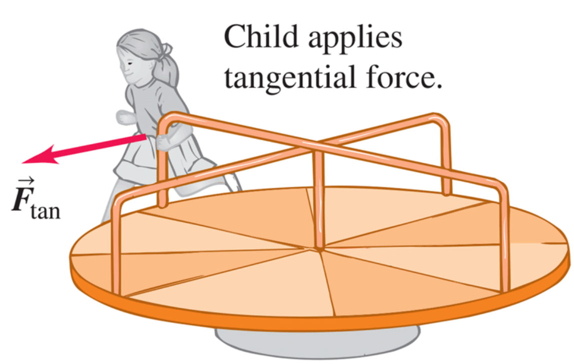
When you pedal a bicycle, you apply forces to a rotating body and do work on it. Similar things happen in many other real-life situations, such as a rotating motor shaft driving a power tool or a car engine propelling the vehicle. Let's see how to apply our ideas about work from Chapter 6 to rotational motion.

Suppose a tangential force $\vec{F}_{\text{tan}}$ acts at the rim of a pivoted disk—for example, a child running while pushing on a playground merry-go-round (Fig. 10.21a). The disk rotates through an infinitesimal angle $d\theta$ about a fixed axis during an infinitesimal time interval dt (Fig. 10.21b). The work dW done by the force $\vec{F}_{\text{tan}}$ while a point on the rim moves a distance ds is $dW = F_{\text{tan}}\,ds$. If $d\theta$ is measured in radians, then $ds = R\,d\theta$ and

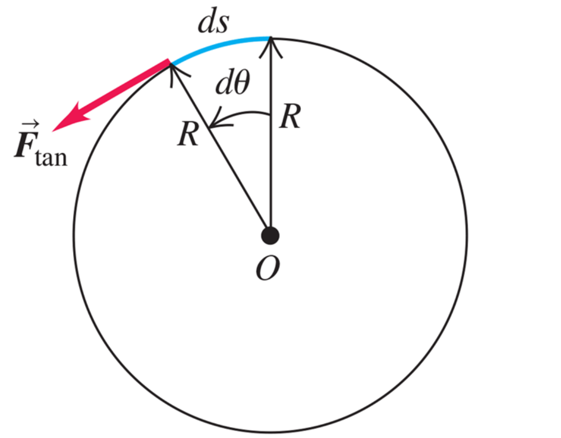
$$dW = F_{\text{tan}}\,R\,d\theta$$

Figure 10.21

(a)



(b) Overhead view of merry-go-round



A tangential force applied to a rotating body does work.

Now $F_{\tan} R$ is the *torque* τ_z due to the force $\vec{F}_{\tan}$, so

(10.19)

$$dW = \tau_z \, d\theta$$

As the disk rotates from θ_1 to θ_2, the total work done by the torque is

(10.20)

Work done by a torque τ_z

Upper limit = final angular position

$$W = \int_{\theta_1}^{\theta_2} \tau_z \, d\theta \qquad (10.20)$$

Integral of the torque with respect to angle

Lower limit = initial angular position

If the torque remains *constant* while the angle changes, then the work is the product of torque and angular displacement:

(10.21)

$$\underbrace{W}_{\text{Work done by a constant torque } \tau_z} = \overbrace{\tau_z}^{\text{Torque}}\underbrace{(\theta_2 - \theta_1)}_{\text{Final minus initial angular position}} = \tau_z \underbrace{\Delta\theta}_{\text{= angular displacement}} \tag{10.21}$$

If torque is expressed in newton-meters ($\mathrm{N \cdot m}$) and angular displacement in radians, the work is in joules. Equation (10.21) is the rotational analog of Eq. (6.1), $W = Fs$, and Eq. (10.20) is the analog of Eq. (6.7), $W = \int F_x\, dx$, for the work done by a force in a straight-line displacement.

If the force in Fig. 10.21 had an axial component (parallel to the rotation axis) or a radial component (directed toward or away from the axis), that component would do no work because the displacement of the point of application has only a tangential component. An axial or radial component of force would also make no contribution to the torque about the axis of rotation. So Eqs. (10.20) and (10.21) are correct for *any* force, no matter what its components.

When a torque does work on a rotating rigid body, the kinetic energy changes by an amount equal to the work done. We can prove this by using exactly the same procedure that we used in Eqs. (6.11), (6.12) and (6.13) for the translational kinetic energy of a particle. Let τ_z represent the *net* torque on the body so that $\tau_z = I\alpha_z$ from Eq. (10.7), and assume that the body is rigid so that the moment of inertia I is constant. We then transform the integrand in Eq. (10.20) into an integrand with respect to ω_z as follows:

$$\tau_z\, d\theta = (I\alpha_z)\, d\theta = I\frac{d\omega_z}{dt}\, d\theta = I\frac{d\theta}{dt}\, d\omega_z = I\omega_z\, d\omega_z$$

Since τ_z is the net torque, the integral in Eq. (10.20) is the *total* work done on the rotating rigid body. This equation then becomes

(10.22)

$$\underbrace{W_{\text{tot}}}_{\substack{\text{Total work done on}\\ \text{a rotating rigid body}\\ =\text{ work done by the}\\ \text{net external torque}}} = \int_{\omega_1}^{\omega_2} I\omega_z\, d\omega_z = \overbrace{\tfrac{1}{2}I\omega_2^2}^{\text{Final rotational kinetic energy}} - \underbrace{\tfrac{1}{2}I\omega_1^2}_{\text{Initial rotational kinetic energy}} \qquad (10.22)$$

The change in the rotational kinetic energy of a *rigid* body equals the work done by forces exerted from outside the body (Fig. 10.22). This equation is analogous to Eq. (6.13), the work–energy theorem for a particle.

Figure 10.22

The rotational kinetic energy of a helicopter's main rotor is equal to the total work done to set it spinning. When it is spinning at a constant rate, positive work is done on the rotor by the engine and negative work is done on it by air resistance. Hence the net work being done is zero and the kinetic energy remains constant.

How does *power* relate to torque? When we divide both sides of Eq. (10.19) by the time interval dt during which the angular displacement occurs, we find

$$\frac{dW}{dt} = \tau_z \frac{d\theta}{dt}$$

But dW/dt is the rate of doing work, or *power* P, and $d\theta/dt$ is angular velocity ω_z:

(10.23)

Torque with respect to rigid body's rotation axis

Power due to a torque acting on a rigid body $P = \tau_z \omega_z$ (10.23)

Angular velocity of rigid body about axis

This is the analog of the relationship $P = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}}$ that we developed in Section 6.4 for particle motion.

Example 10.8 Calculating power from torque

An electric motor exerts a constant $10\ \mathrm{N \cdot m}$ torque on a grindstone, which has a moment of inertia of $2.0\ \mathrm{kg{\cdot}m^2}$ about its shaft. The system starts from rest. Find the work W done by the motor in 8.0 s and the grindstone's kinetic energy K at this time. What average power P_{av} is delivered by the motor?

IDENTIFY and SET UP The only torque acting is that due to the motor. Since this torque is constant, the grindstone's angular acceleration α_z is constant. We'll use Eq. (10.7) to find α_z, and then use this in the kinematics equations from Section 9.2 to calculate the angle $\Delta\theta$ through which the grindstone rotates in 8.0 s and its final angular velocity ω_z. From these we'll calculate W, K, and P_{av}.

EXECUTE We have $\sum \tau_z = 10\ \mathrm{N \cdot m}$ and $I = 2.0\ \mathrm{kg{\cdot}m^2}$, so $\sum \tau_z = I\alpha_z$ yields $\alpha_z = 5.0\ \mathrm{rad/s^2}$. From Eqs. (9.11) and (10.21),

$$\begin{aligned} \Delta\theta &= \tfrac{1}{2}\,\alpha_z t^2 = \tfrac{1}{2}\,(5.0\ \mathrm{rad/s^2})(8.0\ \mathrm{s})^2 = 160\ \mathrm{rad} \\ W &= \tau_z\,\Delta\theta = (10\ \mathrm{N \cdot m})(160\ \mathrm{rad}) = 1600\ \mathrm{J} \end{aligned}$$

From Eqs. (9.7) and (9.17),

$$\begin{aligned} \omega_z &= \alpha_z t = (5.0\ \mathrm{rad/s^2})(8.0\ \mathrm{s}) = 40\ \mathrm{rad/s} \\ K &= \tfrac{1}{2}\,I\omega_z^2 = \tfrac{1}{2}\,(2.0\ \mathrm{kg \cdot m^2})(40\ \mathrm{rad/s})^2 = 1600\ \mathrm{J} \end{aligned}$$

The average power is the work done divided by the time interval:

$$P_{\mathrm{av}} = \frac{1600\ \mathrm{J}}{8.0\ \mathrm{s}} = 200\ \mathrm{J/s} = 200\ \mathrm{W}$$

EVALUATE The initial kinetic energy was zero, so the work done W must equal the final kinetic energy K [Eq. (10.22)]. This is just as we calculated. We can check our result $P_{\mathrm{av}} = 200$ W by considering the *instantaneous* power $P = \tau_z\,\omega_z$. Because ω_z increases continuously, P increases continuously as well; its value increases from zero at $t = 0$ to $(10\ \mathrm{N{\cdot}m})(40\ \mathrm{rad/s}) = 400$ W at $t = 8.0$ s. Both ω_z and P increase *uniformly* with time, so the *average* power is just half this maximum value, or 200 W.

KEY CONCEPT

If a torque acts on a rigid body, the work done equals torque times angular displacement and the power equals torque times angular velocity.

Video Tutor Solution: **Example 10.8**

Test Your Understanding of Section 10.4

You apply equal torques to two different cylinders. Cylinder 1 has a moment of inertia twice as large as cylinder 2. Each cylinder is initially at rest. After one complete rotation, which cylinder has the greater kinetic energy? (i) Cylinder 1; (ii) cylinder 2; (iii) both cylinders have the same kinetic energy.

10.5 Angular Momentum

Every rotational quantity that we have encountered in Chapters 9 and 10 is the analog of some quantity in the translational motion of a particle. The analog of *momentum* of a particle is **angular momentum,** a vector quantity denoted as $\vec{\boldsymbol{L}}$. Its relationship to momentum $\vec{\boldsymbol{p}}$ (which we'll often call *linear momentum* for clarity) is exactly the same as the relationship of torque to force, $\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}}$. For a particle with constant mass m and velocity $\vec{\boldsymbol{v}}$, the angular momentum is

(10.24)

Angular momentum of a particle relative to origin O of an inertial frame of reference

Position vector of particle relative to O

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} \qquad (10.24)$$

Linear momentum of particle = mass times velocity

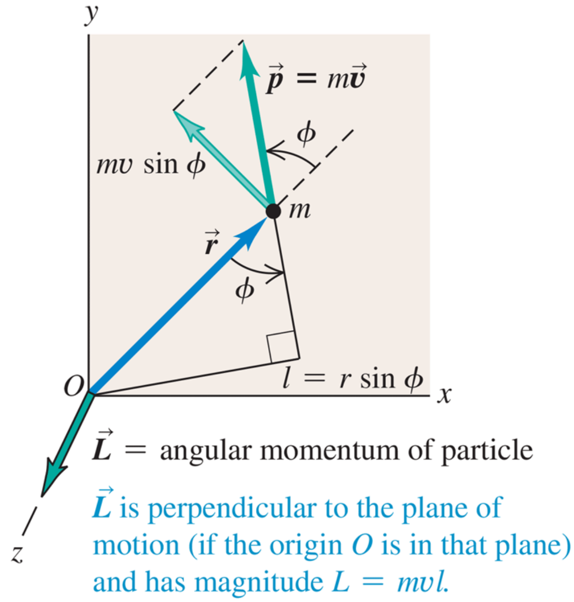
The value of $\vec{\boldsymbol{L}}$ depends on the choice of origin O, since it involves the particle's position vector $\vec{\boldsymbol{r}}$ relative to O. The units of angular momentum are $\text{kg}\cdot\text{m}^2/\text{s}$.

In Fig. 10.23 a particle moves in the xy-plane; its position vector $\vec{\boldsymbol{r}}$ and momentum $\vec{\boldsymbol{p}} = m\vec{\boldsymbol{v}}$ are shown. The angular momentum vector $\vec{\boldsymbol{L}}$ is perpendicular to the xy-plane. The right-hand rule for vector products shows that its direction is along the $+z$-axis, and its magnitude is

(10.25)

$$L = mvr \sin\phi = mvl$$

Figure 10.23



Calculating the angular momentum $\vec{L} = \vec{r} \times m\vec{v} = \vec{r} \times \vec{p}$ of a particle with mass m moving in the xy-plane.

where l is the perpendicular distance from the line of $\vec{v}$ to O. This distance plays the role of "lever arm" for the momentum vector.

When a net force $\vec{F}$ acts on a particle, its velocity and momentum change, so its angular momentum may also change. We can show that the *rate of change* of angular momentum is equal to the torque of the net force. We take the time derivative of Eq. (10.24), using the rule for the derivative of a product:

$$\frac{d\vec{L}}{dt} = \left(\frac{d\vec{r}}{dt} \times m\vec{v}\right) + \left(\vec{r} \times m\frac{d\vec{v}}{dt}\right) = (\vec{v} \times m\vec{v}) + (\vec{r} \times m\vec{a})$$

The first term is zero because it contains the vector product of the vector $\vec{v} = d\vec{r}/dt$ with itself. In the second term we replace $m\vec{a}$ with the net force $\vec{F}$:

(10.26)

$$\frac{d\vec{\boldsymbol{L}}}{dt} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}} = \vec{\boldsymbol{\tau}} \quad \text{(for a particle acted on by net force } \vec{\boldsymbol{F}}\text{)}$$

The rate of change of angular momentum of a particle equals the torque of the net force acting on it. Compare this result to Eq. (8.4): The rate of change $d\vec{\boldsymbol{p}}/dt$ of the *linear* momentum of a particle equals the net force that acts on it.

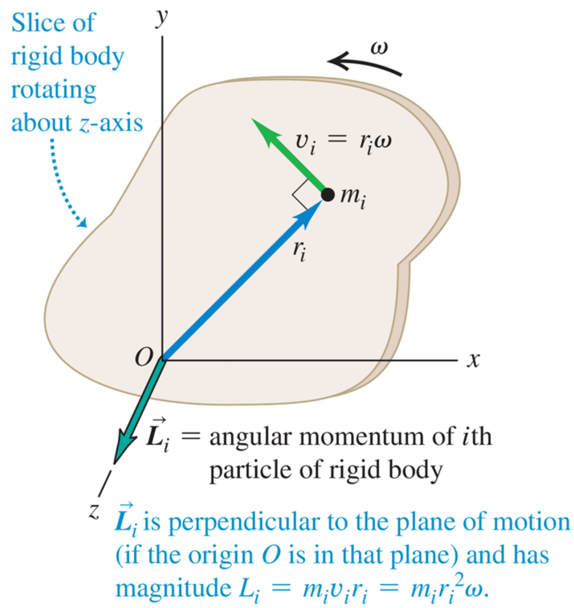
Angular Momentum of a Rigid Body

We can use Eq. (10.25) to find the total angular momentum of a *rigid body* rotating about the z-axis with angular speed ω. First consider a thin slice of the body lying in the xy-plane (Fig. 10.24). Each particle in the slice moves in a circle centered at the origin, and at each instant its velocity $\vec{\boldsymbol{v}}_i$ is perpendicular to its position vector $\vec{\boldsymbol{r}}_i$, as shown. Hence in Eq. (10.25), $\phi = 90^\circ$ for every particle. A particle with mass m_i at a distance r_i from O has a speed v_i equal to $r_i\omega$. From Eq. (10.25) the magnitude L_i of its angular momentum is

(10.27)

$$L_i = m_i\,(r_i\,\omega)\,r_i = m_i\,r_i^2\,\omega$$

Figure 10.24



Calculating the angular momentum of a particle of mass m_i in a rigid body rotating at angular speed ω. (Compare Fig. 10.23.)

The direction of each particle's angular momentum, as given by the right-hand rule for the vector product, is along the $+z$-axis.

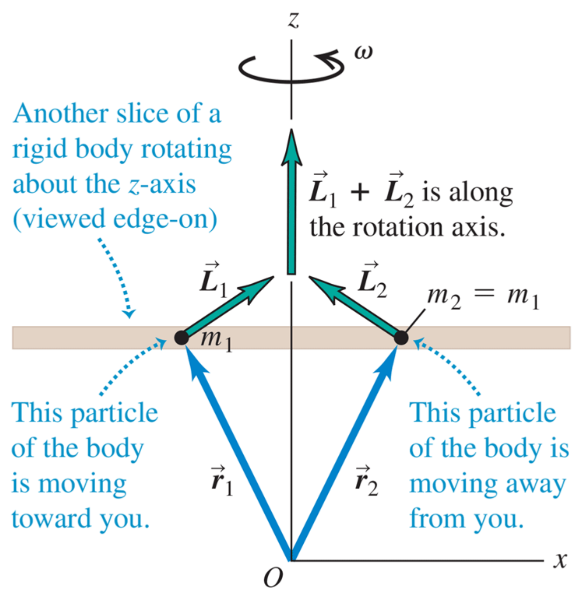
The *total* angular momentum of the slice of the rigid body that lies in the xy-plane is the sum $\sum L_i$ of the angular momenta L_i of all of its particles. From Eq. (10.27),

$$L = \sum L_i = \left(\sum m_i r_i^{\,2}\right)\omega = I\omega$$

where I is the moment of inertia of the slice about the z-axis.

We can do this same calculation for the other slices of the rigid body, all parallel to the xy-plane. For points that do not lie in the xy-plane, a complication arises because the $\vec{r}$ vectors have components in the z-direction as well as in the x- and y-directions; this gives the angular momentum of each particle a component perpendicular to the z-axis. But *if the z-axis is an axis of symmetry*, the perpendicular components for particles on opposite sides of this axis add up to zero (Fig. 10.25). So when a rigid body rotates about an axis of symmetry, its angular momentum vector $\vec{L}$ lies along the symmetry axis, and its magnitude is $L = I\omega$.

Figure 10.25



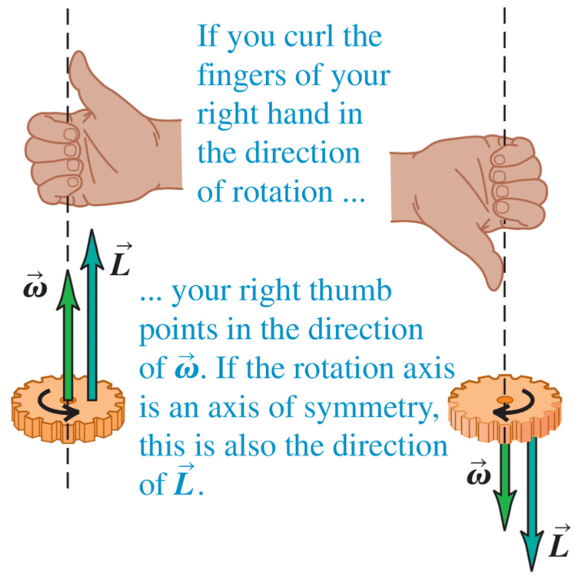
Two particles of the same mass located symmetrically on either side of the rotation axis of a rigid body. The angular momentum vectors $\vec{L}_1$ and $\vec{L}_2$ of the two particles do not lie along the rotation axis, but their vector sum $\vec{L}_1 + \vec{L}_2$ does.

The angular velocity vector $\vec{\omega}$ also lies along the rotation axis, as we saw in Section 9.1. Hence for a rigid body rotating around an axis of symmetry, $\vec{L}$ and $\vec{\omega}$ are in the same direction (Fig. 10.26). So we have the *vector* relationship

(10.28)

$$\underbrace{\vec{L}}_{\substack{\text{Angular momentum of}\\ \text{a rigid body rotating}\\ \text{around a symmetry axis}}} = \overbrace{I}^{\substack{\text{Moment of inertia of rigid}\\ \text{body about symmetry axis}}}\underbrace{\vec{\omega}}_{\text{Angular velocity vector of rigid body}} \tag{10.28}$$

Figure 10.26



For rotation about an axis of symmetry, $\vec{\omega}$ and $\vec{L}$ are parallel and along the axis. The directions of both vectors are given by the right-hand rule (compare Fig. 9.5).

From Eq. (10.26) the rate of change of angular momentum of a particle equals the torque of the net force acting on the particle. For any system of particles (including both rigid and nonrigid bodies), the rate of change of the *total* angular momentum equals the sum of the torques of all forces acting on all the particles. The torques of the *internal* forces add to zero if these forces act along the line from one particle to another, as in Fig. 10.8, and so the sum of the torques includes only the torques of the *external* forces. (We saw a similar cancellation in our discussion of center-of-mass motion in Section 8.5.) So we conclude that

(10.29)

For a system of particles:

$$\underbrace{\sum \vec{\tau}}_{\text{Sum of external torques on the system}} = \underbrace{\frac{d\vec{L}}{dt}}_{\text{Rate of change of total angular momentum } \vec{L} \text{ of system}} \qquad (10.29)$$

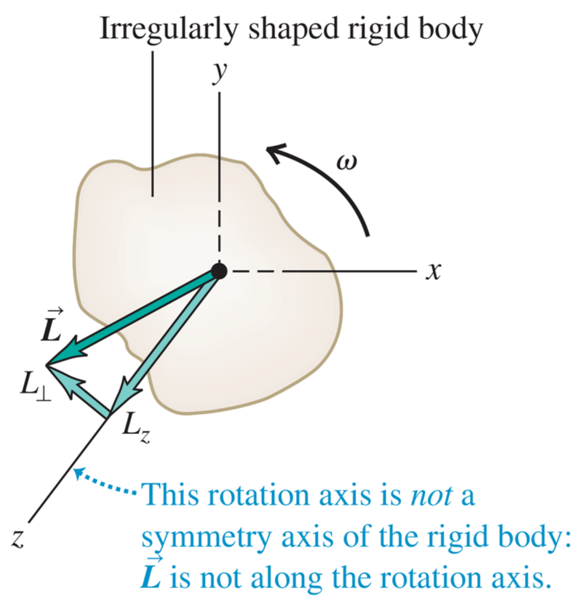
Finally, if the system of particles is a rigid body rotating about a symmetry axis (the z-axis), then $L_z = I\omega_z$ and I is constant. If this axis has a fixed direction in space, then vectors $\vec{\boldsymbol{L}}$ and $\vec{\boldsymbol{\omega}}$ change only in magnitude, not in direction. In that case, $dL_z/dt = I\,d\omega_z/dt = I\alpha_z$, or

$$\sum \tau_z = I\alpha_z$$

which is again our basic relationship for the dynamics of rigid-body rotation. If the body is *not* rigid, I may change; in that case, L changes even when ω is constant. For a nonrigid body, Eq. (10.29) is still valid, even though Eq. (10.7) is not.

When the axis of rotation is *not* a symmetry axis, the angular momentum is in general *not* parallel to the axis (Fig. 10.27). As the rigid body rotates, the angular momentum vector $\vec{\boldsymbol{L}}$ traces out a cone around the rotation axis. Because $\vec{\boldsymbol{L}}$ changes, there must be a net external torque acting on the body even though the angular velocity magnitude ω may be constant. If the body is an unbalanced wheel on a car, this torque is provided by friction in the bearings, which causes the bearings to wear out. "Balancing" a wheel means distributing the mass so that the rotation axis is an axis of symmetry; then $\vec{\boldsymbol{L}}$ points along the rotation axis, and no net torque is required to keep the wheel turning.

Figure 10.27



If the rotation axis of a rigid body is not a symmetry axis, $\vec{L}$ does not in general lie along the rotation axis. Even if $\vec{\omega}$ is constant, the direction of $\vec{L}$ changes and a net torque is required to maintain rotation.

In fixed-axis rotation we often use the term "angular momentum of the body" to refer to only the *component* of $\vec{L}$ along the rotation axis of the body (the z-axis in Fig. 10.27), with a positive or negative sign to indicate the sense of rotation just as with angular velocity.

Example 10.9 Angular momentum and torque

A turbine fan in a jet engine has a moment of inertia of $2.5\ \text{kg}\cdot\text{m}^2$ about its axis of rotation. As the turbine starts up, its angular velocity is given by $\omega_z = (40\ \text{rad/s}^3)t^2$. (a) Find the fan's angular momentum as a function of time, and find its value at $t = 3.0$ s. (b) Find the net torque on the fan as a function of time, and find its value at $t = 3.0$ s.

IDENTIFY and SET UP The fan rotates about its axis of symmetry (the z-axis). Hence the angular momentum vector has only a

z-component L_z, which we can determine from the angular velocity ω_z. Since the direction of angular momentum is constant, the net torque likewise has only a component τ_z along the rotation axis. We'll use Eq. (10.28) to find L_z from ω_z and then Eq. (10.29) to find τ_z.

EXECUTE (a) From Eq. (10.28),

$$\begin{aligned} L_z &= I\omega_z = (2.5 \text{ kg} \cdot \text{m}^2)(40 \text{ rad/s}^3)t^2 \\ &= (100 \text{ kg} \cdot \text{m}^2/\text{s}^3)t^2 \end{aligned}$$

(We dropped the dimensionless quantity "rad" from the final expression.) At $t = 3.0$ s, $L_z = 900$ kg·m^2/s.

(b) From Eq. (10.29),

$$\tau_z = \frac{dL_z}{dt} = (100 \text{ kg·m}^2/\text{s}^3)\,(2t) = (200 \text{ kg·m}^2/\text{s}^3)t$$

At $t = 3.0$ s,

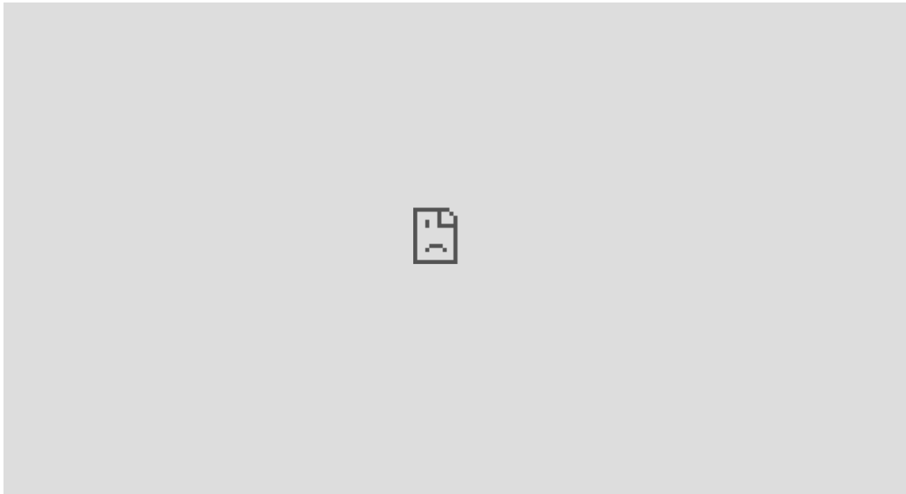
$$\begin{aligned} \tau_z &= (200 \text{ kg} \cdot \text{m}^2/\text{s}^3)(3.0 \text{ s}) \\ &= 600 \text{ kg} \cdot \text{m}^2/\text{s}^2 = 600 \text{ N} \cdot \text{m} \end{aligned}$$

EVALUATE As a check on our expression for τ_z, note that the angular acceleration of the turbine is $\alpha_z = d\omega_z/dt = (40 \text{ rad/s}^3)\,(2t) = (80 \text{ rad/s}^3)t$. Hence from Eq. (10.7), the torque on the fan is $\tau_z = I\alpha_z = (2.5 \text{ kg} \cdot \text{m}^2)(80 \text{ rad/s}^3)t = (200 \text{ kg} \cdot \text{m}^2/\text{s}^3)t$, just as we calculated.

KEY CONCEPT

The angular momentum vector of a rotating rigid body points along the rigid body's rotation axis. The rate of change of angular momentum equals the net external torque on the rigid body.

Video Tutor Solution: **Example 10.9**



Test Your Understanding of Section 10.5

A ball is attached to one end of a piece of string. You hold the other end of the string and whirl the ball in a circle around your hand. (a) If the ball moves at a constant speed, is its linear momentum $\vec{p}$ constant? Why or why not? (b) Is its angular momentum $\vec{L}$ constant? Why or why not?

10.6 Conservation of Angular Momentum

Video Tutor Demo: **Off-Center Collision**

Video Tutor Demo: **Spinning Person Drops Weights**

? We have just seen that angular momentum can be used for an alternative statement of the basic dynamic principle for rotational motion. It also forms the basis for the principle of **conservation of angular momentum.** Like conservation of energy and of linear momentum, this principle is a universal conservation law, valid at all scales from atomic and nuclear systems to the motions of galaxies. This principle follows directly from Eq. (10.29): $\sum \vec{\boldsymbol{\tau}} = d\vec{\boldsymbol{L}}/dt$. If $\sum \vec{\boldsymbol{\tau}} = \mathbf{0}$, then $d\vec{\boldsymbol{L}}/dt = \mathbf{0}$, and $\vec{\boldsymbol{L}}$ is constant.

> ## Conservation of Angular Momentum
> When the net external torque acting on a system is zero, the total angular momentum of the system is constant (conserved).

A circus acrobat, a diver, and an ice skater pirouetting on one skate all take advantage of this principle. Suppose an acrobat has just left a swing; she has her arms and legs extended and is rotating counterclockwise about her center of mass. When she pulls her arms and legs in, her moment of inertia I_{cm} with respect to her center of mass changes from a large value I_1 to a much smaller value I_2. The only external force acting on her is her weight, which has no torque with respect to an axis through her center of mass. So her angular momentum $L_z = I_{\text{cm}}\omega_z$ remains constant, and her angular velocity ω_z increases as I_{cm} decreases. That is,

(10.30)

$$I_1\,\omega_{1z} = I_2\,\omega_{2z} \quad \text{(zero net external torque)}$$

When a skater or ballerina spins with arms outstretched and then pulls her arms in, her angular velocity increases as her moment of inertia decreases. In each case there is conservation of angular momentum in a system in which the net external torque is zero.

When a system has several parts, the internal forces that the parts exert on one another cause changes in the angular momenta of the parts, but the *total* angular momentum doesn't change. Here's an example. Consider two objects A and B that interact with each other but not with anything else, such as the astronauts we discussed in Section 8.2 (see Fig. 8.9). Suppose object A exerts a force $\vec{\boldsymbol{F}}_{A \text{ on B}}$ on object B; the corresponding torque (with respect to whatever point we choose) is $\vec{\boldsymbol{\tau}}_{A \text{ on B}}$. According to Eq. (10.29), this torque is equal to the rate of change of angular momentum of B:

$$\vec{\boldsymbol{\tau}}_{A \text{ on B}} = \frac{d\vec{\boldsymbol{L}}_B}{dt}$$

At the same time, object B exerts a force $\vec{\boldsymbol{F}}_{B \text{ on A}}$ on object A, with a corresponding torque $\vec{\boldsymbol{\tau}}_{B \text{ on A}}$, and

$$\vec{\boldsymbol{\tau}}_{B \text{ on A}} = \frac{d\vec{\boldsymbol{L}}_A}{dt}$$

From Newton's third law, $\vec{\boldsymbol{F}}_{B \text{ on A}} = -\vec{\boldsymbol{F}}_{A \text{ on B}}$. Furthermore, if the forces act along the same line, as in Fig. 10.8, their lever arms with respect to the chosen axis are equal. Thus the *torques* of these two forces are equal and opposite, and $\vec{\boldsymbol{\tau}}_{B \text{ on A}} = -\vec{\boldsymbol{\tau}}_{A \text{ on B}}$. So if we add the two preceding equations, we find

$$\frac{d\vec{\boldsymbol{L}}_A}{dt} + \frac{d\vec{\boldsymbol{L}}_B}{dt} = 0$$

or, because $\vec{\boldsymbol{L}}_A + \vec{\boldsymbol{L}}_B$ is the *total* angular momentum $\vec{\boldsymbol{L}}$ of the system,

(10.31)

$$\frac{d\vec{\boldsymbol{L}}}{dt} = 0 \quad (\textit{zero net external torque})$$

That is, the total angular momentum of the system is constant. The torques of the internal forces can transfer angular momentum from one object to the other, but they can't change the *total* angular momentum of the system (Fig. 10.28).

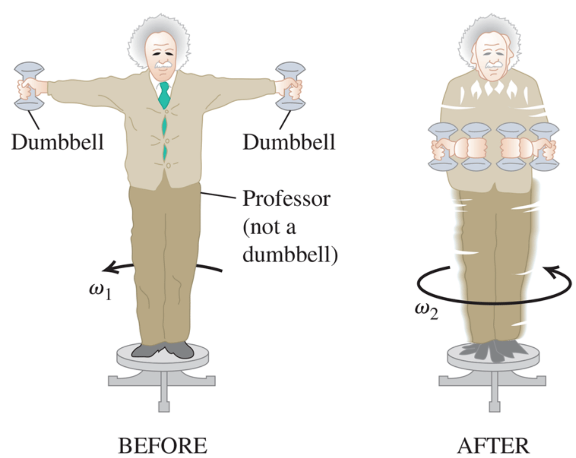
Figure 10.28

A falling cat twists different parts of its body in different directions so that it lands feet first. At all times during this process the angular momentum of the cat as a whole remains zero.

Example 10.10 Anyone can be a ballerina

A physics professor stands at the center of a frictionless turntable with arms outstretched and a 5.0 kg dumbbell in each hand (Fig. 10.29). He is set rotating about the vertical axis, making one revolution in 2.0 s. Find his final angular velocity if he pulls the dumbbells inward to his stomach. His moment of inertia (without the dumbbells) is 3.0 kg·m^2 with arms outstretched and 2.2 kg·m^2 with his hands at his stomach. The dumbbells are 1.0 m from the axis initially and 0.20 m at the end.

Figure 10.29



Fun with conservation of angular momentum.

IDENTIFY, SET UP, and EXECUTE No external torques act about the z-axis, so L_z is constant. We'll use Eq. (10.30) to find the final angular velocity ω_{2z}. The moment of inertia of the system is $I = I_{\text{prof}} + I_{\text{dumbbells}}$. We treat each dumbbell as a particle of mass m that contributes mr^2 to $I_{\text{dumbbells}}$, where r is the perpendicular distance from the axis to the dumbbell. Initially we have

$$\begin{aligned} I_1 &= 3.0\ \text{kg}\cdot\text{m}^2 + 2(5.0\ \text{kg})(1.0\ \text{m})^2 = 13\ \text{kg}\cdot\text{m}^2 \\ \omega_{1z} &= \frac{1\ \text{rev}}{2.0\ \text{s}} = 0.50\ \text{rev/s} \end{aligned}$$

The final moment of inertia is

$$I_2 = 2.2\ \text{kg}\cdot\text{m}^2 + 2(5.0\ \text{kg})(0.20\ \text{m})^2 = 2.6\ \text{kg}\cdot\text{m}^2$$

From Eq. (10.30), the final angular velocity is

$$\omega_{2z} = \frac{I_1}{I_2}\,\omega_{1z} = \frac{13\ \text{kg}\cdot\text{m}^2}{2.6\ \text{kg}\cdot\text{m}^2}(0.50\ \text{rev/s}) = 2.5\ \text{rev/s} = 5\omega_{1z}$$

Can you see why we didn't have to change "revolutions" to "radians" in this calculation?

EVALUATE The angular momentum remained constant, but the angular velocity increased by a factor of 5, from $\omega_{1z} = (0.50\ \text{rev/s})\ \times (2\pi\ \text{rad/rev}) = 3.14\ \text{rad/s}$ to $\omega_{2z} = (2.5\ \text{rev/s})(2\pi\ \text{rad}/rev)\ = 15.7\ \text{rad/s}$. The initial and final kinetic energies are then

$$\begin{aligned} K_1 &= \tfrac{1}{2}\,I_1\,{\omega_{1z}}^2 = \tfrac{1}{2}\,(13\ \text{kg}\cdot\text{m}^2)(3.14\ \text{rad/s})^2 = 64\ \text{J} \\ K_2 &= \tfrac{1}{2}\,I_2\,{\omega_{2z}}^2 = \tfrac{1}{2}\,(2.6\ \text{kg}\cdot\text{m}^2)(15.7\ \text{rad/s})^2 = 320\ \text{J} \end{aligned}$$

The fivefold increase in kinetic energy came from the work that the professor did in pulling his arms and the dumbbells inward.

KEY CONCEPT

If there is zero net external torque on a rigid body, its angular momentum is conserved. If the body changes shape so that its moment of inertia changes, its angular velocity changes to keep the angular momentum constant.

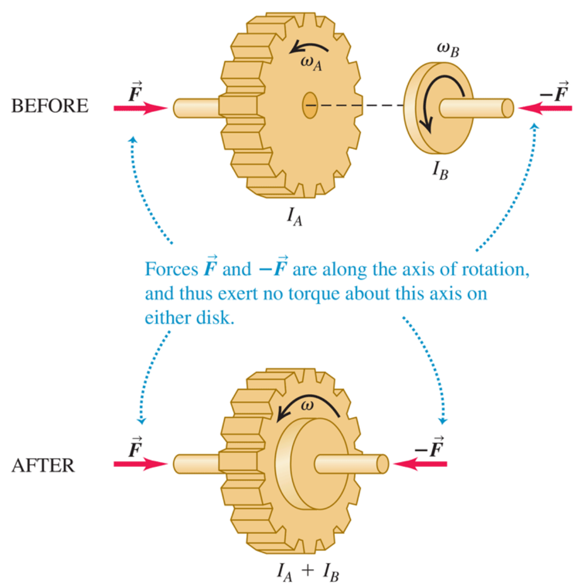
Video Tutor Solution: **Example 10.10**

Example 10.11 A rotational "collision"

WITH VARIATION PROBLEMS

Figure 10.30 shows two disks: an engine flywheel (A) and a clutch plate (B) attached to a transmission shaft. Their moments of inertia are I_A and I_B; initially, they are rotating in the same direction with constant angular speeds ω_A and ω_B, respectively. We push the disks together with forces acting along the axis, so as not to apply any torque on either disk. The disks rub against each other and eventually reach a common angular speed ω. Derive an expression for ω.

Figure 10.30



When the net external torque is zero, angular momentum is conserved.

IDENTIFY, SET UP, and EXECUTE There are no external torques, so the only torque acting on either disk is the torque applied by the other disk. Hence the total angular momentum of the system of two disks is conserved. At the end they rotate together as one object with total moment of inertia $I = I_A + I_B$ and angular speed ω. Figure 10.30 shows that all angular velocities are in the same direction, so we can regard ω_A, ω_B, and ω as components of angular velocity along the rotation axis. Conservation of angular momentum gives

$$\begin{aligned} I_A\omega_A + I_B\omega_B &= (I_A + I_B)\omega \\ \omega &= \frac{I_A\omega_A + I_B\omega_B}{I_A + I_B} \end{aligned}$$

EVALUATE This "collision" is analogous to a completely inelastic collision (see Section 8.3). When two objects in translational motion along the same axis collide and stick, the linear momentum of the system is conserved. Here two objects in *rotational* motion around the same axis "collide" and stick, and the *angular* momentum of the system is conserved.

The kinetic energy of a system decreases in a completely inelastic collision. Here kinetic energy is lost because nonconservative (friction) internal forces act while the two disks rub together. Suppose flywheel A has a mass of 2.0 kg, a radius of 0.20 m, and an initial angular speed of 50 rad/s (about 500 rpm), and clutch plate B has a mass of 4.0 kg, a radius of 0.10 m, and an initial angular speed of 200 rad/s. Can you show that the final kinetic energy is only two-thirds of the initial kinetic energy?

KEY CONCEPT

In processes that conserve angular momentum, the kinetic energy can change if nonconservative forces act.

Video Tutor Solution: Example 10.11

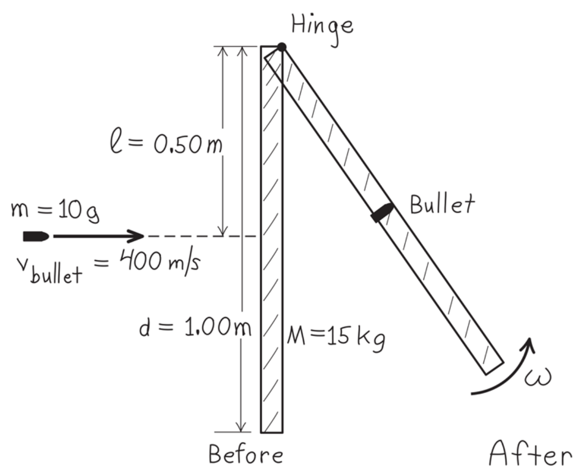
Example 10.12 Angular momentum in a crime bust

WITH VARIATION PROBLEMS

A door 1.00 m wide, of mass 15 kg, can rotate freely about a vertical axis through its hinges. A bullet with a mass of 10 g and a speed of 400 m/s strikes the center of the door, in a direction perpendicular to the plane of the door, and embeds itself there. Find the door's angular speed. Is kinetic energy conserved?

IDENTIFY and SET UP We consider the door and bullet as a system. There is no external torque about the hinge axis, so angular momentum about this axis is conserved. Figure 10.31 shows our sketch. The initial angular momentum is that of the bullet, as given by Eq. (10.25). The final angular momentum is that of a rigid body composed of the door and the embedded bullet. We'll equate these quantities and solve for the resulting angular speed ω of the door and bullet.

Figure 10.31



The swinging door seen from above.

EXECUTE From Eq. (10.25), the initial angular momentum of the bullet is

$$L = mvl = (0.010\ \text{kg})(400\ \text{m/s})(0.50\ \text{m}) = 2.0\ \text{kg}\cdot\text{m}^2/\text{s}$$

The final angular momentum is $I\omega$, where $I = I_{\text{door}} + I_{\text{bullet}}$. From Table 9.2, case (d), for a door of width $d = 1.00$ m,

$$I_{\text{door}} = \frac{Md^2}{3} = \frac{(15\ \text{kg})(1.00\ \text{m})^2}{3} = 5.0\ \text{kg}\cdot\text{m}^2$$

The moment of inertia of the bullet (with respect to the axis along the hinges) is

$$I_{\text{bullet}} = ml^2 = (0.010\ \text{kg})(0.50\ \text{m})^2 = 0.0025\ \text{kg}\cdot\text{m}^2$$

Conservation of angular momentum requires that $mvl = I\omega$, or

$$\omega = \frac{mvl}{I} = \frac{2.0\ \text{kg}\cdot\text{m}^2/\text{s}}{5.0\ \text{kg}\cdot\text{m}^2 + 0.0025\ \text{kg}\cdot\text{m}^2} = 0.40\ \text{rad/s}$$

The initial and final kinetic energies are

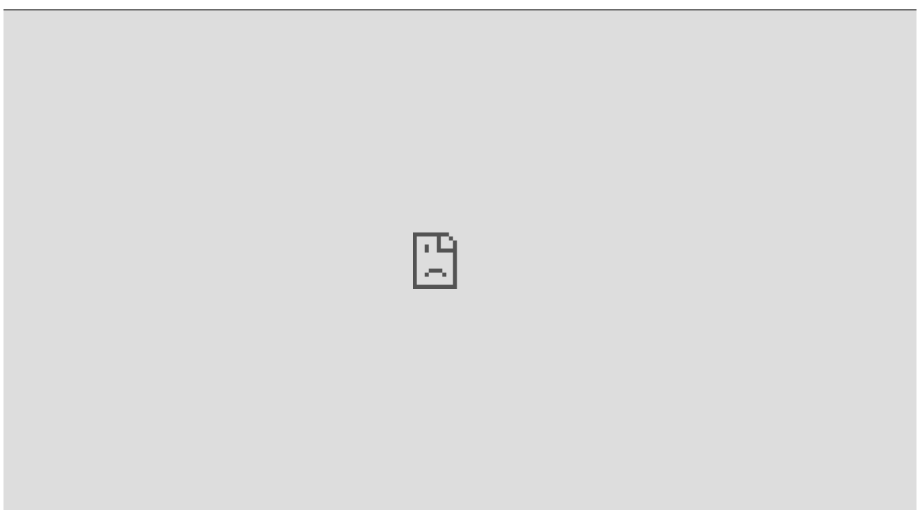
$$\begin{aligned} K_1 &= \tfrac{1}{2}\, m v^2 = \tfrac{1}{2}\,(0.010\ \mathrm{kg})(400\ \mathrm{m/s})^2 = 800\ \mathrm{J} \\ K_2 &= \tfrac{1}{2}\, I\omega^2 = \tfrac{1}{2}\,(5.0025\ \mathrm{kg\cdot m^2})(0.40\ \mathrm{rad/s})^2 = 0.40\ \mathrm{J} \end{aligned}$$

EVALUATE The final kinetic energy is only $\frac{1}{2000}$ of the initial value! We did not expect kinetic energy to be conserved: The collision is inelastic because nonconservative friction forces act during the impact. The door's final angular speed is quite slow: At 0.40 rad/s, it takes 3.9 s to swing through 90° $(\pi/2$ radians).

KEY CONCEPT

The total angular momentum of a system that includes a rigid body and a particle is the sum of the angular momenta for the rigid body and for the particle. You can find the magnitude of the angular momentum of a particle about a rotation axis by multiplying the magnitude of its linear momentum by the perpendicular distance from the axis to the line of the particle's velocity.

Video Tutor Solution: **Example 10.12**



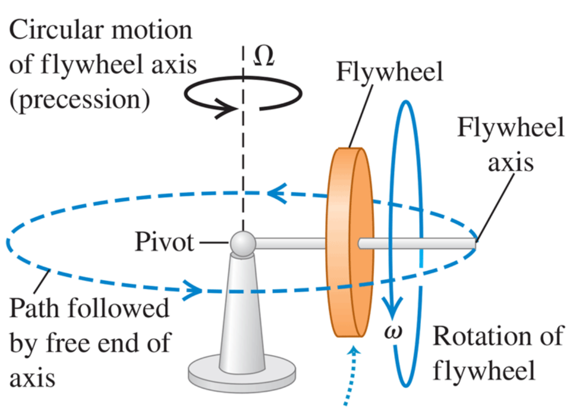
Test Your Understanding of Section 10.6

If the polar ice caps were to melt completely due to global warming, the melted ice would redistribute itself over the earth. This change would cause the length of the day (the time needed for the earth to rotate once on its axis) to (i) increase; (ii) decrease; (iii) remain the same. (*Hint:* Use angular momentum ideas. Assume that the sun, moon, and planets exert negligibly small torques on the earth.)

10.7 Gyroscopes and Precession

In all the situations we've looked at so far in this chapter, the axis of rotation either has stayed fixed or has moved and kept the same direction (such as rolling without slipping). But a variety of new physical phenomena, some quite unexpected, can occur when the axis of rotation changes direction. For example, consider a toy gyroscope that's supported at one end (Fig. 10.32). If we hold it with the flywheel axis horizontal and let go, the free end of the axis simply drops owing to gravity—*if* the flywheel isn't spinning. But if the flywheel *is* spinning, what happens is quite different. One possible motion is a steady circular motion of the axis in a horizontal plane, combined with the spin motion of the flywheel about the axis. This surprising, nonintuitive motion of the axis is called **precession.** Precession is found in nature as well as in rotating machines such as gyroscopes. As you read these words, the earth itself is precessing; its spin axis (through the north and south poles) slowly changes direction, going through a complete cycle of precession every 26,000 years.

Figure 10.32



When the flywheel and its axis are stationary, they will fall to the table surface. When the flywheel spins, it and its axis "float" in the air while moving in a circle about the pivot.

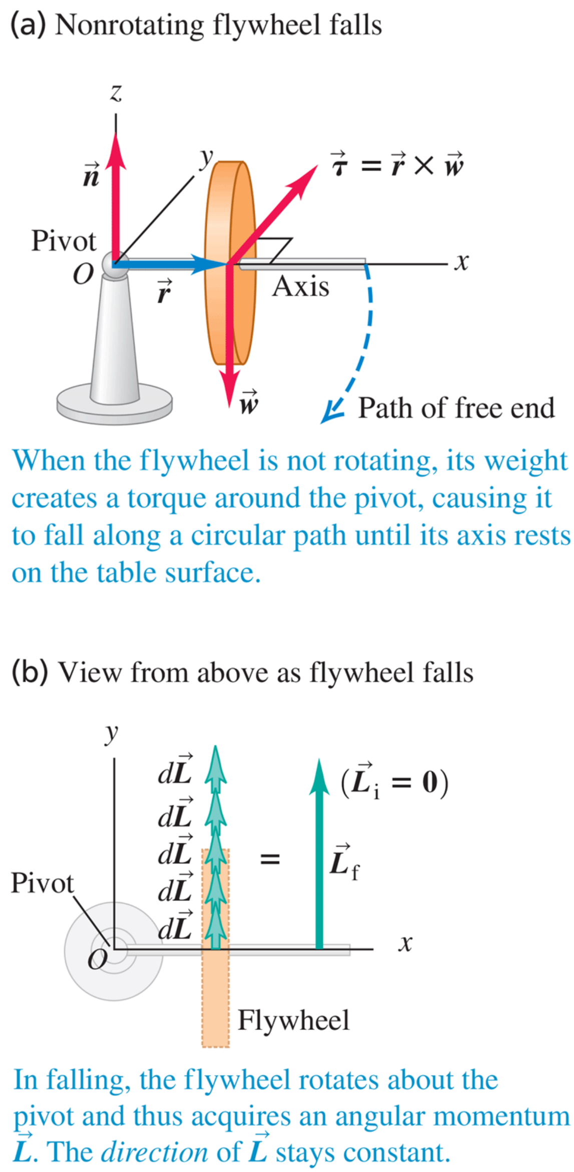
A gyroscope supported at one end. The horizontal circular motion of the flywheel and axis is called precession. The angular speed of precession is Ω.

To study this strange phenomenon of precession, we must remember that angular velocity, angular momentum, and torque are all *vector* quantities. In particular, we need the general relationship between the net torque $\sum \vec{\boldsymbol{\tau}}$ that acts on an object and the rate of change of the object's angular momentum $\vec{\boldsymbol{L}}$, given by Eq. (10.29), $\sum \vec{\boldsymbol{\tau}} = d\vec{\boldsymbol{L}}/dt$. Let's first apply this equation to the case in which the flywheel is *not* spinning (Fig. 10.33a). We take the origin O at the pivot and assume that the flywheel is symmetrical, with mass M and moment of inertia I about the flywheel axis. The flywheel axis is initially along the x-axis. The only external forces on the gyroscope are the normal force $\vec{\boldsymbol{n}}$ acting at the pivot (assumed to be frictionless) and the weight $\vec{\boldsymbol{w}}$ of the flywheel that acts at its center of mass, a distance r from the pivot. The normal force has zero torque with respect to the pivot, and the weight has a torque $\vec{\boldsymbol{\tau}}$ in the y-direction, as shown in Fig. 10.33a. Initially, there is no rotation, and the initial angular momentum $\vec{\boldsymbol{L}}_{\mathrm{i}}$ is zero. From Eq. (10.29) the *change* $d\vec{\boldsymbol{L}}$ in angular momentum in a short time interval dt following this is

(10.32)

$$d\vec{\boldsymbol{L}} = \vec{\boldsymbol{\tau}}\, dt$$

Figure 10.33



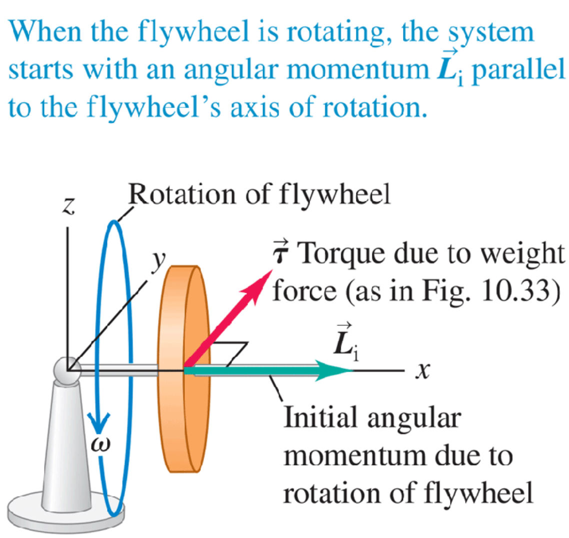
(a) If the flywheel in Fig. 10.32 is initially not spinning, its initial angular momentum is zero. (b) In each successive time interval dt, the torque produces a change $d\vec{L} = \vec{\tau}\, dt$ in the angular momentum. The flywheel acquires an angular momentum $\vec{L}$ in the same direction as $\vec{\tau}$, and the flywheel axis falls.

This change is in the y-direction because $\vec{\tau}$ is. As each additional time interval dt elapses, the angular momentum changes by additional increments $d\vec{L}$ in the y-direction because the direction of the torque is constant (Fig. 10.33b). The steadily increasing horizontal angular momentum means that the gyroscope rotates downward faster and faster around the y-axis until it hits either the stand or the table on which it sits.

Now let's see what happens if the flywheel *is* spinning initially, so the initial angular momentum $\vec{L}_i$ is not zero (Fig. 10.34a). Since the flywheel rotates around its symmetry axis, $\vec{L}_i$ lies along this axis. But each change in angular momentum $d\vec{L}$ is perpendicular to the flywheel axis because the torque $\vec{\tau} = \vec{r} \times \vec{w}$ is perpendicular to that axis (Fig. 10.34b). This causes the *direction* of $\vec{L}$ to change, but not its magnitude. The changes $d\vec{L}$ are always in the horizontal xy-plane, so the angular momentum vector and the flywheel axis with which it moves are always horizontal. That is, the axis doesn't fall—it precesses.

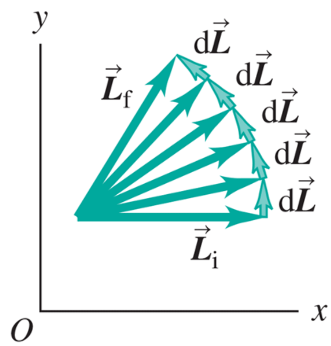
Figure 10.34

(a) Rotating flywheel



(b) View from above

Now the effect of the torque is to cause the angular momentum to precess around the pivot. The gyroscope circles around its pivot without falling.



(a) The flywheel is spinning initially with angular momentum $\vec{L}_i$. The forces (not shown) are the same as those in Fig. 10.33a. (b) Because the initial angular momentum is not zero, each change $d\vec{L} = \vec{\tau}\, dt$ in angular momentum is perpendicular to $\vec{L}$. As a result, the magnitude of $\vec{L}$ remains the same but its direction changes continuously.

If this still seems mystifying to you, think about a ball attached to a string. If the ball is initially at rest and you pull the string toward you, the ball moves toward you also. But if the ball is initially moving and you continuously pull the string in a direction perpendicular to the ball's motion, the ball moves in a circle around your hand; it does not approach your hand at all. In the first case the ball has zero linear momentum $\vec{p}$ to start with; when you apply a force $\vec{F}$ toward you for a time dt, the ball acquires a momentum $d\vec{p} = \vec{F}\, dt$, which is also toward you. But if the ball already has linear momentum $\vec{p}$, a change in momentum $d\vec{p}$ that's perpendicular to $\vec{p}$ changes the direction of motion, not the speed. Replace $\vec{p}$ with $\vec{L}$ and $\vec{F}$ with $\vec{\tau}$ in this argument, and you'll see that precession is simply the rotational analog of uniform circular motion.

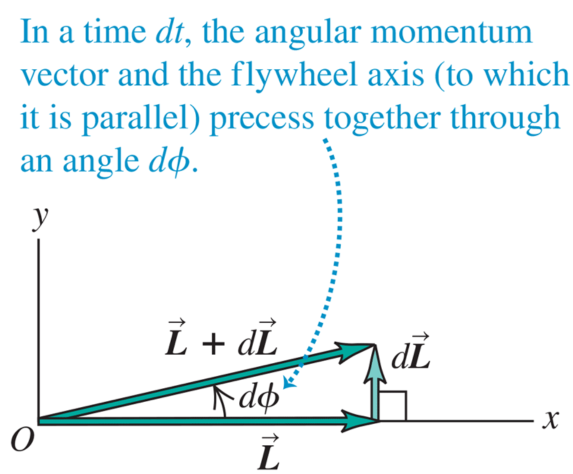
At the instant shown in Fig. 10.34a, the gyroscope has angular momentum $\vec{L}$. A short time interval dt later, the angular momentum is

$\vec{L} + d\vec{L}$; the infinitesimal change in angular momentum is $d\vec{L} = \vec{\tau}\,dt$, which is perpendicular to $\vec{L}$. As the vector diagram in Fig. 10.35 shows, this means that the flywheel axis of the gyroscope has turned through a small angle $d\phi$ given by $d\phi = \left|d\vec{L}\right| / \left|\vec{L}\right|$. The rate at which the axis moves, $d\phi/dt$, is called the **precession angular speed;** denoting this quantity by Ω, we find

(10.33)

$$\Omega = \frac{d\phi}{dt} = \frac{\left|d\vec{L}\right| / \left|\vec{L}\right|}{dt} = \frac{\tau_z}{L_z} = \frac{wr}{I\omega}$$

Figure 10.35



Detailed view of part of Fig. 10.34b.

Thus the precession angular speed is *inversely* proportional to the angular speed of spin about the axis. A rapidly spinning gyroscope precesses slowly; if friction in its bearings causes the flywheel to slow down, the precession angular speed *increases*! The precession angular speed of the earth is very slow (1 rev/26, 000 yr) because its spin angular momentum L_z is large and the torque τ_z, due to the gravitational influences of the moon and sun, is relatively small.

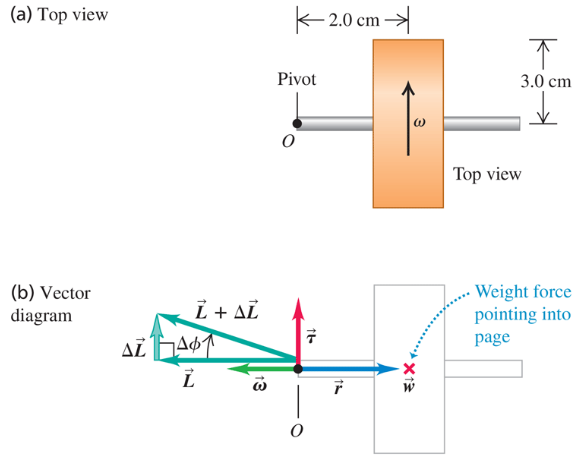
As a gyroscope precesses, its center of mass moves in a circle with radius r in a horizontal plane. Its vertical component of acceleration is zero, so the upward normal force $\vec{n}$ exerted by the pivot must be just equal in magnitude to the weight. The circular motion of the center of mass with angular speed Ω requires a force $\vec{F}$ directed toward the center of the circle, with magnitude $F = M\Omega^2 r$. This force must also be supplied by the pivot.

One key assumption that we made in our analysis of the gyroscope was that the angular momentum vector $\vec{L}$ is associated with only the spin of the flywheel and is purely horizontal. But there will also be a vertical component of angular momentum associated with the precessional motion of the gyroscope. By ignoring this, we've tacitly assumed that the precession is *slow*—that is, that the precession angular speed Ω is very much less than the spin angular speed ω. As Eq. (10.33) shows, a large value of ω automatically gives a small value of Ω, so this approximation is reasonable. When the precession is not slow, additional effects show up, including an up-and-down wobble or *nutation* of the flywheel axis that's superimposed on the precessional motion. You can see nutation occurring in a gyroscope as its spin slows down, so that Ω increases and the vertical component of $\vec{L}$ can no longer be ignored.

Example 10.13 A precessing gyroscope

Figure 10.36a shows a top view of a spinning, cylindrical gyroscope wheel. The pivot is at O, and the mass of the axle is negligible. (a) As seen from above, is the precession clockwise or counterclockwise? (b) If the gyroscope takes 4.0 s for one revolution of precession, what is the angular speed of the wheel?

Figure 10.36



In which direction and at what speed does this gyroscope precess?

IDENTIFY and SET UP We'll determine the direction of precession by using the right-hand rule as in Fig. 10.34, which shows the same kind of gyroscope as Fig. 10.36. We'll use the relationship between precession angular speed Ω and spin angular speed ω, Eq. (10.33), to find ω.

EXECUTE (a) The right-hand rule shows that $\vec{\boldsymbol{\omega}}$ and $\vec{\boldsymbol{L}}$ are to the left in Fig. 10.36b. The weight $\vec{\boldsymbol{w}}$ points into the page in this top view and acts at the center of mass (denoted by $\times$ in the figure). The torque $\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{w}}$ is toward the top of the page, so $d\vec{\boldsymbol{L}}/dt$ is also toward the top of the page. Adding a small $d\vec{\boldsymbol{L}}$ to the initial vector $\vec{\boldsymbol{L}}$ changes the direction of $\vec{\boldsymbol{L}}$ as shown, so the precession is clockwise as seen from above.

(b) Be careful not to confuse ω and Ω! The precession angular speed is $\Omega = (1 \text{ rev})/(4.0 \text{ s}) = (2\pi \text{ rad})/(4.0 \text{ s}) = 1.57 \text{ rad/s}$. The weight is mg, and if the wheel is a solid, uniform cylinder, its moment of inertia about its symmetry axis is $I = \frac{1}{2}mR^2$. From Eq. (10.33),

$$
\begin{aligned}
\omega &= \frac{wr}{I\Omega} = \frac{mgr}{(mR^2/2)\Omega} = \frac{2gr}{R^2\Omega} \\
&= \frac{2(9.8\ \mathrm{m/s}^2)(2.0 \times 10^{-2}\ \mathrm{m})}{(3.0 \times 10^{-2}\ \mathrm{m})^2\ (1.57\ \mathrm{rad/s})} \\
&= 280\ \mathrm{rad/s} = 2600\ \mathrm{rev/min}
\end{aligned}
$$

EVALUATE The precession angular speed Ω is only about 0.6% of the spin angular speed ω, so this is an example of slow precession.

KEY CONCEPT

A spinning rigid body will precess if the net external torque on the rigid body is perpendicular to the body's angular momentum vector.

Video Tutor Solution: **Example 10.13**

Test Your Understanding of Section 10.7

Suppose the mass of the flywheel in Fig. 10.34 is doubled but all other dimensions and the spin angular speed remain the same. What effect would this change have on the precession angular speed Ω? (i) Ω would increase by a factor of 4; (ii) Ω would double; (iii) Ω would be unaffected; (iv) Ω would be one-half as much; (v) Ω would be one-quarter as much.

Chapter 10 Summary

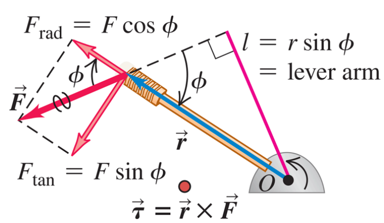
Torque: When a force $\vec{\boldsymbol{F}}$ acts on an object, the torque of that force with respect to a point O has a magnitude given by the product of the force magnitude F and the lever arm l. More generally, torque is a vector $\vec{\boldsymbol{\tau}}$ equal to the vector product of $\vec{\boldsymbol{r}}$ (the position vector of the point at which the force acts) and $\vec{\boldsymbol{F}}$. (See Example 10.1.)

(10.2)

$$\tau = Fl = rF \sin \phi = F_{\tan} r$$

(10.3)

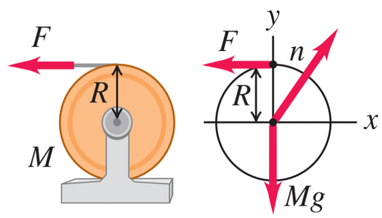
$$\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}}$$



Rotational dynamics: The rotational analog of Newton's second law says that the net torque acting on an object equals the product of the object's moment of inertia and its angular acceleration. (See Examples 10.2 and 10.3.)

(10.7)

$$\sum \tau_z = I\alpha_z$$



Combined translation and rotation: If a rigid body is both moving through space and rotating, its motion can be regarded as translational motion of the center of mass plus rotational motion about an axis through the center of mass. Thus the kinetic energy is a sum of translational and rotational kinetic energies. For dynamics, Newton's second law describes the motion of the center of mass, and the rotational equivalent of Newton's second law describes rotation about the center of mass. In the case of rolling without slipping, there is a special relationship between the motion of the center of mass and the rotational motion. (See Examples 10.4, 10.5, 10.6 and 10.7.)

(10.8)

$$K = \frac{1}{2} M {v_{\mathrm{cm}}}^2 + \frac{1}{2} I_{\mathrm{cm}} \omega^2$$

(10.12)

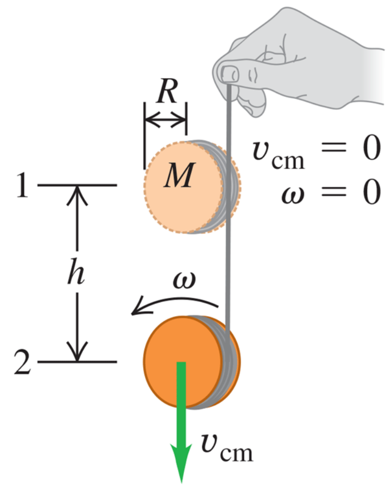
$$\sum \vec{\boldsymbol{F}}_{\mathrm{ext}} = M \vec{\boldsymbol{a}}_{\mathrm{cm}}$$

(10.13)

$$\sum \tau_z = I_{\mathrm{cm}} \alpha_z$$

(10.11)

$$v_{\mathrm{cm}} = R\omega$$
(rolling without slipping)



Work done by a torque: A torque that acts on a rigid body as it rotates does work on that body. The work can be expressed as an integral of the torque. The work–energy theorem says that the total rotational work done on a rigid body is equal to the change in rotational kinetic energy. The power, or rate at which the torque does work, is the product of the torque and the angular velocity (See Example 10.8.)

(10.20)

$$W = \int_{\theta_1}^{\theta_2} \tau_z \, d\theta$$

(10.21)

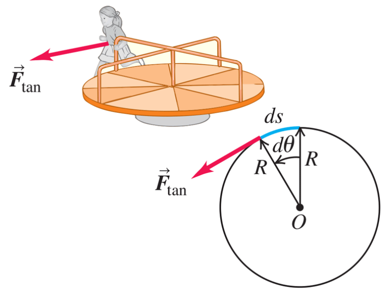
$$W = \tau_z (\theta_2 - \theta_1) = \tau_z \Delta\theta$$
(constant torque only)

(10.22)

$$W_{\text{tot}} = \frac{1}{2} I \omega_2{}^2 - \frac{1}{2} I \omega_1{}^2$$

(10.23)

$$P = \tau_z \omega_z$$



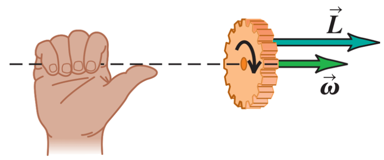
Angular momentum: The angular momentum of a particle with respect to point O is the vector product of the particle's position vector $\vec{r}$ relative to O and its momentum $\vec{p} = m\vec{v}$. When a symmetrical object rotates about a stationary axis of symmetry, its angular momentum is the product of its moment of inertia and its angular velocity vector $\vec{\omega}$. If the object is not symmetrical or the rotation (z) axis is not an axis of symmetry, the component of angular momentum along the rotation axis is $I\omega_z$. (See Example 10.9.)

(10.24)

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$$
(particle)

(10.28)

$$\vec{L} = I\vec{\omega}$$
(rigid body rotating
about axis of symmetry)



Rotational dynamics and angular momentum: The net external torque on a system is equal to the rate of change of its angular momentum. If the net external torque on a system is zero, the total angular momentum of the system is constant (conserved). (See Examples 10.10, 10.11, 10.12 and 10.13.)

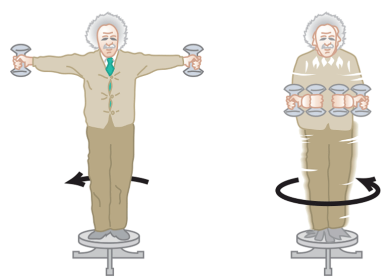
(10.29)

$$\sum \vec{\boldsymbol{\tau}} = \frac{d\vec{\boldsymbol{L}}}{dt}$$

(10.31)

$$\frac{d\vec{\boldsymbol{L}}}{dt} = \mathbf{0}$$

(zero net external torque)



Guided Practice: Dynamics of Rotational Motion

For assigned homework and other learning materials, go to
Mastering Physics.

Key Example Variation Problems

Be sure to review Examples 10.2 and 10.3 (Section 10.2) before attempting these problems.

VP10.3.1 In the cylinder and cable apparatus of Example 10.2, you apply a force to the cable so that a point on the horizontal part of the cable accelerates to the left at $0.60\,\mathrm{m/s^2}$. What are the magnitudes of (a) the angular acceleration of the cylinder, (b) the torque that the cable exerts on the cylinder, and (c) the force that you exert on the cable?

VP10.3.2 In the cylinder, cable, and block apparatus of Example 10.3, you replace the solid cylinder with a thin-walled, hollow cylinder of mass M and radius R. Find (a) the acceleration of the falling block and (b) the tension in the cable as the block falls.

VP10.3.3 A bucket of mass m is hanging from the free end of a rope whose other end is wrapped around a drum (radius R, mass M) that can rotate with negligible friction about a stationary horizontal axis. The drum is not a uniform cylinder and has unknown moment of inertia. When you release the bucket from rest, you find that it has a downward acceleration of magnitude a. What are (a) the tension in the cable between the drum and the bucket and (b) the moment of inertia of the drum about its rotation axis?

VP10.3.4 In the cylinder, cable, and block apparatus of Example10.3, you attach an electric motor to the axis of the cylinder of mass M and radius R and turn the motor on. As a result the block of mass m moves upward with an upward acceleration of magnitude a. What are (a) the tension in the cable between the cylinder and the block, (b) the magnitude of the torque that the cable exerts on the cylinder, and (c) the magnitude of the torque that the motor exerts on the cylinder?

Be sure to review Examples 10.6 and 10.7 (Section 10.3) before attempting these problems.

VP10.7.1 In the primitive yo-yo apparatus of Example10.6, you replace the solid cylinder with a hollow cylinder of mass M, outer radius $R/2$. Find (a) the downward acceleration of the hollow cylinder and (b) the tension in the string.

VP10.7.2 A thin-walled, hollow sphere of mass M rolls without slipping down a ramp that is inclined at an angle β to the horizontal. Find (a) the acceleration of the sphere, (b) the magnitude of the friction force that the ramp exerts on the sphere, and (c) the magnitude of the torque that this force exerts on the sphere.

VP10.7.3 You redo the primitive yo-yo experiment of Example 10.6, but instead of holding the free end of the string stationary, you move your hand vertically so that the tension in the string equals $2Mg/3$. (a) What is the magnitude of the vertical acceleration of the yo-yo's center of mass? Does it accelerate upward or downward? (b) What is the angular acceleration of the yo-yo around its axis?

VP10.7.4 You place a solid cylinder of mass M on a ramp that is inclined at an angle β to the horizontal. The coefficient of static friction for the cylinder on the ramp is μ_s. (a) If the cylinder rolls downhill without slipping, what is the magnitude of the friction force that the ramp exerts on the cylinder? (b) You find by varying the angle of the ramp that the cylinder rolls without slipping if β is less than a certain critical value but the cylinder slips if β is greater than this critical value. What is this critical value of β?

Be sure to review Examples 10.11 and 10.12 (Section 10.6) before attempting these problems.

VP10.12.1 In the situation shown in Example 10.11, suppose disk A has moment of inertia I_A and initial angular speed ω_A, while disk B has moment of inertia $I_A/4$ and initial angular speed $\omega_A/2$. Initially disks A and B are rotating in the *same* direction. (a) What is the final common angular speed of the two disks? (b) What fraction of the initial rotational kinetic energy remains as rotational kinetic energy after the disks have come to their final common angular speed?

VP10.12.2 In the situation shown in Example10.11, suppose disk A has moment of inertia I_A and initial angular speed ω_A, while disk B has moment of inertia $I_A/4$ and initial angular speed $\omega_A/2$. Initially disks A and B are rotating in *opposite* directions. (a) What is the final common angular speed of the two disks? (b) What fraction of the initial rotational kinetic energy remains as rotational kinetic energy after the disks have come to their final common angular speed?

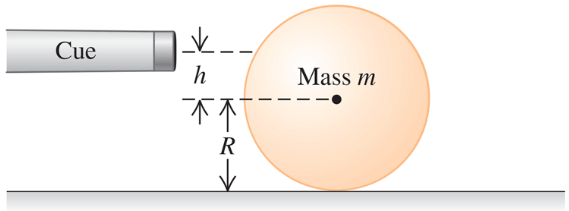
VP10.12.3 Suppose that instead of hitting the center of the door, the bullet in Example10.12 strikes the door at the edge farthest away from the hinge and embeds itself there. (a) What is the angular speed of the door just after the bullet embeds itself? (b) What fraction of the initial kinetic energy of the bullet remains as kinetic energy after the collision?

VP10.12.4 A thin-walled, hollow sphere of mass M and radius R is free to rotate around a vertical shaft that passes through the center of the sphere. Initially the sphere is at rest. A small ball of clay of the same mass M moving horizontally at speed v grazes the surface of the sphere at its equator. After grazing the surface, the ball of clay is moving at speed $v/2$. (a) What is the angular momentum of the ball of clay about the shaft before it grazes the surface? After it grazes the surface? (b) What is the angular speed of the sphere after being grazed by the ball of clay? (c) What fraction of the ball of clay's initial kinetic energy remains as the combined kinetic energy of the sphere and the ball of clay?

Bridging Problem: Billiard Physics

A cue ball (a uniform solid sphere of mass m and radius R) is at rest on a level pool table. Using a pool cue, you give the ball a sharp, horizontal hit of magnitude F at a height h above the center of the ball (Fig. 10.37). The force of the hit is much greater than the friction force f that the table surface exerts on the ball. The hit lasts for a short time Δt. (a) For what value of h will the ball roll without slipping? (b) If you hit the ball dead center ($h = 0$), the ball will slide across the table for a while, but eventually it will roll without slipping. What will the speed of its center of mass be then?

Figure 10.37



Hitting a cue ball with a cue.

Solution Guide

IDENTIFY and SET UP

1. Draw a free-body diagram for the ball for the situation in part (a), including your choice of coordinate axes. Note that the cue exerts both an impulsive force on the ball and an impulsive torque around the center of mass.
2. The cue force applied for a time Δt gives the ball's center of mass a speed v_{cm}, and the cue torque applied for that same time gives the ball an angular speed ω. How must v_{cm} and ω be related for the ball to roll without slipping?
3. Draw two free-body diagrams for the ball in part (b): one showing the forces during the hit and the other showing the forces after the hit but before the ball is rolling without slipping.
4. What is the angular speed of the ball in part (b) just after the hit? While the ball is sliding, does v_{cm} increase or decrease? Does ω increase or decrease? What is the relationship between v_{cm} and ω when the ball is finally rolling without slipping?

EXECUTE

5. In part (a), use the impulse–momentum theorem to find the speed of the ball's center of mass immediately after the hit. Then

use the rotational version of the impulse–momentum theorem to find the angular speed immediately after the hit. (*Hint:* To write the rotational version of the impulse–momentum theorem, remember that the relationship between torque and angular momentum is the same as that between force and linear momentum.)

6. Use your results from step 5 to find the value of h that will cause the ball to roll without slipping immediately after the hit.
7. In part (b), again find the ball's center-of-mass speed and angular speed immediately after the hit. Then write Newton's second law for the translational motion and rotational motion of the ball as it slides. Use these equations to write expressions for v_{cm} and ω as functions of the elapsed time t since the hit.
8. Using your results from step 7, find the time t when v_{cm} and ω have the correct relationship for rolling without slipping. Then find the value of v_{cm} at this time.

EVALUATE

9. If you have access to a pool table, test the results of parts (a) and (b) for yourself!
10. Can you show that if you used a hollow cylinder rather than a solid ball, you would have to hit the top of the cylinder to cause rolling without slipping as in part (a)?

Video Tutor Solution: **Chapter 10 Bridging Problem**

Questions/Exercises/Problems: Dynamics of Rotational Motion

•, **••**, **•••**: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

Discussion Questions

Q10.1 Can a single force applied to an object change both its translational and rotational motions? Explain.

Q10.2 Suppose you could use wheels of any type in the design of a soapbox-derby racer (an unpowered, four-wheel vehicle that coasts from rest down a hill). To conform to the rules on the total weight of the vehicle and rider, should you design with large massive wheels or small light wheels? Should you use solid wheels or wheels with most of the mass at the rim? Explain.

Q10.3 Serious bicyclists say that if you reduce the weight of a bike, it is more effective if you do so in the wheels rather than in the frame. Why would reducing weight in the wheels make it easier on the bicyclist than reducing the same amount in the frame?

Q10.4 The harder you hit the brakes while driving forward, the more the front end of your car will move down (and the rear end move up). Why? What happens when cars accelerate forward? Why do drag racers not use front-wheel drive only?

Q10.5 When an acrobat walks on a tightrope, she extends her arms straight out from her sides. She does this to make it easier for her to catch herself if she should tip to one side or the other. Explain how this works. [*Hint:* Think about Eq. (10.7).]

Q10.6 When you turn on an electric motor, it takes longer to come up to final speed if a grinding wheel is attached to the shaft. Why?

Q10.7 The work done by a force is the product of force and distance. The torque due to a force is the product of force and distance. Does this mean that torque and work are equivalent? Explain.

Q10.8 A valued client brings a treasured ball to your engineering firm, wanting to know whether the ball is solid or hollow. He has tried tapping on it, but that has given insufficient information. Design a simple, inexpensive experiment that you could perform quickly, without injuring the precious ball, to find out whether it is solid or hollow.

Q10.9 You make two versions of the same object out of the same material having uniform density. For one version, all the dimensions are exactly twice as great as for the other one. If the same torque acts on both versions, giving the smaller version angular acceleration α, what will be the angular acceleration of the larger version in terms of α?

Q10.10 Two identical masses are attached to frictionless pulleys by very light strings wrapped around the rim of the pulley and are released from rest. Both pulleys have the same mass and same diameter, but one is solid and the other is a hoop. As the masses fall, in which case is the tension in the string greater, or is it the same in both cases? Justify your answer.

Q10.11 The force of gravity acts on the baton in Fig. 10.11, and forces produce torques that cause a body's angular velocity to change. Why, then, is the angular velocity of the baton in the figure constant?

Q10.12 Without slipping, a certain solid uniform ball rolls at speed v on a horizontal surface and then up a hill to a maximum height h_0. How does the maximum height change (in terms of h_0) if you make only the following changes: (a) double the ball's diameter, (b) double its mass, (c) double both its diameter and mass, (d) double its angular speed at the bottom of the hill?

Q10.13 A wheel is rolling without slipping on a horizontal surface. In an inertial frame of reference in which the surface is at rest, is there any point on the wheel that has a velocity that is purely vertical? Is there any point that has a horizontal velocity component opposite to the velocity of the center of mass? Explain. Do your answers change if the wheel is slipping as it rolls? Why or why not?

Q10.14 A hoop, a uniform solid cylinder, a spherical shell, and a uniform solid sphere are released from rest at the top of an incline. What is the order in which they arrive at the bottom of the incline? Does it matter whether or not the masses and radii of the objects are all the same? Explain.

Q10.15 A ball is rolling along at speed v without slipping on a horizontal surface when it comes to a hill that rises at a constant angle above the horizontal. In which case will it go higher up the hill: if the hill has enough friction to prevent slipping or if the hill is perfectly smooth? Justify your answer in both cases in terms of energy conservation and in terms of Newton's second law.

Q10.16 You are standing at the center of a large horizontal turntable in a carnival funhouse. The turntable is set rotating on frictionless bearings, and it rotates freely (that is, there is no motor driving the turntable). As you walk toward the edge of the turntable, what happens to the combined angular momentum of you and the turntable? What happens to the rotation speed of the turntable? Explain.

Q10.17 **Global Warming.** If the earth's climate continues to warm, ice near the poles will melt, and the water will be added to the oceans. What effect will this have on the length of the day? Justify your answer.

Q10.18 If two spinning objects have the same angular momentum, do they necessarily have the same rotational kinetic energy? If they have the same rotational kinetic energy, do they necessarily have the same angular momentum? Explain.

Q10.19 A student is sitting on a frictionless rotating stool with her arms outstretched as she holds equal heavy weights in each hand. If she suddenly lets go of the weights, will her angular speed increase, stay the same, or decrease? Explain.

Q10.20 A point particle travels in a straight line at constant speed, and the closest distance it comes to the origin of coordinates is a distance l. With respect to this origin, does the particle have nonzero angular momentum? As the particle moves along its straight-line path, does its angular momentum with respect to the origin change?

Q10.21 In Example 10.10 (Section 10.6) the angular speed ω changes, and this must mean that there is nonzero angular acceleration. But there is no torque about the rotation axis if the forces the professor applies to the weights are directly, radially inward. Then, by Eq. (10.7), α_z must be zero. Explain what is wrong with this reasoning that leads to this apparent contradiction.

Q10.22 In Example 10.10 (Section 10.6) the rotational kinetic energy of the professor and dumbbells increases. But since there are no external torques, no work is being done to change the rotational kinetic energy. Then, by Eq. (10.22), the kinetic energy must remain the same! Explain what is wrong with this reasoning, which leads to an apparent contradiction. Where *does* the extra kinetic energy come from?

Q10.23 As discussed in Section 10.6, the angular momentum of a circus acrobat is conserved as she tumbles through the air. Is her *linear* momentum conserved? Why or why not?

Q10.24 If you stop a spinning raw egg for the shortest possible instant and then release it, the egg will start spinning again. If you do the same to a hard-boiled egg, it will remain stopped. Try it. Explain it.

Q10.25 A helicopter has a large main rotor that rotates in a horizontal plane and provides lift. There is also a small rotor on the tail that rotates in a vertical plane. What is the purpose of the tail rotor? (*Hint:* If there were no tail rotor, what would happen when the pilot changed the angular speed of the main rotor?) Some helicopters have no tail rotor, but instead have two large main rotors that rotate in a horizontal plane. Why is it important that the two main rotors rotate in opposite directions?

Q10.26 In a common design for a gyroscope, the flywheel and flywheel axis are enclosed in a light, spherical frame with the flywheel at the center of the frame. The gyroscope is then balanced on top of a pivot so that the flywheel is directly above the pivot. Does the gyroscope precess if it is released while the flywheel is spinning? Explain.

Q10.27 A gyroscope is precessing about a vertical axis. What happens to the precession angular speed if the following changes are made, with all other variables remaining the same? (a) The angular speed of the spinning flywheel is doubled; (b) the total weight is doubled; (c) the moment of inertia about the axis of the spinning flywheel is doubled; (d) the distance from the pivot to the center of gravity is doubled. (e) What happens if all of the variables in parts (a) through (d) are doubled? In each case justify your answer.

Q10.28 A gyroscope takes 3.8 s to precess 1.0 revolution about a vertical axis. Two minutes later, it takes only 1.9 s to precess 1.0 revolution. No one has touched the gyroscope. Explain.

Q10.29 A gyroscope is precessing as in Fig. 10.32. What happens if you gently add some weight to the end of the flywheel axis farthest from the pivot?

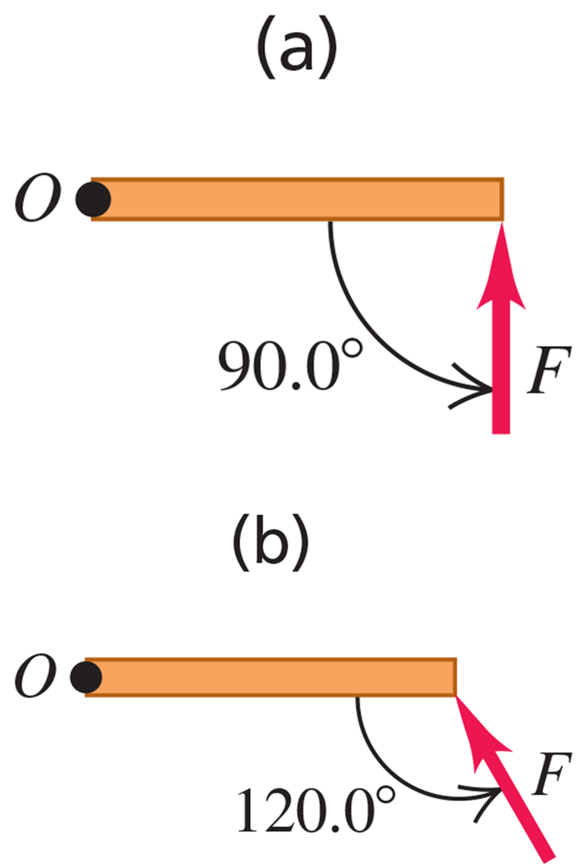
Q10.30 A bullet spins on its axis as it emerges from a rifle. Explain how this prevents the bullet from tumbling and keeps the streamlined end pointed forward.

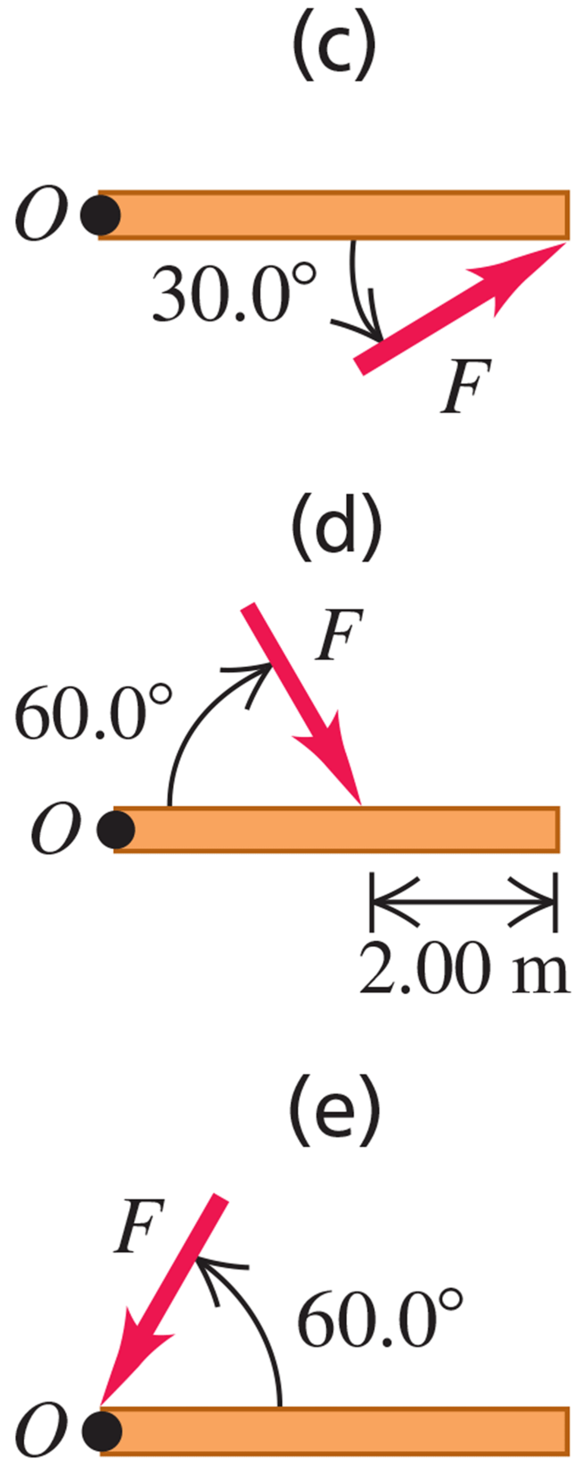
Exercises

Section 10.1 Torque

10.1 • Calculate the torque (magnitude and direction) about point O due to the force $\vec{F}$ in each of the cases sketched in Fig. E10.1. In each case, both the force $\vec{F}$ and the rod lie in the plane of the page, the rod has length 4.00 m, and the force has magnitude $F = 10.0$ N.

Figure E10.1

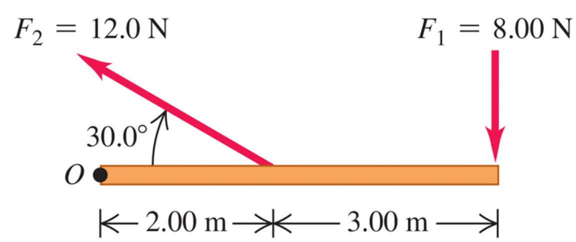



(c)
O
30.0°
F
(d)
F
60.0°
O
2.00 m
(e)
F
60.0°
O

(f)

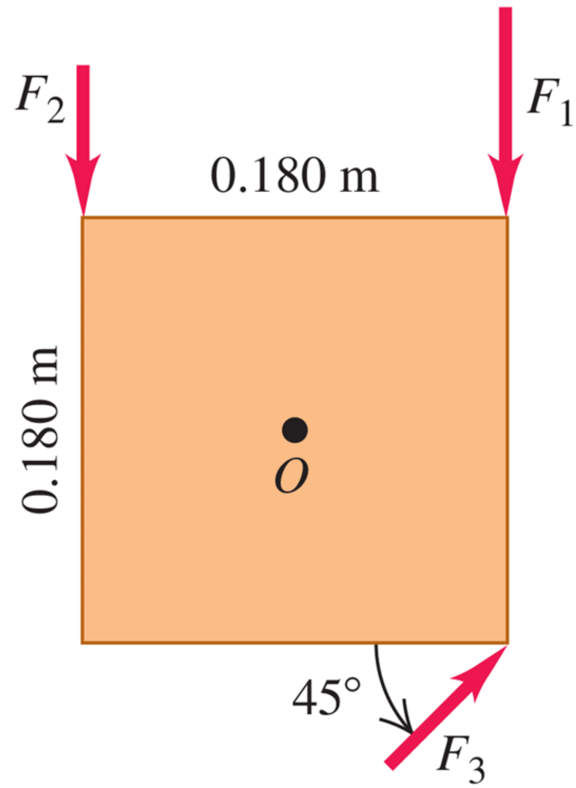
10.2 • Calculate the net torque about point O for the two forces applied as in Fig. E10.2. The rod and both forces are in the plane of the page.

Figure E10.2



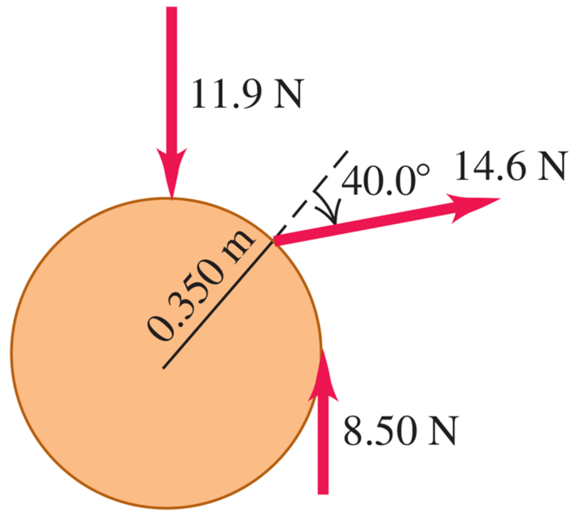
10.3 •• A square metal plate 0.180 m on each side is pivoted about an axis through point O at its center and perpendicular to the plate (Fig. E10.3). Calculate the net torque about this axis due to the three forces shown in the figure if the magnitudes of the forces are $F_1 = 18.0$ N, $F_2 = 26.0$ N, and $F_3 = 14.0$ N. The plate and all forces are in the plane of the page.

Figure E10.3



10.4 • Three forces are applied to a wheel of radius 0.350 m, as shown in Fig. E10.4. One force is perpendicular to the rim, one is tangent to it, and the other one makes a 40.0° angle with the radius. What is the net torque on the wheel due to these three forces for an axis perpendicular to the wheel and passing through its center?

Figure E10.4



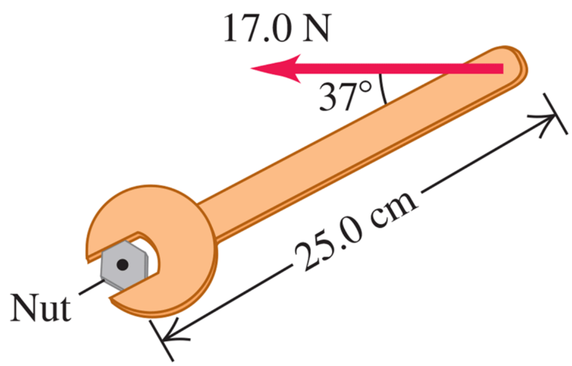
10.5 • One force acting on a machine part is $\overrightarrow{F} = (-5.00\ \mathrm{N})\hat{i} + (4.00\ \mathrm{N})\hat{j}$. The vector from the origin to the point where the force is applied is $\vec{\boldsymbol{r}} = (-0.450\ \mathrm{m})\hat{i} + (0.150\ \mathrm{m})\hat{j}$. (a) In a sketch, show $\vec{\boldsymbol{r}}$, $\vec{\boldsymbol{F}}$, and the origin. (b) Use the right-hand rule to determine the direction of the torque. (c) Calculate the vector torque for an axis at the origin produced by this force. Verify that the direction of the torque is the same as you obtained in part (b).

