Variation of Parameters

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April 23, 2017

1 p. 347 Exercise 47

Find the particular solution of $y'' + 3y' + 2y = 4e^x$.

We start by finding the complementary solution y_c.

$$r^2 + 3r + 2 = 0$$

$$(r+2)(r+1) = 0 \implies r = -1, -2$$

$$y_c = c_1 e^{-x} + c_2 e^{-2x}$$

We then need to find the Wronskian.

$$W = \begin{bmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{bmatrix} = -e^{-3x}$$

With this, we can plug and chug into the Variation of Parameters Formula

$$y_p = -y_1 \int \frac{y_2 f(x)}{W} dx + y_2 \int \frac{y_1 f(x)}{W} dx$$

where y_1 and y_2 are the two solutions in y_c and $f(x)$ is the RHS of the DE.

$$\begin{aligned} y_p &= -e^{-x} \int \frac{4e^{-2x}e^x}{-e^{-3x}} dx + e^{-2x} \int \frac{4e^{-x}e^x}{-e^{-3x}} dx \\ &= 4e^{-x} \int e^{2x} dx - 4e^{-2x} \int e^3 x dx \\ &= 2e^{-x}e^{2x} - \frac{4}{3}e^{-2x}e^{3x} \\ &= 2e^x - \frac{4}{3}e^x \\ &= \boxed{\frac{2}{3}e^x} \end{aligned}$$

2 p. 347 Exercise 49

Find the particular solution of $y'' - 4y' + 4y = 2e^{2x}$.

Finding y_c:

$$r^2 - 4r + 4 = 0 \implies r = 2, 2$$
$$y_c = c_1 e^{2x} + c_2 x e^{2x}$$

Finding W:

$$W = \begin{bmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{bmatrix} = e^{4x}$$

Finding y_p:

$$\begin{aligned} y_p &= -y_1 \int \frac{y_2 f(x)}{W} dx + y_2 \int \frac{y_1 f(x)}{W} dx \\ &= -e^{2x} \int \frac{2xe^{2x}e^{2x}}{e^{4x}} dx + xe^{2x} \int \frac{2e^{2x}e^{2x}}{e^{4x}} dx \\ &= e^{2x} \int 2x dx + xe^{2x} \int 2dx \\ &= 2x^2e^{2x} - x^2e^{2x} \\ &= \boxed{x^2e^{2x}} \end{aligned}$$

3 p. 537 Exercise 58

Find the particular solution of $x^2y'' - 4xy' + 6y = x^3$.

Although the book gives us y_c, it's pretty easy to obtain it ourselves with the substitution $v = \ln x$ given that it's an Euler equation.

$$\begin{aligned} y'' + (-4 - 1)y' + 6y = 0&, \text{ prime} = d/dv \\ y'' - 5y' + 6y &= 0 \\ r^2 - 5r + 6 &= 0 \\ (r - 3)(r - 2) = 0 &\implies r = 2, 3 \end{aligned}$$

$$y_c = c_1 e^{2x} + c_2 e^{3x}$$

Now to apply Variation of Parameters we need to eliminate the leading coefficient in the given DE. So we divide by x^2:

$$y'' - \frac{4}{x}y' + \frac{6}{x^2}y = x$$

Now, we could plug and chug in order to find y_p and call it a day. However, it is useful to know the other (equivalent) way of doing Variation of Parameters, which is shown here.

We are going to find two functions, which we'll call u_1 and u_2, such that $y_p = u_1 y_1 + u_2 y_2$. To do this, we will solve the following system:

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= f(x) \end{aligned}$$

Note that in both this case and the case where we plug and chug, $f(x)$ refers to the RHS *after* we divide by the leading coefficient. Also recall that y_1 and y_2 refer to the solutions contained within y_c.

We now proceed to solve for u_1 and u_2.

$$x^2 u_1' + x^3 u_2' = 0 \tag{1}$$

$$2x u_1' + 3x^2 u_2' = x \tag{2}$$

From (1):

$$u_1' = -x u_2' \tag{3}$$

Substituting (3) into (2):

$$-2x^2 u_2' + 3x^2 u_2' = x \tag{4}$$

$$x^2 u_2' = x \tag{5}$$

$$u_2' = \frac{1}{x} \tag{6}$$

Substituting (6) into (3):

$$u_1' = -1 \tag{7}$$

To obtain u_1 and u_2, we simply integrate (6) and (7).

$$\begin{aligned} u_1 &= \int -1 dx = -x \\ u_2 &= \int \frac{1}{x} dx = \ln|x| \end{aligned}$$

We can now simply plug u_1 and u_2 into $y_p = u_1 y_1 + u_2 y_2$.

$$\boxed{y_p = -x^3 + x^3 \ln|x|}$$

4 p. 537 Exercise 60

Find the particular solution of $4x^2 y'' - 4xy' + 3y = 8x^{4/3}$.

Finding y_c:

$$\begin{aligned} 4y'' + (-4 - 4)y' + 3y &= 0, \text{ prime} = d/dv \text{ and } v = \ln x \\ 4y'' - 8y' + 3y &= 0 \\ 4r^2 - 8r + 3 &= 0 \implies r = 1/2, 3/2 \end{aligned}$$

So

$$\begin{aligned} y_c &= c_1 e^{v/2} + c_2 e^{3v/2} \\ y_c &= c_1\sqrt{x} + c_2 x^{3/2} \text{ (Yes, the book is wrong!)} \end{aligned}$$

Dividing the original DE by leading coefficient:

$$y'' - \frac{y'}{x} + \frac{3y}{4x^2} = 2x^{-2/3}$$

Solving for u_1 and u_2:

$$\sqrt{x}u_1' + x^{3/2}u_2' = 0 \tag{8}$$

$$\frac{u_1'}{2\sqrt{x}} + \frac{3\sqrt{x}u_2'}{2} = 2x^{-2/3} \tag{9}$$

From (8):

$$u_1' = -xu_2' \tag{10}$$

(10) $\rightarrow$ (9):

$$-\frac{\sqrt{x}u_2'}{2} + \frac{3\sqrt{x}u_2'}{2} = 2x^{-2/3} \tag{11}$$

$$\sqrt{x}u_2' = 2x^{-2/3} \tag{12}$$

$$u_2' = 2x^{-7/6} \tag{13}$$

$$u_2 = -12x^{-1/6} \tag{14}$$

(13) $\rightarrow$ (10):

$$\begin{aligned} u_1' &= -2x^{-1/6} \\ u_1 &= -\frac{12x^{5/6}}{5} \end{aligned}$$

Plugging into $y_p = u_1y_1 + u_2y_2$:

$$\begin{aligned} y_p &= -\frac{12x^{5/6}}{5}\sqrt{x} - 12x^{-1/6}x^{3/2} \\ &= -\frac{12x^{4/3}}{5} - 12x^{4/3} \\ &= \boxed{\frac{-72x^{4/3}}{5}} \end{aligned}$$