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Transforming the array manifold Jacobian from polar to Cartesian coordinates

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1. The FIM transform

- Let $\boldsymbol{\theta} = [\phi \ \theta \ r]$ denote the polar parameters *(azimuth, elevation, distance)* and $\boldsymbol{\eta} = [x \ y \ z]$ the corresponding coordinates in Cartesian space
- Assuming the FIM is expressed w.r.t. $\boldsymbol{\theta}$ and we want to transform it to $\boldsymbol{\eta}$, we could use the following transformation:[1]

$$\mathcal{J}(\boldsymbol{\theta}) = \mathcal{P}_{\boldsymbol{\eta}\boldsymbol{\theta}} \mathcal{J}(\boldsymbol{\eta}) \mathcal{P}_{\boldsymbol{\eta}\boldsymbol{\theta}}^{T}$$

$$\Updownarrow$$

$$\mathcal{J}(\boldsymbol{\eta}) = {\mathcal{P}_{\boldsymbol{\eta}\boldsymbol{\theta}}}^{-1} \mathcal{J}(\boldsymbol{\theta}) \mathcal{P}_{\boldsymbol{\eta}\boldsymbol{\theta}}^{-T}$$

[1] Andreas Richter, Estimation of Radio Channel Parameters: Models and Algorithms, 2005, p.57

1. The FIM transform

- Here the transform matrix is

$$\mathcal{P}_{\boldsymbol{\eta}\boldsymbol{\theta}} = \begin{bmatrix} \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \end{bmatrix}$$

- And the parameters are

$$x = r \cdot sin(\theta) cos(\phi)$$

$$y = r \cdot sin(\theta) sin(\phi)$$

$$z = r \cdot cos(\theta)$$

1. The FIM transform

- This leads to the final transformation matrix

$$\mathcal{P}_{\boldsymbol{\eta}\boldsymbol{\theta}} = \begin{bmatrix} -r \cdot sin(\theta) sin(\phi) & r \cdot sin(\theta) cos(\phi) & 0 \\ r \cdot cos(\theta) cos(\phi) & r \cdot cos(\theta) sin(\phi) & -r \cdot sin(\theta) \\ sin(\theta) cos(\phi) & sin(\theta) sin(\phi) & cos(\theta) \end{bmatrix}$$

- Now this can be applied to obtain the CRB in the Cartesian domain by taking the inverse of the transformed FIM

$$CRB = \boldsymbol{\mathcal{J}}(\boldsymbol{\eta})^{-1} = \mathcal{P}_{\boldsymbol{\eta}\boldsymbol{\theta}}{}^{T} \boldsymbol{\mathcal{J}}(\boldsymbol{\theta})^{-1} \mathcal{P}_{\boldsymbol{\eta}\boldsymbol{\theta}}$$

2. The Jacobian Transform - single-path

- Assume a transform of the array steering vector Jacobian is wanted (from $\boldsymbol{\theta}$ to $\boldsymbol{\eta}$), i.e.

$$\mathcal{D}(\boldsymbol{\eta}) = \mathcal{D}(\boldsymbol{\theta})\boldsymbol{Q}$$

- $\boldsymbol{Q}$ is the unkown transformation matrix
- And the Jacobians are

$$\mathcal{D}(\boldsymbol{\eta}) = \left[\frac{\partial \boldsymbol{b}(x,y,z)}{\partial x} \;\; \frac{\partial \boldsymbol{b}(x,y,z)}{\partial y} \;\; \frac{\partial \boldsymbol{b}(x,y,z)}{\partial z}\right]$$

$$\mathcal{D}(\boldsymbol{\theta}) = \left[\frac{\partial \boldsymbol{b}(\phi,\theta,r)}{\partial \phi} \;\; \frac{\partial \boldsymbol{b}(\phi,\theta,r)}{\partial \theta} \;\; \frac{\partial \boldsymbol{b}(\phi,\theta,r)}{\partial r}\right]$$

- Where $\boldsymbol{b}$ is the array steering vector

2. The Jacobian Transform - single-path

- Consider one of the partial derivatives of $\boldsymbol{\theta}$

$$\frac{\partial \boldsymbol{b}(\phi,\theta,r)}{\partial\phi} = \frac{\partial \boldsymbol{b}(x(\phi,\theta,r), y(\phi,\theta,r), z(\phi,\theta,r))}{\partial\phi}$$