10.6 • A metal bar is in the xy-plane with one end of the bar at the origin. A force $\vec{\boldsymbol{F}} = (7.00\ \mathrm{N})\hat{i} + (-3.00\ \mathrm{N})\hat{j}$ is applied to the bar at the point $x = 3.00\ \mathrm{m}$, $y = 4.00\ \mathrm{m}$. (a) In terms of unit vectors $\hat{i}$ and $\hat{j}$, what is the position vector $\vec{\boldsymbol{r}}$ for the point where the force is applied? (b) What are the magnitude and direction of the torque with respect to the origin produced by $\vec{\boldsymbol{F}}$?

10.7 • A machinist is using a wrench to loosen a nut. The wrench is 25.0 cm long, and he exerts a 17.0 N force at the end of the handle at 37° with the handle (Fig. E10.7). (a) What torque does the machinist exert about the center of the nut? (b)

What is the maximum torque he could exert with a force of this magnitude, and how should the force be oriented?

Figure E10.7



Section 10.2 Torque and Angular Acceleration for a Rigid Body

10.8 •• A uniform disk with mass 40.0 kg and radius 0.200 m is pivoted at its center about a horizontal, frictionless axle that is stationary. The disk is initially at rest, and then a constant force $F = 30.0$ N is applied tangent to the rim of the disk. (a) What is the magnitude v of the tangential velocity of a point on the rim of the disk after the disk has turned through 0.200 revolution? (b) What is the magnitude a of the resultant acceleration of a point on the rim of the disk after the disk has turned through 0.200 revolution?

10.9 •• The flywheel of an engine has moment of inertia $1.60\ \text{kg}\cdot\text{m}^2$ about its rotation axis. What constant torque is required to bring it up to an angular speed of 400 rev/min in 8.00 s, starting from rest?

10.10 • A cord is wrapped around the rim of a solid uniform wheel 0.250 m in radius and of mass 9.20 kg. A steady horizontal pull of 40.0 N to the right is exerted on the cord, pulling it off tangentially from the wheel. The wheel is mounted on frictionless bearings on a horizontal axle through its center. (a) Compute the angular acceleration of the wheel and the acceleration of the part of the cord that has already been pulled off the wheel. (b) Find the magnitude and direction of the force that the axle exerts on the wheel. (c) Which of the answers in parts (a) and (b) would change if the pull were upward instead of horizontal?

10.11 •• A machine part has the shape of a solid uniform sphere of mass 225 g and diameter 3.00 cm. It is spinning about a frictionless axle through its center, but at one point on its equator it is scraping against metal, resulting in a friction force of 0.0200 N at that point. (a) Find its angular acceleration. (b) How long will it take to decrease its rotational speed by 22.5 rad/s ?

10.12 •• **CP** A stone is suspended from the free end of a wire that is wrapped around the outer rim of a pulley, similar to what is shown in Fig. 10.10. The pulley is a uniform disk with mass 10.0 kg and radius 30.0 cm and turns on frictionless bearings. You measure that the stone travels 12.6 m in the first 3.00 s starting from rest. Find (a) the mass of the stone and (b) the tension in the wire.

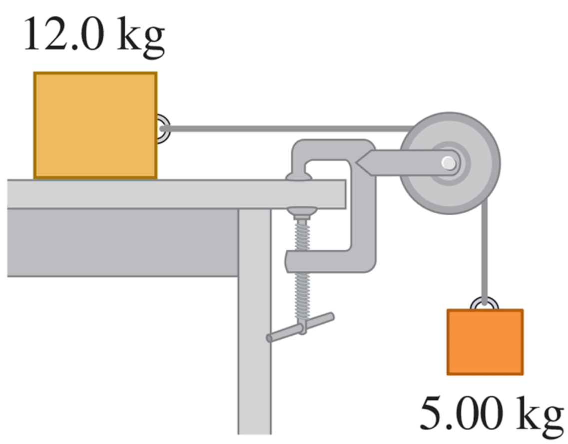
10.13 •• **CP** A 2.00 kg textbook rests on a frictionless, horizontal surface. A cord attached to the book passes over a pulley whose diameter is 0.150 m, to a hanging book with mass 3.00 kg. The system is released from rest, and the books are observed to move 1.20 m in 0.800 s. (a) What is the tension in each part of the cord? (b) What is the moment of inertia of the pulley about its rotation axis?

10.14 •• **CP** A 15.0 kg bucket of water is suspended by a very light rope wrapped around a solid uniform cylinder 0.300 m in diameter with mass 12.0 kg. The cylinder pivots on a frictionless axle through its center. The bucket is released from rest at the top of a well and falls 10.0 m to the water. (a) What is the tension in the rope while the bucket is falling? (b) With what speed does the bucket strike the water? (c) What is the time of fall? (d) While the bucket is falling, what is the force exerted on the cylinder by the axle?

10.15 • A wheel rotates without friction about a stationary horizontal axis at the center of the wheel. A constant tangential force equal to 80.0 N is applied to the rim of the wheel. The wheel has radius 0.120 m. Starting from rest, the wheel has an angular speed of 12.0 rev/s after 2.00 s. What is the moment of inertia of the wheel?

10.16 •• A 12.0 kg box resting on a horizontal, frictionless surface is attached to a 5.00 kg weight by a thin, light wire that passes over a frictionless pulley (Fig. E10.16). The pulley has the shape of a uniform solid disk of mass 2.00 kg and diameter 0.500 m. After the system is released, find (a) the tension in the wire on both sides of the pulley, (b) the acceleration of the box, and (c) the horizontal and vertical components of the force that the axle exerts on the pulley.

Figure E10.16



10.17 •• **CP** A solid cylinder with radius 0.140 m is mounted on a frictionless, stationary axle that lies along the cylinder axis. The cylinder is initially at rest. Then starting at $t = 0$ a constant horizontal force of 3.00 N is applied tangential to the surface of the cylinder. You measure the angular displacement $\theta - \theta_0$ of the cylinder as a function of the time t since the force was first applied. When you plot $\theta - \theta_0$ (in radians) as a function of t^2 (in s^2), your data lie close to a straight line. If the slope of this line is 16.0 rad/s^2, what is the moment of inertia of the cylinder for rotation about the axle?

10.18 •• **CP** Two spheres are rolling without slipping on a horizontal floor. They are made of different materials, but each has mass 5.00 kg and radius 0.120 m. For each the translational speed of the center of mass is 4.00 m/s^2. Sphere A is a uniform solid sphere and sphere B is a thin-walled, hollow sphere. How much work, in joules, must be done on each sphere to bring it to rest? For which sphere is a greater magnitude of work required? Explain. (The spheres continue to roll without slipping as they slow down.)

Section 10.3 Rigid-Body Rotation About a Moving Axis

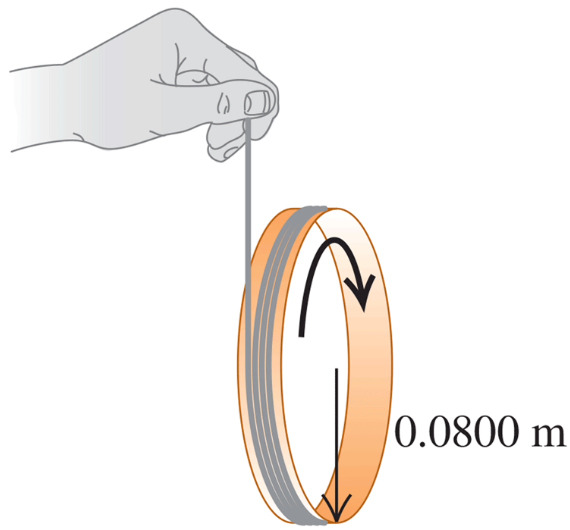
10.19 • A 2.20 kg hoop 1.20 m in diameter is rolling to the right without slipping on a horizontal floor at a steady 2.60 rad/s. (a) How fast is its center moving? (b) What is the total kinetic energy of the hoop? (c) Find the velocity vector of each of the following points, as viewed by a person at rest on the ground: (i) the highest point on the hoop; (ii) the lowest point on the hoop; (iii) a point on the right side of the hoop, midway between the top and the bottom. (d) Find the velocity vector for each of the points in part (c), but this time as viewed by someone moving along with the same velocity as the hoop.

10.20 •• Example 10.7 calculates the friction force needed for a uniform sphere to roll down an incline without slipping. The incline is at an angle β above the horizontal. And the example discusses that the friction is static. (a) If the maximum friction force is given by $f = \mu_s n$, where n is the normal force that the ramp exerts on the sphere, in terms of β what is the minimum coefficient of static friction needed if the sphere is to roll without slipping? (b) Based on your result in part (a), what does the minimum required μ_s become in the limits $\beta \rightarrow 90^\circ$ and $\beta \rightarrow 0^\circ$?

10.21 • What fraction of the total kinetic energy is rotational for the following objects rolling without slipping on a horizontal surface? (a) A uniform solid cylinder; (b) a uniform sphere; (c) a thin-walled, hollow sphere; (d) a hollow cylinder with outer radius R and inner radius $R/2$.

10.22 •• A string is wrapped several times around the rim of a small hoop with radius 8.00 cm and mass 0.180 kg. The free end of the string is held in place and the hoop is released from rest (Fig. E10.22). After the hoop has descended 75.0 cm, calculate (a) the angular speed of the rotating hoop and (b) the speed of its center.

Figure E10.22



10.23 •• A solid ball is released from rest and slides down a hillside that slopes downward at 65.0° from the horizontal. (a) What minimum value must the coefficient of static friction between the hill and ball surfaces have for no slipping to occur? (b) Would the coefficient of friction calculated in part (a) be sufficient to prevent a hollow ball (such as a soccer ball) from slipping? Justify your answer. (c) In part (a), why did we use the coefficient of static friction and not the coefficient of kinetic friction?

10.24 •• A hollow, spherical shell with mass 2.00 kg rolls without slipping down a 38.0° slope. (a) Find the acceleration, the friction force, and the minimum coefficient of static friction needed to prevent slipping. (b) How would your answers to part (a) change if the mass were doubled to 4.00 kg?

10.25 •• A 392 N wheel comes off a moving truck and rolls without slipping along a highway. At the bottom of a hill it is rotating at 25.0 rad/s. The radius of the wheel is 0.600 m, and its moment of inertia about its rotation axis is $0.800MR^2$. Friction does work on the wheel as it rolls up the hill to a stop, a height h above the bottom of the hill; this work has absolute value 2600 J. Calculate h.

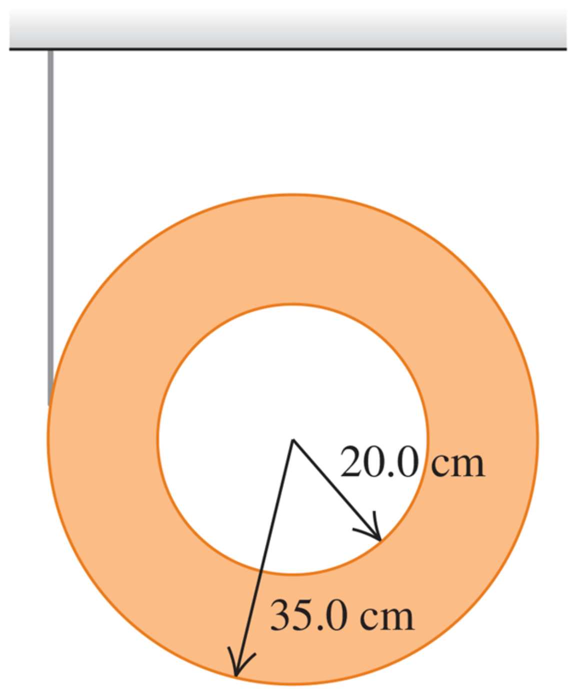
10.26 •• A uniform marble rolls down a symmetrical bowl, starting from rest at the top of the left side. The top of each side is a distance h above the bottom of the bowl. The left half of the bowl is rough enough to cause the marble to roll without slipping, but the right half has no friction because it is coated with oil. (a) How far up the smooth side will the marble go, measured vertically from the bottom? (b) How high would the marble go if both sides were as rough as the left side? (c) How do you account for the fact that the marble goes *higher* with friction on the right side than without friction?

10.27 •• At a typical bowling alley the distance from the line where the ball is released (foul line) to the first pin is 60 ft. Estimate the time it takes the ball to reach the pins after you release it, if it rolls without slipping and has a constant translational speed. Assume the ball weighs 12 lb and has a diameter of 8.5 in. (a) Use your estimate to calculate the rotation rate of the ball, in rev/s. (b) What is its total kinetic energy in joules and what fraction of the total is its rotational kinetic energy? Ignore the finger holes and treat the bowling ball as a uniform sphere.

10.28 • Two uniform solid balls are rolling without slipping at a constant speed. Ball 1 has twice the diameter, half the mass, and one-third the speed of ball 2. The kinetic energy of ball 2 is 27.0 J. What is the kinetic energy of ball 1?

10.29 •• A thin, light string is wrapped around the outer rim of a uniform hollow cylinder of mass 4.75 kg having inner and outer radii as shown in Fig. E10.29. The cylinder is then released from rest. (a) How far must the cylinder fall before its center is moving at 6.66 m/s? (b) If you just dropped this cylinder without any string, how fast would its center be moving when it had fallen the distance in part (a)? (c) Why do you get two different answers when the cylinder falls the same distance in both cases?

Figure E10.29



10.30 •• **A Ball Rolling Uphill.** A bowling ball rolls without slipping up a ramp that slopes upward at an angle β to the horizontal (see Example 10.7 in Section 10.3). Treat the ball as a uniform solid sphere, ignoring the finger holes. (a) Draw the free-body diagram for the ball. Explain why the friction force must be directed *uphill*. (b) What is the acceleration of the center of mass of the ball? (c) What minimum coefficient of static friction is needed to prevent slipping?

10.31 •• A size-5 soccer ball of diameter 22.6 cm and mass 426 g rolls up a hill without slipping, reaching a maximum height of 5.00 m above the base of the hill. We can model this ball as a thin-walled hollow sphere. (a) At what rate was it rotating at the base of the hill? (b) How much rotational kinetic energy did it have then? Neglect rolling friction and assume the system's total mechanical energy is conserved.

Section 10.4 Work and Power in Rotational Motion

10.32 • An engine delivers 175 hp to an aircraft propeller at 2400 rev/min. (a) How much torque does the aircraft engine provide? (b) How much work does the engine do in one revolution of the propeller?

10.33 • A playground merry-go-round has radius 2.40 m and moment of inertia 2100 $kg \cdot m^2$ about a vertical axle through its center, and it turns with negligible friction. (a) A child applies an 18.0 N force tangentially to the edge of the merry-go-round for 15.0 s. If the merry-go-round is initially at rest, what is its angular speed after this 15.0 s interval? (b) How much work did the child do on the merry-go-round? (c) What is the average power supplied by the child?

10.34 •• An electric motor consumes 9.00 kJ of electrical energy in 1.00 min. If one-third of this energy goes into heat and other forms of internal energy of the motor, with the rest going to the motor output, how much torque will this engine develop if you run it at 2500 rpm?

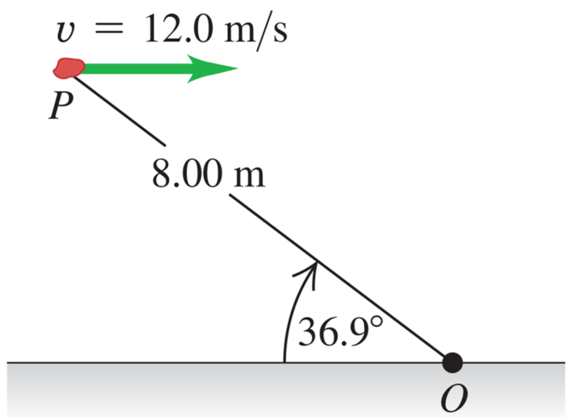
10.35 • A 2.80 kg grinding wheel is in the form of a solid cylinder of radius 0.100 m. (a) What constant torque will bring it from rest to an angular speed of 1200 rev/min in 2.5 s? (b) Through what angle has it turned during that time? (c) Use Eq. (10.21) to calculate the work done by the torque. (d) What is the grinding wheel's kinetic energy when it is rotating at 1200 rev/min? Compare your answer to the result in part (c).

10.36 •• An airplane propeller is 2.08 m in length (from tip to tip) and has a mass of 117 kg. When the airplane's engine is first started, it applies a constant net torque of 1950 $N \cdot m$ to the propeller, which starts from rest. (a) What is the angular acceleration of the propeller? Model the propeller as a slender rod and see Table 9.2. (b) What is the propeller's angular speed after making 5.00 revolutions? (c) How much work is done by the engine during the first 5.00 revolutions? (d) What is the average power output of the engine during the first 5.00 revolutions? (e) What is the instantaneous power output of the motor at the instant that the propeller has turned through 5.00 revolutions?

Section 10.5 Angular Momentum

10.37 • A 2.00 kg rock has a horizontal velocity of magnitude 12.0 m/s when it is at point P in Fig. E10.37. (a) At this instant, what are the magnitude and direction of its angular momentum relative to point O? (b) If the only force acting on the rock is its weight, what is the rate of change (magnitude and direction) of its angular momentum at this instant?

Figure E10.37



10.38 •• A woman with mass 50 kg is standing on the rim of a large horizontal disk that is rotating at 0.80 rev/s about an axis through its center. The disk has mass 110 kg and radius 4.0 m. Calculate the magnitude of the total angular momentum of the woman–disk system. (Assume that you can treat the woman as a point.)

10.39 •• Find the magnitude of the angular momentum of the second hand on a clock about an axis through the center of the clock face. The clock hand has a length of 15.0 cm and a mass of 6.00 g. Take the second hand to be a slender rod rotating with constant angular velocity about one end.

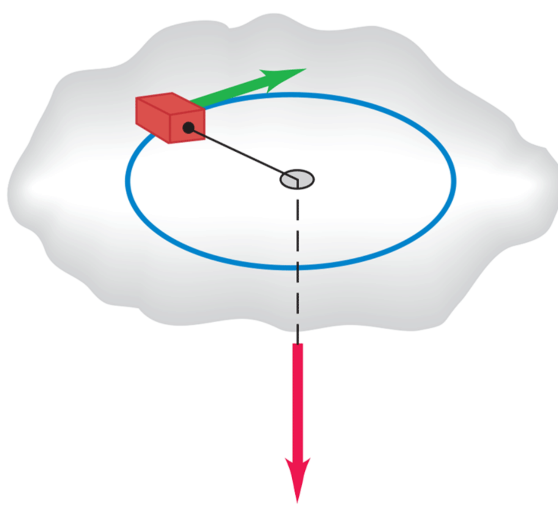
10.40 •• (a) Calculate the magnitude of the angular momentum of the earth in a circular orbit around the sun. Is it reasonable to model it as a particle? (b) Calculate the magnitude of the angular momentum of the earth due to its rotation around an axis through the north and south poles, modeling it as a uniform sphere. Consult Appendix E and the astronomical data in Appendix F.

10.41 •• **CALC** A hollow, thin-walled sphere of mass 12.0 kg and diameter 48.0 cm is rotating about an axle through its center. The angle (in radians) through which it turns as a function of time (in seconds) is given by $\theta(t) = At^2 + Bt^4$, where A has numerical value 1.50 and B has numerical value 1.10. (a) What are the units of the constants A and B? (b) At the time 3.00 s, find (i) the angular momentum of the sphere and (ii) the net torque on the sphere.

Section 10.6 Conservation of Angular Momentum

10.42 • **CP** A small block on a frictionless, horizontal surface has a mass of 0.0250 kg. It is attached to a massless cord passing through a hole in the surface (Fig. E10.42). The block is originally revolving at a distance of 0.300 m from the hole with an angular speed of 2.85 rad/s. The cord is then pulled from below, shortening the radius of the circle in which the block revolves to 0.150 m. Model the block as a particle. (a) Is the angular momentum of the block conserved? Why or why not? (b) What is the new angular speed? (c) Find the change in kinetic energy of the block. (d) How much work was done in pulling the cord?

Figure E10.42

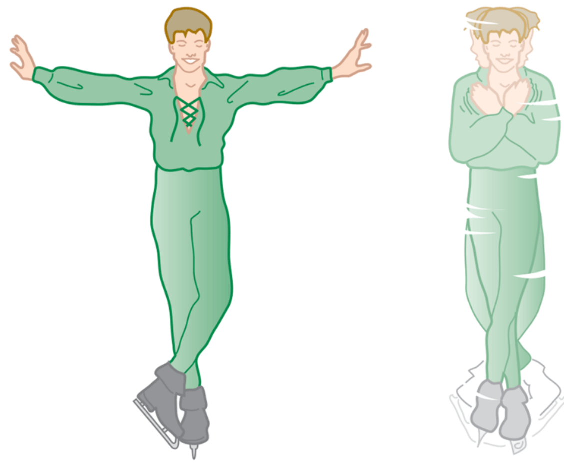


10.43 •• Under some circumstances, a star can collapse into an extremely dense object made mostly of neutrons and called a *neutron star.* The density of a neutron star is roughly 10^{14} times as great as that of ordinary solid matter. Suppose we represent the star as a uniform, solid, rigid sphere, both before and after the collapse. The star's initial radius was 7.0×10^5 km (comparable to our sun); its final radius is 16 km. If the original star rotated once in 30 days, find the angular speed of the neutron star.

10.44 •• A diver comes off a board with arms straight up and legs straight down, giving her a moment of inertia about her rotation axis of 18 $kg \cdot m^2$. She then tucks into a small ball, decreasing this moment of inertia to 3.6 $kg \cdot m^2$. While tucked, she makes two complete revolutions in 1.0 s. If she hadn't tucked at all, how many revolutions would she have made in the 1.5 s from board to water?

10.45 •• **The Spinning Figure Skater.** The outstretched hands and arms of a figure skater preparing for a spin can be considered a slender rod pivoting about an axis through its center (Fig. E10.45). When the skater's hands and arms are brought in and wrapped around his body to execute the spin, the hands and arms can be considered a thin-walled, hollow cylinder. His hands and arms have a combined mass of 8.0 kg. When outstretched, they span 1.8 m; when wrapped, they form a cylinder of radius 25 cm. The moment of inertia about the rotation axis of the remainder of his body is constant and equal to 0.40 $kg \cdot m^2$. If his original angular speed is 0.40 rev/s, what is his final angular speed?

Figure E10.45



10.46 •• A solid wood door 1.00 m wide and 2.00 m high is hinged along one side and has a total mass of 40.0 kg. Initially open and at rest, the door is struck at its center by a handful of sticky mud with mass 0.500 kg, traveling perpendicular to the door at 12.0 m/s just before impact. Find the final angular speed of the door. Does the mud make a significant contribution to the moment of inertia?

10.47 •• A large wooden turntable in the shape of a flat uniform disk has a radius of 2.00 m and a total mass of 120 kg. The turntable is initially rotating at 3.00 rad/s about a vertical axis through its center. Suddenly, a 70.0 kg parachutist makes a soft landing on the turntable at a point near the outer edge. (a) Find the angular speed of the turntable after the parachutist lands. (Assume that you can treat the parachutist as a particle.) (b) Compute the kinetic energy of the system before and after the parachutist lands. Why are these kinetic energies not equal?

10.48 •• **Asteroid Collision!** Suppose that an asteroid traveling straight toward the center of the earth were to collide with our planet at the equator and bury itself just below the surface. What would have to be the mass of this asteroid, in terms of the earth's mass M, for the day to become 25.0%

longer than it presently is as a result of the collision? Assume that the asteroid is very small compared to the earth and that the earth is uniform throughout.

10.49 •• A small 10.0 g bug stands at one end of a thin uniform bar that is initially at rest on a smooth horizontal table. The other end of the bar pivots about a nail driven into the table and can rotate freely, without friction. The bar has mass 50.0 g and is 100 cm in length. The bug jumps off in the horizontal direction, perpendicular to the bar, with a speed of 20.0 cm/s relative to the table. (a) What is the angular speed of the bar just after the frisky insect leaps? (b) What is the total kinetic energy of the system just after the bug leaps? (c) Where does this energy come from?

10.50 •• A thin uniform rod has a length of 0.500 m and is rotating in a circle on a frictionless table. The axis of rotation is perpendicular to the length of the rod at one end and is stationary. The rod has an angular velocity of 0.400 rad/s and a moment of inertia about the axis of 3.00×10^{-3} kg·m^2. A bug initially standing on the rod at the axis of rotation decides to crawl out to the other end of the rod. When the bug has reached the end of the rod and sits there, its tangential speed is 0.160 m/s. The bug can be treated as a point mass. What is the mass of (a) the rod; (b) the bug?

10.51 •• You live on a planet far from ours. Based on extensive communication with a physicist on earth, you have determined that all laws of physics on your planet are the same as ours and you have adopted the same units of seconds and meters as on earth. But you suspect that the value of g, the acceleration of an object in free fall near the surface of your planet, is different from what it is on earth. To test this, you take a solid uniform cylinder and let it roll down an incline. The vertical height h of the top of the incline above the lower end of the incline can be varied. You measure the speed v_{cm} of the center of mass of the cylinder when it reaches the bottom for various values of h. You plot $v^2{}_{\text{cm}}$ (in m^2/s^2) versus h (in m) and find that your data lie close to a straight line with a slope of $6.42\ \text{m/s}^2$. What is the value of g on your planet?

10.52 •• A uniform, 4.5 kg, square, solid wooden gate 1.5 m on each side hangs vertically from a frictionless pivot at the center of its upper edge. A 1.1 kg raven flying horizontally at 5.0 m/s flies into this door at its center and bounces back at 2.0 m/s in the opposite direction. (a) What is the angular speed of the gate just after it is struck by the unfortunate raven? (b) During the collision, why is the angular momentum conserved but not the linear momentum?

10.53 •• A teenager is standing at the rim of a large horizontal uniform wooden disk that can rotate freely about a vertical axis at its center. The mass of the disk (in kg) is M and its radius (in m) is R. The mass of the teenager (in kg) is m. The disk and teenager are initially at rest. The teenager then throws a large rock that has a mass (in kg) of m_{rock}. As it leaves the thrower's hands, the rock is traveling horizontally with speed v (in m/s) relative to the earth in a direction tangent to the rim of the disk. The teenager remains at rest relative to the disk and so rotates with it after throwing the rock. In terms of M, R, m, m_{rock}, and v, what is the angular speed of the disk? Treat the teenager as a point mass.

10.54 •• A uniform solid disk made of wood is horizontal and rotates freely about a vertical axle at its center. The disk has radius 0.600 m and mass 1.60 kg and is initially at rest. A bullet with mass 0.0200 kg is fired horizontally at the disk, strikes the rim of the disk at a point perpendicular to the radius of the disk, and becomes embedded in its rim, a distance of 0.600 m from the axle. After being struck by the bullet, the disk rotates at 4.00 rev/s. What is the horizontal velocity of the bullet just before it strikes the disk?

Section 10.7 Gyroscopes and Precession

10.55 • **Stabilization of the Hubble Space Telescope.** The Hubble Space Telescope is stabilized to within an angle of about 2-millionths of a degree by means of a series of gyroscopes that spin at 19,200 rpm. Although the structure of these gyroscopes is actually quite complex, we can model each of the gyroscopes as a thin-walled cylinder of mass 2.0 kg and diameter 5.0 cm, spinning about its central axis. How large a torque would it take to cause these gyroscopes to precess through an angle of 1.0×10^{-6} degree during a 5.0 hour exposure of a galaxy?

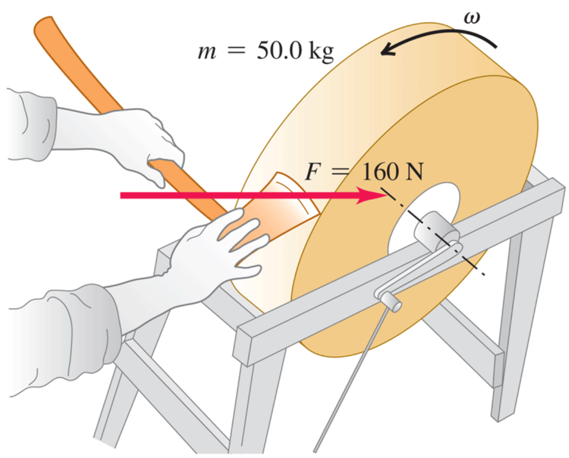
10.56 • **A Gyroscope on the Moon.** A certain gyroscope precesses at a rate of 0.50 rad/s when used on earth. If it were taken to a lunar base, where the acceleration due to gravity is $0.165g$, what would be its precession rate?

Problems

10.57 •• You are riding your bicycle on a city street, and you are staying a constant distance behind a car that is traveling at the speed limit of 30 mph. Estimate the diameters of the bicycle wheels and sprockets and use these estimated quantities to calculate the number of revolutions per minute made by the large sprocket to which the pedals are attached. Do a Web search if you aren’t familiar with the parts of a bicycle.

10.58 •• A 50.0 kg grindstone is a solid disk 0.520 m in diameter. You press an ax down on the rim with a normal force of 160 N (Fig. P10.58). The coefficient of kinetic friction between the blade and the stone is 0.60, and there is a constant friction torque of $6.50\ \mathrm{N \cdot m}$ between the axle of the stone and its bearings. (a) How much force must be applied tangentially at the end of a crank handle 0.500 m long to bring the stone from rest to 120 rev/min in 9.00 s? (b) After the grindstone attains an angular speed of 120 rev/min, what tangential force at the end of the handle is needed to maintain a constant angular speed of 120 rev/min? (c) How much time does it take the grindstone to come from 120 rev/min to rest if it is acted on by the axle friction alone?

Figure P10.58

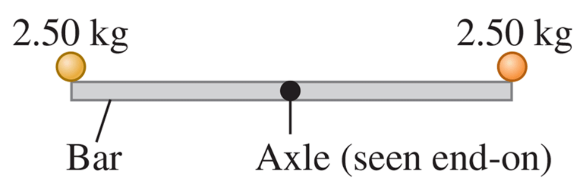


10.59 ••• A grindstone in the shape of a solid disk with diameter 0.520 m and a mass of 50.0 kg is rotating at 850 rev/min. You press an ax against the rim with a normal force of 160 N (Fig. P10.58), and the grindstone comes to rest in 7.50 s. Find the coefficient of friction between the ax and the grindstone. You can ignore friction in the bearings.

10.60 •• **CP** Block A rests on a horizontal tabletop. A light horizontal rope is attached to it and passes over a pulley, and block B is suspended from the free end of the rope. The light rope that connects the two blocks does not slip over the surface of the pulley (radius 0.080 m) because the pulley rotates on a frictionless axle. The horizontal surface on which block A (mass 2.50 kg) moves is frictionless. The system is released from rest, and block B (mass 6.00 kg) moves downward 1.80 m in 2.00 s. (a) What is the tension force that the rope exerts on block B? (b) What is the tension force on block A? (c) What is the moment of inertia of the pulley for rotation about the axle on which it is mounted?

10.61 ••• A thin, uniform, 3.80 kg bar, 80.0 cm long, has very small 2.50 kg balls glued on at either end (Fig. P10.61). It is supported horizontally by a thin, horizontal, frictionless axle passing through its center and perpendicular to the bar. Suddenly the right-hand ball becomes detached and falls off, but the other ball remains glued to the bar. (a) Find the angular acceleration of the bar just after the ball falls off. (b) Will the angular acceleration remain constant as the bar continues to swing? If not, will it increase or decrease? (c) Find the angular velocity of the bar just as it swings through its vertical position.

Figure P10.61

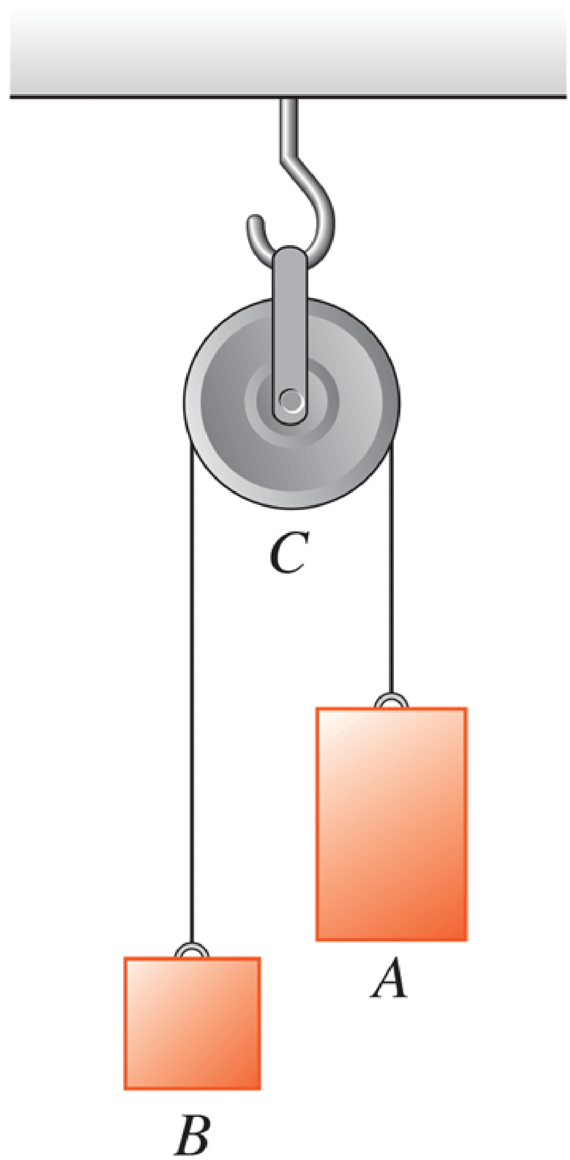


10.62 •• Example 10.7 discusses a uniform solid sphere rolling without slipping down a ramp that is at an angle β above the horizontal. Now consider the same sphere rolling without slipping up the ramp. (a) In terms of g and β, calculate the acceleration of the center of mass of the sphere. Is your result larger or smaller than the acceleration when the sphere rolls down the ramp, or is it the same? (b)

Calculate the friction force (in terms of M, g, and β) for the sphere to roll without slipping as it moves up the incline. Is the result larger, smaller, or the same as the friction force required to prevent slipping as the sphere rolls down the incline?

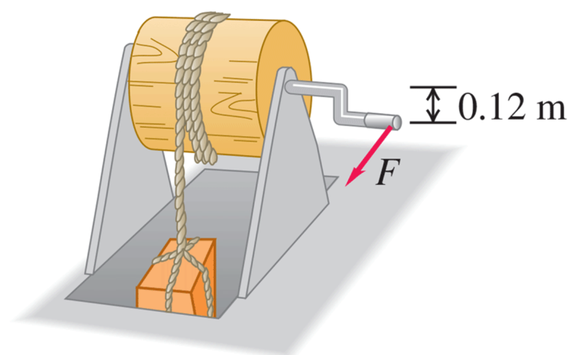
10.63 •• **The Atwood's Machine.** Figure P10.63 illustrates an Atwood's machine. Find the linear accelerations of blocks A and B, the angular acceleration of the wheel C, and the tension in each side of the cord if there is no slipping between the cord and the surface of the wheel. Let the masses of blocks A and B be 4.00 kg and 2.00 kg, respectively, the moment of inertia of the wheel about its axis be 0.220 $kg{\cdot}m^2$, and the radius of the wheel be 0.120 m.

Figure P10.63



10.64 ••• The mechanism shown in Fig. P10.64 is used to raise a crate of supplies from a ship's hold. The crate has total mass 50 kg. A rope is wrapped around a wooden cylinder that turns on a metal axle. The cylinder has radius 0.25 m and moment of inertia $I = 2.9\ \text{kg}{\cdot}\text{m}^2$ about the axle. The crate is suspended from the free end of the rope. One end of the axle pivots on frictionless bearings; a crank handle is attached to the other end. When the crank is turned, the end of the handle rotates about the axle in a vertical circle of radius 0.12 m, the cylinder turns, and the crate is raised. What magnitude of the force $\vec{F}$ applied tangentially to the rotating crank is required to raise the crate with an acceleration of 1.40 $\mathrm{m/s^2}$? (You can ignore the mass of the rope as well as the moments of inertia of the axle and the crank.)

Figure P10.64

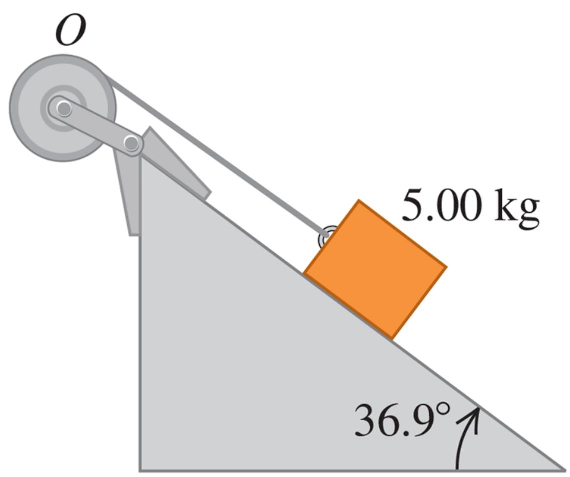


10.65 •• A solid uniform sphere and a thin-walled, hollow sphere have the same mass M and radius R. If they roll without slipping up a ramp that is inclined at an angle β above the horizontal and if both have the same v_{cm} before they start up the incline, calculate the maximum height above their starting point reached by each object. Which object reaches the greater height, or do both of them reach the same height?

10.66 •• A block with mass $m = 5.00$ kg slides down a surface inclined 36.9° to the horizontal (Fig. P10.66). The coefficient of kinetic friction is 0.25. A string attached to the block is wrapped around a flywheel on a fixed axis at O. The flywheel has mass 25.0 kg and moment of inertia 0.500 $\mathrm{kg \cdot m^2}$ with respect to the axis of rotation. The string pulls without slipping at a perpendicular distance of 0.200

m from that axis. (a) What is the acceleration of the block down the plane? (b) What is the tension in the string?

Figure P10.66

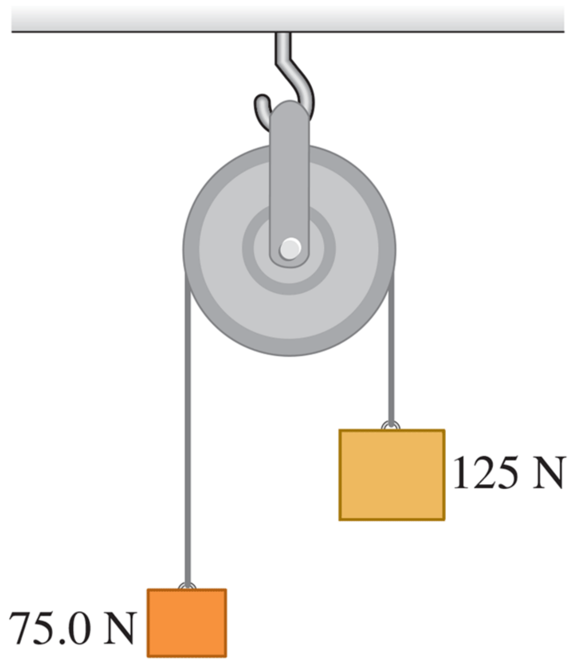


10.67 •• **CP** A wheel with radius 0.0600 m rotates about a horizontal frictionless axle at its center. The moment of inertia of the wheel about the axle is 2.50 $\text{kg}\cdot\text{m}^2$. The wheel is initially at rest. Then at $t = 0$ a force $F = (5.00\ \text{N/s})t$ is applied tangentially to the wheel and the wheel starts to rotate. What is the magnitude of the force at the instant when the wheel has turned through 8.00 revolutions?

10.68 •• A lawn roller in the form of a thin-walled, hollow cylinder with mass M is pulled horizontally with a constant horizontal force F applied by a handle attached to the axle. If it rolls without slipping, find the acceleration and the friction force.

10.69 • Two weights are connected by a very light, flexible cord that passes over an 80.0 N frictionless pulley of radius 0.300 m. The pulley is a solid uniform disk and is supported by a hook connected to the ceiling (Fig. P10.69). What force does the ceiling exert on the hook?

Figure P10.69

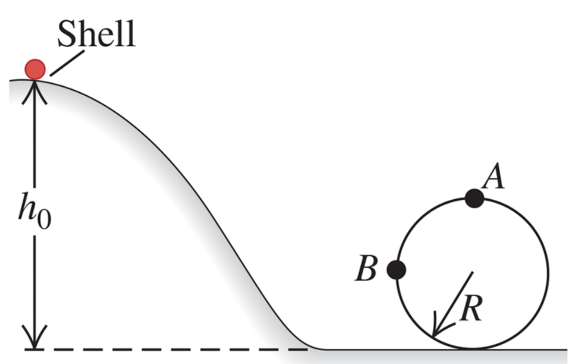


10.70 •• A large uniform horizontal turntable rotates freely about a vertical axle at its center. You measure the radius of the turntable to be 3.00 m. To determine the moment of inertia I of the turntable about the axle, you start the turntable rotating with angular speed ω, which you measure. You then drop a small object of mass m onto the rim of the turntable. After the object has come to rest relative to the turntable, you measure the angular speed ω_{f} of the rotating turntable. You plot the quantity $(\omega - \omega_{\mathrm{f}})/\omega_{\mathrm{f}}$ (with both ω and ω_{f} in rad/s) as a function of m (in kg). You find that your data lie close to a straight line that has slope $0.250\ \mathrm{kg}^{-1}$. What is the moment of inertia I of the turntable?

10.71 • **The Yo-yo.** A yo-yo is made from two uniform disks, each with mass m and radius R, connected by a light axle of radius b. A light, thin string is wound several times around the axle and then held stationary while the yo-yo is released from rest, dropping as the string unwinds. Find the linear acceleration and angular acceleration of the yo-yo and the tension in the string.

10.72 •• **CP** A thin-walled, hollow spherical shell of mass m and radius r starts from rest and rolls without slipping down a track (Fig. P10.72). Points A and B are on a circular part of the track having radius R. The diameter of the shell is very small compared to h_0 and R, and the work done by rolling friction is negligible. (a) What is the minimum height h_0 for which this shell will make a complete loop-the-loop on the circular part of the track? (b) How hard does the track push on the shell at point B, which is at the same level as the center of the circle? (c) Suppose that the track had no friction and the shell was released from the same height h_0 you found in part (a). Would it make a complete loop-the-loop? How do you know? (d) In part (c), how hard does the track push on the shell at point A, the top of the circle? How hard did it push on the shell in part (a)?

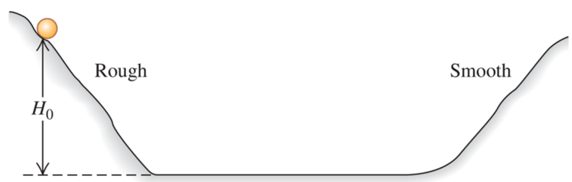
Figure P10.72



10.73 •• A basketball (which can be closely modeled as a hollow spherical shell) rolls down a mountainside into a valley and then up the opposite side, starting from rest at a height H_0

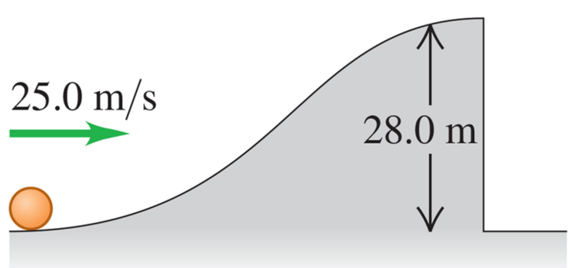
above the bottom. In Fig. P10.73, the rough part of the terrain prevents slipping while the smooth part has no friction. Neglect rolling friction and assume the system's total mechanical energy is conserved. (a) How high, in terms of H_0, will the ball go up the other side? (b) Why doesn't the ball return to height H_0? Has it lost any of its original potential energy?

Figure P10.73



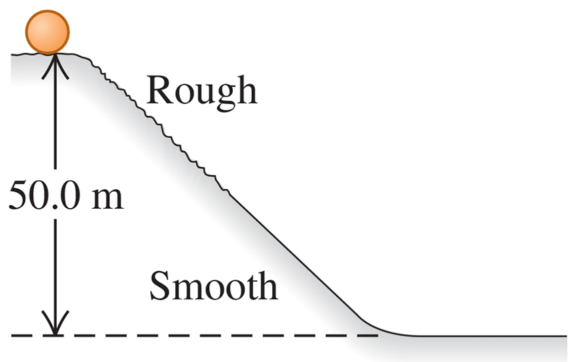
10.74 •• **CP** A solid uniform ball rolls without slipping up a hill (Fig. P10.74). At the top of the hill, it is moving horizontally, and then it goes over the vertical cliff. Neglect rolling friction and assume the system's total mechanical energy is conserved. (a) How far from the foot of the cliff does the ball land, and how fast is it moving just before it lands? (b) Notice that when the balls lands, it has a greater translational speed than when it was at the bottom of the hill. Does this mean that the ball somehow gained energy? Explain!

Figure P10.74



10.75 •• **Rolling Stones.** A solid, uniform, spherical boulder starts from rest and rolls down a 50.0-m-high hill, as shown in Fig. P10.75. The top half of the hill is rough enough to cause the boulder to roll without slipping, but the lower half is covered with ice and there is no friction. What is the translational speed of the boulder when it reaches the bottom of the hill? Neglect rolling friction and assume the system's total mechanical energy is conserved.

Figure P10.75



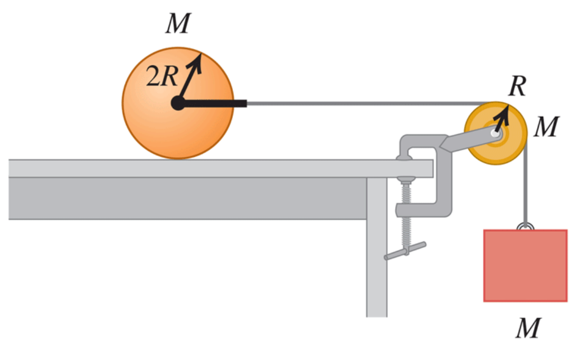
10.76 ••• You are designing a system for moving aluminum cylinders from the ground to a loading dock. You use a sturdy wooden ramp that is 6.00 m long and inclined at 37.0° above the horizontal. Each cylinder is fitted with a light, frictionless yoke through its center, and a light (but strong) rope is attached to the yoke. Each cylinder is uniform and has mass 460 kg and radius 0.300 m. The cylinders are pulled up the ramp by applying a constant force $\vec{F}$ to the free end of the rope. $\vec{F}$ is parallel to the surface of the ramp and exerts no torque on the cylinder. The coefficient of static friction between the ramp surface and the cylinder is 0.120. (a) What is the largest magnitude $\vec{F}$ can have so that the cylinder still rolls without slipping as it moves up the ramp? (b) If the cylinder starts from rest at the bottom of the ramp and rolls without slipping as it moves up the ramp, what is the shortest time it can take the cylinder to reach the top of the ramp?

10.77 •• A 42.0-cm-diameter wheel, consisting of a rim and six spokes, is constructed from a thin, rigid plastic material having a linear mass density of 25.0 g/cm. This wheel is released from rest at the top of a hill 58.0 m high. (a) How fast is it rolling when it reaches the bottom of the hill? (b) How would your answer change if the linear mass density and the diameter of the wheel were each doubled?

10.78 ••• A uniform, 0.0300 kg rod of length 0.400 m rotates in a horizontal plane about a fixed axis through its center and perpendicular to the rod. Two small rings, each with mass 0.0200 kg, are mounted so that they can slide without friction along the rod. They are initially held by catches at positions 0.0500 m on each side of the center of the rod, and the system is rotating at 48.0 rev/min. With no other changes in the system, the catches are released, and the rings slide outward along the rod and fly off at the ends. What is the angular speed (a) of the system at the instant when the rings reach the ends of the rod; (b) of the rod after the rings leave it?

10.79 • A uniform solid cylinder with mass M and radius $2R$ rests on a horizontal tabletop. A string is attached by a yoke to a frictionless axle through the center of the cylinder so that the cylinder can rotate about the axle. The string runs over a disk-shaped pulley with mass M and radius R that is mounted on a frictionless axle through its center. A block of mass M is suspended from the free end of the string (Fig. P10.79). The string doesn't slip over the pulley surface, and the cylinder rolls without slipping on the tabletop. Find the magnitude of the acceleration of the block after the system is released from rest.

Figure P10.79



10.80 ••• A 5.00 kg ball is dropped from a height of 12.0 m above one end of a uniform bar that pivots at its center. The bar has mass 8.00 kg and is 4.00 m in length. At the other end of the bar sits another 5.00 kg ball, unattached to the bar. The dropped ball sticks to the bar after the collision. How high will the other ball go after the collision?

10.81 •• A uniform rod of length L rests on a frictionless horizontal surface. The rod pivots about a fixed frictionless axis at one end. The rod is initially at rest. A bullet traveling parallel to the horizontal surface and perpendicular to the rod with speed v strikes the rod at its center and becomes embedded in it. The mass of the bullet is one-fourth the mass of the rod. (a) What is the final angular speed of the rod? (b) What is the ratio of the kinetic energy of the system after the collision to the kinetic energy of the bullet before the collision?

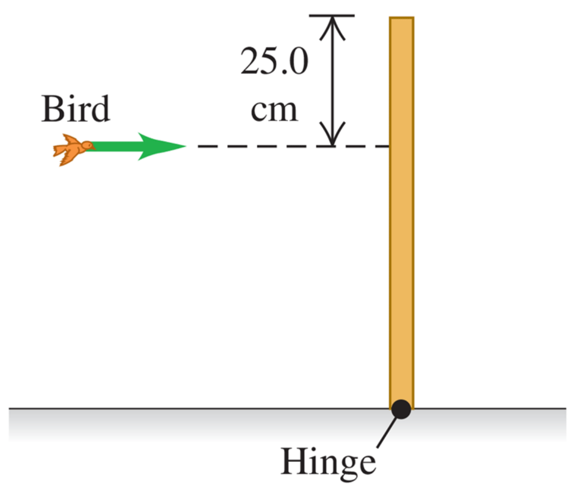
10.82 •• CP A large turntable with radius 6.00 m rotates about a fixed vertical axis, making one revolution in 8.00 s. The moment of inertia of the turntable about this axis is $1200\ \text{kg}\cdot\text{m}^2$. You stand, barefooted, at the rim of the turntable and very slowly walk toward the center, along a radial line painted on the surface of the turntable. Your mass is 70.0 kg. Since the radius of the turntable is large, it is a good approximation to treat yourself as a point mass. Assume that you can maintain your balance by adjusting the positions of your feet. You find that you can reach a point 3.00 m from the center of the turntable before your feet begin to slip. What is the coefficient of static friction between the bottoms of your feet and the surface of the turntable?

10.83 •• In your job as a mechanical engineer you are designing a flywheel and clutch-plate system like the one in Example 10.11. Disk A is made of a lighter material than disk B, and the moment of inertia of disk A about the shaft is one-third that of disk B. The moment of inertia of the shaft is negligible. With the clutch disconnected, A is brought up to an angular speed ω_0; B is initially at rest. The accelerating torque is then removed from A, and A is coupled to B. (Ignore bearing friction.) The design specifications allow for a maximum of 2400 J of thermal energy to be developed when the connection is made. What can be the maximum value of the original kinetic energy of disk A so as not to exceed the maximum allowed value of the thermal energy?

10.84 •• A local ice hockey team has asked you to design an apparatus for measuring the speed of the hockey puck after a slap shot. Your design is a 2.00-m-long, uniform rod pivoted about one end so that it is free to rotate horizontally on the ice without friction. The 0.800 kg rod has a light basket at the other end to catch the 0.163 kg puck. The puck slides across the ice with velocity $\vec{v}$ (perpendicular to the rod), hits the basket, and is caught. After the collision, the rod rotates. If the rod makes one revolution every 0.736 s after the puck is caught, what was the puck's speed just before it hit the rod?

10.85 ••• A 500.0 g bird is flying horizontally at 2.25 m/s, not paying much attention, when it suddenly flies into a stationary vertical bar, hitting it 25.0 cm below the top (Fig. P10.85). The bar is uniform, 0.750 m long, has a mass of 1.50 kg, and is hinged at its base. The collision stuns the bird so that it just drops to the ground afterward (but soon recovers to fly happily away). What is the angular velocity of the bar (a) just after it is hit by the bird and (b) just as it reaches the ground?

Figure P10.85



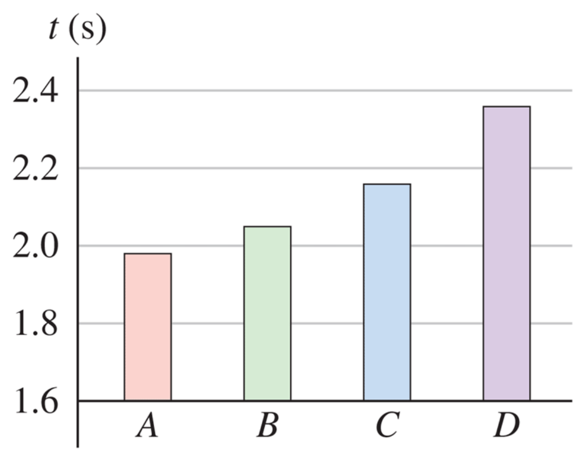
10.86 ••• **CP** A small block with mass 0.130 kg is attached to a string passing through a hole in a frictionless, horizontal surface (see Fig. E10.42). The block is originally revolving in a circle with a radius of 0.800 m about the hole with a tangential speed of 4.00 m/s. The string is then pulled slowly from below, shortening the radius of the circle in which the block revolves. The breaking strength of the string is 30.0 N. What is the radius of the circle when the string breaks?

10.87 • A 55 kg runner runs around the edge of a horizontal turntable mounted on a vertical, frictionless axis through its center. The runner's velocity relative to the earth has magnitude 2.8 m/s. The turntable is rotating in the opposite direction with an angular velocity of magnitude 0.20 rad/s relative to the earth. The radius of the turntable is 3.0 m, and its moment of inertia about the axis of rotation is 80 $kg{\cdot}m^2$. Find the final angular velocity of the system if the runner comes to rest relative to the turntable. (You can model the runner as a particle.)

10.88 •• **DATA** The V6 engine in a 2014 Chevrolet Silverado 1500 pickup truck is reported to produce a maximum power of 285 hp at 5300 rpm and a maximum torque of $305\,\mathrm{ft}\cdot\mathrm{lb}$ at 3900 rpm. (a) Calculate the torque, in both ft · lb and N · m, at 5300 rpm. Is your answer in ft · lb smaller than the specified maximum value? (b) Calculate the power, in both horsepower and watts, at 3900 rpm. Is your answer in hp smaller than the specified maximum value? (c) The relationship between power in hp and torque in ft · lb at a particular angular velocity in rpm is often written as $\mathrm{hp} = [\mathrm{torque}\,(\mathrm{in}\,\mathrm{ft}\cdot\mathrm{lb}) \times \mathrm{rpm}]/c$, where c is a constant. What is the numerical value of c? (d) The engine of a 2012 Chevrolet Camaro ZL1 is reported to produce 580 hp at 6000 rpm. What is the torque (in ft · lb) at 6000 rpm?

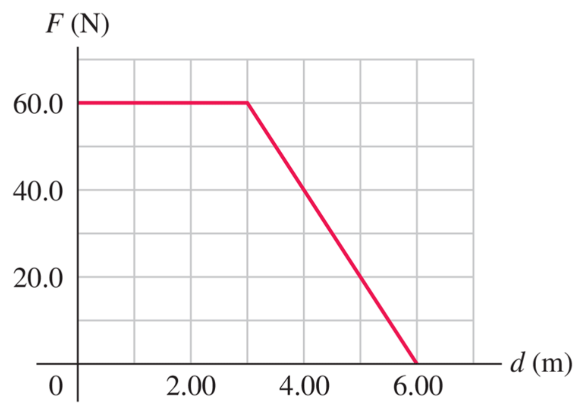
10.89 •• **DATA** You have one object of each of these shapes, all with mass 0.840 kg: a uniform solid cylinder, a thin-walled hollow cylinder, a uniform solid sphere, and a thin-walled hollow sphere. You release each object from rest at the same vertical height h above the bottom of a long wooden ramp that is inclined at 35.0° from the horizontal. Each object rolls without slipping down the ramp. You measure the time t that it takes each one to reach the bottom of the ramp; Fig. P10.89 shows the results. (a) From the bar graphs, identify objects A through D by shape. (b) Which of objects A through D has the greatest total kinetic energy at the bottom of the ramp, or do all have the same kinetic energy? (c) Which of objects A through D has the greatest rotational kinetic energy $\frac{1}{2}I\omega^2$ at the bottom of the ramp, or do all have the same rotational kinetic energy? (d) What minimum coefficient of static friction is required for all four objects to roll without slipping?

Figure P10.89



10.90 ••• **DATA** You are testing a small flywheel (radius 0.166 m) that will be used to store a small amount of energy. The flywheel is pivoted with low-friction bearings about a horizontal shaft through the flywheel's center. A thin, light cord is wrapped multiple times around the rim of the flywheel. Your lab has a device that can apply a specified horizontal force $\vec{F}$ to the free end of the cord. The device records both the magnitude of that force as a function of the horizontal distance the end of the cord has traveled and the time elapsed since the force was first applied. The flywheel is initially at rest. (a) You start with a test run to determine the flywheel's moment of inertia I. The magnitude F of the force is a constant 25.0 N, and the end of the rope moves 8.35 m in 2.00 s. What is I? (b) In a second test, the flywheel again starts from rest but the free end of the rope travels 6.00 m; Fig. P10.90 shows the force magnitude F as a function of the distance d that the end of the rope has moved. What is the kinetic energy of the flywheel when $d = 6.00$ m? (c) What is the angular speed of the flywheel, in rev/min, when $d = 6.00$ m?

Figure P10.90



Challenge Problems

10.91 ••• CP CALC A block with mass m is revolving with linear speed v_1 in a circle of radius r_1 on a frictionless horizontal surface (see Fig. E10.42). The string is slowly pulled from below until the radius of the circle in which the block is revolving is reduced to r_2. (a) Calculate the tension T in the string as a function of r, the distance of the block from the hole. Your answer will be in terms of the initial velocity v_1 and the radius r_1. (b) Use $W = \int_{r1}^{r2} \vec{T}(r) \cdot d\vec{r}$ to calculate the work done by $\vec{T}$ when r changes from r_1 to r_2. (c) Compare the results of part (b) to the change in the kinetic energy of the block.

10.92 ••• When an object is rolling without slipping, the rolling friction force is much less than the friction force when the object is sliding; a silver dollar will roll on its edge much farther than it will slide on its flat side (see Section 5.3). When an object is rolling without slipping on a horizontal surface, we can approximate the friction force to be zero, so that a_x and α_z are approximately zero and v_x and ω_z are approximately constant. Rolling without slipping means $v_x = r\omega_z$ and $a_x = r\alpha_z$. If an object is set in motion on a surface *without* these equalities, sliding (kinetic) friction will act on the object as it slips until rolling without slipping is established. A solid cylinder with mass M and radius R, rotating with angular speed ω_0 about an axis through its center, is set on a horizontal surface for which the kinetic friction coefficient is μ_{k}. (a) Draw a free-body diagram for the cylinder on the surface. Think carefully about the direction of the kinetic friction force on the cylinder. Calculate the accelerations a_x of the center of mass and α_z of rotation about the center of mass. (b) The cylinder is initially slipping completely, so initially $\omega_z = \omega_0$ but $v_x = 0$. Rolling without slipping sets in when $v_x = r\omega_z$. Calculate the *distance* the cylinder rolls before slipping stops. (c) Calculate the work done by the friction force on the cylinder as it moves from where it was set down to where it begins to roll without slipping.

10.93 ••• A demonstration gyroscope wheel is constructed by removing the tire from a bicycle wheel 0.650 m in diameter, wrapping lead wire around the rim, and taping it in place. The shaft projects 0.200 m at each side of the wheel, and a woman holds the ends of the shaft in her hands. The mass of the system is 8.00 kg; its entire mass may be assumed to be located at its rim. The shaft is horizontal, and the wheel is spinning about the shaft at 5.00 rev/s. Find the magnitude and direction of the force each hand exerts on the shaft (a) when the shaft is at rest; (b) when the shaft is rotating in a horizontal plane about its center at 0.050 rev/s; (c) when the shaft is rotating in a horizontal plane about its center at 0.300 rev/s. (d) At what rate must the shaft rotate in order that it may be supported at one end only?

MCAT-Style Passage Problems

BIO **Human Moment of Inertia.** The moment of inertia of the human body about an axis through its center of mass is important in the application of biomechanics to sports such as diving and gymnastics. We can measure the body's moment of inertia in a particular position while a person remains in that position on a horizontal turntable, with the body's center of mass on the turntable's rotational axis. The turntable with the person on it is then accelerated from rest by a torque that is produced by using a rope wound around a pulley on the shaft of the turntable. From the measured tension in the rope and the angular acceleration, we can calculate the body's moment of inertia about an axis through its center of mass.

10.94 The moment of inertia of the empty turntable is $1.5\ \text{kg} \cdot \text{m}^2$. With a constant torque of $2.5\ \text{N} \cdot \text{m}$, the turntable–person system takes 3.0 s to spin from rest to an angular speed of 1.0 rad/s. What is the person's moment of inertia about an axis through her center of mass? Ignore friction in the turntable axle. (a) $2.5\ \text{kg} \cdot \text{m}^2$; (b) $6.0\ \text{kg} \cdot \text{m}^2$; (c) $7.5\ \text{kg} \cdot \text{m}^2$; (d) $9.0\ \text{kg} \cdot \text{m}^2$.

10.95 While the turntable is being accelerated, the person suddenly extends her legs. What happens to the turntable? (a) It suddenly speeds up; (b) it rotates with constant speed; (c) its acceleration decreases; (d) it suddenly stops rotating.

10.96 A doubling of the torque produces a greater angular acceleration. Which of the following would do this, assuming that the tension in the rope doesn't change? (a) Increasing the pulley diameter by a factor of $\sqrt{2}$; (b) increasing the pulley diameter by a factor of 2; (c) increasing the pulley diameter by a factor of 4; (d) decreasing the pulley diameter by a factor of $\sqrt{2}$.

10.97 If the body's center of mass were not placed on the rotational axis of the turntable, how would the person's measured moment of inertia compare to the moment of inertia for rotation about the center of mass? (a) The measured moment of inertia would be too large; (b) the measured moment of inertia would be too small; (c) the two moments of inertia would be the same; (d) it depends on where the body's center of mass is placed relative to the center of the turntable.

Answers: Dynamics of Rotational Motion

Chapter Opening Question ?

(iv) A tossed pin rotates around its center of mass (which is located toward its thick end). This is also the point at which the gravitational force acts on the pin, so this force exerts no torque on the pin. Hence the pin rotates with constant angular momentum, and its angular speed remains the same.

Test Your Understanding

10.1 **(ii)** The force of magnitude P acts along a vertical line, so the lever arm is the horizontal distance from A to the line of action. This is the horizontal component of distance L, which is $L \cos \theta$. Hence the magnitude of the torque is the product of the force magnitude P and the lever arm $L \cos \theta$, or $\tau = PL \cos \theta$.

10.2 **(iii), (ii), (i)** For the hanging object of mass m_2 to accelerate downward, the net force on it must be downward. Hence the magnitude $m_2 g$ of the downward weight force must be greater than the magnitude T_2 of the upward tension force. For the pulley to have a clockwise angular acceleration, the net torque on the pulley must be clockwise. Tension T_2 tends to rotate the pulley clockwise, while tension T_1 tends to rotate the pulley counterclockwise. Both tension forces have the same lever arm R, so there is a clockwise torque $T_2 R$ and a counterclockwise torque $T_1 R$. For the net torque to be clockwise, T_2 must be greater than T_1. Hence $m_2 g > T_2 > T_1$.

10.3 **(a) (ii), (b) (i)** If you redo the calculation of Example 10.6 with a hollow cylinder (moment of inertia $I_{\text{cm}} = MR^2$) instead of a solid cylinder (moment of inertia $I_{\text{cm}} = \frac{1}{2} MR^2$), you'll find $a_{\text{cm}-y} = \frac{1}{2} g$ and $T = \frac{1}{2} Mg$ (instead of $a_{\text{cm}-y} = \frac{2}{3} g$ and $T = \frac{1}{3} Mg$ for a solid cylinder). Hence the acceleration is less but the tension is greater. You can come to the same conclusion without doing the calculation. The greater moment of inertia means that the hollow cylinder will rotate more slowly and hence will roll downward more slowly. To slow the downward motion, a greater upward tension force is needed to oppose the downward force of gravity.

10.4 **(iii)** You apply the same torque over the same angular displacement to both cylinders. Hence, by Eq. (10.21), you do the same amount of work to both cylinders and impart the same kinetic energy to both. (The one with the smaller moment of inertia ends up with a greater angular speed, but that isn't what we are asked. Compare Conceptual Example 6.5 in Section 6.2.)

10.5 (a) **no,** (b) **yes** As the ball goes around the circle, the magnitude of $\vec{\boldsymbol{p}} = m\vec{\boldsymbol{v}}$ remains the same (the speed is constant) but its direction changes, so the linear momentum vector isn't constant. But $\vec{\boldsymbol{L}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{p}}$ *is* constant: It has a constant magnitude (both the speed and the perpendicular distance from your hand to the ball are constant) and a constant direction (along the rotation axis, perpendicular to the plane of the ball's motion). The linear momentum changes because there is a net *force* $\vec{\boldsymbol{F}}$ on the ball (toward the center of the circle). The angular momentum remains constant because there is no net *torque;* the vector $\vec{\boldsymbol{r}}$ points from your hand to the ball and the force $\vec{\boldsymbol{F}}$ on the ball is directed toward your hand, so the vector product $\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}}$ is zero.

10.6 **(i)** In the absence of external torques, the earth's angular momentum $L_z = I\omega_z$ would remain constant. The melted ice would move from the poles toward the equator—that is, away from our planet's rotation axis—and the earth's moment of inertia I would increase slightly. Hence the angular velocity ω_z would decrease slightly and the day would be slightly longer.

10.7 **(iii)** Doubling the flywheel mass would double both its moment of inertia I and its weight w, so the ratio I/w would be unchanged. Equation (10.33) shows that the precession angular speed depends on this ratio, so there would be *no* effect on the value of Ω.

Key Example Variation Problems

VP10.3.1

a. 10 rad/s^2

b. 0.90 N · m

c. 15 N

VP10.3.2

a. $a_y = \dfrac{g}{1 + M/m}$

b. $T = \dfrac{mg}{1 + m/M} = \dfrac{Mg}{1 + M/m}$

VP10.3.3

a. $T = m(g - a)$

b. $I = mR^2\left(\dfrac{g}{a} - 1\right)$

VP10.3.4

a. $T = m(g + a)$

b. $\tau_{\text{cable on cylinder}} = mR(g + a)$

c. $\tau_{\text{motor on cylinder}} = mR(g + a) + \frac{1}{2}MRa$

VP10.7.1

a. $a_{\text{cm}-y} = \frac{8}{13}g$

b. $T = \frac{5}{13}Mg$

VP10.7.2

a. $a_{\text{cm}-x} = \frac{3}{5}g \sin\beta$

b. $f = \frac{2}{5}Mg \sin\beta$

c. $\tau = \frac{2}{5}MgR \sin\beta$

VP10.7.3

a. $a_{\text{cm}} = \frac{1}{3}g$, downward

b. $\alpha_z = \frac{4g}{3R}$

VP10.7.4

a. $f = \frac{1}{3}Mg \sin\beta$

b. $\beta_{\text{critical}} = \arctan 3\mu_{\text{s}}$

VP10.12.1

a. $\frac{9}{10}\omega_A$

b. $\frac{81}{85} = 0.953$

VP10.12.2

a. $\frac{7}{10}\omega_A$

b. $\frac{49}{85} = 0.576$

VP10.12.3

a. 0.80 rad/s

b. $0.0020 = 1/500$

VP10.12.4 **a.** MRv before, $MRv/2$ after

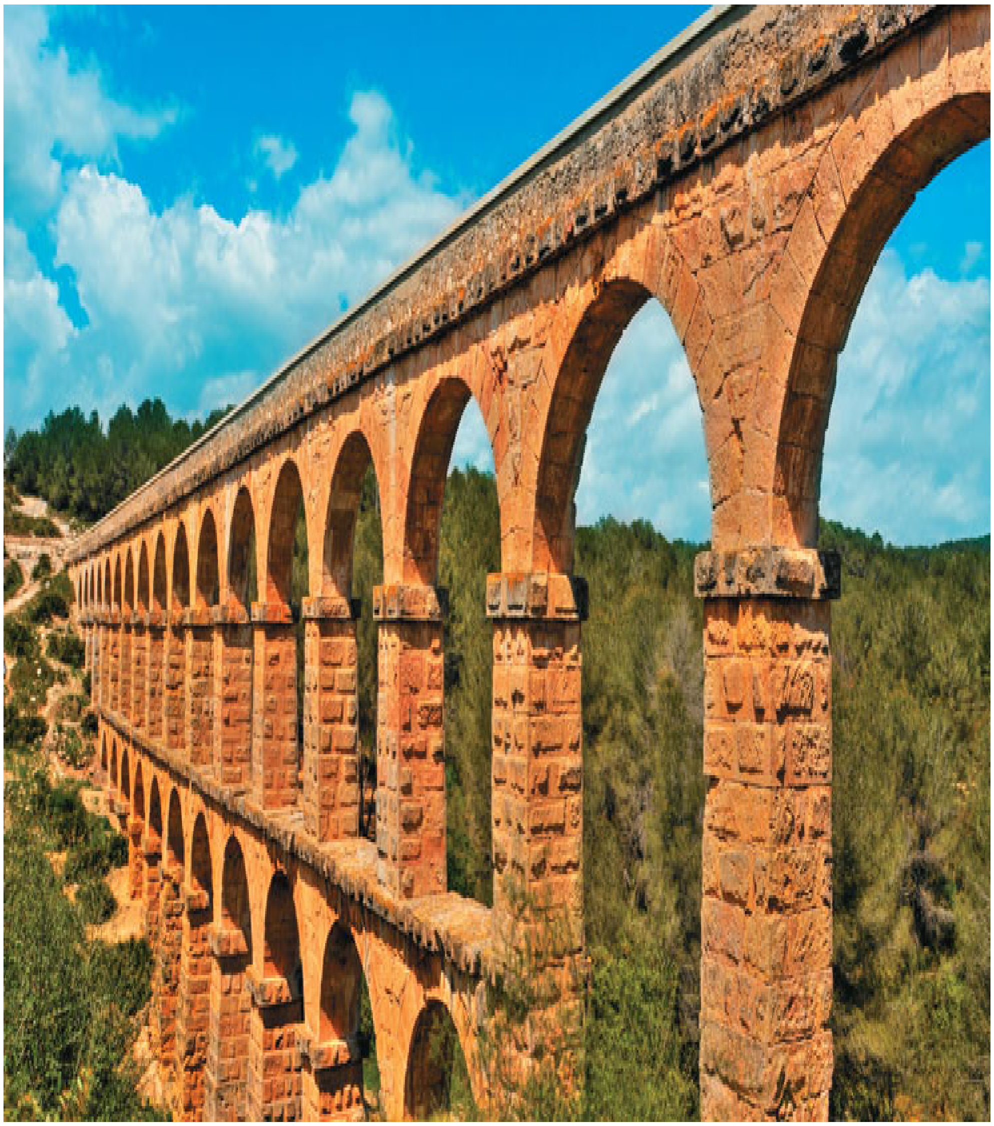
b. $\omega = \dfrac{3v}{4R}$ (c) $\dfrac{5}{8}$

Bridging Problem

(a) $h = \dfrac{2R}{5}$

(b) $\frac{5}{7}$ of the speed it had just after the hit

Chapter 11
Equilibrium and Elasticity



?◻ This Roman aqueduct uses the principle of the arch to sustain the weight of the structure and the water it carries. Are the blocks that make up the arch being (i) compressed, (ii) stretched, (iii) a combination of these, or (iv) neither compressed nor stretched?

Learning Outcomes

In this chapter, you'll learn...

- 11.1 The conditions that must be satisfied for an object or structure to be in equilibrium.
- 11.2 What the center of gravity of an object is and how it relates to the object's stability.
- 11.3 How to solve problems that involve rigid bodies in equilibrium.
- 11.4 How to analyze situations in which an object is deformed by tension, compression, pressure, or shear.
- 11.5 What happens when an object is stretched so much that it deforms or breaks.

You'll need to review...

- 4.2, 5.1 Newton's first law.
- 5.3 Static friction.
- 6.3, 7.2 Hooke's law for an ideal spring.
- 8.5 Center of mass.
- 10.2, 10.5 Torque, rotational dynamics, and angular momentum.

We've devoted a good deal of effort to understanding why and how objects accelerate in response to the forces that act on them. But very often we're interested in making sure that objects *don't* accelerate. Any building, from a multistory skyscraper to the humblest shed, must be designed so that it won't topple over. Similar concerns arise with a suspension bridge, a ladder leaning against a wall, or a crane hoisting a bucket full of concrete.

An object that can be modeled as a *particle* is in equilibrium whenever the vector sum of the forces acting on it is zero. But for the situations we've just described, that condition isn't enough. If forces act at different points on an extended object, an additional requirement must be satisfied to ensure that the object has no tendency to *rotate*: The sum of the *torques* about any point must be zero. This requirement is based on the principles of rotational dynamics developed in Chapter 10. We can compute the torque due to the weight of an object by using the concept of center of gravity, which we introduce in this chapter.

Idealized rigid bodies don't bend, stretch, or squash when forces act on them. But all real materials are *elastic* and do deform to some extent. Elastic properties of materials are tremendously important. You want the wings of an airplane to be able to bend a little, but you'd rather not have them break off. Tendons in your limbs need to stretch when you exercise, but they must return to their relaxed lengths when you stop. Many of the necessities of everyday life, from rubber bands to suspension bridges, depend on the elastic properties of materials. In this chapter we'll introduce the concepts of *stress*, *strain*, and *elastic modulus* and a simple principle called *Hooke's law*, which helps us predict what deformations will occur when forces are applied to a real (not perfectly rigid) object.

11.1 Conditions for Equilibrium

We learned in Sections 4.2 and 5.1 that a particle is in *equilibrium*—that is, the particle does not accelerate—in an inertial frame of reference if the vector sum of all the forces acting on the particle is zero, $\sum \vec{\boldsymbol{F}} = 0$. For an *extended* object, the equivalent statement is that the center of mass of the object has zero acceleration if the vector sum of all external forces acting on the object is zero, as discussed in Section 8.5. This is often called the **first condition for equilibrium**:

(11.1)

First condition for equilibrium: For the center of mass of an object at rest to remain at rest ...

$$\sum \vec{F} = 0 \qquad (11.1)$$

... the *net external force* on the object must be *zero*.

A second condition for an extended object to be in equilibrium is that the object must have no tendency to *rotate*. A rigid body that, in an inertial frame, is not rotating about a certain point has zero angular momentum about that point. If it is not to start rotating about that point, the rate of change of angular momentum must *also* be zero. From the discussion in Section 10.5, particularly Eq. (10.29), this means that the sum of torques due to all the external forces acting on the object must be zero. A rigid body in equilibrium can't have any tendency to start rotating about *any* point, so the sum of external torques must be zero about any point. This is the **second condition for equilibrium**:

(11.2)

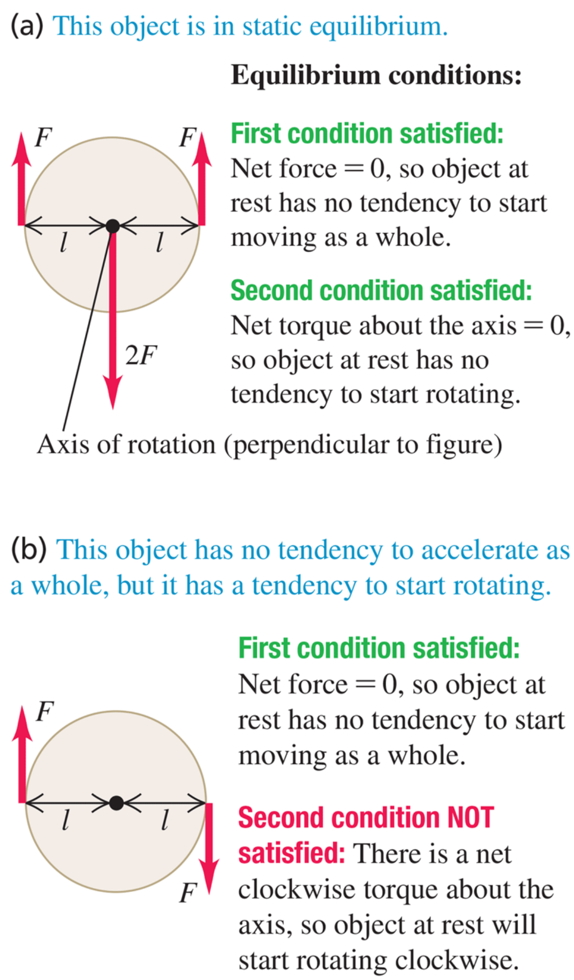
Second condition for equilibrium: For a nonrotating object to remain nonrotating ...

$$\sum \vec{\tau} = 0 \qquad (11.2)$$

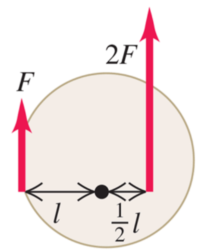
...the *net external torque around any point* on the object must be *zero.*

In this chapter we'll apply the first and second conditions for equilibrium to situations in which a rigid body is at rest (no translation or rotation). Such a rigid body is said to be in **static equilibrium** (Fig. 11.1). But the same conditions apply to a rigid body in uniform *translational* motion (without rotation), such as an airplane in flight with constant speed, direction, and altitude. Such a rigid body is in equilibrium but is not static.

Figure 11.1



(c) This object has a tendency to accelerate as a whole but no tendency to start rotating.



First condition NOT satisfied: There is a net upward force, so object at rest will start moving upward.

Second condition satisfied: Net torque about the axis = 0, so object at rest has no tendency to start rotating.

To be in static equilibrium, an object at rest must satisfy *both* conditions for equilibrium: It can have no tendency to accelerate as a whole or to start rotating.

Test Your Understanding of Section 11.1

Which situation satisfies both the first and second conditions for equilibrium? (i) A seagull gliding at a constant angle below the horizontal and at a constant speed; (ii) an automobile crankshaft turning at an increasing angular speed in the engine of a parked car; (iii) a thrown baseball that does not rotate as it sails through the air.

11.2 Center of Gravity

In most equilibrium problems, one of the forces acting on the object is its weight. We need to be able to calculate the *torque* of this force. The weight doesn't act at a single point; it is distributed over the entire object. But we can always calculate the torque due to the object's weight by assuming that the entire force of gravity (weight) is concentrated at a point called the **center of gravity** (abbreviated "cg"). The acceleration due to gravity decreases with altitude; but if we can ignore this variation over the vertical dimension of the object, then the object's center of gravity is identical to its *center of mass* (abbreviated "cm"), which we defined in Section 8.5. We stated this result without proof in Section 10.2, and now we'll prove it.

First let's review the definition of the center of mass. For a collection of particles with masses $m_1, m_2, \ldots$ and coordinates $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots$, the coordinates x_{cm} , and z_{cm} of the center of mass of the collection are

(11.3)

$$\begin{aligned} x_{\mathrm{cm}} &= \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} &= \frac{\sum_i m_i x_i}{\sum_i m_i} \\ y_{\mathrm{cm}} &= \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} &= \frac{\sum_i m_i y_i}{\sum_i m_i} \quad \text{(center of mass)} \\ z_{\mathrm{cm}} &= \frac{m_1 z_1 + m_2 z_2 + m_3 z_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} &= \frac{\sum_i m_i z_i}{\sum_i m_i} \end{aligned}$$

Also, x_{cm} , y_{cm} , and z_{cm} are the components of the position vector $\vec{\boldsymbol{r}}_{\mathrm{cm}}$ of the center of mass, so Eqs. (11.3) are equivalent to the vector equation

(11.4)

Position vector of center of mass of a system of particles

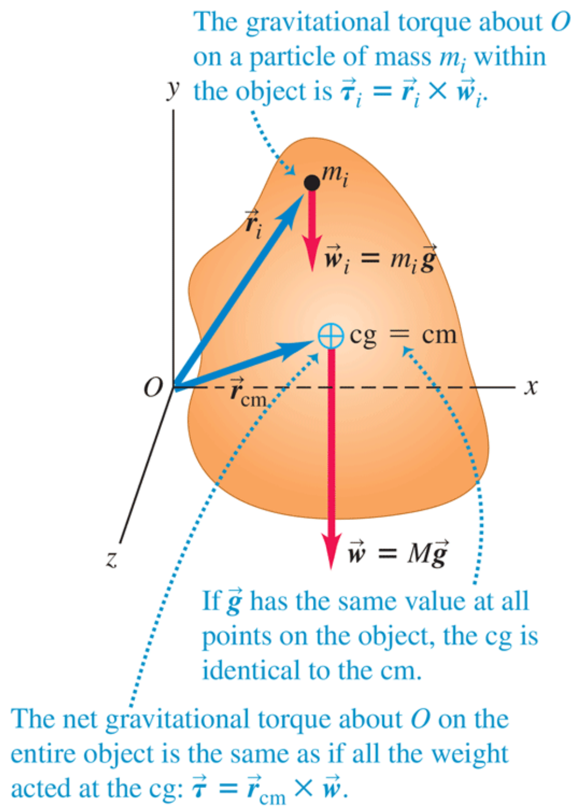
Position vectors of individual particles

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\sum_i m_i\vec{r}_i}{\sum_i m_i} \qquad (11.4)$$

Masses of individual particles

Now consider the gravitational torque on an object of arbitrary shape (Fig. 11.2). We assume that the acceleration due to gravity $\vec{g}$ is the same at every point in the object. Every particle in the object experiences a gravitational force, and the total weight of the object is the vector sum of a large number of parallel forces. A typical particle has mass m_i and weight $\vec{w}_i = m_i\vec{g}$. If $\vec{r}_i$ is the position vector of this particle with respect to an arbitrary origin O, then the torque vector $\vec{\tau}_i$ of the weight $\vec{w}_i$ with respect to O is, from Eq. (10.3),

Figure 11.2



The center of gravity (cg) and center of mass (cm) of an extended object.

$$\vec{\boldsymbol{\tau}}_i = \vec{\boldsymbol{r}}_i \times \vec{\boldsymbol{w}}_i = \vec{\boldsymbol{r}}_i \times m_i\,\vec{\boldsymbol{g}}$$

The *total* torque due to the gravitational forces on all the particles is

$$\begin{aligned}
\overrightarrow{\boldsymbol{\tau}} &= \sum_i \overrightarrow{\boldsymbol{\tau}_i} = \overrightarrow{\boldsymbol{r}}_1 \times m_1 \overrightarrow{\boldsymbol{g}} + \overrightarrow{\boldsymbol{r}}_2 \times m_2 \overrightarrow{\boldsymbol{g}} + \cdots \\
&= \left(m_1 \overrightarrow{\boldsymbol{r}}_1 + m_2 \overrightarrow{\boldsymbol{r}}_2 + \cdots\right) \times \overrightarrow{\boldsymbol{g}} \\
&= \left(\sum_i m_i \overrightarrow{\boldsymbol{r}_i}\right) \times \overrightarrow{\boldsymbol{g}}
\end{aligned}$$

When we multiply and divide this result by the total mass of the object,

$$M = m_1 + m_2 + \cdots = \sum_i m_i$$

we get

$$\overrightarrow{\boldsymbol{\tau}} = \frac{m_1 \overrightarrow{\boldsymbol{r}}_1 + m_2 \overrightarrow{\boldsymbol{r}}_2 + \cdots}{m_1 + m_2 + \cdots} \times M\overrightarrow{g} = \frac{\sum_i m_i \overrightarrow{\boldsymbol{r}}_i}{\sum_i m_i} \times M\overrightarrow{g}$$

The fraction in this equation is just the position vector $\vec{\boldsymbol{r}}_{\mathrm{cm}}$ of the center of mass, with components x_{cm} , y_{cm} , and z_{cm} , as given by Eq. (11.4), and $M\vec{\boldsymbol{g}}$ is equal to the total weight $\vec{\boldsymbol{w}}$ of the object. Thus

(11.5)

$$\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{r}}_{\mathrm{cm}} \times M\vec{\boldsymbol{g}} = \vec{\boldsymbol{r}}_{\mathrm{cm}} \times \vec{\boldsymbol{w}}$$

The total gravitational torque, given by Eq. (11.5), is the same as though the total weight $\vec{\boldsymbol{w}}$ were acting at the position $\vec{\boldsymbol{r}}_{\mathrm{cm}}$ of the center of mass, which we also call the *center of gravity*. **If $\vec{\boldsymbol{g}}$ has the same value at all points on an object, its center of gravity is identical to its center of mass**. Note, however, that the center of mass is defined independently of any gravitational effect.

While the value of $\vec{\boldsymbol{g}}$ varies somewhat with elevation, the variation is extremely slight (Fig. 11.3). We'll assume throughout this chapter that the center of gravity and center of mass are identical unless explicitly stated otherwise.

Figure 11.3

The acceleration due to gravity at the bottom of the 452-m-tall Petronas Towers in Malaysia is only 0.014% greater than at the top. The center of gravity of the towers is only about 2 cm below the center of mass.

Finding and Using the Center of Gravity

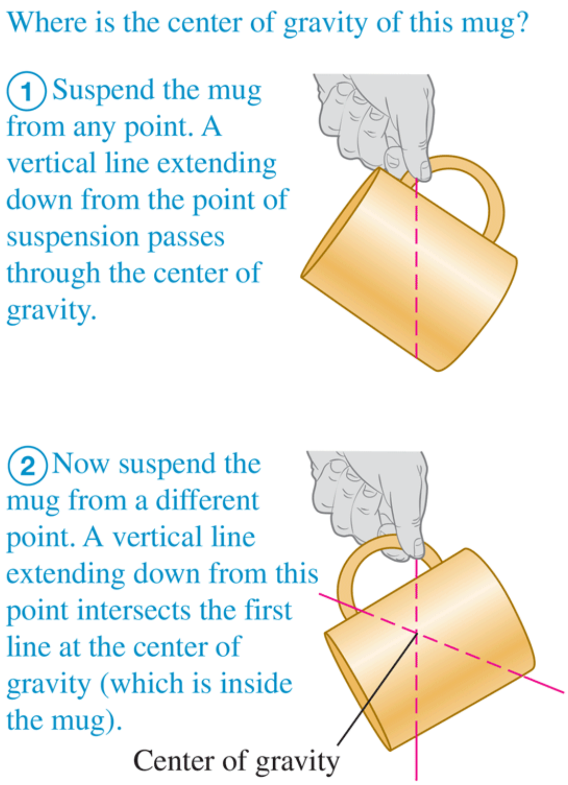
Video Tutor Demo: Balancing a Meter Stick

We can often use symmetry considerations to locate the center of gravity of an object, just as we did for the center of mass. The center of gravity of a homogeneous sphere, cube, or rectangular plate is at its geometric center. The center of gravity of a right circular cylinder or cone is on its axis of symmetry.

For an object with a more complex shape, we can sometimes locate the center of gravity by thinking of the object as being made of symmetrical pieces. For example, we could approximate the human body as a collection of solid cylinders, with a sphere for the head. Then we can locate the center of gravity of the combination with Eqs. (11.3), letting $m_1, m_2, \ \ldots$ be the masses of the individual pieces and $(x_1, y_1, z_1), \ (x_2, y_2, z_2)$,... be the coordinates of their centers of gravity.

When an object in rotational equilibrium and acted on by gravity is supported or suspended at a single point, the center of gravity is always at or directly above or below the point of suspension. If it were anywhere else, the weight would have a torque with respect to the point of suspension, and the object could not be in rotational equilibrium. Figure 11.4 shows an application of this idea.

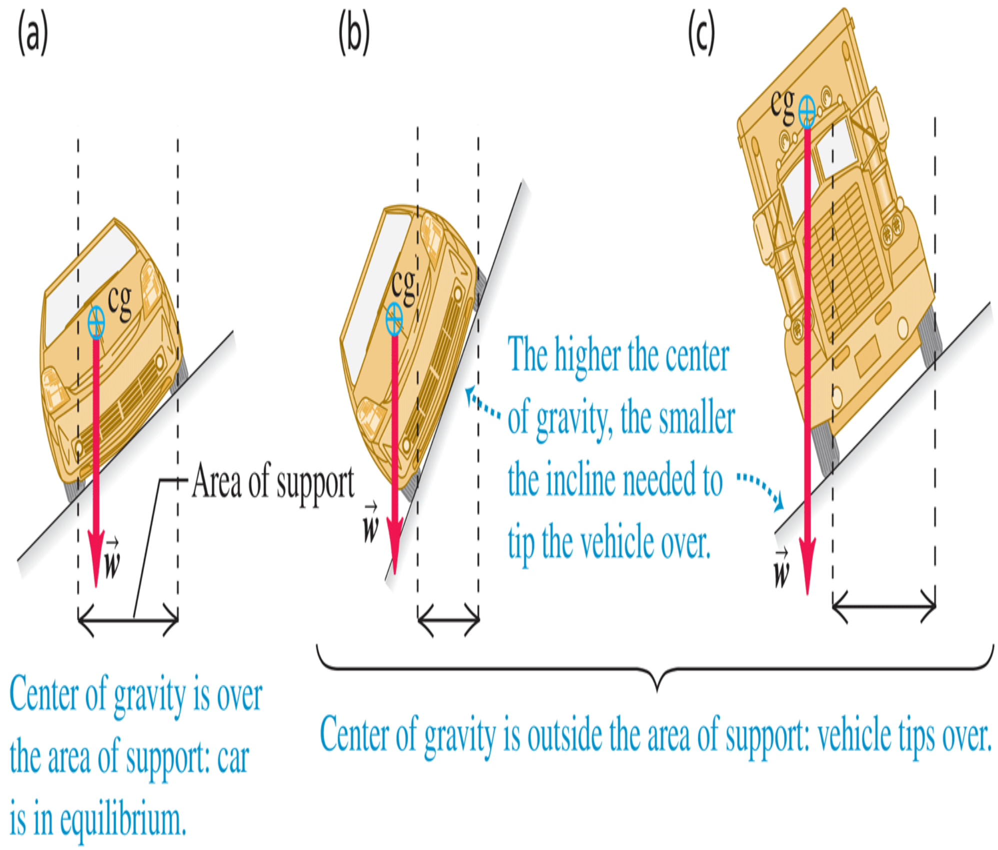
Figure 11.4



Finding the center of gravity of an irregularly shaped object—in this case, a coffee mug.

Using the same reasoning, we can see that an object supported at several points must have its center of gravity somewhere within the area bounded by the supports. This explains why a car can drive on a straight but slanted road if the slant angle is relatively small (Fig. 11.5a) but will tip over if the angle is too steep (Fig. 11.5b). The truck in Fig. 11.5c has a higher center of gravity than the car and will tip over on a shallower incline.

Figure 11.5



In (a) the center of gravity is within the area bounded by the supports, and the car is in equilibrium. The car in (b) and the truck in (c) will tip over because their centers of gravity lie outside the area of support.

The lower the center of gravity and the larger the area of support, the harder it is to overturn an object. Four-legged animals such as deer and horses have a large area of support bounded by their legs; hence they are naturally stable and need only small feet or hooves. Animals that walk on two legs, such as humans and birds, need relatively large feet to give them a reasonable area of support. If a two-legged animal holds its body approximately horizontal, like a chicken or the dinosaur *Tyrannosaurus rex*, it must perform a balancing act as it walks to keep its center of gravity over the foot that is on the ground. A chicken does this by moving its head; *T. rex* probably did it by moving its massive tail.

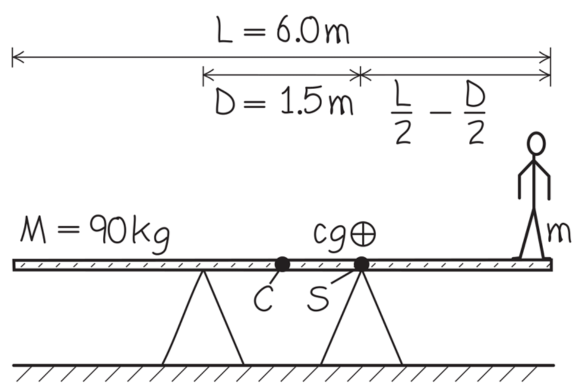
Example 11.1 Walking the plank

WITH VARIATION PROBLEMS

A uniform plank of length $L = 6.0$ m and mass $M = 90$ kg rests on sawhorses separated by $D = 1.5$ m and equidistant from the center of the plank. Cousin Throckmorton wants to stand on the right-hand end of the plank. If the plank is to remain at rest, how massive can Throckmorton be?

