1 Function between Metric Space

Recall a function $f : A \to B$ from a set A to a set B is an assignment to every $a \in A$, an element $f(a) \in B$.

Domain of f: A
Range of f: $f(A) \subset B$

Injective if $\forall x, y \in A, x \neq y$, then $f(x) \neq f(y)$.
Surjective if $f(A) = B$.
Define reverse image $f^{-1}(E) = \{a \in A | f(a) \in E,\ \forall E \subset B\}$.

Lemma: Let $A' \subset A, B' \subset B$, $f : A \to B$. then

$$f(A') \subset B' \iff A' \subset f^{-1}(B')$$

Proof:

$$\begin{aligned} F(A') \subset B' &\iff \forall x \in A',\ f(x) \in B' \\ &\iff \forall x \in A',\ x \in f^{-1}(B') \\ &\iff A' \subset f(B') \end{aligned}$$

2 Metric Space

Let (X, d_X) and (Y, d_Y) be metric spaces.

A function $f : X \to Y$ is *continuous* at $p \in X$ if $\forall \varepsilon > 0$, $\exists \varepsilon > 0$, such that

$$\forall x \in X, \text{ with } d_x(x, p) < \delta \implies d_Y(f(x), f(p)) < \varepsilon.$$

Alternatively, $f(B_\delta(p)) \subset B_\varepsilon(f(p))$.

Theorem: A function $f : X \to Y$ is continuous iff $\forall V \subset Y$ open, $f^{-1}(V)$ is open.

Remark: It only uses the notion of open sets; works for general topological spaces.

Proof: $\implies$: Suppose f is continuous. We need to show $\forall V \subset Y$ open, $f^{-1}(V)$ is open.
That is, $\forall p \in f^{-1}(V)$, we need to show $\exists \delta > 0$ s.t. $B_\delta(p) \subset f^{-1}(V)$.

Since $f(p) \in V$ and V is open, we have $\varepsilon > 0$ and $B_\varepsilon(f(p)) \subset V$. By continuity of f, $\exists B_\delta(p)$, such that $f(B_\delta(p)) \subset B_\varepsilon(f(p))$. Hence

$$f(B_\delta(p)) \subset B_\varepsilon(f(p)) \subset V \implies B_\delta(p) \subset f^{-1}(V)$$

Hence f^{-1} is open.

$\Longleftarrow$: If $f^{-1}(V)$ is open for any $V \subset Y$ open, we need to show that $\forall p \in X$, $\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$$f(B_\delta(p)) \subset B_\varepsilon(f(p)).$$

Since $B_\varepsilon(f(p))$ is open, hence $f^{-1}(B_\varepsilon(f(p)))$ is open, and contains p. By definition of open sets, $\exists \delta > 0$ s.t.

$$B_\delta(p) \subset f^{-1}(B_\varepsilon(f(p)))$$
$$\iff f(B_\delta(p)) \subset B_\varepsilon(f(p))$$

Definition (Limit of a Function)

Given $E \subset X$, let $f : E \to Y$. Suppose p is a limit point of E.

$$\lim_{x \to p} f(x) = q$$

if there is a point $q \in Y$ such that $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$f(B_\delta^x(p) \cap E) \subset B_\varepsilon(q)$$

(Punctured ball, which means that it does not include p).

In other words, $\forall x \in E$ where $0 < d_x(x,p) < \delta$, we have

$$d(f(x), q) < \varepsilon$$

Remark: Recall p is a limit point of E if $\forall \delta > 0$,

$$B_\delta^x(p) \cap E \neq \emptyset$$

E' is the set of limit points of E. E' not necessarily subset or superset of E.

$E = E^{\text{isolated}} \cup E'$ where $E^{\text{isolated}} = \{x \in E \mid \exists \varepsilon > 0, B_\varepsilon(x) \cap E = \{x\}\}$

Theorem

Given $p \in E'$ we have $lim_{x \to p} f(x) = q$ iff for all convergent sequence $p_n \to p$ with $p_n \in E$, $p_n \neq p$, $\lim f(p_n)\ q$.

Proof: $\implies$: Suppose $\lim_{x\to p} f(x) = q$. And suppose $p_n \to p,\ p_n \in E,\ p_n \neq p$. We need to show $\lim_{n\to\infty} f(x) = q$.

For any $\varepsilon > 0$, we need to have an $N > 0$ such that $\forall n > N$, $d(f(p_n), q) < \varepsilon$. By definition of limit of a function, $\varepsilon > 0$ s.t. if $d_x(p_n, p) < \delta$, then $d(f(p_n), q) < \varepsilon$.

By $p_n \to p$, $\exists N > 0$ such that $\forall n > N$, $d(p_n, p) < \delta$. Hence, in summary, $\exists N > 0$ such that $\forall n > N$,

$$d(p_n, p) < \delta \implies d(f(p_n), q) < \varepsilon.$$

$\impliedby$: Suppose $lim_{x\to p} f(x) \neq q$. Then $\exists \varepsilon > 0$ such that $\forall \delta > 0$,

$$f\left(B^x_\delta(p) \cap E\right) \text{ is not contained in } B_\varepsilon(q)$$

That is, $\exists x \in E$, such that $0 < d_x(x, p) < \delta$ such that $d(f(x), q) > \varepsilon$

Let δ take values $\frac{1}{n}$ for $n \in \mathbb{N}$, and obtain a sequence of points x_n such taht $0 < d(x_n, p) < \frac{1}{n}$ and $d(f(x_n), q) > \varepsilon$. This contradicts with the statement that for all sequences $x_n \to p$, $x_n \neq p$, $f(x_n) \to q$.

Theorem Let $f, g : X \to R$. And assume that $\lim_{x\to p} f(x) = A$ and $\lim_{x\to p} g(x) = B$. Then

$$\begin{aligned} \lim(f+g)(x) &= A + B \\ \lim(f \cdot g)(x) &= AB \\ \lim(f/g)(x) &= A/B \end{aligned}$$

If $g(x) \neq 0 \forall x \in X$ and $B \neq 0$.

Definition (Continuity) Given $f : X \to Y$, f is continuous iff for any $p \in X'$ (a limit point of X) we have

$$f(p) = \lim_{x\to p} f(x).$$

i.e.

$$f(\lim_{x\to p} x) = \lim_{x\to p} f(x)$$

Theorem $f + g$, $f \cdot g$, f/g (if $g \neq 0$) are continuous if f, g are continuous.

Proof: Use above.

Theorem if $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous where $(g \circ f)(x) = f(g(x))$.

Proof: Just need to show that $\forall V \subset Z$ open, $(g \circ f)(V)$ is open.

$$\begin{aligned} g^{-1}(V) \text{ open } &\implies f^{-1}\left(g^{-1}(V)\right) \text{ open} \\ &= \implies (g \circ f)^{-1}(V) \text{ open} \\ &= \implies g \circ f \text{continuous.} \end{aligned}$$

Theorem If X, Y are topological spaces, then $X \times Y$ is a topological space with open sets generated by $U \times V$ where $U \subset X$, $V \subset Y$ open.

Given maps $f : Z \to X,\ g : Z \to Y,\ (f, g) : Z \to X \times Y.\ \ (f, g)(z) = (f(z), g(z)).\ \ (f, g)$ is continuous iff f, g are continuous.

Theorem Let $f : X \to \mathbb{R}^n$, with $f(x) = (f_1(x) \dots f_n(x))$, then f is continuous $\iff$ $f_i : X \to \mathbb{R}$ are continuous.