- Using the chain rule yields

$$\begin{aligned}
\frac{\partial \boldsymbol{b}}{\partial\phi} &= \frac{\partial \boldsymbol{b}}{\partial x}\cdot\frac{\partial x}{\partial\phi} + \frac{\partial \boldsymbol{b}}{\partial y}\cdot\frac{\partial y}{\partial\phi} + \frac{\partial \boldsymbol{b}}{\partial z}\cdot\frac{\partial z}{\partial\phi} \\
&= \left[\frac{\partial \boldsymbol{b}}{\partial x}\ \frac{\partial \boldsymbol{b}}{\partial y}\ \frac{\partial \boldsymbol{b}}{\partial z}\right]\cdot\left[\frac{\partial x}{\partial\phi}\ \frac{\partial y}{\partial\phi}\ \frac{\partial z}{\partial\phi}\right]^T \\
&= \boldsymbol{\mathcal{D}}(\boldsymbol{\eta})\cdot\left[\frac{\partial x}{\partial\phi}\ \frac{\partial y}{\partial\phi}\ \frac{\partial z}{\partial\phi}\right]^T
\end{aligned}$$

2. The Jacobian Transform - single-path

- Applying the chain rule on each of the partial derivatives of $\boldsymbol{\theta}$ gives the following expression

$$\mathcal{D}(\boldsymbol{\theta}) = \mathcal{D}(\boldsymbol{\eta}) \cdot \begin{bmatrix} \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial r} \\ \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial r} \\ \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial r} \end{bmatrix}$$

$$= \mathcal{D}(\boldsymbol{\eta}) \cdot \mathcal{P}_{\boldsymbol{\eta}\boldsymbol{\theta}}{}^{T}$$

- And thus the wanted transformation is

$$\mathcal{D}(\boldsymbol{\eta}) = \mathcal{D}(\boldsymbol{\theta}) \cdot \mathcal{P}_{\boldsymbol{\eta}\boldsymbol{\theta}}{}^{-T}$$

- That is, the unkown matrix $\boldsymbol{Q}$ is the inverse transpose of $\mathcal{P}_{\boldsymbol{\eta}\boldsymbol{\theta}}$, the transform matrix derived for the FIM

2. The Jacobian Transform - multi-path

- In case of K paths (or sources) the array steering matrix is

$$\boldsymbol{B} = [\boldsymbol{b}_1 \ \dots \ \boldsymbol{b}_K]$$

- This yields a result similar to the previously derived single-path case:

$$\mathcal{D}(\boldsymbol{\theta}) = \left[\frac{\partial \boldsymbol{B}}{\partial \phi} \ \frac{\partial \boldsymbol{B}}{\partial \theta} \ \frac{\partial \boldsymbol{B}}{\partial r}\right] = \left[\frac{\partial \boldsymbol{B}}{\partial x} \ \frac{\partial \boldsymbol{B}}{\partial y} \ \frac{\partial \boldsymbol{B}}{\partial z}\right] \cdot \begin{bmatrix} \frac{\partial \boldsymbol{x}}{\partial \phi} & \frac{\partial \boldsymbol{x}}{\partial \theta} & \frac{\partial \boldsymbol{x}}{\partial r} \\ \frac{\partial \boldsymbol{y}}{\partial \phi} & \frac{\partial \boldsymbol{y}}{\partial \theta} & \frac{\partial \boldsymbol{y}}{\partial r} \\ \frac{\partial \boldsymbol{z}}{\partial \phi} & \frac{\partial \boldsymbol{z}}{\partial \theta} & \frac{\partial \boldsymbol{z}}{\partial r} \end{bmatrix}$$

$$\Rightarrow \mathcal{D}(\boldsymbol{\eta}) = \mathcal{D}(\boldsymbol{\theta}) \cdot \mathcal{P}_{\boldsymbol{\eta}\boldsymbol{\theta}}{}^{-T}$$

2. The Jacobian Transform - multi-path

- If the actual values of the parameters describing a specific path are

$$\boldsymbol{\theta}_k = [\phi_k \; \theta_k \; r_k], \; \textit{where } k = [1 \ldots K],$$

the partial derivatives contained in the transform matrix, $\mathcal{P}_{\boldsymbol{\eta}\boldsymbol{\theta}}$, are diagonal matrices of the following form:

$$\frac{\partial \mathbf{x}}{\partial \phi} = \begin{bmatrix} \frac{\partial x(\phi_1)}{\partial \phi} & 0 & \ldots & 0 \\ 0 & \frac{\partial x(\phi_2)}{\partial \phi} & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & \ldots & 0 & \frac{\partial x(\phi_K)}{\partial \phi} \end{bmatrix}$$

where the diagonal contains the partial derivative with respect to the parameter in question (here ϕ) evaluated at the actual parameter values