IDENTIFY and SET UP To just balance, Throckmorton's mass m must be such that the center of gravity of the plank–Throcky system is directly over the right-hand sawhorse (Fig. 11.6). We take the origin at C, the geometric center and center of gravity of the plank, and take the positive x-axis horizontally to the right. Then the centers of gravity of the plank and Throcky are at $x_{\text{P}} = 0$ and $x_{\text{T}} = L/2 = 3.0$ m, respectively, and the right-hand sawhorse is at $x_{\text{S}} = D/2$. We'll use Eqs. (11.3) to locate the center of gravity x_{cg} of the plank–Throcky system.

Figure 11.6



Our sketch for this problem.

EXECUTE From the first of Eqs. (11.3),

$$x_{\mathrm{cg}} = \frac{M(0) + m(L/2)}{M + m} = \frac{m}{M + m}\,\frac{L}{2}$$

We set $x_{\mathrm{cg}} = x_{\mathrm{S}}$ and solve for m:

$$\begin{aligned} \frac{m}{M+m}\,\frac{L}{2} &= \frac{D}{2} \\ mL &= (M+m)D \\ m = M\frac{D}{L-D} &= (90\ \mathrm{kg})\,\frac{1.5\ \mathrm{m}}{6.0\ \mathrm{m} - 1.5\ \mathrm{m}} = 30\ \mathrm{kg} \end{aligned}$$

EVALUATE As a check, let's repeat the calculation with the origin at the right-hand sawhorse. Now $x_{\mathrm{S}} = 0$, $x_{\mathrm{P}} = -D/2$, and $x_{\mathrm{T}} = (L/2) - (D/2)$, and we require $x_{\mathrm{cg}} = x_{\mathrm{S}} = 0$:

$$\begin{aligned} x_{\mathrm{cg}} &= \frac{M(-D/2) + m[(L/2) - (D/2)]}{M+m} = 0 \\ m &= \frac{MD/2}{(L/2) - (D/2)} = M\frac{D}{L-D} = 30\ \mathrm{kg} \end{aligned}$$

The result doesn't depend on our choice of origin.

A 60 kg adult could stand only halfway between the right-hand sawhorse and the end of the plank. Can you see why?

KEY CONCEPT

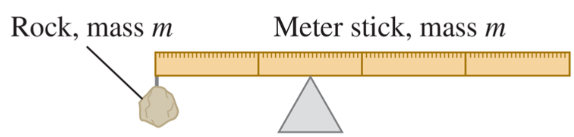
If an extended object supported at two or more points is to be in equilibrium, its center of gravity must be somewhere within the area bounded by the supports. If the object is supported at only one point, its center of gravity must be above that point.

Video Tutor Solution: **Example 11.1**

Test Your Understanding of Section 11.2

A rock is attached to the left end of a uniform meter stick that has the same mass as the rock. In order for the combination of rock and meter stick to balance atop the triangular object in Fig. 11.7, how far from the left end of the stick should the triangular object be placed? (i) Less than 0.25 m; (ii) 0.25 m; (iii) between 0.25 m and 0.50 m; (iv) 0.50 m; (v) more than 0.50 m.

Figure 11.7



At what point will the meter stick with rock attached be in balance?

11.3 Solving Rigid-Body Equilibrium Problems

There are just two key conditions for rigid-body equilibrium: The vector sum of the forces on the object must be zero, and the sum of the torques about any point must be zero. To keep things simple, we'll restrict our attention to situations in which we can treat all forces as acting in a single plane, which we'll call the xy-plane. Then we need consider only the x- and y-components of force in Eq. (11.1), and in Eq. (11.2) we need consider only the z-components of torque (perpendicular to the plane). The first and second conditions for equilibrium are then

(11.6)

$$\sum F_x = 0 \qquad \text{and} \qquad \sum F_y = 0 \, (\text{first condition for equilibrium, forces in xy-plane})$$
$$\sum \tau_z = 0 \qquad (\text{second condition for equilibrium, forces in xy-plane})$$

CAUTION **Choosing the reference point for calculating torques** In equilibrium problems, the choice of reference point for calculating torques in $\sum \tau_z$ is completely arbitrary. But once you make your choice, you must use the *same* point to calculate *all* the torques on an object. Choose the point so as to simplify the calculations as much as possible.

The challenge is to apply these simple conditions to specific problems. Problem-Solving Strategy 11.1 is very similar to the suggestions given in Section 5.1 for the equilibrium of a particle. You should compare it with Problem-Solving Strategy 10.1 (Section 10.2) for rotational dynamics problems.

Problem-Solving Strategy 11.1 Equilibrium of a Rigid Body

IDENTIFY *the relevant concepts*: The first and second conditions for equilibrium $(\sum F_x = 0, \sum F_y = 0,\ \text{and}\ \sum \tau_z = 0)$ are applicable to any rigid body that is not accelerating in space and not rotating.

SET UP *the problem* using the following steps:

1. Sketch the physical situation and identify the object in equilibrium to be analyzed. Sketch the object accurately; do *not* represent it as a point. Include dimensions.
2. Draw a free-body diagram showing all forces acting *on* the object. Show the point on the object at which each force acts.
3. Choose coordinate axes and specify their direction. Specify a positive direction of rotation for torques. Represent forces in terms of their components with respect to the chosen axes.
4. Choose a reference point about which to compute torques. Choose wisely; you can eliminate from your torque equation any force whose line of action goes through the point you choose. The object doesn't actually have to be pivoted about an axis through the reference point.

EXECUTE *the solution* as follows:

1. Write equations expressing the equilibrium conditions. Remember that $\sum F_x = 0$, $\sum F_y = 0$, and $\sum \tau_z = 0$ are *separate* equations. You can compute the torque of a force by finding the torque of each of its components separately, each with its appropriate lever arm and sign, and adding the results.

2. To obtain as many equations as you have unknowns, you may need to compute torques with respect to two or more reference points; choose them wisely, too.

EVALUATE *your answer*: Check your results by writing $\sum \tau_z = 0$ with respect to a different reference point. You should get the same answers.

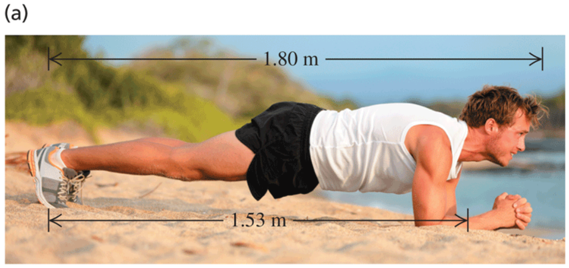
Example 11.2 Locating your center of gravity while you work out

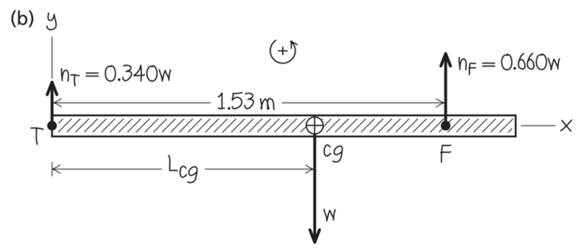
WITH VARIATION PROBLEMS

The *plank* (Fig. 11.8a) is a great way to strengthen abdominal, back, and shoulder muscles. You can also use this exercise position to locate your center of gravity. Holding plank position with a scale under his toes and another under his forearms, one athlete measured that 66.0% of his weight was supported by his forearms and 34.0% by his toes. (That is, the total normal forces on his forearms and toes were $0.660w$ and $0.340w$, respectively, where w is the athlete's weight.) He is 1.80 m tall, and in plank position the distance from his toes to the middle of his forearms is 1.53 m. How far from his toes is his center of gravity?

Figure 11.8

(a)





An athlete in plank position.

IDENTIFY and SET UP We can use the two conditions for equilibrium, Eqs. (11.6), for an athlete at rest. So both the net force and net torque on the athlete are zero. Figure 11.8b shows a free-body diagram, including x- and y-axes and our convention that counterclockwise torques are positive. The weight w acts at the center of gravity, which is between the two supports (as it must be; see Section 11.2). Our target variable is the distance L_{cg}, the lever arm of the weight with respect to the toes T, so it is wise to take torques with respect to T. The torque due to the weight is negative (it tends to cause a clockwise rotation around T), and the torque due to the upward normal force at the forearms F is positive (it tends to cause a counterclockwise rotation around T).

EXECUTE The first condition for equilibrium is satisfied (Fig. 11.8b): $\sum F_x = 0$ because there are no x-components and $\sum F_y = 0$ because $0.340w + 0.660w + (-w) = 0$. We write the torque equation and solve for L_{cg} :

$$\begin{aligned} \sum \tau_z &= 0.340w\,(0) - wL_{cg} + 0.660w(1.53 \text{ m}) = 0 \\ L_{cg} &= 1.01 \text{ m} \end{aligned}$$

EVALUATE The center of gravity is slightly below our athlete's navel (as it is for most people), closer to his head than to his toes. It's also closer to his forearms than to his toes, which is why his forearms support most of his weight. You can check our result by writing the torque equation about the forearms F. You'll find that his center of gravity is 0.52 m from his forearms, or $(1.53\ m) - (0.52\ m) = 1.01\ m$ from his toes.

KEY CONCEPT

For an extended object to be in equilibrium, both the net external *force* and the net external *torque* on the object must be zero. The weight of the object acts at its center of gravity.

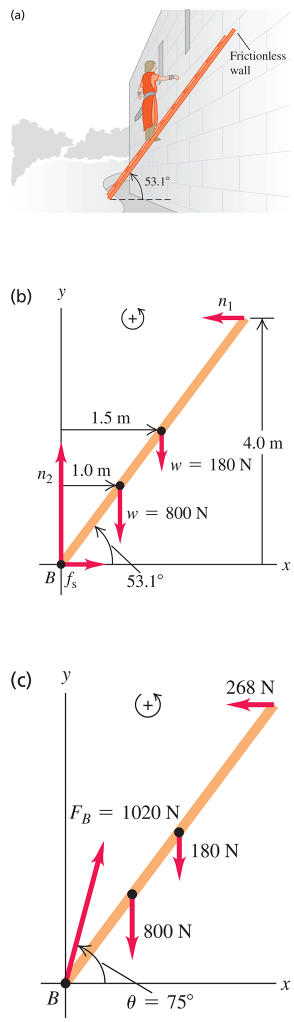
Video Tutor Solution: **Example 11.2**

Example 11.3 Will the ladder slip?

WITH VARIATION PROBLEMS

Sir Lancelot, who weighs 800 N, is assaulting a castle by climbing a uniform ladder that is 5.0 m long and weighs 180 N (Fig. 11.9a). The bottom of the ladder rests on a ledge and leans across the moat in equilibrium against a frictionless, vertical castle wall. The ladder makes an angle of 53.1° with the horizontal. Lancelot pauses one-third of the way up the ladder. (a) Find the normal and friction forces on the base of the ladder. (b) Find the minimum coefficient of static friction needed to prevent slipping at the base. (c) Find the magnitude and direction of the contact force on the base of the ladder.

Figure 11.9



(a) Sir Lancelot pauses a third of the way up the ladder, fearing it will slip. (b) Free-body diagram for the system of Sir Lancelot and the ladder. (c) The contact force at B is the superposition of the normal force and the static friction force.

IDENTIFY and SET UP The ladder–Lancelot system is stationary, so we can use the two conditions for equilibrium to solve part (a). In part (b), we need the relationship among the static friction force, coefficient of static friction, and normal force (see Section 5.3). In part (c), the contact force is the vector sum of the normal and friction forces acting at the base of the ladder, found in part (a). Figure 11.9b shows the free-body diagram, with x- and y-directions as shown and with counterclockwise torques taken to be positive. The ladder's center of gravity is at its geometric center. Lancelot's 800 N weight acts at a point one-third of the way up the ladder.

The wall exerts only a normal force n_1 on the top of the ladder. The forces on the base are an upward normal force n_2 and a static friction force f_s, which must point to the right to prevent slipping. The magnitudes n_2 and f_s are the target variables in part (a). From Eq. (5.4), these magnitudes are related by $f_s \le \mu_s n_2$; the coefficient of static friction μ_s is the target variable in part (b).

EXECUTE

(a) From **Eqs. (11.6)**, the first condition for equilibrium gives

$$\sum F_x = f_s + (-n_1) = 0$$
$$\sum F_y = n_2 + (-800\text{ N}) + (-180\text{ N}) = 0$$

These are two equations for the three unknowns n_1, n_2, and f_s. The second equation gives $n_2 = 980\text{ N}$. To obtain a third equation, we use the second condition for equilibrium. We take torques about point B, about which n_2 and f_s have no torque. The 53.1° angle creates a 3-4-5 right triangle, so from **Fig. 11.9b** the lever arm for the ladder's weight is 1.5 m, the lever arm for Lancelot's weight is 1.0 m, and the lever arm for n_1 is 4.0 m. The torque equation for point B is then

$$\sum \tau_B = n_1(4.0\ \mathrm{m}) - (180\ \mathrm{N})(1.5\ \mathrm{m}) \quad - (800\ \mathrm{N})(1.0\ \mathrm{m}) + n_2\ (0) + f_{\mathrm{s}}$$

Solving for n_1, we get $n_1 = 268\ \mathrm{N}$. We substitute this into the $\sum F_x = 0$ equation and get $f_{\mathrm{s}} = 268\ \mathrm{N}$.

(b) The static friction force f_{s} cannot exceed $\mu_{\mathrm{s}} n_2$, so the *minimum* coefficient of static friction to prevent slipping is

$$(\mu_{\mathrm{s}})_{min} = \frac{f_{\mathrm{s}}}{n_2} = \frac{268\ \mathrm{N}}{980\ \mathrm{N}} = 0.27$$

(c) The components of the contact force $\vec{\boldsymbol{F}}_B$ at the base are the static friction force f_{s} and the normal force n_2 , so

$$\vec{\boldsymbol{F}}_B = f_{\mathrm{s}}\hat{\imath} + n_2\,\hat{\jmath} = (268\ \mathrm{N})\hat{\imath} + (980\ \mathrm{N})\hat{\jmath}$$

The magnitude and direction of $\vec{\boldsymbol{F}}_B$ (**Fig. 11.9c**) are

$$\begin{aligned} F_B &= \sqrt{(268\ \mathrm{N})^2 + (980\ \mathrm{N})^2} = 1020\ \mathrm{N} \\ \theta &= \arctan\frac{980\ \mathrm{N}}{268\ \mathrm{N}} = 75^\circ \end{aligned}$$

EVALUATE As Fig. 11.9c shows, the contact force $\vec{\boldsymbol{F}}_B$ is *not* directed along the length of the ladder. Can you show that if $\vec{\boldsymbol{F}}_B$ were directed along the ladder, there would be a net counterclockwise torque with respect to the top of the ladder, and equilibrium would be impossible?

As Lancelot climbs higher on the ladder, the lever arm and torque of his weight about B increase. This increases the values of n_1, f_s, and the required friction coefficient $(\mu_s)_{min}$, so the ladder is more and more likely to slip as he climbs (see Exercise 11.10). A simple way to make slipping less likely is to use a larger ladder angle (say, 75° rather than 53.1°). This decreases the lever arms with respect to B of the weights of the ladder and Lancelot and increases the lever arm of n_1, all of which decrease the required friction force.

If we had assumed friction on the wall as well as on the floor, the problem would be impossible to solve by using the equilibrium conditions alone. (Try it!) The difficulty is that it's no longer adequate to treat the object as being perfectly rigid. Another problem of this kind is a four-legged table; there's no way to use the equilibrium conditions alone to find the force on each separate leg.

KEY CONCEPT

In an equilibrium problem, you can calculate torques around any point you choose. The torque equation will not include any force whose line of action goes through your chosen point, so choosing the point wisely can simplify your calculations.

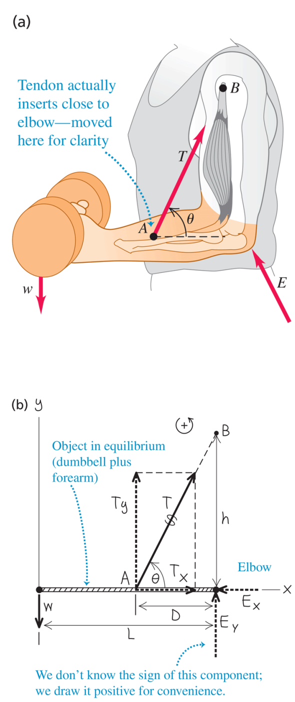
Video Tutor Solution: **Example 11.3**

Example 11.4 Equilibrium and pumping iron

WITH VARIATION PROBLEMS

Figure 11.10a shows a horizontal human arm lifting a dumbbell. The forearm is in equilibrium under the action of the weight $\vec{w}$ of the dumbbell, the tension $\vec{T}$ in the tendon connected to the biceps muscle, and the force $\vec{E}$ exerted on the forearm by the upper arm at the elbow joint. We ignore the weight of the forearm itself. (For clarity, in the drawing we've exaggerated the distance from the elbow to the point A where the tendon is attached.) Given the weight w and the angle θ between the tension force and the horizontal, find T and the two components of $\vec{E}$ (three unknown scalar quantities in all).

Figure 11.10



(a) The situation. (b) Our free-body diagram for the forearm. The weight of the forearm is ignored, and the distance D is greatly exaggerated for clarity.

IDENTIFY and SET UP The system is at rest, so we use the conditions for equilibrium. We represent $\vec{\boldsymbol{T}}$ and $\vec{\boldsymbol{E}}$ in terms of their components (Fig. 11.10b). We guess that the directions of E_x and E_y are as shown; the signs of E_x and E_y as given by our solution will tell us the actual directions. Our target variables are T, E_x , and E_y .

EXECUTE To find T, we take torques about the elbow joint so that the torque equation does not contain E_x , E_y , or T_x , then solve for T_y and hence T:

$$\sum \tau_{\text{elbow}} = Lw - DT_y = 0$$
$$T_y = \frac{Lw}{D} = T \sin \theta \qquad \text{and} \qquad T = \frac{Lw}{D \sin \theta}$$

To find E_x and E_y , we use the first conditions for equilibrium:

$$\begin{aligned} \sum F_x &= T_x + (-E_x) = 0 \\ E_x &= T_x = T \cos \theta = \frac{Lw}{D \sin \theta} \cos \theta = \frac{Lw}{D} \cot \theta = \frac{Lw}{D}\frac{D}{h} = \frac{Lw}{h} \\ \sum F_y &= T_y + E_y + (-w) = 0 \\ E_y &= w - \frac{Lw}{D} = -\frac{(L-D)w}{D} \end{aligned}$$

The negative sign for E_y tells us that it should actually point *down* in Fig. 11.10b.

EVALUATE We can check our results for E_x and E_y by taking torques about points A and B, about both of which T has zero torque:

$$\begin{aligned} \sum \tau_A &= (L-D)w + DE_y = 0 \qquad \text{so} \qquad E_y = -\frac{(L-D)w}{D} \\ \sum \tau_B &= Lw - hE_x = 0 \qquad \text{so} \qquad E_x = \frac{Lw}{h} \end{aligned}$$

As a realistic example, take $w = 200 \text{ N}$, $D = 0.050 \text{ m}$, $L = 0.30 \text{ m}$, and $\theta = 80^\circ$, so that $h = D \tan\theta = (0.050 \text{ m})(5.67) = 0.28 \text{ m}$. Using our results for T, E_x, and E_y, we find

$$\begin{aligned} T &= \frac{Lw}{D \sin\theta} = \frac{(0.30 \text{ m})(200 \text{ N})}{(0.050 \text{ m})\,(0.98)} = 1220 \text{ N} \\ E_y &= -\frac{(L-D)w}{D} = -\frac{(0.30 \text{ m} - 0.050 \text{ m})(200 \text{ N})}{0.050 \text{ m}} = -1000 \text{ N} \\ E_x &= \frac{Lw}{h} = \frac{(0.30 \text{ m})(200 \text{ N})}{0.28 \text{ m}} = 210 \text{ N} \end{aligned}$$

The magnitude of the force at the elbow is

$$E = \sqrt{E^2{}_x + E^2{}_y} = 1020 \text{ N}$$

Note that T and E are *much* larger than the 200 N weight of the dumbbell. A forearm weighs only about 20 N, so it was reasonable to ignore its weight.

KEY CONCEPT

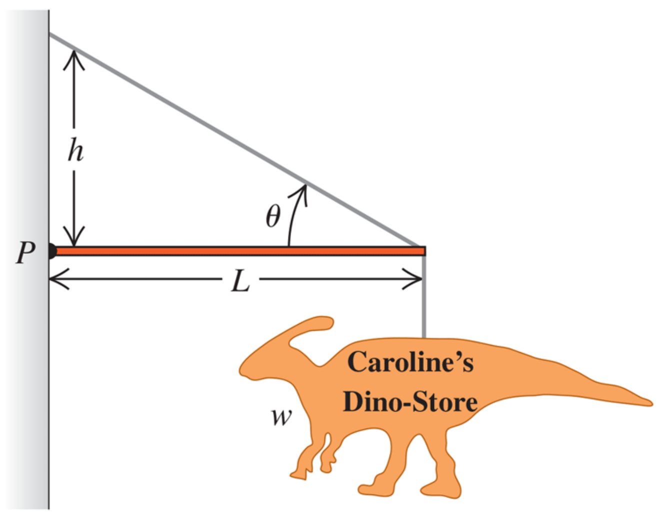
In a two-dimensional equilibrium problem, you have *three* equilibrium equations: two from the condition that the net external force is zero and one from the condition that the net external torque is zero. Use these equations to solve for the unknowns.

Video Tutor Solution: **Example 11.4**

Test Your Understanding of Section 11.3

A metal advertising sign (weight w) for a specialty shop is suspended from the end of a horizontal rod of length L and negligible mass (Fig. 11.11). The rod is supported by a cable at an angle θ from the horizontal and by a hinge at point P. Rank the following force magnitudes in order from greatest to smallest: (i) the weight w of the sign; (ii) the tension in the cable; (iii) the vertical component of force exerted on the rod by the hinge at P.

Figure 11.11



What are the tension in the diagonal cable and the vertical component of force exerted by the hinge at P?

11.4 Stress, Strain, and Elastic Moduli

The rigid body is a useful idealized model, but the stretching, squeezing, and twisting of real objects when forces are applied are often too important to ignore. Figure 11.12 shows three examples. We want to study the relationship between the forces and deformations for each case.

Figure 11.12

(a)

(b)

(c)

Three types of stress. (a) Guitar strings under *tensile stress*, being stretched by forces acting at their ends. (b) A diver under *bulk stress*, being squeezed from all sides by forces due to water pressure. (c) A ribbon under *shear stress*, being deformed and eventually cut by forces exerted by the scissors.

You don’t have to look far to find a deformable object; it’s as plain as the nose on your face (Fig. 11.13). If you grasp the tip of your nose between your index finger and thumb, you’ll find that the harder you pull your nose outward or push it inward, the more it stretches or compresses. Likewise, the harder you squeeze your index finger and thumb together, the more the tip of your nose compresses. If you try to twist the tip of your nose, you’ll get a greater amount of twist if you apply stronger forces.

Figure 11.13

When you pinch your nose, the force per area that you apply to your nose is called *stress*. The fractional change in the size of your nose (the change in size divided by the initial size) is called *strain*. The deformation is *elastic* because your nose springs back to its initial size when you stop pinching.

These observations illustrate a general rule. In each case you apply a **stress** to your nose; the amount of stress is a measure of the forces causing the deformation, on a “force per unit area” basis. And in each case the stress causes a deformation, or **strain**. More careful versions of the experiments with your nose suggest that for relatively small stresses, the resulting strain is proportional to the stress: The greater the deforming forces, the greater the resulting deformation. This proportionality is called **Hooke’s law**, and the ratio of stress to strain is called the **elastic modulus**:

(11.7)

$$\text{Hooke's law: } \frac{\overbrace{\text{Stress}}^{\text{Measure of forces applied to deform an object}}}{\underbrace{\text{Strain}}_{\text{Measure of how much deformation results from stress}}} = \underbrace{\text{Elastic modulus}}_{\text{Property of material of which object is made}} \tag{11.7}$$

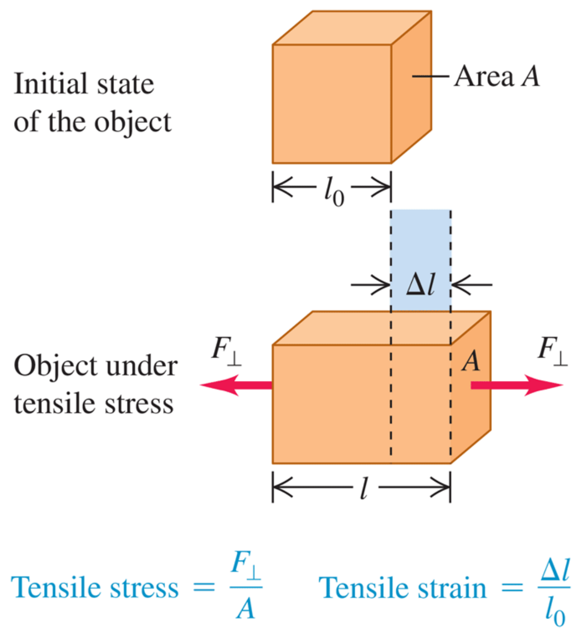
The value of the elastic modulus depends on what the object is made of but not its shape or size. If a material returns to its original state after the stress is removed, it is called **elastic;** Hooke's law is a special case of elastic behavior. If a material instead remains deformed after the stress is removed, it is called **plastic**. Here we'll consider elastic behavior only; we'll return to plastic behavior in Section 11.5.

We used one form of Hooke's law in Section 6.3: The elongation of an ideal spring is proportional to the stretching force. Remember that Hooke's "law" is not really a general law; it is valid over only a limited range of stresses. In Section 11.5 we'll see what happens beyond that limited range.

Tensile and Compressive Stress and Strain

The simplest elastic behavior to understand is the stretching of a bar, rod, or wire when its ends are pulled (Fig. 11.12a). Figure 11.14 shows an object that initially has uniform cross-sectional area A and length l_0. We then apply forces of equal magnitude $F_\perp$ but opposite directions at the ends (this ensures that the object has no tendency to move left or right). We say that the object is in **tension**. We've already talked a lot about tension in ropes and strings; it's the same concept here. The subscript $\perp$ is a reminder that the forces act perpendicular to the cross section.

Figure 11.14



An object in tension. The net force on the object is zero, but the object deforms. The tensile stress (the ratio of the force to the cross-sectional area) produces a tensile strain (the elongation divided by the initial length). The elongation Δl is exaggerated for clarity.

We define the **tensile stress** at the cross section as the ratio of the force $F_\perp$ to the cross-sectional area A:

(11.8)

$$\text{Tensile stress} = \frac{F_\perp}{A}$$

This is a *scalar* quantity because $F_\perp$ is the *magnitude* of the force. The SI unit of stress is the **pascal** (abbreviated Pa and named for the 17th-century French scientist and philosopher Blaise Pascal). Equation (11.8) shows that 1 pascal equals 1 newton per square meter $(\mathrm{N/m^2})$:

$$1 \text{ pascal} = 1 \text{ Pa} = 1 \text{ N/m}^2$$

In the British system the most common unit of stress is the pound per square inch ($\text{lb}/in.^2$ or psi). The conversion factors are

$$1 \text{ psi} = 6895 \text{ Pa} \qquad \text{and} \qquad 1 \text{ Pa} = 1.450 \times 10^{-4} \text{ psi}$$

The units of stress are the same as those of *pressure*, which we'll encounter often in later chapters.

Under tension the object in Fig. 11.14 stretches to a length $l = l_0 + \Delta l$. The elongation Δl does not occur only at the ends; every part of the object stretches in the same proportion. The **tensile strain** of the object equals the fractional change in length, which is the ratio of the elongation Δl to the original length l_0 :

(11.9)

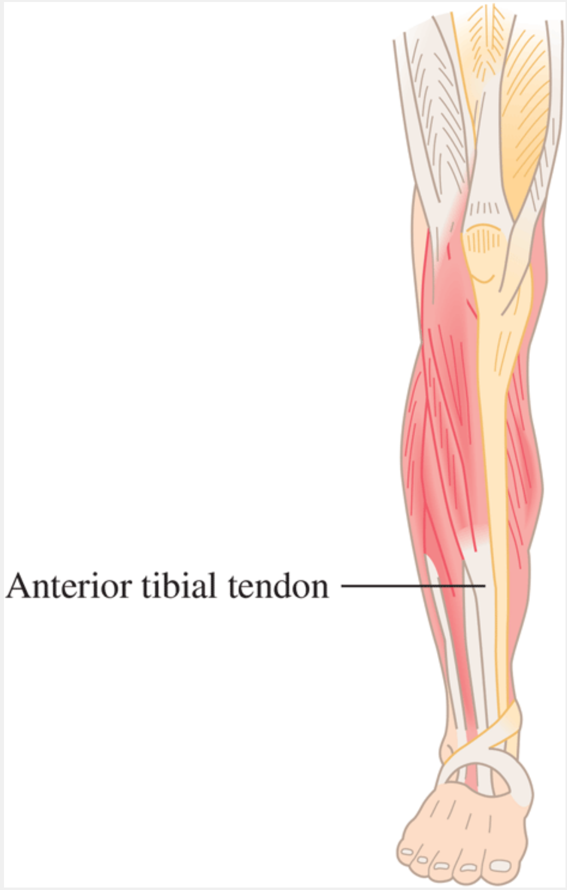
$$\text{Tensile strain} = \frac{l - l_0}{l_0} = \frac{\Delta l}{l_0}$$

Tensile strain is stretch per unit length. It is a ratio of two lengths, always measured in the same units, and so is a pure (dimensionless) number with no units.

BIO Application

Young's Modulus of a Tendon

The anterior tibial tendon connects your foot to the large muscle that runs along the side of your shinbone. (You can feel this tendon at the front of your ankle.) Measurements show that this tendon has a Young's modulus of $1.2 \times 10^9\ Pa$, much less than for the metals listed in Table 11.1. Hence this tendon stretches substantially (up to 2.5% of its length) in response to the stresses experienced in walking and running.



Experiment shows that for a sufficiently small tensile stress, stress and strain are proportional, as in Eq. (11.7). The corresponding elastic modulus is called **Young's modulus**, denoted by Y:

(11.10)

Young's modulus for tension

Force applied perpendicular to cross section

Original length (see Fig. 11.14)

$$Y = \frac{\text{Tensile stress}}{\text{Tensile strain}} = \frac{F_{\perp}/A}{\Delta l/l_0} = \frac{F_{\perp}}{A}\frac{l_0}{\Delta l} \qquad (11.10)$$

Cross-sectional area of object

Elongation (see Fig. 11.14)

Since strain is a pure number, the units of Young's modulus are the same as those of stress: force per unit area. Table 11.1 lists some typical values. (This table also gives values of two other elastic moduli that we'll discuss later in this chapter.) A material with a large value of Y is relatively unstretchable; a large stress is required for a given strain. For example, the value of Y for cast steel (2×10^{11} Pa) is much larger than that for a tendon (1.2×10^{9} Pa).

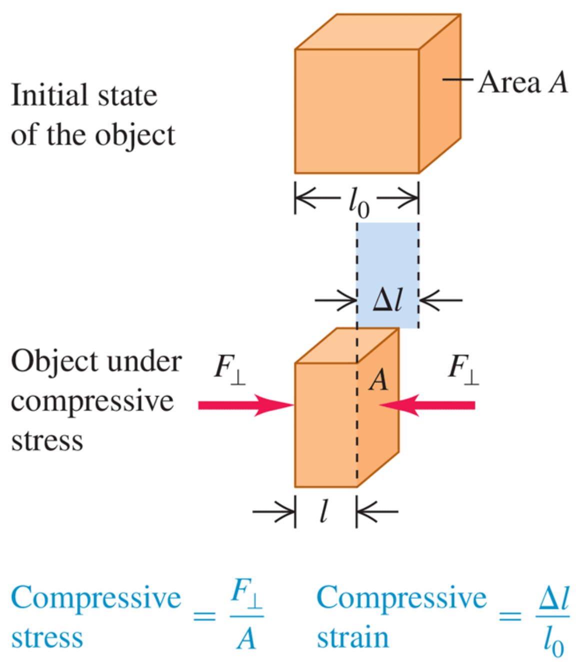
Table 11.1 Approximate Elastic Moduli

Material	Young's Modulus, Y (Pa)	Bulk Modulus, B (Pa)	Shear Modulus, S (Pa)
Aluminum	7.0×10^{10}	7.5×10^{10}	2.5×10^{10}
Brass	9.0×10^{10}	6.0×10^{10}	3.5×10^{10}
Copper	11×10^{10}	14×10^{10}	4.4×10^{10}
Iron	21×10^{10}	16×10^{10}	7.7×10^{10}
Lead	1.6×10^{10}	4.1×10^{10}	0.6×10^{10}
Nickel	21×10^{10}	17×10^{10}	7.8×10^{10}
Silicone rubber	0.001×10^{10}	0.2×10^{10}	0.0002×10^{10}
Steel	20×10^{10}	16×10^{10}	7.5×10^{10}
Tendon (typical)	0.12×10^{10}	—	—

? When the forces on the ends of a bar are pushes rather than pulls (Fig. 11.15), the bar is in **compression** and the stress is a **compressive stress**.

The **compressive strain** of an object in compression is defined in the same way as the tensile strain, but Δl has the opposite direction. Hooke's law and Eq. (11.10) are valid for compression as well as tension if the compressive stress is not too great. For many materials, Young's modulus has the same value for both tensile and compressive stresses. Composite materials such as concrete and stone are an exception; they can withstand compressive stresses but fail under comparable tensile stresses. Stone was the primary building material used by ancient civilizations such as the Babylonians, Assyrians, and Romans, so their structures had to be designed to avoid tensile stresses. Hence they used arches in doorways and bridges, where the weight of the overlying material compresses the stones of the arch together and does not place them under tension.

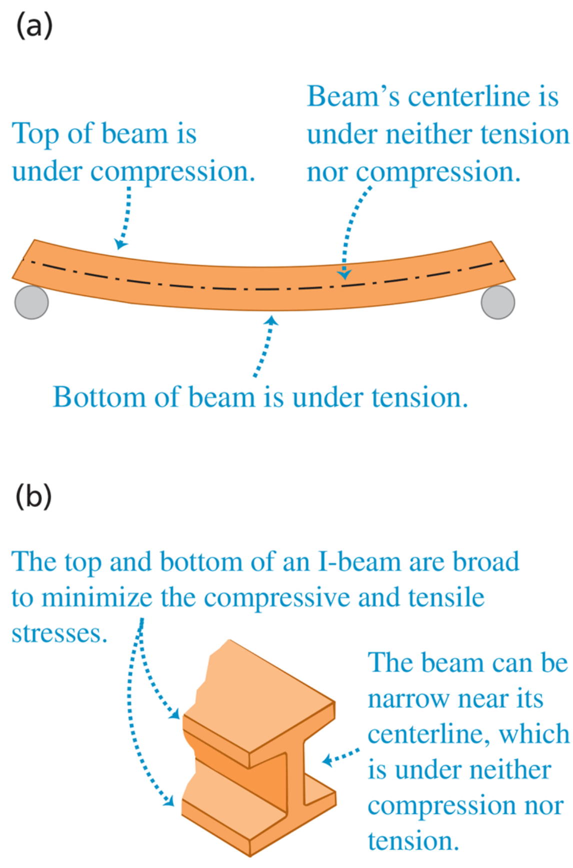
Figure 11.15



An object in compression. The compressive stress and compressive strain are defined in the same way as tensile stress and strain (see Fig. 11.14), except that Δl now denotes the distance that the object contracts.

In many situations, objects can experience both tensile and compressive stresses at the same time. For example, a horizontal beam supported at each end sags under its own weight. As a result, the top of the beam is under compression while the bottom of the beam is under tension (Fig. 11.16a). To minimize the stress and hence the bending strain, the top and bottom of the beam are given a large cross-sectional area. There is neither compression nor tension along the centerline of the beam, so this part can have a small cross section; this helps keep the weight of the beam to a minimum and further helps reduce the stress. The result is an I-beam of the familiar shape used in building construction (Fig. 11.16b).

Figure 11.16



(a) A beam supported at both ends is under both compression and tension. (b) The cross-sectional shape of an I-beam minimizes both stress and weight.

Example 11.5 Tensile stress and strain

WITH VARIATION PROBLEMS

A steel rod 2.0 m long has a cross-sectional area of 0.30 cm^2. It is hung by one end from a support, and a 550 kg milling machine is hung from its other end. Determine the stress on the rod and the resulting strain and elongation.

IDENTIFY, SET UP, and EXECUTE The rod is under tension, so we can use Eq. (11.8) to find the tensile stress; Eq. (11.9), with the value of Young's modulus Y for steel from Table 11.1, to find the corresponding strain; and Eq. (11.10) to find the elongation Δl:

$$\begin{aligned}\text{Tensile stress} &= \frac{F_\perp}{A} = \frac{(550\ \text{kg})(9.8\ \text{m/s}^2)}{3.0\times10^{-5}\ \text{m}^2} = 1.8\times10^8\ \text{Pa}\\ \text{Strain} &= \frac{\Delta l}{l_0} = \frac{\text{Stress}}{Y} = \frac{1.8\times10^8\ \text{Pa}}{20\times10^{10}\ \text{Pa}} = 9.0\times10^{-4}\\ \text{Elongation}=\Delta l &= (\text{Strain})\times l_0\\ &= \left(9.0\times10^{-4}\right)(2.0\ \text{m}) = 0.0018\ \text{m} = 1.8\ \text{mm}\end{aligned}$$

EVALUATE This small elongation, resulting from a load of over half a ton, is a testament to the stiffness of steel. (We've ignored the relatively small stress due to the weight of the rod itself.)

KEY CONCEPT

For an object under tensile or compressive stress, the stress equals the force exerted on either end of the object divided by the cross-sectional area of either end. The strain equals the fractional change in length. The ratio of stress to strain equals Young's modulus for the material of which the object is made.

Video Tutor Solution: Example 11.5

Bulk Stress and Strain

When a scuba diver plunges deep into the ocean, the water exerts nearly uniform pressure everywhere on his surface and squeezes him to a slightly smaller volume (see Fig. 11.12b). This is a different situation from the tensile and compressive stresses and strains we have discussed. The uniform pressure on all sides of the diver is a **bulk stress** (or **volume stress**), and the resulting deformation—a **bulk strain** (or **volume strain**)—is a change in his volume.

BIO Application

Bulk Stress on an Anglerfish

The anglerfish (*Melanocetus johnsonii*) is found in oceans throughout the world at depths as great as 1000 m, where the pressure (that is, the bulk stress) is about 100 atmospheres. Anglerfish are able to withstand such stress because they have no internal air spaces, unlike fish found in the upper ocean, where pressures are lower. The largest anglerfish are about 12 cm (5 in.) long.

If an object is immersed in a fluid (liquid or gas) at rest, the fluid exerts a force on any part of the object's surface; this force is *perpendicular* to the surface. (If we tried to make the fluid exert a force parallel to the surface, the fluid would slip sideways to counteract the effort.) The force $F_{\perp}$ per unit area that the fluid exerts on an immersed object is called the **pressure** p in the fluid:

(11.11)

$$\text{Pressure in a fluid} \rightarrow p = \frac{F_{\perp} \leftarrow \text{Force that fluid applies to surface of an immersed object}}{A \leftarrow \text{Area over which force is exerted}} \qquad (11.11)$$

Pressure has the same units as stress; commonly used units include 1 Pa $(= 1\ N/m^2)$, $1\ lb/in.^2\ (1\ psi)$, and 1 **atmosphere** (1 atm). One atmosphere is the approximate average pressure of the earth's atmosphere at sea level:

$$1 \text{ atmosphere} = 1 \text{ atm} = 1.013 \times 10^5 \text{ Pa} = 14.7 \text{ lb/in.}^2$$

CAUTION Pressure vs. force Unlike force, pressure has no intrinsic direction: The pressure on the surface of an immersed object is the same no matter how the surface is oriented. Hence pressure is a *scalar* quantity, not a vector quantity.

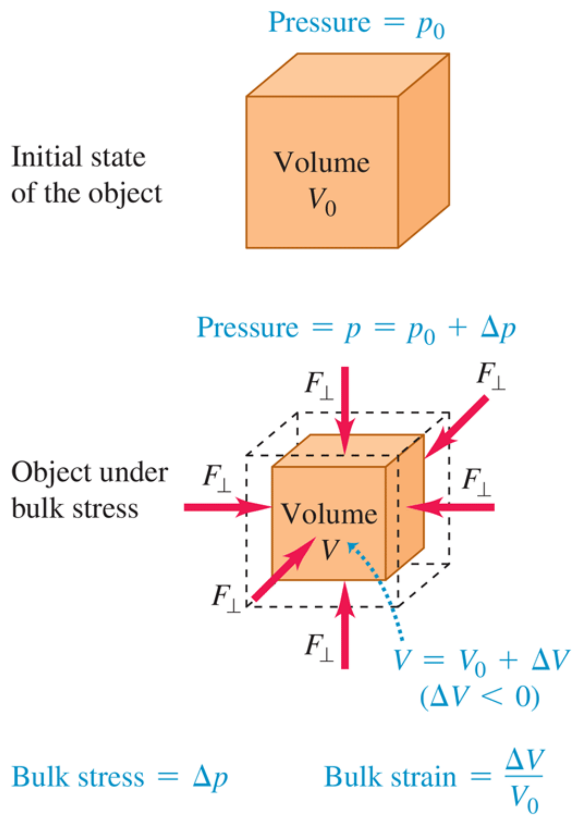
The pressure in a fluid increases with depth. For example, the pressure in the ocean increases by about 1 atm every 10 m. If an immersed object is relatively small, however, we can ignore these pressure differences for purposes of calculating bulk stress. We'll then treat the pressure as having the same value at all points on an immersed object's surface.

Pressure plays the role of stress in a volume deformation. The corresponding strain is the fractional change in volume (Fig. 11.17)—that is, the ratio of the volume change ΔV to the original volume V_0 :

(11.12)

$$\text{Bulk (volume) strain} = \frac{\Delta V}{V_0}$$

Figure 11.17



An object under bulk stress. Without the stress, the cube has volume V_0; when the stress is applied, the cube has a smaller volume V. The volume change ΔV is exaggerated for clarity.

Volume strain is the change in volume per unit volume. Like tensile or compressive strain, it is a pure number, without units.

When Hooke's law is obeyed, an increase in pressure (bulk stress) produces a *proportional* bulk strain (fractional change in volume). The corresponding elastic modulus (ratio of stress to strain) is called the **bulk modulus**, denoted by B. When the pressure on an object changes by a small amount Δp, from p_0 to $p_0 + \Delta p$, and the resulting bulk strain is $\Delta V/V_0$, Hooke's law takes the form

(11.13)

Bulk modulus for compression

Additional pressure on object

$$B = \frac{\text{Bulk stress}}{\text{Bulk strain}} = -\frac{\Delta p}{\Delta V/V_0} \qquad (11.13)$$

Original volume (see Fig. 11.17)

Change in volume (see Fig. 11.17)

We include a minus sign in this equation because an *increase* of pressure always causes a *decrease* in volume. In other words, if Δp is positive, ΔV is negative. The bulk modulus B itself is a positive quantity.

For small pressure changes in a solid or a liquid, we consider B to be constant. The bulk modulus of a *gas*, however, depends on the initial pressure p_0 . Table 11.1 includes values of B for several solid materials. Its units, force per unit area, are the same as those of pressure (and of tensile or compressive stress).

The reciprocal of the bulk modulus is called the **compressibility** and is denoted by k. From Eq. (11.13),

(11.14)

$$k = \frac{1}{B} = -\frac{\Delta V/V_0}{\Delta p} = -\frac{1}{V_0}\frac{\Delta V}{\Delta p} \qquad \text{(compressibility)}$$

Compressibility is the fractional decrease in volume, $-\Delta V/V_0$, per unit increase Δp in pressure. The units of compressibility are those of *reciprocal pressure*, Pa^{-1} or atm^{-1}.

Table 11.2 lists the values of compressibility k for several liquids. For example, the compressibility of water is $46.4 \times 10^{-6}\ \mathrm{atm}^{-1}$, which means that the volume of water decreases by 46.4 parts per million for each 1 atmosphere increase in pressure. Materials with small bulk modulus B and large compressibility k are easiest to compress.

Table 11.2 Compressibilities of Liquids

Liquid	Compressibility, k	
	Pa^{-1}	atm^{-1}
Carbon disulfide	93×10^{-11}	94×10^{-6}
Ethyl alcohol	110×10^{-11}	111×10^{-6}
Glycerin	21×10^{-11}	21×10^{-6}
Mercury	3.7×10^{-11}	3.8×10^{-6}
Water	45.8×10^{-11}	46.4×10^{-6}

Example 11.6 Bulk stress and strain

WITH VARIATION PROBLEMS

A hydraulic press contains $0.25\ \mathrm{m}^3$ (250 L) of oil. Find the decrease in the volume of the oil when it is subjected to a pressure increase $\Delta p = 1.6 \times 10^7$ Pa (about 160 atm or 2300 psi). The bulk modulus of the oil is $B = 5.0 \times 10^9$ Pa (about 5.0×10^4 atm), and its compressibility is $k = 1/B = 20 \times 10^{-6}\ \mathrm{atm}^{-1}$.

IDENTIFY, SET UP, and EXECUTE This example uses the ideas of bulk stress and strain. We are given both the bulk modulus and the compressibility, and our target variable is ΔV. Solving Eq. (11.13) for ΔV, we find

$$\begin{aligned} \Delta V &= -\frac{V_0\,\Delta p}{B} = -\frac{(0.25\ \mathrm{m}^3)(1.6 \times 10^7\ \mathrm{Pa})}{5.0 \times 10^9\ \mathrm{Pa}} \\ = -8.0 \times 10^{-4}\ \mathrm{m}^3 &= -0.80\ \mathrm{L} \end{aligned}$$

Alternatively, we can use Eq. (11.14) with the approximate unit conversions given above:

$$
\begin{aligned}
\Delta V &= -kV_0\,\Delta p = -(20\times 10^{-6}\ \text{atm}^{-1})(0.25\ \text{m}^3)(160\ \text{atm})\\
&= -8.0\times 10^{-4}\ \text{m}^3
\end{aligned}
$$

EVALUATE The negative value of ΔV means that the volume decreases when the pressure increases. The 160 atm pressure increase is large, but the *fractional* volume change is very small:

$$
\frac{\Delta V}{V_0} = \frac{-8.0\times 10^{-4}\ \text{m}^3}{0.25\ \text{m}^3} = -0.0032 \qquad \text{or} \qquad -0.32\%
$$

KEY CONCEPT

For an object under bulk stress, the stress equals the additional pressure applied to all sides of the object. The strain equals the fractional change in volume. The ratio of stress to strain equals the bulk modulus for the material of which the object is made.

Video Tutor Solution: **Example 11.6**

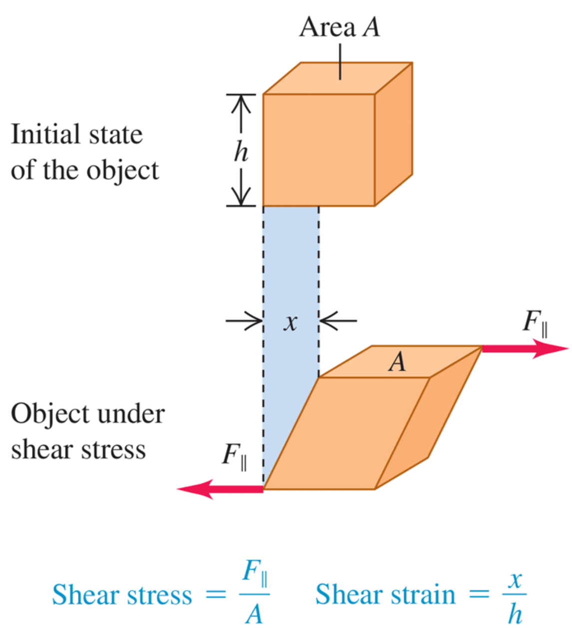
Shear Stress and Strain

The third kind of stress-strain situation is called *shear*. The ribbon in Fig. 11.12c is under **shear stress**: One part of the ribbon is being pushed up while an adjacent part is being pushed down, producing a deformation of the ribbon. Figure 11.18 shows an object being deformed by a shear stress. In the figure, forces of equal magnitude but opposite direction act *tangent* to the surfaces of opposite ends of the object. We define the shear stress as the force $F_{||}$ acting tangent to the surface divided by the area A on which it acts:

(11.15)

$$\text{Shear stress} = \frac{F_{||}}{A}$$

Figure 11.18



An object under shear stress. Forces are applied tangent to opposite surfaces of the object (in contrast to the situation in Fig. 11.14, in which the forces act perpendicular to the surfaces). The deformation x is exaggerated for clarity.

Shear stress, like the other two types of stress, is a force per unit area.

Figure 11.18 shows that one face of the object under shear stress is displaced by a distance x relative to the opposite face. We define **shear strain** as the ratio of the displacement x to the transverse dimension h:

(11.16)

$$\text{Shear strain} = \frac{x}{h}$$

In real-life situations, x is typically much smaller than h. Like all strains, shear strain is a dimensionless number; it is a ratio of two lengths.

If the forces are small enough that Hooke's law is obeyed, the shear strain is *proportional* to the shear stress. The corresponding elastic modulus (ratio of shear stress to shear strain) is called the **shear modulus**, denoted by S:

(11.17)

Force applied tangent to surface of object

Transverse dimension (see Fig. 11.18)

Shear modulus for shear

$$S = \frac{\text{Shear stress}}{\text{Shear strain}} = \frac{F_{\parallel}/A}{x/h} = \frac{F_{\parallel}}{A}\frac{h}{x} \qquad (11.17)$$

Deformation (see Fig. 11.18)

Area over which force is exerted

Table 11.1 gives several values of shear modulus. For a given material, S is usually one-third to one-half as large as Young's modulus Y for tensile stress. Keep in mind that the concepts of shear stress, shear strain, and shear modulus apply to *solid* materials only. The reason is that *shear* refers to deforming an object that has a definite shape (see Fig. 11.18). This concept doesn't apply to gases and liquids, which do not have definite shapes.

Example 11.7 Shear stress and strain

WITH VARIATION PROBLEMS

Suppose the object in Fig. 11.18 is the brass base plate of an outdoor sculpture that experiences shear forces in an earthquake. The plate is 0.80 m square and 0.50 cm thick. What is the force exerted on each of its edges if the resulting displacement x is 0.16 mm?

IDENTIFY and SET UP This example uses the relationship among shear stress, shear strain, and shear modulus. Our target variable is the force $F_{||}$ exerted parallel to each edge, as shown in Fig. 11.18. We'll find the shear strain from Eq. (11.16), the shear stress from Eq. (11.17), and $F_{||}$ from Eq. (11.15). Table 11.1 gives the shear modulus of brass. In Fig. 11.18, h represents the 0.80 m length of each side of the plate. The area A in Eq. (11.15) is the product of the 0.80 m length and the 0.50 cm thickness.

EXECUTE From Eq. (11.16),

$$\text{Shear strain} = \frac{x}{h} = \frac{1.6 \times 10^{-4}\ \text{m}}{0.80\ \text{m}} = 2.0 \times 10^{-4}$$

From Eq. (11.17),

$$\begin{aligned}\text{Shear stress} &= (\textit{Shear strain}) \times S \\ &= \left(2.0 \times 10^{-4}\right)\left(3.5 \times 10^{10}\ \text{Pa}\right) = 7.0 \times 10^{6}\ \text{Pa}\end{aligned}$$

Finally, from Eq. (11.15),

$$\begin{aligned}F_{||} &= (\text{Shear stress}) \times A \\ &= (7.0 \times 10^{6}\ \text{Pa})(0.80\ \text{m})(0.0050\ \text{m}) = 2.8 \times 10^{4}\ \text{N}\end{aligned}$$

EVALUATE The shear force supplied by the earthquake is more than 3 tons! The large shear modulus of brass makes it hard to deform.

Further, the plate is relatively thick (0.50 cm), so the area A is relatively large and a substantial force $F_{||}$ is needed to provide the necessary stress $F_{||}/A$.

KEY CONCEPT

For an object under shear stress, the stress equals the force exerted tangent to either of two opposite surfaces of the object divided by the area of that surface. The strain equals the deformation divided by the distance between the two surfaces. The ratio of stress to strain equals the shear modulus for the material of which the object is made.

Video Tutor Solution: **Example 11.7**

Test Your Understanding of Section 11.4

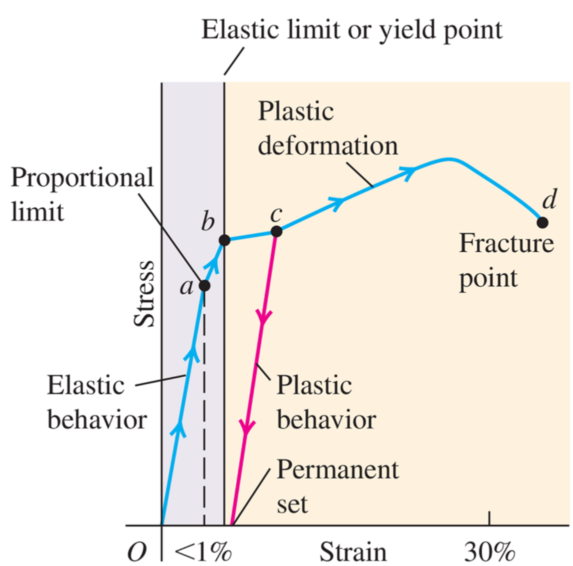
A copper rod of cross-sectional area $0.500\ \text{cm}^2$ and length $1.00\ \text{m}$ is elongated by $2.00 \times 10^{-2}\ \text{mm}$, and a steel rod of the same cross-sectional area but $0.100\ \text{m}$ in length is elongated by $2.00 \times 10^{-3}\ \text{mm}$. (a) Which rod has greater tensile *strain*? (i) The copper rod; (ii) the steel rod; (iii) the strain is the same for both. (b) Which rod is under greater tensile *stress*? (i) The copper rod; (ii) the steel rod; (iii) the stress is the same for both.

11.5 Elasticity and Plasticity

Hooke's law—the proportionality of stress and strain in elastic deformations—has a limited range of validity. In the preceding section we used phrases such as "if the forces are small enough that Hooke's law is obeyed." Just what *are* the limitations of Hooke's law? What's more, if you pull, squeeze, or twist *anything* hard enough, it will bend or break. Can we be more precise than that?

To address these questions, let's look at a graph of tensile stress as a function of tensile strain. Figure 11.19 shows a typical graph of this kind for a metal such as copper or soft iron. The strain is shown as the *percent* elongation; the horizontal scale is not uniform beyond the first portion of the curve, up to a strain of less than 1%. The first portion is a straight line, indicating Hooke's law behavior with stress directly proportional to strain. This straight-line portion ends at point ; the stress at this point is called the *proportional limit*.

Figure 11.19



Typical stress-strain diagram for a ductile metal under tension.

From to , stress and strain are no longer proportional, and Hooke's law is *not* obeyed. However, from to (and to), the behavior of the material is *elastic*: If the load is gradually removed starting at any point between and , the curve is retraced until the material returns to its original length. This elastic deformation is *reversible*.

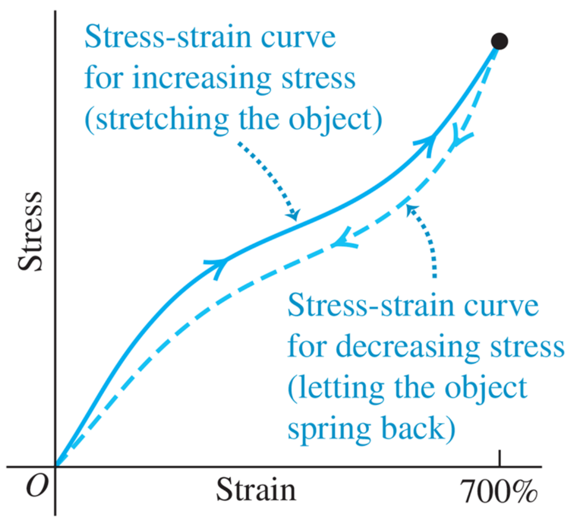
Point , the end of the elastic region, is called the *yield point;* the stress at the yield point is called the *elastic limit*. When we increase the stress beyond point , the strain continues to increase. But if we remove the load at a point like beyond the elastic limit, the material does *not* return to its original length. Instead, it follows the red line in Fig. 11.19. The material has deformed *irreversibly* and acquired a *permanent set*. This is the *plastic* behavior mentioned in Section 11.4.

Once the material has become plastic, a small additional stress produces a relatively large increase in strain, until a point is reached at which *fracture* takes place. That's what happens if a steel guitar string in Fig. 11.12a is tightened too much: The string breaks at the fracture point. Steel is *brittle* because it breaks soon after reaching its elastic limit; other materials, such as soft iron, are *ductile*—they can be given a large permanent stretch without breaking. (The material depicted in Fig. 11.19 is ductile, since it can stretch by more than 30% before breaking.)

Unlike uniform materials such as metals, stretchable biological materials such as tendons and ligaments have no true plastic region. That's because these materials are made of a collection of microscopic fibers; when stressed beyond the elastic limit, the fibers tear apart from each other. (A torn ligament or tendon is one that has fractured in this way.)

If a material is still within its elastic region, something very curious can happen when it is stretched and then allowed to relax. Figure 11.20 is a stress-strain curve for vulcanized rubber that has been stretched by more than seven times its original length. The stress is not proportional to the strain, but the behavior is elastic because when the load is removed, the material returns to its original length. However, the material follows *different* curves for increasing and decreasing stress. This is called *elastic hysteresis*. The work done by the material when it returns to its original shape is less than the work required to deform it; that's due to internal friction. Rubber with large elastic hysteresis is very useful for absorbing vibrations, such as in engine mounts and shock-absorber bushings for cars. Tendons display similar behavior.

Figure 11.20



Typical stress-strain diagram for vulcanized rubber. The curves are different for increasing and decreasing stress, a phenomenon called elastic hysteresis.

The stress required to cause actual fracture of a material is called the *breaking stress*, the *ultimate strength*, or (for tensile stress) the *tensile strength*. Two materials, such as two types of steel, may have very similar elastic constants but vastly different breaking stresses. Table 11.3 gives typical values of breaking stress for several materials in tension. Comparing Tables 11.1 and 11.3 shows that iron and steel are comparably *stiff* (they have almost the same value of Young's modulus), but steel is *stronger* (it has a larger breaking stress than does iron).

Table 11.3 **Approximate Breaking Stresses**

Material	Breaking Stress (Pa or N/m^2)
Aluminum	2.2×10^8
Brass	4.7×10^8
Glass	10×10^8
Iron	3.0×10^8
Steel	$5-20 \times 10^8$
Tendon (typical)	1×10^8

Test Your Understanding of Section 11.5

While parking your car, you accidentally back into a steel post. You pull forward until the car no longer touches the post and then get out to inspect the damage. What does your rear bumper look like if the strain in the impact was (a) less than at the proportional limit; (b) greater than at the proportional limit but less than at the yield point; (c) greater than at the yield point but less than at the fracture point; and (d) greater than at the fracture point?

Chapter 11 Summary

Conditions for equilibrium: For a rigid body to be in equilibrium, two conditions must be satisfied. First, the vector sum of forces must be zero. Second, the sum of torques about any point must be zero. The torque due to the weight of an object can be found by assuming the entire weight is concentrated at the center of gravity, which is at the same point as the center of mass if $\vec{\boldsymbol{g}}$ has the same value at all points. (See Examples 11.1, 11.2, 11.3 and 11.4.)

(11.1)

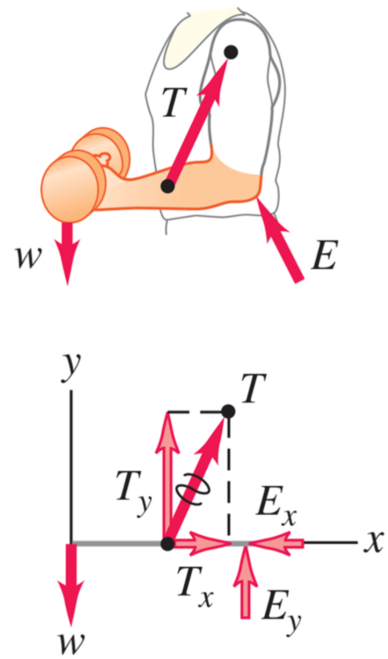
$$\boldsymbol{F}$$

(11.2)

$$\vec{\boldsymbol{\tau}}$$

(11.4)

$$\vec{\boldsymbol{r}} \qquad \frac{\vec{\boldsymbol{r}} \qquad \vec{\boldsymbol{r}} \qquad \vec{\boldsymbol{r}}}{\quad}$$

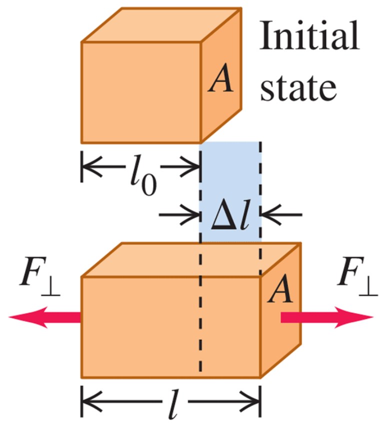


Stress, strain, and Hooke's law: Hooke's law states that in elastic deformations, stress (force per unit area) is proportional to strain (fractional deformation). The proportionality constant is called the elastic modulus.

(11.7)

Tensile and compressive stress: Tensile stress is tensile force per unit area, Tensile strain is fractional change in length, The elastic modulus for tension is called Young's modulus Y. Compressive stress and strain are defined in the same way. (See Example 11.5.)

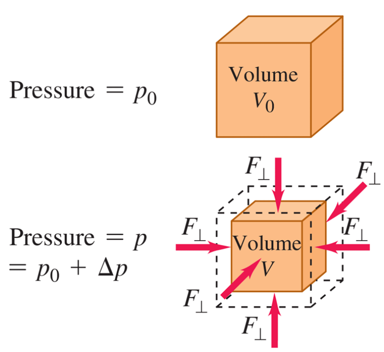
(11.10)



Bulk stress: Pressure in a fluid is force per unit area. Bulk stress is pressure change, and bulk strain is fractional volume change, The elastic modulus for compression is called the bulk modulus, B. Compressibility, k, is the reciprocal of bulk modulus: (See Example 11.6.)

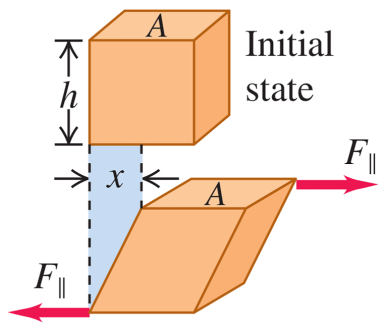
(11.11)

(11.13)



Shear stress: Shear stress is force per unit area, for a force applied tangent to a surface. Shear strain is the displacement x of one side divided by the transverse dimension h. The elastic modulus for shear is called the shear modulus, S. (See Example 11.7.)

(11.17)



The limits of Hooke's law: The proportional limit is the maximum stress for which stress and strain are proportional. Beyond the proportional limit, Hooke's law is not valid. The elastic limit is the stress beyond which irreversible deformation occurs. The breaking stress, or ultimate strength, is the stress at which the material breaks.

Guided Practice: Equilibrium and Elasticity

For assigned homework and other learning materials, go to **Mastering Physics**.

Key Example Variation Problems

Be sure to review Example 11.1 (Section 11.2) before attempting these problems.

VP11.1.1 A uniform plank 8.00 m in length with mass 40.0 kg is supported at two points located 1.00 m and 5.00 m, respectively, from the left-hand end. What is the maximum additional mass you could place on the right-hand end of the plank and have the plank still be at rest?

VP11.1.2 A bowling ball (which we can regard as a uniform sphere) has a mass of 7.26 kg and a radius of 0.216 m. A baseball has a mass of 0.145 kg. If you connect these two balls with a lightweight rod, what must be the distance between the center of the bowling ball and the center of the baseball so that the system of the two balls and the rod will balance at the point where the rod touches the surface of the bowling ball?

VP11.1.3 Three small objects are arranged along a uniform rod of mass m and length L: one of mass m at the left end, one of mass m at the center, and one of mass $2m$ at the right end. How far to the left or right of the rod's center should you place a support so that the rod with the attached objects will balance there?

VP11.1.4 A small airplane with full fuel tanks, but no occupants or baggage, has a mass of 1.17×10^3 kg and a center of gravity that is 2.58 m behind the nose of the airplane. The pilot's seat is 2.67 m behind the nose, and the baggage compartment is 4.30 m behind the nose. A 75.0 kg pilot boards the plane and is the only occupant. If the center of gravity of the airplane with pilot can be no more than 2.76 m behind the nose for in-flight stability, what is the maximum mass that the baggage compartment can hold?

Be sure to review Examples 11.2, 11.3, and 11.4 (Section 11.3) before attempting these problems.

VP11.4.1 The rear wheels of a truck support 57.0% of the weight of the truck, while the front wheels support 43.0% of the weight. The center of gravity of the truck is 2.52 m in front of the rear wheels. What is the wheelbase of the truck (the distance between the front and rear wheels)?

VP11.4.2 A small airplane is sitting at rest on the ground. Its center of gravity is 2.58 m behind the nose of the airplane, the front wheel (nose wheel) is 0.800 m behind the nose, and the main wheels are 3.02 m behind the nose. What percentage of the airplane's weight is supported by the nose wheel, and what percentage is supported by the main wheels?

VP11.4.3 Figure11.11 shows a metal advertising sign of weight w suspended from the end of a horizontal rod of negligible mass and length L. The end of the rod with the sign is supported by a cable at an angle θ from the horizontal, and the other end is supported by a hinge at point P. (a) Using the idea that there is zero net torque about the end of the rod with the attached sign, find the vertical component of the force $\vec{F}_{\text{hinge}}$ exerted by the hinge. (b) Using the idea that there is zero net vertical force on the rod with the attached sign, find the tension in the cable. (c) Using the idea that there is zero net horizontal force on the rod with the attached sign, find the horizontal component of the force exerted by the hinge.

VP11.4.4 Suppose that in Figure11.11 the horizontal rod has weight w (the same as the hanging sign). (a) Using the idea that there is zero net torque about the end of the rod with the attached sign, find the vertical component of the force $\vec{F}_{\text{hinge}}$ exerted by the hinge. (b) Using the idea that there is zero net vertical force on the rod with the attached sign, find the tension in the cable. (c) Using the idea that there is zero net horizontal force on the rod with the attached sign, find the horizontal component of the force exerted by the hinge.

Be sure to review Examples 11.5, 11.6, and 11.7 (Section 11.4) before attempting these problems.

VP11.7.1 A copper wire has a radius of 4.5 mm. When forces of a certain equal magnitude but opposite directions are applied to the ends of the wire, the wire stretches by 5.0×10^{-3} of its original length. (a) What is the tensile stress on the wire? (b) What is the magnitude of the force on either end?

VP11.7.2 An aluminum cylinder with a radius of 2.5 cm and a height of 82 cm is used as one leg of a workbench. The workbench pushes down on the cylinder with a force of 3.2×10^4N. (a) What is the compressive strain of the cylinder? (b) By what distance does the cylinder's height decrease as a result of the forces on it?

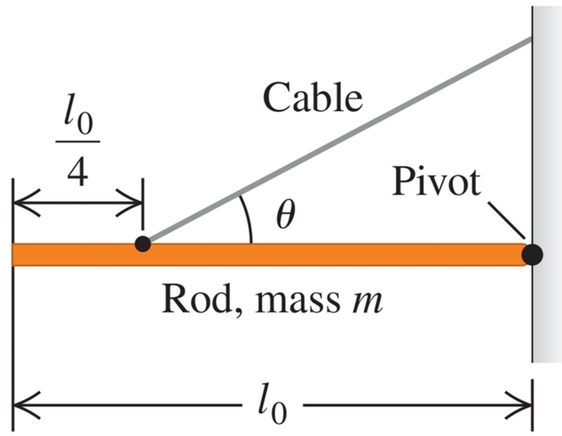
VP11.7.3 The pressure on the surface of a sphere of radius 1.2 cm is increased by 2.5×10^7 Pa. Calculate the resulting decrease in volume of the sphere if it is made (a) of lead and (b) of mercury.

VP11.7.4 You apply forces of magnitude 4.2×10^4 N to the top and bottom surfaces of a brass cube. The forces are tangent to each surface and parallel to the sides of each surface. If the cube is 2.5 cm on a side, what is the resulting shear displacement?

Bridging Problem: In Equilibrium and Under Stress

A horizontal, uniform, solid copper rod has an original length l_0, cross-sectional area A, Young's modulus Y, bulk modulus B, shear modulus S, and mass m. It is supported by a frictionless pivot at its right end and by a cable a distance $l_0/4$ from its left end (Fig. 11.21). Both pivot and cable are attached so that they exert their forces uniformly over the rod's cross section. The cable makes an angle θ with the rod and compresses it. (a) Find the tension in the cable. (b) Find the magnitude and direction of the force exerted by the pivot on the right end of the rod. How does this magnitude compare to the cable tension? How does this angle compare to θ? (c) Find the change in length of the rod due to the stresses exerted by the cable and pivot on the rod. (The length change is small compared to the original length l_0.) (d) By what factor would your answer in part (c) increase if the solid copper rod were twice as long but had the same cross-sectional area?

Figure 11.21



What are the forces on the rod? What are the stress and strain?

Solution Guide

IDENTIFY and SET UP

1. Draw a free-body diagram for the rod. Be careful to place each force in the correct location.
2. List the unknown quantities, and decide which are the target variables.
3. What conditions must be met so that the rod remains at rest? What kind of stress (and resulting strain) is involved? Use your answers to select the appropriate equations.

EXECUTE

4. Use your equations to solve for the target variables. (*Hint:* You can make the solution easier by carefully choosing the point around which you calculate torques.)
5. Use trigonometry to decide whether the pivot force or the cable tension has the greater magnitude and whether the angle of the pivot force is greater than, less than, or equal to θ.

EVALUATE

6. Check whether your answers are reasonable. Which force, the cable tension or the pivot force, holds up more of the weight of the rod? Does this make sense?

Video Tutor Solution: **Chapter 11 Bridging Problem**

Questions/Exercises/Problems: Equilibrium and Elasticity

•, **••**, **•••**: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

Discussion Questions

Q11.1 Does a rigid object in uniform rotation about a fixed axis satisfy the first and second conditions for equilibrium? Why? Does it then follow that every particle in this object is in equilibrium? Explain.

Q11.2 (a) Is it possible for an object to be in translational equilibrium (the first condition) but *not* in rotational equilibrium (the second condition)? Illustrate your answer with a simple example. (b) Can an object be in rotational equilibrium yet *not* in translational equilibrium? Justify your answer with a simple example.

Q11.3 Car tires are sometimes "balanced" on a machine that pivots the tire and wheel about the center. Weights are placed around the wheel rim until it does not tip from the horizontal plane. Discuss this procedure in terms of the center of gravity.

Q11.4 Does the center of gravity of a solid object always lie within the material of the object? If not, give a counterexample.

Q11.5 In Section 11.2 we always assumed that the value of g was the same at all points on the object. This is *not* a good approximation if the dimensions of the object are great enough, because the value of g decreases with altitude. If this is taken into account, will the center of gravity of a long, vertical rod be above, below, or at its center of mass? Explain how this can be used to keep the long axis of an orbiting spacecraft pointed toward the earth. (This would be useful for a weather satellite that must always keep its camera lens trained on the earth.) The moon is not exactly spherical but is somewhat elongated. Explain why this same effect is responsible for keeping the same face of the moon pointed toward the earth at all times.

Q11.6 You are balancing a wrench by suspending it at a single point. Is the equilibrium stable, unstable, or neutral if the point is above, at, or below the wrench's center of gravity? In each case give the reasoning behind your answer. (For rotation, a rigid object is in *stable* equilibrium if a small rotation of the object produces a torque that tends to return the object to equilibrium; it is in *unstable* equilibrium if a small rotation produces a torque that tends to take the object farther from equilibrium; and it is in *neutral* equilibrium if a small rotation produces no torque.)

Q11.7 You can probably stand flatfooted on the floor and then rise up and balance on your tiptoes. Why are you unable do it if your toes are touching the wall of your room? (Try it!)

Q11.8 You freely pivot a horseshoe from a horizontal nail through one of its nail holes. You then hang a long string with a weight at its bottom from the same nail, so that the string hangs vertically in front of the horseshoe without touching it. How do you know that the horseshoe's center of gravity is along the line behind the string? How can you locate the center of gravity by repeating the process at another nail hole? Will the center of gravity be within the solid material of the horseshoe?

Q11.9 An object consists of a ball of weight W glued to the end of a uniform bar also of weight W. If you release it from rest, with the bar horizontal, what will its behavior be as it falls if air resistance is negligible? Will it (a) remain horizontal; (b) rotate about its center of gravity; (c) rotate about the ball; or (d) rotate so that the ball swings downward? Explain your reasoning.

Q11.10 Suppose that the object in Question 11.9 is released from rest with the bar tilted at 60° above the horizontal with the ball at the upper end. As it is falling, will it (a) rotate about its center of gravity until it is horizontal; (b) rotate about its center of gravity until it is vertical with the ball at the bottom; (c) rotate about the ball until it is vertical with the ball at the bottom; or (d) remain at 60° above the horizontal?

Q11.11 Why must a water skier moving with constant velocity lean backward? What determines how far back she must lean? Draw a free-body diagram for the water skier to justify your answers.

Q11.12 In pioneer days, when a Conestoga wagon was stuck in the mud, people would grasp the wheel spokes and try to turn the wheels, rather than simply pushing the wagon. Why?

Q11.13 The mighty Zimbo claims to have leg muscles so strong that he can stand flat on his feet and lean forward to pick up an apple on the floor with his teeth. Should you pay to see him perform, or do you have any suspicions about his claim? Why?

Q11.14 Why is it easier to hold a 10 kg dumbbell in your hand at your side than it is to hold it with your arm extended horizontally?

Q11.15 Certain features of a person, such as height and mass, are fixed (at least over relatively long periods of time). Are the following features also fixed? (a) location of the center of gravity of the body; (b) moment of inertia of the body about an axis through the person's center of mass. Explain your reasoning.

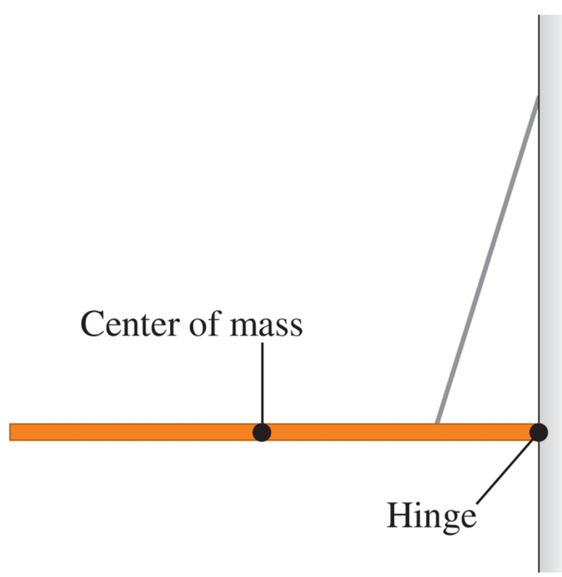
Q11.16 During pregnancy, women often develop back pains from leaning backward while walking. Why do they have to walk this way?

Q11.17 Why is a tapered water glass with a narrow base easier to tip over than a glass with straight sides? Does it matter whether the glass is full or empty?

Q11.18 When a tall, heavy refrigerator is pushed across a rough floor, what factors determine whether it slides or tips?

Q11.19 A uniform beam is suspended horizontally and attached to a wall by a small hinge (Fig. Q11.19). What are the directions (upward or downward, and to the left or the right) of the components of the force that the hinge exerts *on the beam*? Explain.

Figure Q11.19



Q11.20 If a metal wire has its length doubled and its diameter tripled, by what factor does its Young's modulus change?

Q11.21 A metal wire of diameter D stretches by 0.100 mm when supporting a weight W. If the same-length wire is used to support a weight three times as heavy, what would its diameter have to be (in terms of D) so it still stretches only 0.100 mm?

Q11.22 Compare the mechanical properties of a steel cable, made by twisting many thin wires together, with the properties of a solid steel rod of the same diameter. What advantages does each have?

Q11.23 The material in human bones and elephant bones is essentially the same, but an elephant has much thicker legs. Explain why, in terms of breaking stress.

Q11.24 There is a small but appreciable amount of elastic hysteresis in the large tendon at the back of a horse's leg. Explain how this can cause damage to the tendon if a horse runs too hard for too long a time.

Q11.25 When rubber mounting blocks are used to absorb machine vibrations through elastic hysteresis, as mentioned in Section 11.5, what becomes of the energy associated with the vibrations?

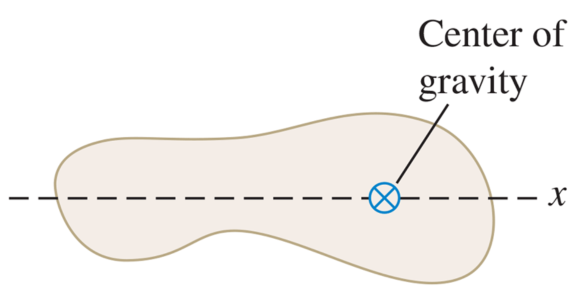
Exercises

Section 11.2 Center of Gravity

11.1 •• A 0.120 kg, 50.0-cm-long uniform bar has a small 0.055 kg mass glued to its left end and a small 0.110 kg mass glued to the other end. The two small masses can each be treated as point masses. You want to balance this system horizontally on a fulcrum placed just under its center of gravity. How far from the left end should the fulcrum be placed?

11.2 •• The center of gravity of a 5.00 kg irregular object is shown in Fig. E11.2. You need to move the center of gravity 2.20 cm to the left by gluing on a 1.50 kg mass, which will then be considered as part of the object. Where should the center of gravity of this additional mass be located?

Figure E11.2



11.3 • A uniform rod is 2.00 m long and has mass 1.80 kg. A 2.40 kg clamp is attached to the rod. How far should the center of gravity of the clamp be from the left-hand end of the rod in order for the center of gravity of the composite object to be 1.20 m from the left-hand end of the rod?

11.4 • Consider the free-body diagram shown in Fig. 11.9b. (a) What is the horizontal distance of the center of gravity of the person–ladder system from the point where the ladder touches the ground? (b) What is the torque about the rotation axis shown in the figure (point B) computed by taking the total weight of the person plus ladder acting at the center of gravity? (c) How does the result in part (b) compare to the sum of the individual torques computed in Example 11.3 for the weight of the person and the weight of the ladder?

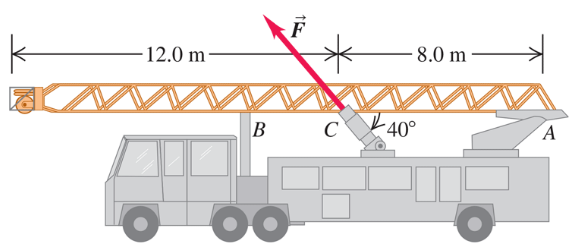
11.5 •• A uniform steel rod has mass 0.300 kg and length 40.0 cm and is horizontal. A uniform sphere with radius 8.00 cm and mass 0.900 kg is welded to one end of the bar, and a uniform sphere with radius 6.00 cm and mass 0.380 kg is welded to the other end of the bar. The centers of the rod and of each sphere all lie along a horizontal line. How far is the center of gravity of the combined object from the center of the rod?

Section 11.3 Solving Rigid-Body Equilibrium Problems

11.6 • A uniform 300 N trapdoor in a floor is hinged at one side. Find the net upward force needed to begin to open it and the total force exerted on the door by the hinges (a) if the upward force is applied at the center and (b) if the upward force is applied at the center of the edge opposite the hinges.

11.7 •• **Raising a Ladder**. A ladder carried by a fire truck is 20.0 m long. The ladder weighs 3400 N and its center of gravity is at its center. The ladder is pivoted at one end (A) about a pin (Fig. E11.7); ignore the friction torque at the pin. The ladder is raised into position by a force applied by a hydraulic piston at C. Point C is 8.0 m from A, and the force $\vec{F}$ exerted by the piston makes an angle of 40° with the ladder. What magnitude must $\vec{F}$ have to just lift the ladder off the support bracket at B? Start with a free-body diagram of the ladder.

Figure E11.7

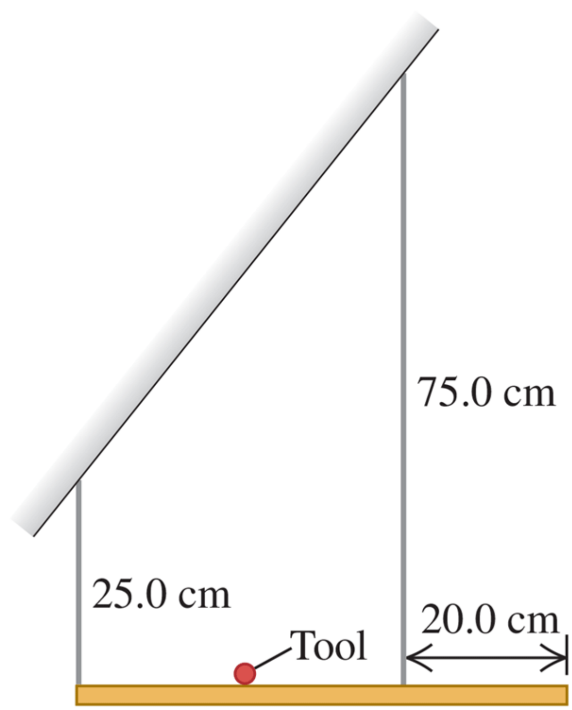


11.8 •• Two people are carrying a uniform wooden board that is 3.00 m long and weighs 160 N. If one person applies an upward force equal to 60 N at one end, at what point does the other person lift? Begin with a free-body diagram of the board.

11.9 •• Two people carry a heavy electric motor by placing it on a light board 2.00 m long. One person lifts at one end with a force of 400 N, and the other lifts the opposite end with a force of 600 N. (a) What is the weight of the motor, and where along the board is its center of gravity located? (b) Suppose the board is not light but weighs 200 N, with its center of gravity at its center, and the two people exert the same forces as before. What is the weight of the motor in this case, and where is its center of gravity located?

11.10 •• A 60.0 cm, uniform, 50.0 N shelf is supported horizontally by two vertical wires attached to the sloping ceiling (Fig. E11.10). A very small 25.0 N tool is placed on the shelf midway between the points where the wires are attached to it. Find the tension in each wire. Begin by making a free-body diagram of the shelf.

Figure E11.10

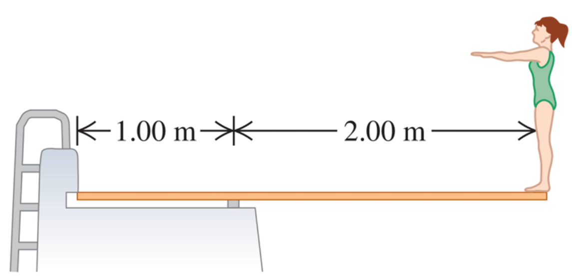


11.11 •• A 350 N, uniform, 1.50 m bar is suspended horizontally by two vertical cables at each end. Cable A can support a maximum tension of 500.0 N without breaking, and cable B can support up to 400.0 N. You want to place a small weight on this bar. (a) What is the heaviest weight you can put on without breaking either cable, and (b) where should you put this weight?

11.12 •• A uniform ladder 5.0 m long rests against a frictionless, vertical wall with its lower end 3.0 m from the wall. The ladder weighs 160 N. The coefficient of static friction between the foot of the ladder and the ground is 0.40. A man weighing 740 N climbs slowly up the ladder. Start by drawing a free-body diagram of the ladder. (a) What is the maximum friction force that the ground can exert on the ladder at its lower end? (b) What is the actual friction force when the man has climbed 1.0 m along the ladder? (c) How far along the ladder can the man climb before the ladder starts to slip?

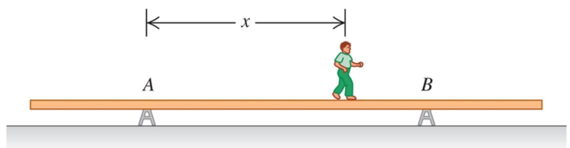
11.13 • A diving board 3.00 m long is supported at a point 1.00 m from the end, and a diver weighing 500 N stands at the free end (Fig. E11.13). The diving board is of uniform cross section and weighs 280 N. Find (a) the force at the support point and (b) the force at the left-hand end.

Figure E11.13



11.14 • A uniform aluminum beam 9.00 m long, weighing 300 N, rests symmetrically on two supports 5.00 m apart (Fig. E11.14). A boy weighing 600 N starts at point A and walks toward the right. (a) In the same diagram construct two graphs showing the upward forces F_A and F_B exerted on the beam at points A and B, as functions of the coordinate x of the boy. Let 1 cm $=$ 100 N vertically and 1 cm $=$ 1.00 m horizontally. (b) From your diagram, how far beyond point B can the boy walk before the beam tips? (c) How far from the right end of the beam should support B be placed so that the boy can walk just to the end of the beam without causing it to tip?

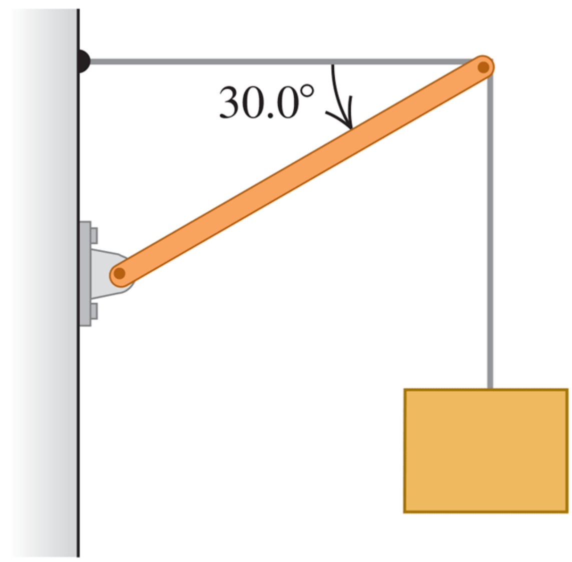
Figure E11.14



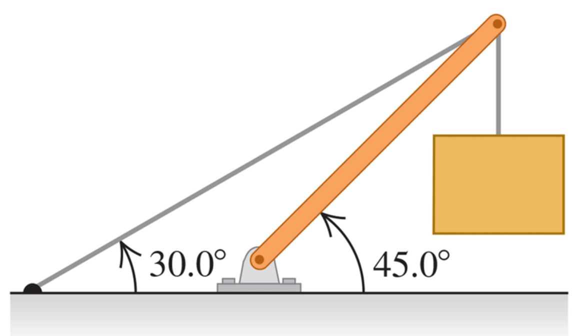
11.15 • Find the tension T in each cable and the magnitude and direction of the force exerted on the strut by the pivot in each of the arrangements in Fig. E11.15. In each case let w be the weight of the suspended crate full of priceless art objects. The strut is uniform and also has weight w. Start each case with a free-body diagram of the strut.

Figure E11.15

(a)

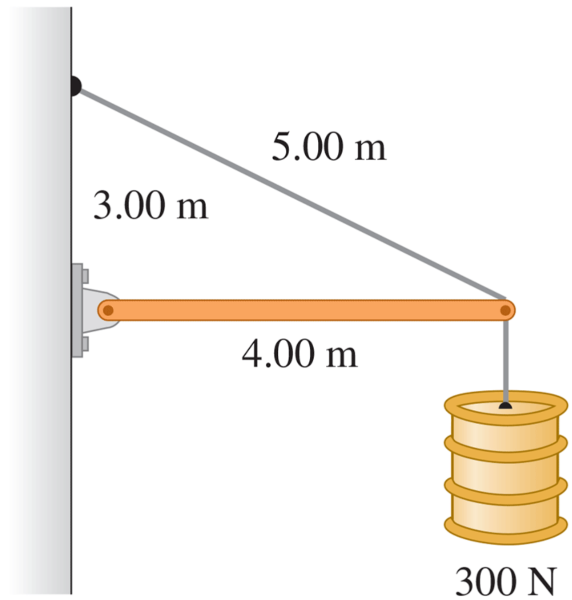


(b)



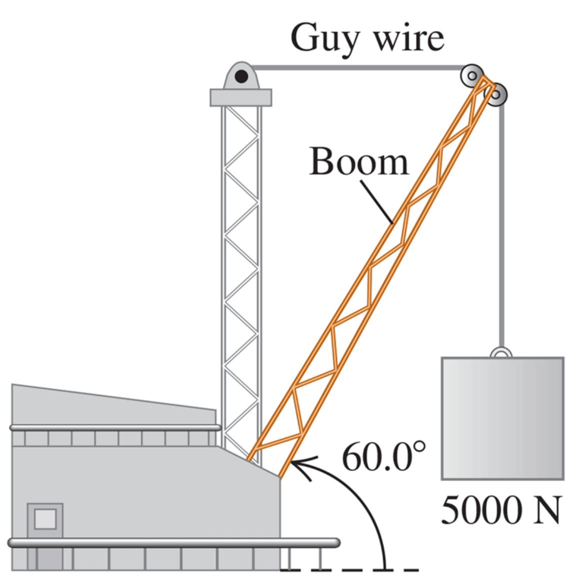
11.16 • The horizontal beam in Fig. E11.16 weighs 190 N, and its center of gravity is at its center. Find (a) the tension in the cable and (b) the horizontal and vertical components of the force exerted on the beam at the wall.

Figure E11.16



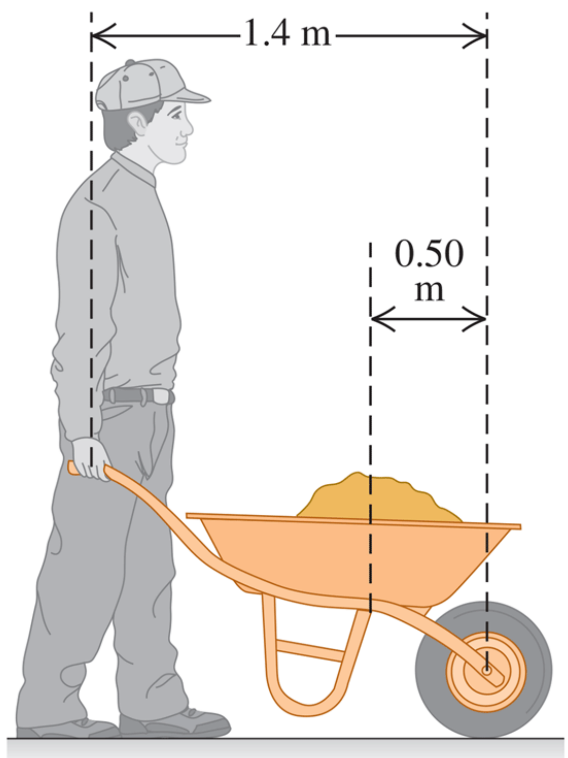
11.17 •• The boom shown in Fig. E11.17 weighs 2600 N and is attached to a frictionless pivot at its lower end. It is not uniform; the distance of its center of gravity from the pivot is 35% of its length. Find (a) the tension in the guy wire and (b) the horizontal and vertical components of the force exerted on the boom at its lower end. Start with a free-body diagram of the boom.

Figure E11.17



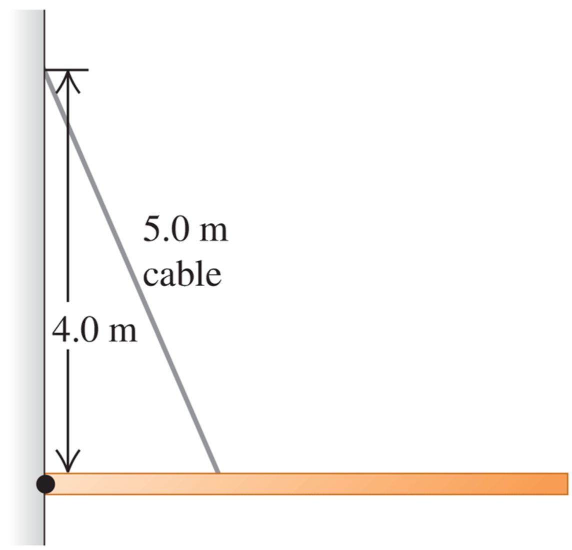
11.18 •• You are pushing an 80.0 N wheelbarrow as shown in **Fig. E11.18**. You lift upward on the handle of the wheelbarrow so that the only point of contact between the wheelbarrow and the ground is at the front wheel. Assume the distances are as shown in the figure, where 0.50 m is the horizontal distance from the center of the wheel to the center of gravity of the wheelbarrow. The center of gravity of the dirt in the wheelbarrow is assumed to also be a horizontal distance of 0.50 m from the center of the wheel. Estimate the maximum total upward force that you can apply to the wheelbarrow handles. (a) If you apply this estimated force, what is the maximum weight of dirt that you can carry in the wheelbarrow? Express your answer in pounds. (b) If the dirt has the weight you calculated in part (a), what upward force does the ground apply to the wheel of the wheelbarrow?

Figure E11.18



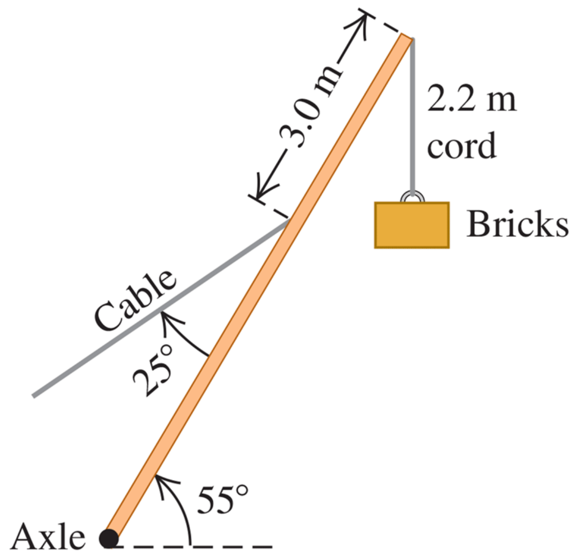
11.19 •• A 9.00-m-long uniform beam is hinged to a vertical wall and held horizontally by a 5.00-m-long cable attached to the wall 4.00 m above the hinge (Fig. E11.19). The metal of this cable has a test strength of 1.00 kN, which means that it will break if the tension in it exceeds that amount. (a) Draw a free-body diagram of the beam. (b) What is the heaviest beam that the cable can support in this configuration? (c) Find the horizontal and vertical components of the force the hinge exerts on the beam. Is the vertical component upward or downward?

Figure E11.19



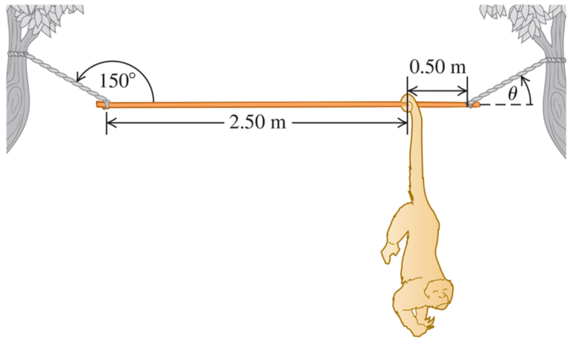
11.20 •• A 15,000 N crane pivots around a friction-free axle at its base and is supported by a cable making a 25° angle with the crane (Fig. E11.20). The crane is 16 m long and is not uniform, its center of gravity being 7.0 m from the axle as measured along the crane. The cable is attached 3.0 m from the upper end of the crane. When the crane is raised to 55° above the horizontal holding an 11,000 N pallet of bricks by a 2.2 m, very light cord, find (a) the tension in the cable and (b) the horizontal and vertical components of the force that the axle exerts on the crane. Start with a free-body diagram of the crane.

Figure E11.20



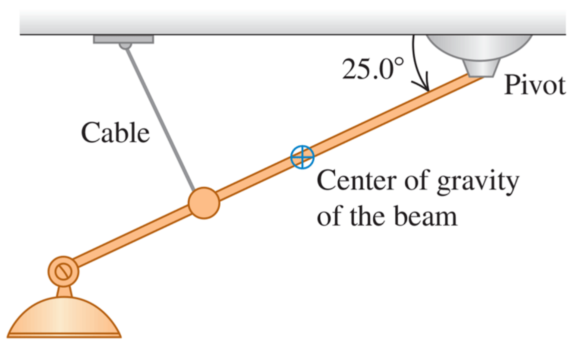
11.21 •• A 3.00-m-long, 190 N, uniform rod at the zoo is held in a horizontal position by two ropes at its ends (Fig. E11.21). The left rope makes an angle of 150° with the rod, and the right rope makes an angle θ with the horizontal. A 90 N howler monkey (*Alouatta seniculus*) hangs motionless 0.50 m from the right end of the rod as he carefully studies you. Calculate the tensions in the two ropes and the angle θ. First make a free-body diagram of the rod.

Figure E11.21



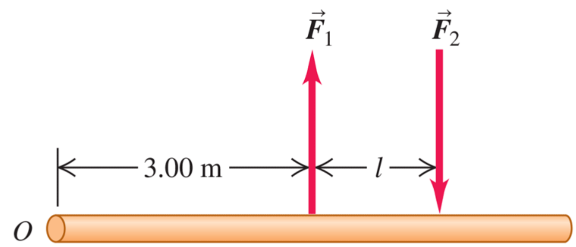
11.22 •• A nonuniform beam 4.50 m long and weighing 1.40 kN makes an angle of 25.0° below the horizontal. It is held in position by a frictionless pivot at its upper right end and by a cable 3.00 m farther down the beam and perpendicular to it (Fig. E11.22). The center of gravity of the beam is 2.00 m down the beam from the pivot. Lighting equipment exerts a 5.00 kN downward force on the lower left end of the beam. Find the tension T in the cable and the horizontal and vertical components of the force exerted on the beam by the pivot. Start by sketching a free-body diagram of the beam.

Figure E11.22



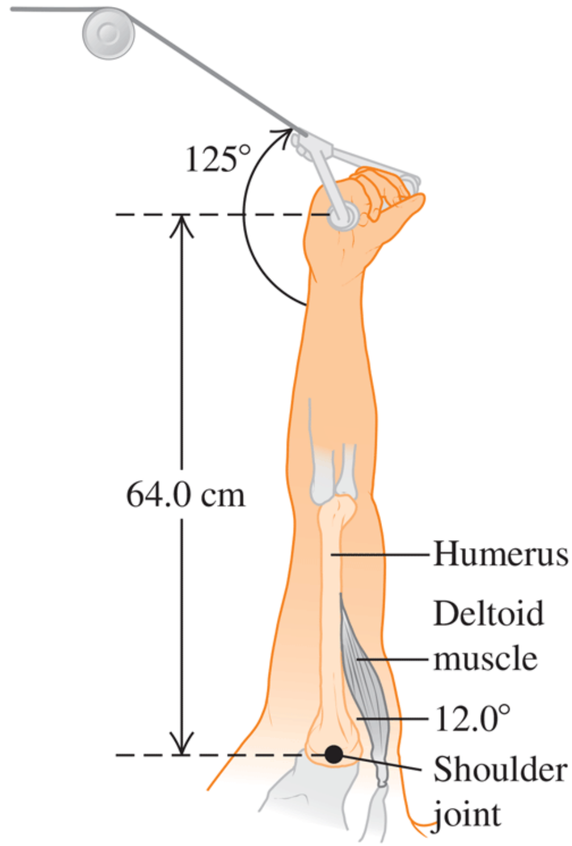
11.23 • **A Couple.** Two forces equal in magnitude and opposite in direction, acting on an object at two different points, form what is called a *couple*. Two antiparallel forces with equal magnitudes $F_1 = F_2 = 8.00$ N are applied to a rod as shown in Fig. E11.23. (a) What should the distance l between the forces be if they are to provide a net torque of $6.40\ \mathrm{N} \cdot m$ about the left end of the rod? (b) Is the sense of this torque clockwise or counterclockwise? (c) Repeat parts (a) and (b) for a pivot at the point on the rod where $\vec{F}_2$ is applied.

Figure E11.23



11.24 •• BIO **A Good Workout.** You are doing exercises on a Nautilus machine in a gym to strengthen your deltoid (shoulder) muscles. Your arms are raised vertically and can pivot around the shoulder joint, and you grasp the cable of the machine in your hand 64.0 cm from your shoulder joint. The deltoid muscle is attached to the humerus 15.0 cm from the shoulder joint and makes a 12.0° angle with that bone (Fig. E11.24). If you have set the tension in the cable of the machine to 36.0 N on each arm, what is the tension in each deltoid muscle if you simply hold your outstretched arms in place? (*Hint:* Start by making a clear free-body diagram of your arm.)

Figure E11.24



11.25 •• A uniform rod has one end attached to a vertical wall by a frictionless hinge. A horizontal wire runs from the other end of the rod to a point on the wall above the hinge and holds the rod at an angle θ above the horizontal. You vary the angle θ by changing the length of the wire, and for each θ you measure the tension T in the wire. You plot your data in the form of a T-versus-cot θ graph. The data lie close to a straight line that has slope 30.0 N. What is the mass of the rod?

Section 11.4 Stress, Strain, and Elastic Moduli

11.26 • BIO **Biceps Muscle**. A relaxed biceps muscle requires a force of 25.0 N for an elongation of 3.0 cm; the same muscle under maximum tension requires a force of 500 N for the same elongation. Find Young's modulus for the muscle tissue under each of these conditions if the muscle is assumed to be a uniform cylinder with length 0.200 m and cross-sectional area 50.0 cm^2.

11.27 •• A circular steel wire 2.00 m long must stretch no more than 0.25 cm when a tensile force of 700 N is applied to each end of the wire. What minimum diameter is required for the wire?

11.28 •• Two cylindrical rods, one steel and the other copper, are joined end to end. Each rod is 0.750 m long and 1.50 cm in diameter. The combination is subjected to a tensile force with magnitude 4000 N. For each rod, what are (a) the strain and (b) the elongation?

11.29 •• A metal rod that is 4.00 m long and 0.50 cm^2 in cross-sectional area is found to stretch 0.20 cm under a tension of 5000 N. What is Young's modulus for this metal?

11.30 •• **Stress on a Mountaineer's Rope**. A nylon rope used by mountaineers elongates 1.10 m under the weight of a 65.0 kg climber. If the rope is 45.0 m in length and 7.0 mm in diameter, what is Young's modulus for nylon?

11.31 • A lead sphere has volume 6.0 cm^3 when it is resting on a lab table, where the pressure applied to the sphere is atmospheric pressure. The sphere is then placed in the fluid of a hydraulic press. What increase in the pressure above atmospheric pressure produces a 0.50% decrease in the volume of the sphere?

11.32 •• A vertical, solid steel post 25 cm in diameter and 2.50 m long is required to support a load of 8000 kg. You can ignore the weight of the post. What are (a) the stress in the post; (b) the strain in the post; and (c) the change in the post's length when the load is applied?

11.33 •• BIO **Compression of Human Bone**. The bulk modulus for bone is 15 GPa. (a) If a diver-in-training is put into a pressurized suit, by how much would the pressure have to be raised (in atmospheres) above atmospheric pressure to compress her bones by 0.10% of their original volume? (b) Given that the pressure in the ocean increases by 1.0×10^4 Pa for every meter of depth below the surface, how deep would this diver have to go for her bones to compress by 0.10%? Does it seem that bone compression is a problem she needs to be concerned with when diving?

11.34 • A solid gold bar is pulled up from the hold of the sunken RMS *Titanic*. (a) What happens to its volume as it goes from the pressure at the ship to the lower pressure at the ocean's surface? (b) The pressure difference is proportional to the depth. How many times greater would the volume change have been had the ship been twice as deep? (c) The bulk modulus of lead is one-fourth that of gold. Find the ratio of the volume change of a solid lead bar to that of a gold bar of equal volume for the same pressure change.

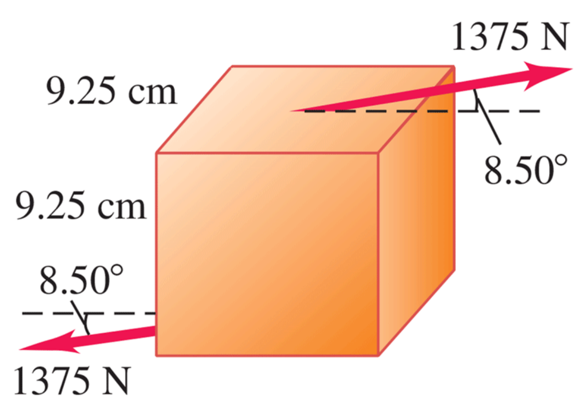
11.35 • A specimen of oil having an initial volume of 600 cm^3 is subjected to a pressure increase of 3.6×10^6 Pa, and the volume is found to decrease by 0.45 cm^3. What is the bulk modulus of the material? The compressibility?

11.36 •• In the Challenger Deep of the Marianas Trench, the depth of seawater is 10.9 km and the pressure is 1.16×10^8 Pa (about 1.15×10^3 atm). (a) If a cubic meter of water is taken from the surface to this depth, what is the change in its volume? (Normal atmospheric pressure is about 1.0×10^5 Pa. Assume that k for seawater is the same as the freshwater value given in Table 11.2.) (b) What is the density of seawater at this depth? (At the surface, seawater has a density of $1.03 \times 10^3\ \mathrm{kg/m^3}$.)

11.37 •• A square steel plate is 10.0 cm on a side and 0.500 cm thick. (a) Find the shear strain that results if two forces, each of magnitude 9.0×10^5 N and in opposite directions, act tangent to the surfaces of a pair of opposite sides of the object, as in Fig. E11.18. (b) Find the displacement x in centimeters.

11.38 • In lab tests on a 9.25 cm cube of a certain material, a force of 1375 N directed at 8.50° to the cube (Fig. E11.38) causes the cube to deform through an angle of 1.24°. What is the shear modulus of the material?

Figure E11.38



11.39 •• A steel wire with radius r_{steel} has a fractional increase in length of $(\Delta l/l_0)_{\mathrm{steel}}$ when the tension in the wire is increased from zero to T_{steel}. An aluminum wire has radius r_{al} that is twice the radius of the steel wire: $r_{\mathrm{al}} = 2r_{\mathrm{steel}}$. In terms of T_{steel}, what tension in the aluminum wire produces the same fractional change in length as in the steel wire?

11.40 •• You apply a force of magnitude $F_\perp$ to one end of a wire and another force $F_\perp$ in the opposite direction to the other end of the wire. The cross-sectional area of the wire is $8.00\ \text{mm}^2$. You measure the fractional change in the length of the wire, $\Delta l/l_0$, for several values of $F_\perp$. When you plot your data with $\Delta l/l_0$ on the vertical axis and $F_\perp$ (in units of N) on the horizontal axis, the data lie close to a line that has slope $8.0 \times 10^{-7}\ \text{N}^{-1}$. What is the value of Young's modulus for this wire?

11.41 • An increase in applied pressure Δp_1 produces a fractional volume change of $(\Delta V/V_0)_1$ for a sample of glycerin. In terms of Δp_1, what pressure increase above atmospheric pressure is required to produce the same volume change $(\Delta V/V_0)_1$ for a sample of ethyl alcohol?

Section 11.5 Elasticity and Plasticity

11.42 •• A brass wire is to withstand a tensile force of 350 N without breaking. What minimum diameter must the wire have?

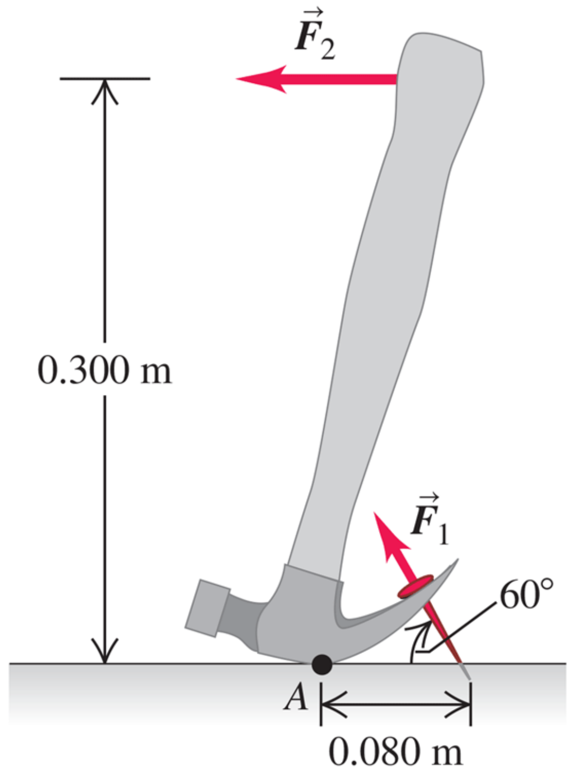
11.43 •• In a materials testing laboratory, a metal wire made from a new alloy is found to break when a tensile force of 90.8 N is applied perpendicular to each end. If the diameter of the wire is 1.84 mm, what is the breaking stress of the alloy?

11.44 •• **CP** A steel cable with cross-sectional area $3.00\ \text{cm}^2$ has an elastic limit of 2.40×10^8 Pa. Find the maximum upward acceleration that can be given a 1200 kg elevator supported by the cable if the stress is not to exceed one-third of the elastic limit.

Problems

11.45 •• You are using a hammer to pull a nail from the floor, as shown in Fig. P11.45. Force $\vec{F}_2$ is the force you apply to the handle, and force $\vec{F}_1$ is the force the hammer applies to the nail. Estimate the maximum magnitude of the force $\vec{F}_2$ that you could apply to the hammer. If you apply this force, what is the magnitude of the force $\vec{F}_1$ that the hammer applies to the nail?

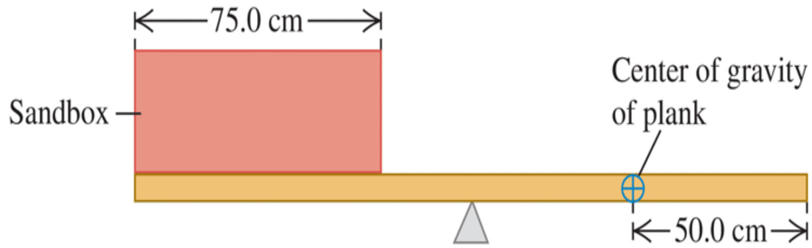
Figure P11.45



11.46 ••• A door 1.00 m wide and 2.00 m high weighs 330 N and is supported by two hinges, one 0.50 m from the top and the other 0.50 m from the bottom. Each hinge supports half the total weight of the door. Assuming that the door's center of gravity is at its center, find the horizontal components of force exerted on the door by each hinge.

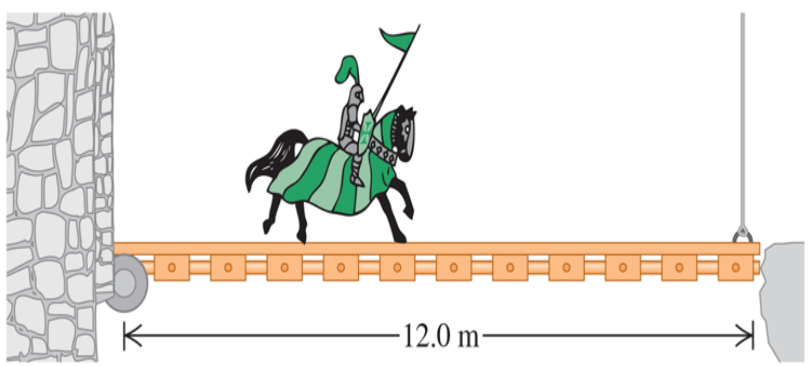
11.47 ••• A box of negligible mass rests at the left end of a 2.00 m, 25.0 kg plank (Fig. P11.47). The width of the box is 75.0 cm, and sand is to be distributed uniformly throughout it. The center of gravity of the nonuniform plank is 50.0 cm from the right end. What mass of sand should be put into the box so that the plank balances horizontally on a fulcrum placed just below its midpoint?

Figure P11.47



11.48 • Sir Lancelot rides slowly out of the castle at Camelot and onto the 12.0-m-long drawbridge that passes over the moat (Fig. P11.48). Unbeknownst to him, his enemies have partially severed the vertical cable holding up the front end of the bridge so that it will break under a tension of 5.80×10^3 N. The bridge has mass 200 kg and its center of gravity is at its center. Lancelot, his lance, his armor, and his horse together have a combined mass of 600 kg. Will the cable break before Lancelot reaches the end of the drawbridge? If so, how far from the castle end of the bridge will the center of gravity of the horse plus rider be when the cable breaks?

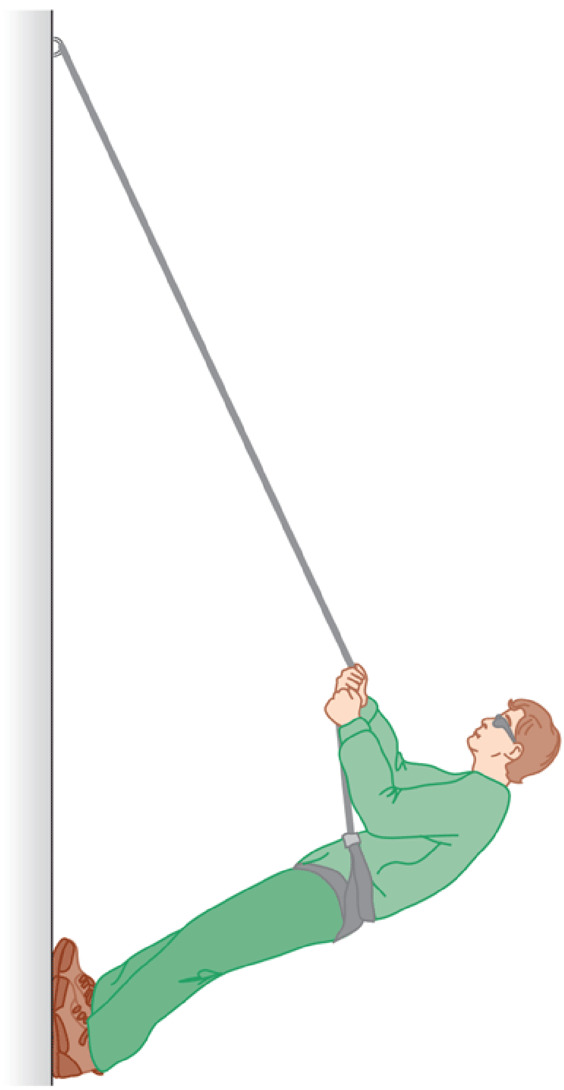
Figure P11.48



11.49 ••• **Mountain Climbing.** Mountaineers often use a rope to lower themselves down the face of a cliff (this is called *rappelling*). They do this with their body nearly horizontal and their feet pushing against the cliff (Fig. P11.49).

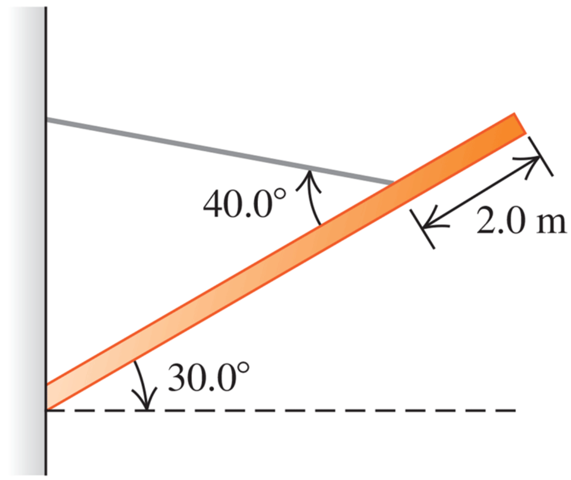
Suppose that an 82.0 kg climber, who is 1.90 m tall and has a center of gravity 1.1 m from his feet, rappels down a vertical cliff with his body raised 35.0° above the horizontal. He holds the rope 1.40 m from his feet, and it makes a 25.0° angle with the cliff face. (a) What tension does his rope need to support? (b) Find the horizontal and vertical components of the force that the cliff face exerts on the climber's feet. (c) What minimum coefficient of static friction is needed to prevent the climber's feet from slipping on the cliff face if he has one foot at a time against the cliff?

Figure P11.49



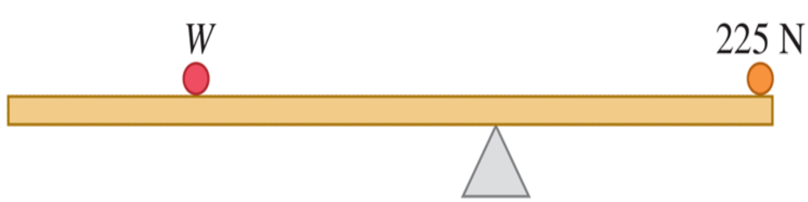
11.50 •• A uniform, 8.0 m, 1150 kg beam is hinged to a wall and supported by a thin cable attached 2.0 m from the free end of the beam (Fig. P11.50). The beam is supported at an angle of 30.0° above the horizontal. (a) Draw a free-body diagram of the beam. (b) Find the tension in the cable. (c) How hard does the beam push inward on the wall?

Figure P11.50



11.51 •• A uniform, 255 N rod that is 2.00 m long carries a 225 N weight at its right end and an unknown weight W toward the left end (Fig. P11.51). When W is placed 50.0 cm from the left end of the rod, the system just balances horizontally when the fulcrum is located 75.0 cm from the right end. (a) Find W. (b) If W is now moved 25.0 cm to the right, how far and in what direction must the fulcrum be moved to restore balance?

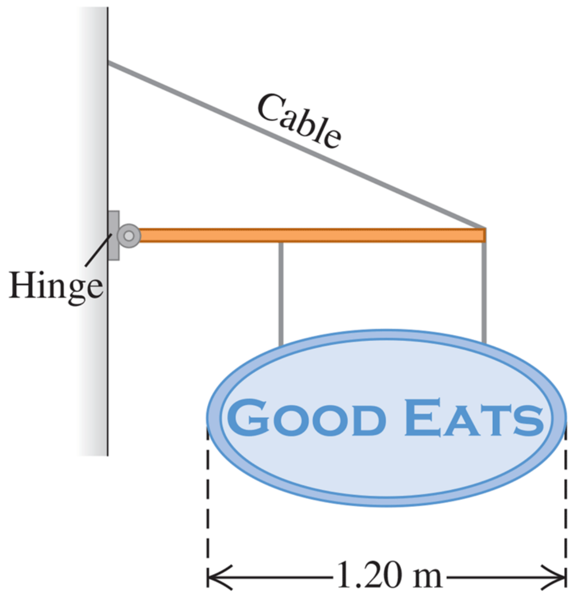
Figure P11.51



11.52 ••• A claw hammer is used to pull a nail out of a board (see Fig. P11.45). The nail is at an angle of 60° to the board, and a force $\vec{F}_1$ of magnitude 400 N applied to the nail is required to pull it from the board. The hammer head contacts the board at point A, which is 0.080 m from where the nail enters the board. A horizontal force $\vec{F}_2$ is applied to the hammer handle at a distance of 0.300 m above the board. What magnitude of force $\vec{F}_2$ is required to apply the required 400 N force (F_1) to the nail? (Ignore the weight of the hammer.)

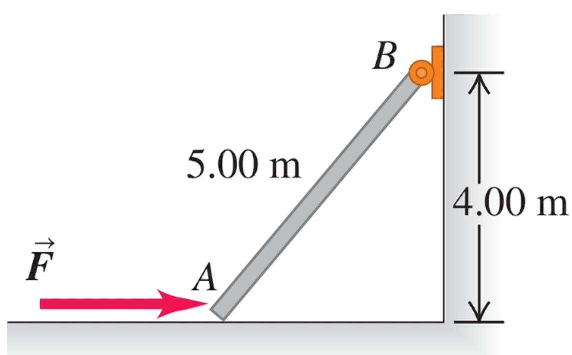
11.53 •• You open a restaurant and hope to entice customers by hanging out a sign (Fig. P11.53). The uniform horizontal beam supporting the sign is 1.50 m long, has a mass of 16.0 kg, and is hinged to the wall. The sign itself is uniform with a mass of 28.0 kg and overall length of 1.20 m. The two wires supporting the sign are each 32.0 cm long, are 90.0 cm apart, and are equally spaced from the middle of the sign. The cable supporting the beam is 2.00 m long. (a) What minimum tension must your cable be able to support without having your sign come crashing down? (b) What minimum vertical force must the hinge be able to support without pulling out of the wall?

Figure P11.53



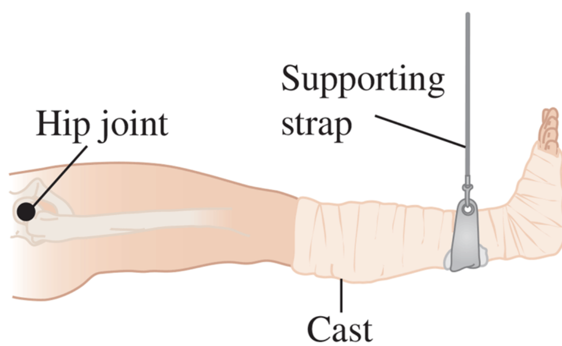
11.54 • End A of the bar AB in Fig. P11.54 rests on a frictionless horizontal surface, and end B is hinged. A horizontal force $\vec{F}$ of magnitude 220 N is exerted on end A. Ignore the weight of the bar. What are the horizontal and vertical components of the force exerted by the bar on the hinge at B?

Figure P11.54



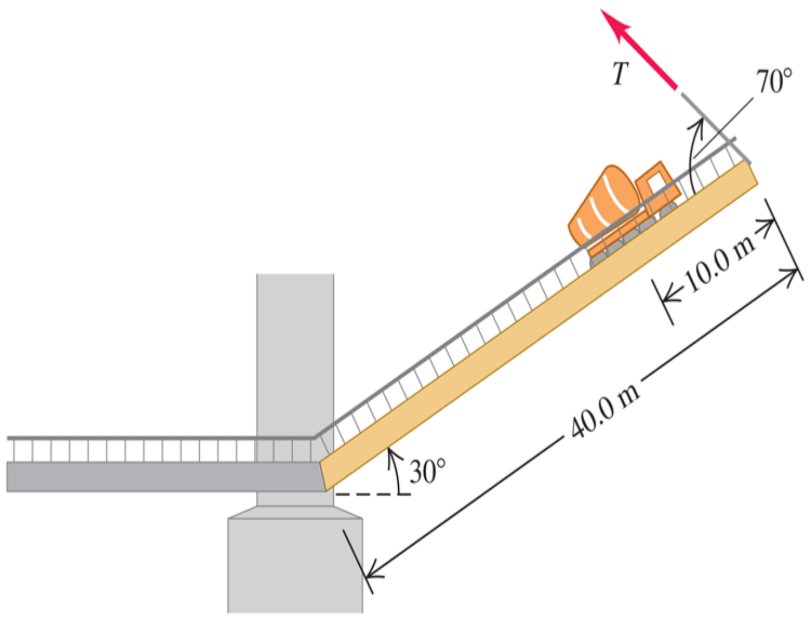
11.55 •• BIO **Supporting a Broken Leg.** A therapist tells a 74 kg patient with a broken leg that he must have his leg in a cast suspended horizontally. For minimum discomfort, the leg should be supported by a vertical strap attached at the center of mass of the leg–cast system (Fig. P11.55). To comply with these instructions, the patient consults a table of typical mass distributions and finds that both upper legs (thighs) together typically account for 21.5% of body weight and the center of mass of each thigh is 18.0 cm from the hip joint. The patient also reads that the two lower legs (including the feet) are 14.0% of body weight, with a center of mass 69.0 cm from the hip joint. The cast has a mass of 5.50 kg, and its center of mass is 78.0 cm from the hip joint. How far from the hip joint should the supporting strap be attached to the cast?

Figure P11.55



11.56 • **A Truck on a Drawbridge.** A loaded cement mixer drives onto an old drawbridge, where it stalls with its center of gravity three-quarters of the way across the span. The truck driver radios for help, sets the handbrake, and waits. Meanwhile, a boat approaches, so the drawbridge is raised by means of a cable attached to the end opposite the hinge (Fig. P11.56). The drawbridge is 40.0 m long and has a mass of 18,000 kg; its center of gravity is at its midpoint. The cement mixer, with driver, has mass 30,000 kg. When the drawbridge has been raised to an angle of 30° above the horizontal, the cable makes an angle of 70° with the surface of the bridge. (a) What is the tension T in the cable when the drawbridge is held in this position? (b) What are the horizontal and vertical components of the force the hinge exerts on the span?

Figure P11.56

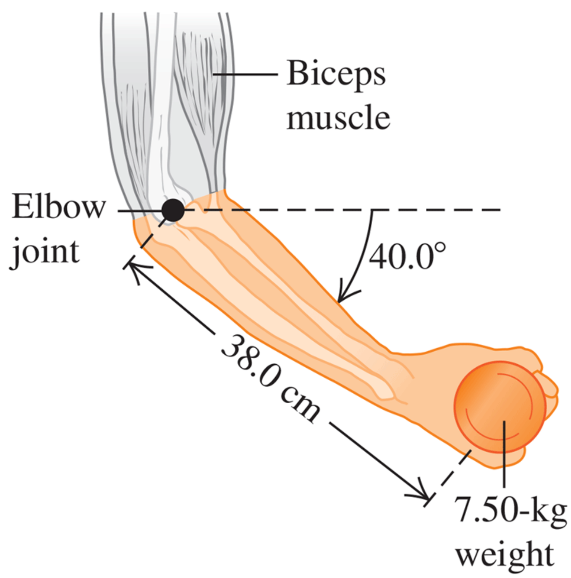


11.57 •• **CP** The left-hand end of a uniform rod of length L and mass m is attached to a vertical wall by a frictionless hinge. The rod is held at an angle θ above the horizontal by a horizontal wire that runs between the wall and the right-hand end of the rod. (a) If the tension in the wire is T, what is the magnitude of the angle θ that the rod makes with the horizontal? (b) The wire breaks and the rod rotates about the hinge. What is the angular speed of the rod as the rod passes through a horizontal position?

11.58 •• **BIO** **Pumping Iron.** A 72.0 kg weightlifter doing arm raises holds a 7.50 kg weight. Her arm pivots around the elbow joint, starting 40.0° below the horizontal (Fig.

P11.58). Biometric measurements have shown that, together, the forearms and the hands account for 6.00% of a person's weight. Since the upper arm is held vertically, the biceps muscle always acts vertically and is attached to the bones of the forearm 5.50 cm from the elbow joint. The center of mass of this person's forearm–hand combination is 16.0 cm from the elbow joint, along the bones of the forearm, and she holds the weight 38.0 cm from her elbow joint. (a) Draw a free-body diagram of the forearm. (b) What force does the biceps muscle exert on the forearm? (c) Find the magnitude and direction of the force that the elbow joint exerts on the forearm. (d) As the weightlifter raises her arm toward a horizontal position, will the force in the biceps muscle increase, decrease, or stay the same? Why?

Figure P11.58



11.59 •• The left-hand end of a light rod of length L is attached to a vertical wall by a frictionless hinge. An object of mass m is suspended from the rod at a point a distance αL from the hinge, where $0 < \alpha \leq 1.00$. The rod is held in a horizontal position by a light wire that runs from the right-hand end of the rod to the wall. The wire makes an angle θ with the rod. (a) What is the angle β that the net force exerted by the hinge on the rod makes with the horizontal? (b) What is the value of α for which $\beta = \theta$? (c) What is β when $\alpha = 1.00$?

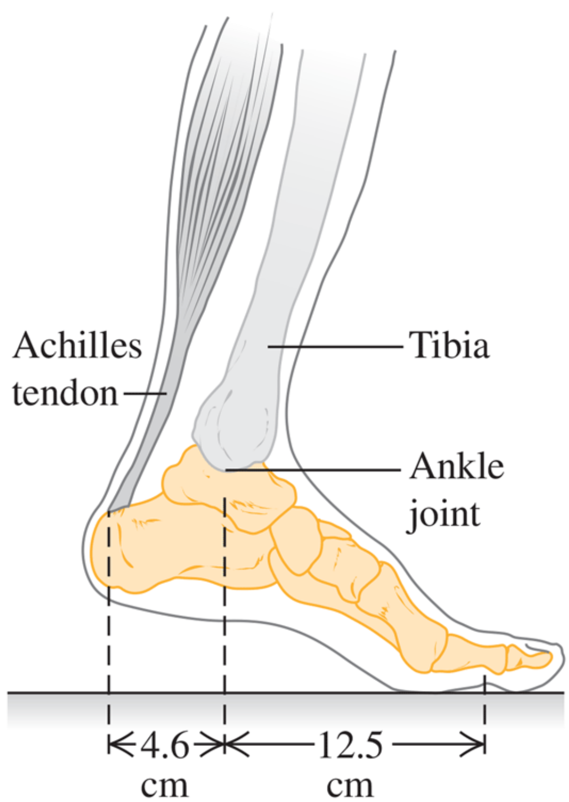
11.60 •• The left-hand end of a slender uniform rod of mass m is placed against a vertical wall. The rod is held in a horizontal position by friction at the wall and by a light wire that runs from the right-hand end of the rod to a point on the wall above the rod. The wire makes an angle θ with the rod. (a) What must the magnitude of the friction force be in order for the rod to remain at rest? (b) If the coefficient of static friction between the rod and the wall is μ_s, what is the maximum angle between the wire and the rod at which the rod doesn't slip at the wall?

11.61 •• A uniform, 7.5-m-long beam weighing 6490 N is hinged to a wall and supported by a thin cable attached 1.5 m from the free end of the beam. The cable runs between the beam and the wall and makes a 40° angle with the beam. What is the tension in the cable when the beam is at an angle of 30° above the horizontal?

11.62 •• **CP** A uniform drawbridge must be held at a 37° angle above the horizontal to allow ships to pass underneath. The drawbridge weighs 45,000 N and is 14.0 m long. A cable is connected 3.5 m from the hinge where the bridge pivots (measured along the bridge) and pulls horizontally on the bridge to hold it in place. (a) What is the tension in the cable? (b) Find the magnitude and direction of the force the hinge exerts on the bridge. (c) If the cable suddenly breaks, what is the magnitude of the angular acceleration of the drawbridge just after the cable breaks? (d) What is the angular speed of the drawbridge as it becomes horizontal?

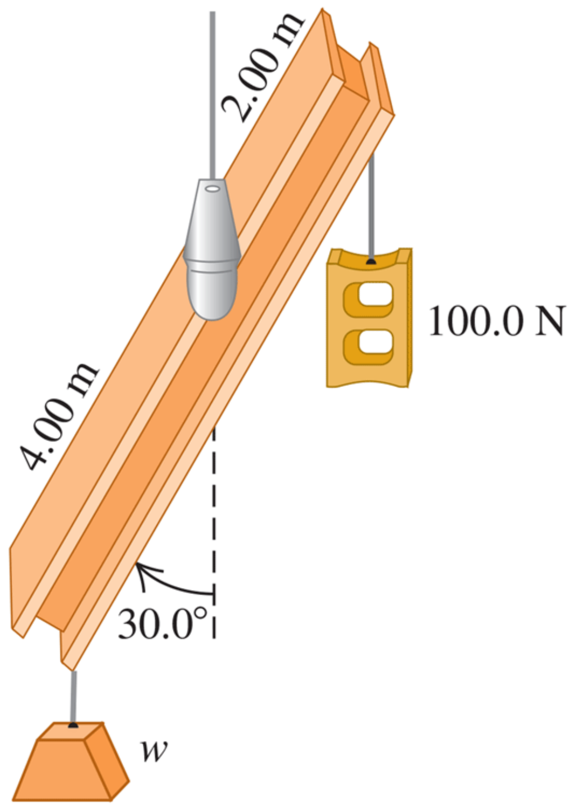
11.63 •• BIO **Tendon-Stretching Exercises.** As part of an exercise program, a 75 kg person does toe raises in which he raises his entire body weight on the ball of one foot (Fig. P11.63). The Achilles tendon pulls straight upward on the heel bone of his foot. This tendon is 25 cm long and has a cross-sectional area of 78 mm^2 and a Young's modulus of 1470 MPa. (a) Draw a free-body diagram of the person's foot (everything below the ankle joint). Ignore the weight of the foot. (b) What force does the Achilles tendon exert on the heel during this exercise? Express your answer in newtons and in multiples of his weight. (c) By how many millimeters does the exercise stretch his Achilles tendon?

Figure P11.63



11.64 •• (a) In Fig. P11.64 a 6.00-m-long, uniform beam is hanging from a point 1.00 m to the right of its center. The beam weighs 140 N and makes an angle of 30.0° with the vertical. At the right-hand end of the beam a 100.0 N weight is hung; an unknown weight w hangs at the left end. If the system is in equilibrium, what is w? You can ignore the thickness of the beam. (b) If the beam makes, instead, an angle of 45.0° with the vertical, what is w?

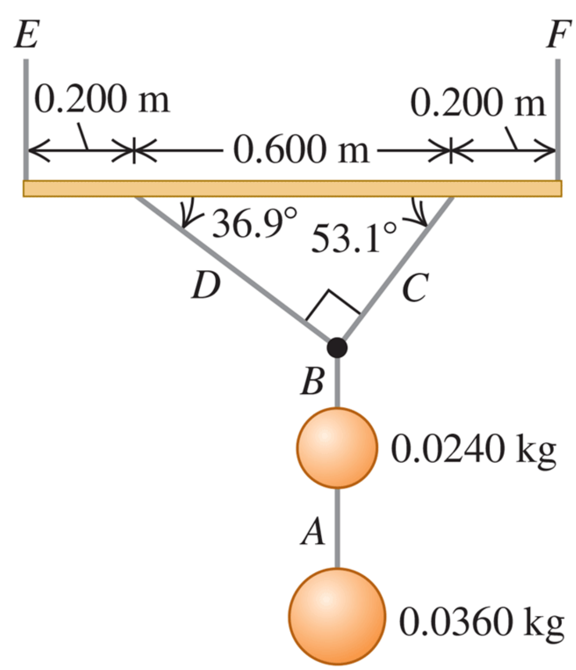
Figure P11.64



11.65 •• The left-hand end of a uniform rod of mass 2.00 kg and length 1.20 m is attached to a vertical wall by a frictionless hinge. The rod is held in a horizontal position by an aluminum wire that runs between the right-hand end of the rod and a point on the wall that is above the hinge. The cross-sectional radius of the wire is 2.50 mm, and the wire makes an angle of 30.0° with the rod. (a) What is the length of the wire? (b) An object of mass 90.0 kg is suspended from the right-hand end of the rod. What is the increase in the length of the wire when this object is added? In your analysis do you need to be concerned that the lengthening of the wire means that the rod is no longer horizontal?

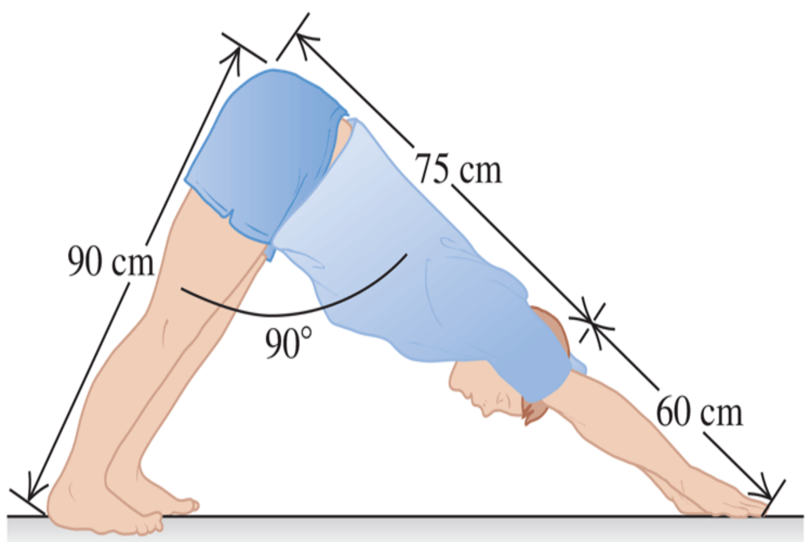
11.66 • A holiday decoration consists of two shiny glass spheres with masses 0.0240 kg and 0.0360 kg suspended from a uniform rod with mass 0.120 kg and length 1.00 m (Fig. P11.66). The rod is suspended from the ceiling by a vertical cord at each end, so that it is horizontal. Calculate the tension in each of the cords A through F.

Figure P11.66



11.67 •• BIO **Downward-Facing Dog.** The yoga exercise "Downward-Facing Dog" requires stretching your hands straight out above your head and bending down to lean against the floor. This exercise is performed by a 750 N person as shown in Fig. P11.67. When he bends his body at the hip to a 90° angle between his legs and trunk, his legs, trunk, head, and arms have the dimensions indicated. Furthermore, his legs and feet weigh a total of 277 N, and their center of mass is 41 cm from his hip, measured along his legs. The person's trunk, head, and arms weigh 473 N, and their center of gravity is 65 cm from his hip, measured along the upper body. (a) Find the normal force that the floor exerts on each foot and on each hand, assuming that the person does not favor either hand or either foot. (b) Find the friction force on each foot and on each hand, assuming that it is the same on both feet and on both hands (but not necessarily the same on the feet as on the hands). [*Hint:* First treat his entire body as a system; then isolate his legs (or his upper body).]

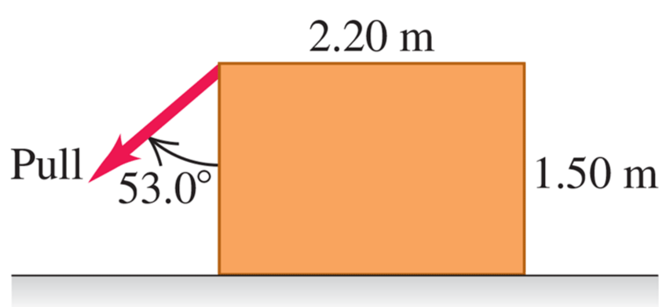
Figure P11.67



11.68 •• **CP** A brass wire is 1.40 m long and has a cross-sectional area of 6.00 mm^2. A small steel ball with mass 0.0800 kg is attached to the end of the wire. You hold the other end of the wire and whirl the ball in a vertical circle of radius 1.40 m. What speed must the ball have at the lowest point of its path if its fractional change in length of the brass wire at this point from its unstretched length is 2.0×10^{-5}? Treat the ball as a point mass.

11.69 • A worker wants to turn over a uniform, 1250 N, rectangular crate by pulling at 53.0° on one of its vertical sides (Fig. P11.69). The floor is rough enough to prevent the crate from slipping. (a) What pull is needed to just start the crate to tip? (b) How hard does the floor push upward on the crate? (c) Find the friction force on the crate. (d) What is the minimum coefficient of static friction needed to prevent the crate from slipping on the floor?

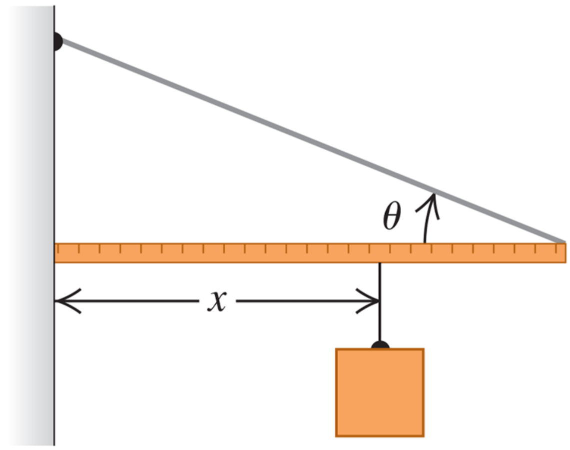
Figure P11.69



11.70 ••• One end of a uniform meter stick is placed against a vertical wall (Fig. P11.70). The other end is held by a lightweight cord that makes an angle θ with the stick. The coefficient of static friction between the end of the meter stick and the wall is 0.40. (a) What is the maximum value the angle θ can have if the stick is to remain in equilibrium? (b) Let the angle θ be 15°. A block of the same weight as the meter stick is suspended from the stick, as shown, at a distance x from the wall. What is the minimum value of x for which the stick will remain in equilibrium? (c) When

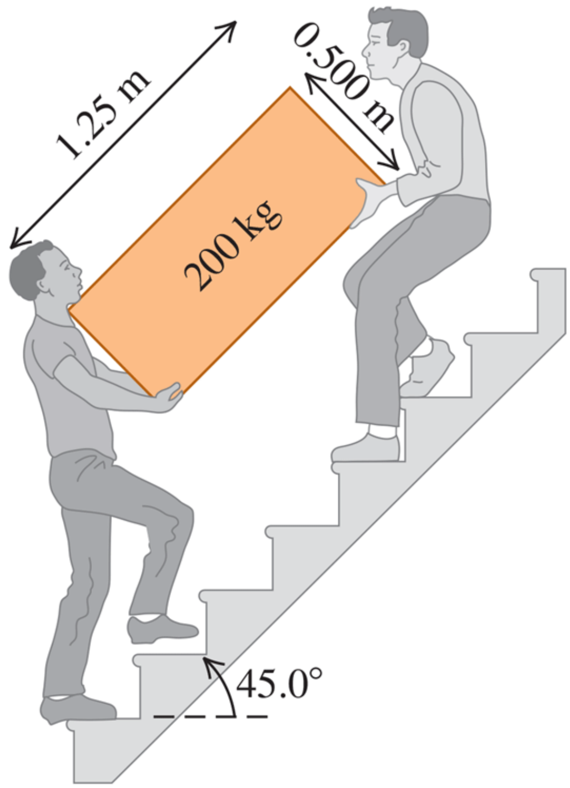
$\theta = 15^\circ$, how large must the coefficient of static friction be so that the block can be attached 10 cm from the left end of the stick without causing it to slip?

Figure P11.70



11.71 •• Two friends are carrying a 200 kg crate up a flight of stairs. The crate is 1.25 m long and 0.500 m high, and its center of gravity is at its center. The stairs make a 45.0° angle with respect to the floor. The crate also is carried at a 45.0° angle, so that its bottom side is parallel to the slope of the stairs (Fig. P11.71). If the force each person applies is vertical, what is the magnitude of each of these forces? Is it better to be the person above or below on the stairs?

Figure P11.71

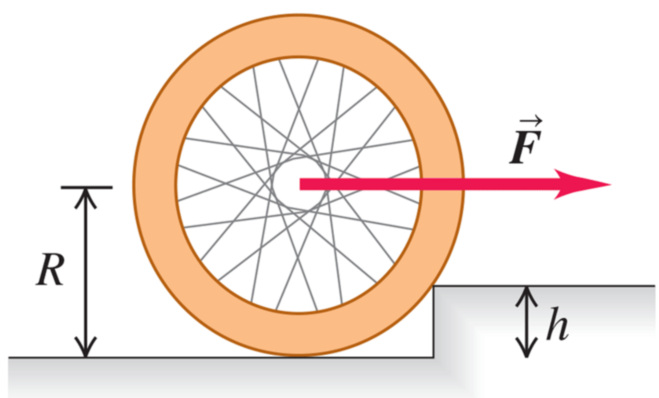


11.72 ••• In a city park a nonuniform wooden beam 4.00 m long is suspended horizontally by a light steel cable at each end. The cable at the left-hand end makes an angle of 30.0° with the vertical and has tension 620 N. The cable at the right-hand end of the beam makes an angle of 50.0° with the vertical. As an employee of the Parks and Recreation Department, you are asked to find the weight of the beam and the location of its center of gravity.

11.73 •• **CALC** **BIO** Refer to the discussion of holding a dumbbell in Example 11.4 (Section 11.3). The maximum weight that can be held in this way is limited by the maximum allowable tendon tension T (determined by the strength of the tendons) and by the distance D from the elbow to where the tendon attaches to the forearm. (a) Let T_{max} represent the maximum value of the tendon tension. Use the results of Example 11.4 to express w_{max} (the maximum weight that can be held) in terms of $T_{\max}$, L, D, and h. Your expression should *not* include the angle θ. (b) The tendons of different primates are attached to the forearm at different values of D. Calculate the derivative of $w_{\max}$ with respect to D, and determine whether the derivative is positive or negative. (c) A chimpanzee tendon is attached to the forearm at a point farther from the elbow than for humans. Use this to explain why chimpanzees have stronger arms than humans. (The disadvantage is that chimpanzees have less flexible arms than do humans.)

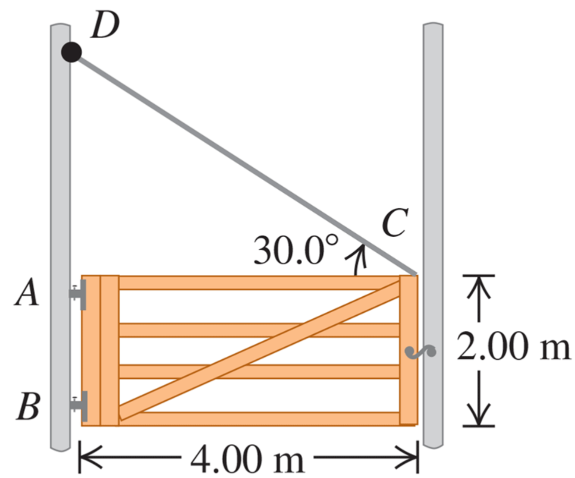
11.74 •• You are trying to raise a bicycle wheel of mass m and radius R up over a curb of height h. To do this, you apply a horizontal force $\vec{F}$ (**Fig. P11.74**). What is the smallest magnitude of the force $\vec{F}$ that will succeed in raising the wheel onto the curb when the force is applied (a) at the center of the wheel and (b) at the top of the wheel? (c) In which case is less force required?

Figure P11.74



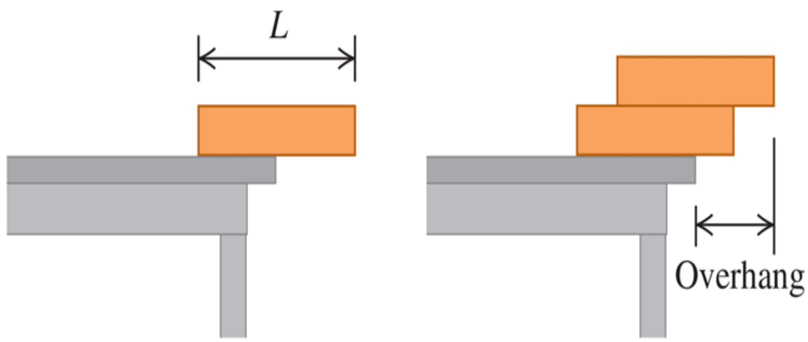
11.75 • **The Farmyard Gate.** A gate 4.00 m wide and 2.00 m high weighs 700 N. Its center of gravity is at its center, and it is hinged at A and B. To relieve the strain on the top hinge, a wire CD is connected as shown in Fig. P11.75. The tension in CD is increased until the horizontal force at hinge A is zero. What are (a) the tension in the wire CD; (b) the magnitude of the horizontal component of the force at hinge B; (c) the combined vertical force exerted by hinges A and B?

Figure P11.75



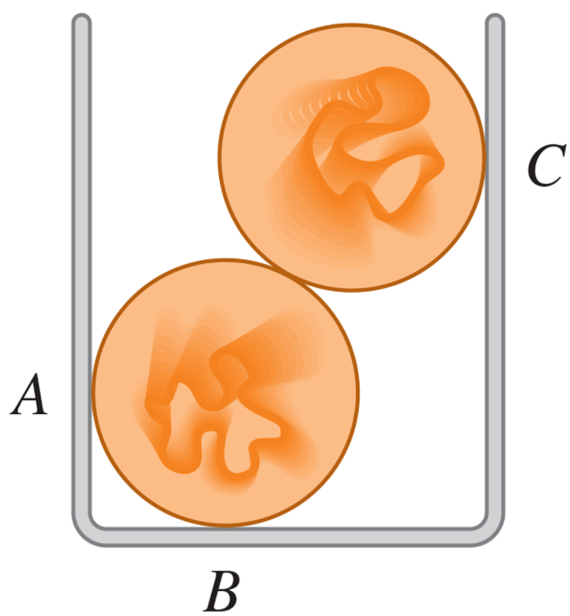
11.76 • If you put a uniform block at the edge of a table, the center of the block must be over the table for the block not to fall off. (a) If you stack two identical blocks at the table edge, the center of the top block must be over the bottom block, and the center of gravity of the two blocks together must be over the table. In terms of the length L of each block, what is the maximum overhang possible (Fig. P11.76)? (b) Repeat part (a) for three identical blocks and for four identical blocks. (c) Is it possible to make a stack of blocks such that the uppermost block is not directly over the table at all? How many blocks would it take to do this? (Try.)

Figure P11.76



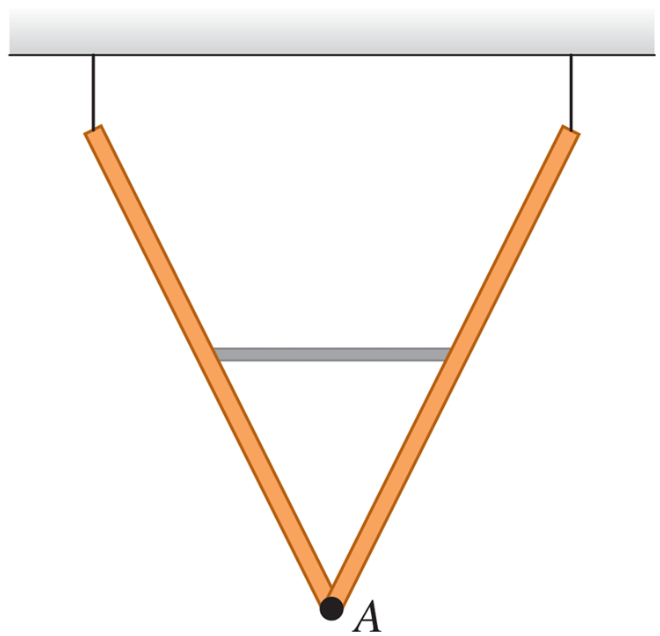
11.77 ••• Two uniform, 75.0 g marbles 2.00 cm in diameter are stacked as shown in Fig. P11.77 in a container that is 3.00 cm wide. (a) Find the force that the container exerts on the marbles at the points of contact A, B, and C. (b) What force does each marble exert on the other?

Figure P11.77



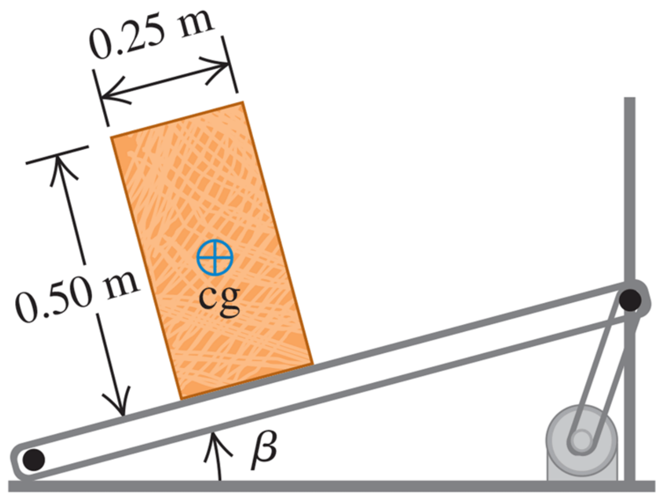
11.78 •• Two identical, uniform beams weighing 260 N each are connected at one end by a frictionless hinge. A light horizontal crossbar attached at the midpoints of the beams maintains an angle of 53.0° between the beams. The beams are suspended from the ceiling by vertical wires such that they form a “V” (Fig. P11.78). (a) What force does the crossbar exert on each beam? (b) Is the crossbar under tension or compression? (c) What force (magnitude and direction) does the hinge at point A exert on each beam?

Figure P11.78



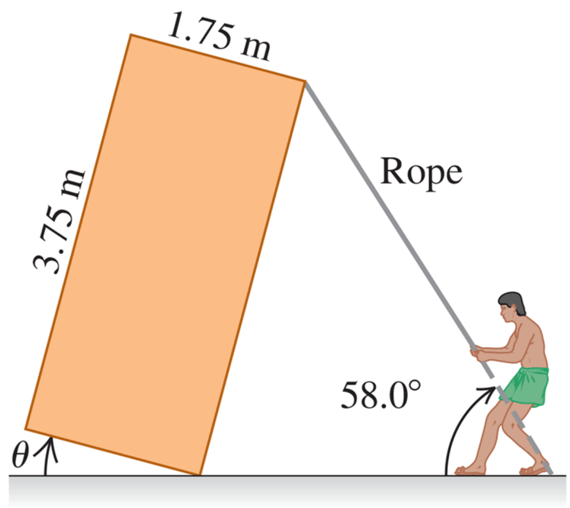
11.79 • An engineer is designing a conveyor system for loading hay bales into a wagon (Fig. P11.79). Each bale is 0.25 m wide, 0.50 m high, and 0.80 m long (the dimension perpendicular to the plane of the figure), with mass 30.0 kg. The center of gravity of each bale is at its geometrical center. The coefficient of static friction between a bale and the conveyor belt is 0.60, and the belt moves with constant speed. (a) The angle β of the conveyor is slowly increased. At some critical angle a bale will tip (if it doesn’t slip first), and at some different critical angle it will slip (if it doesn’t tip first). Find the two critical angles and determine which happens at the smaller angle. (b) Would the outcome of part (a) be different if the coefficient of friction were 0.40?

Figure P11.79



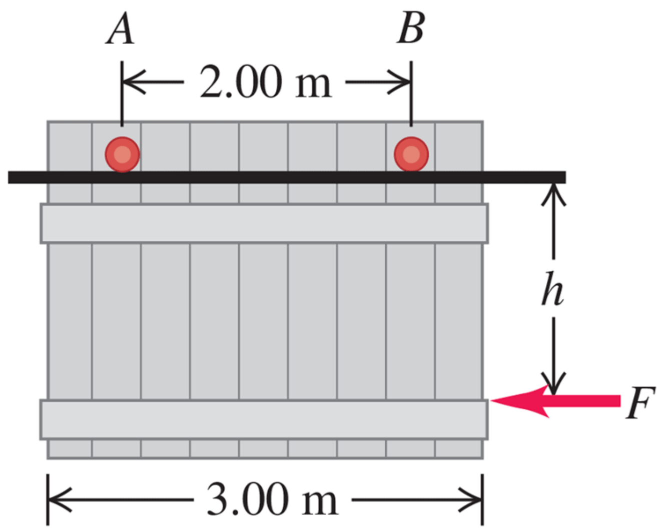
11.80 ••• **Pyramid Builders.** Ancient pyramid builders are balancing a uniform rectangular stone slab of weight w, tipped at an angle θ above the horizontal, using a rope (Fig. P11.80). The rope is held by five workers who share the force equally. (a) If $\theta = 20.0^\circ$, what force does each worker exert on the rope? (b) As θ increases, does each worker have to exert more or less force than in part (a), assuming they do not change the angle of the rope? Why? (c) At what angle do the workers need to exert *no force* to balance the slab? What happens if θ exceeds this value?

Figure P11.80



11.81 •• A garage door is mounted on an overhead rail (Fig. P11.81). The wheels at A and B have rusted so that they do not roll, but rather slide along the track. The coefficient of kinetic friction is 0.52. The distance between the wheels is 2.00 m, and each is 0.50 m from the vertical sides of the door. The door is uniform and weighs 950 N. It is pushed to the left at constant speed by a horizontal force $\vec{F}$, that is applied as shown in the figure. (a) If the distance h is 1.60 m, what is the vertical component of the force exerted on each wheel by the track? (b) Find the maximum value h can have without causing one wheel to leave the track.

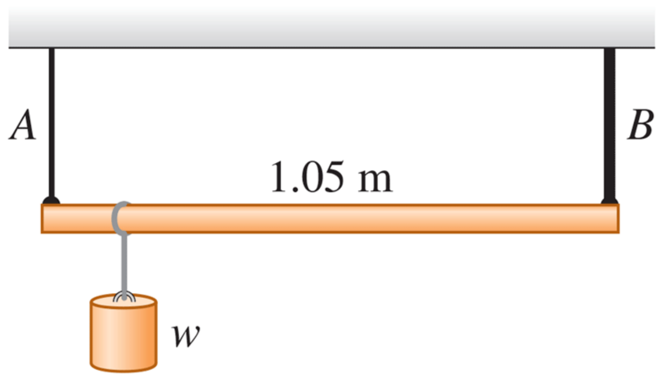
Figure P11.81



11.82 ••• **CP** A 12.0 kg mass, fastened to the end of an aluminum rod with an unstretched length of 0.70 m, is whirled in a vertical circle with a constant angular speed of 120 rev/min. The cross-sectional area of the rod is 0.014 cm^2. Calculate the elongation of the rod when the mass is (a) at the lowest point of the path and (b) at the highest point of its path.

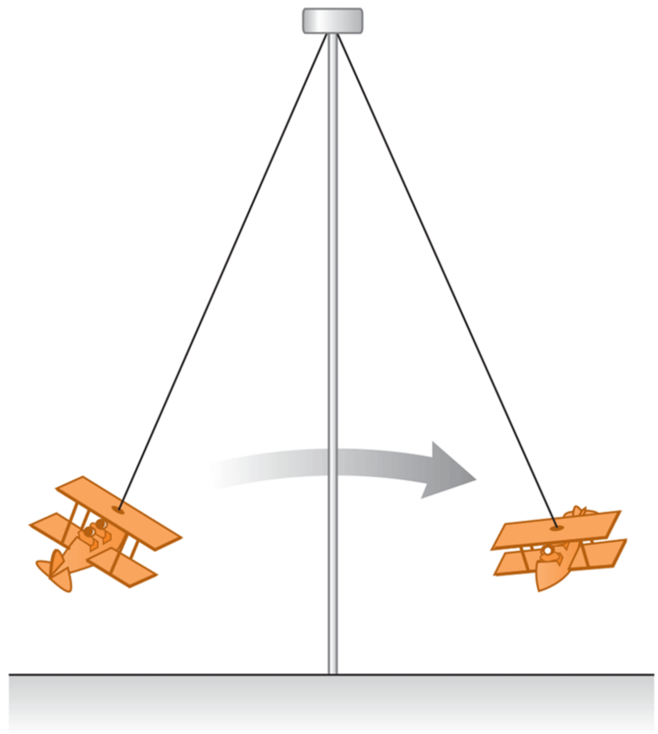
11.83 ••• A 1.05-m-long rod of negligible weight is supported at its ends by wires A and B of equal length (Fig. P11.83). The cross-sectional area of A is 2.00 mm^2 and that of B is 4.00 mm^2. Young's modulus for wire A is 1.80×10^{11} Pa; that for B is 1.20×10^{11} Pa. At what point along the rod should a weight w be suspended to produce (a) equal stresses in A and B and (b) equal strains in A and B?

Figure P11.83



11.84 ••• **CP** An amusement park ride consists of airplane-shaped cars attached to steel rods (Fig. P11.84). Each rod has a length of 15.0 m and a cross-sectional area of 8.00 cm^2. The rods are attached to a frictionless hinge at the top, so that the cars can swing outward when the ride rotates. (a) How much is each rod stretched when it is vertical and the ride is at rest? (Assume that each car plus two people seated in it has a total weight of 1900 N.) (b) When operating, the ride has a maximum angular speed of 12.0 rev/min. How much is the rod stretched then?

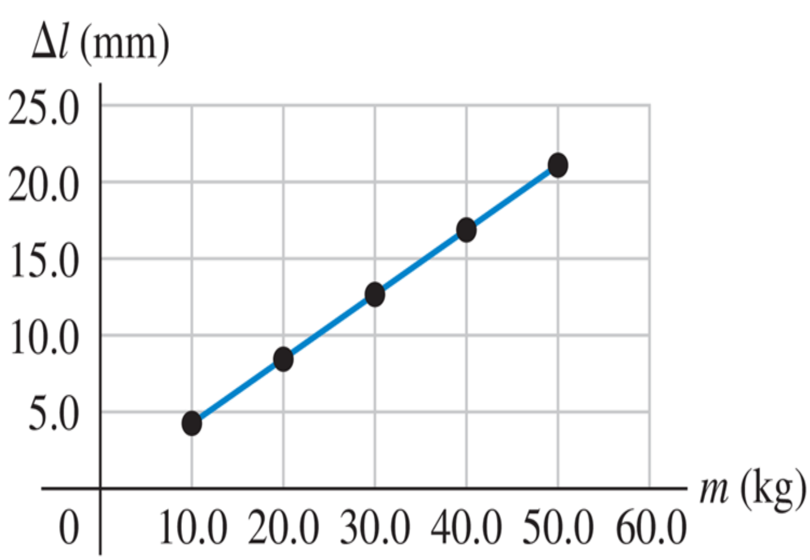
Figure P11.84



11.85 ••• CP BIO **Stress on the Shin Bone.** The compressive strength of our bones is important in everyday life. Young's modulus for bone is about 1.4×10^{10} Pa. Bone can take only about a 1.0% change in its length before fracturing. (a) What is the maximum force that can be applied to a bone whose minimum cross-sectional area is 3.0 cm^2? (This is approximately the cross-sectional area of a tibia, or shin bone, at its narrowest point.) (b) Estimate the maximum height from which a 70 kg man could jump and not fracture his tibia. Take the time between when he first touches the floor and when he has stopped to be 0.030 s, and assume that the stress on his two legs is distributed equally.

11.86 •• DATA You are to use a long, thin wire to build a pendulum in a science museum. The wire has an unstretched length of 22.0 m and a circular cross section of diameter 0.860 mm; it is made of an alloy that has a large breaking stress. One end of the wire will be attached to the ceiling, and a 9.50 kg metal sphere will be attached to the other end. As the pendulum swings back and forth, the wire's maximum angular displacement from the vertical will be 36.0°. You must determine the maximum amount the wire will stretch during this motion. So, before you attach the metal sphere, you suspend a test mass (mass m) from the wire's lower end. You then measure the increase in length Δl of the wire for several different test masses. Figure P11.86, a graph of Δl versus m, shows the results and the straight line that gives the best fit to the data. The equation for this line is $\Delta l = (0.422\ \text{mm/kg})m$. (a) Assume that $g = 9.80\ \text{m/s}^2$, and use Fig. P11.86 to calculate Young's modulus Y for this wire. (b) You remove the test masses, attach the 9.50 kg sphere, and release the sphere from rest, with the wire displaced by 36.0°. Calculate the amount the wire will stretch as it swings through the vertical. Ignore air resistance.

Figure P11.86



11.87 •• **DATA** You need to measure the mass M of a 4.00-m-long bar. The bar has a square cross section but has some holes drilled along its length, so you suspect that its center of gravity isn't in the middle of the bar. The bar is too long for you to weigh on your scale. So, first you balance the bar on a knife-edge pivot and determine that the bar's center of gravity is 1.88 m from its left-hand end. You then place the bar on the pivot so that the point of support is 1.50 m from the left-hand end of the bar. Next you suspend a 2.00 kg mass (m_1) from the bar at a point 0.200 m from the left-hand end. Finally, you suspend a mass $m_2 = 1.00$ kg from the bar at a distance x from the left-hand end and adjust x so that the bar is balanced. You repeat this step for other values of m_2 and record each corresponding value of x. The table gives your results.

m_2 (kg)	1.00	1.50	2.00	2.50	3.00	4.00
x (m)	3.50	2.83	2.50	2.32	2.16	2.00

(a) Draw a free-body diagram for the bar when m_1 and m_2 are suspended from it. (b) Apply the static equilibrium equation $\sum \tau_z = 0$ with the axis at the location of the knife-edge pivot. Solve the equation for x as a function of m_2. (c) Plot x versus $1/m_2$. Use the slope of the best-fit straight line and the equation you derived in part (b) to calculate that bar's mass M. Use $g = 9.80 \text{ m/s}^2$. (d) What is the y-intercept of the straight line that fits the data? Explain why it has this value.

11.88 ••• **DATA** You are a construction engineer working on the interior design of a retail store in a mall. A 2.00-m-long uniform bar of mass 8.50 kg is to be attached at one end to a wall, by means of a hinge that allows the bar to rotate freely with very little friction. The bar will be held in a horizontal position by a light cable from a point on the bar (a distance x from the hinge) to a point on the wall above the hinge. The cable makes an angle θ with the bar. The architect has proposed four possible ways to connect the cable and asked you to assess them:

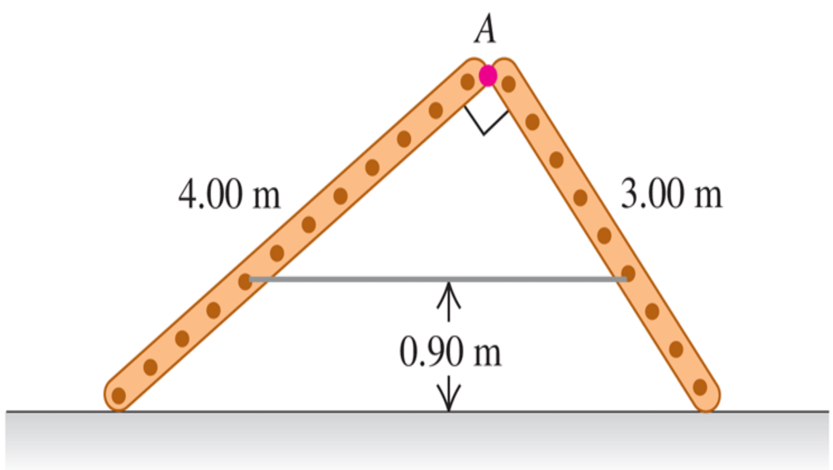
Alternative	A	B	C	D
x (m)	2.00	1.50	0.75	0.50
θ (degrees)	30	60	37	75

(a) There is concern about the strength of the cable that will be required. Which set of x and θ values in the table produces the smallest tension in the cable? The greatest? (b) There is concern about the breaking strength of the sheetrock wall where the hinge will be attached. Which set of x and θ values produces the smallest horizontal component of the force the bar exerts on the hinge? The largest? (c) There is also concern about the required strength of the hinge and the strength of its attachment to the wall. Which set of x and θ values produces the smallest magnitude of the vertical component of the force the bar exerts on the hinge? The largest? (*Hint:* Does the direction of the vertical component of the force the hinge exerts on the bar depend on where along the bar the cable is attached?) (d) Is one of the alternatives given in the table preferable? Should any of the alternatives be avoided? Discuss.

Challenge Problems

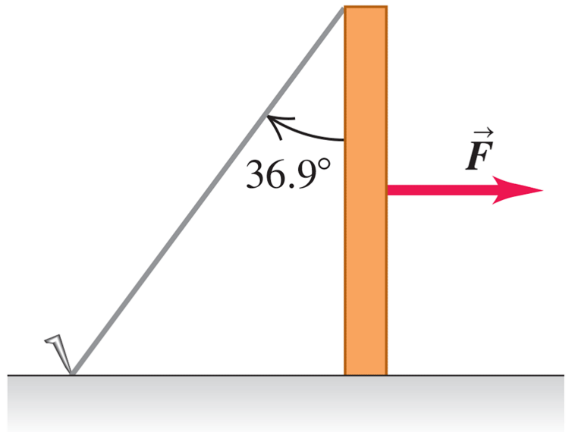
11.89 ••• Two ladders, 4.00 m and 3.00 m long, are hinged at point A and tied together by a horizontal rope 0.90 m above the floor (Fig. P11.89). The ladders weigh 480 N and 360 N, respectively, and the center of gravity of each is at its center. Assume that the floor is freshly waxed and frictionless. (a) Find the upward force at the bottom of each ladder. (b) Find the tension in the rope. (c) Find the magnitude of the force one ladder exerts on the other at point A. (d) If an 800 N painter stands at point A, find the tension in the horizontal rope.

Figure P11.89



11.90 ••• **Knocking Over a Post.** One end of a post weighing 400 N and with height h rests on a rough horizontal surface with $\mu_s = 0.30$. The upper end is held by a rope fastened to the surface and making an angle of 36.9° with the post (Fig. P11.90). A horizontal force $\vec{F}$ is exerted on the post as shown. (a) If the force $\vec{F}$ is applied at the midpoint of the post, what is the largest value it can have without causing the post to slip? (b) How large can the force be without causing the post to slip if its point of application is $\frac{6}{10}$ of the way from the ground to the top of the post? (c) Show that if the point of application of the force is too high, the post cannot be made to slip, no matter how great the force. Find the critical height for the point of application.

Figure P11.90



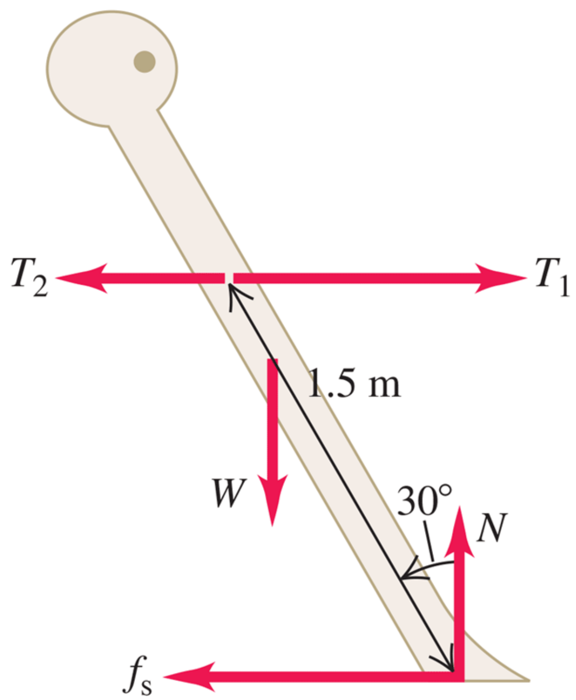
11.91 ••• CP An angler hangs a 4.50 kg fish from a vertical steel wire 1.50 m long and 5.00×10^{-3} cm^2 in cross-sectional area. The upper end of the wire is securely fastened to a support. (a) Calculate the amount the wire is stretched by the hanging fish. The angler now applies a varying force $\vec{F}$ at the lower end of the wire, pulling it very slowly downward by 0.500 mm from its equilibrium position. For this downward motion, calculate (b) the work done by gravity; (c) the work done by the force $\vec{F}$; (d) the work done by the force the wire exerts on the fish; and (e) the change in the elastic potential energy (the potential energy associated with the tensile stress in the wire). Compare the answers in parts (d) and (e).

MCAT-Style Passage Problems

BIO **Torques and Tug-of-War**. In a study of the biomechanics of the tug-of-war, a 2.0-m-tall, 80.0 kg competitor in the middle of the line is considered to be a rigid body leaning back at an angle of 30.0° to the vertical. The competitor is pulling on a rope that is held horizontal a distance of 1.5 m from his feet (as measured along the line of the body).

At the moment shown in the figure, the man is stationary and the tension in the rope in front of him is $T_1 = 1160$ N. Since there is friction between the rope and his hands, the tension in the rope behind him, T_2, is not equal to T_1. His center of mass is halfway between his feet and the top of his head. The coefficient of static friction between his feet and the ground is 0.65.

Figure 11.92



Side view.

11.92 What is tension T_2 in the rope behind him? (a) 590 N; (b) 650 N; (c) 860 N; (d) 1100 N.

11.93 If he leans slightly farther back (increasing the angle between his body and the vertical) but remains stationary in this new position, which of the following statements is true? Assume that the rope remains horizontal. (a) The difference between T_1 and T_2 will increase, balancing the increased torque about his feet that his weight produces when he leans farther back; (b) the difference between T_1 and T_2 will decrease, balancing the increased torque about his feet that his weight produces when he leans farther back; (c) neither T_1 nor T_2 will change, because no other forces are changing; (d) both T_1 and T_2 will change, but the difference between them will remain the same.

11.94 His body is again leaning back at 30.0° to the vertical, but now the height at which the rope is held above—but still parallel to—the ground is varied. The tension in the rope in front of the competitor (T_1) is measured as a function of the shortest distance between the rope and the ground (the holding height). Tension T_1 is found to decrease as the holding height increases. What could explain this observation? As the holding height increases, (a) the moment arm of the rope about his feet decreases due to the angle that his body makes with the vertical; (b) the moment arm of the weight about his feet decreases due to the angle that his body makes with the vertical; (c) a smaller tension in the rope is needed to produce a torque sufficient to balance the torque of the weight about his feet; (d) his center of mass moves down to compensate, so less tension in the rope is required to maintain equilibrium.

11.95 His body is leaning back at 30.0° to the vertical, but the coefficient of static friction between his feet and the ground is suddenly reduced to 0.50. What will happen? (a) His entire body will accelerate forward; (b) his feet will slip forward; (c) his feet will slip backward; (d) his feet will not slip.

Answers: Equilibrium and Elasticity

Chapter Opening Question ?

(i) Each stone in the arch is under compression, not tension. This is because the forces on the stones tend to push them inward toward the center of the arch and thus squeeze them together. Compared to a solid supporting wall, a wall with arches is just as strong yet much more economical to build.

Test Your Understanding

11.1 **(i)** Situation (i) satisfies both equilibrium conditions because the seagull has zero acceleration (so $\sum \vec{F} = 0$) and no tendency to start rotating (so $\sum \vec{\tau} = 0$). Situation (ii) satisfies the first condition because the crankshaft as a whole does not accelerate through space, but it does not satisfy the second condition; the crankshaft has an angular acceleration, so $\sum \vec{\tau}$ is not zero. Situation (iii) satisfies the second condition (there is no tendency to rotate) but not the first one; the baseball accelerates in its flight (due to gravity and air resistance), so $\sum \vec{F}$ is not zero.

11.2 **(ii)** In equilibrium, the center of gravity must be at the point of support. Since the rock and meter stick have the same mass and hence the same weight, the center of gravity of the system is midway between their respective centers. The center of gravity of the meter stick alone is 0.50 m from the left end (that is, at the middle of the meter stick), so the center of gravity of the combination of rock and meter stick is 0.25 m from the left end.

11.3 **(ii), (i), (iii)** This is the same situation described in Example 11.4, with the rod replacing the forearm, the hinge replacing the elbow, and the cable replacing the tendon. The only difference is that the cable attachment point is at the end of the rod, so the distances D and L are identical. From Example 11.4, the tension is

$$T = \frac{Lw}{L\sin\theta} = \frac{w}{\sin\theta}$$

Since $\sin\theta$ is less than 1, the tension T is greater than the weight w. The vertical component of the force exerted by the hinge is

$$E_y = -\frac{(L - L)w}{L} = 0$$

In this situation, the hinge exerts *no* vertical force. To see this, calculate torques around the right end of the horizontal rod: The only force that exerts a torque around this point is the vertical component of the hinge force, so this force component must be zero.

11.4 (a) **(iii)**, (b) **(ii)** In (a), the copper rod has 10 times the elongation Δl of the steel rod, but it also has 10 times the original length l_0. Hence the tensile strain $\Delta l/l_0$ is the same for both rods. In (b), the stress is equal to Young's modulus Y multiplied by the strain. From Table 11.1, steel has a larger value of Y, so a greater stress is required to produce the same strain.

11.5 In (a) and (b), the bumper will have sprung back to its original shape (although the paint may be scratched). In (c), the bumper will have a permanent dent or deformation. In (d), the bumper will be torn or broken.

Key Example Variation Problems

VP11.1.1 13.3 kg

VP11.1.2 11.0 m

VP11.1.3 $L/10$ to the right of center

VP11.1.4 141 kg

VP11.4.1 5.86 m

VP11.4.2 nose wheel: 19.8%, main wheels: 80.2%

VP11.4.3 **a.** $F_{\text{hinge}-y} = 0$

b. $T = w/\sin\theta$

c. $F_{\text{hinge}-x} = w/\tan\theta$

VP11.4.4 **a.** $F_{\text{hinge}-y} = w/2$

b. $T = 3w/2\sin\theta$

c. $F_{\text{hinge}-x} = 3w/2\tan\theta$

VP11.7.1 **a.** 5.5×10^{8} Pa

b. 3.5×10^{4} N

VP11.7.2 **a.** 2.3×10^{-4}

b. 0.19 mm

VP11.7.3 **a.** $4.4 \times 10^{-9}\ \text{m}^3$

b. $6.7 \times 10^{-9}\ \text{m}^3$

VP11.7.4 4.8×10^{-5} m

Bridging Problem

(a) $$T = \frac{2mg}{3\sin\theta}$$

(b) $$F = \frac{2mg}{3\sin\theta}\sqrt{\cos^2\theta + \frac{1}{4}\sin^2\theta}, \quad \phi = \arctan\left(\frac{1}{2}\tan\theta\right)$$

(c) $$\Delta l = \frac{mgl_0}{2AY\tan\theta}$$

(d) 4

Chapter 12
Fluid Mechanics

? A colorful tail-spot wrasse (*Halichoeres melanurus*) is about 10 cm long and can float in the ocean with little effort, while a manta ray (*Manta birostris*) is more than 5 m across and must "flap" its fins continuously to keep from sinking. Which of these best explains the difference? A manta ray has (i) a different shape; (ii) greater mass; (iii) greater volume; (iv) a greater product of mass and volume; (v) a greater ratio of mass to volume.

Learning Outcomes

In this chapter, you'll learn...

12.1 The meaning of the density of a material and the average density of an object.

12.2 What is meant by the pressure in a fluid, and how it is measured.

12.3 How to calculate the buoyant force that a fluid exerts on an object immersed in it.

12.4 The significance of laminar versus turbulent fluid flow, and how the speed of flow in a tube depends on the tube's size.

12.5 How to use Bernoulli's equation to relate pressure and flow speed at different points in certain types of flow.

12.6 How viscous flow and turbulent flow differ from ideal flow.

You'll need to review...

7.1 Mechanical energy change when forces other than gravity do work.

11.4 Pressure and its units.

Fluids play a vital role in many aspects of everyday life. We drink them, breathe them, swim in them. They circulate through our bodies and control our weather. The physics of fluids is therefore crucial to our understanding of both nature and technology.

We begin our study with **fluid statics,** the study of fluids at rest in equilibrium situations. Like other equilibrium situations, it is based on Newton's first and third laws. We'll explore the key concepts of density, pressure, and buoyancy. **Fluid dynamics,** the study of fluids in motion, is much more complex; indeed, it is one of the most complex branches of mechanics. Fortunately, we can analyze many important situations by using simple idealized models and familiar principles such as Newton's laws and conservation of energy. Even so, we'll barely scratch the surface of this broad and interesting topic.

12.1 Gases, Liquids, and Density

A **fluid** is any substance that can flow and change the shape of the volume that it occupies. (By contrast, a solid tends to maintain its shape.) We use the term “fluid” for both gases and liquids. The key difference between them is that a liquid has *cohesion*, while a gas does not. The molecules in a liquid are close to one another, so they can exert attractive forces on each other and thus tend to stay together (that is, to cohere). That’s why a quantity of liquid maintains the same volume as it flows: If you pour 500 mL of water into a pan, the water will still occupy a volume of 500 mL. The molecules of a gas, by contrast, are separated on average by distances far larger than the size of a molecule. Hence the forces between molecules are weak, there is little or no cohesion, and a gas can easily change in volume. If you open the valve on a tank of compressed oxygen that has a volume of 500 mL, the oxygen will expand to a far greater volume.

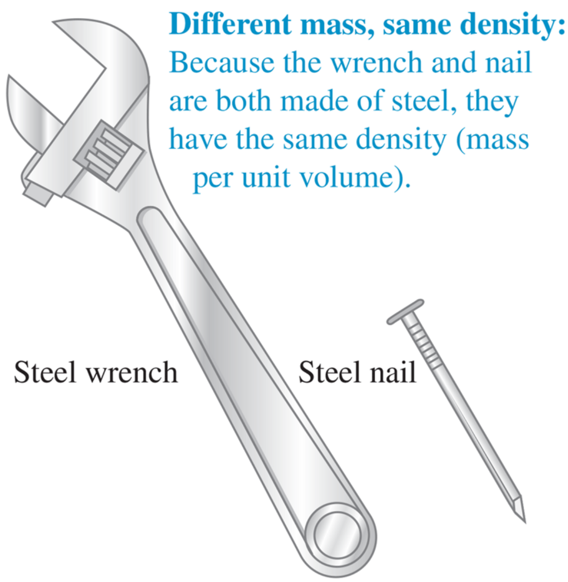
An important property of *any* material, fluid or solid, is its **density,** defined as its mass per unit volume. A homogeneous material such as ice or iron has the same density throughout. We use ρ (the Greek letter rho) for density. For a homogeneous material,

(12.1)

$$\underbrace{\rho}_{\text{Density of a homogeneous material}} = \frac{\overbrace{m}^{\text{Mass of material}}}{\underbrace{V}_{\text{Volume occupied by material}}} \qquad (12.1)$$

Two objects made of the same material have the same density even though they may have different masses and different volumes. That’s because the *ratio* of mass to volume is the same for both objects (Fig. 12.1).

Figure 12.1



Two objects with different masses and different volumes but the same density.

The SI unit of density is the kilogram per cubic meter ($1\ \mathrm{kg/m^3}$). The cgs unit, the gram per cubic centimeter ($1\ \mathrm{g/cm^3}$), is also widely used:

$$1\ \mathrm{g/cm^3} = 1000\ \mathrm{kg/m^3}$$

The densities of some common substances at ordinary temperatures are given in Table 12.1. Note the wide range of magnitudes. The densest material found on earth is the metal osmium ($\rho = 22,500\ \mathrm{kg/m^3}$), but its density pales by comparison to the densities of exotic astronomical objects, such as white dwarf stars and neutron stars.

Table 12.1 Densities of Some Common Substances

Material	Density (kg/m^3)*	Material	Density (kg/m^3)*
Air (1 atm, 20°C)	1.20	Iron, steel	7.8×10^3
Ethanol	0.81×10^3	Brass	8.6×10^3
Benzene	0.90×10^3	Copper	8.9×10^3
Ice	0.92×10^3	Silver	10.5×10^3
Water	1.00×10^3	Lead	11.3×10^3
Seawater	1.03×10^3	Mercury	13.6×10^3
Blood	1.06×10^3	Gold	19.3×10^3
Glycerin	1.26×10^3	Platinum	21.4×10^3
Concrete	2×10^3	White dwarf star	10^{10}
Aluminum	2.7×10^3	Neutron star	10^{18}

*To obtain the densities in grams per cubic centimeter, simply divide by 10^3.

The **specific gravity** of a material is the ratio of its density to the density of water at $4.0\,^\circ\mathrm{C}$, $1000\ \mathrm{kg/m^3}$; it is a pure number without units. For example, the specific gravity of aluminum is 2.7. “Specific gravity” is a poor term, since it has nothing to do with gravity; “relative density” would have been a better choice.

The density of some materials varies from point to point within the material. One example is the material of the human body, which includes low-density fat (about $940\ \mathrm{kg/m^3}$) and high-density bone (from 1700 to $2500\ \mathrm{kg/m^3}$). Two others are the earth’s atmosphere (which is less dense at high altitudes) and oceans (which are denser at greater depths). For these materials, Eq. (12.1) describes the **average density.** In general, the density of a material depends on environmental factors such as temperature and pressure.

BIO Application

Liquid Cohesion in Trees

How do trees—some of which grow to heights greater than 100 m —supply water to their highest leaves? The answer lies in the strong cohesive forces between molecules of liquid water. Narrow pipes within the tree extend upward from the roots to the leaves. As water evaporates from the leaves, cohesive forces pull replacement water upward through these pipes.

Example 12.1 The weight of a roomful of air

Find the mass and weight of the air at $20^\circ\mathrm{C}$ in a living room with a $4.0\ \mathrm{m} \times 5.0\ \mathrm{m}$ floor and a ceiling 3.0 m high, and the mass and weight of an equal volume of water.

IDENTIFY and SET UP

We assume that the air density is the same throughout the room. (Air is less dense at high elevations than near sea level, but the density varies negligibly over the room's 3.0 m height; see Section 12.2.) We use Eq. (12.1) to relate the mass m_{air} to the room's volume V (which we'll calculate) and the air density ρ_{air} (given in Table 12.1).