2. The Jacobian Transform - multi-path

- Thus, in the general (multi-path) reciever only case, the transform matrix, $\mathcal{P}_{\eta\theta}$, is a 3K-by-3K matrix consisting of 9 diagonal matrices, each corresponding to the partial derivative of a Cartesian parameter with respect to a polar parameter

3. The Jacobian transform for a MIMO system - single-path

- The previous cases assumed a single receiving array
- However, if the system is modeled with both a transmitting and a receiving array, the combined steering vector is the Kronecker product of the Tx and Rx steering vectors:

$$\boldsymbol{b}(\phi_R, \theta_R, r_R, \phi_T, \theta_T, r_T) = \boldsymbol{b}_R(\phi_R, \theta_R, r_R) \otimes \boldsymbol{b}_T(\phi_T, \theta_T, r_T)$$

- Here $\boldsymbol{\theta}_R = [\phi_R \ \theta_R \ r_R]$ *and* $\boldsymbol{\theta}_T = [\phi_T \ \theta_T \ r_T]$ define a single point in Cartesian space, $\boldsymbol{\eta} = [x \ y \ z]$

3. The Jacobian transform for a MIMO system - single-path

- The whole system's Jacobian is defined as

$$\mathcal{D}(\boldsymbol{\theta_R}, \boldsymbol{\theta_T}) = \left[\frac{\partial \boldsymbol{b_R} \otimes \boldsymbol{b_T}}{\partial \phi_R} \ \frac{\partial \boldsymbol{b_R} \otimes \boldsymbol{b_T}}{\partial \theta_R} \ \cdots \ \frac{\partial \boldsymbol{b_R} \otimes \boldsymbol{b_T}}{\partial \theta_T} \ \frac{\partial \boldsymbol{b_R} \otimes \boldsymbol{b_T}}{\partial r_T} \right],$$

  or more conveniently as

$$\mathcal{D}(\boldsymbol{\eta}) = \left[\frac{\partial \boldsymbol{b_R} \otimes \boldsymbol{b_T}}{\partial x} \ \frac{\partial \boldsymbol{b_R} \otimes \boldsymbol{b_T}}{\partial y} \ \frac{\partial \boldsymbol{b_R} \otimes \boldsymbol{b_T}}{\partial z} \right]$$

- If the Rx array has M_R elements and the Tx array M_T elements, the whole system's Cartesian Jacobian is a $M_T M_R$-by-3 matrix

3. The Jacobian transform for a MIMO system - single-path

- Using the chain rule, the Jacobian can be written as

$$\mathcal{D}(\eta) = \left[\frac{\partial \boldsymbol{b_R}}{\partial x} \otimes \boldsymbol{b_T} + \boldsymbol{b_R} \otimes \frac{\partial \boldsymbol{b_T}}{\partial x} \;\; \ldots \frac{\partial \boldsymbol{b_R}}{\partial z} \otimes \boldsymbol{b_T} + \boldsymbol{b_R} \otimes \frac{\partial \boldsymbol{b_T}}{\partial z} \;\right]$$

$$= \left[\frac{\partial \boldsymbol{b_R}}{\partial x} \; \frac{\partial \boldsymbol{b_R}}{\partial y} \; \frac{\partial \boldsymbol{b_R}}{\partial z}\right] \otimes \boldsymbol{b_T} + \boldsymbol{b_R} \otimes \left[\frac{\partial \boldsymbol{b_T}}{\partial x} \; \frac{\partial \boldsymbol{b_T}}{\partial y} \; \frac{\partial \boldsymbol{b_T}}{\partial z}\right]$$

$$= \mathcal{D}(\eta_{\boldsymbol{R}}) \otimes \boldsymbol{b_T} + \boldsymbol{b_R} \otimes \mathcal{D}(\eta_{\boldsymbol{T}})$$

- And the final transform becomes:

$$\mathcal{D}(\eta) = (\mathcal{D}(\theta_{\boldsymbol{R}}) \cdot \mathcal{P}_{\eta\theta_{\boldsymbol{R}}}{}^{-T}) \otimes \boldsymbol{b_T} + \boldsymbol{b_R} \otimes (\mathcal{D}(\theta_{\boldsymbol{T}}) \cdot \mathcal{P}_{\eta\theta_{\boldsymbol{T}}}{}^{-T})$$