EXECUTE We have $V = (4.0\ \mathrm{m})(5.0\ \mathrm{m})(3.0\ \mathrm{m}) = 60\ \mathrm{m}^3$, so from Eq. (12.1),

$$\begin{aligned} m_{\mathrm{air}} &= \rho_{air} V = (1.20\ \mathrm{kg/m^3})(60\ \mathrm{m^3}) = 72\ \mathrm{kg} \\ w_{\mathrm{air}} &= m_{\mathrm{air}} g = (72\ \mathrm{kg})(9.8\ \mathrm{m/s^2}) = 700\ \mathrm{N} = 160\ \mathrm{lb} \end{aligned}$$

The mass and weight of an equal volume of water are

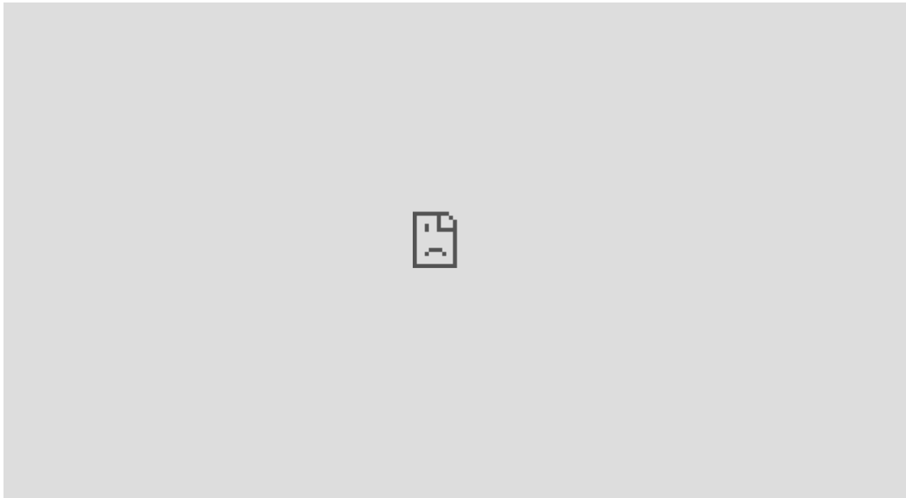
$$\begin{aligned} m_{\mathrm{water}} &= \rho_{\mathrm{water}} V = (1000\ \mathrm{kg/m^3})(60\ \mathrm{m^3}) = 6.0 \times 10^4\ \mathrm{kg} \\ w_{\mathrm{water}} &= m_{\mathrm{water}} g = (6.0 \times 10^4\ \mathrm{kg})(9.8\ \mathrm{m/s^2}) \\ &= 5.9 \times 10^5\ \mathrm{N} = 1.3 \times 10^5\ \mathrm{lb} = 66\ \mathrm{tons} \end{aligned}$$

EVALUATE A roomful of air weighs about the same as an average adult. Water is nearly a thousand times denser than air, so its mass and weight are larger by the same factor. The weight of a roomful of water would collapse the floor of an ordinary house.

KEY CONCEPT

To find the density of a uniform substance, divide the mass of the substance by the volume that it occupies.

Video Tutor Solution: **Example 12.1**



Test Your Understanding of Section 12.1

Rank the following objects in order from highest to lowest average density: (i) mass $m = 4.00$ kg, volume $V = 1.60 \times 10^{-3}\ \mathrm{m}^3$; (ii) $m = 8.00$ kg, $V = 1.60 \times 10^{-3}\ m^3$; (iii) $m = 8.00$ kg, $V = 3.20 \times 10^{-3}\ m^3$; (iv) $m = 2560$ kg, $V = 0.640\ m^3$; (v) $m = 2560$ kg, $V = 1.28\ m^3$.

12.2 Pressure in a Fluid

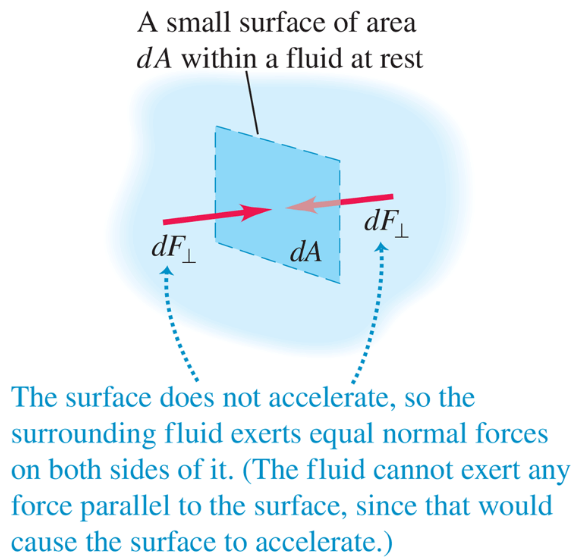
A fluid exerts a force perpendicular to any surface in contact with it, such as a container wall or an object immersed in the fluid. This is the force that you feel pressing on your legs when you dangle them in a swimming pool. Even when a fluid as a whole is at rest, the molecules that make up the fluid are in motion; the force exerted by the fluid is due to molecules colliding with their surroundings.

Imagine a surface *within* a fluid at rest. For this surface and the fluid to remain at rest, the fluid must exert forces of equal magnitude but opposite direction on the surface's two sides. Consider a small surface of area dA centered on a point in the fluid; the normal force exerted by the fluid on each side is $dF_{\perp}$ (Fig. 12.2). We define the **pressure** p at that point as the normal force per unit area—that is, the ratio of $dF_{\perp}$ to dA (Fig. 12.3):

(12.2)

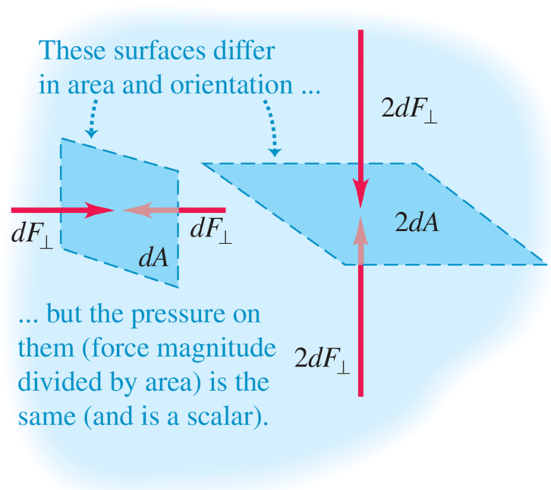
$$\underbrace{p}_{\text{Pressure at a point in a fluid}} = \frac{\overbrace{dF_{\perp}}^{\text{Normal force exerted by fluid on a small surface at that point}}}{\underbrace{dA}_{\text{Area of surface}}} \qquad (12.2)$$

Figure 12.2



Forces acting on a small surface within a fluid at rest.

Figure 12.3



Pressure is a scalar with units of newtons per square meter. By contrast, force is a vector with units of newtons.

If the pressure is the same at all points of a finite plane surface with area A, then

(12.3)

$$p = \frac{F_{\perp}}{A}$$

where $F_{\perp}$ is the net normal force on one side of the surface. The SI unit of pressure is the **pascal,** where

$$1 \text{ pascal} = 1 \text{ Pa} = 1 \text{ N/m}^2$$

We introduced the pascal in Chapter 11. Two related units, used principally in meteorology, are the *bar,* equal to 10^5 Pa, and the *millibar,* equal to 100 Pa.

Atmospheric pressure p_a is the pressure of the earth's atmosphere, the pressure at the bottom of this sea of air in which we live. This pressure varies with weather changes and with elevation. Normal atmospheric pressure at sea level (an average value) is 1 *atmosphere* (atm), defined to be exactly 101,325 Pa. To four significant figures,

$$\begin{aligned}(p_a)_{av} &= 1 \text{ atm} = 1.013 \times 10^5 \text{ Pa} \\ &= 1.013 \text{ bar} = 1013 \text{ millibar} = 14.70 \text{ Ib/in.}^2\end{aligned}$$

CAUTION **Don't confuse pressure and force** In everyday language "pressure" and "force" mean pretty much the same thing. In fluid mechanics, however, these words describe very different quantities. Pressure acts perpendicular to any surface in a fluid, no matter how that surface is oriented (Fig. 12.3). Hence pressure has no direction of its own; it's a scalar. By contrast, force is a vector with a definite direction. Remember, too, that pressure is force per unit area. As Fig. 12.3 shows, a surface with twice the area has twice as much force exerted on it by the fluid, so the pressure is the same.

Example 12.2 The force of air

In the room described in Example 12.1, what is the total downward force on the floor due to an air pressure of 1.00 atm?

IDENTIFY and SET UP

This example uses the relationship among the pressure p of a fluid (air), the area A subjected to that pressure, and the resulting normal force $F_\perp$ the fluid exerts. The pressure is uniform, so we use Eq. (12.3), $F_\perp = pA$, to determine $F_\perp$. The floor is horizontal, so $F_\perp$ is vertical (downward).

EXECUTE We have $A = (4.0\ \mathrm{m})(5.0\ \mathrm{m}) = 20\ \mathrm{m}^2$, so from Eq. (12.3),

$$\begin{aligned} F_\perp = &= pA = (1.013 \times 10^5\ \mathrm{N/m^2})(20\ \mathrm{m^2}) \\ &= 2.0 \times 10^6\ \mathrm{N} = 4.6 \times 10^5\ \mathrm{Ib} = 230\ \mathrm{tons} \end{aligned}$$

EVALUATE Unlike the water in Example 12.1, $F_\perp$ will not collapse the floor here, because there is an *upward* force of equal magnitude on the floor's underside. If the house has a basement, this upward force is exerted by the air underneath the floor. In this case, if we ignore the thickness of the floor, the *net* force due to air pressure is zero.

KEY CONCEPT

To find the force exerted by a fluid perpendicular to a surface, multiply the pressure of the fluid by the surface's area. This relationship comes from the definition of pressure as the normal force per unit area within the fluid.

Video Tutor Solution: **Example 12.2**

Pressure, Depth, and Pascal's Law

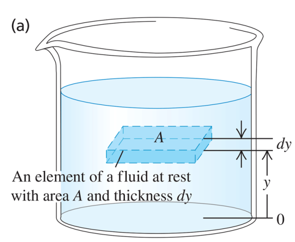
Video Tutor Demo: **Water Level in Pascal's Vases**

Video Tutor Demo: **Pressure in Water and Alcohol**

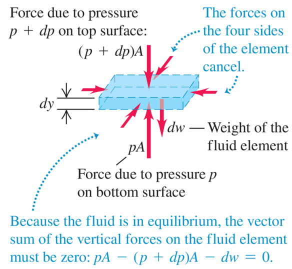
If the weight of the fluid can be ignored, the pressure in a fluid is the same throughout its volume. We used that approximation in our discussion of bulk stress and strain in Section 11.4. But often the fluid's weight is *not* negligible, and pressure variations are important. Atmospheric pressure is less at high altitude than at sea level, which is why airliner cabins have to be pressurized. When you dive into deep water, you can feel the increased pressure on your ears.

We can derive a relationship between the pressure p at any point in a fluid at rest and the elevation y of the point. We'll assume that the density ρ has the same value throughout the fluid (that is, the density is *uniform*), as does the acceleration due to gravity g. If the fluid is in equilibrium, any thin element of the fluid with thickness dy is also in equilibrium (Fig. 12.4a). The bottom and top surfaces each have area A, and they are at elevations y and $y + dy$ above some reference level where $y = 0$. The fluid element has volume $dV = A\,dy$, mass $dm = \rho\,dV = \rho A\,dy$, and weight $dw = dm\,g = \rho g A\,dy$.

Figure 12.4



(b)



The forces on an element of fluid in equilibrium.

What are the other forces on this fluid element (Fig. 12.4b)? Let's call the pressure at the bottom surface p; then the total $y-$component of upward force on this surface is pA. The pressure at the top surface is $p + dp$, and the total $y-$component of (downward) force on the top surface is $-(p + dp)A$. The fluid element is in equilibrium, so the total $y-$component of force, including the weight and the forces at the bottom and top surfaces, must be zero:

$$\sum F_y = 0$$

so

$$pA - (p + dp)A - \rho g A\, dy = 0$$

When we divide out the area A and rearrange, we get

(12.4)

$$\frac{dp}{dy} = -\rho g$$

This equation shows that when y increases, p decreases; that is, as we move upward in the fluid, pressure decreases, as we expect. If p_1 and p_2 are the pressures at elevations y_1 and y_2, respectively, and if ρ and g are constant, then

(12.5)

Pressure difference between two points in a fluid of uniform density

Uniform density of fluid

$$p_2 - p_1 = -\rho g(y_2 - y_1) \tag{12.5}$$

Heights of the two points

Acceleration due to gravity ($g > 0$)

It's often convenient to express Eq. (12.5) in terms of the *depth* below the surface of a fluid (Fig. 12.5). Take point 1 at any level in the fluid and let p represent the pressure at this point. Take point 2 at the *surface* of the fluid, where the pressure is p_0 (subscript zero for zero depth). The depth of point 1 below the surface is $h = y_2 - y_1$, and Eq. (12.5) becomes

$$p_0 - p = -\rho g\,(y_2 - y_1) = -\rho gh \qquad \text{or}$$

(12.6)

Pressure at depth h in a fluid of uniform density

Uniform density of fluid

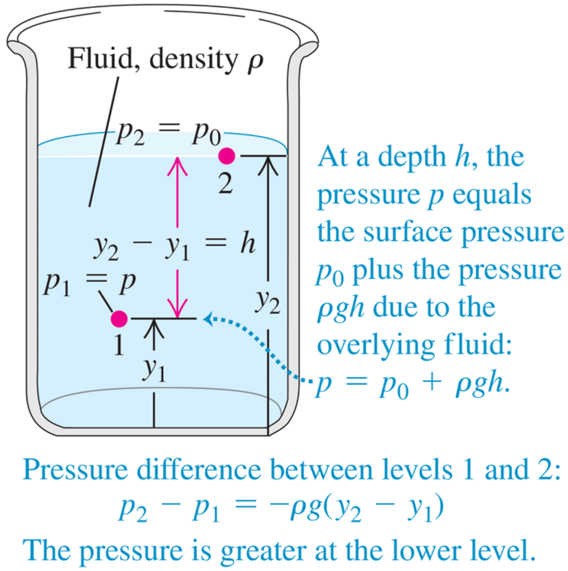
$$p = p_0 + \rho gh \tag{12.6}$$

Depth below surface

Pressure at surface of fluid

Acceleration due to gravity ($g > 0$)

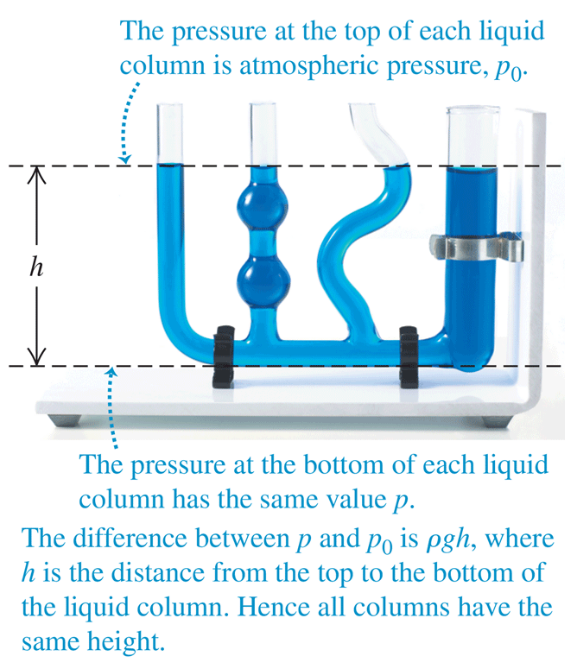
Figure 12.5



How pressure varies with depth in a fluid with uniform density.

The pressure p at a depth h is greater than the pressure p_0 at the surface by an amount ρgh. Note that the pressure is the same at any two points at the same level in the fluid. The *shape* of the container does not matter (Fig. 12.6).

Figure 12.6



Each fluid column has the same height, no matter what its shape.

Equation (12.6) shows that if we increase the pressure p_0 at the top surface, possibly by using a piston that fits tightly inside the container to push down on the fluid surface, the pressure p at any depth increases by exactly the same amount. This observation is called *Pascal's law*.

Pascal's law

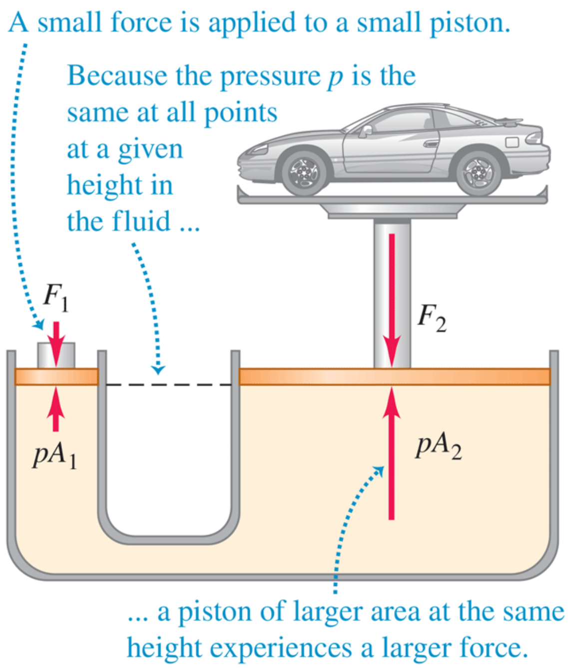
Pressure applied to an enclosed fluid is transmitted undiminished to every portion of the fluid and the walls of the containing vessel.

The hydraulic lift (Fig. 12.7) illustrates Pascal's law. A piston with small cross-sectional area A_1 exerts a force F_1 on the surface of a liquid such as oil. The applied pressure $p = F_1/A_1$ is transmitted through the connecting pipe to a larger piston of area A_2. The applied pressure is the same in both cylinders, so

(12.7)

$$p = \frac{F_1}{A_1} = \frac{F_2}{A_2} \qquad \text{and} \qquad F_2 = \frac{A_2}{A_1} F_1$$

Figure 12.7



The hydraulic lift is an application of Pascal's law. The size of the fluid-filled container is exaggerated for clarity.

The hydraulic lift is a force-multiplying device with a multiplication factor equal to the ratio of the areas of the two pistons. Dentist's chairs, car lifts and jacks, many elevators, and hydraulic brakes all use this principle.

For gases the assumption that the density ρ is uniform is realistic over only short vertical distances. In a room with a ceiling height of 3.0 m filled with air of uniform density $1.2\ \text{kg/m}^3$, the difference in pressure between floor and ceiling, given by Eq. (12.6), is

$$\rho g h = (1.2\ \text{kg/m}^3)(9.8\ \text{m/s}^2)(3.0\ \text{m}) = 35\ \text{Pa}$$

or about 0.00035 atm, a very small difference. But between sea level and the summit of Mount Everest (8882 m) the density of air changes by nearly a factor of 3, and in this case we cannot use Eq. (12.6). Liquids, by contrast, are nearly incompressible, and it is usually a very good approximation to regard their density as independent of pressure.

Absolute Pressure and Gauge Pressure

If the pressure inside a car tire is equal to atmospheric pressure, the tire is flat. The pressure has to be *greater* than atmospheric to support the car, so the significant quantity is the *difference* between the inside and outside pressures. When we say that the pressure in a car tire is "32 pounds" (actually 32 lb/in.^2, equal to 220 kPa or 2.2×10^5 Pa), we mean that it is *greater* than atmospheric pressure (14.7 lb/in.^2 or 1.01×10^5 Pa) by this amount. The *total* pressure in the tire is then 47 lb/in.^2 or 320 kPa. The excess pressure above atmospheric pressure is usually called **gauge pressure,** and the total pressure is called **absolute pressure.** Engineers use the abbreviations psig and psia for "pounds per square inch gauge" and "pounds per square inch absolute," respectively. If the pressure is *less* than atmospheric, as in a partial vacuum, the gauge pressure is negative.

Example 12.3 Finding absolute and gauge pressures

WITH VARIATION PROBLEMS

Water stands 12.0 m deep in a storage tank whose top is open to the atmosphere. What are the absolute and gauge pressures at the bottom of the tank?

IDENTIFY and SET UP Table 11.2 indicates that water is nearly incompressible, so we can treat it as having uniform density. The level of the top of the tank corresponds to point 2 in Fig. 12.5, and the level of the bottom of the tank corresponds to point 1. Our target variable is p in Eq. (12.6). We have $h = 12.0$ m and $p_0 = 1 \text{ atm} = 1.01 \times 10^5$ Pa.

EXECUTE From Eq. (12.6), the pressures are

$$
\begin{aligned}
\text{absolute}:\quad p &= p_0 + \rho g h\\
&= \left(1.01 \times 10^5\,\mathrm{Pa}\right) + \left(1000\,\mathrm{kg/m^3}\right)\left(9.80\mathrm{m/s^2}\right)(12.0\mathrm{m})\\
&= 2.19 \times 10^5\,\mathrm{Pa} = 2.16\,\mathrm{atm} = 31.8\,\mathrm{Ib/in.}^2\\
\text{gauge}:\quad p - p_0 &= (2.19 - 1.01) \times 10^5\,\mathrm{Pa}\\
&= 1.18 \times 10^5\,\mathrm{Pa} = 1.16\,\mathrm{atm} = 17.1\,\mathrm{Ib/in.}^2
\end{aligned}
$$

EVALUATE A pressure gauge at the bottom of such a tank would probably be calibrated to read gauge pressure rather than absolute pressure.

KEY CONCEPT

Absolute pressure is the total pressure at a given point in a fluid. Gauge pressure is the difference between absolute pressure and atmospheric pressure.

Video Tutor Solution: **Example 12.3**

Pressure Gauges

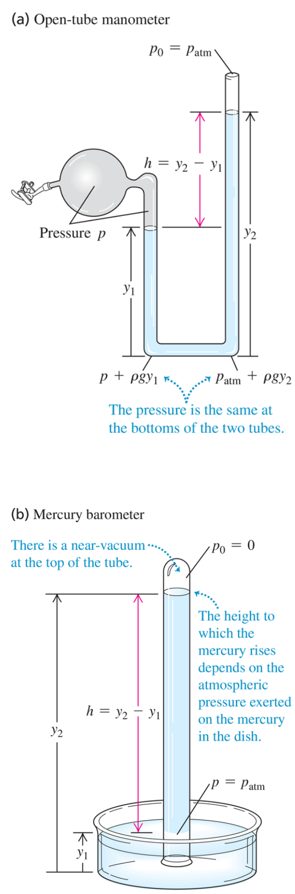
The simplest pressure gauge is the open-tube *manometer* (Fig. 12.8a). The U-shaped tube contains a liquid of density ρ, often mercury or water.

The left end of the tube is connected to the container where the pressure p is to be measured, and the right end is open to the atmosphere at pressure $p_0 = p_{\text{atm}}$. The pressure at the bottom of the tube due to the fluid in the left column is $p + \rho g y_1$, and the pressure at the bottom due to the fluid in the right column is $p_{\text{atm}} + \rho g y_2$. These pressures are measured at the same level, so they must be equal:

(12.8)

$$\begin{aligned} p + \rho g y_1 &= p_{\text{atm}} + \rho g y_2 \\ p - \rho_{\text{atm}} &= \rho g \left(y_2 - y_1\right) = pgh \end{aligned}$$

Figure 12.8



Two types of pressure gauge.

In Eq. (12.8), p is the *absolute pressure*, and the difference $p - p_{\text{atm}}$ between absolute and atmospheric pressure is the gauge pressure. Thus the gauge pressure is proportional to the difference in height $h = y_2 - y_1$ of the liquid columns.

BIO Application

Gauge Pressure of Blood

Blood-pressure readings, such as $130/80$, give the maximum and minimum gauge pressures in the arteries, measured in mm Hg or torr. Blood pressure varies with vertical position within the body; the standard reference point is the upper arm, level with the heart.

Another common pressure gauge is the **mercury barometer.** It consists of a long glass tube, closed at one end, that has been filled with mercury and then inverted in a dish of mercury (Fig. 12.8b). The space above the mercury column contains only mercury vapor; its pressure is negligibly small, so the pressure p_0 at the top of the mercury column is practically zero. From Eq. (12.6),

(12.9)

$$p_{\mathrm{atm}} = p = 0 + \rho g\left(y_2 - y_1\right) = \rho g h$$

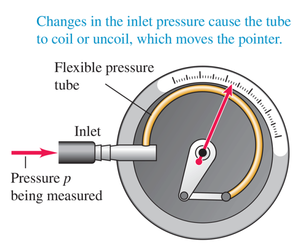
So the height h of the mercury column indicates the atmospheric pressure p_{atm}.

Pressures are often described in terms of the height of the corresponding mercury column, as so many "inches of mercury" or "millimeters of mercury" (abbreviated mm Hg). A pressure of 1 mm Hg is called *1 torr*, after Evangelista Torricelli, inventor of the mercury barometer. But these units depend on the density of mercury, which varies with temperature, and on the value of g, which varies with location, so the pascal is the preferred unit of pressure.

Many types of pressure gauges use a flexible sealed tube (Fig. 12.9). A change in the pressure either inside or outside the tube causes a change in its dimensions. This change is detected optically, electrically, or mechanically.

Figure 12.9

(a)



(b)

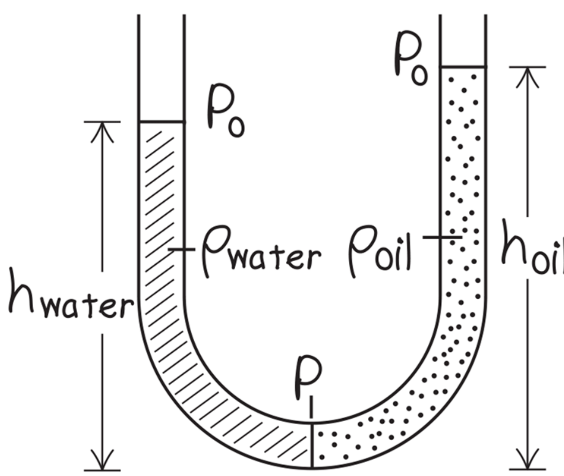
(a) A Bourdon pressure gauge. When the pressure inside the flexible tube increases, the tube straightens out a little, deflecting the attached pointer. (b) This Bourdon-type pressure gauge is connected to a high-pressure gas line. The gauge pressure shown is just over 5 bars (1 bar $= 10^5\ Pa$).

Example 12.4 A tale of two fluids

WITH VARIATION PROBLEMS

A manometer tube is partially filled with water. Oil (which does not mix with water) is poured into the left arm of the tube until the oil–water interface is at the midpoint of the tube as shown in Fig. 12.10. Both arms of the tube are open to the air. Find a relationship between the heights h_{oil} and h_{water}.

Figure 12.10



Our sketch for this problem.

IDENTIFY and SET UP Figure 12.10 shows our sketch. The relationship between pressure and depth given by Eq. (12.6) applies to fluids of uniform density only; we have two fluids of different densities, so we must write a separate pressure–depth relationship for each. Both fluid columns have pressure p at the bottom (where they are in contact and in equilibrium), and both are at atmospheric pressure p_0 at the top (where both are in contact with and in equilibrium with the air).

EXECUTE Writing Eq. (12.6) for each fluid gives

$$\begin{aligned} p &= p_0 + \rho_{\text{water}} g h_{\text{water}} \\ p &= p_0 + \rho_{\text{oil}} g h_{\text{oil}} \end{aligned}$$

Since the pressure p at the bottom of the tube is the same for both fluids, we set these two expressions equal to each other and solve for h_{oil} in terms of h_{water} :

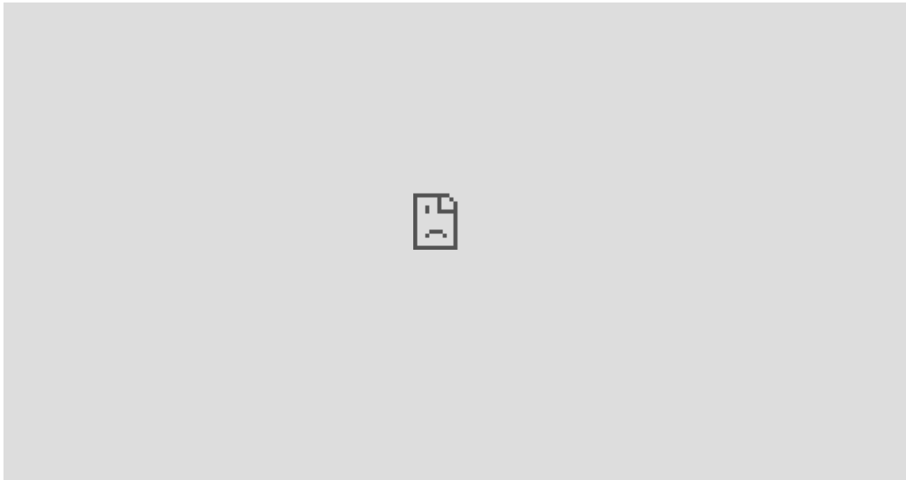
$$h_{\text{oil}} = \frac{\rho_{\text{water}}}{\rho_{\text{oil}}} h_{\text{water}}$$

EVALUATE Water ($\rho_{\text{water}} = 1000 \text{ kg}/m^3$) is denser than oil ($\rho_{\text{oil}} \approx 850 \text{ kg}/m^3$), so h_{oil} is greater than h_{water} as Fig. 12.10 shows. It takes a greater height of low-density oil to produce the same pressure p at the bottom of the tube.

KEY CONCEPT

The pressure is the same at all points *at the same level* in a fluid at rest. This is true even if the fluid contains different substances with different densities.

Video Tutor Solution: **Example 12.4**



Test Your Understanding of Section 12.2

Mercury is less dense at high temperatures than at low temperatures. Suppose you move a mercury barometer from the cold interior of a tightly sealed refrigerator to outdoors on a hot summer day. You find that the column of mercury remains at the same height in the tube. Compared to the air pressure inside the refrigerator, is the air pressure outdoors (i) higher, (ii) lower, or (iii) the same? (Ignore the very small change in the dimensions of the glass tube due to the temperature change.)

12.3 Buoyancy

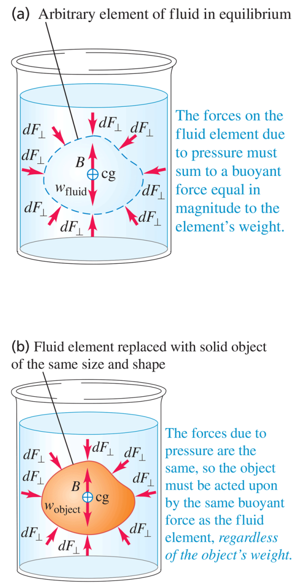
Video Tutor Demo: **Weighing Weights in Water**

An object immersed in water seems to weigh less than when it is in air. When the object is less dense than the fluid, it floats. The human body usually floats in water, and a helium-filled balloon floats in air. These are examples of **buoyancy,** a phenomenon described by *Archimedes's principle:*

> **Archimedes's principle**
>
> When an object is completely or partially immersed in a fluid, the fluid exerts an upward force on the object equal to the weight of the fluid displaced by the object.

To prove this principle, we consider an arbitrary element of fluid at rest. The dashed curve in Fig. 12.11a outlines such an element. The arrows labeled $dF_{\perp}$ represent the forces exerted on the element's surface by the surrounding fluid.

Figure 12.11



Archimedes's principle.

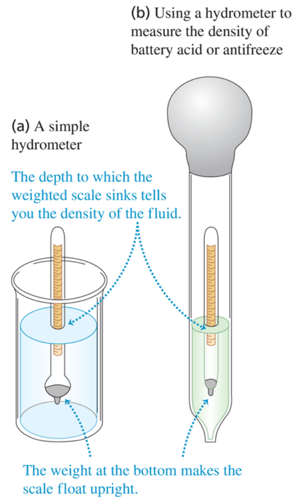
The entire fluid is in equilibrium, so the sum of all the $y-$components of force on this element of fluid is zero. Hence the sum of the $y-$components of the *surface* forces must be an upward force equal in magnitude to the weight mg of the fluid inside the surface. Also, the sum of the torques on the element of fluid must be zero, so the line of action of the resultant $y-$component of surface force must pass through the center of gravity of this element of fluid.

Now we replace the fluid inside the surface with a solid object that has exactly the same shape (Fig. 12.11b). The pressure at every point is the same as before. So the total upward force exerted on the object by the fluid is also the same, again equal in magnitude to the weight mg of the fluid displaced to make way for the object. We call this upward force the **buoyant force** on the solid object. The line of action of the buoyant force again passes through the center of gravity of the displaced fluid (which doesn't necessarily coincide with the center of gravity of the object).

? When a balloon floats in equilibrium in air, its weight (including the gas inside it) must be the same as the weight of the air displaced by the balloon. A fish's flesh is denser than water, yet many fish can float while submerged. These fish have a gas-filled cavity within their bodies, which makes the fish's *average* density the same as water's. So the net weight of the fish is the same as the weight of the water it displaces. An object whose average density is *less* than that of a liquid can float partially submerged at the free upper surface of the liquid. A ship made of steel (which is much denser than water) can float because the ship is hollow, with air occupying much of its interior volume, so its average density is less than that of water. The greater the density of the liquid, the less of the object is submerged. When you swim in seawater (density 1030 kg/m^3), your body floats higher than in freshwater (1000 kg/m^3).

A practical example of buoyancy is the hydrometer, used to measure the density of liquids (Fig. 12.12a). The calibrated float sinks into the fluid until the weight of the fluid it displaces is exactly equal to its own weight. The hydrometer floats *higher* in denser liquids than in less dense liquids, and a scale in the top stem permits direct density readings. Hydrometers like this are used in medical diagnosis to measure the density of urine (which depends on a patient's level of hydration). Figure 12.12b shows a type of hydrometer used to measure the density of battery acid or antifreeze. The bottom of the large tube is immersed in the liquid; the bulb is squeezed to expel air and is then released, like a giant medicine dropper. The liquid rises into the outer tube, and the hydrometer floats in this liquid.

Figure 12.12



Measuring the density of a fluid.

CAUTION The buoyant force depends on the fluid density The buoyant force on an object is proportional to the density of the *fluid* in which the object is immersed, *not* the density of the object. If a wooden block and an iron block have the same volume and both are submerged in water, both experience the same buoyant force. The wooden block rises and the iron block sinks because this buoyant force is greater than the weight of the wooden block but less than the weight of the iron block.

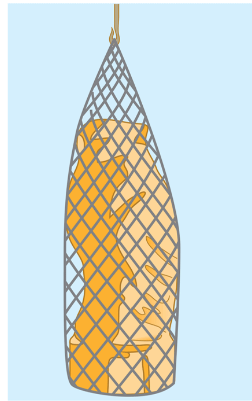
Example 12.5 Buoyancy

WITH VARIATION PROBLEMS

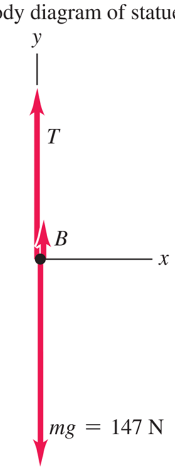
A 15.0 kg solid gold statue is raised from the sea bottom (Fig. 12.13a). What is the tension in the hoisting cable (assumed massless) when the statue is (a) at rest and completely underwater and (b) at rest and completely out of the water?

Figure 12.13

(a) Immersed statue in equilibrium



(b) Free-body diagram of statue



What is the tension in the cable hoisting the statue?

IDENTIFY and SET UP In both cases the statue is in equilibrium and experiences three forces: its weight, the cable tension, and a buoyant force equal in magnitude to the weight of the fluid displaced by the statue (seawater in part (a), air in part (b)). Figure 12.13b shows the free-body diagram for the statue. Our target variables are the values of the tension in seawater (T_{sw}) and in air (T_{air}). We are given the mass m_{statue} , and we can calculate the buoyant force in seawater (B_{sw}) and in air (B_{air}) by using Archimedes's principle.

EXECUTE

(a) To find B_{sw}, we first find the statue's volume V by using the density of gold from Table 12.1:

$$V = \frac{m_{\text{statue}}}{\rho_{\text{gold}}} = \frac{15.0\text{ kg}}{19.3 \times 10^3\text{ kg/m}^3} = 7.77 \times 10^{-4}\text{ m}^3$$

The buoyant force B_{sw} equals the weight of this same volume of seawater. Using Table 12.1 again:

$$\begin{aligned} B_{\text{sw}} &= w_{\text{sw}} = m_{\text{sw}}g = \rho_{\text{sw}}Vg \\ &= \left(1.03 \times 10^3\,\text{kg/m}^3\right)\left(7.77 \times 10^{-4}\text{m}^3\right)\left(9.80\,\text{m/s}^2\right) \\ &= 7.84\text{N} \end{aligned}$$

The statue is at rest, so the net external force acting on it is zero. From Fig. 12.13b,

$$\begin{aligned} \sum F_y &= B_{\text{sw}} + T_{\text{sw}} + (-m_{\text{statue}}g) = 0 \\ T_{sw} &= m_{stature}g - B_{\text{sw}} = (15.0\,\text{kg})\left(9.80\,\text{m/s}^2\right) - 7.84\,\text{N} \\ &= 147\,\text{N} - 7.84\,\text{N} = 139\,\text{N} \end{aligned}$$

A spring scale attached to the upper end of the cable will indicate a tension 7.84 N less than the statue's actual weight $m_{\text{statue}}g = 147\text{ N}$.

(b) The density of air is about 1.2 kg/m^3, so the buoyant force of air on the statue is

$$\begin{aligned} B_{\text{air}} &= p_{\text{air}}Vg \\ &= \left(1.2\,\text{kg/m}^3\right)\left(7.77 \times 10^{-4}\text{m}^3\right)\left(9.80\,\text{m/s}^2\right) \\ &= 9.1 \times 10^{-3}\,\text{N} \end{aligned}$$

This is negligible compared to the statue's actual weight $m_{\text{statue}}g = 147\text{ N}$. So within the precision of our data, the tension in the cable with the statue in air is $T_{\text{air}} = m_{\text{statue}}g = 147\text{ N}$.

EVALUATE The denser the fluid, the greater the buoyant force and the smaller the cable tension. If the fluid had the same density as the statue, the buoyant force would be equal to the statue's weight and the tension would be zero (the cable would go slack). If the fluid were denser than the statue, the tension would be *negative:* The buoyant force would be greater than the statue's weight, and a downward force would be required to keep the statue from rising upward.

KEY CONCEPT

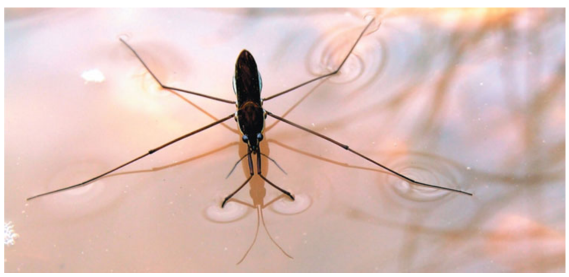
The buoyant force on an object immersed in a fluid is equal to the weight of the fluid that the object displaces (Archimedes's principle). The greater the density of the fluid, the greater the buoyant force that the fluid exerts.

Video Tutor Solution: Example 12.5

Surface Tension

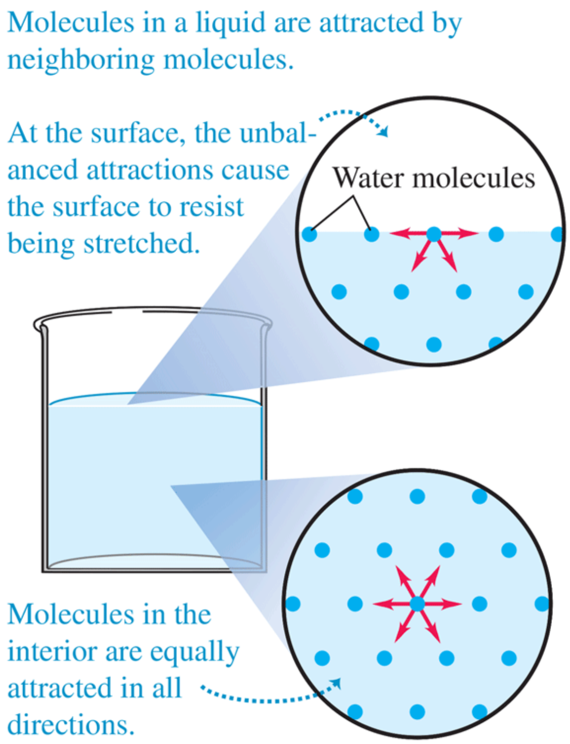
We've seen that if an object is less dense than water, it will float partially submerged. But a paper clip can rest *atop* a water surface even though its density is several times that of water. This is an example of **surface tension:** The surface of the liquid behaves like a membrane under tension (Fig. 12.14). Surface tension arises because the molecules of the liquid exert attractive forces on each other. There is zero net force on a molecule within the interior of the liquid, but a surface molecule is drawn into the interior (Fig. 12.15). Thus the liquid tends to minimize its surface area, just as a stretched membrane does.

Figure 12.14



The surface of the water acts like a membrane under tension, allowing this water strider to "walk on water."

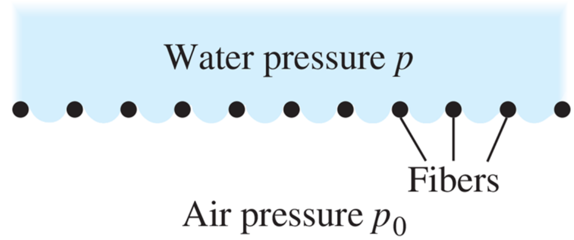
Figure 12.15



A molecule at the surface of a liquid is attracted into the bulk liquid, which tends to reduce the liquid's surface area.

Surface tension explains why raindrops are spherical (*not* teardrop-shaped): A sphere has a smaller surface area for its volume than any other shape. It also explains why hot, soapy water is used for washing. To wash clothing thoroughly, water must be forced through the tiny spaces between the fibers (Fig. 12.16). This requires increasing the surface area of the water, which is difficult to achieve because of surface tension. The job is made easier by increasing the temperature of the water and adding soap, both of which decrease the surface tension.

Figure 12.16



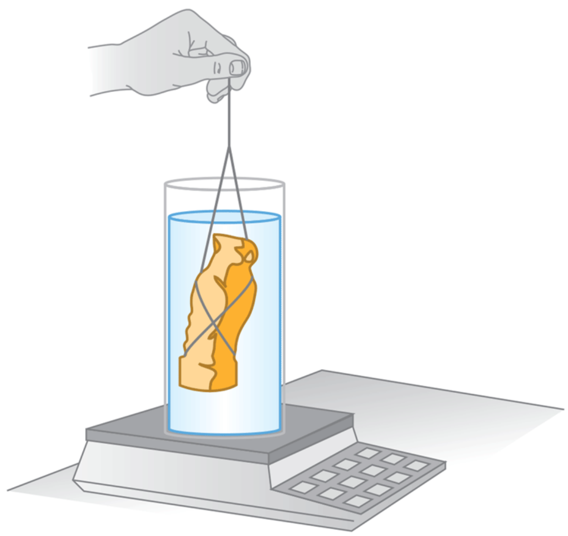
Surface tension makes it difficult to force water through small crevices. The required water pressure p can be reduced by using hot, soapy water, which has less surface tension.

Surface tension is important for a millimeter-sized water drop, which has a relatively large surface area for its volume. (A sphere of radius r has surface area $4\pi r^2$ and volume $(4\pi/3)r^3$. The ratio of surface area to volume is $3/r$, which increases with decreasing radius.) But for large quantities of liquid, the ratio of surface area to volume is relatively small, and surface tension is negligible compared to pressure forces. For the remainder of this chapter, we'll consider only fluids in bulk and ignore the effects of surface tension.

Test Your Understanding of Section 12.3

You place a container of seawater on a scale and note the reading on the scale. You now suspend the statue of Example 12.5 in the water (Fig. 12.17). How does the scale reading change? (i) It increases by 7.84 N; (ii) it decreases by 7.84 N; (iii) it remains the same; (iv) none of these.

Figure 12.17



How does the scale reading change when the statue is immersed in water?

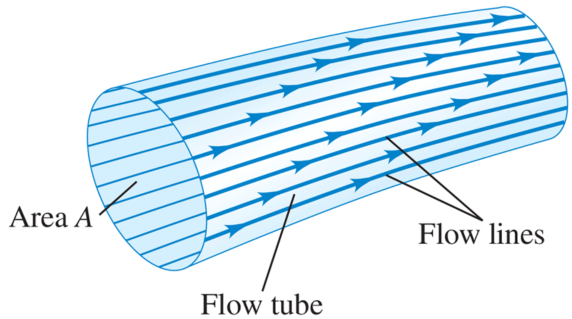
12.4 Fluid Flow

We are now ready to consider *motion* of a fluid. Fluid flow can be extremely complex, as shown by the currents in river rapids or the swirling flames of a campfire. But we can represent some situations by relatively simple idealized models. An **ideal fluid** is a fluid that is *incompressible* (that is, its density cannot change) and has no internal friction (called **viscosity**). Liquids are approximately incompressible in most situations, and we may also treat a gas as incompressible if the pressure differences from one region to another are not too great. Internal friction in a fluid causes shear stresses when two adjacent layers of fluid move relative to each other, as when fluid flows inside a tube or around an obstacle. In some cases we can ignore these shear forces in comparison with forces arising from gravitation and pressure differences.

The path of an individual particle in a moving fluid is called a **flow line.** In **steady flow,** the overall flow pattern does not change with time, so every element passing through a given point follows the same flow line. In this case the "map" of the fluid velocities at various points in space remains constant, although the velocity of a particular particle may change in both magnitude and direction during its motion. A **streamline** is a curve whose tangent at any point is in the direction of the fluid velocity at that point. When the flow pattern changes with time, the streamlines do not coincide with the flow lines. We'll consider only steady-flow situations, for which flow lines and streamlines are identical.

The flow lines passing through the edge of an imaginary element of area, such as area A in Fig. 12.18, form a tube called a **flow tube.** From the definition of a flow line, in steady flow no fluid can cross the side walls of a given flow tube.

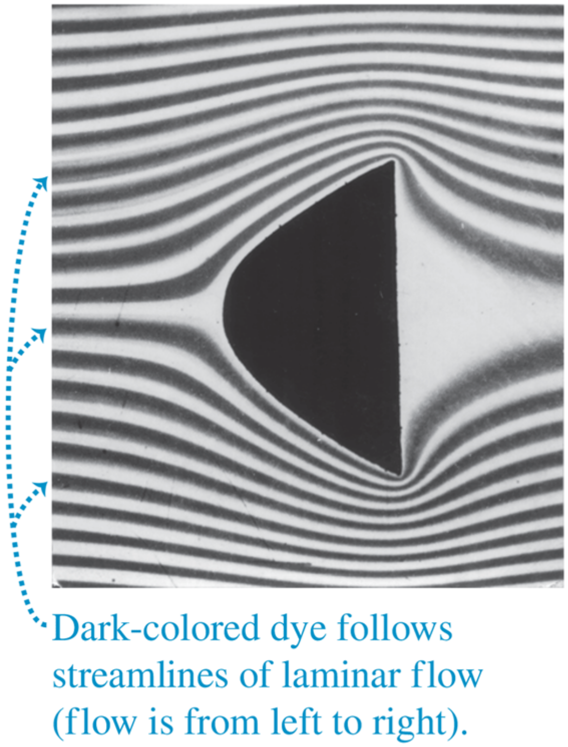
Figure 12.18



A flow tube bounded by flow lines. In steady flow, fluid cannot cross the walls of a flow tube.

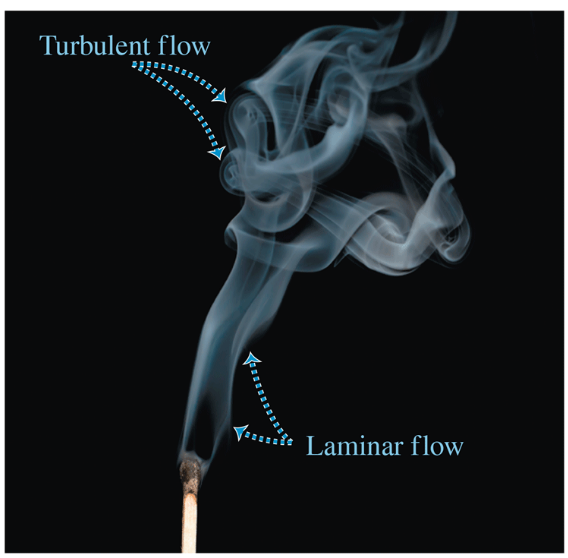
Figure 12.19 shows the pattern of fluid flow from left to right around an obstacle. The photograph was made by injecting dye into water flowing between two closely spaced glass plates. This pattern is typical of **laminar flow,** in which adjacent layers of fluid slide smoothly past each other and the flow is steady. (A *lamina* is a thin sheet.) At sufficiently high flow rates, or when boundary surfaces cause abrupt changes in velocity, the flow can become irregular and chaotic. This is called **turbulent flow** (Fig. 12.20). In turbulent flow there is no steady-state pattern; the flow pattern changes continuously.

Figure 12.19



Laminar flow around an obstacle.

Figure 12.20

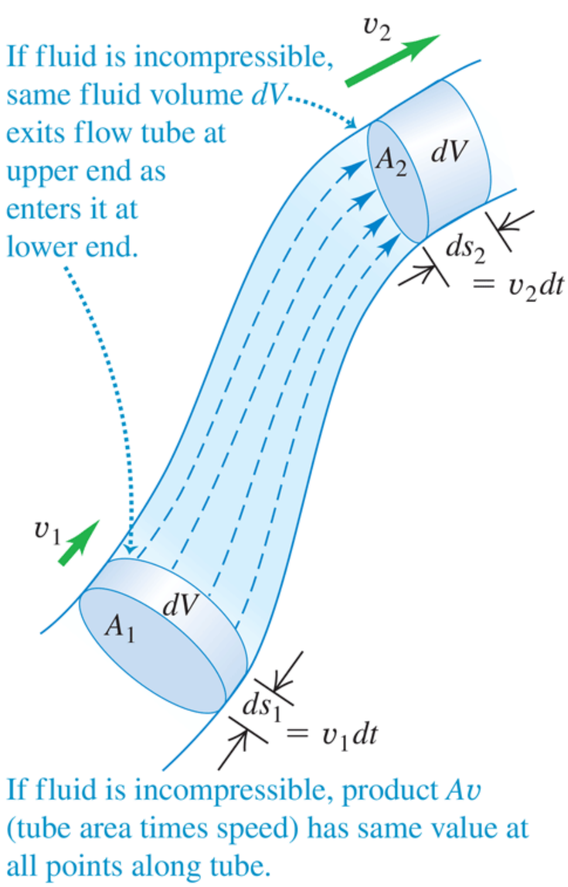


The flow of smoke rising from this burnt match is laminar up to a certain point, and then becomes turbulent.

The Continuity Equation

The mass of a moving fluid doesn't change as it flows. This leads to an important relationship called the **continuity equation.** Consider a portion of a flow tube between two stationary cross sections with areas A_1 and A_2 (Fig. 12.21). The fluid speeds at these sections are v_1 and v_2, respectively. As we mentioned above, no fluid flows in or out across the side walls of such a tube. During a small time interval dt, the fluid at A_1 moves a distance $ds_1 = v_1\,dt$, so a cylinder of fluid with height $v_1\,dt$ and volume $dV_1 = A_1 v_1\,dt$ flows into the tube across A_1. During this same interval, a cylinder of volume $dV_2 = A_2 v_2\,dt$ flows out of the tube across A_2.

Figure 12.21



A flow tube with changing cross-sectional area.

Let's first consider the case of an incompressible fluid so that the density ρ has the same value at all points. The mass dm_1 flowing into the tube across A_1 in time dt is $dm_1 = \rho A_1 v_1 \, dt$. Similarly, the mass dm_2 that flows out across A_2 in the same time is $dm_2 = \rho A_2 v_2 \, dt$. In steady flow the total mass in the tube is constant, so $dm_1 = dm_2$ and

$$\rho A_1 v_1 \, dt = \rho A_2 v_2 \, dt \qquad \text{or}$$

(12.10)

Continuity equation for an **incompressible fluid:**

$$A_1 v_1 = A_2 v_2 \qquad (12.10)$$

Cross-sectional area of flow tube at two points (see Fig. 12.21)

Speed of flow at the two points

The product Av is the *volume flow rate* dV/dt, the rate at which volume crosses a section of the tube:

(12.11)

Volume flow rate of a fluid

$$\frac{dV}{dt} = Av \qquad (12.11)$$

Cross-sectional area of flow tube

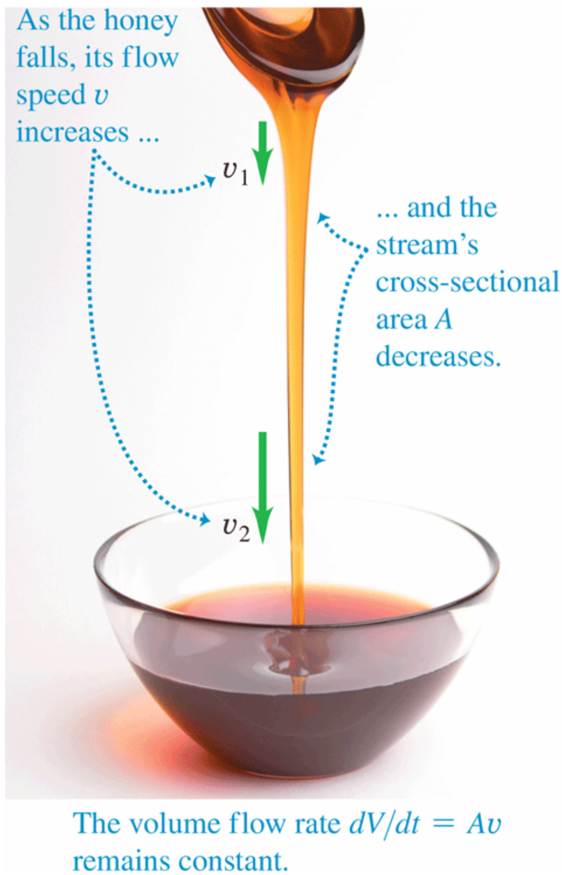
Speed of flow

The *mass flow rate* is the mass flow per unit time through a cross section. This is equal to the density ρ times the volume flow rate dV/dt.

Equation (12.10) shows that the volume flow rate has the same value at all points along any flow tube (Fig. 12.22). When the cross section of a flow tube decreases, the speed increases, and vice versa. A broad, deep part of a river has a larger cross section and slower current than a narrow, shallow part, but the volume flow rates are the same in both. This is the essence of the familiar maxim, "Still waters run deep." If a water pipe with

2 cm diameter is connected to a pipe with 1 cm diameter, the flow speed is four times as great in the 1 cm part as in the 2 cm part.

Figure 12.22



The continuity equation, Eq. (12.10), helps explain the shape of a stream of honey poured from a spoon.

We can generalize Eq. (12.10) for the case in which the fluid is *not* incompressible. If ρ_1 and ρ_2 are the densities at sections 1 and 2, then

(12.12)

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2 \quad \text{(continuity equation, compressible fluid)}$$

If the fluid is denser at point 2 than at point 1 ($\rho_2 > \rho_1$), the volume flow rate at point 2 will be less than at point 1 ($A_2 v_2 < A_1 v_1$). We leave the details to you. If the fluid is incompressible so that ρ_1 and ρ_2 are always equal, Eq. (12.12) reduces to Eq. (12.10).

Example 12.6 Flow of an incompressible fluid

Incompressible oil of density 850 kg/m^3 is pumped through a cylindrical pipe at a rate of 9.5 liters per second. (a) The first section of the pipe has a diameter of 8.0 cm. What is the flow speed of the oil? What is the mass flow rate? (b) The second section of the pipe has a diameter of 4.0 cm. What are the flow speed and mass flow rate in that section?

IDENTIFY and SET UP Since the oil is incompressible, the volume flow rate has the *same* value ($9.5 \text{ L}/s$) in both sections of pipe. The mass flow rate (the density times the volume flow rate) also has the same value in both sections. (This is just the statement that no fluid is lost or added anywhere along the pipe.) We use the volume flow rate equation, Eq. (12.11), to determine the speed v_1 in the 8.0-cm-diameter section and the continuity equation for incompressible flow, Eq. (12.10), to find the speed v_2 in the 4.0-cm-diameter section.

EXECUTE

(a) From Eq. (12.11) the volume flow rate in the first section is $dV/dt = A_1|\text{vecv}|_1$, where A_1 is the cross-sectional area of the pipe of diameter 8.0 cm and radius 4.0 cm. Hence

$$v_1 = \frac{dV/dt}{A_1} = \frac{(9.5 \text{ L/s})(10^{-3} \text{ m}^3/\text{L})}{\pi(4.0 \times 10^{-2} \text{ m})^2} = 1.9 \text{ m}/s$$

The mass flow rate is

$\rho \, dV/dt = (850 \text{ kg/m}^3)(9.5 \times 10^{-3} \text{ m}^3/\text{s}) = \ 8.1 \text{ kg/s}.$

(b) From the continuity equation, Eq. (12.10),

$$v_2 = \frac{A_1}{A_2} v_1 = \frac{\pi(4.0 \times 10^{-2} \text{ m})^2}{\pi(2.0 \times 10^{-2} \text{ m})^2} (1.9 \text{ m/s}) = 7.6 \text{ m/s} = 4v_1$$

The volume and mass flow rates are the same as in part (a).

EVALUATE The second section of pipe has one-half the diameter and one-fourth the cross-sectional area of the first section. Hence the speed must be four times greater in the second section, which is just what our result shows.

KEY CONCEPT

The continuity equation states that as an incompressible fluid moves through a flow tube, the volume flow rate (the flow tube's cross-sectional area times the flow speed) is the same *at all points* along the flow tube. The narrower the flow tube, the faster the flow speed.

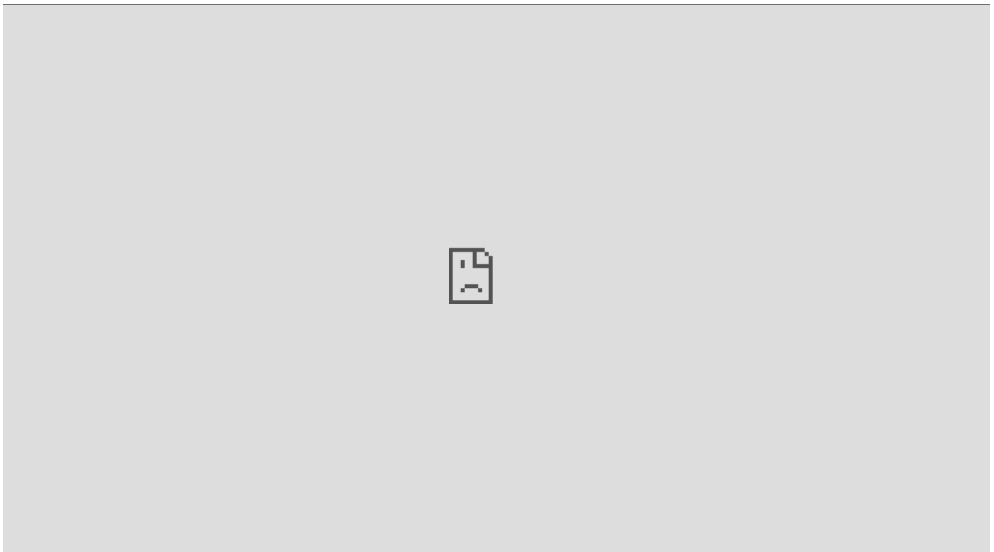
Video Tutor Solution: **Example 12.6**

Test Your Understanding of Section 12.4

A maintenance crew is working on a section of a three-lane highway, leaving only one lane open to traffic. The result is much slower traffic flow (a traffic jam). Do cars on a highway behave like the molecules of (i) an incompressible fluid or (ii) a compressible fluid?

12.5 Bernoulli's Equation

Video Tutor Demo: Air Jet Blows between Bowling Balls



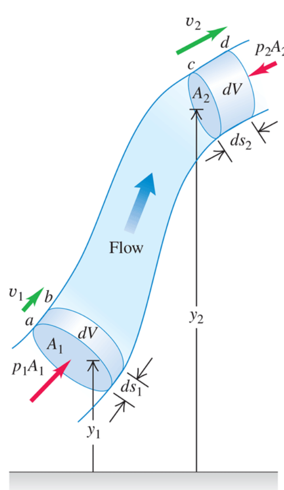
According to the continuity equation, the speed of fluid flow can vary along the paths of the fluid. The pressure can also vary; it depends on height as in the static situation (see Section 12.2), and it also depends on the speed of flow. We can derive an important relationship called *Bernoulli's equation,* which relates the pressure, flow speed, and height for flow of an ideal, incompressible fluid. Bernoulli's equation is useful in analyzing many kinds of fluid flow.

The dependence of pressure on speed follows from the continuity equation, Eq. (12.10). When an incompressible fluid flows along a flow tube with varying cross section, its speed *must* change, and so an element of fluid must have an acceleration. If the tube is horizontal, the force that causes this acceleration has to be applied by the surrounding fluid. This means that the pressure *must* be different in regions of different cross section; if it were the same everywhere, the net force on every fluid element would be zero. When a horizontal flow tube narrows and a fluid element speeds up, it must be moving toward a region of lower pressure in order to have a net forward force to accelerate it. If the elevation also changes, this causes an additional pressure difference.

Deriving Bernoulli's Equation

To derive Bernoulli's equation, we apply the work–energy theorem to the fluid in a section of a flow tube. In Fig. 12.23 we consider the element of fluid that at some initial time lies between the two cross sections a and c. The speeds at the lower and upper ends are v_1 and v_2. In a small time interval dt, the fluid that is initially at a moves to b, a distance $ds_1 = v_1\,dt$, and the fluid that is initially at c moves to d, a distance $ds_2 = v_2\,dt$. The cross-sectional areas at the two ends are A_1 and A_2, as shown. The fluid is incompressible; hence by the continuity equation, Eq. (12.10), the volume of fluid dV passing *any* cross section during time dt is the same. That is, $dV = A_1\,ds_1 = A_2\,ds_2$.

Figure 12.23



Deriving Bernoulli's equation. The net work done on a fluid element by the pressure of the surrounding fluid equals the change in the kinetic energy plus the change in the gravitational potential energy.

Let's compute the *work* done on this fluid element during dt. If there is negligible internal friction in the fluid (i.e., no viscosity), the only nongravitational forces that do work on the element are due to the pressure of the surrounding fluid. The pressures at the two ends are p_1 and p_2 ; the force on the cross section at a is $p_1 A_1$, and the force at c is $p_2 A_2$. The net work dW done on the element by the surrounding fluid during this displacement is therefore

(12.13)

$$dW = p_1 A_1 \, ds_1 - p_2 A_2 \, ds_2 = (p_1 - p_2) dV$$

The term $p_2 A_2 ds_2$ has a negative sign because the force at c opposes the displacement of the fluid.

The work dW is due to forces other than the conservative force of gravity, so it equals the change in the total mechanical energy (kinetic energy plus gravitational potential energy) associated with the fluid element. The mechanical energy for the fluid between sections b and c does not change. At the beginning of dt the fluid between a and b has volume $A_1 \, ds_1$, mass $\rho A_1 \, ds_1$, and kinetic energy $\frac{1}{2} \rho \left(A_1 \, ds_1\right) v_1^2$. At the end of dt the fluid between c and d has kinetic energy $\frac{1}{2} \rho \left(A_2 \, ds_2\right) v_2^2$. The net change in kinetic energy dK during time dt is

(12.14)

$$dK = \frac{1}{2} \rho \, dV \left(v_2^2 - v_1^2\right)$$

What about the change in gravitational potential energy? At the beginning of time interval dt, the potential energy for the mass between a and b is $dm \, g y_1 = \rho \, dV \, g y_1$. At the end of dt, the potential energy for the mass between c and d is $dm \, g y_2 = \rho \, dV \, g y_2$. The net change in potential energy dU during dt is

(12.15)

$$dU = \rho\, dV\, g\,(y_2 - y_1)$$

Combining Eqs. (12.13), (12.14), and (12.15) in the energy equation $dW = dK + dU$, we obtain

(12.16)

$$\begin{aligned}(p_1 - p_2)dV &= \tfrac{1}{2}\,\rho\, dV(v_2^2 - v_1^2) + \rho\, dV\, g\,(y_2 - y_1)\\ p_1 - p_2 &= \tfrac{1}{2}p(v_2^2 - v_1^2) + \rho g\,(y_2 - y_1)\end{aligned}$$

This is **Bernoulli's equation.** It states that the work done on a unit volume of fluid by the surrounding fluid is equal to the sum of the changes in kinetic and potential energies per unit volume that occur during the flow. We may also interpret Eq. (12.16) in terms of pressures. The first term on the right is the pressure difference associated with the change of speed of the fluid. The second term on the right is the additional pressure difference caused by the weight of the fluid and the difference in elevation of the two ends.

BIO Application

Why Healthy Giraffes Have High Blood Pressure

Bernoulli's equation suggests that as blood flows upward at roughly constant speed v from the heart to the brain, the pressure p will drop as the blood's height y increases. For blood to reach the brain with the required minimal pressure, the human heart provides a maximum (systolic) gauge pressure of about 120 mm Hg. The vertical distance from heart to brain is much larger for a giraffe, so its heart must produce a much greater maximum gauge pressure (about 280 mm Hg).

We can also express Eq. (12.16) in a more convenient form as

(12.17)

$$p_1 + \rho g y_1 + \frac{1}{2}\,\rho v_1^2 = p_2 + \rho g y_2 + \frac{1}{2}\,\rho v_2^2$$

Subscripts 1 and 2 refer to *any* two points along the flow tube, so we can write

(12.18)

Bernoulli's equation for an ideal, incompressible fluid:

$$\overset{\text{Pressure}}{p} + \overset{\text{Fluid density}}{\rho}\,\underset{\text{Acceleration due to gravity}}{g}\,\underset{\text{Elevation}}{y} + \frac{1}{2}\rho\,\underset{\text{Flow speed}}{v}^{2} = \overset{\text{Value is same at all points in flow tube.}}{\text{constant}} \tag{12.18}$$

Note that when the fluid is *not* moving (so $v_1 = v_2 = 0$), Eq. (12.17) reduces to the pressure relationship we derived for a fluid at rest, Eq. (12.5).

CAUTION Bernoulli's equation applies in certain situations only We stress again that Bernoulli's equation is valid for only incompressible, steady flow of a fluid with no internal friction (no viscosity). It's a simple equation, but don't be tempted to use it in situations in which it doesn't apply!

Problem-Solving Strategy 12.1 Bernoulli's Equation

Bernoulli's equation is derived from the work–energy theorem, so much of Problem-Solving Strategy 7.1 (Section 7.1) applies here.

IDENTIFY *the relevant concepts:* Bernoulli's equation is applicable to steady flow of an incompressible fluid that has no internal friction (see Section 12.6). It is generally applicable to flows through large pipes and to flows within bulk fluids (e.g., air flowing around an airplane or water flowing around a fish).

SET UP *the problem* using the following steps:

1. Identify the points 1 and 2 referred to in Bernoulli's equation, Eq. (12.17).
2. Define your coordinate system, particularly the level at which $y = 0$. Take the positive $y-$direction to be upward.
3. List the unknown and known quantities in Eq. (12.17). Decide which unknowns are the target variables.

EXECUTE *the solution* as follows: Write Bernoulli's equation and solve for the unknowns. You may need the continuity equation, Eq. (12.10), to relate the two speeds in terms of cross-sectional areas of pipes or containers. You may also need Eq. (12.11) to find the volume flow rate.

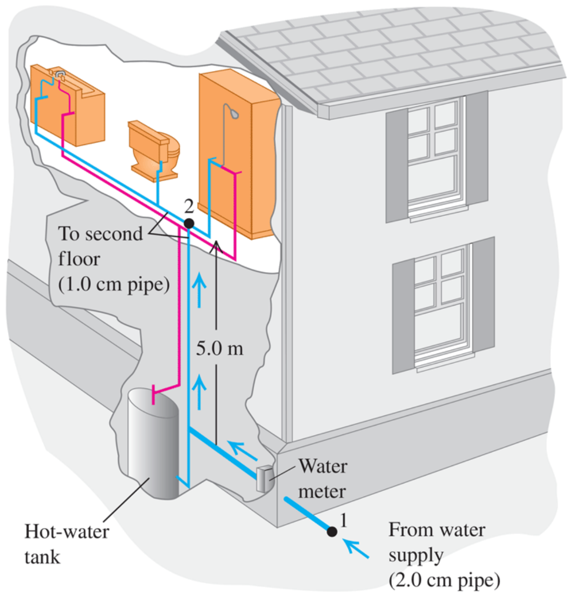
EVALUATE *your answer:* Verify that the results make physical sense. Check that you have used consistent units: In SI units, pressure is in pascals, density in kilograms per cubic meter, and speed in meters per second. The pressures must be either *all* absolute pressures or *all* gauge pressures.

Example 12.7 Water pressure in the home

WITH VARIATION PROBLEMS

Water enters a house (Fig. 12.24) through a pipe with an inside diameter of 2.0 cm at an absolute pressure of 4.0×10^5 Pa (about 4 atm). A 1.0-cm-diameter pipe leads to the second-floor bathroom 5.0 m above. When the flow speed at the inlet pipe is 1.5 m/s, find the flow speed, pressure, and volume flow rate in the bathroom.

Figure 12.24



What is the water pressure in the second-story bathroom of this house?

IDENTIFY and SET UP We assume that the water flows at a steady rate. Water is effectively incompressible, so we can use the continuity equation. It's reasonable to ignore internal friction because the pipe has a relatively large diameter, so we can also use Bernoulli's equation. Let points 1 and 2 be at the inlet pipe and at the bathroom, respectively. We are given the pipe diameters at points 1 and 2, from which we calculate the areas A_1 and A_2, as well as the speed $v_1 = 1.5\ \mathrm{m}/s$ and pressure $p_1 = 4.0 \times 10^5$ Pa at the inlet pipe. We take $y_1 = 0$ and $y_2 = 5.0$ m. We find the speed v_2 from the continuity equation and the pressure p_2 from Bernoulli's equation. Knowing v_2, we calculate the volume flow rate $v_2 A_2$.

EXECUTE From the continuity equation, Eq. (12.10),

$$v_2 = \frac{A_1}{A_2} v_1 = \frac{\pi(1.0\ \mathrm{cm})^2}{\pi(0.50\ \mathrm{cm})^2} (1.5\ \mathrm{m/s}) = 6.0\ \mathrm{m}/s$$

From Bernoulli's equation, Eq. (12.16),

$$
\begin{aligned}
p_2 &= p_1 - \tfrac{1}{2}\,\rho(v_2^2 - v_1^2) - \rho g\,(y_2 - y_1) \\
&= 4.0 \times 10^5\,\mathrm{Pa} - \tfrac{1}{2}\left(1.0 \times 10^3\,\mathrm{kg/m^3}\right)\left(36\,\mathrm{m^2/s^2} - 2.25\,\mathrm{m^2/s^2}\right) \\
&\quad - \left(1.0 \times 10^3\,\mathrm{kg/m^3}\right)\left(9.8\,\mathrm{m/s^2}\right)(5.0\,\mathrm{m}) \\
&= 4.0 \times 10^5\,\mathrm{Pa} - 0.17 \times 10^5\,\mathrm{Pa} - 0.49 \times 10^5\,\mathrm{Pa} \\
&= 3.3 \times 10^5\,\mathrm{Pa} = 3.3\ \mathrm{atm} = 48\,\mathrm{Ib/in.}^2
\end{aligned}
$$

The volume flow rate is

$$
\begin{aligned}
\frac{dV}{dt} &= A_2\,v_2 = \pi(0.50 \times 10^{-2}\ \mathrm{m})^2(6.0\ \mathrm{m}/s) \\
&= 4.7 \times 10^{-4}\mathrm{m^3/s} = 0.47\,\mathrm{L/s}
\end{aligned}
$$

EVALUATE This is a reasonable flow rate for a bathroom faucet or shower. Note that if the water is turned off, both v_1 and v_2 are zero, the term $\frac{1}{2}\,\rho\left(v_2^2 - v_1^2\right)$ in Bernoulli's equation vanishes, and p_2 rises from 3.3×10^5 Pa to 3.5×10^5 Pa.

KEY CONCEPT

Bernoulli's equation allows you to relate the flow speeds at two different points in a fluid to the pressures and heights at those two points.

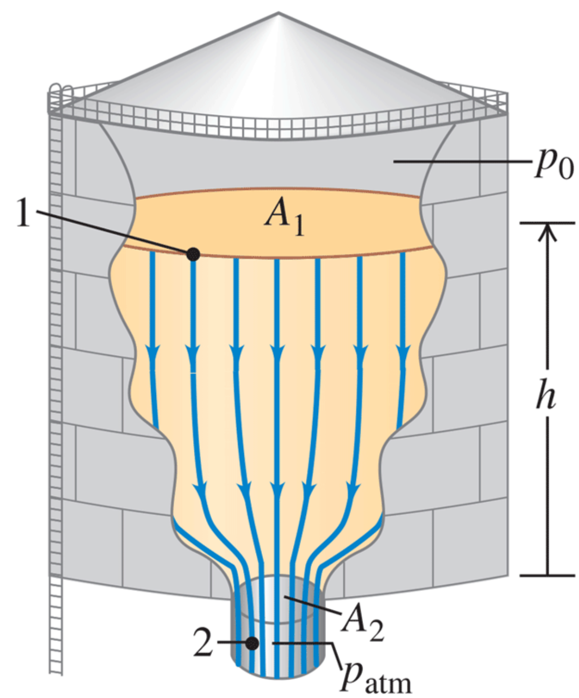
Video Tutor Solution: **Example 12.7**

Example 12.8 Speed of efflux

WITH VARIATION PROBLEMS

Figure 12.25 shows a gasoline storage tank with cross-sectional area A_1, filled to a depth h. The space above the gasoline contains air at pressure p_0 , and the gasoline flows out the bottom of the tank through a short pipe with cross-sectional area A_2 . Derive expressions for the flow speed in the pipe and the volume flow rate.

Figure 12.25



Calculating the speed of efflux for gasoline flowing out the bottom of a storage tank.

IDENTIFY and SET UP We consider the entire volume of moving liquid as a single flow tube of an incompressible fluid with negligible internal friction. Hence, we can use Bernoulli's equation. Points 1 and 2 are at the surface of the gasoline and at the exit pipe, respectively. At point 1 the pressure is p_0, which we assume to be fixed; at point 2 it is atmospheric pressure p_{atm}. We take $y = 0$ at the exit pipe, so $y_1 = h$ and $y_2 = 0$. Because A_1 is very much larger than A_2, the upper surface of the gasoline will drop very slowly and so v_1 is essentially zero. We find v_2 from Eq. (12.17) and the volume flow rate from Eq. (12.11).

EXECUTE We apply Bernoulli's equation to points 1 and 2:

$$p_0 + \tfrac{1}{2}\,\rho v_1^2 + \rho g h = p_{\text{atm}} + \tfrac{1}{2}\,\rho v_2^2 + \rho g\,(0)$$
$$v_2^2 \;=\; v_1^2 \;+\; 2\left(\tfrac{p_0 - p_{atm}}{p}\right) \;+\; 2gh$$

Using $v_1 = 0$, we find

$$v_2 = \sqrt{2\left(\frac{p_0 - p_{\text{atm}}}{\rho}\right) + 2gh}$$

From Eq. (12.11), the volume flow rate is $dV/dt = v_2\,A_2$.

EVALUATE The speed v_2, sometimes called the *speed of efflux,* depends on both the pressure difference $(p_0 - p_{\text{atm}})$ and the height h of the liquid level in the tank. If the top of the tank is vented to the atmosphere, $p_0 = p_{\text{atm}}$ and $p_0 - p_{\text{atm}} = 0$. Then

$$v_2 = \sqrt{2gh}$$

That is, the speed of efflux from an opening at a distance h below the top surface of the liquid is the *same* as the speed an object would acquire in falling freely through a height h. This result is called *Torricelli's theorem.* It is valid also for a hole in a side wall at a depth h below the surface. If $p_0 = p_{\text{atm}}$, the volume flow rate is

$$\frac{dV}{dt} = A_2\sqrt{2gh}$$

KEY CONCEPT

When solving problems about the flow of an incompressible fluid with negligible internal friction, you can use both Bernoulli's equation and the continuity equation.

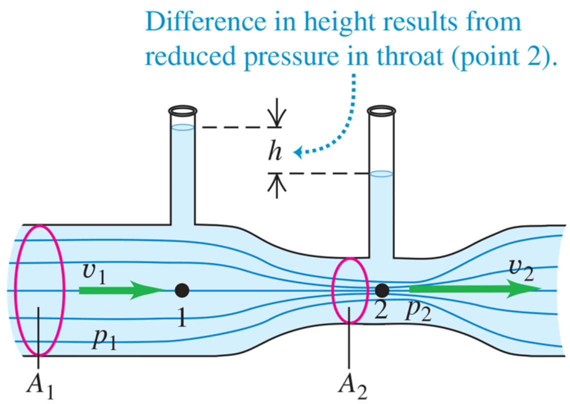
Video Tutor Solution: Example 12.8

Example 12.9 The Venturi meter

WITH VARIATION PROBLEMS

Figure 12.26 shows a *Venturi meter,* used to measure flow speed in a pipe. Derive an expression for the flow speed v_1 in terms of the cross-sectional areas A_1 and A_2.

Figure 12.26



The Venturi meter.

IDENTIFY and SET UP The flow is steady, and we assume the fluid is incompressible and has negligible internal friction. Hence we can apply Bernoulli's equation to the wide part (point 1) and narrow part (point 2, the *throat*) of the pipe. Equation (12.6) relates h to the pressure difference $p_1 - p_2$.

EXECUTE Points 1 and 2 have the same vertical coordinate $y_1 = y_2$, so Eq. (12.17) says

$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$$

From the continuity equation, $v_2 = (A_1/A_2)v_1$. Substituting this and rearranging, we get

$$p_1 - p_2 = \frac{1}{2}\rho v_1^2\left[\left(\frac{A_1}{A_2}\right)^2 - 1\right]$$

From Eq. (12.6), the pressure difference $p_1 - p_2$ is also equal to $\rho g h$. Substituting this and solving for v_1, we get

$$v_1 = \sqrt{\frac{2gh}{(A_1/A_2)^2 - 1}}$$

EVALUATE Because A_1 is greater than A_2, v_2 is greater than v_1 and the pressure p_2 in the throat is *less* than p_1. Those pressure differences produce a net force to the right that makes the fluid speed up as it enters the throat, and a net force to the left that slows it as it leaves.

KEY CONCEPT

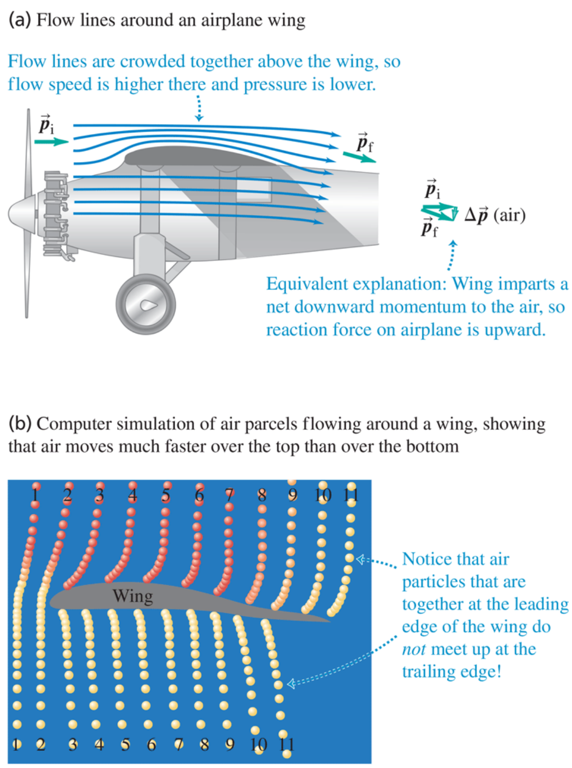
When an incompressible fluid with negligible internal friction flows through a pipe of varying size, the pressure and flow speed both change. Where the cross-sectional area is small, the pressure is low and the speed is high; where the cross-sectional area is large, the pressure is high and the speed is low.

Video Tutor Solution: **Example 12.9**

Conceptual Example 12.10 Lift on an airplane wing

Figure 12.27a shows flow lines around a cross section of an airplane wing. The flow lines crowd together above the wing, corresponding to increased flow speed and reduced pressure, just as in the Venturi throat in Example 12.9. Hence the downward force of the air on the top side of the wing is less than the upward force of the air on the underside of the wing, and there is a net upward force or *lift*. Lift is not simply due to the impulse of air striking the underside of the wing; in fact, the reduced pressure on the upper wing surface makes the greatest contribution to the lift. (This simplified discussion ignores the formation of vortices.)

Figure 12.27



Flow around an airplane wing.

We can understand the lift force on the basis of momentum changes instead. The vector diagram in Fig. 12.27a shows that there is a net *downward* change in the vertical component of momentum of the air flowing past the wing, corresponding to the downward force the wing exerts on the air. The reaction force *on* the wing is *upward,* as we concluded above.

Similar flow patterns and lift forces are found in the vicinity of any humped object in a wind. A moderate wind makes an umbrella "float"; a strong wind can turn it inside out. At high speed, lift can reduce traction on a car's tires; a "spoiler" at the car's tail, shaped like an upside-down wing, provides a compensating downward force.

CAUTION **A misconception about wings** Some discussions of lift claim that air travels faster over the top of a wing because "it has farther to travel." This claim assumes that air molecules that part company at the front of the wing, one traveling over the wing and one under it, must meet again at the wing's trailing edge. Not so! Figure 12.27b shows a computer simulation of parcels of air flowing around an airplane wing. Parcels that are adjacent at the front of the wing do *not* meet at the trailing edge; the flow over the top of the wing is much faster than if the parcels had to meet. In accordance with Bernoulli's equation, this faster speed means that there is even lower pressure above the wing (and hence greater lift) than the "farther-to-travel" claim would suggest.

KEY CONCEPT

The pressure in a flowing incompressible fluid with negligible internal friction is low at points where the flow lines are crowded together, such as above the upper surface of an airplane wing.

Video Tutor Solution: **Example 12.10**

Test Your Understanding of Section 12.5

Which is the most accurate statement of Bernoulli's principle? (i) Fast-moving air causes lower pressure; (ii) lower pressure causes fast-moving air; (iii) both (i) and (ii) are equally accurate.

12.6 Viscosity and Turbulence

In our discussion of fluid flow we assumed that the fluid had no internal friction and that the flow was laminar. While these assumptions are often quite valid, in many important physical situations the effects of viscosity (internal friction) and turbulence (nonlaminar flow) are extremely important. Let's take a brief look at some of these situations.

Viscosity

Viscosity is internal friction in a fluid. Viscous forces oppose the motion of one portion of a fluid relative to another. Viscosity is the reason it takes effort to paddle a canoe through calm water, but it is also the reason the paddle works. Viscous effects are important in the flow of fluids in pipes, the flow of blood, the lubrication of engine parts, and many other situations.

Fluids that flow readily, such as water or gasoline, have smaller viscosities than do "thick" liquids such as honey or motor oil. Viscosities of all fluids are strongly temperature dependent, increasing for gases and decreasing for liquids as the temperature increases (Fig. 12.28). Oils for engine lubrication must flow equally well in cold and warm conditions, and so are designed to have as *little* temperature variation of viscosity as possible.

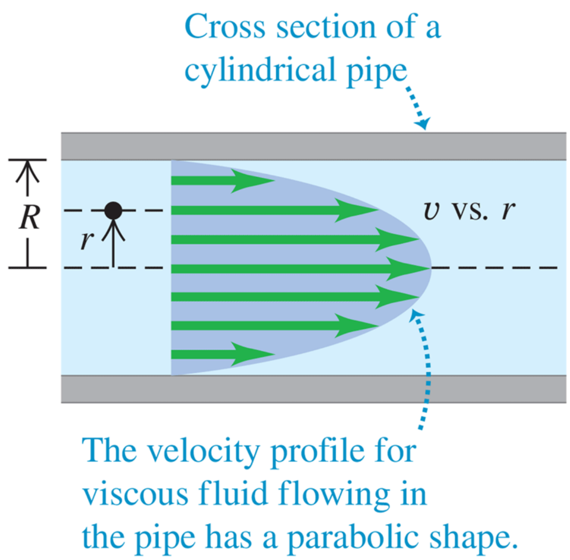
Figure 12.28

Lava is an example of a viscous fluid. The viscosity decreases with increasing temperature: The hotter the lava, the more easily it can flow.

A viscous fluid always tends to cling to a solid surface in contact with it. There is always a thin *boundary layer* of fluid near the surface, in which the fluid is nearly at rest with respect to the surface. That's why dust particles can cling to a fan blade even when it is rotating rapidly, and why you can't get all the dirt off your car by just squirting a hose at it.

Viscosity has important effects on the flow of liquids through pipes, including the flow of blood in the circulatory system. First think about a fluid with zero viscosity so that we can apply Bernoulli's equation, Eq. (12.17). If the two ends of a long cylindrical pipe are at the same height ($y_1 = y_2$) and the flow speed is the same at both ends ($v_1 = v_2$), Bernoulli's equation tells us that the pressure is the same at both ends of the pipe. But this isn't true if we account for viscosity. To see why, consider Fig. 12.29, which shows the flow-speed profile for laminar flow of a viscous fluid in a long cylindrical pipe. Due to viscosity, the speed is *zero* at the pipe walls (to which the fluid clings) and is greatest at the center of the pipe. The motion is like a lot of concentric tubes sliding relative to one another, with the central tube moving fastest and the outermost tube at rest. Viscous forces between the tubes oppose this sliding, so to keep the flow going we must apply a greater pressure at the back of the flow than at the front. That's why you have to keep squeezing a tube of toothpaste or a packet of ketchup (both viscous fluids) to keep the fluid coming out of its container. Your fingers provide a pressure at the back of the flow that is far greater than the atmospheric pressure at the front of the flow.

Figure 12.29



Velocity profile for a viscous fluid in a cylindrical pipe.

The pressure difference required to sustain a given volume flow rate through a cylindrical pipe of length L and radius R turns out to be proportional to L/R^4. If we decrease R by one-half, the required pressure increases by $2^4 = 16$; decreasing R by a factor of 0.90 (a 10% reduction) increases the required pressure difference by a factor of $(1/0.90)^4 = 1.52$ (a 52% increase). This simple relationship explains the connection between a high-cholesterol diet (which tends to narrow the arteries) and high blood pressure. Due to the R^4 dependence, even a small narrowing of the arteries can result in substantially elevated blood pressure and added strain on the heart muscle.

Turbulence

When the speed of a flowing fluid exceeds a certain critical value, the flow is no longer laminar. Instead, the flow pattern becomes extremely irregular and complex, and it changes continuously with time; there is no steady-state pattern. This irregular, chaotic flow is called **turbulence.** Figure 12.20 shows the contrast between laminar and turbulent flow for smoke rising in air. Bernoulli's equation is *not* applicable to regions where turbulence occurs because the flow is not steady.

BIO Application

Listening for Turbulent Flow

Normal blood flow in the human aorta is laminar, but a small disturbance such as a heart pathology can cause the flow to become turbulent. Turbulence makes noise, which is why listening to blood flow with a stethoscope is a useful diagnostic technique.

Whether a flow is laminar or turbulent depends in part on the fluid's viscosity. The greater the viscosity, the greater the tendency for the fluid to flow in sheets (laminae) and the more likely the flow is to be laminar. (When we discussed Bernoulli's equation in Section 12.5, we assumed that the flow was laminar and that the fluid had zero viscosity. In fact, a *little* viscosity is needed to ensure that the flow is laminar.)

For a fluid of a given viscosity, flow speed is a determining factor for the onset of turbulence. A flow pattern that is stable at low speeds suddenly becomes unstable when a critical speed is reached. Irregularities in the flow pattern can be caused by roughness in the pipe wall, variations in the density of the fluid, and many other factors. At low flow speeds, these disturbances damp out; the flow pattern is *stable* and tends to maintain its laminar nature (Fig. 12.30a). When the critical speed is reached, however, the flow pattern becomes unstable. The disturbances no longer damp out but grow until they destroy the entire laminar-flow pattern (Fig. 12.30b).

Figure 12.30

(a) Low speed: laminar flow

(b) High speed:
turbulent flow

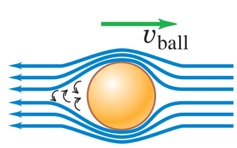
The flow of water from a faucet can be (a) laminar or (b) turbulent.

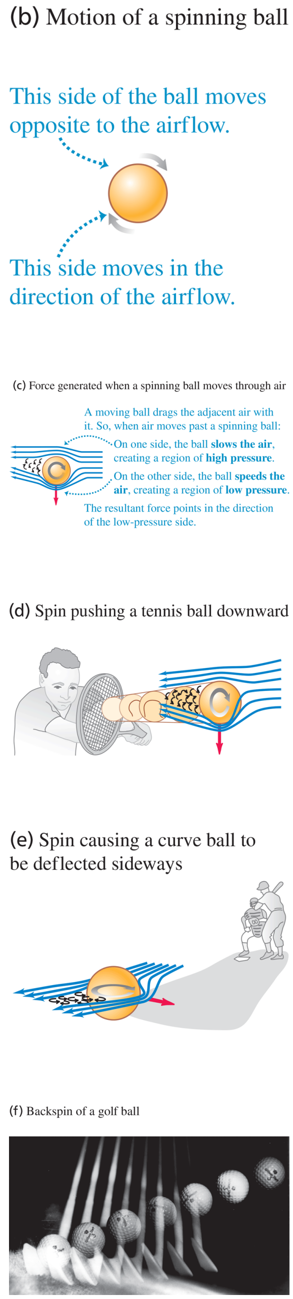
Conceptual Example 12.11 The curve ball

Does a curve ball *really* curve? Yes, it does, and the reason is turbulence. Figure 12.31a shows a nonspinning ball moving through the air from left to right. The flow lines show that to an observer moving with the ball, the air stream appears to move from right to left. Because of the high speeds involved (typically near 35 m/s, or 75 mi/h), there is a region of *turbulent* flow behind the ball.

Figure 12.31

(a) Motion of air relative
to a nonspinning ball





(a)–(e) Analyzing the motion of a spinning ball through the air. (f) Stroboscopic photograph of a golf ball being struck by a club. The picture was taken at 1000 flashes per second. The ball rotates about once in eight flashes, corresponding to an angular speed of 125 rev/s, or 7500 rpm. Source: Harold Edgerton at MIT, copyright 2014. Courtesy of Palm Press, Inc.

Figure 12.31b shows a *spinning* ball with "top spin." Layers of air near the ball's surface are pulled around in the direction of the spin by friction between the ball and air and by the air's internal friction (viscosity). Hence air moves relative to the ball's surface more slowly at the top of the ball than at the bottom, and turbulence occurs farther forward on the top side than on the bottom. As a result, the average pressure at the top of the ball is now greater than that at the bottom, and the resulting net force deflects the ball downward (Fig. 12.31c). "Top spin" is used in tennis to keep a fast serve in the court (Fig. 12.31d).

In baseball, a curve ball spins about a nearly *vertical* axis and the resulting deflection is sideways. In that case, Fig. 12.31c is a *top* view of the situation. A curve ball thrown by a left-handed pitcher spins as shown in Fig. 12.31e and will curve *toward* a right-handed batter, making it harder to hit.

A similar effect occurs when golf balls acquire "backspin" from impact with the grooved, slanted club face. Figure 12.31f shows the backspin of a golf ball just after impact. The resulting pressure difference between the top and bottom of the ball causes a *lift* force that keeps the ball in the air longer than would be possible without spin. A well-hit drive appears, from the tee, to "float" or even curve *upward* during the initial portion of its flight. This is a real effect, not an illusion. The dimples on the golf ball play an essential role; the viscosity of air gives a dimpled ball a much longer trajectory than an undimpled one with the same initial velocity and spin.

KEY CONCEPT

Even in a fluid with low viscosity (that is, little internal friction) such as air, the effects of viscosity can be important for determining how that fluid flows around objects.

Video Tutor Solution: **Example 12.11**

Test Your Understanding of Section 12.6

How much more thumb pressure must a nurse use to administer an injection with a hypodermic needle of inside diameter 0.30 mm compared to one with inside diameter 0.60 mm? Assume that the two needles have the same length and that the volume flow rate is the same in both cases. (i) Twice as much; (ii) 4 times as much; (iii) 8 times as much; (iv) 16 times as much; (v) 32 times as much.

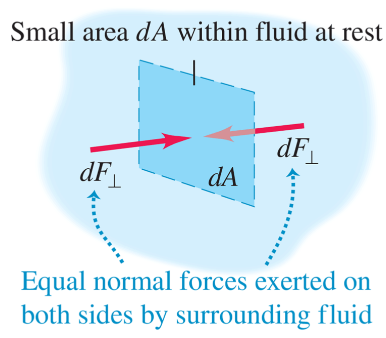
Chapter 12 Summary

Density and pressure: Density is mass per unit volume. If a mass m of homogeneous material has volume V, its density is the ratio Specific gravity is the ratio of the density of a material to the density of water. (See Example 12.1.)

Pressure is normal force per unit area. Pascal's law states that pressure applied to an enclosed fluid is transmitted undiminished to every portion of the fluid. Absolute pressure is the total pressure in a fluid; gauge pressure is the difference between absolute pressure and atmospheric pressure. The SI unit of pressure is the pascal (Pa): (See Example 12.2.)

(12.1)

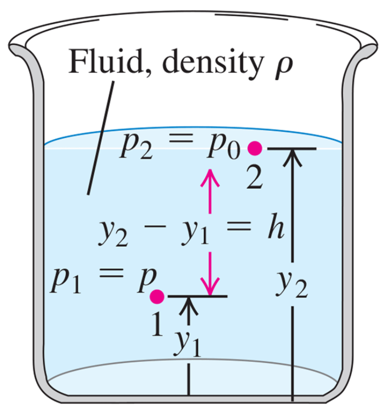
(12.2)



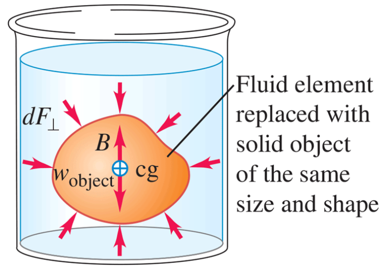
Pressures in a fluid at rest: The pressure difference between points 1 and 2 in a static fluid of uniform density (an incompressible fluid) is proportional to the difference between the elevations and If the pressure at the surface of an incompressible liquid at rest is then the pressure at a depth h is greater by an amount (See Examples 12.3 and 12.4.)

(12.5)

(12.6)



Buoyancy: Archimedes's principle states that when an object is immersed in a fluid, the fluid exerts an upward buoyant force on the object equal to the weight of the fluid that the object displaces. (See Example 12.5.)



Fluid flow: An ideal fluid is incompressible and has no viscosity (no internal friction). A flow line is the path of a fluid particle; a streamline is a curve tangent at each point to the velocity vector at that point. A flow tube is a tube bounded at its sides by flow lines. In laminar flow, layers of fluid slide smoothly past each other. In turbulent flow, there is great disorder and a constantly changing flow pattern.

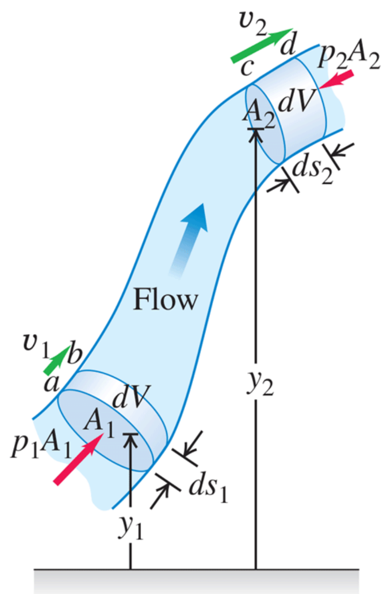
Conservation of mass in an incompressible fluid is expressed by the continuity equation, which relates the flow speeds and for two cross sections and in a flow tube. The product equals the volume flow rate, the rate at which volume crosses a section of the tube. (See Example 12.6.)

Bernoulli's equation states that a quantity involving the pressure p, flow speed , and elevation y has the same value anywhere in a flow tube, assuming steady flow in an ideal fluid. This equation can be used to relate the properties of the flow at any two points. (See Examples 12.7, 12.8, 12.9 and 12.10.)

(12.10)

(12.11)

(12.18)



Guided Practice: Fluid Mechanics

For assigned homework and other learning materials, go to
Mastering Physics.

Key Example Variation Problems

Be sure to review Examples 12.3 and 12.4 (Section 12.2) before attempting these problems.

VP12.4.1 Liquefied natural gas (LNG) in a vertical storage tank has a density of 455 kg/m^3. The depth of LNG in the tank is 2.00 m, and the absolute pressure of the air inside the tank above the upper surface of the LNG is 1.22×10^5 Pa. What is the absolute pressure at the bottom of the tank?

VP12.4.2 You fill a vertical glass tube to a depth of 15.0 cm with freshwater. You then pour on top of the water an additional 15.0 cm of gasoline (density $7.40 \times 10^2 kg/m^3$), which does not mix with water. The upper surface of the gasoline is exposed to the air. Find the gauge pressure (a) at the interface between the gasoline and water and (b) at the bottom of the tube.

VP12.4.3 In an open-tube manometer (see Fig. 12.8a), the absolute pressure in the container of gas on the left is 2.10×10^5 Pa. If atmospheric pressure is 1.01×10^5 Pa and the liquid in the manometer is mercury (density $1.36 \times 10^4 \text{kg/m}^3$), what will be the difference in the heights of the left and right columns of liquid?

VP12.4.4 In the manometer tube shown in Fig. 12.10, the oil in the right-hand arm is olive oil of density 916 kg/m^3. (a) If the top of the oil is 25.0 cm above the bottom of the tube, what is the height of the top of the water above the bottom of the tube? (b) What is the gauge pressure 15.0 cm beneath the surface of the water? (c) At what depth below the surface of the oil is the gauge pressure the same as in part (b)?

Be sure to review Example 12.5 (Section 12.3) before attempting these problems.

VP12.5.1 An object of volume $7.50 \times 10^{-4} \text{m}^3$ and density $1.15 \times 10^3 \text{kg/m}^3$ is completely submerged in a fluid. (a) Calculate the weight of this object. (b) Calculate the buoyant force on this object if the fluid is (i) air (density 1.20 kg/m^3), (ii) water (density $1.00 \times 10^3 \text{kg/m}^3$), and (iii) glycerin (density $1.26 \times 10^3 \text{kg/m}^3$). In each case state whether the object will rise or sink if released while submerged in the fluid.

VP12.5.2 A sphere of volume $1.20 \times 10^{-3} \text{m}^3$ hangs from a cable. When the sphere is completely submerged in water, the tension in the cable is 29.4 N. (a) What is the buoyant force on the submerged sphere? (b) What is the weight of the sphere? (c) What is the density of the sphere?

VP12.5.3 A cube of volume $5.50 \times 10^{-3} \text{m}^3$ and density $7.50 \times 10^3 \text{kg/m}^3$ hangs from a cable. When the cube has the lower half of its volume submerged in an unknown liquid, the tension in the cable is 375 N. What is the density of the liquid? (Ignore the small buoyant force exerted by the air on the upper half of the cube.)

VP12.5.4 A wooden cylinder of length L and cross-sectional area A is partially submerged in a liquid with the axis of the cylinder oriented straight up and down. The density of the liquid is ρ_L. (a) If the length of the cylinder that is below the surface of the liquid is d, what is the buoyant force that the liquid exerts on the cylinder? (b) If the cylinder floats in the position described in part (a), what is the density of the cylinder? (Ignore the small buoyant force exerted by the air on the part of the cylinder above the surface of the liquid.)

Be sure to review Examples 12.7, 12.8, and 12.9 (Section 12.5) before attempting these problems.

VP12.9.1 A pipe leads from a storage tank on the roof of a building to the ground floor. The absolute pressure of the water in the storage tank where it connects to the pipe is 3.0×10^5 Pa, the pipe has a radius of 1.0 cm where it connects to the storage tank, and the speed of flow in this pipe is 1.6 m/s. The pipe on the ground floor has a radius of 0.50 cm and is 9.0 m below the storage tank. Find (a) the speed of flow and (b) the pressure in the pipe on the ground floor.

VP12.9.2 The storage tank in Fig. 12.25 contains ethanol (density $8.1 \times 10^2 \text{kg/m}^3$). The tank is 4.0 m in radius, the short pipe at the bottom of the tank is 1.0 cm in radius, and the height of ethanol in the tank is 3.2 m. The volume flow rate of ethanol from the short pipe is $4.4 \times 10^{-3} \text{m}^3/\text{s}$. (a) What is the speed at which ethanol flows out of the short pipe? (b) What is the gauge pressure of the air inside the tank in the space above the ethanol?

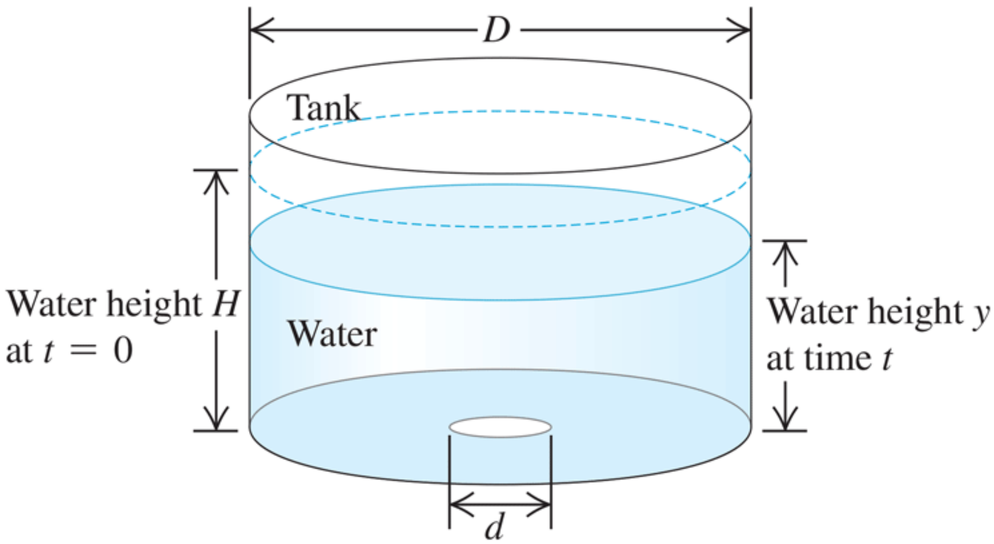
VP12.9.3 In a Venturi meter (see Fig. 12.26) that uses freshwater, the pressure difference between points 1 and 2 is 8.1×10^2 Pa, and the wide and narrow parts of the pipe have radii 2.5 cm and 1.2 cm, respectively. Find (a) the difference in the heights of the liquid levels in the two vertical tubes and (b) the volume flow rate through the horizontal pipe.

VP12.9.4 In the storage tank shown in Fig. 12.25, suppose area A_1 is *not* very much larger than area A_2. In this case we cannot treat the speed v_1 of the upper surface of the gasoline as zero. Derive an expression for the flow speed in the pipe in this situation in terms of p_0, p_{atm} g, h, A_1, A_2 and ρ.

Bridging Problem: How Long to Drain?

A large cylindrical tank with diameter D is open to the air at the top. The tank contains water to a height H. A small circular hole with diameter d, where $d \ll D$, is then opened at the bottom of the tank (Fig. 12.32). Ignore any effects of viscosity. (a) Find y, the height of water in the tank a time t after the hole is opened, as a function of t. (b) How long does it take to drain the tank completely? (c) If you double height H, by what factor does the time to drain the tank increase?

Figure 12.32



A water tank that is open at the top and has a hole at the bottom.

Solution Guide

IDENTIFY and SET UP

1. Draw a sketch of the situation that shows all of the relevant dimensions.
2. List the unknown quantities, and decide which of these are the target variables.
3. At what speed does water flow out of the bottom of the tank? How is this related to the volume flow rate of water out of the tank? How is the volume flow rate related to the rate of change of y?

EXECUTE

4. Use your results from step 3 to write an equation for dy/dt.
5. Your result from step 4 is a relatively simple differential equation. With your knowledge of calculus, you can integrate it to find y as

a function of t. (*Hint:* Once you've done the integration, you'll still have to do a little algebra.)

6. Use your result from step 5 to find the time when the tank is empty. How does your result depend on the initial height H?

EVALUATE

7. Check whether your answers are reasonable. A good check is to draw a graph of y versus t. According to your graph, what is the algebraic sign of dy/dt at different times? Does this make sense?

Video Tutor Solution: **Chapter 12 Bridging Problem**

Questions/Exercises/Problems: Fluid Mechanics

•, **••**, **•••**: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

Discussion Questions

Q12.1 A cube of oak wood with very smooth faces normally floats in water. Suppose you submerge it completely and press one face flat against the bottom of a tank so that no water is under that face. Will the block float to the surface? Is there a buoyant force on it? Explain.

Q12.2 A rubber hose is attached to a funnel, and the free end is bent around to point upward. When water is poured into the funnel, it rises in the hose to the same level as in the funnel, even though the funnel has a lot more water in it than the hose does. Why? What supports the extra weight of the water in the funnel?

Q12.3 Comparing Example 12.1 (Section 12.1) and Example 12.2 (Section 12.2), it seems that 700 N of air is exerting a downward force of on the floor. How is this possible?

Q12.4 Equation (12.7) shows that an area ratio of 100 to 1 can give 100 times more output force than input force. Doesn't this violate conservation of energy? Explain.

Q12.5 You have probably noticed that the lower the tire pressure, the larger the contact area between the tire and the road. Why?

Q12.6 In hot-air ballooning, a large balloon is filled with air heated by a gas burner at the bottom. Why must the air be heated? How does the balloonist control ascent and descent?

Q12.7 In describing the size of a large ship, one uses such expressions as "it displaces 20,000 tons." What does this mean? Can the weight of the ship be obtained from this information?

Q12.8 You drop a solid sphere of aluminum in a bucket of water that sits on the ground. The buoyant force equals the weight of water displaced; this is less than the weight of the sphere, so the sphere sinks to the bottom. If you take the bucket with you on an elevator that accelerates upward, the apparent weight of the water increases and the buoyant force on the sphere increases. Could the acceleration of the elevator be great enough to make the sphere pop up out of the water? Explain.

Q12.9 A rigid, lighter-than-air dirigible filled with helium cannot continue to rise indefinitely. Why? What determines the maximum height it can attain?

Q12.10 Which has a greater buoyant force on it: a piece of wood floating with part of its volume above water or a piece of submerged iron? Or, must you know their masses before you can answer? Explain.

Q12.11 The purity of gold can be tested by weighing it in air and in water. How? Do you think you could get away with making a fake gold brick by gold-plating some cheaper material?

Q12.12 During the Great Mississippi Flood of 1993, the levees in St. Louis tended to rupture first at the bottom. Why?

Q12.13 A cargo ship travels from the Atlantic Ocean (salt water) to Lake Ontario (freshwater) via the St. Lawrence River. The ship rides several centimeters lower in the water in Lake Ontario than it did in the ocean. Explain.

Q12.14 You push a piece of wood under the surface of a swimming pool. After it is completely submerged, you keep pushing it deeper and deeper. As you do this, what will happen to the buoyant force on it? Will the force keep increasing, stay the same, or decrease? Why?

Q12.15 An old question is "Which weighs more, a pound of feathers or a pound of lead?" If the weight in pounds is the gravitational force, will a pound of feathers balance a pound of lead on opposite pans of an equal-arm balance? Explain, taking into account buoyant forces.

Q12.16 Suppose the door that opens into a room makes an airtight but frictionless fit in its frame. Do you think you could open the door if the air pressure outside the room were standard atmospheric pressure and the air pressure inside the room was 1% greater? Explain.

Q12.17 At a certain depth in an incompressible liquid, the absolute pressure is At twice this depth, will the absolute pressure be equal to greater than or less than Justify your answer.

Q12.18 A piece of iron is glued to the top of a block of wood. When the block is placed in a bucket of water with the iron on top, the block floats. The block is now turned over so that the iron is submerged beneath the wood. Does the block float or sink? Does the water level in the bucket rise, drop, or stay the same? Explain.

Q12.19 You take an empty glass jar and push it into a tank of water with the open mouth of the jar downward, so that the air inside the jar is trapped and cannot get out. If you push the jar deeper into the water, does the buoyant force on the jar stay the same? If not, does it increase or decrease? Explain.

Q12.20 You are floating in a canoe in the middle of a swimming pool. Your friend is at the edge of the pool, carefully noting the level of the water on the side of the pool. You have a bowling ball with you in the canoe. If you carefully drop the bowling ball over the side of the canoe and it sinks to the bottom of the pool, does the water level in the pool rise or fall?

Q12.21 You are floating in a canoe in the middle of a swimming pool. A large bird flies up and lights on your shoulder. Does the water level in the pool rise or fall?

Q12.22 Two identical buckets are filled to the brim with water, but one of them has a piece of wood floating in it. Which bucket of water weighs more? Explain.

Q12.23 An ice cube floats in a glass of water. When the ice melts, will the water level in the glass rise, fall, or remain unchanged? Explain.

Q12.24 A helium-filled balloon is tied to a light string inside a car at rest. The other end of the string is attached to the floor of the car, so the balloon pulls the string vertical. The car's windows are closed. Now the car accelerates forward. Does the balloon move? If so, does it move forward or backward? Justify your reasoning with reference to buoyancy. (If you have a chance, try this experiment yourself—but with someone else driving!)

Q12.25 If the velocity at each point in space in steady-state fluid flow is constant, how can a fluid particle accelerate?

Q12.26 In a store-window vacuum cleaner display, a table-tennis ball is suspended in midair in a jet of air blown from the outlet hose of a tank-type vacuum cleaner. The ball bounces around a little but always moves back toward the center of the jet, even if the jet is tilted from the vertical. How does this behavior illustrate Bernoulli's equation?

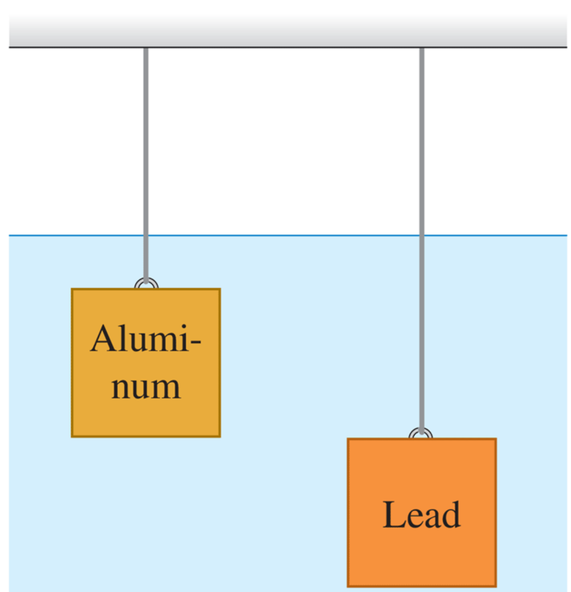
Q12.27 A tornado consists of a rapidly whirling air vortex. Why is the pressure always much lower in the center than at the outside? How does this condition account for the destructive power of a tornado?

Q12.28 Airports at high elevations have longer runways for takeoffs and landings than do airports at sea level. One reason is that aircraft engines develop less power in the thin air well above sea level. What is another reason?

Q12.29 When a smooth-flowing stream of water comes out of a faucet, it narrows as it falls. Explain.

Q12.30 Identical-size lead and aluminum cubes are suspended at different depths by two wires in a large vat of water (Fig. Q12.30). (a) Which cube experiences a greater buoyant force? (b) For which cube is the tension in the wire greater? (c) Which cube experiences a greater force on its lower face? (d) For which cube is the difference in pressure between the upper and lower faces greater?

Figure Q12.30



Exercises

Section 12.1 Gases, Liquids, and Density

12.1 •• On a part-time job, you are asked to bring a cylindrical iron rod of length 85.8 cm and diameter 2.85 cm from a storage room to a machinist. Will you need a cart? (To answer, calculate the weight of the rod.)

12.2 •• A cube 5.0 cm on each side is made of a metal alloy. After you drill a cylindrical hole 2.0 cm in diameter all the way through and perpendicular to one face, you find that the cube weighs 6.30 N. (a) What is the density of this metal? (b) What did the cube weigh before you drilled the hole in it?

12.3 • You purchase a rectangular piece of metal that has dimensions and mass 0.0158 kg. The seller tells you that the metal is gold. To check this, you compute the average density of the piece. What value do you get? Were you cheated?

12.4 • (a) What is the average density of the sun? (b) What is the average density of a neutron star that has the same mass as the sun but a radius of only 20.0 km?

12.5 •• A uniform lead sphere and a uniform aluminum sphere have the same mass. What is the ratio of the radius of the aluminum sphere to the radius of the lead sphere?

12.6 •• A hollow cylindrical copper pipe is 1.50 m long and has an outside diameter of 3.50 cm and an inside diameter of 2.50 cm. How much does it weigh?

Section 12.2 Pressure in a Fluid

12.7 •• **Oceans on Mars.** Scientists have found evidence that Mars may once have had an ocean 0.500 km deep. The acceleration due to gravity on Mars is (a) What would be the gauge pressure at the bottom of such an ocean, assuming it was freshwater? (b) To what depth would you need to go in the earth's ocean to experience the same gauge pressure?

12.8 •• **BIO** (a) Calculate the difference in blood pressure between the feet and top of the head for a person who is 1.65 m tall. (b) Consider a cylindrical segment of a blood vessel 2.00 cm long and 1.50 mm in diameter. What *additional* outward force would such a vessel need to withstand in the person's feet compared to a similar vessel in her head?

12.9 • **BIO** In intravenous feeding, a needle is inserted in a vein in the patient's arm and a tube leads from the needle to a reservoir of fluid located at height above the arm. The top of the reservoir is open to the air. If the gauge pressure inside the vein is 5980 Pa, what is the minimum value of that allows fluid to enter the vein? Assume the needle diameter is large enough that you can ignore the viscosity (see Section 12.6) of the fluid.

12.10 • A barrel contains a 0.120 m layer of oil floating on water that is 0.250 m deep. The density of the oil is (a) What is the gauge pressure at the oil–water interface? (b) What is the gauge pressure at the bottom of the barrel?

12.11 • A U-shaped tube with both arms open to the air has a 35.0 cm column of liquid of unknown density in its right arm. Beneath this liquid and not mixing with it is glycerin that extends into the left arm of the tube. The surface of the glycerin in the left arm is 12.0 cm below the surface of the unknown liquid in the right arm. What is the density of the unknown liquid?

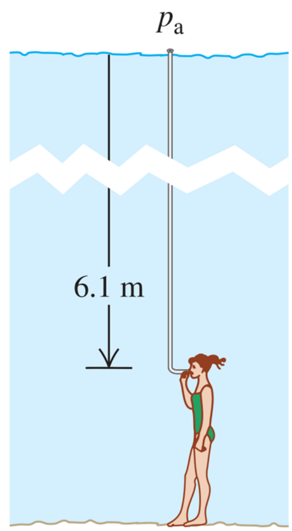
12.12 •• You are designing a diving bell to withstand the pressure of seawater at a depth of 250 m. (a) What is the gauge pressure at this depth? (You can ignore changes in the density of the water with depth.) (b) At this depth, what is the net force due to the water outside and the air inside the bell on a circular glass window 30.0 cm in diameter if the pressure inside the diving bell equals the pressure at the surface of the water? (Ignore the small variation of pressure over the surface of the window.)

12.13 •• BIO **Ear Damage from Diving.** If the force on the tympanic membrane (eardrum) increases by about 1.5 N above the force from atmospheric pressure, the membrane can be damaged. When you go scuba diving in the ocean, below what depth could damage to your eardrum start to occur? The eardrum is typically 8.2 mm in diameter. (Consult Table 12.1.)

12.14 •• The liquid in the open-tube manometer in Fig. 12.8a is mercury, and Atmospheric pressure is 980 millibars. What are (a) the absolute pressure at the bottom of the U-shaped tube; (b) the absolute pressure in the open tube at a depth of 4.00 cm below the free surface; (c) the absolute pressure of the gas in the container; (d) the gauge pressure of the gas in pascals?

12.15 • BIO There is a maximum depth at which a diver can breathe through a snorkel tube (Fig. E12.15) because as the depth increases, so does the pressure difference, which tends to collapse the diver's lungs. Since the snorkel connects the air in the lungs to the atmosphere at the surface, the pressure inside the lungs is atmospheric pressure. What is the external–internal pressure difference when the diver's lungs are at a depth of 6.1 m (about 20 ft)? Assume that the diver is in freshwater. (A scuba diver breathing from compressed air tanks can operate at greater depths than can a snorkeler, since the pressure of the air inside the scuba diver's lungs increases to match the external pressure of the water.)

Figure E12.15



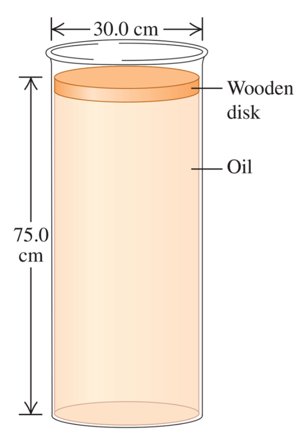
12.16 •• BIO The lower end of a long plastic straw is immersed below the surface of the water in a plastic cup. An average person sucking on the upper end of the straw can pull water into the straw to a vertical height of 1.1 m above the surface of the water in the cup. (a) What is the lowest gauge pressure that the average person can achieve inside his lungs? (b) Explain why your answer in part (a) is negative.

12.17 •• An electrical short cuts off all power to a submersible diving vehicle when it is 30 m below the surface of the ocean. The crew must push out a hatch of area and weight 300 N on the bottom to escape. If the pressure inside is 1.0 atm, what downward force must the crew exert on the hatch to open it?

12.18 •• A tall cylinder with a cross-sectional area is partially filled with mercury; the surface of the mercury is 8.00 cm above the bottom of the cylinder. Water is slowly poured in on top of the mercury, and the two fluids don't mix. What volume of water must be added to double the gauge pressure at the bottom of the cylinder?

12.19 •• A cylindrical disk of wood weighing 45.0 N and having a diameter of 30.0 cm floats on a cylinder of oil of density (Fig. E12.19). The cylinder of oil is 75.0 cm deep and has a diameter the same as that of the wood. (a) What is the gauge pressure at the top of the oil column? (b) Suppose now that someone puts a weight of 83.0 N on top of the wood, but no oil seeps around the edge of the wood. What is the *change* in pressure at (i) the bottom of the oil and (ii) halfway down in the oil?

Figure E12.19



12.20 • You are doing experiments from a research ship in the Atlantic Ocean. On a day when the atmospheric pressure at the surface of the water is , at what depth below the surface of the water is the absolute pressure (a) twice the pressure at the surface and (b) four times the pressure at the surface?

12.21 •• **Hydraulic Lift I.** For the hydraulic lift shown in Fig. 12.7, what must be the ratio of the diameter of the vessel at the car to the diameter of the vessel where the force is applied so that a 1520 kg car can be lifted with a force of just 125 N?

12.22 • **Hydraulic Lift II.** The piston of a hydraulic automobile lift is 0.30 m in diameter. What gauge pressure, in pascals, is required to lift a car with a mass of 1200 kg? Also express this pressure in atmospheres.

12.23 • Early barometers contained wine instead of mercury. On a day when the height of the liquid in the barometer tube is 750 mm for a mercury barometer, what is for a barometer that uses wine with density ?

Section 12.3 Buoyancy

12.24 • Estimate the fraction of your body's total volume that is above the surface of the water when you float in seawater (density). (a) Use this estimate and your weight to calculate the total volume of your body. (b) What is your average density? How does your average density compare to the density of seawater?

12.25 •• A 900 N athlete in very good condition does not float in a freshwater pool. To keep him from sinking to the bottom, an upward force of 20 N must be applied to him. What are his volume and his average density?

12.26 •• A rock has mass 1.80 kg. When the rock is suspended from the lower end of a string and totally immersed in water, the tension in the string is 12.8 N. What is the smallest density of a liquid in which the rock will float?

12.27 • A 950 kg cylindrical buoy floats vertically in seawater. The diameter of the buoy is 0.900 m. Calculate the additional distance the buoy will sink when an 80.0 kg man stands on top of it.

12.28 •• A slab of ice floats on a freshwater lake. What minimum volume must the slab have for a 65.0 kg woman to be able to stand on it without getting her feet wet?

12.29 •• An ore sample weighs 17.50 N in air. When the sample is suspended by a light cord and totally immersed in water, the tension in the cord is 11.20 N. Find the total volume and the density of the sample.

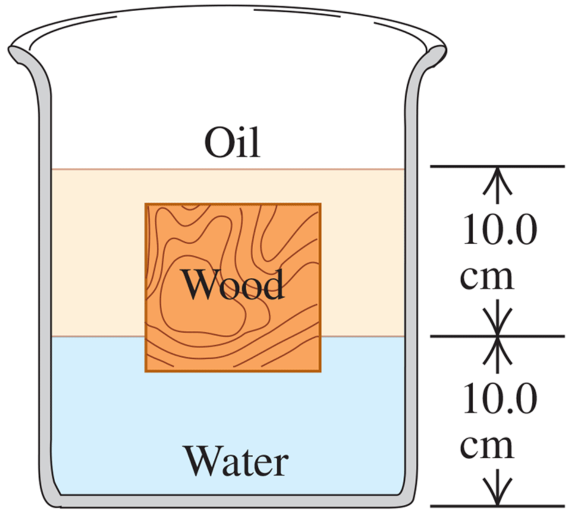
12.30 •• You are preparing some apparatus for a visit to a newly discovered planet Caasi having oceans of glycerin and a surface acceleration due to gravity of If your apparatus floats in the oceans on earth with 25.0% of its volume submerged, what percentage will be submerged in the glycerin oceans of Caasi?

12.31 •• A rock with density is suspended from the lower end of a light string. When the rock is in air, the tension in the string is 28.0 N. What is the tension in the string when the rock is totally immersed in a liquid with density ?

12.32 • A hollow plastic sphere is held below the surface of a freshwater lake by a cord anchored to the bottom of the lake. The sphere has a volume of and the tension in the cord is 1120 N. (a) Calculate the buoyant force exerted by the water on the sphere. (b) What is the mass of the sphere? (c) The cord breaks and the sphere rises to the surface. When the sphere comes to rest, what fraction of its volume will be submerged?

12.33 •• A cubical block of wood, 10.0 cm on a side, floats at the interface between oil and water with its lower surface 1.50 cm below the interface (Fig. E12.33). The density of the oil is (a) What is the gauge pressure at the upper face of the block? (b) What is the gauge pressure at the lower face of the block? (c) What are the mass and density of the block?

Figure E12.33



12.34 • A solid aluminum ingot weighs 89 N in air. (a) What is its volume? (b) The ingot is suspended from a rope and totally immersed in water. What is the tension in the rope (the *apparent* weight of the ingot in water)?

12.35 •• A rock is suspended by a light string. When the rock is in air, the tension in the string is 39.2 N. When the rock is totally immersed in water, the tension is 28.4 N. When the rock is totally immersed in an unknown liquid, the tension is 21.5 N. What is the density of the unknown liquid?

12.36 • A uniform plastic block floats in water with 30.0% of its volume above the surface of the water. The block is placed in a second liquid and floats with 20.0% of its volume above the surface of the liquid. What is the density of the second liquid?

12.37 •• A large plastic cylinder with mass 30.0 kg and density 370 is in the water of a lake. A light vertical cable runs between the bottom of the cylinder and the bottom of the lake and holds the cylinder so that 30.0% of its volume is above the surface of the water. What is the tension in the cable?

Section 12.4 Fluid Flow

12.38 •• Water runs into a fountain, filling all the pipes, at a steady rate of (a) How fast will it shoot out of a hole 4.50 cm in diameter? (b) At what speed will it shoot out if the diameter of the hole is three times as large?

12.39 •• A shower head has 20 circular openings, each with radius 1.0 mm. The shower head is connected to a pipe with radius 0.80 cm. If the speed of water in the pipe is what is its speed as it exits the shower-head openings?

12.40 • Water is flowing in a pipe with a varying cross-sectional area, and at all points the water completely fills the pipe. At point 1 the cross-sectional area of the pipe is and the magnitude of the fluid velocity is (a) What is the fluid speed at points in the pipe where the cross-sectional area is (a) and (b) (c) Calculate the volume of water discharged from the open end of the pipe in 1.00 hour.

12.41 • Water is flowing in a pipe with a circular cross section but with varying cross-sectional area, and at all points the water completely fills the pipe. (a) At one point in the pipe the radius is 0.150 m. What is the speed of the water at this point if water is flowing into this pipe at a steady rate of (b) At a second point in the pipe the water speed is What is the radius of the pipe at this point?

12.42 •• On another planet that you are exploring, a large tank is open to the atmosphere and contains ethanol. A horizontal pipe of cross-sectional area has one end inserted into the tank just above the bottom of the tank. The other end of the pipe is open to the atmosphere. The viscosity of the ethanol can be neglected. You measure the volume flow rate of the ethanol from the tank as a function of the depth of the ethanol in the tank. If you graph the volume flow rate squared as a function of your data lie close to a straight line that has slope . What is the value of the acceleration of a free-falling object at the surface of the planet?

12.43 •• A large, cylindrical water tank with diameter 3.00 m is on a platform 2.00 m above the ground. The vertical tank is open to the air and the depth of the water in the tank is 2.00 m. There is a hole with diameter 0.500 cm in the side of the tank just above the bottom of the tank. The hole is plugged with a cork. You remove the cork and collect in a bucket the water that flows out the hole. (a) When 1.00 gal of water flows out of the tank, what is the change in the height of the water in the tank? (b) How long does it take you to collect 1.00 gal of water in the bucket? Based on your answer in part (a), is it reasonable to ignore the change in the depth of the water in the tank as 1.00 gal of water flows out?

Section 12.5 Bernoulli's Equation

12.44 • A soft drink (mostly water) flows in a pipe at a beverage plant with a mass flow rate that would fill 220 0.355-L cans per minute. At point 2 in the pipe, the gauge pressure is 152 kPa and the cross-sectional area is At point 1, 1.35 m above point 2, the cross-sectional area is Find the (a) mass flow rate; (b) volume flow rate; (c) flow speeds at points 1 and 2; (d) gauge pressure at point 1.

12.45 •• A sealed tank containing seawater to a height of 11.0 m also contains air above the water at a gauge pressure of 3.00 atm. Water flows out from the bottom through a small hole. How fast is this water moving?

12.46 •• BIO **Artery Blockage.** A medical technician is trying to determine what percentage of a patient's artery is blocked by plaque. To do this, she measures the blood pressure just before the region of blockage and finds that it is while in the region of blockage it is Furthermore, she knows that blood flowing through the normal artery just before the point of blockage is traveling at 30.0 , and the specific gravity of this patient's blood is 1.06. What percentage of the cross-sectional area of the patient's artery is blocked by the plaque?

12.47 • What gauge pressure is required in the city water mains for a stream from a fire hose connected to the mains to reach a vertical height of 15.0 m? (Assume that the mains have a much larger diameter than the fire hose.)

12.48 • A small circular hole 6.00 mm in diameter is cut in the side of a large water tank, 14.0 m below the water level in the tank. The top of the tank is open to the air. Find (a) the speed of efflux of the water and (b) the volume discharged per second.

12.49 • At a certain point in a horizontal pipeline, the water's speed is and the gauge pressure is Find the gauge pressure at a second point in the line if the cross-sectional area at the second point is twice that at the first.

12.50 •• At one point in a pipeline the water's speed is and the gauge pressure is Find the gauge pressure at a second point in the line, 11.0 m lower than the first, if the pipe diameter at the second point is twice that at the first.

12.51 •• A golf course sprinkler system discharges water from a horizontal pipe at the rate of At one point in the pipe, where the radius is 4.00 cm, the water's absolute pressure is At a second point in the pipe, the water passes through a constriction where the radius is 2.00 cm. What is the water's absolute pressure as it flows through this constriction?

Section 12.6 Viscosity and Turbulence

12.52 • A pressure difference of is required to maintain a volume flow rate of for a viscous fluid flowing through a section of cylindrical pipe that has radius 0.210 m. What pressure difference is required to maintain the same volume flow rate if the radius of the pipe is decreased to 0.0700 m?

12.53 •• BIO **Clogged Artery.** Viscous blood is flowing through an artery partially clogged by cholesterol. A surgeon wants to remove enough of the cholesterol to double the flow rate of blood through this artery. If the original diameter of the artery is what should be the new diameter (in terms of D) to accomplish this for the same pressure gradient?

Problems

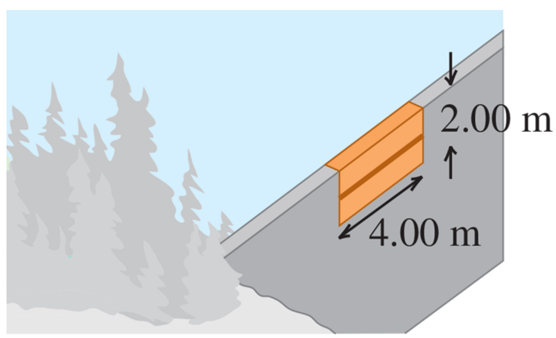
12.54 •• CP The deepest point known in any of the earth's oceans is in the Marianas Trench, 10.92 km deep. (a) Assuming water is incompressible, what is the pressure at this depth? Use the density of seawater. (b) The actual pressure is your calculated value will be less because the density actually varies with depth. Using the compressibility of water and the actual pressure, find the density of the water at the bottom of the Marianas Trench. What is the percent change in the density of the water?

12.55 ••• CALC A swimming pool is 5.0 m long, 4.0 m wide, and 3.0 m deep. Compute the force exerted by the water against (a) the bottom and (b) either end. (*Hint:* Calculate the force on a thin, horizontal strip at a depth and integrate this over the end of the pool.) Do not include the force due to air pressure.

12.56 •• A rock is suspended from the lower end of a light string. When the rock is totally immersed in water, the tension in the string is 1.20 N. When the rock is totally immersed in ethanol, the tension in the string is 1.60 N. Use the densities in Table 12.1 to calculate the weight of the rock.

12.57 ••• **CP CALC** The upper edge of a gate in a dam runs along the water surface. The gate is 2.00 m high and 4.00 m wide and is hinged along a horizontal line through its center (Fig. P12.57). Calculate the torque about the hinge arising from the force due to the water. (*Hint:* Use a procedure similar to that used in Problem 12.55; calculate the torque on a thin, horizontal strip at a depth and integrate this over the gate.)

Figure P12.57



12.58 ••• A large tank has a 40.0-cm-deep layer of water floating on top of a 60.0-cm-deep layer of glycerin. A dense wooden cube has a side length of 2.00 cm and a mass of . When the cube floats in the tank, what fraction of its volume is below the surface of the glycerin?

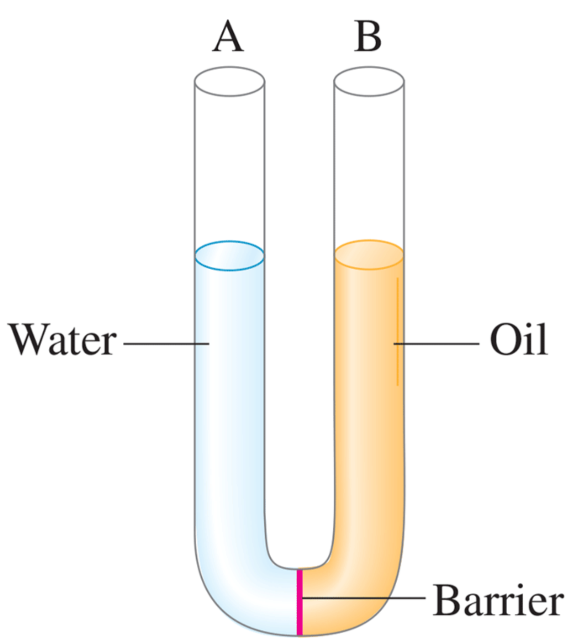
12.59 •• A block of plastic in the shape of a rectangular solid that has height 8.00 cm and area for its top and bottom surfaces is floating in water. You place coins on the top surface of the block (at the center, so the top surface of the block remains horizontal). By measuring the height of the block above the surface of the water, you can determine the height below the surface. You measure *h* for various values of the total mass of the coins that you have placed on the block. You plot versus and find that your data lie close to a straight line that has slope 0.0390 m/kg and intercept 0.0312 m. What is the mass of the block?

12.60 •• **Ballooning on Mars.** It has been proposed that we could explore Mars using inflated balloons to hover just above the surface. The buoyancy of the atmosphere would keep the balloon aloft. The density of the Martian atmosphere is (although this varies with temperature). Suppose we construct these balloons of a thin but tough plastic having a density such that each square meter has a mass of 5.00 g. We inflate them with a very light gas whose mass we can ignore. (a) What should be the radius and mass of these balloons so they just hover above the surface of Mars? (b) If we released one of the balloons from part (a) on earth, where the atmospheric density is what would be its initial acceleration assuming it was the same size as on Mars? Would it go up or down? (c) If on Mars these balloons have five times the radius found in part (a), how heavy an instrument package could they carry?

12.61 •• A 0.180 kg cube of ice (frozen water) is floating in glycerin. The gylcerin is in a tall cylinder that has inside radius 3.50 cm. The level of the glycerin is well below the top of the cylinder. If the ice completely melts, by what distance does the height of liquid in the cylinder change? Does the level of liquid rise or fall? That is, is the surface of the water above or below the original level of the glycerin before the ice melted?

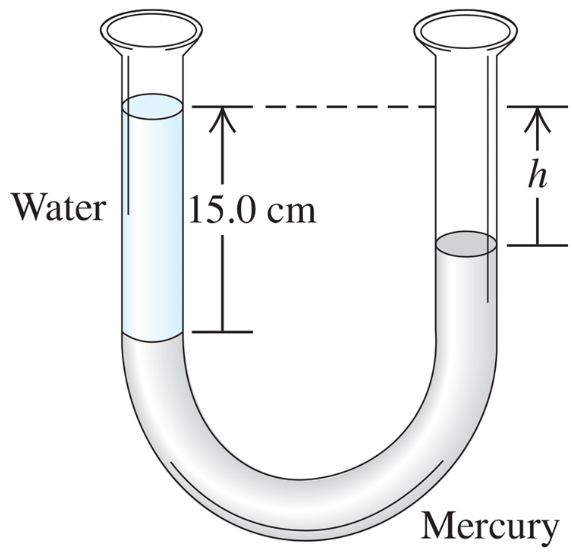
12.62 •• A narrow, U-shaped glass tube with open ends is filled with 25.0 cm of oil (of specific gravity 0.80) and 25.0 cm of water on opposite sides, with a barrier separating the liquids (Fig. P12.62). (a) Assume that the two liquids do not mix, and find the final heights of the columns of liquid in each side of the tube after the barrier is removed. (b) For the following cases, arrive at your answer by simple physical reasoning, not by calculations: (i) What would be the height on each side if the oil and water had equal densities? (ii) What would the heights be if the oil's density were much less than that of water?

Figure P12.62



12.63 • A U-shaped tube open to the air at both ends contains some mercury. A quantity of water is carefully poured into the left arm of the U-shaped tube until the vertical height of the water column is 15.0 cm (Fig. P12.63). (a) What is the gauge pressure at the water–mercury interface? (b) Calculate the vertical distance from the top of the mercury in the right-hand arm of the tube to the top of the water in the left-hand arm.

Figure P12.63



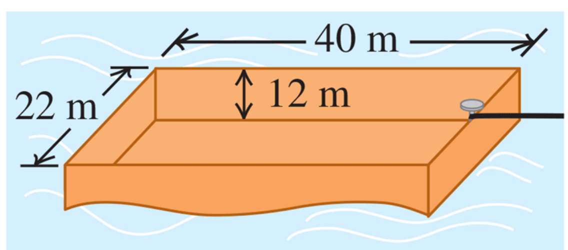
12.64 •• CALC **The Great Molasses Flood.** On the afternoon of January 15, 1919, an unusually warm day in Boston, a 17.7-m-high, 27.4-m-diameter cylindrical metal tank used for storing molasses ruptured. Molasses flooded into the streets in a 5-m-deep stream, killing pedestrians and horses and knocking down buildings. The molasses had a density of If the tank was full before the accident, what was the total outward force the molasses exerted on its sides? (*Hint:* Consider the outward force on a circular ring of the tank wall of width and at a depth below the surface. Integrate to find the total outward force. Assume that before the tank ruptured, the pressure at the surface of the molasses was equal to the air pressure outside the tank.)

12.65 •• A large, 40.0 kg cubical block of wood with uniform density is floating in a freshwater lake with 20.0% of its volume above the surface of the water. You want to load bricks onto the floating block and then push it horizontally through the water to an island where you are building an outdoor grill. (a) What is the volume of the block? (b) What is the maximum mass of bricks that you can place on the block without causing it to sink below the water surface?

12.66 ••• A hot-air balloon has a volume of The balloon fabric (the envelope) weighs 900 N. The basket with gear and full propane tanks weighs 1700 N. If the balloon can barely lift an additional 3200 N of passengers, breakfast, and champagne when the outside air density is what is the average density of the heated gases in the envelope?

12.67 • An open barge has the dimensions shown in Fig. P12.67. If the barge is made out of 4.0-cm-thick steel plate on each of its four sides and its bottom, what mass of coal can the barge carry in freshwater without sinking? Is there enough room in the barge to hold this amount of coal? (The density of coal is about)

Figure P12.67



12.68 ••• A piece of wood is 0.600 m long, 0.250 m wide, and 0.080 m thick. Its density is What volume of lead must be fastened underneath it to sink the wood in calm water so that its top is just even with the water level? What is the mass of this volume of lead?

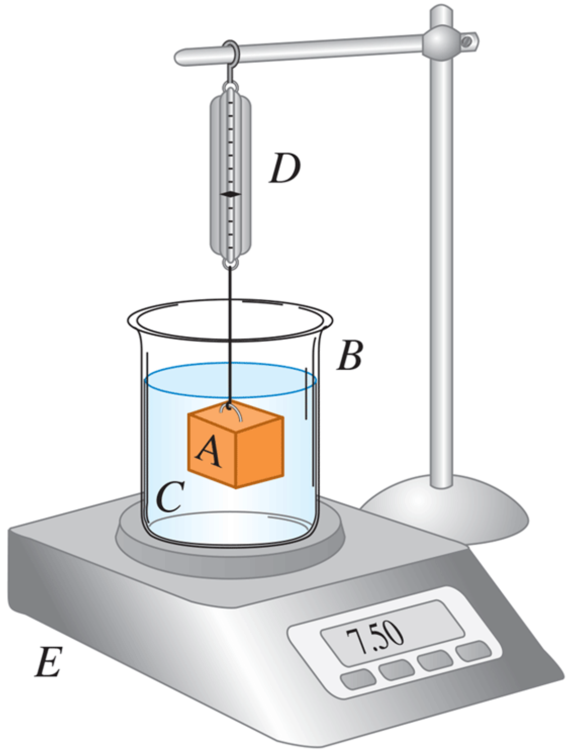
12.69 •• A gallon of milk in a full plastic jug is sitting on the edge of your kitchen table. Estimate the vertical distance between the top surface of the milk and the bottom of the jug. Also estimate the distance from the tabletop to the floor. You punch a small hole in the side of the jug just above the bottom of the jug, and milk flows out the hole. When the milk first starts to flow out the hole, what horizontal distance does it travel before reaching the floor? Assume the milk is in free fall after it has passed through the hole, and neglect the viscosity of the milk.

12.70 •• Ethanol is flowing in a pipe and at all points completely fills the pipe. At point in the pipe the gauge pressure is and the cross-sectional area of the pipe is . The other end of the pipe (point) is open to the air, has cross-sectional area , and is at a vertical height of 3.00 m above point . What must the gauge pressure be at if the volume flow rate out of the pipe at point is ?

12.71 •• CP A firehose must be able to shoot water to the top of a building 28.0 m tall when aimed straight up. Water enters this hose at a steady rate of and shoots out of a round nozzle. Neglect air resistance. (a) What is the maximum diameter this nozzle can have? (b) If the only nozzle available has a diameter twice as great, what is the highest point the water can reach?

12.72 •• Block in Fig. P12.72 hangs by a cord from spring balance and is submerged in a liquid contained in beaker The mass of the beaker is 1.00 kg; the mass of the liquid is 1.80 kg. Balance reads 3.50 kg, and balance reads 7.50 kg. The volume of block is (a) What is the density of the liquid? (b) What will each balance read if block is pulled up out of the liquid?

Figure P12.72



12.73 ••• **CALC** A closed and elevated vertical cylindrical tank with diameter 2.00 m contains water to a depth of 0.800 m. A worker accidently pokes a circular hole with diameter 0.0200 m in the bottom of the tank. As the water drains from the tank, compressed air above the water in the tank maintains a gauge pressure of at the surface of the water. Ignore any effects of viscosity. (a) Just after the hole is made, what is the speed of the water as it emerges from the hole? What is the ratio of this speed to the efflux speed if the top of the tank is open to the air? (b) How much time does it take for all the water to drain from the tank? What is the ratio of this time to the time it takes for the tank to drain if the top of the tank is open to the air?

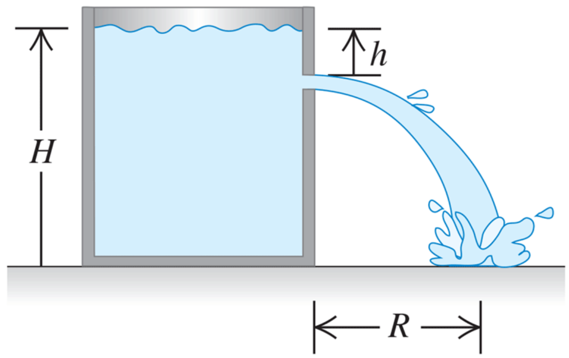
12.74 •• A plastic ball has radius 12.0 cm and floats in water with 24.0% of its volume submerged. (a) What force must you apply to the ball to hold it at rest totally below the surface of the water? (b) If you let go of the ball, what is its acceleration the instant you release it?

12.75 ••• A cubical block of density and with sides of length floats in a liquid of greater density (a) What fraction of the block's volume is above the surface of the liquid? (b) The liquid is denser than water and does not mix with it. If water is poured on the surface of that liquid, how deep must the water layer be so that the water surface just rises to the top of the block? Express your answer in terms of and (c) Find the depth of the water layer in part (b) if the liquid is mercury, the block is made of iron, and cm.

12.76 •• A barge is in a rectangular lock on a freshwater river. The lock is 60.0 m long and 20.0 m wide, and the steel doors on each end are closed. With the barge floating in the lock, a load of scrap metal is put onto the barge. The metal has density (a) When the load of scrap metal, initially on the bank, is placed onto the barge, what vertical distance does the water in the lock rise? (b) The scrap metal is now pushed overboard into the water. Does the water level in the lock rise, fall, or remain the same? If it rises or falls, by what vertical distance does it change?

12.77 • **CP** Water stands at a depth in a large, open tank whose side walls are vertical (Fig. P12.77). A hole is made in one of the walls at a depth below the water surface. (a) At what distance from the foot of the wall does the emerging stream strike the floor? (b) How far above the bottom of the tank could a second hole be cut so that the stream emerging from it could have the same range as for the first hole?

Figure P12.77



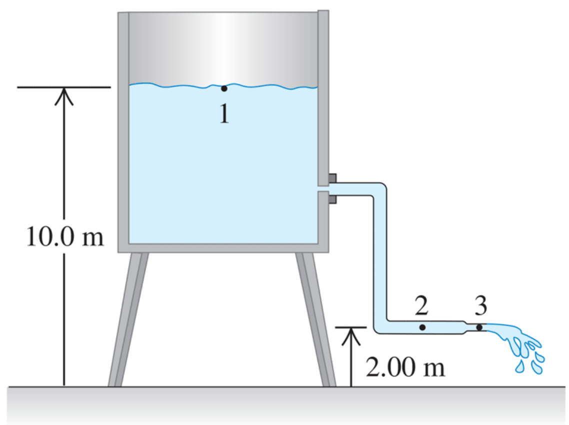
12.78 •• Your uncle is in the below-deck galley of his boat while you are spear fishing in the water nearby. An errant spear makes a small hole in the boat's hull, and water starts to leak into the galley. (a) If the hole is 0.900 m below the water surface and has area , how long does it take 10.0 L of water to leak into the boat? (b) Do you need to take into consideration the fact that the boat sinks lower into the water as water leaks in?

12.79 •• CP You hold a hose at waist height and spray water horizontally with it. The hose nozzle has a diameter of 1.80 cm, and the water splashes on the ground a distance of 0.950 m horizontally from the nozzle. If you constrict the nozzle to a diameter of 0.750 cm, how far from the nozzle, horizontally, will the water travel before it hits the ground? (Ignore air resistance.)

12.80 ••• A cylindrical bucket, open at the top, is 25.0 cm high and 10.0 cm in diameter. A circular hole with a cross-sectional area is cut in the center of the bottom of the bucket. Water flows into the bucket from a tube above it at the rate of How high will the water in the bucket rise?

12.81 • Water flows steadily from an open tank as in Fig. P12.81. The elevation of point 1 is 10.0 m, and the elevation of points 2 and 3 is 2.00 m. The cross-sectional area at point 2 is at point 3 it is The area of the tank is very large compared with the cross-sectional area of the pipe. Assuming that Bernoulli's equation applies, compute (a) the discharge rate in cubic meters per second and (b) the gauge pressure at point 2.

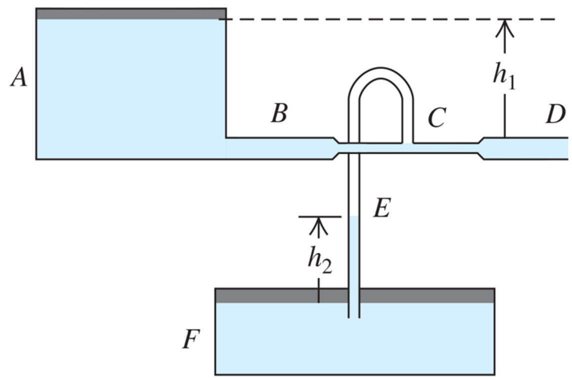
Figure P12.81



12.82 •• CP In 1993 the radius of Hurricane Emily was about 350 km. The wind speed near the center ("eye") of the hurricane, whose radius was about 30 km, reached about As air swirled in from the rim of the hurricane toward the eye, its angular momentum remained roughly constant. Estimate (a) the wind speed at the rim of the hurricane; (b) the pressure difference at the earth's surface between the eye and the rim. (*Hint:* See Table 12.1.) Where is the pressure greater? (c) If the kinetic energy of the swirling air in the eye could be converted completely to gravitational potential energy, how high would the air go? (d) In fact, the air in the eye is lifted to heights of several kilometers. How can you reconcile this with your answer to part (c)?

12.83 •• Two very large open tanks and (Fig. P12.83) contain the same liquid. A horizontal pipe *BCD*, having a constriction at and open to the air at leads out of the bottom of tank , and a vertical pipe opens into the constriction at and dips into the liquid in tank Assume streamline flow and no viscosity. If the cross-sectional area at is one-half the area at and if is a distance below the level of the liquid in , to what height will liquid rise in pipe Express your answer in terms of

Figure P12.83

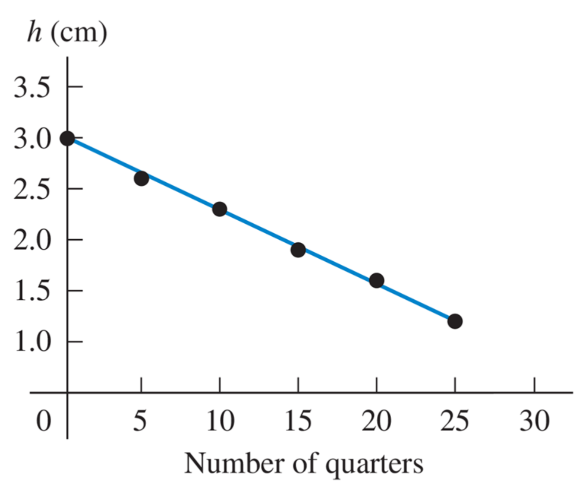


12.84 • A liquid flowing from a vertical pipe has a definite shape as it flows from the pipe. To get the equation for this shape, assume that the liquid is in free fall once it leaves the pipe. Just as it leaves the pipe, the liquid has speed and the radius of the stream of liquid is (a) Find an equation for the speed of the liquid as a function of the distance it has fallen. Combining this with the equation of continuity, find an expression for the radius of the stream as a function of (b) If water flows out of a vertical pipe at a speed of how far below the outlet will the radius be one-half the original radius of the stream?

12.85 •• **DATA** The density values in Table 12.1 are listed in increasing order. A chemistry student notices that the first four chemical elements that are included are also listed in order of increasing atomic mass. (a) See whether there is a simple relationship between density and atomic mass by plotting a graph of density (in) versus atomic mass for all eight elements in that table. (See Appendix D for their atomic masses in grams per mole.) (b) Can you draw a straight line or simple curve through the points to find a "simple" relationship? (c) Explain why "More massive atoms result in more dense solids" does not tell the whole story.

12.86 •• **DATA** You have a bucket containing an unknown liquid. You also have a cube-shaped wooden block that you measure to be 8.0 cm on a side, but you don't know the mass or density of the block. To find the density of the liquid, you perform an experiment. First you place the wooden block in the liquid and measure the height of the top of the floating block above the liquid surface. Then you stack various numbers of U.S. quarter-dollar coins onto the block and measure the new value of . The straight line that gives the best fit to the data you have collected is shown in Fig. P12.86. Find the mass of one quarter (see www.usmint.gov for quarters dated 2012). Use this information and the slope and intercept of the straight-line fit to your data to calculate (a) the density of the liquid (in) and (b) the mass of the block (in kg).

Figure P12.86



12.87 ••• **DATA** The Environmental Protection Agency is investigating an abandoned chemical plant. A large, closed cylindrical tank contains an unknown liquid. You must determine the liquid's density and the height of the liquid in the tank (the vertical distance from the surface of the liquid to the bottom of the tank). To maintain various values of the gauge pressure in the air that is above the liquid in the tank, you can use compressed air. You make a small hole at the bottom of the side of the tank, which is on a concrete platform—so the hole is 50.0 cm above the ground. The table gives your measurements of the horizontal distance that the initially horizontal stream of liquid pouring out of the tank travels before it strikes the ground and the gauge pressure of the air in the tank.

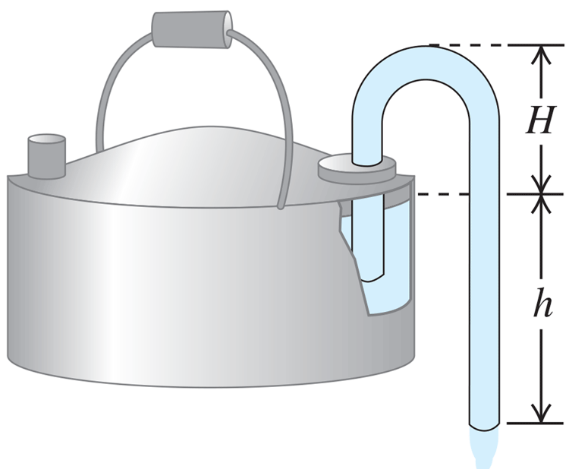
p_g (atm)	0.50	1.00	2.00	3.00	4.00
R (m)	5.4	6.5	8.2	9.7	10.9

(a) Graph as a function of . Explain why the data points fall close to a straight line. Find the slope and intercept of that line. (b) Use the slope and intercept found in part (a) to calculate the height (in meters) of the liquid in the tank and the density of the liquid (in). Use . Assume that the liquid is nonviscous and that the hole is small enough compared to the tank's diameter so that the change in during the measurements is very small.

Challenge Problem

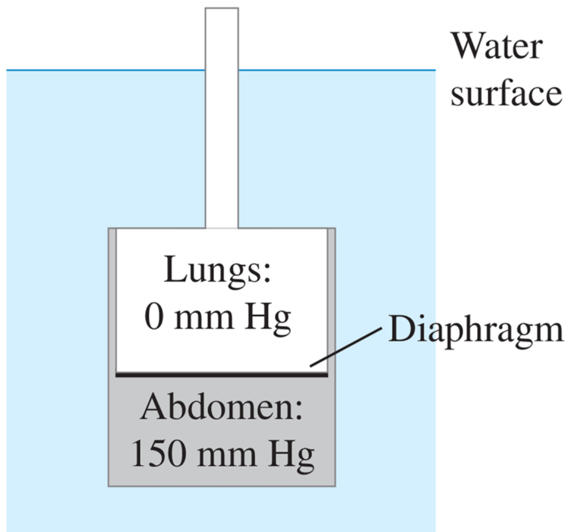
12.88 ••• A *siphon* (Fig. P12.88) is a convenient device for removing liquids from containers. To establish the flow, the tube must be initially filled with fluid. Let the fluid have density and let the atmospheric pressure be Assume that the cross-sectional area of the tube is the same at all points along it. (a) If the lower end of the siphon is at a distance below the surface of the liquid in the container, what is the speed of the fluid as it flows out the lower end of the siphon? (Assume that the container has a very large diameter, and ignore any effects of viscosity.) (b) A curious feature of a siphon is that the fluid initially flows “uphill.” What is the greatest height that the high point of the tube can have if flow is still to occur?

Figure P12.88



MCAT-Style Passage Problems

BIO **Elephants Under Pressure.** An elephant can swim or walk with its chest several meters underwater while the animal breathes through its trunk, which remains above the water surface and acts like a snorkel. The elephant’s tissues are at an increased pressure due to the surrounding water, but the lungs are at atmospheric pressure because they are connected to the air through the trunk. The figure shows the gauge pressures in an elephant’s lungs and abdomen when the elephant’s chest is submerged to a particular depth in a lake. In this situation, the elephant's diaphragm, which separates the lungs from the abdomen, must sustain the difference in pressure between the lungs and the abdomen. The diaphragm of an elephant is typically 3.0 cm thick and 120 cm in diameter. (See "Why Doesn't the Elephant Have a Pleural Space?" by John B. West, *Physiology*, Vol. 17:47–50, April 1, 2002.)



12.89 For the situation shown, the tissues in the elephant's abdomen are at a gauge pressure of 150 mm Hg. This pressure corresponds to what distance below the surface of a lake? (a) 1.5 m; (b) 2.0 m; (c) 3.0 m; (d) 15 m.

12.90 The maximum force the muscles of the diaphragm can exert is 24,000 N. What maximum pressure difference can the diaphragm withstand? (a) 160 mm Hg; (b) 760 mm Hg; (c) 920 mm Hg; (d) 5000 mm Hg.

12.91 How does the force the diaphragm experiences due to the difference in pressure between the lungs and abdomen depend on the abdomen's distance below the water surface? The force (a) increases linearly with distance; (b) increases as distance squared; (c) increases as distance cubed; (d) increases exponentially with distance.

12.92 If the elephant were to snorkel in salt water, which is more dense than freshwater, would the maximum depth at which it could snorkel be different from that in freshwater? (a) Yes—that depth would increase, because the pressure would be lower at a given depth in salt water than in freshwater; (b) yes—that depth would decrease, because the pressure would be higher at a given depth in salt water than in freshwater; (c) no, because pressure differences within the submerged elephant depend on only the density of air, not the density of the water; (d) no, because the buoyant force on the elephant would be the same in both cases.

Answers: Fluid Mechanics

Chapter Opening Question ?

(v) The ratio of mass to volume is density. The flesh of both the wrasse and the ray is denser than seawater, but a wrasse has a gas-filled body cavity called a swimbladder. Hence the *average* density of the wrasse's body is the same as that of seawater, and the fish neither sinks nor rises. Rays have no such cavity, so they must swim continuously to avoid sinking: Their fins provide lift, much like the wings of a bird or airplane (see Section 12.5).

Test Your Understanding

12.1 **(ii), (iv), (i) and (iii) (tie), (v)** In each case the average density equals the mass divided by the volume:

i.
ii.
iii.
iv.
v.

Note that compared to object (i), object (ii) has double the mass but the same volume and so has double the average density. Object (iii) has double the mass and double the volume of object (i), so (i) and (iii) have the same average density. Finally, object (v) has the same mass as object (iv) but double the volume, so (v) has half the average density of (iv).

12.2 (ii) From Eq. (12.9), the pressure outside the barometer is equal to the product When the barometer is taken out of the refrigerator, the density decreases while the height of the mercury column remains the same. Hence the air pressure must be lower outdoors than inside the refrigerator.

12.3 (i) Consider the water, the statue, and the container together as a system; the total weight of the system does not depend on whether the statue is immersed. The total supporting force, including the tension and the upward force of the scale on the container (equal to the scale reading), is the same in both cases. But we saw in Example 12.5 that decreases by 7.84 N when the statue is immersed, so the scale reading must *increase* by 7.84 N. An alternative viewpoint is that the water exerts an upward buoyant force of 7.84 N on the statue, so the statue must exert an equal downward force on the water, making the scale reading 7.84 N greater than the weight of water and container.

12.4 (ii) A highway that narrows from three lanes to one is like a pipe whose cross-sectional area narrows to one-third of its value. If cars behaved like the molecules of an incompressible fluid, then as the cars encountered the one-lane section, the spacing between cars (the “density”) would stay the same but the cars would triple their speed. This would keep the “volume flow rate” (number of cars per second passing a point on the highway) the same. In real life cars behave like the molecules of a *compressible* fluid: They end up packed closer (the “density” increases) and fewer cars per second pass a point on the highway (the “volume flow rate” decreases).

12.5 **(ii)** Newton’s second law tells us that an object accelerates (its velocity changes) in response to a net force. In fluid flow, a pressure difference between two points means that fluid particles moving between those two points experience a force, and this force causes the fluid particles to accelerate and change speed.

12.6 **(iv)** The required pressure is proportional to where is the inside radius of the needle (half the inside diameter). With the smaller-diameter needle, the pressure is greater by a factor of

Key Example Variation Problems

VP12.4.1 Pa

VP12.4.2 **a.** Pa

b. Pa

VP12.4.3 81.8 cm

VP12.4.4 **a.** 22.9 cm

b. Pa

c. 16.4 cm

VP12.5.1 (a) 8.45 N (b) (i) , sink; (ii) 7.35 N, sink; (iii) 9.26 N, rise

VP12.5.2 **a.** 11.8 N

b. 41.2 N

c.

VP12.5.3

VP12.5.4 **a.**

b. ρ

VP12.9.1 **a.**

b. Pa

VP12.9.2 **a.**

b. Pa

VP12.9.3 **a.** 8.3 cm

b.

VP12.9.4

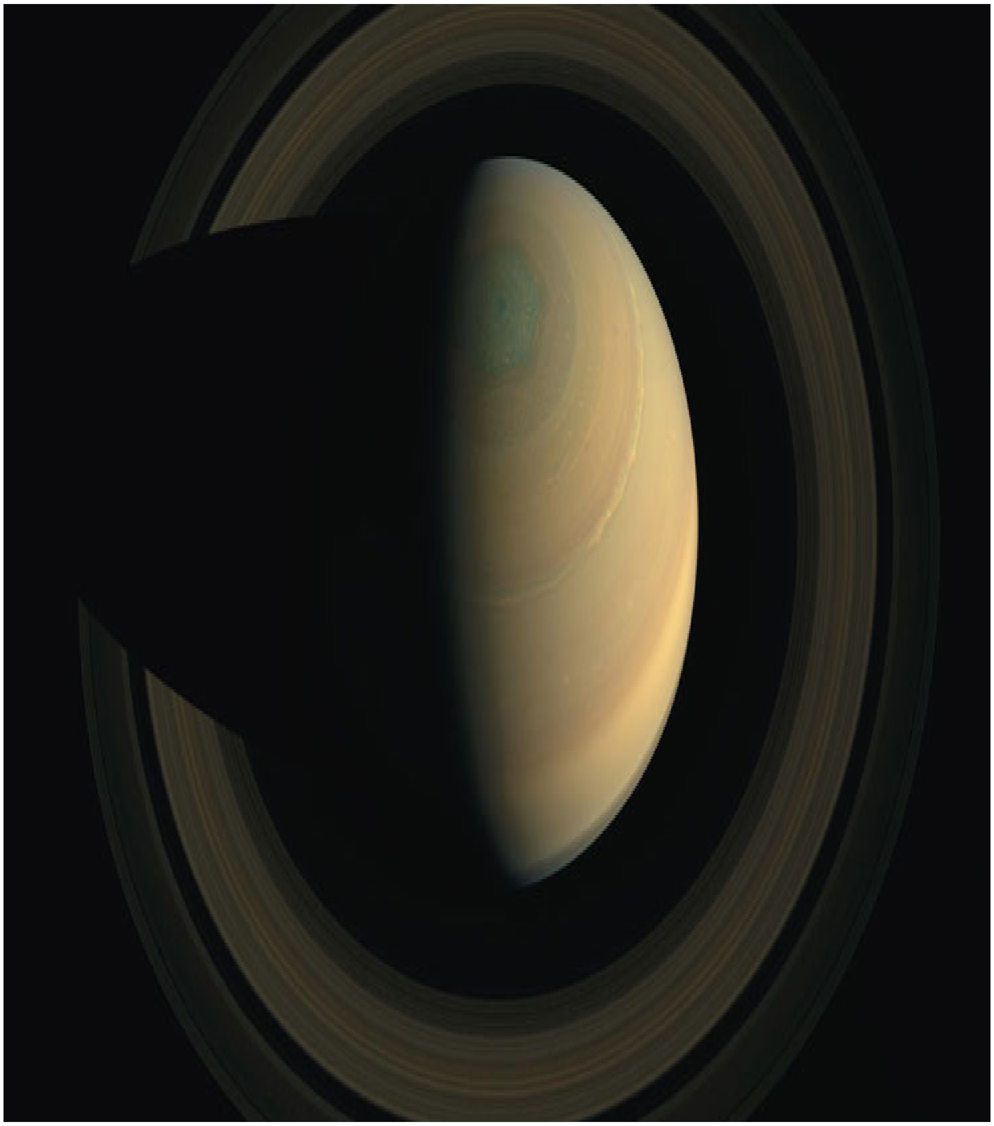
Bridging Problem

(a)

(b)

(c)

Chapter 13
Gravitation



?◰ The rings of Saturn are made of countless individual orbiting particles. Compared with a ring particle that orbits far from Saturn, does a ring particle close to Saturn orbit with (i) the same speed and greater acceleration; (ii) a faster speed and the same acceleration; (iii) a slower speed and the same acceleration; (iv) a faster speed and greater acceleration; or (v) none of these?

Learning Outcomes

In this chapter, you'll learn...

13.1 How to calculate the gravitational forces that any two objects exert on each other.

13.2 How to relate the weight of an object to the general expression for gravitational force.

13.3 How to use and interpret the generalized expression for gravitational potential energy.

13.4 How to calculate the speed, orbital period, and total mechanical energy of a satellite in a circular orbit.

13.5 How to apply and interpret the three laws that describe the motion of planets.

13.6 Why the gravitational force exerted by a spherically symmetric planet is the same as if all of the planet's mass were concentrated at its center.

13.7 How the earth's rotation affects the apparent weight of an object at different latitudes.

13.8 What black holes are, how to calculate their properties, and how astronomers discover them.

You'll need to review...

3.3 Projectile motion.

4.4 Weight.

5.2 Apparent weightlessness.

5.4 Force and acceleration in uniform circular motion.

5.5 The fundamental forces of nature.

7.1 Gravitational potential energy and total mechanical energy conservation.

7.4 Force and potential energy.

Some of the earliest investigations in physical science started with questions that people asked about the night sky. Why doesn't the moon fall to earth? Why do the planets move across the sky? Why doesn't the earth fly off into space rather than remaining in orbit around the sun? The study of gravitation provides the answers to these and many related questions.

As we remarked in Chapter 5, gravitation is one of the four classes of interactions found in nature, and it was the earliest of the four to be studied extensively. Newton discovered in the 17th century that the same interaction that makes an apple fall out of a tree also keeps the planets in their orbits around the sun. This was the beginning of *celestial mechanics,* the study of the dynamics of objects in space. Today, our knowledge of celestial mechanics allows us to determine how to put a satellite into any desired orbit around the earth or to choose just the right trajectory to send a spacecraft to another planet.

In this chapter you'll learn the basic law that governs gravitational interactions. This law is *universal:* Gravity acts in the same fundamental way between the earth and your body, between the sun and a planet, and between a planet and one of its moons. We'll apply the law of gravitation to phenomena such as the variation of weight with altitude, the orbits of satellites around the earth, and the orbits of planets around the sun.

13.1 Newton’s Law of Gravitation

The gravitational attraction that’s most familiar to you is your *weight,* the force that attracts you toward the earth. By studying the motions of the moon and planets, Newton discovered a fundamental **law of gravitation** that describes the gravitational attraction between *any* two objects. Newton published this law in 1687 along with his three laws of motion. In modern language, it says

> ### Newton's Law of Gravitation
> Every particle of matter in the universe attracts every other particle with a force that is directly proportional to the product of the masses of the particles and inversely proportional to the square of the distance between them.

Figure 13.1 depicts this law, which we can express as an equation:

(13.1)

Newton’s **law of gravitation:** Magnitude of attractive gravitational force between any two particles

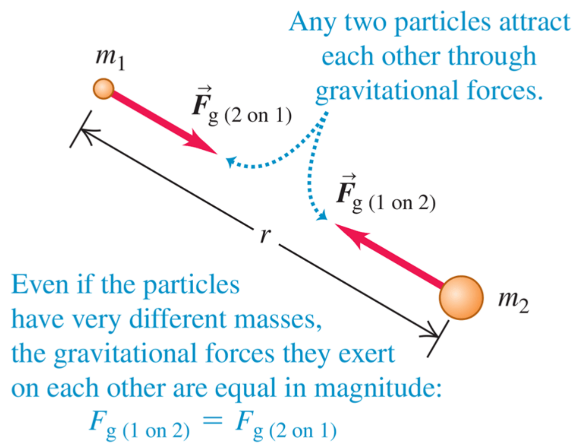
Gravitational constant (same for any two particles)

Masses of particles

$$F_g = \frac{Gm_1m_2}{r^2} \qquad (13.1)$$

Distance between particles

Figure 13.1



The gravitational forces that two particles of masses m_1 and m_2 exert on each other.

The **gravitational constant** G in Eq. (13.1) is a fundamental physical constant that has the same value for *any* two particles. We'll see shortly what the value of G is and how this value is measured.

Equation (13.1) tells us that the gravitational force between two particles decreases with increasing distance r: If the distance is doubled, the force is only one-fourth as great, and so on. Although many of the stars in the night sky are far more massive than the sun, they are so far away that their gravitational force on the earth is negligibly small.

CAUTION Don't confuse g and G The symbols g and G are similar, but they represent two very different gravitational quantities. Lowercase g is the acceleration due to gravity, which relates the weight w of an object to its mass m : $w = mg$. The value of g is different at different locations on the earth's surface and on the surfaces of other planets. By contrast, capital G relates the gravitational force between any two objects to their masses and the distance between them. We call G a *universal* constant because it has the same value for any two objects, no matter where in space they are located. We'll soon see how the values of g and G are related.

Gravitational forces always act along the line joining the two particles and form an action–reaction pair. Even when the masses of the particles are different, the two interaction forces have equal magnitude (Fig. 13.1). The attractive force that your body exerts on the earth has the same magnitude as the force that the earth exerts on you. When you fall from a diving board into a swimming pool, the entire earth rises up to meet you! (You don't notice this because the earth's mass is greater than yours by a factor of about 10^{23}. Hence the earth's acceleration is only 10^{-23} as great as yours.)

Gravitation and Spherically Symmetric Objects

We have stated the law of gravitation in terms of the interaction between two *particles*. It turns out that the gravitational interaction of any two objects that have *spherically symmetric* mass distributions (such as solid spheres or spherical shells) is the same as though we concentrated all the mass of each at its center, as in Fig. 13.2. Thus, if we model the earth as a spherically symmetric object with mass m_{E}, the force it exerts on a particle or on a spherically symmetric object with mass m, at a distance r between centers, is

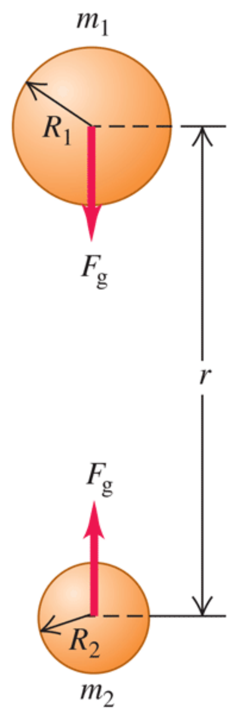
(13.2)

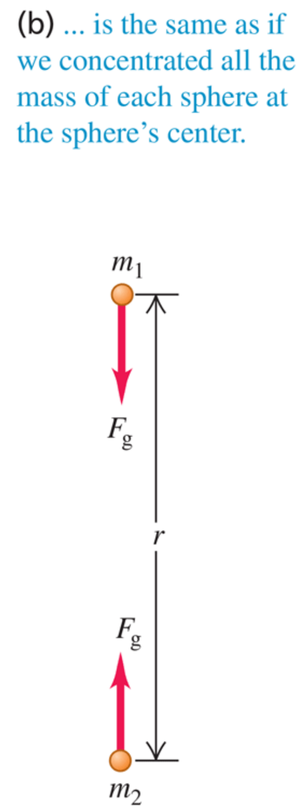
$$F_{\mathrm{g}} = \frac{G m_{\mathrm{E}} m}{r^2}$$

provided that the object lies outside the earth. A force of the same magnitude is exerted *on* the earth by the object. (We'll prove these statements in Section 13.6.)

Figure 13.2

(a) The gravitational force between two spherically symmetric masses m_1 and m_2 ...



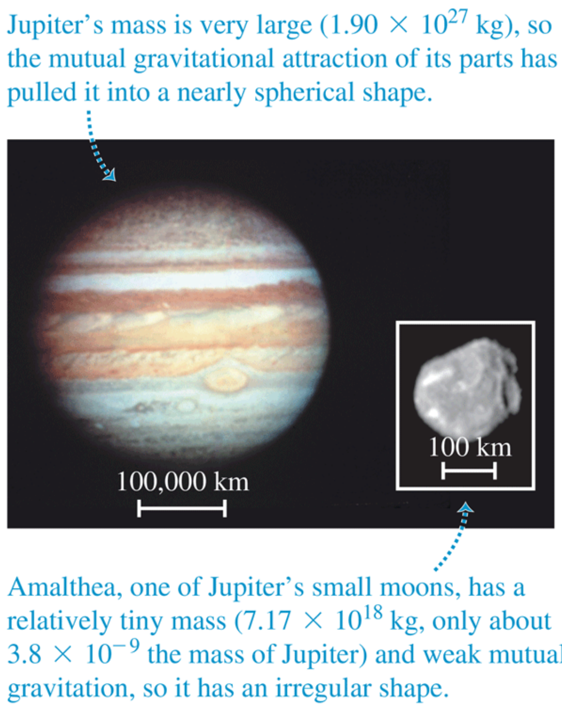


The gravitational effect *outside* any spherically symmetric mass distribution is the same as though all of the mass were concentrated at its center.

At points *inside* the earth the situation is different. If we could drill a hole to the center of the earth and measure the gravitational force on an object at various depths, we would find that toward the center of the earth the force *decreases,* rather than increasing as $1/r^2$. As the object enters the interior of the earth (or other spherical object), some of the earth's mass is on the side of the object opposite from the center and pulls in the opposite direction. Exactly at the center, the earth's gravitational force on the object is zero.

Spherically symmetric objects are an important case because moons, planets, and stars all tend to be spherical. Since all particles in an object gravitationally attract each other, the particles tend to move to minimize the distance between them. As a result, the object naturally tends to assume a spherical shape, just as a lump of clay forms into a sphere if you squeeze it with equal forces on all sides. This effect is greatly reduced in celestial objects of low mass, since the gravitational attraction is less, and these objects tend *not* to be spherical (Fig. 13.3).

Figure 13.3



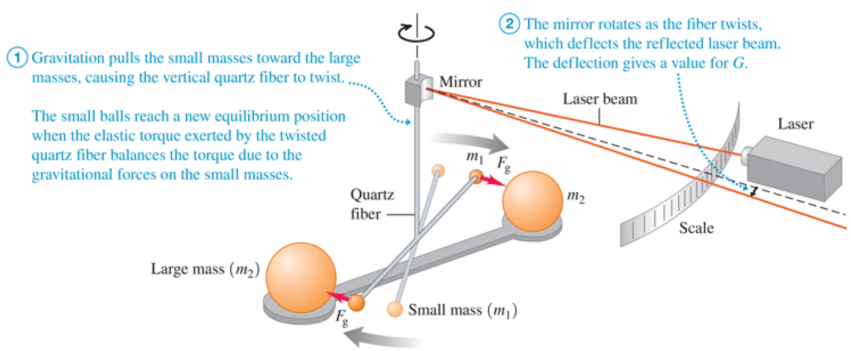
Spherical and nonspherical objects: the planet Jupiter and one] of Jupiter's small moons, Amalthea.

Determining the Value of G

To determine the value of the gravitational constant G, we have to *measure* the gravitational force between two objects of known masses m_1 and m_2 at a known distance r. The force is extremely small for objects that are small enough to be brought into the laboratory, but it can be measured with an instrument called a *torsion balance,* which Sir Henry Cavendish used in 1798 to determine G.

Figure 13.4 shows a modern version of the Cavendish torsion balance. A light, rigid rod shaped like an inverted T is supported by a very thin, vertical quartz fiber. Two small spheres, each of mass m_1 , are mounted at the ends of the horizontal arms of the T. When we bring two large spheres, each of mass m_2 , to the positions shown, the attractive gravitational forces twist the T through a small angle. To measure this angle, we shine a beam of light on a mirror fastened to the T. The reflected beam strikes a scale, and as the T twists, the reflected beam moves along the scale.

Figure 13.4



The principle of the Cavendish balance, used for determining the value of G. The angle of deflection has been exaggerated here for clarity.

After calibrating the Cavendish balance, we can measure gravitational forces and thus determine G. The accepted value as of this writing (2018) is

$$G = 6.67408(31) \times 10^{-11}\ \mathrm{N{\cdot}m^2/kg^2}$$

To three significant figures, $G = 6.67 \times 10^{-11}\ \mathrm{N{\cdot}m^2/kg^2}$. Because $1\ \mathrm{N} = 1\ \mathrm{kg{\cdot}m/s^2}$, the units of G can also be expressed as $\mathrm{m^3/(kg{\cdot}s^2)}$.

Gravitational forces combine vectorially. If each of two masses exerts a force on a third, the *total* force on the third mass is the vector sum of the individual forces of the first two. Example 13.3 makes use of this property, which is often called *superposition of forces* (see Section 4.1).

Example 13.1 Calculating gravitational force

The mass m_1 of one of the small spheres of a Cavendish balance is 0.0100 kg, the mass m_2 of the nearest large sphere is 0.500 kg, and the center-to-center distance between them is 0.0500 m. Find the gravitational force F_g on each sphere due to the other.

IDENTIFY, SET UP, and EXECUTE Because the spheres are spherically symmetric, we can calculate F_g by treating them as *particles* separated by 0.0500 m, as in Fig. 13.2. Each sphere experiences the same magnitude of force from the other sphere. We use Newton's law of gravitation, Eq. (13.1), to determine F_g :

$$\begin{aligned} F_g &= \frac{(6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2})(0.0100\ \mathrm{kg})(0.500\ \mathrm{kg})}{(0.0500\ \mathrm{m})^2} \\ &= 1.33 \times 10^{-10}\mathrm{N} \end{aligned}$$

EVALUATE It's remarkable that such a small force could be measured —or even detected—more than 200 years ago. Only a very massive object such as the earth exerts a gravitational force we can feel.

CAUTION Newton's third law applies to gravitational forces, too Even though the large sphere in this example has 50 times the mass of the small sphere, each sphere feels the *same* magnitude of force from the other. But because their masses are different, the accelerations of the two objects in response to that magnitude of force are different (see the following example).

KEY CONCEPT

Any two objects exert attractive gravitational forces on each other that are proportional to the product of the masses of the two objects and inversely proportional to the square of the distance between their centers (Newton's law of gravitation).

Video Tutor Solution: Example 13.1

Example 13.2 Acceleration due to gravitational attraction

Suppose the two spheres in Example 13.1 are placed with their centers 0.0500 m apart at a point in space far removed from all other objects. What is the magnitude of the acceleration of each, relative to an inertial system?

IDENTIFY, SET UP, and EXECUTE Each sphere exerts on the other a gravitational force of the same magnitude F_g, which we found in Example 13.1. We can ignore any other forces. The *acceleration* magnitudes a_1 and a_2 are different because the masses are different. To determine these we'll use Newton's second law:

$$
\begin{aligned}
a_1 &= \frac{F_g}{m_1} = \frac{1.33\times10^{-10}\ \mathrm{N}}{0.0100\ \mathrm{kg}} = 1.33 \times 10^{-8}\ \mathrm{m/s^2} \\
a_2 &= \frac{F_g}{m_2} = \frac{1.33\times10^{-10}\ \mathrm{N}}{0.500\ \mathrm{kg}} = 2.66 \times 10^{-10}\ \mathrm{m/s^2}
\end{aligned}
$$

EVALUATE The larger sphere has 50 times the mass of the smaller one and hence has $\frac{1}{50}$ the acceleration. These accelerations are *not* constant; the gravitational forces increase as the spheres move toward each other.

KEY CONCEPT

The gravitational forces that two objects exert on each other are always equal in magnitude. The accelerations caused by these forces, however, will be different in magnitude for the two objects if they have different masses.

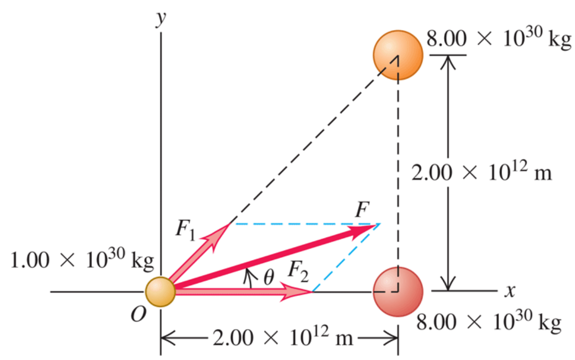
Video Tutor Solution: **Example 13.2**

Example 13.3 Superposition of gravitational forces

Many stars belong to *systems* of two or more stars held together by their mutual gravitational attraction. Figure 13.5 shows a three-star system at an instant when the stars are at the vertices of a 45°

right triangle. Find the total gravitational force exerted on the small star by the two large ones.

Figure 13.5



The total gravitational force on the small star (at O) is the vector sum of the forces exerted on it by the two larger stars. (For comparison, the mass of the sun—a rather ordinary star—is 1.99×10^{30} kg and the earth–sun distance is 1.50×10^{11} m.)

IDENTIFY, SET UP, and EXECUTE We use the principle of superposition: The total force $\vec{F}$ on the small star is the vector sum of the forces $\vec{\boldsymbol{F}}_1$ and $\vec{\boldsymbol{F}}_2$ due to each large star, as Fig. 13.5 shows. We assume that the stars are spheres as in Fig. 13.2. We first calculate the magnitudes F_1 and F_2 from Eq. (13.1) and then compute the vector sum by using components:

$$F_1 = \frac{\begin{gathered}\left(6.67\times10^{-11}\mathrm{N{\cdot}m^2/kg^2}\right)\\ \times\left(8.00\times10^{30}\mathrm{kg}\right)\left(1.00\times10^{30}\mathrm{kg}\right)\end{gathered}}{\left(2.00\times10^{12}\mathrm{m}\right)^2+\left(2.00\times10^{12}\mathrm{m}\right)^2} = 6.67\times10^{25}\mathrm{N}$$

$$F_2 = \frac{\begin{gathered}\left(6.67\times10^{-11}\mathrm{N{\cdot}m^2/kg^2}\right)\\ \times\left(8.00\times10^{30}\mathrm{kg}\right)\left(1.00\times10^{30}\mathrm{kg}\right)\end{gathered}}{\left(2.00\times10^{12}\mathrm{m}\right)^2} = 1.33\times10^{26}\mathrm{N}$$

The x- and y-components of these forces are

$$
\begin{aligned}
F_{1x} &= \left(6.67 \times 10^{25}\text{N}\right)\left(\cos 45^{\circ}\right) = 4.72 \times 10^{25}\text{N} \\
F_{1y} &= \left(6.67 \times 10^{25}\text{N}\right)\left(\sin 45^{\circ}\right) = 4.72 \times 10^{25}\text{N} \\
F_{2x} &= 1.33 \times 10^{26}\text{N} \\
F_{2y} &= 0
\end{aligned}
$$

The components of the total force $\vec{F}$ on the small star are

$$
\begin{aligned}
F_x = F_{1x} + F_{2x} = 1.81 \times 10^{26}\text{N} \\
F_y = F_{1y} + F_{2y} = 4.72 \times 10^{25}\text{N}
\end{aligned}
$$

The magnitude of $\vec{F}$ and its angle θ (see Fig. 13.5) are

$$
\begin{aligned}
F &= \sqrt{{F_x}^2 + {F_y}^2} = \sqrt{\left(1.81 \times 10^{26}\ \text{N}\right)^2 + \left(4.72 \times 10^{25}\ \text{N}\right)^2} \\
&= 1.87 \times 10^{26}\ \text{N} \\
\theta &= \arctan\frac{F_y}{F_x} = \arctan\frac{4.72 \times 10^{25}\ \text{N}}{1.81 \times 10^{26}\ \text{N}} = 14.6^{\circ}
\end{aligned}
$$

EVALUATE While the force magnitude F is tremendous, the magnitude of the resulting acceleration is not: $a = F/m = (1.87 \times 10^{26}\ \text{N})/\ (1.00 \times 10^{30}\ \text{kg}) = 1.87 \times 10^{-4}\ \text{m/s}^2$. Furthermore, the force $\vec{F}$ is *not* directed toward the center of mass of the two large stars.

KEY CONCEPT

When there are two objects that exert gravitational forces on a third object, use vector addition to find the magnitude and direction of the net gravitational force on the third object.

Video Tutor Solution: **Example 13.3**

Why Gravitational Forces Are Important

Comparing Examples 13.1 and 13.3 shows that gravitational forces are negligible between ordinary household-sized objects but very substantial between objects that are the size of stars. Indeed, gravitation is *the* most important force on the scale of planets, stars, and galaxies (Fig. 13.6). It is responsible for holding our earth together and for keeping the planets in orbit about the sun. The mutual gravitational attraction between different parts of the sun compresses material at the sun's core to very high densities and temperatures, making it possible for nuclear reactions to take place there. These reactions generate the sun's energy output, which makes it possible for life to exist on earth and for you to read these words.

Figure 13.6

Our solar system is part of a spiral galaxy like this one, which contains roughly 10^{11} stars as well as gas, dust, and other matter. The entire assemblage is held together by the mutual gravitational attraction of all the matter in the galaxy.

The gravitational force is so important on the cosmic scale because it acts *at a distance*, without any direct contact between objects. Electric and magnetic forces have this same remarkable property, but they are less important on astronomical scales because large accumulations of matter are electrically neutral; that is, they contain equal amounts of positive and negative charge. As a result, the electric and magnetic forces between stars or planets are very small or zero. The strong and weak interactions that we discussed in Section 5.5 also act at a distance, but their influence is negligible at distances much greater than the diameter of an atomic nucleus (about 10^{-14} m).

A useful way to describe forces that act at a distance is in terms of a *field*. One object sets up a disturbance or field at all points in space, and the force that acts on a second object at a particular point is its response to the first object's field at that point. There is a field associated with each force that acts at a distance, and so we refer to gravitational fields, electric fields, magnetic fields, and so on. We won't need the field concept for our study of gravitation in this chapter, so we won't discuss it further here. But in later chapters we'll find that the field concept is an extraordinarily powerful tool for describing electric and magnetic interactions.

Test Your Understanding of Section 13.1

The planet Saturn has about 100 times the mass of the earth and is about 10 times farther from the sun than the earth is. Compared to the acceleration of the earth caused by the sun's gravitational pull, how great is the acceleration of Saturn due to the sun's gravitation? (i) 100 times greater; (ii) 10 times greater; (iii) the same; (iv) $\frac{1}{10}$ as great; (v) $\frac{1}{100}$ as great.

13.2 Weight

We defined the *weight* of an object in Section 4.4 as the attractive gravitational force exerted on it by the earth. We can now broaden our definition and say that *the weight of an object is the total gravitational force exerted on the object by all other objects in the universe*. When the object is near the surface of the earth, we can ignore all other gravitational forces and consider the weight as just the earth's gravitational attraction. At the surface of the *moon* we consider an object's weight to be the gravitational attraction of the moon, and so on.

Application

Walking and Running on the Moon

You automatically transition from a walk to a run when the vertical force you exert on the ground—which, by Newton's third law, equals the vertical force the ground exerts on you—exceeds your weight. This transition from walking to running happens at much lower speeds on the moon, where objects weigh only 17% as much as on earth. Hence, the Apollo astronauts found themselves running even when moving relatively slowly during their moon "walks."