3. The Jacobian transform for a MIMO system - multi-path

- In the multi-path case the combined steering matrix is the columnwise Kronecker product of the Tx and Rx steering matrices - that is, their Khatri-Rao product

$$\boldsymbol{B}(\phi_R, \theta_R, r_R, \phi_T, \theta_T, r_T) = \boldsymbol{B}_R(\phi_R, \theta_R, r_R) \diamondsuit \boldsymbol{B}_T(\phi_T, \theta_T, r_T)$$

- Using previous results, the general MIMO Jacobian transform can be expressed as

$$\boldsymbol{\mathcal{D}}(\boldsymbol{\eta}) = \boldsymbol{\mathcal{D}}(\boldsymbol{\eta_R}) \diamondsuit [\boldsymbol{B_T}\ \boldsymbol{B_T}\ \boldsymbol{B_T}] + [\boldsymbol{B_R}\ \boldsymbol{B_R}\ \boldsymbol{B_R}] \diamondsuit \boldsymbol{\mathcal{D}}(\boldsymbol{\eta_T})$$

where $\diamondsuit$ denotes the Khatri-Rao product and $\boldsymbol{B_T}$ / $\boldsymbol{B_R}$ are the Tx / Rx steering matrices

3. The Jacobian transform for a MIMO system - multi-path

- Proof:

$$\boldsymbol{B} = \boldsymbol{B_R} \diamondsuit \boldsymbol{B_T},$$

$$\Rightarrow \mathcal{D}(\eta) = \left[\frac{\partial \boldsymbol{B_R} \diamondsuit \boldsymbol{B_T}}{\partial x} \; \frac{\partial \boldsymbol{B_R} \diamondsuit \boldsymbol{B_T}}{\partial y} \; \frac{\partial \boldsymbol{B_R} \diamondsuit \boldsymbol{B_T}}{\partial z} \right]$$

$$= \left[\frac{\partial \boldsymbol{B_R}}{\partial x} \diamondsuit \boldsymbol{B_T} \quad \frac{\partial \boldsymbol{B_R}}{\partial y} \diamondsuit \boldsymbol{B_T} \quad \frac{\partial \boldsymbol{B_R}}{\partial z} \diamondsuit \boldsymbol{B_T} \right] +$$

$$\left[\boldsymbol{B_R} \diamondsuit \frac{\partial \boldsymbol{B_T}}{\partial x} \quad \boldsymbol{B_R} \diamondsuit \frac{\partial \boldsymbol{B_T}}{\partial y} \quad \boldsymbol{B_R} \diamondsuit \frac{\partial \boldsymbol{B_T}}{\partial z} \right]$$

$$= \left[\frac{\partial \boldsymbol{B_R}}{\partial x} \; \frac{\partial \boldsymbol{B_R}}{\partial y} \; \frac{\partial \boldsymbol{B_R}}{\partial z} \right] \diamondsuit \left[\boldsymbol{B_T} \; \boldsymbol{B_T} \; \boldsymbol{B_T} \right] +$$

$$\left[\boldsymbol{B_R} \; \boldsymbol{B_R} \; \boldsymbol{B_R} \right] \diamondsuit \left[\frac{\partial \boldsymbol{B_T}}{\partial x} \; \frac{\partial \boldsymbol{B_T}}{\partial y} \; \frac{\partial \boldsymbol{B_T}}{\partial z} \right] \square$$

4. Summary

- The transformation matrix:

$$\mathcal{P}_{\boldsymbol{\eta\theta}} = \begin{bmatrix} \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \end{bmatrix}$$

- The FIM transform:

$$\mathcal{J}(\boldsymbol{\eta}) = \mathcal{P}_{\boldsymbol{\eta\theta}}{}^{-1} \mathcal{J}(\boldsymbol{\theta}) \mathcal{P}_{\boldsymbol{\eta\theta}}^{-T}$$

- The Jacobian transform (receiver only):

$$\mathcal{D}(\boldsymbol{\eta}) = \mathcal{D}(\boldsymbol{\theta}) \cdot \mathcal{P}_{\boldsymbol{\eta\theta}}{}^{-T}$$

- The Jacobian transform (MIMO):

$$\mathcal{D}(\boldsymbol{\eta}) = \mathcal{D}(\boldsymbol{\eta_R}) \diamondsuit [\boldsymbol{B_T} \; \boldsymbol{B_T} \; \boldsymbol{B_T}] + [\boldsymbol{B_R} \; \boldsymbol{B_R} \; \boldsymbol{B_R}] \diamondsuit \mathcal{D}(\boldsymbol{\eta_T})$$