If we again model the earth as a spherically symmetric object with radius R_{E}, the weight of a small object at the earth's surface (a distance R_{E} from its center) is

(13.3)

Gravitational constant

Mass of the earth

Weight of an object at the earth's surface ...

$$w = F_{\mathrm{g}} = \frac{Gm_{\mathrm{E}}m}{R_{\mathrm{E}}^2} \tag{13.3}$$

Mass of object

... equals gravitational force the earth exerts on object.

Radius of the earth

But we also know from Section 4.4 that the weight w of an object is the force that causes the acceleration g of free fall, so by Newton's second law, $w = mg$. Equating this with Eq. (13.3) and dividing by m, we find

(13.4)

Gravitational constant Mass of the earth

Acceleration due to gravity at the earth's surface

$$g = \frac{G m_{\mathrm{E}}}{R_{\mathrm{E}}^{2}} \qquad (13.4)$$

Radius of the earth

The acceleration due to gravity g is independent of the mass m of the object because m doesn't appear in this equation. We already knew that, but we can now see how it follows from the law of gravitation.

We can *measure* all the quantities in Eq. (13.4) except for m_{E}, so this relationship allows us to compute the mass of the earth. Solving Eq. (13.4) for m_{E} and using $R_{\mathrm{E}} = 6370\ \mathrm{km} = 6.37 \times 10^{6}\ \mathrm{m}$ and $g = 9.80\ \mathrm{m/s^{2}}$, we find

$$m_{\mathrm{E}} = \frac{g R_{\mathrm{E}}{}^{2}}{G} = 5.96 \times 10^{24}\ \mathrm{kg}$$

This is very close to the currently accepted value of $5.972 \times 10^{24}\ \mathrm{kg}$. Once Cavendish had measured G, he computed the mass of the earth in just this way.

At a point above the earth's surface a distance r from the center of the earth (a distance $r - R_{\mathrm{E}}$ above the surface), the weight of an object is given by Eq. (13.3) with R_{E} replaced by r:

(13.5)

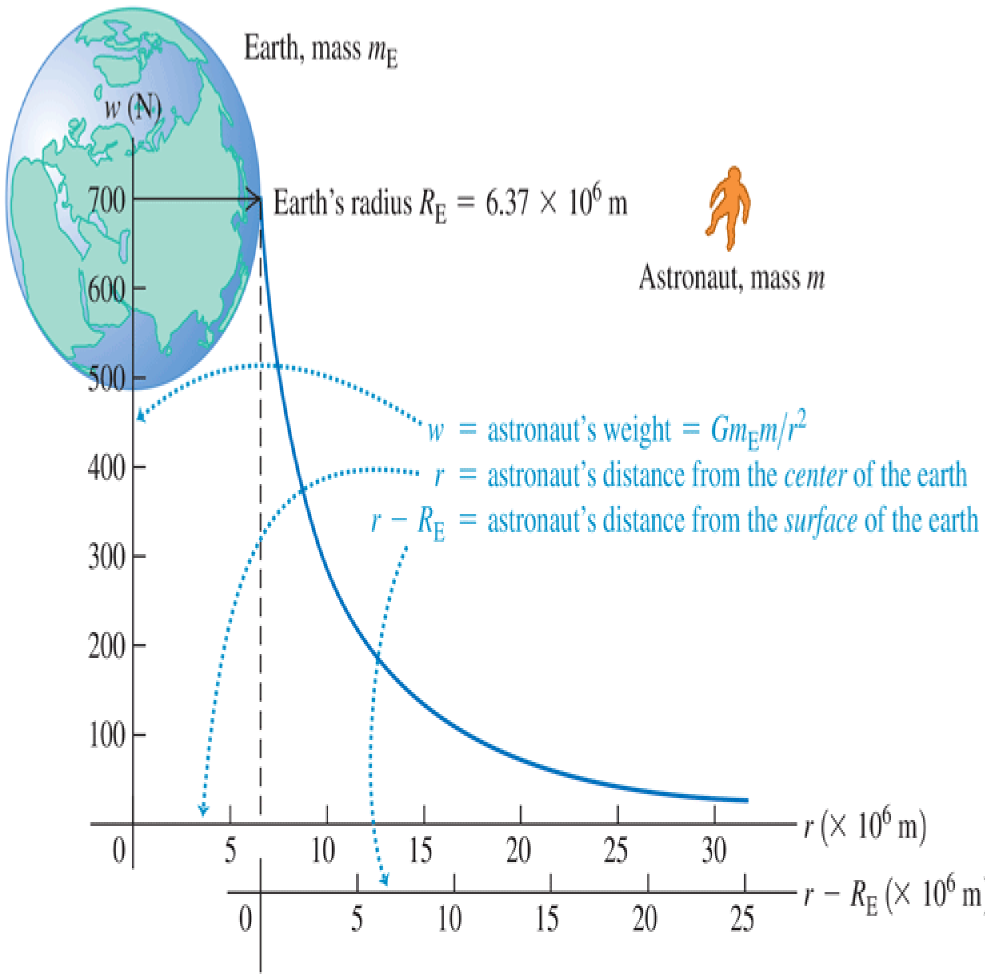
$$w = F_{\mathrm{g}} = \frac{G m_{\mathrm{E}}\, m}{r^{2}}$$

The weight of an object decreases inversely with the square of its distance from the earth's center (Fig. 13.7). Figure 13.8 shows how the weight varies with height above the earth for an astronaut who weighs 700 N at the earth′s surface.

Figure 13.7

In an airliner at high altitude, you are farther from the center of the earth than when on the ground and hence weigh slightly less. Can you show that at an altitude of 10 km above the surface, you weigh 0.3% less than you do on the ground?

Figure 13.8



An astronaut who weighs 700 N at the earth's surface experiences less gravitational attraction when above the surface. The relevant distance r is from the astronaut to the *center* of the earth (*not* from the astronaut to the earth's surface).

The *apparent* weight of an object on earth differs slightly from the earth's gravitational force because the earth rotates and is therefore not precisely an inertial frame of reference. We've ignored this relatively small effect in our discussion but will consider it carefully in Section 13.7.

While the earth is an approximately spherically symmetric distribution of mass, it is *not* uniform throughout its volume. To demonstrate this, let's first calculate the average *density*, or mass per unit volume, of the earth. If we assume a spherical earth, the volume is

$$\begin{aligned} V_{\mathrm{E}} &= \tfrac{4}{3}\,\pi R_{\mathrm{E}}{}^{3} = \tfrac{4}{3}\,\pi\left(6.37\times 10^{6}\ \mathrm{m}\right)^{3} \\ &= 1.08\times 10^{21}\ \mathrm{m}^{3} \end{aligned}$$

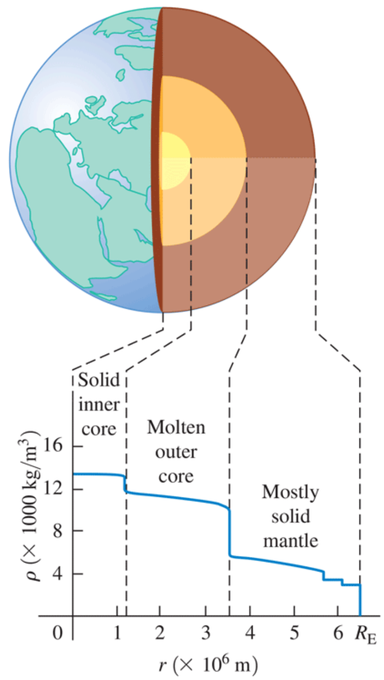
The average density ρ (the Greek letter rho) of the earth is the total mass divided by the total volume:

$$\begin{aligned} \rho &= \frac{m_{\mathrm{E}}}{V_{\mathrm{E}}} = \frac{5.97\times 10^{24}\ \mathrm{kg}}{1.08\times 10^{21}\ \mathrm{m}^{3}} \\ &= 5500\,\mathrm{kg/m}^{3} = 5.5\,\mathrm{g/cm}^{3} \end{aligned}$$

$$= 5500\ \mathrm{kg/m}^{3} = 5.5\ \mathrm{g/cm}^{3}$$

(Compare to the density of water, $1000\ \mathrm{kg/m}^{3} = 1.00\ \mathrm{g/cm}^{3}$.) If the earth were uniform, rocks near the earth's surface would have this same density. In fact, the density of surface rocks is substantially lower, ranging from about $2000\ \mathrm{kg/m}^{3}$ for sedimentary rocks to about $3300\ \mathrm{kg/m}^{3}$ for basalt. So the earth *cannot* be uniform, and its interior must be much more dense than its surface in order that the *average* density be $5500\ \mathrm{kg/m}^{3}$. According to geophysical models of the earth's interior, the maximum density at the center is about $13,000\ \mathrm{kg/m}^{3}$. Figure 13.9 is a graph of density as a function of distance from the center.

Figure 13.9



The density ρ of the earth decreases with increasing distance r from its center.

Example 13.4 Gravity on Mars

A robotic lander with an earth weight of 3430 N is sent to Mars, which has radius $R_M = 3.39 \times 10^6$ m and mass $m_M = 6.42 \times 10^{23}$ kg (see Appendix F). Find the weight F_g of the lander on the Martian surface and the acceleration there due to gravity, g_M.

IDENTIFY and SET UP To find F_g we use Eq. (13.3), replacing m_E and R_E with m_M and R_M. We determine the lander mass m from the lander's earth weight w and then find g_M from $F_g = mg_M$.

EXECUTE The lander's earth weight is $w = mg$, so

$$m = \frac{w}{g} = \frac{3430\ \text{N}}{9.80\ \text{m/s}^2} = 350\ \text{kg}$$

The mass is the same no matter where the lander is. From Eq. (13.3), the lander's weight on Mars is

$$\begin{aligned} F_g &= \frac{Gm_M m}{R_M{}^2} \\ &= \frac{\left(6.67\times10^{-11}\mathrm{N}\times\mathrm{m}^2/\mathrm{kg}^2\right)\left(6.42\times10^{23}\mathrm{kg}\right)(350\,\mathrm{kg})}{\left(3.39\times10^6\mathrm{m}\right)^2} \\ &= 1.30\times10^3\mathrm{N} \end{aligned}$$

The acceleration due to gravity on Mars is

$$g_M = \frac{F_g}{m} = \frac{1.30\times10^3\ \mathrm{N}}{350\ \mathrm{kg}} = 3.7\ \mathrm{m}/s^2$$

EVALUATE Even though Mars has just 11% of the earth's mass (6.42×10^{23} kg versus 5.97×10^{24} kg), the acceleration due to gravity g_M (and hence an object's weight F_g) is roughly 40% as large as on earth. That's because g_M is also inversely proportional to the square of the planet's radius, and Mars has only 53% the radius of earth (3.39×10^6 m versus 6.37×10^6 m).

You can check our result for g_M by using Eq. (13.4), with appropriate replacements. Do you get the same answer?

KEY CONCEPT

An object's weight at the surface of a planet equals the gravitational force exerted on the object by that planet. The distance between the planet and the object equals the planet's radius. The acceleration due to gravity at a planet's surface is proportional to the planet's mass and inversely proportional to the square of its radius.

Video Tutor Solution: **Example 13.4**

Test Your Understanding of Section 13.2

Rank the following hypothetical planets in order from highest to lowest value of g at the surface. (Here m_{P} and R_{P} are the mass and radius of the planet, and m_{E} and R_{E} are the mass and radius of the earth.)

(i) **i.** $m_{\mathrm{P}} = 2m_{\mathrm{E}},\ R_{\mathrm{P}} = 2R_{\mathrm{E}}$;

(ii) **ii.** $m_{\mathrm{P}} = 4m_{\mathrm{E}},\ R_{\mathrm{P}} = 4R_{\mathrm{E}}$;

(iii) **iii.** $m_{\mathrm{P}} = 4m_{\mathrm{E}},\ R_{\mathrm{P}} = 2R_{\mathrm{E}}$;

(iv) **iv.** $m_{\mathrm{P}} = 2m_{\mathrm{E}},\ R_{\mathrm{P}} = 4R_{\mathrm{E}}$.

13.3 Gravitational Potential Energy

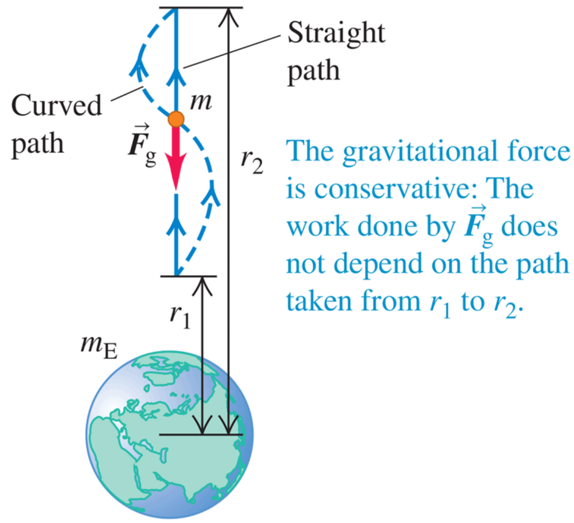
When we first introduced gravitational potential energy in Section 7.1, we assumed that the earth's gravitational force on an object of mass m doesn't depend on the object's height. This led to the expression $U \quad mgy$ But Eq. (13.2), $F \quad Gm \quad m \; r$ shows that the gravitational force exerted by the earth (mass m) *does* in general depend on the distance r from the object to the earth's center. For problems in which an object can be far from the earth's surface, we need a more general expression for gravitational potential energy.

To find this expression, we follow the same steps as in Section 7.1. We consider an object of mass m outside the earth, and first compute the work W done by the gravitational force when the object moves directly away from or toward the center of the earth from $r \quad r$ to $r \quad r$ as in Fig. 13.10. This work is given by

(13.6)

$$W \qquad {}_{r}^{r} F_r \, dr$$

Figure 13.10



Calculating the work done on an object by the gravitational force as the object moves from radial coordinate r to r

---

where F_r is the radial component of the gravitational force F—that is, the component in the direction *outward* from the center of the earth. Because F points directly *inward* toward the center of the earth, F_r is negative. It differs from Eq. (13.2), the magnitude of the gravitational force, by a minus sign:

(13.7)

$$F_r \qquad \frac{Gm \quad m}{r}$$

Substituting Eq. (13.7) into Eq. (13.6), we see that W is given by

(13.8)

$$W \qquad Gm \quad m \int_{r}^{r} \frac{dr}{r} \qquad \frac{Gm \quad m}{r} \qquad \frac{Gm \quad m}{r}$$

The path doesn't have to be a straight line; it could also be a curve like the one in Fig. 13.10. By an argument similar to that in Section 7.1, this work depends on only the initial and final values of r not on the path taken. This also proves that the gravitational force is always *conservative.*

We now define the corresponding potential energy U so that
W U U as in Eq. (7.3). Comparing this with Eq. (13.8), we see that the appropriate definition for **gravitational potential energy** is

(13.9)

Gravitational constant

Mass of the earth

Gravitational potential energy
(general expression)

$$U = -\frac{Gm_{\mathrm{E}}m}{r} \tag{13.9}$$

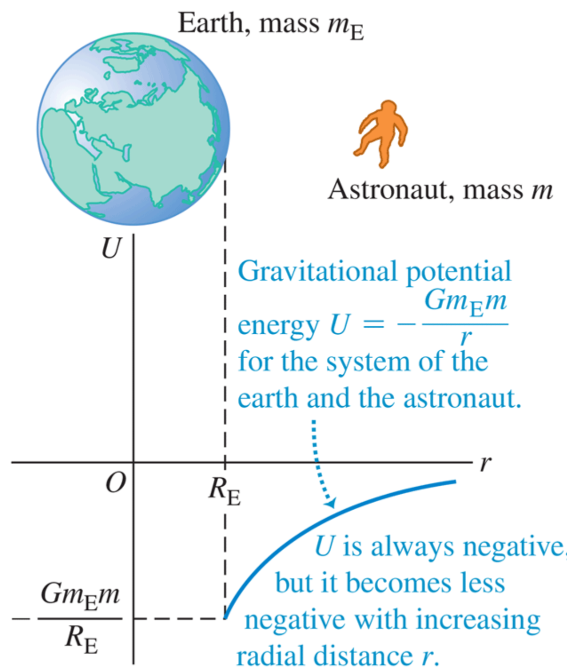
Mass of object

Distance of object from the earth's center

CAUTION Gravitational force vs. gravitational potential energy Don't confuse the expressions for gravitational force, Eq. (13.7), and gravitational potential energy, Eq. (13.9). The force F_r is proportional to r while potential energy U is proportional to r

Figure 13.11 shows how the gravitational potential energy depends on the distance r between the object of mass m and the center of the earth. When the object moves away from the earth, r increases, the gravitational force does negative work, and U increases (i.e., becomes less negative). When the object "falls" toward earth, r decreases, the gravitational work is positive, and the potential energy decreases (i.e., becomes more negative).

Figure 13.11



A graph of the gravitational potential energy U for the system of the earth (mass m and an astronaut (mass m) versus the astronaut's distance r from the center of the earth.

CAUTION Don't worry about gravitational potential energy being negative You may be troubled by Eq. (13.9) because it states that gravitational potential energy is always negative. But in fact you've seen negative values of U before. In using the formula U mgy in Section 7.1, we found that U was negative whenever the object of mass m was at a value of y below the arbitrary height we chose to be y —that is, whenever the object and the earth were closer together than some arbitrary distance. (See, for instance, Example 7.2 in Section 7.1.) In defining U by Eq. (13.9), we have chosen U to be zero when the object of mass m is infinitely far from the earth r As the object moves toward the earth, gravitational potential energy decreases and so becomes negative.

If we wanted, we could make U at the earth's surface, where r R by adding the quantity Gm m R to Eq. (13.9). This would make U

positive when r R We won't do this for two reasons: One, it would complicate the expression for U; two, the added term would not affect the *difference* in potential energy between any two points, which is the only physically significant quantity.

If the earth's gravitational force on an object is the only force that does work, then the total mechanical energy of the system of the earth and object is constant, or *conserved*. In the following example we'll use this principle to calculate **escape speed,** the speed required for an object to escape completely from a planet.

Example 13.5 "From the earth to the moon"

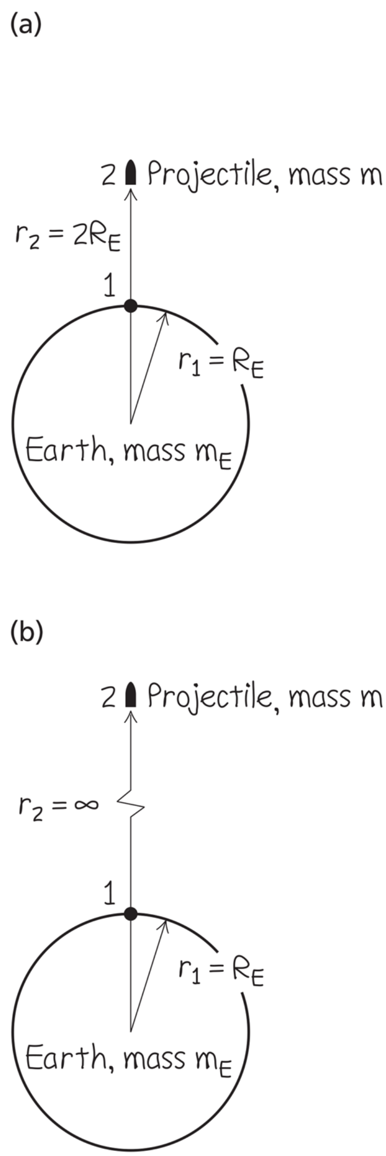
With VARIATION PROBLEMS

In Jules Verne's 1865 story with this title, three men went to the moon in a shell fired from a giant cannon sunk in the earth in Florida. (a) Find the minimum muzzle speed needed to shoot a shell straight up to a height above the earth equal to the earth's radius R . (b) Find the minimum muzzle speed that would allow a shell to escape from the earth completely (the *escape speed*). Neglect air resistance, the earth's rotation, and the gravitational pull of the moon. The earth's radius and mass are R and m .

IDENTIFY and SET UP Once the shell leaves the cannon muzzle, only the (conservative) gravitational force does work. Hence we can use conservation of total mechanical energy to find the speed at which the shell must leave the muzzle so as to come to a halt (a) at two earth radii from the earth's center and (b) at an infinite distance from earth. The energy-conservation equation is K U K U with U given by Eq. (13.9).

In Fig. 13.12 point 1 is at r R , where the shell leaves the cannon with speed v (the target variable). Point 2 is where the shell reaches its maximum height; in part (a) r R (Fig. 13.12a), and in part (b) r (Fig 13.12b). In both cases v and K Let m be the mass of the shell (with passengers).

Figure 13.12



Our sketches for this problem.

EXECUTE (a) We solve the energy-conservation equation for v

$$K \quad U \qquad K \quad U$$

$$-mv \qquad \frac{Gm\ m}{R} \qquad\qquad \frac{Gm\ m}{R}$$

$$v \qquad \sqrt{\frac{Gm}{R}}$$

(b) Now r so U (see Fig. 13.11). Since K , the total mechanical energy K U is zero in this case. Again we solve the energy-conservation equation for v :

$$-\ mv \qquad \frac{Gm\ \ m}{R}$$

$$v \qquad \sqrt{\frac{Gm}{R}}$$

EVALUATE Our results don't depend on the mass of the shell or the direction of launch. A modern spacecraft launched from Florida must attain essentially the speed found in part (b) to escape the earth; however, before launch it's already moving at to the east because of the earth's rotation. Launching to the east takes advantage of this "free" contribution toward escape speed.

To generalize, the initial speed v needed for an object to escape from the surface of a spherical object of mass M and radius R (ignoring air resistance) is v $\sqrt{GM\ R}$ (escape speed). This equation yields for Mars, for Jupiter, and for the sun.

If the initial speed is greater than the speed v $\sqrt{GM\ \ R}$ that we found in part (a), the shell will still be moving when it gets to

$r \qquad R$. In the same way, if the initial speed is greater than the escape speed $v \qquad \overline{GM \quad R}$ that we found in part (b), the shell will still be moving even when it is very far from the earth (at $r \qquad$.

KEY CONCEPT

When an object moves far away from a planet's surface, use Eq. (13.11) for the gravitational potential energy. The potential energy for the system of planet and object increases as the object moves farther away; it reaches its maximum value of zero when the object is infinitely far away.

Video Tutor Solution: **Example 13.5**

More on Gravitational Potential Energy

As a final note, let's show that when we are close to the earth's surface, Eq. (13.9) reduces to the familiar $U \quad mgy$ from Chapter 7. We first rewrite Eq. (13.8) as

$$W \qquad Gm \;\; m\frac{r \qquad r}{r \;\; r}$$

If the object stays close to the earth, then in the denominator we may replace r and r by R the earth's radius, so

$$W \qquad Gm \;\; m\frac{r \qquad r}{R}$$

According to Eq. (13.4), $g \quad Gm \quad R$ so

$$W \qquad mg \;\; r \qquad r$$

If we replace the r's by y's, this is just Eq. (7.1) for the work done by a constant gravitational force. In Section 7.1 we used this equation to derive Eq. (7.2), $U \quad mgy$ so we may consider Eq. (7.2) for gravitational potential energy to be a special case of the more general Eq. (13.9).

Test Your Understanding of Section 13.3

If a planet has the same surface gravity as the earth (that is, the same value of g at the surface), what is its escape speed? (i) The same as the earth's; (ii) less than the earth's; (iii) greater than the earth's; (iv) any of these are possible.

13.4 The Motion of Satellites

Artificial satellites orbiting the earth are a familiar part of technology (Fig. 13.13). But how do they stay in orbit, and what determines the properties of their orbits? We can use Newton's laws and the law of gravitation to provide the answers. In the next section we'll analyze the motion of planets in the same way.

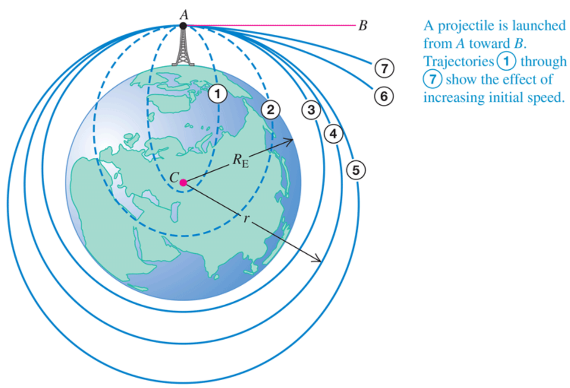
Figure 13.13

With a mass of approximately 4.5×10^5 kg and a width of over 108 m, the International Space Station is the largest satellite ever placed in orbit.

To begin, think back to the discussion of projectile motion in Section 3.3. In Example 3.6 a motorcycle rider rides horizontally off the edge of a cliff, launching himself into a parabolic path that ends on the flat ground at the base of the cliff. If he survives and repeats the experiment with increased launch speed, he will land farther from the starting point. We can imagine him launching himself with great enough speed that the earth's curvature becomes significant. As he falls, the earth curves away beneath him. If he is going fast enough, and if his launch point is high enough that he clears the mountaintops, he may be able to go right on around the earth without ever landing.

Figure 13.14 shows a variation on this theme. We launch a projectile from point A in the direction AB, tangent to the earth's surface. Trajectories 1 through 7 show the effect of increasing the initial speed. In trajectories 3 through 5 the projectile misses the earth and becomes a satellite. If there is no retarding force such as air resistance, the projectile's speed when it returns to point A is the same as its initial speed and it repeats its motion indefinitely.

Figure 13.14



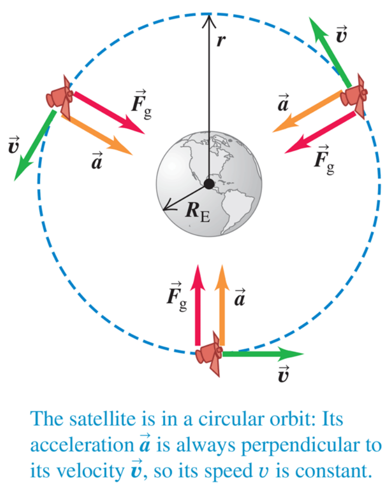
Trajectories of a projectile launched from a great height (ignoring air resistance). Orbits 1 and 2 would be completed as shown if the earth were a point mass at C. (This illustration is based on one in Isaac Newton's *Principia*.)

Trajectories 1 through 5 close on themselves and are called **closed orbits.** All closed orbits are ellipses or segments of ellipses; trajectory 4 is a circle, a special case of an ellipse. (We'll discuss the properties of an ellipse in Section 13.5.) Trajectories 6 and 7 are **open orbits.** For these paths the projectile never returns to its starting point but travels ever farther away from the earth.

Satellites: Circular Orbits

A *circular* orbit, like trajectory 4 in Fig. 13.14, is the simplest case. It is also an important case, since many artificial satellites have nearly circular orbits and the orbits of the planets around the sun are also fairly circular. The only force acting on a satellite in circular orbit around the earth is the earth's gravitational attraction, which is directed toward the center of the earth and hence toward the center of the orbit (Fig. 13.15). As we discussed in Section 5.4, this means that the satellite is in *uniform* circular motion and its speed is constant. The satellite isn't falling *toward* the earth; rather, it's constantly falling *around* the earth. In a circular orbit the speed is just right to keep the distance from the satellite to the center of the earth constant.

Figure 13.15



The force $\vec{F}_g$ due to the earth's gravitational attraction provides the centripetal acceleration that keeps a satellite in orbit. Compare to Fig. 5.28.

? Let's see how to find the constant speed v of a satellite in a circular orbit. The radius of the orbit is r, measured from the *center* of the earth; the acceleration of the satellite has magnitude $a_{rad} = v^2/r$ and is always directed toward the center of the circle. By the law of gravitation, the net force (gravitational force) on the satellite of mass m has magnitude

$F_g = Gm_E m/r^2$ and is in the same direction as the acceleration. Newton's second law $\left(\sum \vec{F} = m\vec{a}\right)$ then tells us that

$$\frac{Gm_E m}{r^2} = \frac{mv^2}{r}$$

Solving this for v, we find

(13.10)

Speed of satellite in a circular orbit around the earth

Gravitational constant

Mass of the earth

$$v = \sqrt{\frac{Gm_E}{r}} \qquad (13.10)$$

Radius of orbit

This relationship shows that we can't choose the orbit radius r and the speed v independently; for a given radius r, the speed v for a circular orbit is determined.

The satellite's mass m doesn't appear in Eq. (13.10), which shows that the motion of a satellite does not depend on its mass. An astronaut on board an orbiting space station is herself a satellite of the earth, held by the earth's gravity in the same orbit as the station. The astronaut has the same velocity and acceleration as the station, so nothing is pushing her against the station's floor or walls. She is in a state of *apparent weightlessness*, as in a freely falling elevator; see the discussion following Example 5.9 in Section 5.2. (*True* weightlessness would occur only if the astronaut were infinitely far from any other masses, so that the gravitational force on her would be zero.) Indeed, every part of her body is apparently weightless; she feels nothing pushing her stomach against her intestines or her head against her shoulders (Fig. 13.16).

Figure 13.16

These astronauts are in a state of apparent weightlessness. Who is right side up?

Apparent weightlessness is not just a feature of circular orbits; it occurs whenever gravity is the only force acting on a spacecraft. Hence it occurs for orbits of any shape, including open orbits such as trajectories 6 and 7 in Fig. 13.14.

We can derive a relationship between the radius r of a circular orbit and the period T, the time for one revolution. The speed v is the distance $2\pi r$ traveled in one revolution, divided by the period:

(13.11)

$$v = \frac{2\pi r}{T}$$

We solve Eq. (13.11) for T and substitute v from Eq. (13.10):

(13.12)

Radius of orbit

Period of a circular orbit around the earth

$$T = \frac{2\pi r}{v} = 2\pi r\sqrt{\frac{r}{Gm_E}} = \frac{2\pi r^{3/2}}{\sqrt{Gm_E}} \qquad (13.12)$$

Orbital speed

Gravitational constant

Mass of the earth

Equations (13.10) and (13.12) show that larger orbits correspond to slower speeds and longer periods. As an example, the International Space Station (Fig. 13.13) orbits 6800 km from the center of the earth (400 km above the earth's surface) with an orbital speed of 7.7 km/s and an orbital period of 93 min. The moon orbits the earth in a much larger orbit of radius 384,000 km, and so has a much slower orbital speed (1.0 km/s) and a much longer orbital period (27.3 days).

It's interesting to compare Eq. (13.10) to the calculation of escape speed in Example 13.5. We see that the escape speed from a spherical object with radius R is $\sqrt{2}$ times greater than the speed of a satellite in a circular orbit at that radius. If our spacecraft is in circular orbit around *any* planet, we have to multiply our speed by a factor of $\sqrt{2}$ to escape to infinity, regardless of the planet's mass.

Since the speed v in a circular orbit is determined by Eq. (13.10) for a given orbit radius r, the total mechanical energy $E = K + U$ is determined as well. Using Eqs. (13.9) and (13.10), we have

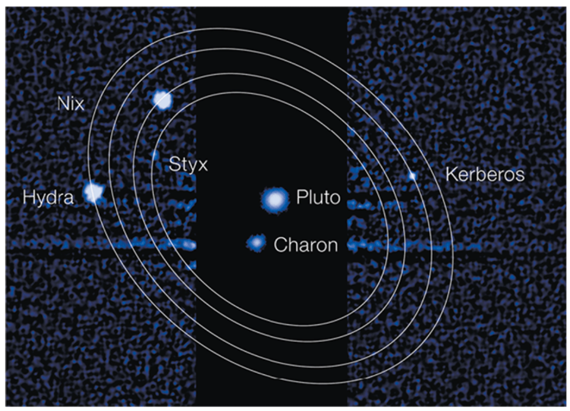
(13.13)

$$\begin{aligned} E &= K + U = \tfrac{1}{2}mv^2 + \left(-\frac{Gm_{\mathrm{E}}m}{r}\right) \\ &= \tfrac{1}{2}m\left(\frac{Gm_{\mathrm{E}}}{r}\right) - \frac{Gm_{\mathrm{E}}m}{r} \\ &= -\frac{Gm_{\mathrm{E}}m}{2r} \quad \text{(circular orbit)} \end{aligned}$$

The total mechanical energy in a circular orbit is negative and equal to one-half the potential energy. Increasing the orbit radius r means increasing the total mechanical energy (that is, making E less negative). If the satellite is in a relatively low orbit that encounters the outer fringes of earth's atmosphere, the total mechanical energy decreases due to negative work done by the force of air resistance; as a result, the orbit radius decreases until the satellite hits the ground or burns up in the atmosphere.

We have talked mostly about earth satellites, but we can apply the same analysis to the circular motion of *any* object under its gravitational attraction to a stationary object. Figure 13.17 shows an example.

Figure 13.17



The dwarf planet Pluto is known to have at least five satellites. In accordance with Eqs. (13.10) and (13.12), the larger the satellite's orbit around Pluto, the slower the satellite's orbital speed and the longer its orbital period.

Example 13.6 A satellite orbit

WITH VARIATION PROBLEMS

You wish to put a 1000 kg satellite into a circular orbit 340 km above the earth's surface. (a) What speed, period, and radial acceleration will it have? (b) How much work must be done to the satellite to put it in orbit? (c) How much additional work would have to be done to make the satellite escape the earth? The earth's radius and mass are given in Example 13.5 (Section 13.3).

IDENTIFY and SET UP The satellite is in a circular orbit, so we can use the equations derived in this section. In part (a), we first find the radius r of the satellite's orbit from its altitude. We then calculate the speed v and period T from Eqs. (13.10) and (13.12) and the acceleration from $a_{\text{rad}} = v^2/r$. In parts (b) and (c), the work required is the difference between the initial and final total mechanical energies, which for a circular orbit is given by Eq. (13.13).

EXECUTE (a) The radius of the satellite's orbit is $r = 6370\text{ km} + 340\text{ km} = 6710\text{ km} = 6.71 \times 10^6\text{ m}$. From Eq. (13.10), the orbital speed is

$$\begin{aligned} v &= \sqrt{\frac{Gm_{\text{E}}}{r}} = \sqrt{\frac{(6.67 \times 10^{-11}\text{ N} \cdot \text{m}^2/\text{kg}^2)(5.97 \times 10^{24}\text{ kg})}{6.71 \times 10^6\text{ m}}} \\ &= 7700\,\text{m/s} \end{aligned}$$

We find the orbital period from Eq. (13.12):

$$T = \frac{2\pi r}{v} = \frac{2\pi(6.67 \times 10^6\text{ m})}{7700\text{ m/s}} = 5470\text{ s} = 91.2\text{ min}$$

Finally, the radial acceleration is

$$a_{\text{rad}} = \frac{v^2}{r} = \frac{(7700\text{ m/s})^2}{6.71 \times 10^6\text{ m}} = 8.84\text{ m/s}^2$$

This is the value of g at a height of 340 km above the earth's surface; it is about 10% less than the value of g at the surface.

(b) The work required is the difference between E_2, the total mechanical energy when the satellite is in orbit, and E_1, the total mechanical energy when the satellite was at rest on the launch pad. From Eq. (13.13), the energy in orbit is

$$
\begin{aligned}
E_2 &= -\frac{Gm_E m}{2r} \\
&= \frac{\left(6.71\times10^{-11}\text{N}\cdot\text{m}^2/\text{kg}^2\right)\left(5.97\times10^{24}\text{kg}\right)(1000\,\text{kg})}{2\left(6.71\times10^6\text{m}\right)} \\
&= -2.97\times10^{10}\,\text{J}
\end{aligned}
$$

The satellite's kinetic energy is zero on the launch pad ($r = R_E$), so

$$
\begin{aligned}
E_1 &= K_1 + U_1 = 0 + \left(-\frac{Gm_E m}{R_E}\right) \\
&= \frac{\left(6.67\times10^{-11}\text{N}\cdot\text{m}^2/\text{kg}^2\right)\left(5.97\times10^{24}\text{kg}\right)(1000\,\text{kg})}{6.37\times10^6\text{m}} \\
&= -6.25\times10^{10}\,\text{J}
\end{aligned}
$$

Hence the work required is

$$
\begin{aligned}
W_{\text{required}} &= E_2 - E_1 = (-2.97\times10^{10}\ \text{J}) - (-6.25\times10^{10}\ \text{J}) \\
&= 3.28\times10^{10}\,\text{J}
\end{aligned}
$$

(c) We saw in part (b) of Example 13.5 that the minimum total mechanical energy for a satellite to escape to infinity is zero. Here, the total mechanical energy in the circular orbit is $E_2 = -2.97 \times 10^{10}$ J; to increase this to zero, an amount of work equal to 2.97×10^{10} J would have to be done on the satellite, presumably by rocket engines attached to it.

EVALUATE In part (b) we ignored the satellite's initial kinetic energy (while it was still on the launch pad) due to the rotation of the earth. How much difference does this make? (See Example 13.5 for useful data.)

KEY CONCEPT

The larger the radius of a satellite's orbit, the greater (less negative) the total mechanical energy, the greater (less negative) the potential energy, and the smaller the kinetic energy (the satellite moves more slowly in a larger orbit). The radius of a satellite's orbit is always measured from the center of the earth.

Video Tutor Solution: **Example 13.6**

Test Your Understanding of Section 13.4

Your personal spacecraft is in a low-altitude circular orbit around the earth. Air resistance from the outer regions of the atmosphere does negative work on the spacecraft, causing the radius of the circular orbit to decrease slightly. Does the speed of the spacecraft (i) remain the same, (ii) increase, or (iii) decrease?

13.5 Kepler's Laws and the Motion of Planets

The name *planet* comes from a Greek word meaning "wanderer," and indeed the planets continuously change their positions in the sky relative to the background of stars. One of the great intellectual accomplishments of the 16th and 17th centuries was the threefold realization that the earth is also a planet, that all planets orbit the sun, and that the apparent motions of the planets as seen from the earth can be used to determine their orbits precisely.

The first and second of these ideas were published by Nicolaus Copernicus in Poland in 1543. The nature of planetary orbits was deduced between 1601 and 1619 by the German astronomer and mathematician Johannes Kepler, using precise data on apparent planetary motions compiled by his mentor, the Danish astronomer Tycho Brahe. By trial and error, Kepler discovered three empirical laws that accurately described the motions of the planets:

Kepler's Laws

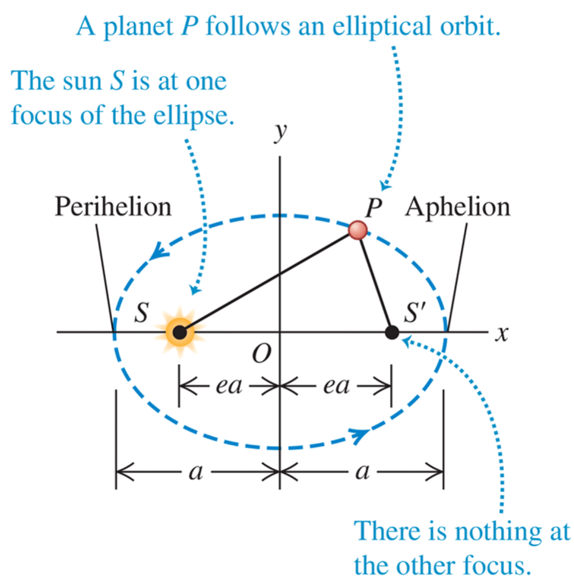
1. Each planet moves in an elliptical orbit, with the sun at one focus of the ellipse.
2. A line from the sun to a given planet sweeps out equal areas in equal times.
3. The periods of the planets are proportional to the $\frac{3}{2}$ powers of the major axis lengths of their orbits.

Kepler did not know *why* the planets moved in this way. Three generations later, when Newton turned his attention to the motion of the planets, he discovered that each of Kepler's laws can be *derived;* they are consequences of Newton's laws of motion and the law of gravitation. Let's see how each of Kepler's laws arises.

Kepler's First Law

First consider the elliptical orbits described in Kepler's first law. Figure 13.18 shows the geometry of an ellipse. The longest dimension is the *major axis*, with half-length a; this half-length is called the **semi-major axis.** The sum of the distances from S to P and from S' to P is the same for all points on the curve. S and S' are the *foci* (plural of *focus*). The sun is at S (*not* at the center of the ellipse) and the planet is at P; we think of both as points because the size of each is very small in comparison to the distance between them. There is nothing at the other focus, S'.

Figure 13.18



Geometry of an ellipse. The sum of the distances SP and $S'P$ is the same for every point on the curve. The sizes of the sun (S) and planet (P) are exaggerated for clarity.

The distance of each focus from the center of the ellipse is ea, where e is a dimensionless number between 0 and 1 called the **eccentricity.** If $e = 0$, the two foci coincide and the ellipse is a circle. The actual orbits of the planets are fairly circular; their eccentricities range from 0.007 for Venus to 0.206 for Mercury. (The earth's orbit has $e = 0.017$.) The point in the planet's orbit closest to the sun is the *perihelion,* and the point most distant is the *aphelion.*

Newton showed that for an object acted on by an attractive force proportional to $1/r^2$, the only possible closed orbits are a circle or an ellipse; he also showed that open orbits (trajectories 6 and 7 in Fig. 13.14) must be parabolas or hyperbolas. These results can be derived from Newton's laws and the law of gravitation, together with a lot more differential equations than we're ready for.

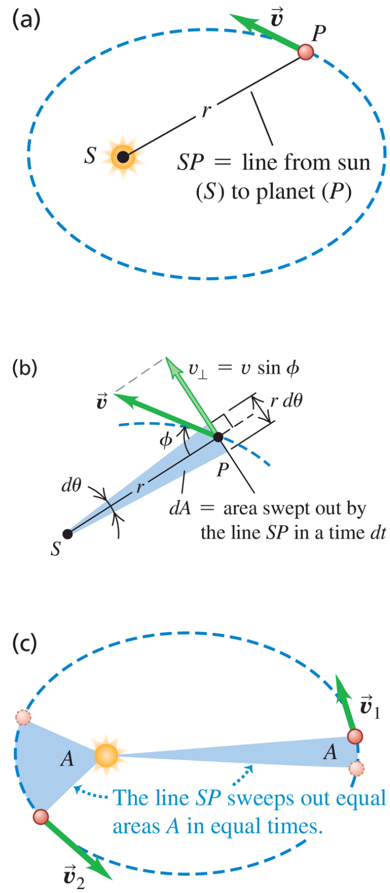
Kepler's Second Law

Figure 13.19 shows Kepler's second law. In a small time interval dt, the line from the sun S to the planet P turns through an angle $d\theta$. The area swept out is the colored triangle with height r, base length $r\, d\theta$, and area $dA = \frac{1}{2} r^2\, d\theta$ in Fig. 13.19b. The rate at which area is swept out, dA/dt, is called the *sector velocity:*

(13.14)

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$

Figure 13.19



(a) The planet (P) moves about the sun (S) in an elliptical orbit. (b) In a time dt the line SP sweeps out an area $dA = \frac{1}{2}\,(r\,d\theta)r = \frac{1}{2}\,r^2\,d\theta$. (c) The planet's speed varies so that the line SP sweeps out the same area A in a given time t regardless of the planet's position in its orbit.

The essence of Kepler's second law is that the sector velocity has the same value at all points in the orbit. When the planet is close to the sun, r is small and $d\theta/dt$ is large; when the planet is far from the sun, r is large and $d\theta/dt$ is small.

To see how Kepler's second law follows from Newton's laws, we express dA/dt in terms of the velocity vector $\vec{v}$ of the planet P. The component of $\vec{v}$ perpendicular to the radial line is $v_\perp = v \sin\phi$. From Fig. 13.19b the displacement along the direction of $v_\perp$ during time dt is $r\,d\theta$, so we also have $v_\perp = r\,d\theta/dt$. Using this relationship in Eq. (13.14), we find

(13.15)

$$\frac{dA}{dt} = \frac{1}{2}\,rv\sin\phi \qquad \text{(sector velocity)}$$

Now $rv\sin\phi$ is the magnitude of the vector product $\vec{r}\times\vec{v}$, which in turn is $1/m$ times the angular momentum $\vec{L} = \vec{r}\times m\vec{v}$ of the planet with respect to the sun. So we have

(13.16)

$$\frac{dA}{dt} = \frac{1}{2m}\,\left|\vec{r}\times m\vec{v}\right| = \frac{L}{2m}$$

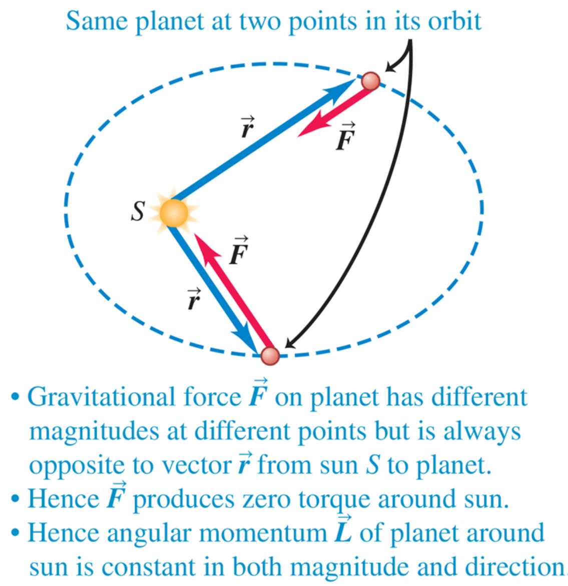
Thus Kepler's second law—that sector velocity is constant—means that angular momentum is constant!

It is easy to see why the angular momentum of the planet *must* be constant. According to Eq. (10.26), the rate of change of $\vec{L}$ equals the torque of the gravitational force $\vec{F}$ acting on the planet:

$$\frac{d\vec{L}}{dt} = \vec{\tau} = \vec{r}\times\vec{F}$$

In our situation, $\vec{r}$ is the vector from the sun to the planet, and the force $\vec{F}$ is directed from the planet to the sun (Fig. 13.20). So these vectors always lie along the same line, and their vector product $\vec{r}\times\vec{F}$ is zero. Hence $d\vec{L}/dt = 0$. This conclusion does not depend on the $1/r^2$ behavior of the force; angular momentum is conserved for *any* force that acts always along the line joining the particle to a fixed point. Such a force is called a *central force.* (Kepler's first and third laws are valid for a $1/r^2$ force *only*.)

Figure 13.20

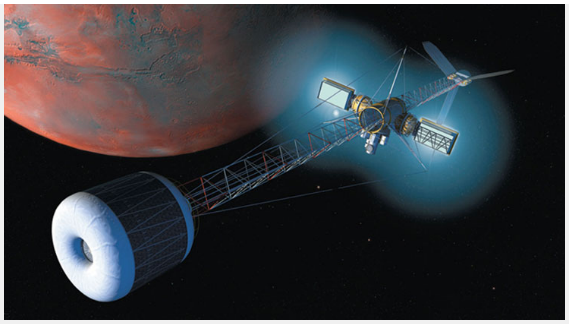


Because the gravitational force that the sun exerts on a planet produces zero torque around the sun, the planet's angular momentum around the sun remains constant.

BIO Application

Biological Hazards of Interplanetary Travel

A spacecraft sent from earth to another planet spends most of its journey coasting along an elliptical orbit with the sun at one focus. Rockets are used at only the start and end of the journey, and even the trip to a nearby planet like Mars takes several months. During its journey, the spacecraft is exposed to cosmic rays— radiation that emanates from elsewhere in our galaxy. (On earth we're shielded from this radiation by our planet's magnetic field, as we'll describe in Chapter 27.) This poses no problem for a robotic spacecraft but would be a severe medical hazard for astronauts undertaking such a voyage.



Conservation of angular momentum also explains why the orbit lies in a plane. The vector $\vec{L} = \vec{r} \times m\vec{v}$ is always perpendicular to the plane of the vectors $\vec{r}$ and $\vec{v}$; since $\vec{L}$ is constant in magnitude *and* direction, $\vec{r}$ and $\vec{v}$ always lie in the same plane, which is just the plane of the planet's orbit.

Kepler's Third Law

We have already derived Kepler's third law for the particular case of circular orbits. Equation (13.12) shows that the period of a satellite or planet in a circular orbit is proportional to the $\frac{3}{2}$ power of the orbit radius. Newton was able to show that this same relationship holds for an *elliptical* orbit, with the orbit radius r replaced by the semi-major axis a:

(13.17)

$$T = \frac{2\pi a^{3/2}}{\sqrt{Gm_{\text{S}}}} \qquad \text{(elliptical orbit around the sun)}$$

Since the planet orbits the sun, not the earth, we have replaced the earth's mass m_{E} in Eq. (13.12) with the sun's mass m_{S}. Note that the period does not depend on the eccentricity e. An asteroid in an elongated elliptical orbit with semi-major axis a will have the same orbital period as a planet in a circular orbit of radius a. The key difference is that the asteroid moves at different speeds at different points in its elliptical orbit (Fig. 13.19c), while the planet's speed is constant around its circular orbit.

Conceptual Example 13.7 Orbital speeds

WITH VARIATION PROBLEMS

At what point in an elliptical orbit (see Fig. 13.19) does a planet move the fastest? The slowest?

SOLUTION Total mechanical energy is conserved as a planet moves in its orbit. The planet's kinetic energy $K = \frac{1}{2}mv^2$ is maximum when the potential energy $U = -Gm_S m/r$ is minimum (that is, most negative; see Fig. 13.11), which occurs when the sun–planet distance r is a minimum. Hence the speed v is greatest at perihelion. Similarly, K is minimum when r is maximum, so the speed is slowest at aphelion.

Your intuition about falling objects is helpful here. As the planet falls inward toward the sun, it picks up speed, and its speed is maximum when closest to the sun. The planet slows down as it moves away from the sun, and its speed is minimum at aphelion.

KEY CONCEPT

Total mechanical energy remains constant for a planet in an elliptical orbit. The planet's kinetic energy (and speed) therefore increases as it moves closer to the sun and the gravitational potential energy decreases (becomes more negative).

Video Tutor Solution: Example 13.7

Example 13.8 Kepler's third law

WITH VARIATION PROBLEMS

The asteroid Pallas has an orbital period of 4.62 years and an orbital eccentricity of 0.233. Find the semi-major axis of its orbit.

IDENTIFY and SET UP We need Kepler's third law, which relates the period T and the semi-major axis a for an orbiting object (such as an asteroid). We use Eq. (13.17) to determine a; from Appendix F we have $m_S = 1.99 \times 10^{30}$ kg, and a conversion factor from Appendix E gives $T = (4.62\ \text{y})(3.156 \times 10^7\ \text{s/y}) = 1.46 \times 10^8\ \text{s}$. Note that we don't need the value of the eccentricity.

EXECUTE From Eq. (13.17), $a^{3/2} = \left[(Gm_S)^{1/2}T\right]/2\pi$. To solve for a, we raise both sides of this expression to the $\dfrac{2}{3}$ power and then substitute the values of G, m_S, and T:

$$a = \left(\frac{Gm_S T^2}{4\pi^2}\right)^{1/3} = 4.15 \times 10^{11}\ \text{m}$$

(Plug in the numbers yourself to check.)

EVALUATE Our result is intermediate between the semi-major axes of Mars and Jupiter (see Appendix F). Most known asteroids orbit in an “asteroid belt” between the orbits of these two planets.

KEY CONCEPT

Kepler’s third law allows you to relate the period of an orbit, the semi-major axis of that orbit, and the mass of the object being orbited.

Video Tutor Solution: **Example 13.8**

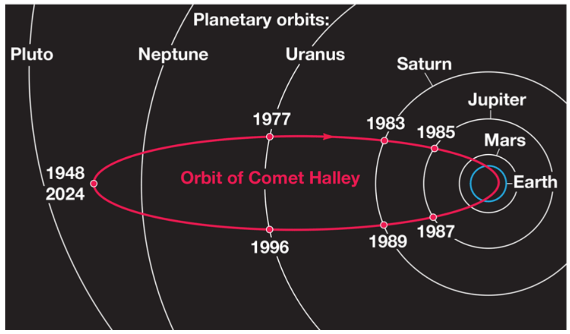
Example 13.9 Comet Halley

WITH VARIATION PROBLEMS

Comet Halley (Fig. 13.21) moves in an elongated elliptical orbit around the sun. Its distances from the sun at perihelion and aphelion are 8.80×10^{7} km and 5.25×10^{9} km, respectively. Find the orbital semi-major axis, eccentricity, and period in years.

Figure 13.21

(a)



(b)

(a) The orbit of Comet Halley. (b) Comet Halley as it appeared in 1986. At the heart of the comet is an icy object, called the nucleus, that is about 10 km across. When the comet's orbit carries it close to the sun, the heat of sunlight causes the nucleus to partially evaporate. The evaporated material forms the tail, which can be tens of millions of kilometers long.

IDENTIFY and SET UP We are to find the semi-major axis a, eccentricity e, and orbital period T. We can use Fig. 13.18 to find a and e from the given perihelion and aphelion distances. Knowing a, we can find T from Kepler's third law, Eq. (13.17).

EXECUTE From Fig. 13.18, the length $2a$ of the major axis equals the sum of the comet–sun distance at perihelion and the comet–sun distance at aphelion. Hence

$$a = \frac{(8.80 \times 10^7 \text{ km}) + (5.25 \times 10^9 \text{ km})}{2} = 2.67 \times 10^9 \text{ km}$$

Figure 13.18 also shows that the comet–sun distance at perihelion is $a - ea = a(1 - e)$. This distance is 8.80×10^7 km, so

$$e = 1 - \frac{8.80 \times 10^7 \text{ km}}{a} = 1 - \frac{8.80 \times 10^7 \text{ km}}{2.67 \times 10^9 \text{ km}} = 0.967$$

From Eq. (13.17), the period is

$$\begin{aligned} T &= \frac{2\pi a^{3/2}}{\sqrt{Gm_S}} = \frac{2\pi(2.67 \times 10^{12} \text{ m})^{3/2}}{\sqrt{(6.67 \times 10^{-11} \text{ N}\cdot\text{m}^2/\text{kg}^2)(1.99 \times 10^{30} \text{ kg})}} \\ &= 2.38 \times 10^9 \text{s} = 75.3 \text{ y} \end{aligned}$$

EVALUATE The eccentricity is close to 1, so the orbit is very elongated (see Fig. 13.21a). Comet Halley was at perihelion in early 1986 (Fig. 13.21b); it will next reach perihelion one period later, in 2061.

KEY CONCEPT

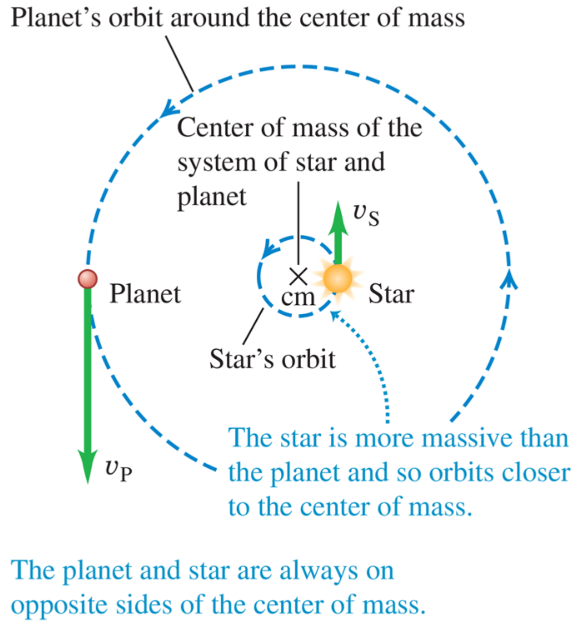
Kepler's third law applies to orbits of any eccentricity *e* between 0 and 1. You can determine *e* from the semi-major axis of the orbit and the distance at perihelion from the orbiting object to the object being orbited.

Video Tutor Solution: **Example 13.9**

Planetary Motions and the Center of Mass

We have assumed that as a planet or comet orbits the sun, the sun remains absolutely stationary. This can't be correct; because the sun exerts a gravitational force on the planet, the planet exerts a gravitational force on the sun of the same magnitude but opposite direction. In fact, *both* the sun and the planet orbit around their common center of mass (Fig. 13.22). We've made only a small error by ignoring this effect, however; the sun's mass is about 750 times the total mass of all the planets combined, so the center of mass of the solar system is not far from the center of the sun. Remarkably, astronomers have used this effect to detect the presence of planets orbiting other stars. Sensitive telescopes are able to detect the apparent "wobble" of a star as it orbits the common center of mass of the star and an unseen companion planet. (The planets are too faint to observe directly.) By analyzing these "wobbles," astronomers have discovered planets in orbit around hundreds of other stars.

Figure 13.22



Both a star and its planet orbit about their common center of mass.

The most remarkable result of Newton's analysis of planetary motion is that objects in the heavens obey the *same* laws of motion as do objects on the earth. This *Newtonian synthesis,* as it has come to be called, is one of the great unifying principles of science. It has had profound effects on the way that humanity looks at the universe—not as a realm of impenetrable mystery, but as a direct extension of our everyday world, subject to scientific study and calculation.

Test Your Understanding of Section 13.5

The orbit of Comet X has a semi-major axis that is four times longer than the semi-major axis of Comet Y. What is the ratio of the orbital period of X to the orbital period of Y? (i) 2; (ii) 4; (iii) 8; (iv) 16; (v) 32; (vi) 64.

13.6 Spherical Mass Distributions

We have stated without proof that the gravitational interaction between two spherically symmetric mass distributions is the same as though all the mass of each were concentrated at its center. Now we're ready to prove this statement. Newton searched for a proof for several years, and he delayed publication of the law of gravitation until he found one.

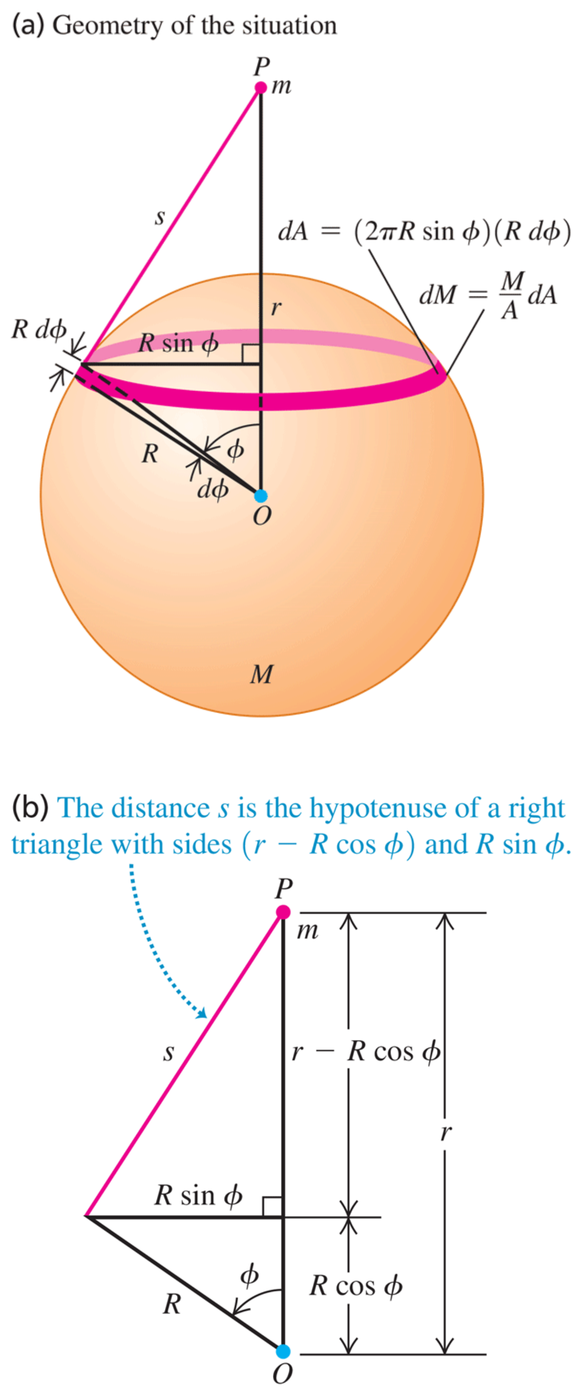
Rather than starting with two spherically symmetric masses, we'll tackle the simpler problem of a point mass m interacting with a thin spherical shell with total mass M. We'll show that when m is outside the sphere, the *potential energy* associated with this gravitational interaction is the same as though M were concentrated in a point at the center of the sphere. We learned in Section 7.4 that the force is the negative derivative of the potential energy, so the *force* on m is also the same as for a point mass M. Our result will also hold for *any* spherically symmetric mass distribution, which we can think of as being made of many concentric spherical shells.

A Point Mass Outside a Spherical Shell

We start by considering a ring on the surface of a shell (Fig. 13.23a, next page), centered on the line from the center of the shell to m. We do this because all of the particles that make up the ring are the same distance s from the point mass m. From Eq. (13.9) the potential energy of interaction between the earth (mass m_{E}) and a point mass m, separated by a distance r, is $U = -Gm_{\mathrm{E}}\, m/r$. From this expression, we see that the potential energy of interaction between the point mass m and a particle of mass m_i within the ring is

$$U_i = -\frac{Gmm_i}{s}$$

Figure 13.23



Calculating the gravitational potential energy of interaction between a point mass m outside a spherical shell and a ring on the surface of the shell of mass M.

To find the potential energy dU of interaction between m and the entire ring of mass $dM = \sum\limits_i m_i$, we sum this expression for U_i over all particles in the ring:

(13.18)

$$dU = \sum U_i = \sum_i \left(-\frac{Gmm_i}{s} \right) = -\frac{Gm}{s} \sum_i m_i = -\frac{Gm\,dM}{s}$$

To proceed, we need to know the mass dM of the ring. We can find this with the aid of a little geometry. The radius of the shell is R, so in terms of the angle ϕ shown in the figure, the radius of the ring is $R \sin \phi$, and its circumference is $2\pi R \sin \phi$. The width of the ring is $R\, d\phi$, and its area dA is approximately equal to its width times its circumference:

$$dA = 2\pi R^2 \sin \phi \, d\phi$$

The ratio of the ring mass dM to the total mass M of the shell is equal to the ratio of the area dA of the ring to the total area $A = 4\pi R^2$ of the shell:

(13.19)

$$\frac{dM}{M} = \frac{2\pi R^2 \sin \phi \, d\phi}{4\pi R^2} = \frac{1}{2} \sin \phi \, d\phi$$

Now we solve Eq. (13.19) for dM and substitute the result into Eq. (13.18) to find the potential energy of interaction between point mass m and the ring:

(13.20)

$$dU = -\frac{GMm \sin \phi \, d\phi}{2s}$$

The total potential energy of interaction between the point mass and the *shell* is the integral of Eq. (13.20) over the whole sphere as ϕ varies from 0 to π (*not* 2π*!*) and s varies from $r - R$ to $r + R$. To carry out the integration, we have to express the integrand in terms of a single variable; we choose s. To express ϕ and $d\phi$ in terms of s, we have to do a little more geometry. Figure 13.23b shows that s is the hypotenuse of a right triangle with sides $(r - R\cos\phi)$ and $R\sin\phi$, so the Pythagorean theorem gives

(13.21)

$$\begin{aligned} s^2 &= (r - R\cos\phi)^2 + (R\sin\phi)^2 \\ &= r^2 - 2rR\cos\phi + R^2 \end{aligned}$$

We take differentials of both sides:

$$2s\,ds = 2rR\sin\phi\,d\phi$$

Next we divide this by $2rR$ and substitute the result into Eq. (13.20):

(13.22)

$$dU = -\frac{GMm}{2s}\,\frac{s\,ds}{rR} = -\frac{GMm}{2rR}\,ds$$

We can now integrate Eq. (13.22), recalling that s varies from $r - R$ to $r + R$:

(13.23)

$$U = -\frac{GMm}{2rR}\int_{r-R}^{r+R} ds = -\frac{GMm}{2rR}\Big[(r+R) - (r-R)\Big]$$

Finally, we have

(13.24)

$$U = -\frac{GMm}{r} \qquad \text{(point mass } m \text{ outside spherical shell } M\text{)}$$

This is equal to the potential energy of two point masses m and M at a distance r. So we have proved that the gravitational potential energy of spherical shell M and point mass m at any distance r is the same as though they were point masses. Because the force is given by $F_r = -dU/dr$, the force is also the same.

The Gravitational Force Between Spherical Mass Distributions

Any spherically symmetric mass distribution can be thought of as a combination of concentric spherical shells. Because of the principle of superposition of forces, what is true of one shell is also true of the combination. So we have proved half of what we set out to prove: that the gravitational interaction between any spherically symmetric mass distribution and a point mass is the same as though all the mass of the spherically symmetric distribution were concentrated at its center.

The other half is to prove that *two* spherically symmetric mass distributions interact as though both were points. That's easier. In Fig. 13.23a the forces the two objects exert on each other are an action–reaction pair, and they obey Newton's third law. So we have also proved that the force that m exerts *on* sphere M is the same as though M were a point. But now if we replace m with a spherically symmetric mass distribution centered at m's location, the resulting gravitational force on any part of M is the same as before, and so is the total force. This completes our proof.

A Point Mass Inside a Spherical Shell

We assumed at the beginning that the point mass m was outside the spherical shell, so our proof is valid only when m is outside a spherically symmetric mass distribution. When m is *inside* a spherical shell, the geometry is as shown in Fig. 13.24. The entire analysis goes just as before; Eqs. (13.18), (13.19), (13.20), (13.21) and (13.22) are still valid. But when we get to Eq. (13.23), the limits of integration have to be changed to $R - r$ and $R + r$. We then have

(13.25)

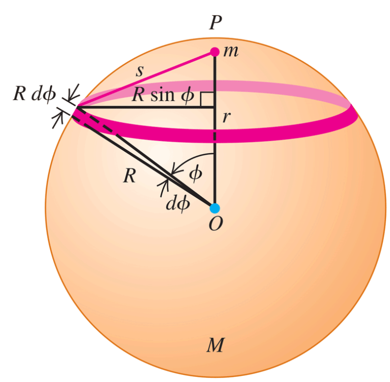
$$U = -\frac{GMm}{2rR}\int_{R-r}^{R+r} ds = -\frac{GMm}{2rR}\Big[(R+r)-(R-r)\Big]$$

and the final result is

(13.26)

$$U = -\frac{GMm}{R} \qquad \text{(point mass } m \text{ inside spherical shell } M\text{)}$$

Figure 13.24



When a point mass m is *inside* a uniform spherical shell of mass M, the potential energy is the same no matter where inside the shell the point mass is located. The force from the masses' mutual gravitational interaction is zero.

Compare this result to Eq. (13.24): Instead of having r, the distance between m and the center of M, in the denominator, we have R, the radius of the shell. This means that U in Eq. (13.26) doesn't depend on r and thus has the same value everywhere inside the shell. When m moves around inside the shell, no work is done on it, so the force on m at any point inside the shell must be zero.

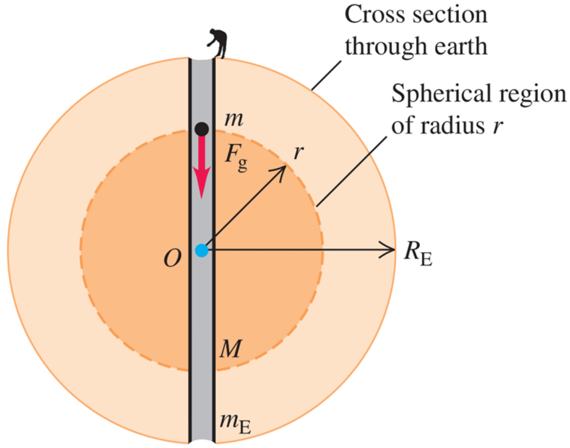
More generally, at any point in the interior of any spherically symmetric mass distribution (not necessarily a shell), at a distance r from its center, the gravitational force on a point mass m is the same as though we removed all the mass at points farther than r from the center and concentrated all the remaining mass at the center.

Example 13.10 "Journey to the center of the earth"

Imagine that we drill a hole through the earth along a diameter and drop a mail pouch down the hole. Derive an expression for the gravitational force F_g on the pouch as a function of its distance from the earth's center. Assume that the earth's density is uniform (not a very realistic model; see Fig. 13.9).

IDENTIFY and SET UP From the discussion immediately above, the value of F_g at a distance r from the earth's center is determined by only the mass M within a spherical region of radius r (Fig. 13.25). Hence F_g is the same as if all the mass within radius r were concentrated at the center of the earth. The mass of a uniform sphere is proportional to the volume of the sphere, which is $\frac{4}{3}\pi r^3$ for a sphere of arbitrary radius r and $\frac{4}{3}\pi R^3{}_E$ for the entire earth.

Figure 13.25



A hole through the center of the earth (assumed to be uniform). When an object is a distance r from the center, only the mass inside a sphere of radius r exerts a net gravitational force on it.

EXECUTE The ratio of the mass M of the sphere of radius r to the mass m_{E} of the earth is

$$\frac{M}{m_{\mathrm{E}}} = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3{}_E} = \frac{r^3}{R^3{}_E} \qquad \text{so} \qquad M = m_{\mathrm{E}}\frac{r^3}{R^3{}_E}$$

The magnitude of the gravitational force on m is then

$$F_{\mathrm{g}} = \frac{GMm}{r^2} = \frac{Gm}{r^2}\left(m_{\mathrm{E}}\frac{r^3}{R^3{}_E}\right) = \frac{Gm_{\mathrm{E}}m}{R^3{}_E}r$$

EVALUATE Inside this uniform-density sphere, F_{g} is *directly proportional* to the distance r from the center, rather than to $1/r^2$ as it is outside the sphere. At the surface $r = R_{\mathrm{E}}$, we have $F_{\mathrm{g}} = Gm_{\mathrm{E}}m/R^2{}_E$, as we should. In the next chapter we'll learn how to compute the time it would take for the mail pouch to emerge on the other side of the earth.

KEY CONCEPT

The gravitational force on a particle at a point a radius r from the center of a spherically symmetric mass distribution is caused solely by the mass *inside* a sphere of radius r. The mass outside radius r has no effect on the particle.

Video Tutor Solution: **Example 13.10**

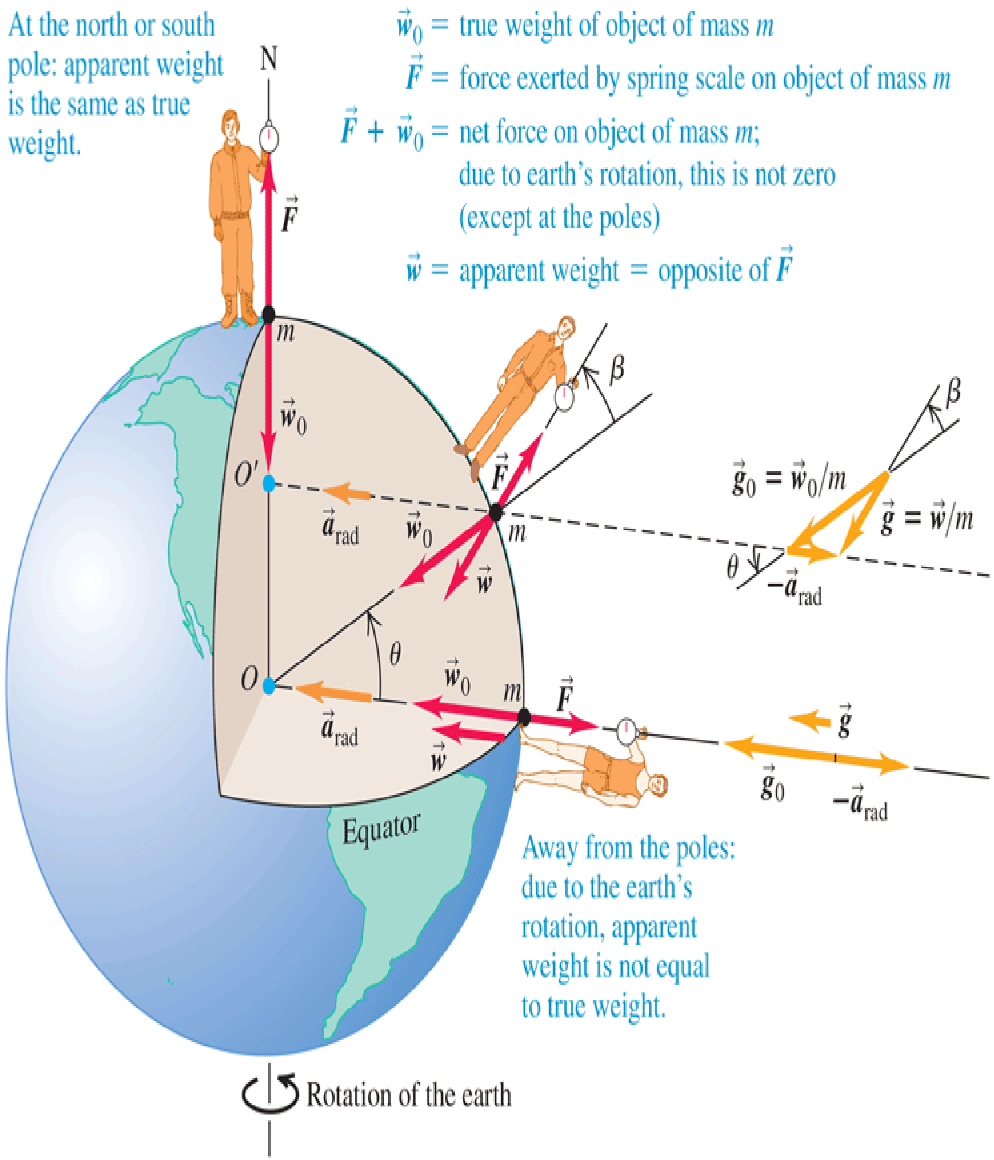
Test Your Understanding of Section 13.6

In the classic 1913 science-fiction novel *At the Earth's Core,* by Edgar Rice Burroughs, explorers discover that the earth is a hollow sphere and that an entire civilization lives on the inside of the sphere. Would it be possible to stand and walk on the inner surface of a hollow, nonrotating planet?

13.7 Apparent Weight and the Earth's Rotation

Because the earth rotates on its axis, it is not precisely an inertial frame of reference. For this reason the apparent weight of an object on earth is not precisely equal to the earth's gravitational attraction, which we'll call the **true weight** $\vec{\boldsymbol{w}}_0$ of the object. Figure 13.26 is a cutaway view of the earth, showing three observers. Each one holds a spring scale with an object of mass m hanging from it. Each scale applies a tension force $\vec{F}$ to the object hanging from it, and the reading on each scale is the magnitude F of this force. If the observers are unaware of the earth's rotation, each one *thinks* that the scale reading equals the weight of the object because he thinks the object on his spring scale is in equilibrium. So each observer thinks that the tension $\vec{F}$ must be opposed by an equal and opposite force $\vec{w}$, which we call the **apparent weight.** But if the objects are rotating with the earth, they are *not* precisely in equilibrium. Our problem is to find the relationship between the apparent weight $\vec{w}$ and the true weight $\vec{\boldsymbol{w}}_0$.

Figure 13.26



Except at the poles, the reading for an object being weighed on a scale (the *apparent weight*) is less than the gravitational force of attraction on the object (the *true weight*). The reason is that a net force is needed to provide a centripetal acceleration as the object rotates with the earth. For clarity, the illustration greatly exaggerates the angle β between the true and apparent weight vectors.

If we assume that the earth is spherically symmetric, then the true weight $\vec{w}_0$ has magnitude $Gm_E\, m/R_E{}^2$, where m_E and R_E are the mass and radius of the earth. This value is the same for all points on the earth's surface. If the center of the earth can be taken as the origin of an inertial coordinate system, then the object at the north pole really *is* in equilibrium in an inertial system, and the reading on that observer's spring scale is equal to w_0. But the object at the equator is moving in a circle of radius R_{E} with speed v, and there must be a net inward force equal to the mass times the centripetal acceleration:

$$w_0 - F = \frac{mv^2}{R_{\mathrm{E}}}$$

So the magnitude of the apparent weight (equal to the magnitude of *F*) is

(13.27)

$$w = w_0 - \frac{mv^2}{R_{\mathrm{E}}} \qquad \text{(at the equator)}$$

If the earth were not rotating, the object when released would have a free-fall acceleration $g_0 = w_0/m$. Since the earth *is* rotating, the falling object's actual acceleration relative to the observer at the equator is $g = w/m$. Dividing Eq. (13.27) by m and using these relationships, we find

$$g = g_0 - \frac{v^2}{R_{\mathrm{E}}} \qquad \text{(at the equator)}$$

To evaluate v^2/R_{E}, we note that in 86,164 s a point on the equator moves a distance equal to the earth's circumference, $2\pi R_{\mathrm{E}} = 2\pi(6.37 \times 10^6\ \mathrm{m})$. (The solar day, 86,400 s, is $\frac{1}{365}$ longer than this because in one day the earth also completes $\frac{1}{365}$ of its orbit around the sun.) Thus we find

$$\begin{aligned} v &= \frac{2\pi(6.37\times10^6\,\mathrm{m})}{86{,}164\,\mathrm{s}} = 465\,\mathrm{m/s} \\ \frac{v^2}{R_{\mathrm{E}}} &= \frac{(465\,\mathrm{m/s})^2}{6.37\times10^6\,\mathrm{m}} = 0.0339\,\mathrm{m/s^2} \end{aligned}$$

So for a spherically symmetric earth the acceleration due to gravity should be about 0.03 m/s^2 less at the equator than at the poles.

At locations intermediate between the equator and the poles, the true weight $\vec{\boldsymbol{w}}_0$ and the centripetal acceleration are not along the same line, and we need to write a vector equation corresponding to Eq. (13.27). From Fig. 13.26 we see that the appropriate equation is

(13.28)

$$\vec{\boldsymbol{w}} = \vec{\boldsymbol{w}}_0 - m\vec{\boldsymbol{a}}_{\text{rad}} = m\vec{\boldsymbol{g}}_0 - m\vec{\boldsymbol{a}}_{\text{rad}}$$

The difference in the magnitudes of g and g_0 lies between zero and 0.0339 m/s^2. As Fig. 13.26 shows, the *direction* of the apparent weight differs from the direction toward the center of the earth by a small angle β, which is 0.1° or less.

Table 13.1 gives the values of g at several locations. In addition to moderate variations with latitude, there are small variations due to elevation, differences in local density, and the earth's deviation from perfect spherical symmetry.

Table 13.1 Variations of *g* with Latitude and Elevation

Station	North Latitude	Elevation (m)	g (m/s^2)
Canal Zone	09°	0	9.78243
Jamaica	18°	0	9.78591
Bermuda	32°	0	9.79806
Denver, CO	40°	1638	9.79609
Pittsburgh, PA	40.5°	235	9.80118
Cambridge, MA	42°	0	9.80398
Greenland	70°	0	9.82534

Test Your Understanding of Section 13.7

Imagine a planet that has the same mass and radius as the earth but that makes 10 rotations during the time the earth makes one rotation. What would be the difference between the acceleration due to gravity at the planet's equator and the acceleration due to gravity at its poles? (i) 0.00339 m/s^2; (ii) 0.0339 m/s^2; (iii) 0.339 m/s^2; (iv) 3.39 m/s^2.

13.8 Black Holes

In 1916 Albert Einstein presented his general theory of relativity, which included a new concept of the nature of gravitation. In his theory, a massive object actually changes the geometry of the space around it. Other objects sense this altered geometry and respond by being attracted to the first object. The general theory of relativity is beyond our scope in this chapter, but we can look at one of its most startling predictions: the existence of **black holes,** objects whose gravitational influence is so great that nothing—not even light—can escape them. We can understand the basic idea of a black hole by using Newtonian principles.

The Escape Speed from a Star

Think first about the properties of our own sun. Its mass $M = 1.99 \times 10^{30}$ kg and radius $R = 6.96 \times 10^{8}$ m are much larger than those of any planet, but compared to other stars, our sun is not exceptionally massive. You can find the sun's average density ρ in the same way we found the average density of the earth in Section 13.2:

$$\rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3} = \frac{1.99 \times 10^{30}\ \mathrm{kg}}{\frac{4}{3}\pi(6.96 \times 10^{8}\ \mathrm{m})^3} = 1410\ \mathrm{kg/m^3}$$

The sun's temperatures range from 5800 K (about $5500^\circ\mathrm{C}$ or $10,000^\circ\mathrm{F}$) at the surface up to 1.5×10^{7} K (about $2.7 \times 10^{7\,\circ}\mathrm{F}$) in the interior, so it surely contains no solids or liquids. Yet gravitational attraction pulls the sun's gas atoms together until the sun is, on average, 41% denser than water and about 1200 times as dense as the air we breathe.

Now think about the escape speed for an object at the surface of the sun. In Example 13.5 (Section 13.3) we found that the escape speed from the surface of a spherical mass M with radius R is $v = \sqrt{2GM/R}$. Substituting $M = \rho V = \rho\left(\frac{4}{3}\pi R^3\right)$ into the expression for escape speed gives

(13.29)

$$v = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{8\pi G\rho}{3}}\,R$$

Using either form of this equation, you can show that the escape speed for an object at the surface of our sun is $v = 6.18 \times 10^5\ \mathrm{m/s}$ (about 2.2 million km/h, or 1.4 million mi/h). This value, roughly $\frac{1}{500}$ the speed of light in vacuum, is independent of the mass of the escaping object; it depends on only the mass and radius (or average density and radius) of the sun.

Now consider various stars with the same average density ρ and different radii R. Equation (13.29) shows that for a given value of density ρ, the escape speed v is directly proportional to R. In 1783 the Rev. John Michell noted that if an object with the same average density as the sun had about 500 times the radius of the sun, its escape speed would be greater than the speed of light in vacuum, c. With his statement that "all light emitted from such a body would be made to return towards it," Michell became the first person to suggest the existence of what we now call a black hole.

Black Holes, the Schwarzschild Radius, and the Event Horizon

The first expression for escape speed in Eq. (13.29) suggests that an object of mass M will act as a black hole if its radius R is less than or equal to a certain critical radius. How can we determine this critical radius? You might think that you can find the answer by simply setting $v = c$ in Eq. (13.29). As a matter of fact, this does give the correct result, but only because of two compensating errors. The kinetic energy of light is *not* $mc^2/2$, and the gravitational potential energy near a black hole is *not* given by Eq. (13.9). In 1916, Karl Schwarzschild used Einstein's general theory of relativity to derive an expression for the critical radius R_S, now called the **Schwarzschild radius.** The result turns out to be the same as though we had set $v = c$ in Eq. (13.29), so

$$c = \sqrt{\frac{2GM}{R_S}}$$

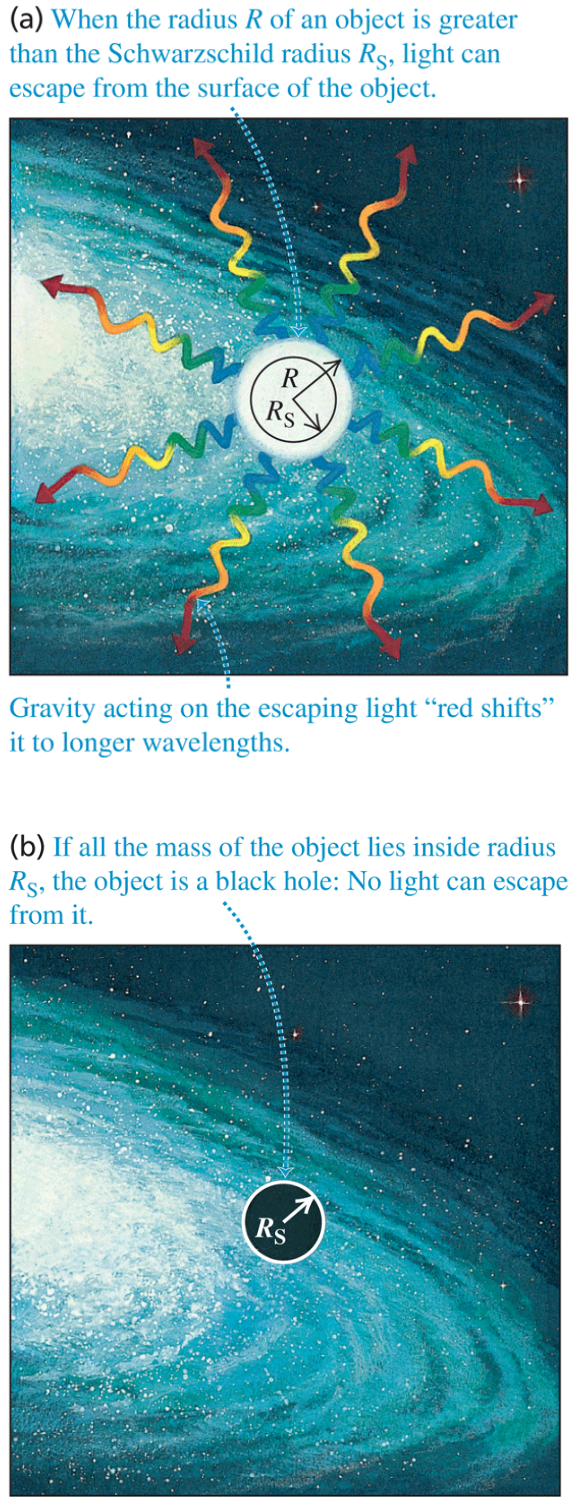
Solving for the Schwarzschild radius R_S, we find

(13.30)

$$\underset{\text{Schwarzschild radius of a black hole}}{R_S} = \frac{\overset{\text{Gravitational constant}}{2G}\,\overset{\text{Mass of black hole}}{M}}{\underset{\text{Speed of light in vacuum}}{c^2}} \qquad (13.30)$$

If a spherical, nonrotating object with mass M has a radius less than R_S, then *nothing* (not even light) can escape from the surface of the object, and the object is a black hole (Fig. 13.27). In this case, any other object within a distance R_S of the center of the black hole is trapped by the gravitational attraction of the black hole and cannot escape from it.

Figure 13.27



(a) An object with a radius R greater than the Schwarzschild radius R_S . (b) If the object collapses to a radius smaller than R_S , it is a black hole with an escape speed greater than the speed of light. The surface of the sphere of radius R_S is called the event horizon of the black hole.

The surface of the sphere with radius R_S surrounding a black hole is called the **event horizon:** Since light can't escape from within that sphere, we can't see events occurring inside. All that an observer outside the event horizon can know about a black hole is its mass (from its gravitational effects on other objects), its electric charge (from the electric forces it exerts on other charged objects), and its angular momentum (because a rotating black hole tends to drag space—and everything in that space—around with it). All other information about the object is irretrievably lost when it collapses inside its event horizon.

Example 13.11 Black hole calculations

Astrophysical theory suggests that a burned-out star whose mass is at least three solar masses will collapse under its own gravity to form a black hole. If it does, what is the radius of its event horizon?

IDENTIFY, SET UP, and EXECUTE The radius in question is the Schwarzschild radius. We use Eq. (13.30) with a value of M equal to three solar masses, or $M = 3(1.99 \times 10^{30}\ \text{kg}) = 6.0 \times 10^{30}\ \text{kg}$:

$$\begin{aligned} R_S &= \frac{2GM}{c^2} = \frac{2\left(6.67\times10^{-11}\text{N}\cdot\text{m}^2/\text{kg}^2\right)\left(6.0\times10^{30}\text{kg}\right)}{\left(3.00\times10^{8}\text{m/s}\right)^2} \\ &= 8.9 \times 10^3\text{m} = 8.9\,\text{km} = 5.5\,\text{mi} \end{aligned}$$

EVALUATE The average density of such an object is

$$\rho = \frac{M}{\frac{4}{3}\pi R^3} = \frac{6.0 \times 10^{30}\ \text{kg}}{\frac{4}{3}\pi(8.9 \times 10^3\ \text{m})^3} = 2.0 \times 10^{18}\text{kg/m}^3$$

This is about 10^{15} times as great as the density of familiar matter on earth and is comparable to the densities of atomic nuclei. In fact, once the object collapses to a radius of R_S, nothing can prevent it from collapsing further. All of the mass ends up being crushed down to a single point called a *singularity* at the center of the event horizon. This point has zero volume and so has *infinite* density.

KEY CONCEPT

The event horizon of a black hole has a radius of R_S, the Schwarzschild radius, which is proportional to the mass of the black hole. Nothing can escape from inside the event horizon.

Video Tutor Solution: **Example 13.11**

A Visit to a Black Hole

At points far from a black hole, its gravitational effects are the same as those of any normal object with the same mass. If the sun collapsed to form a black hole, the orbits of the planets would be unaffected. But things get dramatically different close to the black hole. If you decided to become a martyr for science and jump into a black hole, the friends you left behind would notice several odd effects as you moved toward the event horizon, most of them associated with effects of general relativity.

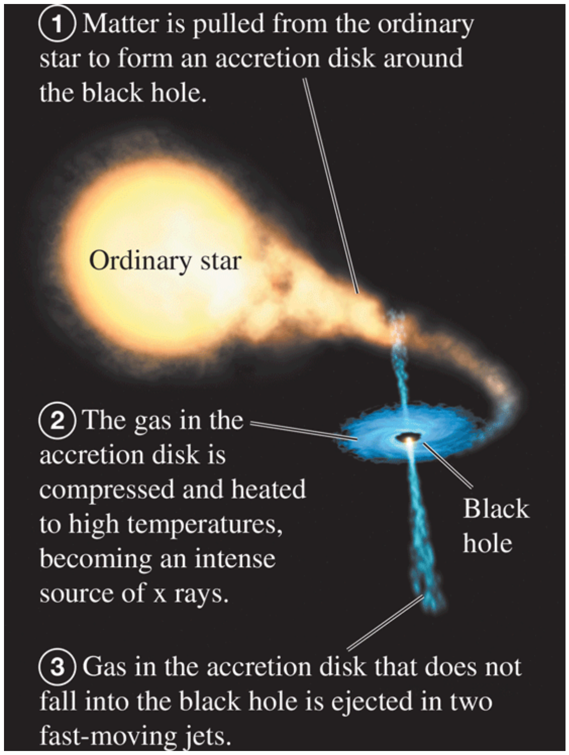
If you carried a radio transmitter to send back your comments on what was happening, your friends would have to retune their receiver continuously to lower and lower frequencies, an effect called the *gravitational red shift*. Consistent with this shift, they would observe that your clocks (electronic or biological) would appear to run more and more slowly, an effect called *time dilation*. In fact, during their lifetimes they would never see you make it to the event horizon.

In your frame of reference, you would make it to the event horizon in a rather short time but in a rather disquieting way. As you fell feet first into the black hole, the gravitational pull on your feet would be greater than that on your head, which would be slightly farther away from the black hole. The *differences* in gravitational force on different parts of your body would be great enough to stretch you along the direction toward the black hole and compress you perpendicular to it. These effects (called *tidal forces*) would rip you to atoms, and then rip your atoms apart, before you reached the event horizon.

Detecting Black Holes

If light cannot escape from a black hole and if black holes are as small as Example 13.11 suggests, how can we know that such things exist? The answer is that any gas or dust near the black hole tends to be pulled into an *accretion disk* that swirls around and into the black hole, rather like a whirlpool (Fig. 13.28). Friction within the accretion disk's gas causes it to lose mechanical energy and spiral into the black hole; as it moves inward, it is compressed together. This causes heating of the gas, just as air compressed in a bicycle pump gets hotter. Temperatures in excess of 10^6 K can occur in the accretion disk, so hot that the disk emits not just visible light (as do objects that are "red-hot" or "white-hot") but x rays. Astronomers look for these x rays (emitted by the gas material *before* it crosses the event horizon) to signal the presence of a black hole. Several promising candidates have been found, and astronomers now express considerable confidence in the existence of black holes.

Figure 13.28

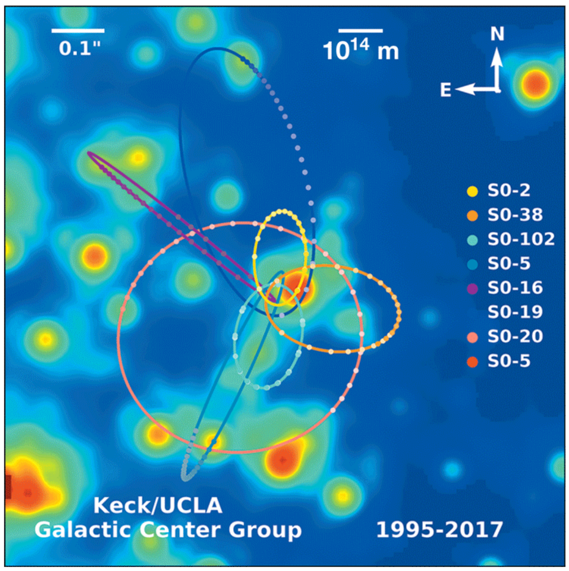


In a *binary* star system, two stars orbit each other; in the special case shown here, one of the stars is a black hole. The black hole itself cannot be seen, but the x rays from its accretion disk can be detected.

Black holes in binary star systems like the one depicted in Fig. 13.28 have masses a few times greater than the sun's mass. There is also mounting evidence for the existence of much larger *supermassive black holes*. One example lies at the center of our Milky Way galaxy, some 26,000 light-years from earth in the direction of the constellation Sagittarius. High-resolution images of the galactic center reveal stars moving at speeds greater than 1500 km/s about an unseen object that lies at the position of a source of radio waves called Sgr A* (Fig. 13.29). By analyzing these motions, astronomers can infer the period T and semi-major axis a of each star's orbit. The mass m_X of the unseen object can be calculated from Kepler's third law in the form given in Eq. (13.17), with the mass of the sun m_S replaced by m_X :

$$T = \frac{2\pi a^{3/2}}{\sqrt{Gm_X}} \qquad \text{so} \qquad m_X = \frac{4\pi^2 a^3}{GT^2}$$

Figure 13.29



This false-color image shows the motions of stars at the center of our galaxy over a 17-year period. Analysis of these orbits by using Kepler's third law indicates that the stars are moving about an unseen object that is some 4.1×10^6 times the mass of the sun. The scale bar indicates a length of 10^{14} m (670 times the distance from the earth to the sun).

The conclusion is that the mysterious dark object at the galactic center has a mass of 8.2×10^{36} kg, or 4.1 *million* times the mass of the sun. Yet observations with radio telescopes show that it has a radius no more than 4.4×10^{10} m, about one-third of the distance from the earth to the sun. These observations suggest that this massive, compact object is a black hole with a Schwarzschild radius of 1.1×10^{10} m. Astronomers hope to improve the resolution of their observations so that they can actually see the event horizon of this black hole.

Additional evidence for the existence of black holes has come from observations of *gravitational radiation*. Einstein's general theory of relativity, which we'll discuss in Chapter 37, predicts that space itself is curved by the presence of massive objects like a planet, star, or black hole. If a massive object accelerates in a certain manner, it produces ripples in the curvature of space that radiate outward from the object. The disturbances caused by such gravitational radiation are incredibly feeble, but can be measured with sensitive detectors if the objects that produce them are very massive and have tremendous accelerations. This can happen when two massive black holes are in close orbits around each other. Each black hole has an acceleration as it moves around its curved orbit, so it emits gravitational radiation that carries away energy. This makes the orbits of the black holes smaller, so they move faster and emit gravitational radiation at an ever faster rate. The orbits keep shrinking until the two black holes finally merge. Since 2015 scientists have detected the gravitational radiation from several such black hole mergers. From the data, they conclude that the merging black holes have masses from 7 to 36 times the mass of the sun. (Rainer Weiss, Kip Thorne, and Barry Barish were awarded the 2017 Nobel Prize in Physics for their contributions to these discoveries.)

Other lines of research suggest that even larger black holes, in excess of 10^9 times the mass of the sun, lie at the centers of other galaxies. Observational and theoretical studies of black holes of all sizes continue to be an exciting area of research in both physics and astronomy.

Test Your Understanding of Section 13.8

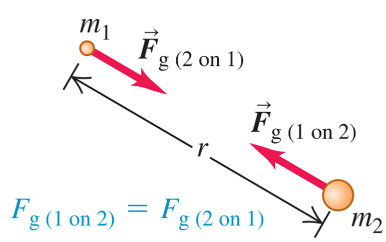
If the sun somehow collapsed to form a black hole, what effect would this event have on the orbit of the earth? The orbit would (i) shrink; (ii) expand; (iii) remain the same size.

Chapter 13 Summary

Newton's law of gravitation: *Any* two particles with masses m_1 and m_2 , a distance r apart, attract each other with forces inversely proportional to r^2. These forces form an action–reaction pair and obey Newton's third law. When two or more objects exert gravitational forces on a particular object, the total gravitational force on that individual object is the vector sum of the forces exerted by the other objects. The gravitational interaction between spherical mass distributions, such as planets or stars, is the same as if all the mass of each distribution were concentrated at the center. (See Examples 13.1, 13.2 and 13.3 and 13.10.)

(13.1)

$$F_g = \frac{Gm_1m_2}{r^2}$$



Gravitational force, weight, and gravitational potential energy: The weight w of an object is the total gravitational force exerted on it by all other objects in the universe. Near the surface of the earth (mass m_E and radius R_E), the weight is essentially equal to the gravitational force of the earth alone. The gravitational potential energy U of two masses m and m_E separated by a distance r is inversely proportional to r. The potential energy is never positive; it is zero only when the two objects are infinitely far apart. (See Examples 13.4 and 13.5.)

(13.3)

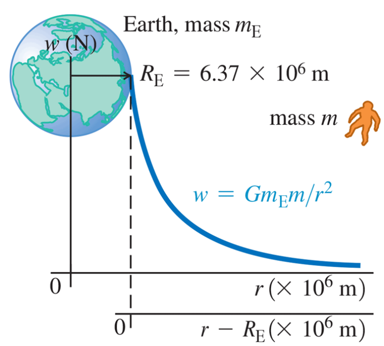
$$w = F_g = \frac{Gm_E m}{R_E^2}$$
(weight at earth′s surface)

(13.4)

$$g = \frac{Gm_E}{R_E^2}$$
(acceleration due to gravity at earth′s surface)

(13.9)

$$U = -\frac{Gm_E m}{r}$$



Orbits: When a satellite moves in a circular orbit, the centripetal acceleration is provided by the gravitational attraction of the earth. Kepler's three laws describe the more general case: an elliptical orbit of a planet around the sun or a satellite around a planet. (See Examples 13.6, 13.7, 13.8 and 13.9.)

(13.10)

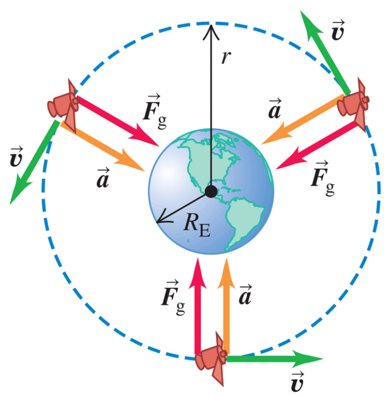
$$v = \sqrt{\frac{Gm_E}{r}}$$

(speed in circular orbit)

(13.12)

$$T = \frac{2\pi r}{v} = 2\pi r \sqrt{\frac{r}{Gm_E}} = \frac{2\pi r^{3/2}}{\sqrt{Gm_E}}$$

(period in circular orbit)



Black holes: If a nonrotating spherical mass distribution with total mass M has a radius less than its Schwarzschild radius R_S , it is called a black hole. The gravitational interaction prevents anything, including light, from escaping from within a sphere with radius R_S . (See Example 13.11.)

(13.30)

$$R_S = \frac{2GM}{c^2}$$

(Schwarzschild radius)

If all of the object is inside its Schwarzschild radius $R_S = 2GM/c^2$, the object is a black hole.

Guided Practice: Gravitation

For assigned homework and other learning materials, go to **Mastering Physics**.

Key Example Variation Problems

Be sure to review Example 13.5 (Section 13.3) before attempting these problems.

VP13.5.1 The *New Horizons* spacecraft (mass 478 kg) was launched from earth in 2006 and flew past Pluto in 2015. What minimum amount of work has to be done on a spacecraft of this mass to send it from the earth's surface to infinitely far away from the earth? Neglect air resistance and the gravitational effects of the sun, moon, and other planets.

VP13.5.2 A piece of spacecraft debris initially at rest falls to the earth's surface from a height above the earth equal to one-half of the earth's radius. Find the speed at which the piece of debris hits the surface. Neglect air resistance and the gravitational pull of the moon.

VP13.5.3 An astronaut on Mars (mass radius) launches a probe straight upward from the surface at What is the maximum height above the surface that the probe reaches? Neglect air resistance and the gravitational pull of the two small moons of Mars.

VP13.5.4 In order to rendezvous with an asteroid passing close to the earth, a spacecraft must be moving at relative to the earth at a distance of from the center of the earth. At what speed must this spacecraft be launched from the earth's surface? Neglect air resistance and the gravitational pull of the moon.

Be sure to review Example 13.6 (Section 13.4) before attempting these problems.

VP13.6.1 You wish to place a spacecraft in a circular orbit around the earth so that its orbital speed will be What are this orbit's (a) radius, (b) altitude above the earth's surface, and (c) period (in hours)?

VP13.6.2 You are designing a spacecraft intended to monitor a human expedition to Mars (mass radius). This spacecraft will orbit around the Martian equator with an orbital period of 24.66 h, the same as the rotation period of Mars, so that it will always be above the same point on the equator. (a) What must be the radius of the orbit? (b) What will be the speed of the spacecraft in its orbit?

VP13.6.3 A spacecraft of mass 1.00×10^3 kg orbits the sun (mass) in a circular orbit of radius (equal to the average distance from the sun to the earth). You wish to move the spacecraft into a smaller circular orbit around the sun of radius (equal to the average distance from the sun to Venus). In doing this, what will be the changes in (a) the spacecraft's kinetic energy, (b) the gravitational potential energy of the sun–

spacecraft system, and (c) the total mechanical energy of the sun–spacecraft system? Neglect the gravitational pulls of the planets on the spacecraft.

VP13.6.4 A satellite of mass is at point at an altitude of above the earth's surface, and traveling at (a) If you wanted to put the satellite into a circular orbit at this altitude above the earth's surface, how much work would you have to do on it at point *P*? (b) If instead you wanted to make the satellite escape the earth, how much work would you have to do on it at point *P*?

Be sure to review Examples 13.7, 13.8, and 13.9 (Section 13.5) before attempting these problems.

VP13.9.1 Some comets are in highly elongated orbits that come very close to the sun at perihelion. The distance from one such comet to the center of the sun is at perihelion and at aphelion. For this comet's orbit, find (a) the semi-major axis, (b) the eccentricity, and (c) the period (in years).

VP13.9.2 The orbit of a certain asteroid around the sun has period 7.85 y and eccentricity 0.250. Find (a) the semi-major axis and the distances from the sun to the asteroid at (b) perihelion and (c) aphelion.

VP13.9.3 The minor planet (33340) 1998 VG_{44} makes exactly two orbits of the sun during the time that Neptune makes three orbits. The eccentricity of the orbit of this minor planet is 0.25, and the orbit radius of Neptune is (a) Find the semi-major axis of this minor planet. (b) Find the distance between the sun and the minor planet at perihelion. At perihelion, is this minor planet inside or outside the orbit of Neptune?

VP13.9.4 In 2017 astronomers discovered a planet orbiting the star HATS-43. The orbit of the planet around HATS-43 has semi-major axis eccentricity 0.173, and period 4.39 days. Find (a) the distance between HATS-43 and the planet at its closest approach and (b) the mass of HATS-43. (For comparison, the distance between the sun and Mercury at its closest approach is and the mass of the sun is)

Bridging Problem: Speeds in an Elliptical Orbit

A comet orbits the sun (mass) in an elliptical orbit of semi-major axis a and eccentricity e. (a) Find expressions for the speeds of the comet at perihelion and aphelion. (b) Evaluate these expressions for Comet Halley (see Example 13.9), and find the kinetic energy, gravitational potential energy, and total mechanical energy for this comet at perihelion and aphelion. Take the mass of Comet Halley to be

Solution Guide

IDENTIFY and SET UP

1. Sketch the situation; show all relevant dimensions. Label the perihelion and aphelion. (See Figure 13.18.)
2. List the unknown quantities, and identify the target variables.
3. Just as for a satellite orbiting the earth, the total mechanical energy is conserved for a comet orbiting the sun. (Why?) What other quantity is conserved as the comet moves in its orbit? (*Hint:* See Section 13.5.)

EXECUTE

4. You'll need at least two equations that involve the two unknown speeds, and you'll need expressions for the sun–comet distances at perihelion and aphelion. (**HINT:** See Fig. 13.18.)
5. Solve the equations for your target variables. Compare your expressions: Which speed is lower? Does this make sense?
6. Use your expressions from step 5 to find the perihelion and aphelion speeds for Comet Halley. (*Hint:* See Appendix F.)
7. Use your results from step 6 to find the kinetic energy gravitational potential energy and total mechanical energy for Comet Halley at perihelion and aphelion.

EVALUATE

8. Check whether your results from part (a) make sense for the special case of a circular orbit
9. In part (b), how do your calculated values of at perihelion and aphelion compare? Does this make sense? What does it mean that is negative?

Video Tutor Solution: **Chapter 13 Bridging Problem**

Questions/Exercises/Problems: Gravitation

•, ••, •••: Difficulty levels. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **DATA**: Problems involving real data, scientific evidence, experimental design, and/or statistical reasoning. **BIO**: Biosciences problems.

Discussion Questions

Q13.1 A student wrote: "The only reason an apple falls downward to meet the earth instead of the earth rising upward to meet the apple is that the earth is much more massive and so exerts a much greater pull." Please comment.

Q13.2 If all planets had the same average density, how would the acceleration due to gravity at the surface of a planet depend on its radius?

Q13.3 Is a pound of butter on the earth the same amount as a pound of butter on Mars? What about a kilogram of butter? Explain.

Q13.4 Example 13.2 (Section 13.1) shows that the acceleration of each sphere caused by the gravitational force is inversely proportional to the mass of that sphere. So why does the force of gravity give all masses the same acceleration when they are dropped near the surface of the earth?

Q13.5 When will you attract the sun more: today at noon, or tonight at midnight? Explain.

Q13.6 Since the moon is constantly attracted toward the earth by the gravitational interaction, why doesn't it crash into the earth?

Q13.7 A spaceship makes a circular orbit with period T around a star. If it were to orbit, at the same distance, a star with three times the mass of the original star, would the new period (in terms of T) be (a) $3T$, (b) $T\sqrt{3}$, (c) T, (d) $T/\sqrt{3}$, or (e) $T/3$?

Q13.8 A planet makes a circular orbit with period T around a star. If the planet were to orbit at the same distance around this star, but the planet had three times as much mass, what would the new period (in terms of T) be: (a) $3T$, (b) $T\sqrt{3}$, (c) T, (d) $T/\sqrt{3}$, or (e) $T/3$?

Q13.9 The sun pulls on the moon with a force that is more than twice the magnitude of the force with which the earth attracts the moon. Why, then, doesn't the sun take the moon away from the earth?

Q13.10 Which takes more fuel: a voyage from the earth to the moon or from the moon to the earth? Explain.

Q13.11 A planet is moving at constant speed in a circular orbit around a star. In one complete orbit, what is the net amount of work done on the planet by the star's gravitational force: positive, negative, or zero? What if the planet's orbit is an ellipse, so that the speed is not constant? Explain your answers.

Q13.12 Does the escape speed for an object at the earth's surface depend on the direction in which it is launched? Explain. Does your answer depend on whether or not you include the effects of air resistance?

Q13.13 If a projectile is fired straight up from the earth's surface, what would happen if the total mechanical energy (kinetic plus potential) is (a) less than zero, and (b) greater than zero? In each case, ignore air resistance and the gravitational effects of the sun, the moon, and the other planets.

Q13.14 Discuss whether this statement is correct: "In the absence of air resistance, the trajectory of a projectile thrown near the earth's surface is an *ellipse*, not a parabola."

Q13.15 The earth is closer to the sun in November than in May. In which of these months does it move faster in its orbit? Explain why.

Q13.16 A communications firm wants to place a satellite in orbit so that it is always directly above the earth's 45th parallel (latitude 45° north). This means that the plane of the orbit will not pass through the center of the earth. Is such an orbit possible? Why or why not?

Q13.17 At what point in an elliptical orbit is the acceleration maximum? At what point is it minimum? Justify your answers.

Q13.18 What would Kepler's third law be for circular orbits if an amendment to Newton's law of gravitation made the gravitational force inversely proportional to r^3? Would this change affect Kepler's other two laws? Explain.

Q13.19 In the elliptical orbit of Comet Halley shown in Fig. 13.21a, the sun's gravity is responsible for making the comet fall inward from aphelion to perihelion. But what is responsible for making the comet move from perihelion back outward to aphelion?

Q13.20 Many people believe that orbiting astronauts feel weightless because they are "beyond the pull of the earth's gravity." How far from the earth would a spacecraft have to travel to be truly beyond the earth's gravitational influence? If a spacecraft were really unaffected by the earth's gravity, would it remain in orbit? Explain. What is the real reason astronauts in orbit feel weightless?

Q13.21 As part of their training before going into orbit, astronauts ride in an airliner that is flown along the same parabolic trajectory as a freely falling projectile. Explain why this gives the same experience of apparent weightlessness as being in orbit.

Exercises

Section 13.1 Newton's Law of Gravitation

13.1 • What is the ratio of the gravitational pull of the sun on the moon to that of the earth on the moon? (Assume the distance of the moon from the sun can be approximated by the distance of the earth from the sun.) Use the data in Appendix F. Is it more accurate to say that the moon orbits the earth, or that the moon orbits the sun?

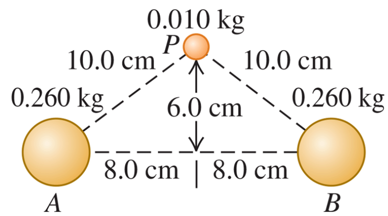
13.2 • You are sitting in the front row in your physics class. Estimate the gravitational force that the instructor exerts on you. Identify the assumptions and approximations you made to reach your answer. How does the magnitude that you estimate for this force compare with the gravity force exerted on you by the earth?

13.3 • **Rendezvous in Space!** A couple of astronauts agree to rendezvous in space after hours. Their plan is to let gravity bring them together. One of them has a mass of 65 kg and the other a mass of 72 kg, and they start from rest 20.0 m apart. (a) Make a free-body diagram of each astronaut, and use it to find his or her initial acceleration. As a rough approximation, we can model the astronauts as uniform spheres. (b) If the astronauts' acceleration remained constant, how many days would they have to wait before reaching each other? (Careful! They *both* have acceleration toward each other.) (c) Would their acceleration, in fact, remain constant? If not, would it increase or decrease? Why?

13.4 •• Two uniform spheres, each with mass M and radius R, touch each other. What is the magnitude of their gravitational force of attraction?

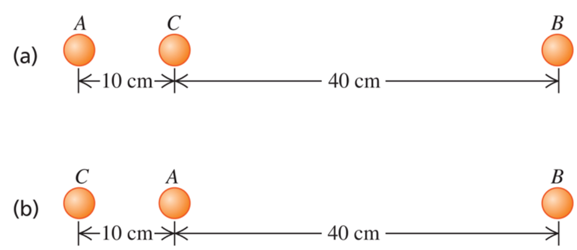
13.5 • Two uniform spheres, each of mass 0.260 kg, are fixed at points A and B (Fig. E13.5). Find the magnitude and direction of the initial acceleration of a uniform sphere with mass 0.010 kg if released from rest at point P and acted on only by forces of gravitational attraction of the spheres at A and B.

Figure E13.5



13.6 •• Find the magnitude and direction of the net gravitational force on mass A due to masses B and C in Fig. E13.6. Each mass is 2.00 kg.

Figure E13.6



13.7 • A typical adult human has a mass of about 70 kg. (a) What force does a full moon exert on such a human when it is directly overhead with its center 378,000 km away? (b) Compare this force with the force exerted on the human by the earth.

13.8 •• An 8.00 kg point mass and a 12.0 kg point mass are held in place 50.0 cm apart. A particle of mass m is released from a point between the two masses 20.0 cm from the 8.00 kg mass along the line connecting the two fixed masses. Find the magnitude and direction of the acceleration of the particle.

13.9 •• A particle of mass $3m$ is located 1.00 m from a particle of mass m. (a) Where should you put a third mass M so that the net gravitational force on M due to the two masses is exactly zero? (b) Is the equilibrium of M at this point stable or unstable (i) for points along the line connecting m and $3m$, and (ii) for points along the line passing through M and perpendicular to the line connecting m and $3m$?

13.10 •• The point masses m and $2m$ lie along the x-axis, with m at the origin and $2m$ at $x = L$. A third point mass M is moved along the x-axis. (a) At what point is the net gravitational force on M due to the other two masses equal to zero? (b) Sketch the x-component of the net force on M due to m and 2 m, taking quantities to the right as positive. Include the regions $x < 0$, $0 < x < L$, and $x > L$. Be especially careful to show the behavior of the graph on either side of $x = 0$ and $x = L$.

Section 13.2 Weight

13.11 •• At what distance above the surface of the earth is the acceleration due to the earth's gravity 0.980 m/s^2 if the acceleration due to gravity at the surface has magnitude 9.80 m/s^2?

13.12 • The mass of Venus is 81.5% that of the earth, and its radius is 94.9% that of the earth. (a) Compute the acceleration due to gravity on the surface of Venus from these data. (b) If a rock weighs 75.0 N on earth, what would it weigh at the surface of Venus?

13.13 • Titania, the largest moon of the planet Uranus, has $\frac{1}{8}$ the radius of the earth and $\frac{1}{1700}$ the mass of the earth. (a) What is the acceleration due to gravity at the surface of Titania? (b) What is the average density of Titania? (This is less than the density of rock, which is one piece of evidence that Titania is made primarily of ice.)

13.14 • Rhea, one of Saturn's moons, has a radius of 764 km and an acceleration due to gravity of 0.265 m/s^2 at its surface. Calculate its mass and average density.

13.15 • A science-fiction author asks for your help. He wants to write about a newly discovered spherically symmetric planet that has the same average density as the earth but with a 25% larger radius. (a) What is g on this planet? (b) If he decides to have his explorers weigh the same on this planet as on earth, how should he change its average density?

Section 13.3 Gravitational Potential Energy

13.16 •• **Volcanoes on Io.** Jupiter's moon Io has active volcanoes (in fact, it is the most volcanically active object in the solar system) that eject material as high as 500 km (or even higher) above the surface. Io has a mass of 8.93×10^{22} kg and a radius of 1821 km. For this calculation, ignore any variation in gravity over the 500 km range of the debris. How high would this material go on earth if it were ejected with the same speed as on Io?

13.17 • Use the results of Example 13.5 (Section 13.3) to calculate the escape speed for a spacecraft (a) from the surface of Mars and (b) from the surface of Jupiter. Use the data in Appendix F. (c) Why is the escape speed for a spacecraft independent of the spacecraft's mass?

13.18 •• Ten days after it was launched toward Mars in December 1998, the *Mars Climate Orbiter* spacecraft (mass 629 kg) was 2.87×10^6 km from the earth and traveling at 1.20×10^4 km$/h$ relative to the earth. At this time, what were (a) the spacecraft's kinetic energy relative to the earth and (b) the potential energy of the earth–spacecraft system?

13.19 •• A planet orbiting a distant star has radius 3.24×10^6 m. The escape speed for an object launched from this planet's surface is 7.65×10^3 m/s. What is the acceleration due to gravity at the surface of the planet?

13.20 •• Estimate the cruising altitude of an airplane during a transatlantic flight. If you are a passenger in the plane when it is at this altitude, what is the percentage change in the gravitational energy of the system of you and the earth compared to when you were in the airport waiting to board the plane? For which of your two locations is the gravitational potential energy greater? (***HINT:*** Use the power series for $(1 + x)^n$ given in Appendix B.)

13.21 • You are standing on the surface of a planet that has spherical symmetry and a radius of 5.00×10^6 m. The gravitational potential energy U of the system composed of you and the planet is $-1.20 \times 10^{+9}$ J if we choose U to be zero when you are very far from the planet. What is the magnitude of the gravity force that the planet exerts on you when you are standing on its surface?

Section 13.4 The Motion of Satellites

13.22 • A satellite of mass m is in a circular orbit around a spherical planet of mass m_{p}. The kinetic energy of the satellite is K_A when its orbit radius is r_A. In terms of r_A, what must the orbit radius be in order for the kinetic energy of the satellite to be $2K_A$?

13.23 • Let the gravitational potential energy of the earth–satellite system be zero in the limit that the orbit radius is very large, so the total mechanical energy of the satellite–earth system is given by Eq. (13.13). If the kinetic energy of the satellite is 2.00×10^6 J, what are the total energy and the gravitational potential energy of the satellite–earth system?

13.24 • An earth satellite moves in a circular orbit with an orbital speed of 6200 m/s. Find (a) the time of one revolution of the satellite; (b) the radial acceleration of the satellite in its orbit.

13.25 • For a satellite to be in a circular orbit 890 km above the surface of the earth, (a) what orbital speed must it be given, and (b) what is the period of the orbit (in hours)?

13.26 •• **Aura Mission.** On July 15, 2004 NASA launched the *Aura* spacecraft to study the earth's climate and atmosphere. This satellite was injected into an orbit 705 km above the earth's surface. Assume a circular orbit. (a) How many hours does it take this satellite to make one orbit? (b) How fast (in km/s) is the *Aura* spacecraft moving?

13.27 •• Two satellites are in circular orbits around a planet that has radius 9.00×10^6 m. One satellite has mass 68.0 kg, orbital radius 7.00×10^7 m, and orbital speed 4800 m/s. The second satellite has mass 84.0 kg and orbital radius 3.00×10^7 m. What is the orbital speed of this second satellite?

13.28 •• **International Space Station.** In its orbit each day, the International Space Station makes 15.65 revolutions around the earth. Assuming a circular orbit, how high is this satellite above the surface of the earth?

13.29 • Deimos, a moon of Mars, is about 12 km in diameter with mass 1.5×10^{15} kg. Suppose you are stranded alone on Deimos and want to play a one-person game of baseball. You would be the pitcher, and you would be the batter! (a) With what speed would you have to throw a baseball so that it would go into a circular orbit just above the surface and return to you so you could hit it? Do you think you could actually throw it at this speed? (b) How long (in hours) after throwing the ball should you be ready to hit it? Would this be an action-packed baseball game?

Section 13.5 Kepler's Laws and the Motion of Planets

13.30 •• **Hot Jupiters.** In 2004 astronomers reported the discovery of a large Jupiter-sized planet orbiting very close to the star HD 179949 (hence the term "hot Jupiter"). The orbit was just $\frac{1}{9}$ the distance of Mercury from our sun, and it takes the planet only 3.09 days to make one orbit (assumed to be circular). (a) What is the mass of the star? Express your answer in kilograms and as a multiple of our sun's mass. (b) How fast (in km/s) is this planet moving?

13.31 •• The star Rho^1 Cancri is 57 light-years from the earth and has a mass 0.85 times that of our sun. A planet has been detected in a circular orbit around Rho^1 Cancri with an orbital radius equal to 0.11 times the radius of the earth's orbit around the sun. What are (a) the orbital speed and (b) the orbital period of the planet of Rho^1 Cancri?

13.32 •• In March 2006, two small satellites were discovered orbiting Pluto, one at a distance of 48,000 km and the other at 64,000 km. Pluto already was known to have a large satellite Charon, orbiting at 19,600 km with an orbital period of 6.39 days. Assuming that the satellites do not affect each other, find the orbital periods of the two small satellites *without* using the mass of Pluto.

13.33 • The dwarf planet Pluto has an elliptical orbit with a semi-major axis of 5.91×10^{12} m and eccentricity 0.249. (a) Calculate Pluto's orbital period. Express your answer in seconds and in earth years. (b) During Pluto's orbit around the sun, what are its closest and farthest distances from the sun?

13.34 •• **Planets Beyond the Solar System.** On October 15, 2001, a planet was discovered orbiting around the star HD 68988. Its orbital distance was measured to be 10.5 million kilometers from the center of the star, and its orbital period was estimated at 6.3 days. What is the mass of HD 68988? Express your answer in kilograms and in terms of our sun’s mass. (Consult Appendix F.)

Section 13.6 Spherical Mass Distributions

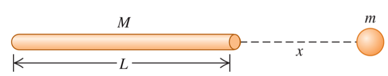
13.35 •• A thin spherical shell has radius $r_A = 4.00$ m and mass $m_A = 20.0$ kg. It is concentric with a second thin spherical shell that has radius $r_B = 6.00$ m and mass $m_B = 40.0$ kg. What is the net gravitational force that the two shells exert on a point mass of 0.0200 kg that is a distance r from the common center of the two shells, for (a) $r = 2.00$ m (inside both shells), (b) $r = 5.00$ m (in the space between the two shells), and (c) $r = 8.00$ m (outside both shells)?

13.36 • A uniform, spherical, 1000.0 kg shell has a radius of 5.00 m. (a) Find the gravitational force this shell exerts on a 2.00 kg point mass placed at the following distances from the center of the shell: (i) 5.01 m, (ii) 4.99 m, (iii) 2.72 m. (b) Sketch a qualitative graph of the magnitude of the gravitational force this sphere exerts on a point mass m as a function of the distance r of m from the center of the sphere. Include the region from $r = 0$ to $r \to \infty$.

13.37 •• A uniform, solid, 1000.0 kg sphere has a radius of 5.00 m. (a) Find the gravitational force this sphere exerts on a 2.00 kg point mass placed at the following distances from the center of the sphere: (i) 5.01 m, (ii) 2.50 m. (b) Sketch a qualitative graph of the magnitude of the gravitational force this sphere exerts on a point mass m as a function of the distance r of m from the center of the sphere. Include the region from $r = 0$ to $r \to \infty$.

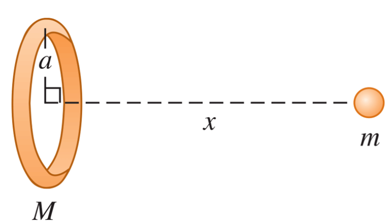
13.38 • **CALC** A thin, uniform rod has length L and mass M. A small uniform sphere of mass m is placed a distance x from one end of the rod, along the axis of the rod (Fig. E13.38). (a) Calculate the gravitational potential energy of the rod–sphere system. Take the potential energy to be zero when the rod and sphere are infinitely far apart. Show that your answer reduces to the expected result when x is much larger than L. (***HINT:***Use the power series expansion for $\ln(1 + x)$ given in Appendix B.) (b) Use $F_x = -dU/dx$ to find the magnitude and direction of the gravitational force exerted on the sphere by the rod (see Section 7.4). Show that your answer reduces to the expected result when x is much larger than L.

Figure E13.38



13.39 • **CALC** Consider the ring- shaped object in Fig. E13.39. A particle with mass m is placed a distance x from the center of the ring, along the line through the center of the ring and perpendicular to its plane. (a) Calculate the gravitational potential energy U of this system. Take the potential energy to be zero when the two objects are far apart. (b) Show that your answer to part (a) reduces to the expected result when x is much larger than the radius a of the ring. (c) Use $F_x = -dU/dx$ to find the magnitude and direction of the force on the particle (see Section 7.4). (d) Show that your answer to part (c) reduces to the expected result when x is much larger than a. (e) What are the values of U and F_x when $x = 0$? Explain why these results make sense.

Figure E13.39



13.40 • Define the gravitational field $\vec{g}$ at some point to be equal to the gravitational force $\vec{F}_{\mathrm{g}}$ on a small object placed at that point divided by the mass m of the object, so $\vec{g} = \vec{F}_{\mathrm{g}}/m$. A spherical shell has mass M and radius R. What is the magnitude of the gravitational field at the following distances from the center of the shell: (a) $r < R$ and (b) $r > R$?

Section 13.7 Apparent Weight and the Earth's Rotation

13.41 • The acceleration due to gravity at the north pole of Neptune is approximately 11.2 m/s^2. Neptune has mass 1.02×10^{26} kg and radius 2.46×10^4 km and rotates once around its axis in about 16 h. (a) What is the gravitational force on a 3.00 kg object at the north pole of Neptune? (b) What is the apparent weight of this same object at Neptune's equator? (Note that Neptune's "surface" is gaseous, not solid, so it is impossible to stand on it.)

13.42 •• **A Visit to Santa.** You decide to visit Santa Claus at the north pole to put in a good word about your splendid behavior throughout the year. While there, you notice that the elf Sneezy, when hanging from a rope, produces a tension of 395.0 N in the rope. If Sneezy hangs from a similar rope while delivering presents at the earth's equator, what will the tension in it be? (Recall that the earth is rotating about an axis through its north and south poles.) Consult Appendix F and start with a free-body diagram of Sneezy at the equator.

Section 13.8 Black Holes

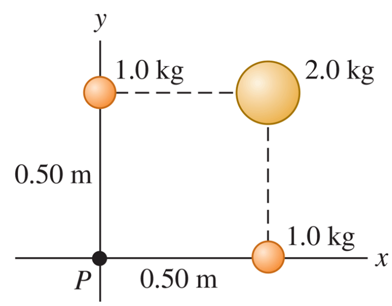
13.43 •• **At the Galaxy's Core.** Astronomers have observed a small, massive object at the center of our Milky Way galaxy (see Section 13.8). A ring of material orbits this massive object; the ring has a diameter of about 15 light-years and an orbital speed of about 200 km/s. (a) Determine the mass of the object at the center of the Milky Way galaxy. Give your answer both in kilograms and in solar masses (one solar mass is the mass of the sun). (b) Observations of stars, as well as theories of the structure of stars, suggest that it is impossible for a single star to have a mass of more than about 50 solar masses. Can this massive object be a single, ordinary star? (c) Many astronomers believe that the massive object at the center of the Milky Way galaxy is a black hole. If so, what must the Schwarzschild radius of this black hole be? Would a black hole of this size fit inside the earth's orbit around the sun?

13.44 • In 2005 astronomers announced the discovery of a large black hole in the galaxy Markarian 766 having clumps of matter orbiting around once every 27 hours and moving at $30,000$ km/s. (a) How far are these clumps from the center of the black hole? (b) What is the mass of this black hole, assuming circular orbits? Express your answer in kilograms and as a multiple of our sun's mass. (c) What is the radius of its event horizon?

Problems

13.45 • Three uniform spheres are fixed at the positions shown in Fig. P13.45. (a) What are the magnitude and direction of the force on a 0.0150 kg particle placed at *P*? (b) If the spheres are in deep outer space and a 0.0150 kg particle is released from rest 300 m from the origin along a line 45° below the $-x-$ axis, what will the particle's speed be when it reaches the origin?

Figure P13.45



13.46 ••• **CP** **Exploring Europa.** There is strong evidence that Europa, a satellite of Jupiter, has a liquid ocean beneath its icy surface. Many scientists think we should land a vehicle there to search for life. Before launching it, we would want to test such a lander under the gravity conditions at the surface of Europa. One way to do this is to put the lander at the end of a rotating arm in an orbiting earth satellite. If the arm is 4.25 m long and pivots about one end, at what angular speed (in rpm) should it spin so that the acceleration of the lander is the same as the acceleration due to gravity at the surface of Europa? The mass of Europa is 4.80×10^{22} kg and its diameter is 3120 km.

13.47 ••• **CP** An experiment is performed in deep space with two uniform spheres, one with mass 50.0 kg and the other with mass 100.0 kg. They have equal radii, $r = 0.20$ m. The spheres are released from rest with their centers 40.0 m apart. They accelerate toward each other because of their mutual gravitational attraction. You can ignore all gravitational forces other than that between the two spheres. (a) Explain why linear momentum is conserved. (b) When their centers are 20.0 m apart, find (i) the speed of each sphere and (ii) the magnitude of the relative velocity with which one sphere is approaching the other. (c) How far from the initial position of the center of the 50.0 kg sphere do the surfaces of the two spheres collide?

13.48 •• **Mission to Titan.** On December 25, 2004, the *Huygens* probe separated from the *Cassini* spacecraft orbiting Saturn and began a 22 day journey to Saturn's moon Titan, on whose surface it landed. Besides the data in Appendix F, it is useful to know that Titan is 1.22×10^6 km from the center of Saturn and has a mass of 1.35×10^{23} kg and a diameter of 5150 km. At what distance from Titan should the gravitational pull of Titan the gravitational pull of Saturn?

13.49 • **Geosynchronous Satellites.** Many satellites are moving in a circle in the earth's equatorial plane. They are at such a height above the earth's surface that they always remain above the same point. (a) Find the altitude of these satellites above the earth's surface. (Such an orbit is said to be *geosynchronous*.) (b) Explain, with a sketch, why the radio signals from these satellites cannot directly reach receivers on earth that are north of 81.3° N latitude.

13.50 •• Two spherically symmetric planets with no atmosphere have the same average density, but planet B has twice the radius of planet A. A small satellite of mass m_A has period T_A when it orbits planet A in a circular orbit that is just above the surface of the planet. A small satellite of mass m_B has period T_B when it orbits planet B in a circular orbit that is just above the surface of the planet. How does T_B compare to T_A?

13.51 ••• What is the escape speed from a 300-km-diameter asteroid with a density of 2500 kg/m^3?

13.52 • • • A landing craft with mass 12,500 kg is in a circular orbit 5.75×10^5 m above the surface of a planet. The period of the orbit is 5800 s. The astronauts in the lander measure the diameter of the planet to be 9.60×10^6 m. The lander sets down at the north pole of the planet. What is the weight of an 85.6 kg astronaut as he steps out onto the planet's surface?

13.53 • • Planet X rotates in the same manner as the earth, around an axis through its north and south poles, and is perfectly spherical. An astronaut who weighs 943.0 N on the earth weighs 915.0 N at the north pole of Planet X and only 850.0 N at its equator. The distance from the north pole to the equator is 18,850 km, measured along the surface of Planet X. (a) How long is the day on Planet X? (b) If a 45,000 kg satellite is placed in a circular orbit 2000 km above the surface of Planet X, what will be its orbital period?

13.54 • • • **CALC** Example 13.10 assumes a constant density for the earth, but Fig. 13.9 shows that this is not accurate. (a) Approximate the density graph in Fig. 13.9 by a straight line running from $13,000\ \mathrm{kg/m^3}$ at $r = 0$ to $3000\ \mathrm{kg/m^3}$ at $r = R_\mathrm{E}$. Divide the earth into concentric shells of width dr and therefore volume $dV = 4\pi r^2 dr$. Integrate to find the mass enclosed as a function of the distance from the earth's center. (b) Find the expression for the magnitude of the gravitational force on a mass m at a distance r from the earth's center.

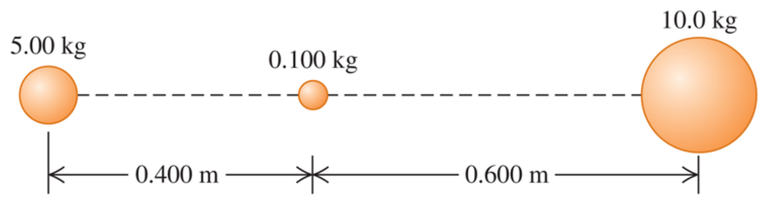
13.55 • • **CP** An astronaut, whose mission is to go where no one has gone before, lands on a spherical planet in a distant galaxy. As she stands on the surface of the planet, she releases a small rock from rest and finds that it takes the rock 0.480 s to fall 1.90 m. If the radius of the planet is 8.60×10^7 m, what is the mass of the planet?

13.56 ••• **CP** Your starship, the *Aimless Wanderer*, lands on the mysterious planet Mongo. As chief scientist-engineer, you make the following measurements: A 2.50 kg stone thrown upward from the ground at 12.0 m/s returns to the ground in 4.80 s; the circumference of Mongo at the equator is 2.00×10^5 km; and there is no appreciable atmosphere on Mongo. The starship commander, Captain Confusion, asks for the following information: (a) What is the mass of Mongo? (b) If the *Aimless Wanderer* goes into a circular orbit 30,000 km above the surface of Mongo, how many hours will it take the ship to complete one orbit?

13.57 •• **CP** You are exploring a distant planet. When your spaceship is in a circular orbit at a distance of 630 km above the planet's surface, the ship's orbital speed is 4900 m/s. By observing the planet, you determine its radius to be 4.48×10^6 m. You then land on the surface and, at a place where the ground is level, launch a small projectile with initial speed 12.6 m/s at an angle of 30.8° above the horizontal. If resistance due to the planet's atmosphere is negligible, what is the horizontal range of the projectile?

13.58 •• The 0.100 kg sphere in Fig. P13.58 is released from rest at the position shown in the sketch, with its center 0.400 m from the center of the 5.00 kg mass. Assume that the only forces on the 0.100 kg sphere are the gravitational forces exerted by the other two spheres and that the 5.00 kg and 10.0 kg spheres are held in place at their initial positions. What is the speed of the 0.100 kg sphere when it has moved 0.400 m to the right from its initial position?

Figure P13.58



13.59 • The magnitude of the gravitational flux Φ_g through a surface with area A is defined such that it is equal to gA when the gravitational field $\vec{g}$ (see Exercise 13.40) is at all points on the surface constant and perpendicular to the surface. For a spherically symmetric object with radius R and mass M, this is the case for a surface that is spherical with radius r and concentric with the object. Consider the case where $r > R$. For this surface calculate the magnitude of the gravitational flux through the surface due to the gravitation field of the object. Does your answer depend on the radius r of the spherical surface? (We'll consider a similar result for the electric field in Chapter 22.)

13.60 ••• A narrow uniform rod has length $2a$. The linear mass density of the rod is ρ, so the mass m of a length l of the rod is ρl. (a) A point mass is located a perpendicular distance r from the center of the rod. Calculate the magnitude and direction of the force that the rod exerts on the point mass. (***HINT:*** Let the rod be along the y-axis with the center of the rod at the origin, and divide the rod into infinitesimal segments that have length dy and that are located at coordinate y. The mass of the segment is $dm = \rho\, dy$. Write expressions for the x- and y-components of the force on the point mass, and integrate from $-a$ to $+a$ to find the components of the total force. Use the integrals in Appendix B.) (b) What does your result become for $a \gg r$? (***HINT:*** Use the power series for $(1 + x)^n$ given in Appendix B.) (c) For $a \gg r$, what is the gravitational field

$\vec{g} = \vec{F}_g/m$ at a distance r from the rod? (d) Consider a cylinder of radius r and length L whose axis is along the rod. As in part (c), let the length of the rod be much greater than both the radius and length of the cylinder. Then the gravitational field is constant on the curved side of the cylinder and perpendicular to it, so the gravitational flux Φ_g through this surface is equal to gA, where $A = 2\pi rL$ is the area of the curved side of the cylinder (see Problem 13.59). Calculate this flux. Write your result in terms of the mass M of the portion of the rod that is inside the cylindrical surface. How does your result depend on the radius of the cylindrical surface?

13.61 • **Falling Hammer.** A hammer with mass m is dropped from rest from a height h above the earth's surface. This height is not necessarily small compared with the radius R_E of the earth. Ignoring air resistance, derive an expression for the speed v of the hammer when it reaches the earth's surface. Your expression should involve h, R_E, and m_E (the earth's mass).

13.62 • (a) Calculate how much work is required to launch a spacecraft of mass m from the surface of the earth (mass m_E, radius R_E) and place it in a circular *low earth orbit*—that is, an orbit whose altitude above the earth's surface is much less than R_E. (As an example, the International Space Station is in low earth orbit at an altitude of about 400 km, much less than $R_E = 6370$ km.) Ignore the kinetic energy that the spacecraft has on the ground due to the earth's rotation. (b) Calculate the minimum amount of additional work required to move the spacecraft from low earth orbit to a very great distance from the earth. Ignore the gravitational effects of the sun, the moon, and the other planets. (c) Justify the statement “In terms of energy, low earth orbit is halfway to the edge of the universe.”

13.63 • **Binary Star—Equal Masses.** Two identical stars with mass M orbit around their center of mass. Each orbit is circular and has radius R, so that the two stars are always on opposite sides of the circle. (a) Find the gravitational force of one star on the other. (b) Find the orbital speed of each star and the period of the orbit. (c) How much energy would be required to separate the two stars to infinity?

13.64 •• CP **Binary Star—Different Masses.** Two stars, with masses M_1 and M_2, are in circular orbits around their center of mass. The star with mass M_1 has an orbit of radius R_1; the star with mass M_2 has an orbit of radius R_2. (a) Show that the ratio of the orbital radii of the two stars equals the reciprocal of the ratio of their masses—that is, $R_1/R_2 = M_2/M_1$. (b) Explain why the two stars have the same orbital period, and show that the period T is given by $T = 2\pi(R_1 + R_2)^{3/2}/\sqrt{G\,(M_1 + M_2)}$. (c) The two stars in a certain binary star system move in circular orbits. The first star, Alpha, has an orbital speed of 36.0 km/s. The second star, Beta, has an orbital speed of 12.0 km/s. The orbital period is 137 d. What are the masses of each of the two stars? (d) One of the best candidates for a black hole is found in the binary system called A0620-0090. The two objects in the binary system are an orange star, V616 Monocerotis, and a compact object believed to be a black hole (see Fig. 13.28). The orbital period of A0620-0090 is 7.75 hours, the mass of V616 Monocerotis is estimated to be 0.67 times the mass of the sun, and the mass of the black hole is estimated to be 3.8 times the mass of the sun. Assuming that the orbits are circular, find the radius of each object’s orbit and the orbital speed of each object. Compare these answers to the orbital radius and orbital speed of the earth in its orbit around the sun.

13.65 ••• Comets travel around the sun in elliptical orbits with large eccentricities. If a comet has speed 2.0×10^4 m/s when at a distance of 2.5×10^{11} m from the center of the sun, what is its speed when at a distance of 5.0×10^{10} m?

13.66 • The planet Uranus has a radius of 25,360 km and a surface acceleration due to gravity of 9.0 $\mathrm{m/s^2}$ at its poles. Its moon Miranda (discovered by Kuiper in 1948) is in a circular orbit about Uranus at an altitude of 104,000 km above the planet's surface. Miranda has a mass of 6.6×10^{19} kg and a radius of 236 km. (a) Calculate the mass of Uranus from the given data. (b) Calculate the magnitude of Miranda's acceleration due to its orbital motion about Uranus. (c) Calculate the acceleration due to Miranda's gravity at the surface of Miranda. (d) Do the answers to parts (b) and (c) mean that an object released 1 m above Miranda's surface on the side toward Uranus will fall *up* relative to Miranda? Explain.

13.67 ••• **CP** Consider a spacecraft in an elliptical orbit around the earth. At the low point, or perigee, of its orbit, it is 400 km above the earth's surface; at the high point, or apogee, it is 4000 km above the earth's surface. (a) What is the period of the spacecraft's orbit? (b) Using conservation of angular momentum, find the ratio of the spacecraft's speed at perigee to its speed at apogee. (c) Using conservation of energy, find the speed at perigee and the speed at apogee. (d) It is necessary to have the spacecraft escape from the earth completely. If the spacecraft's rockets are fired at perigee, by how much would the speed have to be increased to achieve this? What if the rockets were fired at apogee? Which point in the orbit is more efficient to use?

13.68 •• A rocket with mass 5.00×10^3 kg is in a circular orbit of radius 7.20×10^6 m around the earth. The rocket's engines fire for a period of time to increase that radius to 8.80×10^6 m, with the orbit again circular. (a) What is the change in the rocket's kinetic energy? Does the kinetic energy increase or decrease? (b) What is the change in the rocket's gravitational potential energy? Does the potential energy increase or decrease? (c) How much work is done by the rocket engines in changing the orbital radius?

13.69 ••• A 5000 kg spacecraft is in a circular orbit 2000 km above the surface of Mars. How much work must the spacecraft engines perform to move the spacecraft to a circular orbit that is 4000 km above the surface?

13.70 •• A satellite with mass 848 kg is in a circular orbit with an orbital speed of 9640 m/s around the earth. What is the new orbital speed after friction from the earth's upper atmosphere has done -7.50×10^9 J of work on the satellite? Does the speed increase or decrease?

13.71 ••• **CALC** Planets are not uniform inside. Normally, they are densest at the center and have decreasing density outward toward the surface. Model a spherically symmetric planet, with the same radius as the earth, as having a density that decreases linearly with distance from the center. Let the density be 15.0×10^3 $\mathrm{kg/m^3}$ at the center and 2.0×10^3 $\mathrm{kg/m^3}$ at the surface. What is the acceleration due to gravity at the surface of this planet?

13.72 •• One of the brightest comets of the 20th century was Comet Hyakutake, which passed close to the sun in early 1996. The orbital period of this comet is estimated to be about 30,000 years. Find the semi-major axis of this comet's orbit. Compare it to the average sun–Pluto distance and to the distance to Alpha Centauri, the nearest star to the sun, which is 4.3 light-years distant.

13.73 • • • **CALC** An object in the shape of a thin ring has radius a and mass M. A uniform sphere with mass m and radius R is placed with its center at a distance x to the right of the center of the ring, along a line through the center of the ring, and perpendicular to its plane (see Fig. E13.39). What is the gravitational force that the sphere exerts on the ring-shaped object? Show that your result reduces to the expected result when x is much larger than a.

13.74 • • **CALC** A uniform wire with mass M and length L is bent into a semicircle. Find the magnitude and direction of the gravitational force this wire exerts on a point with mass m placed at the center of curvature of the semicircle.

13.75 • **CALC** A shaft is drilled from the surface to the center of the earth (see Fig. 13.25). As in Example 13.10 (Section 13.6), make the unrealistic assumption that the density of the earth is uniform. With this approximation, the gravitational force on an object with mass m, that is inside the earth at a distance r from the center, has magnitude $F_g = Gm_E mr/R_E^3$ (as shown in Example 13.10) and points toward the center of the earth. (a) Derive an expression for the gravitational potential energy $U(r)$ of the object–earth system as a function of the object's distance from the center of the earth. Take the potential energy to be zero when the object is at the center of the earth. (b) If an object is released in the shaft at the earth's surface, what speed will it have when it reaches the center of the earth?

13.76 • • **DATA** For each of the eight planets Mercury to Neptune, the semi-major axis a of their orbit and their orbital period T are as follows:

Planet	Semi-major Axis (10^6 km)	Orbital Period (days)
Mercury	57.9	88.0
Venus	108.2	224.7
Earth	149.6	365.2
Mars	227.9	687.0
Jupiter	778.3	4331
Saturn	1426.7	10,747
Uranus	2870.7	30,589
Neptune	4498.4	59,800

(a) Explain why these values, when plotted as T^2 versus a^3, fall close to a straight line. Which of Kepler's laws is being tested? However, the values of T^2 and a^3 cover such a wide range that this plot is not a very practical way to graph the data. (Try it.) Instead, plot $\log(T)$ (with T in seconds) versus $\log(a)$ (with a in meters). Explain why the data should also fall close to a straight line in such a plot. (b) According to Kepler's laws, what should be the slope of your $\log(T)$ versus $\log(a)$ graph in part (a)? Does your graph have this slope? (c) Using $G = 6.674 \times 10^{-11}\ \mathrm{N{\cdot}m^2/kg^2}$, calculate the mass of the sun from the y-intercept of your graph. How does your calculated value compare with the value given in Appendix F? (d) The only asteroid visible to the naked eye (and then only under ideal viewing conditions) is Vesta, which has an orbital period of 1325.4 days. What is the length of the semi-major axis of Vesta's orbit? Where does this place Vesta's orbit relative to the orbits of the eight major planets? Some scientists argue that Vesta should be called a minor planet rather than an asteroid.

13.77 •• **DATA** For a spherical planet with mass M, volume V, and radius R, derive an expression for the acceleration due to gravity at the planet's surface, g, in terms of the average density of the planet, $\rho = M/V$, and the planet's diameter, $D = 2R$. The table gives the values of D and g for the eight major planets:

Planet	D (km)	g (m/s^2)
Mercury	4879	3.7
Venus	12,104	8.9
Earth	12,756	9.8
Mars	6792	3.7
Jupiter	142,984	23.1
Saturn	120,536	9.0
Uranus	51,118	8.7
Neptune	49,528	11.0

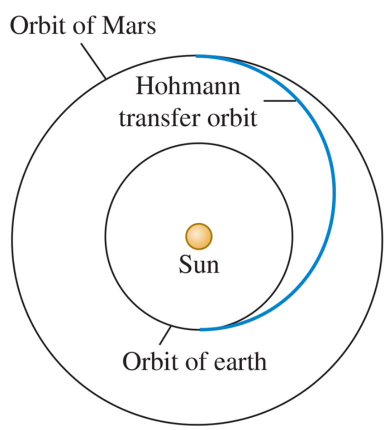
(a) Treat the planets as spheres. Your equation for g as a function of ρ and D shows that if the average density of the planets is constant, a graph of g versus D will be well represented by a straight line. Graph g as a function of D for the eight major planets. What does the graph tell you about the variation in average density? (b) Calculate the average density for each major planet. List the planets in order of decreasing density, and give the calculated average density of each. (c) The earth is not a uniform sphere and has greater density near its center. It is reasonable to assume this might be true for the other planets. Discuss the effect this has on your analysis. (d) If Saturn had the same average density as the earth, what would be the value of g at Saturn's surface?

13.78 ••• DATA For a planet in our solar system, assume that the axis of orbit is at the sun and is circular. Then the angular momentum about that axis due to the planet's orbital motion is $L = Mv$R. (a) Derive an expression for L in terms of the planet's mass M, orbital radius R, and period T of the orbit. (b) Using Appendix F, calculate the magnitude of the orbital angular momentum for each of the eight major planets. (Assume a circular orbit.) Add these values to obtain the total angular momentum of the major planets due to their orbital motion. (All the major planets orbit in the same direction in close to the same plane, so adding the magnitudes to get the total is a reasonable approximation.) (c) The rotational period of the sun is 24.6 days. Using Appendix F, calculate the angular momentum the sun has due to the rotation about its axis. (Assume that the sun is a uniform sphere.) (d) How does the rotational angular momentum of the sun compare with the total orbital angular momentum of the planets? How does the mass of the sun compare with the total mass of the planets? The fact that the sun has most of the mass of the solar system but only a small fraction of its total angular momentum must be accounted for in models of how the solar system formed. (e) The sun has a density that decreases with distance from its center. Does this mean that your calculation in part (c) overestimates or underestimates the rotational angular momentum of the sun? Or doesn't the nonuniform density have any effect?

Challenge Problems

13.79 ••• **Interplanetary Navigation.** The most efficient way to send a spacecraft from the earth to another planet is to use a *Hohmann transfer orbit* (Fig. P13.79). If the orbits of the departure and destination planets are circular, the Hohmann transfer orbit is an elliptical orbit whose perihelion and aphelion are tangent to the orbits of the two planets. The rockets are fired briefly at the departure planet to put the spacecraft into the transfer orbit; the spacecraft then coasts until it reaches the destination planet. The rockets are then fired again to put the spacecraft into the same orbit about the sun as the destination planet. (a) For a flight from earth to Mars, in what direction must the rockets be fired at the earth and at Mars: in the direction of motion or opposite the direction of motion? What about for a flight from Mars to the earth? (b) How long does a one-way trip from the earth to Mars take, between the firings of the rockets? (c) To reach Mars from the earth, the launch must be timed so that Mars will be at the right spot when the spacecraft reaches Mars's orbit around the sun. At launch, what must the angle between a sun–Mars line and a sun–earth line be? Use Appendix F.

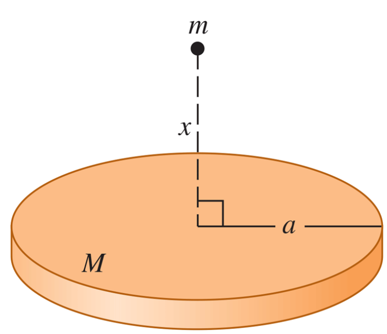
Figure P13.79



13.80 ••• CP **Tidal Forces near a Black Hole.** An astronaut inside a spacecraft, which protects her from harmful radiation, is orbiting a black hole at a distance of 120 km from its center. The black hole is 5.00 times the mass of the sun and has a Schwarzschild radius of 15.0 km. The astronaut is positioned inside the spaceship such that one of her 0.030 kg ears is 6.0 cm farther from the black hole than the center of mass of the spacecraft and the other ear is 6.0 cm closer. (a) What is the tension between her ears? Would the astronaut find it difficult to keep from being torn apart by the gravitational forces? (Since her whole body orbits with the same angular velocity, one ear is moving too slowly for the radius of its orbit and the other is moving too fast. Hence her head must exert forces on her ears to keep them in their orbits.) (b) Is the center of gravity of her head at the same point as the center of mass? Explain.

13.81 ••• **CALC** Mass M is distributed uniformly over a disk of radius a. Find the gravitational force (magnitude and direction) between this disk-shaped mass and a particle with mass m located a distance x above the center of the disk (Fig. P13.81). Does your result reduce to the correct expression as x becomes very large? (*Hint:* Divide the disk into infinitesimally thin concentric rings, use the expression derived in Exercise 13.39 for the gravitational force due to each ring, and integrate to find the total force.)

Figure P13.81



MCAT-Style Passage Problems

Exoplanets. As planets with a wide variety of properties are being discovered outside our solar system, astrobiologists are considering whether and how life could evolve on planets that might be very different from earth. One recently discovered extrasolar planet, or exoplanet, orbits a star whose mass is 0.70 times the mass of our sun. This planet has been found to have 2.3 times the earth's diameter and 7.9 times the earth's mass. For planets in this size range, computer models indicate a relationship between the planet's density and composition:

Density Compared with That of the Earth	Composition
2–3 times	Mostly iron
0.9–2 times	Iron core with a rock mantle
0.4–0.9 times	Iron core with a rock mantle and some lighter elements, such as (water) ice
$<$0.4 times	Hydrogen and/or helium gas

Based on S. Seager et al., "Mass–Radius Relationships for Solid Exoplanets"; arXiv:0707.2895 [astro-ph].

13.82 Based on these data, what is the most likely composition of this planet? (a) Mostly iron; (b) iron and rock; (c) iron and rock with some lighter elements; (d) hydrogen and helium gases.

13.83 How many times the acceleration due to gravity g near the earth's surface is the acceleration due to gravity near the surface of this exoplanet? (a) About $0.29g$; (b) about $0.65g$; (c) about $1.5g$; (d) about $7.9g$.

13.84 Observations of this planet over time show that it is in a nearly circular orbit around its star and completes one orbit in only 9.5 days. How many times the orbital radius r of the earth around our sun is this exoplanet's orbital radius around its sun? Assume that the earth is also in a nearly circular orbit. (a) $0.026r$; (b) $0.078r$; (c) $0.70r$; (d) $2.3r$.

Answers: Gravitation

Chapter Opening Question ?

(iv) For a satellite a distance r from the center of its planet, the orbital speed is proportional to $\sqrt{1/r}$ and the acceleration due to gravity is proportional to $1/r^2$ (see Section 13.4). Hence a particle that orbits close to Saturn has a faster speed and a greater acceleration than one that orbits farther away.

Test Your Understanding

13.1 **(v)** From Eq. (13.1), the gravitational force of the sun (mass m_1) on a planet (mass m_2) a distance r away has magnitude $F_g = Gm_1 m_2/r^2$. Compared to the earth, Saturn has a value of r^2 that is $10^2 = 100$ times greater and a value of m_2 that is also 100 times greater. Hence the *force* that the sun exerts on Saturn has the same magnitude as the force that the sun exerts on earth. The *acceleration* of a planet equals the net force divided by the planet's mass: Since Saturn has 100 times more mass than the earth, its acceleration is $\frac{1}{100}$ as great as that of the earth.

13.2 **(iii)**, **(i)**, **(ii)**, **(iv)** From Eq. (13.4), the acceleration due to gravity at the surface of a planet of mass m_P and radius R_P is $g_P = Gm_P/R_P{}^2$. That is, g_P is directly proportional to the planet's mass and inversely proportional to the square of its radius. It follows that compared to the value of g at the earth's surface, the value of g_P on each planet is (i) $2/2^2 = \frac{1}{2}$ as great; (ii) $4/4^2 = \frac{1}{4}$ as great; (iii) $4/2^2 = 1$ time as great—that is, the same as on earth; and (iv) $2/4^2 = \frac{1}{8}$ as great.

13.3 **(iv)** For a planet of mass m_P and radius R_P, the surface gravity is $g = Gm_P/R_P{}^2$ while the escape speed is $v_{esc} = \sqrt{2Gm_P/R_P}$. Comparing these two expressions, you get $v_{esc} = \sqrt{2gR_P}$. So even if a planet has the same value of g as the earth, its escape speed can be different, depending on how its radius R_P compares with the earth's radius. For the planet Saturn, for example, m_P is about 100 times the earth's mass and R_P is about 10 times the earth's radius. The value of g is different than on earth by a factor of $(100)/(10)^2 = 1$ (i.e., it is the same as on earth), while the escape speed is greater by a factor of $\sqrt{100/10} = 3.2$.

13.4 **(ii)** Equation (13.10) shows that in a smaller-radius orbit, the spacecraft has a faster speed. The negative work done by air resistance decreases the *total* mechanical energy $E = K + U$; the kinetic energy K increases (becomes more positive), but the gravitational potential energy U decreases (becomes more negative) by a greater amount.

13.5 **(iii)** Equation (13.17) shows that the orbital period T is proportional to the $\frac{3}{2}$ power of the semi-major axis a. Hence the orbital period of Comet X is longer than that of Comet Y by a factor of $4^{3/2} = 8$.

13.6 **No.** Our analysis shows that there is *zero* gravitational force inside a hollow spherical shell. Hence visitors to the interior of a hollow planet would find themselves weightless, and they could not stand or walk on the planet's inner surface.

13.7 **(iv)** The discussion following Eq. (13.27) shows that the difference between the acceleration due to gravity at the equator and at the poles is v^2/R_E. Since this planet has the same radius and hence the same circumference as the earth, the speed v at its equator must be 10 times the speed at the earth's equator. Hence v^2/R_E is $10^2 = 100$ times greater than for the earth, or $100(0.0339 \text{ m/s}^2) = 3.39 \text{ m/s}^2$. The acceleration due to gravity at the poles is 9.80 m/s^2, while at the equator it is dramatically less, $9.80 \text{ m/s}^2 - 3.39 \text{ m/s}^2 = 6.41 \text{ m/s}^2$. You can show that if this planet were to rotate 17.0 times faster than the earth, the acceleration due to gravity at the equator would be *zero* and loose objects would fly off the equator's surface!

13.8 **(iii)** If the sun collapsed into a black hole (which, according to our understanding of stars, it cannot do), the sun would have a much smaller radius but the same mass. The sun's gravitational force on the earth doesn't depend on the sun's radius, so the earth's orbit would be unaffected.

Key Example Variation Problems

VP13.5.1 2.99×10^{10} J

VP13.5.2 6.46×10^{3} m/s

VP13.5.3 1.88×10^{6} m

VP13.5.4 1.39×10^{4} m/s

VP13.6.1
a. 2.49×10^{7} m
b. 1.85×10^{7} m
c. 10.9 h

VP13.6.2
a. 2.04×10^{7} m
b. 1.45×10^{3} m/s

VP13.6.3
a. $+1.72 \times 10^{11}$ J
b. -3.44×10^{11} J
c. -1.72×10^{11} J

VP13.6.4
a. -1.19×10^{10} J
b. $+1.83 \times 10^{10}$ J

VP13.9.1
a. 1.50×10^{12} m
b. 0.996
c. 31.8 y

VP13.9.2
a. 5.91×10^{11} m
b. 4.43×10^{11} m
c. 7.39×10^{11} m

VP13.9.3
a. 5.9×10^{12} m
b. 4.4×10^{12} m; inside

VP13.9.4
a. 6.13×10^{9} m
b. 1.67×10^{30} kg

Bridging Problem

(a) Perihelion: $v_{\mathrm{P}} = \sqrt{\frac{Gm_{\mathrm{S}}}{a}\frac{(1+e)}{(1-e)}}$

aphelion: $v_{\mathrm{A}} = \sqrt{\frac{Gm_{\mathrm{S}}}{a}\frac{(1-e)}{(1+e)}}$

(b) $v_{\mathrm{P}} = 54.4\ \mathrm{km/s},\ v_A = 0.913\ \mathrm{km}/s$; $K_{\mathrm{P}} = 3.26 \times 10^{23}$ J,

$U_{\mathrm{P}} = -3.31 \times 10^{23}$ J, $E_{\mathrm{P}} = -5.47 \times 10^{21}$ J;

$K_{\mathrm{A}} = 9.17 \times 10^{19}$ J, $U_{\mathrm{A}} = -5.56 \times 10^{21}$ J,

$E_{\mathrm{A}} = -5.47 \times 10^{21}$ J

Chapter 14
Periodic Motion

? Dogs walk with much quicker strides than do humans. Is this primarily because, compared to human legs, dogs′ legs (i) are shorter; (ii) are less massive; (iii) have a higher ratio of muscle to fat; (iv) have paws rather than toes; or (v) more than one of these?

Learning Outcomes

In this chapter, you'll learn...

- 14.1 How to describe oscillations in terms of amplitude, period, frequency, and angular frequency.
- 14.2 How to do calculations with simple harmonic motion, an important type of oscillation.
- 14.3 How to use energy concepts to analyze simple harmonic motion.
- 14.4 How to apply the ideas of simple harmonic motion to different physical situations.
- 14.5 How to analyze the motions of a simple pendulum.
- 14.6 What a physical pendulum is, and how to calculate the properties of its motion.
- 14.7 What determines how rapidly an oscillation dies out.
- 14.8 How a driving force applied to an oscillator at a particular frequency can cause a very large response, or resonance.

You'll need to review...

- 1.3 Time standards.
- 3.4 Uniform circular motion.
- 6.3 Hooke's law.
- 7.2, 7.4 Elastic potential energy; relating force and potential energy.
- 9.3 Relating angular motion and linear motion.

10.2 Newton's second law for rotational motion.

Many kinds of motion repeat themselves over and over: the vibration of a quartz crystal in a watch, the swinging pendulum of a grandfather clock, the sound vibrations produced by a clarinet or an organ pipe, and the back-and-forth motion of the pistons in a car engine. This kind of motion, called **periodic motion** or **oscillation**, is the subject of this chapter. Understanding periodic motion will be essential for our later study of waves, sound, alternating electric currents, and light.

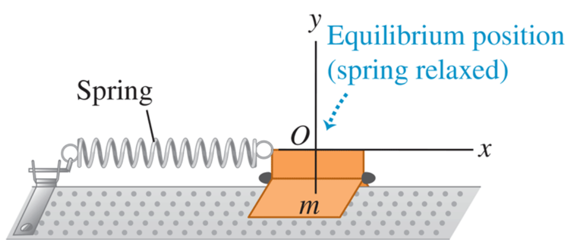
An object that undergoes periodic motion always has a stable equilibrium position. When it is moved away from this position and released, a force or torque comes into play to pull it back toward equilibrium. But by the time it gets there, it has picked up some kinetic energy, so it overshoots, stopping somewhere on the other side, and is again pulled back toward equilibrium. Picture a ball rolling back and forth in a round bowl or a pendulum that swings back and forth past its straight-down position.

In this chapter we'll concentrate on two simple examples of systems that can undergo periodic motions: spring-mass systems and pendulums. We'll also study why oscillations often tend to die out with time and why some oscillations can build up to greater and greater displacements from equilibrium when periodically varying forces act.

14.1 Describing Oscillation

Figure 14.1 shows one of the simplest systems that can have periodic motion. An object with mass m rests on a frictionless horizontal guide system, such as a linear air track, so it can move along the x-axis only. The object is attached to a spring of negligible mass that can be either stretched or compressed. The left end of the spring is held fixed, and the right end is attached to the object. The spring force is the only horizontal force acting on the object; the vertical normal and gravitational forces always add to zero.

Figure 14.1

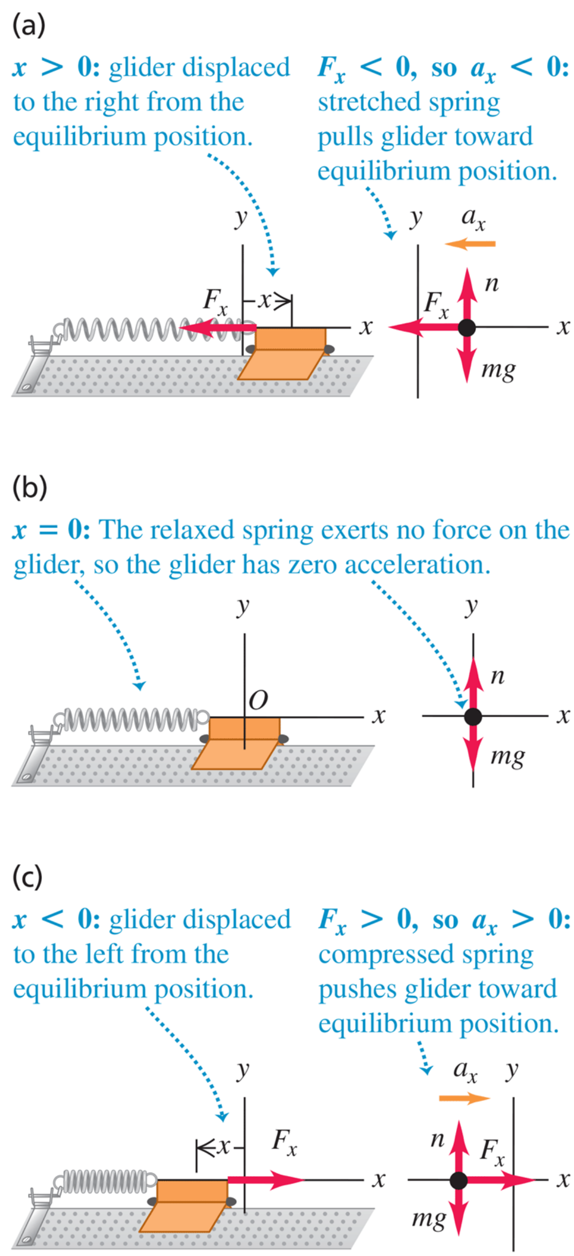


A system that can have periodic motion.

It's simplest to define our coordinate system so that the origin O is at the equilibrium position, where the spring is neither stretched nor compressed. Then x is the x-component of the **displacement** of the object from equilibrium and is also the change in the length of the spring. The spring exerts a force on the object with x-component F_x , and the x-component of acceleration is $a_x = F_x$ /m.

Figure 14.2 shows the object for three different displacements of the spring. Whenever the object is displaced from its equilibrium position, the spring force tends to restore it to the equilibrium position. We call a force with this character a **restoring force**. Oscillation can occur only when there is a restoring force tending to return the system to equilibrium.

Figure 14.2



Model for periodic motion. When the object is displaced from its equilibrium position at $x = 0$, the spring exerts a restoring force back toward the equilibrium position.

Let's analyze how oscillation occurs in this system. If we displace the object to the right to $x = A$ and then let go, the net force and the acceleration are to the left (Fig. 14.2a). The speed increases as the object approaches the equilibrium position O. When the object is at O, the net force acting on it is zero (Fig. 14.2b), but because of its motion it *overshoots* the equilibrium position. On the other side of the equilibrium position the object is still moving to the left, but the net force and the acceleration are to the right (Fig. 14.2c); hence the speed decreases until the object comes to a stop. We'll show later that with an ideal spring, the stopping point is at $x = -A$. The object then accelerates to the right, overshoots equilibrium again, and stops at the starting point $x = A$, ready to repeat the whole process. The object is oscillating! If there is no friction or other force to remove mechanical energy from the system, this motion repeats forever; the restoring force perpetually draws the object back toward the equilibrium position, only to have the object overshoot time after time.

In different situations the force may depend on the displacement x from equilibrium in different ways. But oscillation *always* occurs if the force is a *restoring* force that tends to return the system to equilibrium.

Amplitude, Period, Frequency, and Angular Frequency

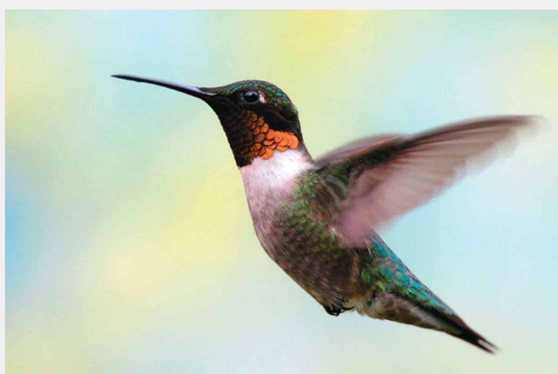
Here are some terms that we'll use in discussing periodic motions of all kinds:

The **amplitude** of the motion, denoted by A, is the maximum magnitude of displacement from equilibrium—that is, the maximum value of $|x|$. It is always positive. If the spring in Fig. 14.2 is an ideal one, the total overall range of the motion is $2A$. The SI unit of A is the meter. A complete vibration, or **cycle**, is one complete round trip—say, from A to $-A$ and back to A, or from O to A, back through O to $-A$, and back to O. Note that motion from one side to the other (say, $-A$ to A) is a half-cycle, not a whole cycle.

BIO Application

Wing Frequencies

The ruby-throated hummingbird (*Archilochus colubris*) normally flaps its wings at about 50 Hz, producing the sound that gives hummingbirds their name. Insects can flap their wings at even faster rates, from 330 Hz for a house fly and 600 Hz for a mosquito to an amazing 1040 Hz for the tiny biting midge.



The **period**, T, is the time to complete one cycle. It is always positive. The SI unit is the second, but it is sometimes expressed as "seconds per cycle."

The **frequency**, f, is the number of cycles in a unit of time. It is always positive. The SI unit of frequency is the *hertz*, named for the 19th-century German physicist Heinrich Hertz:

$$1 \text{ hertz} = 1 \text{ Hz} = 1 \text{ cycle/s} = 1 \text{ s}^{-1}$$

The **angular frequency**, ω, is 2π times the frequency:

$$\omega = 2\pi f$$

We'll learn shortly why ω is a useful quantity. It represents the rate of change of an angular quantity (not necessarily related to a rotational motion) that is always measured in radians, so its units are rad/s. Since f is in cycle/s, we may regard the number 2π as having units rad/cycle.

By definition, period and frequency are reciprocals of each other:

(14.1)

Period

In periodic motion
frequency and period
are reciprocals of each other.

$$f = \frac{1}{T} \qquad T = \frac{1}{f} \tag{14.1}$$

Frequency

Also, from the definition of ω,

(14.2)

Angular frequency
related to frequency
and period

Frequency

$$\omega = 2\pi f = \frac{2\pi}{T} \tag{14.2}$$

Period

CAUTION One period spans a complete cycle Keep in mind that the period of an oscillation is the time for a complete cycle—for example, the time to travel from $x = -A$ to $x = +A$ *and* back again to $x = -A$.

Example 14.1 Period, frequency, and angular frequency

An ultrasonic transducer used for medical diagnosis oscillates at $6.7\ \mathrm{MHz} = 6.7 \times 10^6\ \mathrm{Hz}$. How long does each oscillation take, and what is the angular frequency?

IDENTIFY and SET UP The target variables are the period T and the angular frequency ω. We can find these from the given frequency f in Eqs. (14.1) and (14.2).

EXECUTE From Eqs. (14.1) and (14.2),

$$\begin{aligned} T &= \frac{1}{f} = \frac{1}{6.7 \times 10^6\ \mathrm{Hz}} = 1.5 \times 10^{-7}\ \mathrm{s} = 0.15\ \mu\mathrm{s} \\ \omega &= 2\pi f = (6.7 \times 10^6\ \mathrm{Hz}) \\ &= (2\pi\ \mathrm{rad/cycle})(6.7 \times 10^6\ \mathrm{cycle/s}) = 4.2 \times 10^7\ \mathrm{rad/s} \end{aligned}$$

EVALUATE This is a very rapid vibration, with large f and ω and small T. A slow vibration has small f and ω and large T.

KEY CONCEPT

The period of an oscillation is the reciprocal of the oscillation frequency. The angular frequency equals the frequency multiplied by 2π.

Video Tutor Solution: Example 14.1

Test Your Understanding of Section 14.1

An object like that shown in Fig. 14.2 oscillates back and forth. For each of the following values of the object's x-velocity v_x and x-acceleration a_x, state whether its displacement x is positive, negative, or zero. (a) $v_x > 0$ and $a_x > 0$; (b) $v_x > 0$ and $a_x < 0$; (c) $v_x < 0$ and $a_x > 0$; (d) $v_x < 0$ and $a_x < 0$; (e) $v_x = 0$ and $a_x < 0$; (f) $v_x > 0$ and $a_x = 0$.

14.2 Simple Harmonic Motion

The simplest kind of oscillation occurs when the restoring force F_x is *directly proportional* to the displacement from equilibrium x This happens if the spring in Figs. 14.1 and 14.2 is an ideal one that obeys *Hooke's law* (see Section 6.3). The constant of proportionality between F_x and x is the force constant k On either side of the equilibrium position, F_x and x always have opposite signs. In Section 6.3 we represented the force acting *on* a stretched ideal spring as F_x kx The x-component of force the spring exerts *on the object* is the negative of this, so

(14.3)

Restoring force exerted by an ideal spring

x-component of force

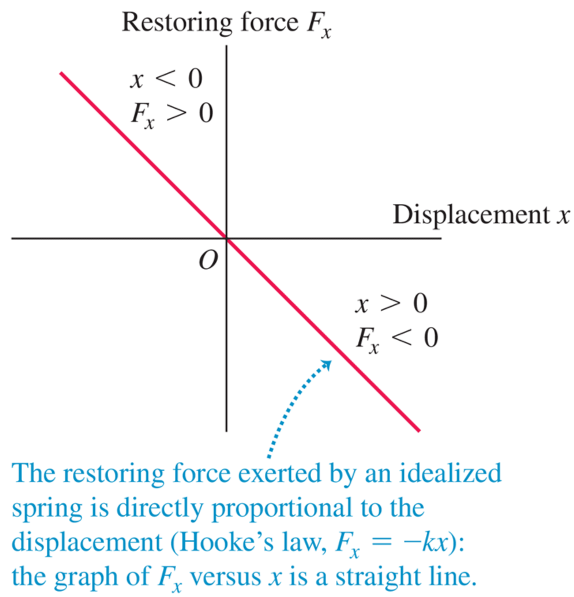
$$F_x = -kx \qquad (14.3)$$

Displacement

Force constant of spring

This equation gives the correct magnitude and sign of the force, whether x is positive, negative, or zero (Fig. 14.3). The force constant k is always positive and has units of (a useful alternative set of units is We are assuming that there is no friction, so Eq. (14.3) gives the *net* force on the object.

Figure 14.3



An idealized spring exerts a restoring force that obeys Hooke's law, $F_x \qquad kx$ Oscillation with such a restoring force is called simple harmonic motion.

When the restoring force is directly proportional to the displacement from equilibrium, as given by Eq. (14.3)*, the oscillation is called* **simple harmonic motion (SHM)**. The acceleration $a_x \quad d \; x \; dt \quad F_x \; m$ of an object in SHM is

(14.4)

Equation for simple harmonic motion

x-component of acceleration

Force constant of restoring force

$$a_x = \underbrace{\frac{d^2x}{dt^2}}_{\text{Second derivative of displacement}} = -\frac{k}{m}x \qquad (14.4)$$

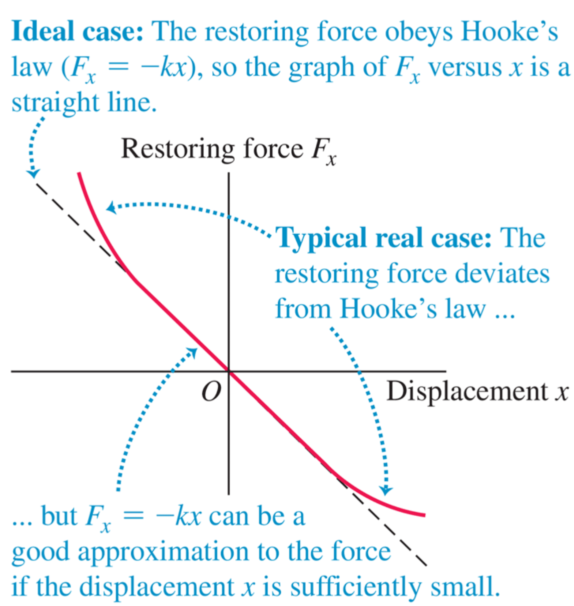
Displacement

Mass of object

The minus sign means that, in SHM, the acceleration and displacement always have opposite signs. This acceleration is *not* constant, so don't even think of using the constant-acceleration equations from Chapter 2. We'll see shortly how to solve this equation to find the displacement x as a function of time. An object that undergoes simple harmonic motion is called a **harmonic oscillator**.

Why is simple harmonic motion important? Not all periodic motions are simple harmonic; in periodic motion in general, the restoring force depends on displacement in a more complicated way than in Eq. (14.3). But in many systems the restoring force is *approximately* proportional to displacement if the displacement is sufficiently small (Fig. 14.4). That is, if the amplitude is small enough, the oscillations of such systems are approximately simple harmonic and therefore approximately described by Eq. (14.4). Thus we can use SHM as an approximate model for many different periodic motions, such as the vibration of a tuning fork, the electric current in an alternating-current circuit, and the oscillations of atoms in molecules and solids.

Figure 14.4



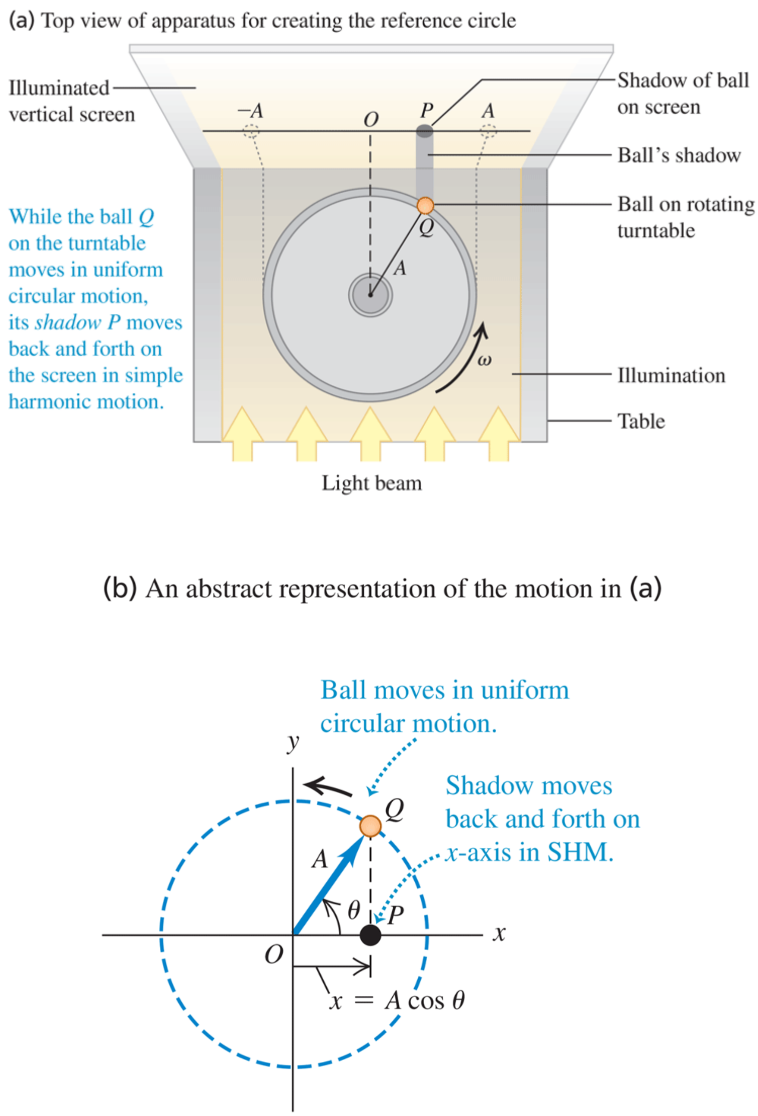
In most real oscillations Hooke's law applies provided the object doesn't move too far from equilibrium. In such a case small-amplitude oscillations are approximately simple harmonic.

Circular Motion and the Equations of SHM

To explore the properties of simple harmonic motion, we must express the displacement x of the oscillating object as a function of time, x t The second derivative of this function, d x dt must be equal to k m times the function itself, as required by Eq. (14.4). As we mentioned, the formulas for constant acceleration from Section 2.4 are no help because the acceleration changes constantly as the displacement x changes. Instead, we'll find x t by noting that SHM is related to *uniform circular motion*, which we studied in Section 3.4.

Figure 14.5a shows a top view of a horizontal disk of radius A with a ball attached to its rim at point Q The disk rotates with constant angular speed ω (measured in so the ball moves in uniform circular motion. A horizontal light beam casts a shadow of the ball on a screen. The shadow at point P oscillates back and forth as the ball moves in a circle. We then arrange an object attached to an ideal spring, like the combination shown in Figs. 14.1 and 14.2, so that the object oscillates parallel to the shadow. We'll prove that the motions of the object and of the ball's shadow are *identical* if the amplitude of the object's oscillation is equal to the disk radius A and if the angular frequency πf of the oscillating object is equal to the angular speed ω of the rotating disk. That is, *simple harmonic motion is the projection of uniform circular motion onto a diameter*.

Figure 14.5



(a) Relating uniform circular motion and simple harmonic motion. (b) The ball's shadow moves exactly like an object oscillating on an ideal spring.

We can verify this remarkable statement by finding the acceleration of the shadow at P and comparing it to the acceleration of an object undergoing SHM, given by Eq. (14.4). The circle in which the ball moves so that its projection matches the motion of the oscillating object is called the **reference circle;** we'll call the point Q the *reference point*. We take the reference circle to lie in the xy-plane, with the origin O at the center of the circle (Fig. 14.5b). At time t the vector OQ from the origin to reference point Q makes an angle θ with the positive x-axis. As point Q moves around the reference circle with constant angular speed ω vector OQ rotates with the same angular speed. Such a rotating vector is called a **phasor**. (This term was in use long before the invention of the *Star Trek* stun gun with a similar name.) We'll use phasors again when we study alternating-current circuits in Chapter 31 and the interference of light in Chapters 35 and 36.

The x-component of the phasor at time t is just the x-coordinate of the point Q

(14.5)

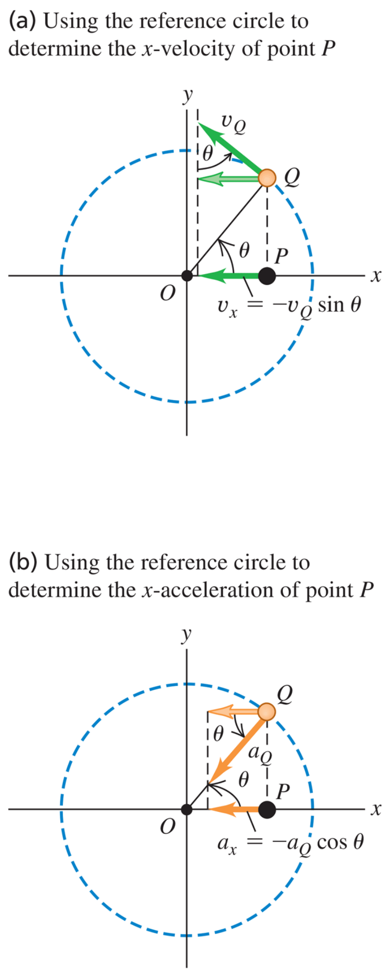
$$x \quad A \quad \theta$$

This is also the x-coordinate of the shadow P which is the *projection* of Q onto the x-axis. Hence the x-velocity of the shadow P along the x-axis is equal to the x-component of the velocity vector of point Q (Fig. 14.6a), and the x-acceleration of P is equal to the x-component of the acceleration vector of Q (Fig. 14.6b). Since point Q is in uniform circular motion, its acceleration vector a_Q is always directed toward O Furthermore, the magnitude of a_Q is constant and given by the angular speed squared times the radius of the circle (see Section 9.3):

(14.6)

$$a_Q \quad \omega \; A$$

Figure 14.6



The (a) x-velocity and (b) x-acceleration of the ball's shadow P (see Fig. 14.5) are the x-components of the velocity and acceleration vectors, respectively, of the ball Q

Figure 14.6b shows that the x-component of a_Q is $a_x \quad a_Q \quad \theta$

Combining this with Eqs. (14.5) and (14.6), we get that the acceleration of point P is

(14.7)

$$a_x \quad a_Q \quad \theta \quad \omega \; A \quad \theta$$

(14.8)

$$a_x \qquad \omega \;\; x$$

The acceleration of point P is directly proportional to the displacement x and always has the opposite sign. These are precisely the hallmarks of simple harmonic motion.

Equation (14.8) is *exactly* the same as Eq. (14.4) for the acceleration of a harmonic oscillator, provided that the angular speed ω of the reference point Q is related to the force constant k and mass m of the oscillating object by

(14.9)

$$\omega \qquad \frac{k}{m} \qquad\qquad \omega \qquad \overline{\frac{k}{m}}$$

We have been using the same symbol ω for the angular *speed* of the reference point Q and the angular *frequency* of the oscillating point P The reason is that these quantities are equal! If point Q makes one complete revolution in time T then point P goes through one complete cycle of oscillation in the same time; hence T is the period of the oscillation. During time T the point Q moves through π radians, so its angular speed is $\omega \qquad \pi \;\; T$ But this is the same as Eq. (14.2) for the angular frequency of the point P which verifies our statement about the two interpretations of ω This is why we introduced angular frequency in Section 14.1; this quantity makes the connection between oscillation and circular motion. So we reinterpret Eq. (14.9) as an expression for the angular frequency of simple harmonic motion:

(14.10)

Angular frequency for simple harmonic motion

$$\omega = \sqrt{\frac{k}{m}} \qquad (14.10)$$

Force constant of restoring force

Mass of object

When you start an object oscillating in SHM, the value of ω is not yours to choose; it is predetermined by the values of k and m The units of k are or so k m is in When we take the square root in Eq. (14.10), we get or more properly because this is an *angular* frequency (recall that a radian is not a true unit).

CAUTION Don't confuse frequency and angular frequency You can run into trouble if you don't make the distinction between frequency f and angular frequency ω πf Frequency tells you how many cycles of oscillation occur per second, while angular frequency tells you how many radians per second this corresponds to on the reference circle. In solving problems, pay careful attention to whether the goal is to find f or ω

According to Eqs. (14.1) and (14.2), the frequency f and period T are

(14.11)

Frequency for simple harmonic motion

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad (14.11)$$

Angular frequency

Force constant of restoring force

Mass of object

(14.12)

Period for simple harmonic motion

$$T = \frac{1}{f} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \qquad (14.12)$$

Mass of object

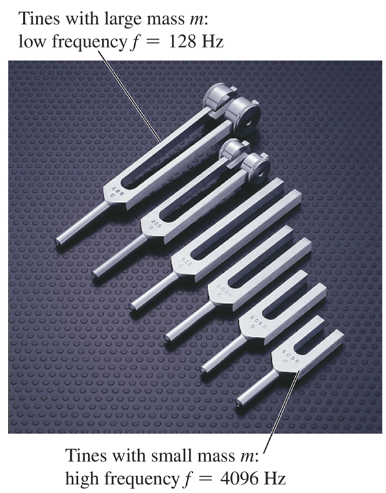
Force constant of restoring force

Frequency

Angular frequency

We see from Eq. (14.12) that a larger mass m will have less acceleration and take a longer time for a complete cycle (Fig. 14.7). A stiffer spring (one with a larger force constant k) exerts a greater force at a given deformation x causing greater acceleration and a shorter time T per cycle.

Figure 14.7



The greater the mass m in a tuning fork's tines, the lower the frequency of oscillation $f = \frac{1}{2\pi}\sqrt{k/m}$ and the lower the pitch of the sound that the tuning fork produces.

Period and Amplitude in SHM

Equations (14.11) and (14.12) show that the period and frequency of simple harmonic motion are completely determined by the mass m and the force constant k *In simple harmonic motion the period and frequency do not depend on the amplitude A*. For given values of m and k the time of one complete oscillation is the same whether the amplitude is large or small. Equation (14.3) shows why we should expect this. Larger A means that the object reaches larger values of x and is subjected to larger restoring forces. This increases the average speed of the object over a complete cycle; this exactly compensates for having to travel a larger distance, so the same total time is involved.

The oscillations of a tuning fork are essentially simple harmonic motion, so it always vibrates with the same frequency, independent of amplitude. This is why a tuning fork can be used as a standard for musical pitch. If it were not for this characteristic of simple harmonic motion, it would be impossible to play most musical instruments in tune. If you encounter an oscillating object with a period that *does* depend on the amplitude, the oscillation is *not* simple harmonic motion.

Example 14.2 Angular frequency, frequency, and period in SHM

WITH VARIATION PROBLEMS

A spring is mounted horizontally, with its left end fixed. A spring balance attached to the free end and pulled toward the right (Fig. 14.8a) indicates that the stretching force is proportional to the displacement, and a force of 6.0 N causes a displacement of 0.030 m. We replace the spring balance with a 0.50 kg glider, pull it 0.020 m to the right along a frictionless air track, and release it from rest (Fig. 14.8b). (a) Find the force constant k of the spring. (b) Find the angular frequency ω frequency f and period T of the resulting oscillation.

Figure 14.8